STAT 536: Class 2.

Last class: History of MCMC

Two main objectives of this class are:

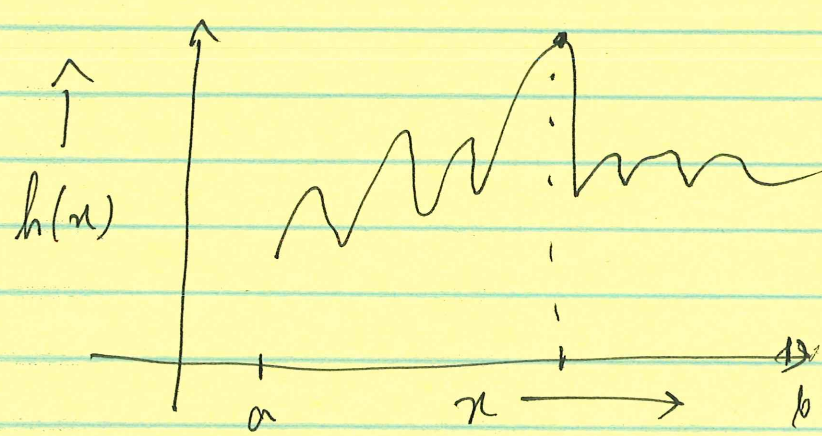
1) Evaluate integrals numerically:

e.g: $h(x) = (\cos 50x + \sin 50x)^2$.

$$\int_a^b h(x)\,dx = ?$$

2) Optimization:

$h(x) \rightarrow$ Same as above.



to compute $\max_{a \le x \le b} h(x)$. or

$$\underset{a \le x \le b}{\operatorname{argmax}}\ h(x).$$

These objectives are deterministic. however, we can induce randomness to achieve these objectives.

<u>Tool</u>: to be able to sample from any given distribution.

Review of Basics:

1) Random Variable: a variable whose possible values are outcomes of a random experiment.

e.g: Coin toss:

$$X = \begin{cases} 0 & \text{tails} \\ 1 & \text{heads.} \end{cases}$$

$$P(X=0) = \frac{1}{2} \qquad P(X=1) = \frac{1}{2}.$$

2) Cumulative distribution function (cdf):

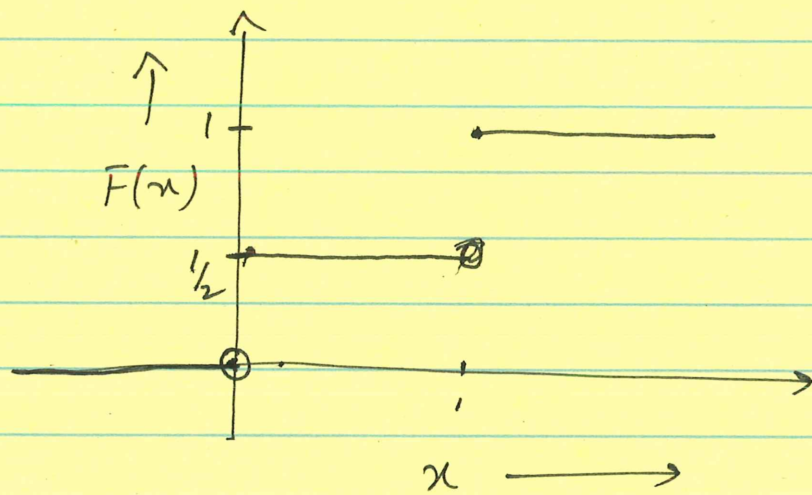
for a r.v. X, the cdf

$$F(x) = P[X \leq x]$$

$x \to$ any given number.

e.g : Coin Toss :

$$X = \begin{cases} 0 & \text{w.p } 1/2 \\ 1 & \text{w.p } 1/2 \end{cases}$$



$$F(x) = \begin{cases} 0 & x < 0 \\ 1/2 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

Two key properties of cdf:

a) Non-decreasing:

if $x_1 \leq x_2$

then $F(x_1) \leq F(x_2)$

b) Right-continuous:

$$\lim_{x \to x_0^+} F(x) = F(x_0).$$

3) Probability density function (pdf):

~~ia~~ For any cdf F, a function f such that

$$\left.\begin{aligned} F(x) &= \int_{-\infty}^{x} f(x)\,dx \\ \text{and } \int_{-\infty}^{\infty} f(x)\,dx &= 1 \end{aligned}\right\} \text{when } X \text{ is continuous}$$

$$F(x) = \sum_{-\infty}^{x} f(x). \quad \text{and}$$

$$\left.\sum_{-\infty}^{\infty} f(x) = 1 \quad \right\} X \text{ is discrete.}$$

4). Mean and Variance of a r.v. X:

(First moment, population mean, expected Value)

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx. \rightarrow X \text{ is continuous}$$

$$= \sum_{-\infty}^{\infty} x f(x) \rightarrow X \text{ is discrete}$$

Mean of a function of X:

$$E(h(X)) = \int_{-\infty}^{\infty} h(x) f(x)\, dx \, .$$

$\rightarrow$ X is cont.

$$= \sum_{-\infty}^{\infty} h(x) f(x) \cancel{dx}$$

$\rightarrow$ X is discrete.

Variance of X:

Var (X)

$$V(X) = E(X-\mu)^2 \quad : \mu = E(X)$$

$$= EX^2 - (EX)^2$$

Some common distributions:

1) Discrete distributions:

(a) Bernoulli distribution (p)

$$X = \begin{cases} 0 & \text{w.p.} \quad (1-p) \\ 1 & \text{w.p.} \quad p \end{cases}$$

$P(X=0) = (1-p)$, $P(X=1) = p$

$E(X) = \sum x f(x)$

$= 0(1-p) + 1(p)$

$= p$

$Var(X) = EX^2 - (EX)^2$

$= p - p^2$

$= p(1-p)$.

$EX^2 =$

$0^2(1-p) + 1^2(p)$

$= p$

b) Poisson distribution (λ)

$$X \sim P(\lambda)$$

$$X \in \{0, 1, 2 \ldots \infty\}.$$

$$f(x) = P[X = x] = \underbrace{\frac{\lambda^x e^{-\lambda}}{x!}}$$

$$E(X) = \sum_{0}^{\infty} x f(x)$$

$$= \sum_{0}^{\infty} x \cdot \frac{\lambda^x e^{-\lambda}}{x!}$$

$x! = x(x-1)!$

$$= \sum_{1}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x-1)!}$$

$$= \lambda \underbrace{\sum_{1}^{\infty} \frac{\lambda^{(x-1)} e^{-\lambda}}{(x-1)!}}_{1} = \lambda$$

$V(X) = \lambda \rightarrow$ HW.

c) Binomial distribution :

$$X \sim Bin(n, p).$$

$$B(n, p)$$

$$P[X = x] = \binom{n}{x} p^x (1-p)^{n-x}$$

$X \rightarrow$ Counts the number of successes in n independent trials, where $p \rightarrow$ probability of each success.

e.g: toss two coins :

$X \to$ number of heads :

$X \in \{0, 1, 2\}$.

$$X \sim Bin\left(2, \frac{1}{2}\right)$$

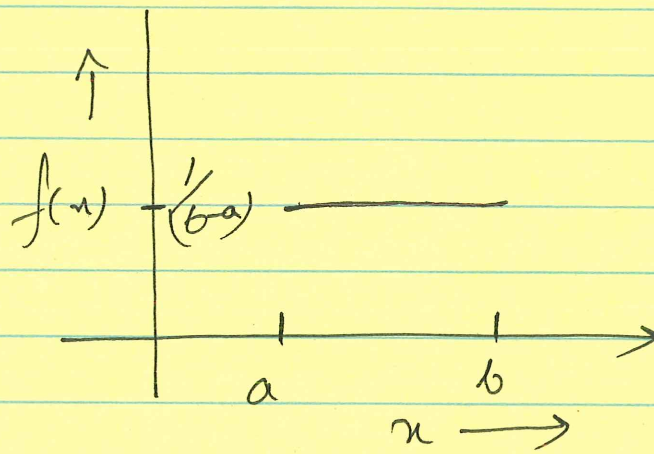
$E(X) = np$, $V(X) = np(1-p)$.

2) Continuous distributions :

(a) Uniform distribution :

$$X \sim U(a, b).$$

$$f(x) = \begin{cases} \dfrac{1}{(b-a)} & a < x < b. \\ 0 & \text{else} \end{cases}$$



$$E(X) = \frac{a+b}{2}, \quad V(X) = \frac{(b-a)^2}{12}.$$

(b) Normal distribution

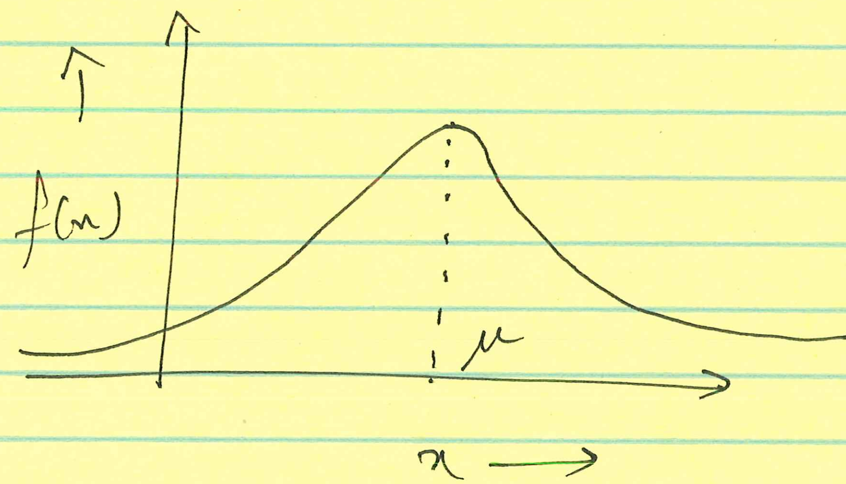
$$X \sim N(\mu, \sigma^2).$$

$$-\infty < \mu < \infty$$

$$\sigma^2 > 0$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right].$$

$$-\infty < x < \infty$$



standard normal dist : $N(0,1)$

useful property:

1) if $Z \sim N(0,1)$ then

$$\mu + \sigma Z \sim N(\mu, \sigma^2).$$

2) $X \sim N(\mu, \sigma^2)$

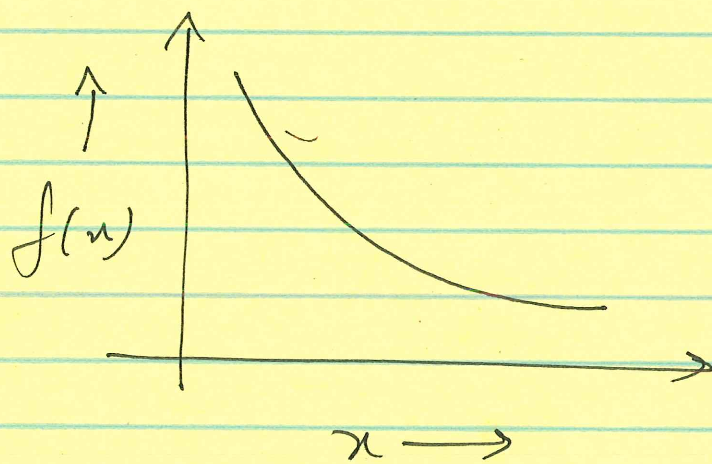
$$E(X) = \mu, \quad V(X) = \sigma^2.$$

c) Exponential distribution:

$X \sim Exp(\lambda)$, $X \in (0, \infty)$

$\lambda > 0$, $supp(X) = (0, \infty)$.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , \quad x > 0 \\ 0 & \quad \text{else} \end{cases}$$



$$E(X) = \frac{1}{\lambda} \quad , \quad V(X) = \frac{1}{\lambda^2}$$