STAT 536

→ HW1 is on BB.

→ Missed last friday:

→ wed: Lab.

→ Friday: 1hr → Lec.
2 hr → Quiz 1

Last class:
→ AR algorithm.

i) $X \sim g$, $U \sim U(0,1)$

ii) $Y = X$ if $U \leq \frac{f(x)}{Mg(x)}$.

iii) otherwise revert to (i).

Ex: Simulate Gamma from Gamma.

Objective : simulate $Ga(\alpha, \beta)$
$\sim f$: target density
$\alpha \to$ not an integer.

$g \sim Ga(a, b)$: instrument.

$$a = \lfloor \alpha \rfloor$$

$$h(x) = \underbrace{\frac{f(x)}{g(x)}}. \quad : \quad x = \frac{\alpha - a}{\beta - b}$$

$$h(x) \le M(b)$$

$$= \underbrace{\frac{\beta^\alpha}{\Gamma(\alpha)} \Gamma(a)\, b^{-a} \left(\frac{\alpha - a}{\beta - b}\right)^{\alpha - a} e^{-(\alpha - a)}}$$

we want to choose b so that

$\underline{\underline{M(b)}}$ is smallest.

$\log \underline{\underline{M(b)}}$ $\qquad \frac{d}{db} \log M(b) = 0.$

$\rightarrow$ Solve for b.

$$b = \frac{a\beta}{\alpha}$$

$$M(b)\Big|_{b=\frac{a\beta}{\alpha}} = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \Gamma(a) \cdot \left(\frac{a\beta}{\alpha}\right)^{-a} \left(\frac{\alpha-a}{\beta-b}\right)^{\alpha-a} e^{-(\alpha-a)}.$$

$$\leq \beta^{\alpha} \left(\frac{a\beta}{\alpha}\right)^{-a} \left(\frac{\alpha-a}{\beta-b}\right)^{\alpha-a} e^{-(\alpha-a)}$$

$$\left[\, \Gamma(a) \leq \Gamma(\alpha) \,\right]$$

$$a = \lfloor \alpha \rfloor$$

(target)

Ex: Normal distribution ($N(0,1)$) from double exponential dist (Laplace dist) (instrument):

$$X \sim DE(\alpha), \quad \alpha > 0$$

$$g(x) = \frac{\alpha}{2} \exp\left(-\alpha |x|\right), \quad x \in (-\infty, \infty).$$

Result: X_1 and X_2 are ind $Exp(\alpha)$

$$X_1 - X_2 \sim DE(\alpha)$$

→ HW2.

→ use AR algorithm to simulate from the target ($N(0,1)$)

$Y \sim N(0,1)$.

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right].$$

$$h(x) = \frac{f(x)}{g(x)} = \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-x^2}{2}\right] \cdot \frac{2}{\alpha} e^{\alpha|x|}$$

Maximize $\log h(x)$ in x.

$$\max_x h(x) = \sqrt{\frac{2}{\pi}}\ \alpha^{-1} e^{\alpha^2/2}$$

$$= M(\alpha).$$

minimize $M(\alpha)$ in α .

$\rightarrow$ derivative and set to 0.

$$\min_{\alpha} M(\alpha) = \sqrt{\frac{2e}{\pi}} \quad [$$

Proof of AR algorithm:

To show that $Y \sim f$

equivalently $P[Y \leq y] = \int_{-\infty}^{y} f(x)\, dx$.

$$\underbrace{P[Y \le y]}_{} = P\left[X \le y \;\middle|\; U \le \frac{f(X)}{Mg(X)}\right].$$

$$= \frac{P\left[X \le y,\ \underbrace{U \le \frac{f(X)}{Mg(X)}}_{}\right]}{P\left[U \le \frac{f(X)}{Mg(X)}\right]}.$$

$$\left[P[A|B] = \frac{P[A \cap B]}{P(B)}\right]$$

$$= \frac{\displaystyle\int_{-\infty}^{y} \underbrace{\int_{0}^{f(x)/Mg(x)} du}_{}\ \underline{g(x)\,dx}.}{\displaystyle\int_{-\infty}^{\infty} \int_{0}^{f(x)/Mg(x)} du\ g(x)\,dx.}$$

$$= \int_{-\infty}^{y} \frac{f(x)}{\cancel{M} \cancel{g(x)}} \cancel{g(x)}\, dx .$$

$$\frac{\displaystyle\int_{-\infty}^{y} \frac{f(x)}{M g(x)} g(x)\, dx}{\frac{1}{M}\left[\displaystyle\int_{-\infty}^{\infty} f(x)\, dx\right]} .$$

$$= \frac{\frac{1}{\cancel{M}} \displaystyle\int_{-\infty}^{y} f(x)\, dx}{\frac{1}{\cancel{M}}} .$$

$$= \underbrace{\int_{-\infty}^{y} f(x)\, dx}$$

$$\therefore P[Y \leq y] = \int_{-\infty}^{y} f(x)\, dx .$$

$$Y \sim f$$

Two consequences of the AR algorithm:

1) provides a method to simulate any distribution know upto its normalising constant.

Suppose $\underline{f(x)} = \underline{c_1 f_1(x)}$

Suppose.

$$\underbrace{\frac{f_1(x)}{g(x)} \le M}$$

then $\frac{f(x)}{g(x)} \le C \cdot M = M_2$.

e.g: $N(0,1)$.

$$f(x) = \underbrace{\frac{1}{\sqrt{2\pi}}} e^{-x^2/2}$$

$$\propto e^{-x^2/2}$$

$$\sqrt{2\pi} = \int e^{-x^2/2} dx$$

$$\therefore \underline{\underline{\frac{f(x)}{M_2\, g(x)}}} = \frac{c \cdot f_1(x)}{c \cdot M \cdot g(x)} = \frac{f_1(x)}{M g(x)}.$$

$\therefore$ we can implement AR algorithm

i) $X \sim g$, $U \sim U(0,1)$.

ii) $Y = X$ if $U \leq \dfrac{f_1(x)}{M g(x)}$.

where
$$\sup_{\substack{x \\ f(x)>0}} \frac{f_1(x)}{g(x)} \leq M.$$

iii) otherwise return to (i)