# STAT 536 : Class 5

Last class : → Generating random Variables.
→ Probability integral transform

$U \sim U(0,1)$.

$X = F^{-1}(U)$, then $X \sim F(x)$.

Ex: $U \sim U(0,1)$.

$$X = -\frac{1}{\lambda}\log(1-U).$$

then $X \sim Exp(\lambda)$.

Proof: To show:
$X = F^{-1}(U)$, then

$$F(x) = P[\underbrace{X}_{F^{-1}(U)} \le x] = P[U \le F(x)].$$

In other words,

$$\underbrace{\{(u,x) : F^{-1}(u) \le x\}}_{A} = \underbrace{\{(u,x) : F(x) \ge u\}}_{B}.$$

[Two sets $A$ and $B$, T.S. $A = B$].

i.e $\underline{A \subseteq B}$

$B \subseteq A$

(i) $A \subseteq B$

$\underline{(u_1, x_1)} \in A = \{(u,x) : \underline{F^{-1}(u) \le x}\}$.

$$F^{-1}(u_1) \le x_1$$

$$\underline{F(F^{-1}(u_1)) \le F(x_1)}.$$ [$F$ is non-decreasing]

Recall def. of generalized inv.

$F^{-1}(y) = \inf\{x : \cancel{F^{-1}(x) \ge y}$
$F(x) \ge y\}$.

verify $\underline{F(F^{-1}(u)) \geq u}$.

[using definition of $F^{-1}$].

$\therefore \quad u_1 \leq F(x_1)$.

$\therefore (u_1, x_1) \in B$

$\Rightarrow \quad A \subseteq B$

Similarly verify that $B \subseteq A$.
$\rightarrow$ HW.

hence $A = B$.

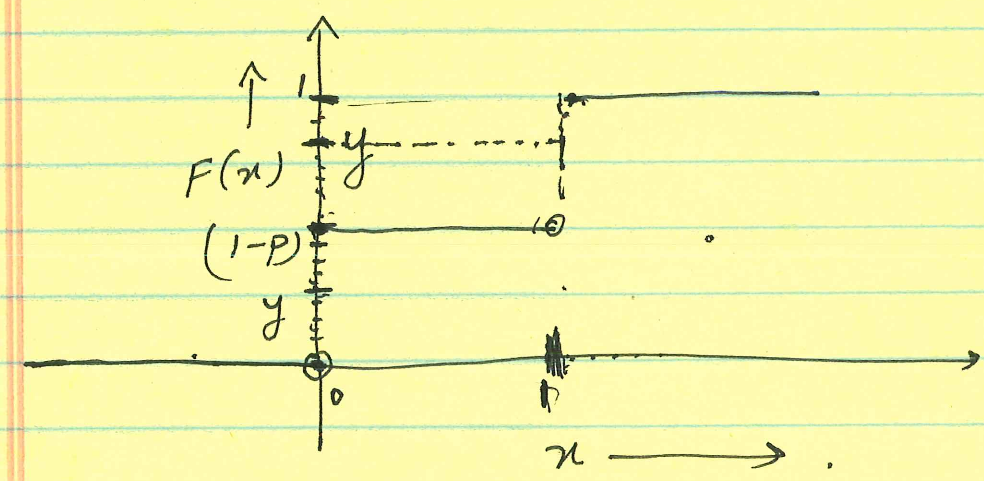
$\therefore P[X \leq x] = P[U \leq F(x)] = F(x)$

Ex: Show how to transform

$U \sim U(0,1)$.

into $X \sim Ber(p)$ r.v.

$$F(x) = \begin{cases} 0 & x < 0 \\ 1-p & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}$$



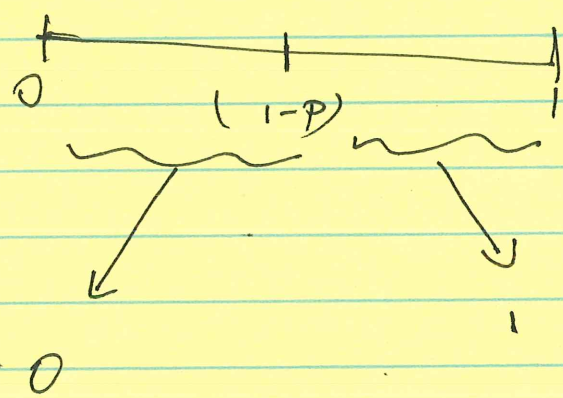
$$F^{-1}: [0,1] \longrightarrow \mathbb{R}$$

$$y \in (1-p, 1].$$

$$F^{-1}(y) = 1$$

$$y \in (0, 1-p).$$

$$F^{-1}(y) = 0$$




$\therefore$ define $X = F^{-1}(U)$

$$U \sim U(0,1).$$

then

$U \sim U(0,1)$

then $F^{-1}(U) = \begin{cases} 0 & \text{if } u \in (0, 1-p). \\ 1 & \text{if } u \in [1-p, 1) \end{cases}$

by the earlier theorem

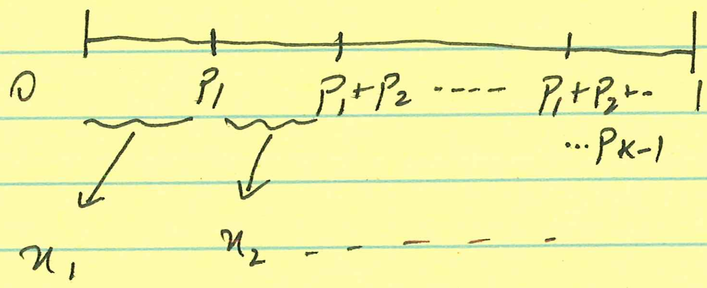
$$X = F^{-1}(U) \sim Ber(p).$$

In general, for any discrete distribution with a finite support.

$$X \in \{x_1, x_2, \ldots x_K\}.$$

$$P[X = x_j] = p_j \quad , j = 1 \ldots K.$$

How would you generate X using a uniform random variable?




$$F^{-1}(U) = \begin{cases} x_1 & \text{if } u \in (0, p_1) \\ x_2 & \text{if } u \in [p_1, p_1+p_2). \\ \vdots & \\ x_k & \text{if } u \in [p_1+\cdots+p_{k-1}, 1) \end{cases}$$

(2) Transformation methods:

Simulate a distribution through another distribution that is easier to simulate.

Recall:

(1) Gamma distribution: (cont.).

$\alpha, \beta > 0$ , $x \in (0, \infty)$.

$X \sim Ga(\alpha, \beta)$.

$$\underline{f_1(x)} = \frac{\beta^{\alpha}}{\underline{\Gamma(\alpha)}} \underline{x^{\alpha-1} e^{-\beta x}}$$

$\Gamma(\alpha) \to$ gamma function.

Properties:

(i) $X_1 \sim Ga(\alpha_1, \beta)$, $X_2 \sim Ga(\alpha_2, \beta)$

and $X_1$ and $X_2$ are independent. then.

$$X_1 + X_2 \sim Ga(\alpha_1 + \alpha_2, \beta)$$

ii) if $X_1 \sim Ga(1, 1)$.

then for any constant $\beta > 0$

$$\beta X_1 \sim Ga(1, \beta).$$

iii). if $X \sim Exp(\lambda)$ then.

$$X \sim Ga(1, \lambda).$$

(2) Chi-sq. dist ($\chi^2_\nu$).

$\nu \to$ degrees of freedom.

$X \sim \chi^2_\nu$

$$f(x) = \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \underline{x^{\frac{\nu}{2}-1} e^{-\frac{x}{2}}}, \quad x > 0.$$

(i) $\chi^2_\nu \sim Ga(2\nu, \frac{1}{2})$.

ii) $\chi^2_2 \sim Exp(\frac{1}{2})$.

iii) $X \sim \chi^2_{\nu_1}$, $Y \sim \chi^2_{\nu_2}$,

and X and Y are independent.

then $\frac{X}{X+Y} \sim Beta(\frac{\nu_1}{2}, \frac{\nu_2}{2})$.