STAT 536 : L7

Last class :

Uniform → Gamma
→ Chi-sq
→ Beta-dist .

Uniform → Normal dist
(Box mueller transformation).

→ Inverse method
→ Tranformation method.

<u>Note</u>: Recall that if $Z \sim N(0,1)$

then for any μ, $\sigma^2 > 0$ then .

$$X = \mu + \sigma Z \sim N(\mu, \sigma^2).$$

∴ to simulate a $N(\mu, \sigma^2)$ dist for any μ, σ^2 it is sufficient to simulate a $N(0,1)$.

Generating a multivariate normal.
distribution:

Note: if $X = (X_1, X_2 \ldots X_p)' \sim N_p(\mu, \Sigma)$

where $\mu = (\mu_1 \ldots \mu_p)'$

transpose.

$\Sigma_{p \times p}$

$\Sigma \Rightarrow$ positive definite.

Now, we also have ~~chto~~ choleski
decomposition

$$\Sigma = \Gamma \Gamma'$$

[we can find a Γ so that this relation holds],

[in R: $chol(\Sigma) = \Gamma$]

if we have

$$Z = (Z_1 \cdots Z_p) \sim N(0, I_{\cdot}).$$

$$I_{p \times p} \rightarrow \text{identity matrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix}$$

then

$$\underset{\cdot}{\mu} + \underset{\sqcup}{\Gamma} \underset{\cdot}{Z} \sim N(\mu, \Gamma\Gamma').$$

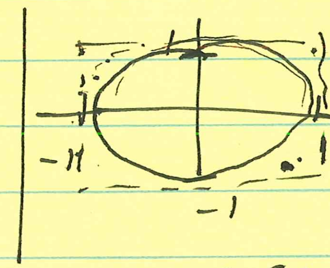
$$= N(\mu, \Sigma).$$

~~Marsat~~

Marsaglia's polar method (alternative to Box-Mueller).

i) Generate $U_1, U_2 \overset{i.i.d}{\sim} U(-1,1)$.

Until
$$S = U_1^2 + U_2^2 < 1$$



$U_1 = 0.91$

$U_2 = -0.92$

ii) Define.

$$X_1 = \sqrt{-2\log S}\ \frac{U_1}{\sqrt{S}}$$

$$X_2 = \sqrt{-2\log S}\ \frac{U_2}{\sqrt{S}}$$

then X_1 and X_2 are independent $N(0,1)$ r.v.'s.

→ Proof: Extra credit HW.

→ This method may be slighty more computationally efficient.

→ avoids computing cos, sin.

(1) Inverse method
(2) Transformation method.
(3) Accept - reject methods (AR).

Accept reject Algorithm

Th: Let $f_X(x)$ be a density from which we want to simulate r.v.'s.

Let $g_Y(y)$ be another density from which we know how to simulate r.v.'s, such that.

$$f(x) \le M g(x) \quad \forall\, x \text{ in the support of } f.$$

(1) — $\left[\text{i.e. } \sup\limits_{\substack{x \\ f(x)>0}} \dfrac{f(x)}{g(x)} \le M \right].$

$M \to$ Constant.

then the AR algorithm is given as

i) Generate $X \sim g$, $U \sim U(0,1)$.

ii) Accept $Y = X$ if $U \leq \frac{f(X)}{M g(X)}$.

iii) Return to (i) otherwise.

Produces a random Variable Y distributed according to f.

Note 1: What happen if support of f is not included in support of g?

→ Cannot use AR.

$$\begin{cases} f \sim N(0,1) \\ g \sim U(-1,1) \end{cases}$$

$x = 2$ $\quad \frac{f(x)}{g(x)}$?

If

Support of $g \supseteq$ support of f.

[otherwise the relation ① cannot be satisfied].

$f \rightarrow$ target density

$g \rightarrow$ instrumental density

Note 2: Although AR algorithm is valid for any constant M that satisfies ①, however choosing M to be too large may result in an inefficient algorithm.

Ex: Simulate Gamma from Gamma:

$X \sim \text{Gamma}(\alpha, \beta)$.

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0.$$

Case I) if α is an integer:

$U_1, \ldots, U_\alpha \overset{iid}{\sim} U(0,1)$.

$$Y = -\beta \sum_{j=1}^{\alpha} \log U_j \sim \text{Gamma}(\alpha, \beta).$$

Case II) if α is not an integer:

choose the instrumental density to be

Gamma(a, b). $\sim g$.

with $a = \lfloor \alpha \rfloor \rightarrow$ floor function.

Then we want to simulate

$Y \sim$ Gamma(α, β). $\sim f$

To implement AR alg. we need an upper bound for $h(x) = \frac{f(x)}{g(x)}$.

$$\log h(x) = c + (\alpha - a)\log x - (\beta - b)x.$$

$$\frac{d}{dx}\log h(x) = \frac{(\alpha - a)}{x} - (\beta - b)$$

Set $= 0$ and solve

$$x = \frac{(\alpha - a)}{(\beta - b)}$$

$$h(x) \leq \left.\frac{f(x)}{g(x)}\right|_{x = \frac{(\alpha - a)}{(\beta - b)}}$$

$$= \underbrace{\frac{\beta^{\alpha}}{\Gamma(\alpha)}\Gamma(a)\, b^{-a}\left(\frac{\alpha - a}{\beta - b}\right)^{(\alpha - a)} e^{-(\alpha - a)}}.$$

$$= \underbrace{M(b)}$$