STAT 536:

Last class:

→ Probability integral transform
→ Transformation methods.
→ some properties of gamma dist.

Common transformations:

(1) Gamma dist.

$\alpha \to$ positive integer,
$\beta \to$ positive number.

if $U_1, U_2 \ldots U_\alpha \sim U(0,1)$.

then
$$Y = -\beta \sum_{j=1}^{\alpha} \log(U_j)$$
$$\sim Ga(\alpha, \beta)$$

~~Recat~~
$U \sim U(0,1)$
$(1-U) \sim U(0,1)$

2) chi-sq dist:

$\nu \to$ integer(+)

if $\underline{U_1, U_2 \ldots U_\nu} \sim U(0,1)$.

then $Y = -2 \sum_{j=1}^{\nu} \log(U_j)$

$\sim \chi^2_{2\nu}$

3) Beta distribution (a, b).

$a, b \to (+)$ integers.

$\underline{U_1, U_2 \ldots\ldots U_{a+b} \sim U(0,1)}$.

$$Y = \frac{\sum_{j=1}^{a} \log(U_j)}{\underbrace{\sum_{j=1}^{a+b} \log(U_j)}_{\sum_{j=1}^{a} \log U_i \;+\; \sum_{j=a+1}^{b} \log(U_i)}} \sim \text{Beta}(a, b).$$

Generating normal r.v.'s:

Box Mueller transformation:

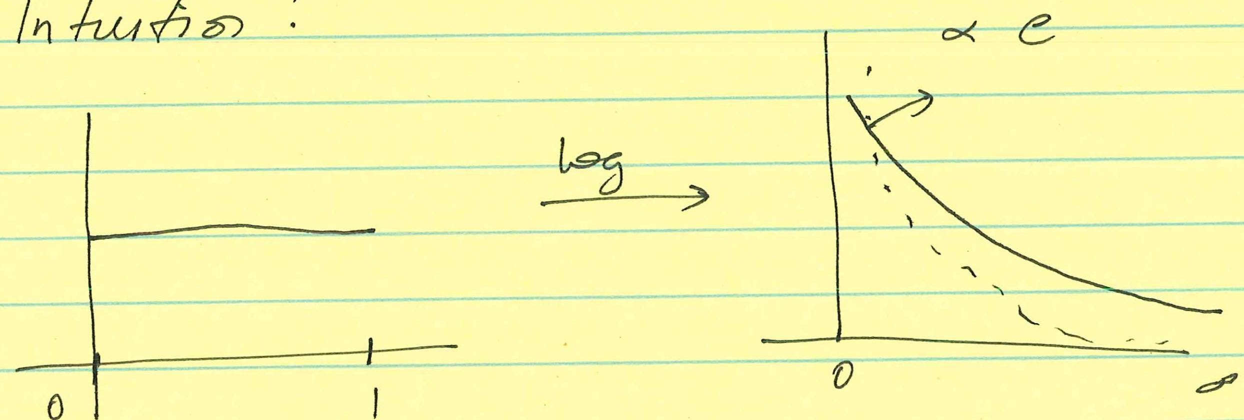
Th: Suppose U_1 and U_2 are independent $U(0,1)$ r.v.'s

Let $X_1 = \sqrt{-2\log U_1}\ \cos(2\pi U_2)$

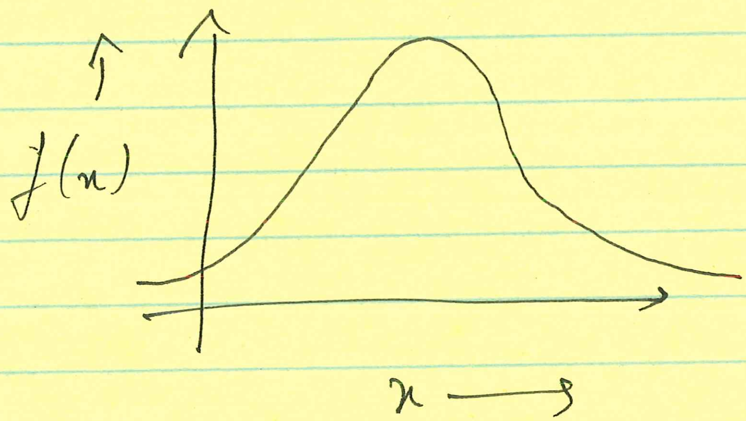
$X_2 = \sqrt{-2\log U_1}\ \sin(2\pi U_2)$.

then (X_1, X_2) are independent r.v's. with standard normal distribution.

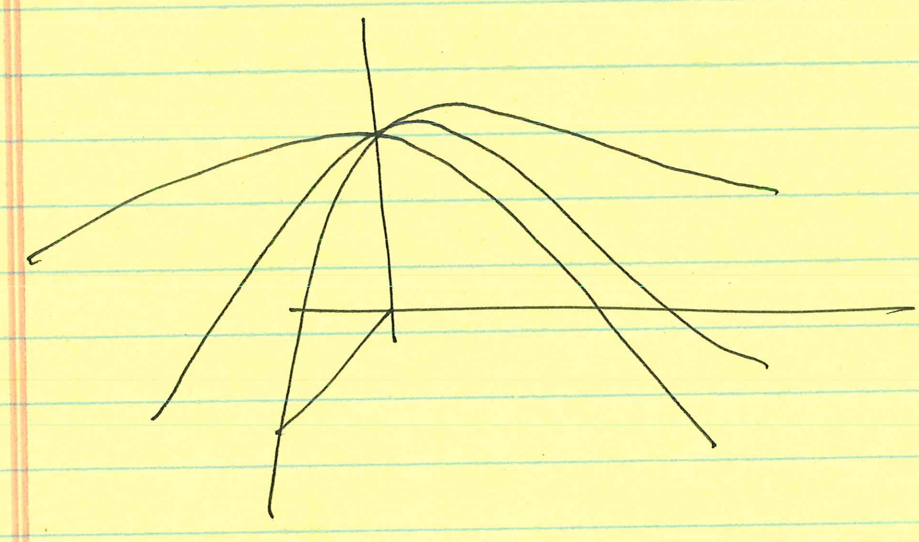
Intuition:



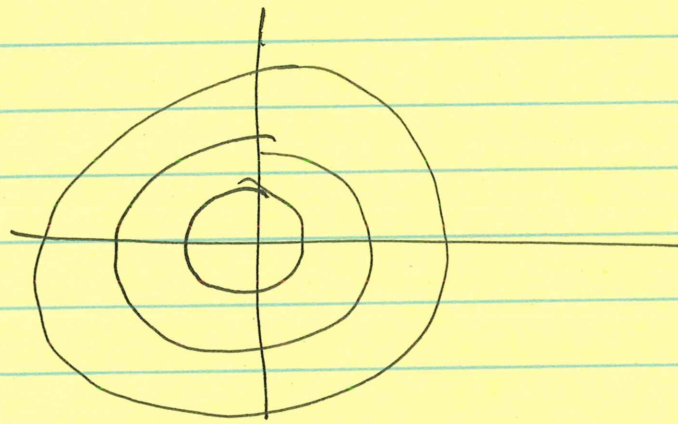
Normal dist (1d) $N(0,1)$.



(2d) : $N_2(0, I_{2\times2})$



Contour plot (under independent)

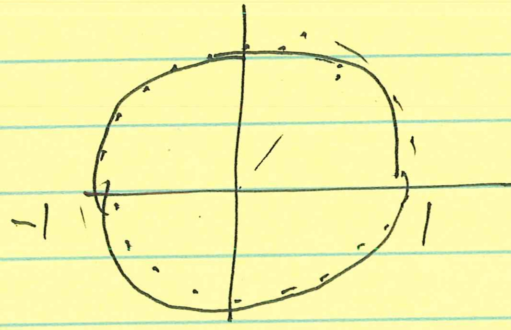


Now,

$0 < a_1 < a_2 \ldots a_n < 1$.

$x_i = \log(2\pi a_i)$ $\quad y_i = \sin(2\pi a_i)$.

plot (x_i, y_i).



$$f(u) = \{ 1 \quad u \in (0,1) \}.$$

Proof: $U_1 \sim U(0,1)$, $U_2 \sim U(0,1)$.

U_1, U_2 are independent.

$$f_{U_1, U_2}(u_1, u_2) = 1 \quad , u_1, u_2 \in (0,1)$$

Jacobian method: (method that allows us to compute density of a transformed r.v.)

$$f_{X_1, X_2}(x_1, x_2) = |J| \, f_{U_1, U_2}\left(g_1^{-1}(x_1, x_2), g_2^{-1}(x_1, x_2) \right).$$

where

$$\rightarrow \quad J = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{bmatrix}.$$

and $u_1 = g_1^{-1}(x_1, x_2)$

$u_2 = g_2^{-1}(x_1, x_2)$.

$$X_1 = \sqrt{-2\log U_1} \cdot \cos(2\pi U_2)$$

$$X_2 = \sqrt{-2\log U_1} \cdot \sin(2\pi U_2).$$

$$X_1^2 + X_2^2 = -2\log U_1$$

$$U_1 = \exp\left[-\frac{(X_1^2 + X_2^2)}{2}\right]$$

$X_1^2 = -2\log U_1 \cos^2(2\pi U_2)$ $\quad X_2^2 = -2\log U_1 \sin^2 2\pi U_2$.

$= -2\log U_1 \left[\cos^2(\,) + \sin^2(\,)\right]$

and

$$\frac{X_2}{X_1} \; \cancel{\frac{X_2}{X_1}} = \tan\left(2\pi U_2\right)$$

$$\underline{U_2 = \frac{1}{2\pi} \tan^{-1}\left(\frac{X_2}{X_1}\right)}$$

upon computing derivatives :

$$J = \begin{bmatrix} -x_1 \exp\left[-\frac{(x_1^2 + x_2^2)}{2}\right] , & -x_2 \exp\left[\quad\right] \\ -\frac{1}{2\pi} \frac{x_2}{x_1^2 + x_2^2} , & \frac{1}{2\pi} \frac{x_1}{x_1^2 + x_2^2} \end{bmatrix}$$

$$|J| = \frac{1}{2\pi} \frac{(x_1^2 + x_2^2)}{(x_1^2 + x_2^2)} \cdot \exp\left[-\frac{(x_1^2 + x_2^2)}{2}\right].$$

$$f_{X_1, X_2}(x_1, x_2) = \frac{1}{2\pi} \exp\left[-\frac{(x_1^2 + x_2^2)}{2}\right].$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x_1^2}{2}\right] \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x_2^2}{2}\right]$$

$\sim N(0,1)\ N(0,1)$

$\rightarrow$ Joint density of ind. std. normal r.v.'s.

$\cancel{X}$

$Y \cancel{= f(X)}$

$\downarrow$

use

$X \sim f(x)$

$Y = g(X).$

use Jacobian method to compute density of Y from the density of X $(f(x))$.