

STAT 536

→ HW1 is on BB.

→ Missed last Friday:

→ wed: lab.

→ Friday: 1 hr \rightarrow dec.

2 hr \rightarrow Quiz 1

Last class:

→ AR algorithm.

i) $X \sim g$, $U \sim U(0,1)$

ii) $Y = X$ if $U \leq \frac{f(x)}{Mg(x)}$.

iii) otherwise revert to (i).

Ex: Simulate Gamma from Gamma.

Objective : simulate $\text{Ga}(\alpha, \beta)$
 $\sim f$: target density
 $\alpha \rightarrow$ not an integer.

$g \sim \text{Ga}(\alpha, \beta)$: instrument.

$$a = L\alpha \lfloor$$

$$h(x) = \frac{f(x)}{g(x)} \quad : \quad x = \frac{\alpha - a}{\beta - b}$$

$$h(x) \leq M(b)$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \left[\Gamma(\alpha) b^{-\alpha} \left(\frac{\alpha - a}{\beta - b} \right)^{\alpha - a} e^{-(\beta - b)} \right]$$

we want to choose b so that
 $\underline{\underline{M(b)}}$ is smallest.

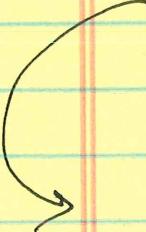
$$\log \underline{\underline{M(b)}} \quad \frac{d}{db} \log M(b) = 0 .$$

→ solve for b .

$$b = \frac{\alpha \beta}{\alpha}$$

$$M(b) \Big|_{b=\frac{\alpha \beta}{\alpha}} = \frac{\beta^\alpha}{\Gamma(\alpha)} \Gamma(\alpha) \cdot \left(\frac{\alpha \beta}{\alpha}\right)^{-\alpha}$$

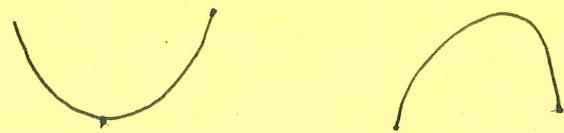
$$\left(\frac{\alpha-a}{\beta-b}\right)^{\alpha-a} e^{-(\alpha-a)} .$$



$$\leq \beta^\alpha \left(\frac{\alpha}{\alpha}\right)^{-\alpha} \left(\frac{\alpha-a}{\beta-b}\right)^{\alpha-a} e^{-(\alpha-a)}$$

$$[\Gamma(a) \leq \Gamma(\alpha)]$$

$a = \lfloor \alpha \rfloor$



(target)
Ex: Normal distribution ($N(0,1)$) from
double exponential dist (Laplace
(instrument) dist):

$$x \sim \underline{DE(\alpha)}, \quad \alpha > 0$$

$$g(x) = \frac{\alpha}{2} \exp(-\alpha|x|),$$

$x \in (-\infty, \infty).$

Result: ~~X_1 and X_2 are ind~~ $\exp(\alpha)$

$$\underline{x_1 - x_2 \sim DE(\alpha)}$$

\rightarrow HW 2.

→ use AR algorithm ~~to~~ simulate
from the target $N(0, 1)$

$$Y \sim N(0, 1).$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

$$h(x) = \frac{f(x)}{g(x)} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{2}{\alpha} e^{\alpha/x}$$

Maximize $\log h(x)$ in x .

$$\max_x h(x) = \sqrt{\frac{2}{\pi}} \alpha^{-1} e^{-\alpha^2/2}$$
$$= M(\alpha).$$

minimize $M(\alpha)$ in α

→ derivative and set to 0.

$$\min_{\alpha} M(\alpha) = \sqrt{\frac{2e}{\pi}} \quad [$$

Proof of AR algorithm:

To show that $Y \sim f$

equivalently $P[Y \leq y] = \int_{-\infty}^y f(x) dx$.

$$\underline{P[Y \leq y]} := P\left[X \leq y \mid U \leq \frac{f(x)}{Mg(x)}\right].$$

$$= P\left[X \leq y, \underbrace{U \leq \frac{f(x)}{Mg(x)}}_{\text{---}}\right]$$

$$\overbrace{\quad\quad\quad}^{\text{---}} P\left[U \leq \frac{f(x)}{Mg(x)}\right].$$

$$[P[A|B] = \frac{P[A \cap B]}{P(B)}]$$

$$= \int_{-\infty}^y \int_0^{f(n)/Mg(n)} du g(n) dx.$$

$$\overbrace{\quad\quad\quad}^{\text{---}} \int_{-\infty}^0 \int_0^{f(n)/Mg(n)} du g(n) dx.$$

$$= \int_{-\infty}^y \frac{f(u)}{\cancel{Mg(u)}} g(u) du .$$

—————
 M. $\int_{-\infty}^{\infty} f u g du .$
 [—————]

$$= \cancel{\frac{1}{M}} \int_{-\infty}^y f(u) du .$$

—————
 M.
 $\int_{-\infty}^y f(u) du .$

$$\therefore P[Y \leq y] = \int_{-\infty}^y f(u) du .$$

$$Y \sim f$$

Two consequences of the AR algorithm:

- 1) provides a method to simulate any distribution known up to its normalising constant.

Suppose $\underline{f(n)} = \frac{c_i f_i(n)}{g(n)}$

Suppose

$$\frac{f_i(n)}{g(n)} \leq M.$$

$$g(n)$$

then $\frac{f(n)}{g(n)} \leq c \cdot M = M_2$

e.g.: $N(0,1)$.

$$f(n) = \frac{1}{\sqrt{2\pi}} e^{-\frac{n^2}{2}}$$

$$\propto e^{-\frac{n^2}{2}}$$

$$\sqrt{2\pi} = \int e^{-\frac{n^2}{2}} dn$$

$$\therefore \frac{\underline{f(n)}}{\underline{Mg(n)}} = \frac{c \cdot \underline{f_1(n)}}{c \cdot M \cdot \underline{g(n)}} = \frac{\underline{f_1(n)}}{\underline{Mg(n)}}.$$

\therefore we can implement AR algorithm

i) $X \sim g$, $U \sim U(0, 1)$.

ii) $Y = X$ if $U \leq \frac{\underline{f_1(n)}}{\underline{Mg(n)}}$.

where

$$\sup_{\substack{n \\ f(n) > 0}} \frac{\underline{f_1(n)}}{\underline{g(n)}} \leq M.$$

iii) otherwise return to (i)