EconS 305: Intermediate Microeconomics w/o Calculus
Homework 4:
Pricing, Imperfect Competition and Game Theory

Due: Friday, June 12th, 2020 at 5:00pm via Blackboard

- Please submit all homework solutions in the order the questions are presented and as **one .PDF**.

- Please **show all calculations** as these exercises are meant to refine your quantitative tool set. If I can not follow your calculations or it seems as you just "copy and pasted" answers from the internet, I will be deducting half the points from that solution.

1. **Price Discrimination: Block Pricing (2nd Degree Price Discrimination)**

   Consider a monopolist looking to price discriminate by setting a price for the first units sold, and a different price for the remaining units sold. The first q_1 units will be sold to the first consumers, until the stock runs out, and the remaining $(q_2 - q_1)$ units will be the quantity provided to the remaining consumers. The firm faces the same inverse demand curve of $p(q_i) = a - b(q_i)$ for all of the consumers, where $a > c$ and $i \in \{1, 2\}$, and has a constant marginal cost of c. Notice that the first consumers will be paying a different price per unit for the quantity they consume, and that the remaining consumers will be paying a different price per unit for the quantity they consume.

   We represent the Profit Maximization Problem (PMP_i) for the firm as a two part problem where they maximize profits for each consumer i as:

$$\max_{q_1 \geq 0} \ \pi_1 = p(q_1)q_1 - cq_1$$
$$\max_{q_2 \geq 0} \ \pi_2 = p(q_2)(q_2 - q_1) - c(q_2 - q_1)$$

   Where, we can combine these two PMPs into one function such that the monopolist maximizes the two choice variables of q_1 & q_2 in one problem

$$\max_{q_1, q_2 \geq 0} \ \pi = \pi_1 + \pi_2$$
$$= p(q_1)q_1 - cq_1 + p(q_2)(q_2 - q_1) - c(q_2 - q_1)$$

$$= (a - b(q_1))q_1 - cq_1 + (a - b(q_2))(q_2 - q_1) - c(q_2 - q_1)$$

CALCULUS PART:

From here, we can take our derivatives and set them equal to zero

$$\frac{\partial \pi(q_1, q_2)}{\partial q_1} = a - 2bq_1 - c - a + bq_2 + c = 0 \tag{1}$$

$$\frac{\partial \pi(q_1, q_2)}{\partial q_2} = a - 2bq_2 - c + bq_1 = 0 \tag{2}$$

where we now have two equations ((1) and (2)), and two choice variable (q_1, q_2) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

(a) Find the firm's optimal allocation of quantities (q_1, q_2) to maximize its profit in equilibrium (i.e. find q_1^*, q_2^*). Do any of these quantities look familiar? Identify it and state why it is familiar to you as an economic analyst.

We find the equilibrium allocation by simultaneously solving for q_1^* and q_2^*, and we get

$$q_1^*, q_2^* = \left(\frac{(a - c)}{3b}, \frac{2(a - c)}{3b}\right)$$

Notice that the q_1^* is exactly the same equilibrium quantity we derived for the Cournot Duopoly model where the firms were competing in quantities. Also, notice that $q_2^* > q_1^*$, which means that, by the law of demand, price for q_2^* should be lower than the price for q_1^*.

(b) Find the the equilibrium prices the firm will be charging for each one of the quantities (i.e. find $p(q_1^*), p(q_2^*)$). Which one is larger? Does it have the same relationship as the two equilibrium quantities derived in part a?

Plugging q_i^* into it's appropriate inverse demand function $p(q_i^*) = a - bq_i^*$ we get

$$p(q_1^*), p(q_2^*) = \left(\frac{(2a + c)}{3}, \frac{(a + 2c)}{3}\right)$$

Knowing that $a > c$ by assumption, we can see that the price for the first q_1 units consumed is unambiguously larger than the remaining units sold $(q_2 - q_1)$ (i.e. $p(q_1^*) > p(q_2^*)$). This is not the same relationship we found in part a, as the law of demand says we should have an inverse relationship in prices to quantities.

(c) Find both of the firm's profits in equilibrium (i.e. find π_1^* and π_2^*). Do you notice anything familiar about the profits coming from π_2^*? Finally, add these profits up to get total profits of the price discriminating firm (i.e. find $\pi^* = \pi_1^* + \pi_2^*$).

$$\pi_1^* = \frac{2(a-c)^2}{9b} \quad \equiv \quad 2\pi^{Duopoly}$$

$$\pi_2^* = \frac{(a-c)^2}{9b} \quad \equiv \quad \pi^{Duopoly}$$

$$\pi^* = \pi_1^* + \pi_2^* = \frac{2(a-c)^2}{9b} + \frac{(a-c)^2}{9b}$$

$$\implies \pi^{Price\ Discrimination} = \frac{(a-c)^2}{3b}$$

(d) Derive the profits for the standard monopolist model (as we did in HW 3-Question 2) without fixed costs, and compare these profits to the price discriminating monopolist profits you got in part c. Which profit is larger, and what does this tell you about a monopolist's incentive to price discriminate?

With fixed costs equal to 0 ($F = 0$), we compare both profits such that

$$\pi^{Price\ Discrimination} \quad \equiv \quad \frac{(a-c)^2}{3b} > \frac{(a-c)^2}{4b} \quad \equiv \quad \pi^{Monopolist}$$

This means that a monopolist has an incentive to price discriminate when it has market power.

2. **Modeling a Cartel Between Two Firms**

Consider two firms acting as a cartel. In economics, we typically model a cartel as N firms collectively acting as a monopoly, and then evenly dividing the profits among each firm participating in the cartel. This implies that we can calculate each firm's profits from participating in a cartel as $\pi^{Cartel} = \frac{\pi^{Monopolist}}{N}$. So, we consider first the monopolist's problem where they face a linear inverse demand function of $p(Q) = a - b(Q)$, where $a > c$, and a total cost function of $TC(Q) = cQ + F$. We can interpret the fixed cost (F) as perhaps some "entry" fee, and we interpret c as the cost the firm has to pay per each unit of output produced (i.e. the marginal cost).

We can represent the Profit Maximization Problem (PMP) for firm as:

$$\max_{Q \geq 0} \ \pi = p(Q)Q - (cQ + F)$$

$$\implies \max_{Q \geq 0} \ \pi = [a - bQ]\, Q - (cQ + F)$$

CALCULUS PART:

From here, we can take our derivatives and set them equal to zero

$$\frac{\partial \pi(Q)}{\partial Q} = a - 2bQ - c = 0 \tag{3}$$

where we now have one equation ((3)), and one choice variable (Q) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

(a) Find the firm's optimal allocation of production (Q) to maximize its profit in equilibrium (i.e. find Q^*). After getting your Q^*, please divide this by $N = 2$ as this is the amount produced by each firm (i.e. $q_i^{Cartel} = \frac{Q^*}{N}$).
The optimal allocation of joint quantity produced is

$$Q^* = \frac{(a-c)}{2b}$$

and the optimal quantitiy produced by each firm is

$$q_i^{Cartel} = \frac{Q^*}{N}$$

$$= \frac{\frac{(a-c)}{2b}}{2}$$

$$\implies q_i^{Cartel} = \frac{(a-c)}{4b}$$

(b) What is the equilibrium price the firm will receive (i.e. find $p(Q^*)$?

$$p(Q^*) = \frac{(a+c)}{2}$$

(c) What is the optimal profit function of the firm (i.e. find $\pi^*(Q^*)$

$$\pi^*(Q^*) = p(Q^*)Q^* - cQ^* - F$$

$$\implies \pi^*(Q^*) = \frac{(a-c)^2}{4b} - F$$

(d) Calculate the profits that each firm gets from participating in the cartel (i.e. find π^{Cartel} where $N = 2$).

$$\pi^{Cartel} = \frac{\pi^{Monopolist}}{N}$$

$$\pi^{Cartel} = \frac{\frac{(a-c)^2}{4b} - F}{2}$$

$$\implies \pi^{Cartel} = \frac{(a-c)^2}{8b} - \frac{F}{2}$$

Which is the profit that each firm will receive by participating in the cartel.

3. **A Cournot Game of Competing in Quantities w/ Fixed Costs - A Game Theory Extension**

Consider two firms competing a la Cournot in a market with an inverse demand function of $p(Q) = a - b(Q)$, where $Q = q_i + q_j$ and $a > c$, and a total cost function of $TC_i(q_i) = F + c_i q_i$. Notice that each firm has the same fixed cost (F) but their marginal costs (c_i) are not equal to each other (i.e. $c_i \neq c_j$). This means these homogeneous product producing firms have asymmetric costs, and we can represent the Profit Maximization Problem (PMP_i) for firm i as:

CALCULUS PART:

$$\max_{q_i \geq 0} \ \ \pi_i = [a - b(q_i + q_j)]\, q_i - (F + c_i q_i)$$

$$\frac{\partial \pi_i(q_i, q_j)}{\partial q_i} = a - 2bq_i - bq_j - c_i = 0 \tag{4}$$

And through symmetry we know that firm j's PMP is

$$\max_{q_j \geq 0} \ \ \pi_j = [a - b(q_i + q_j)]\, q_j - (F + c_j q_j)$$

$$\frac{\partial \pi_j(q_i, q_j)}{\partial q_j} = a - 2bq_j - bq_i - c_j = 0 \tag{5}$$

where we now have two equations ((2) and (3)), and two choice variables (q_i and q_j) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

(a) Find the optimal equilibrium allocation for each firm when they are competing a la Cournot. That is, find q_i^* and q_j^*. Note that this solution is analogous to the answer you derived in Question 3 of Homework 3.

We find the equilibrium allocation by simultaneously solving for q_i^* and q_j^* using each firm's BRFs derived from equations (4) and (5)

$$BRF_i \equiv q_i(q_j) = \frac{(a - c_i)}{2b} - \frac{1}{2} q_j \qquad\qquad BRF_j \equiv q_j(q_i) = \frac{(a - c_j)}{2b} - \frac{1}{2} q_i$$

$$q_i(q_j) = \frac{(a - c_i)}{2b} - \frac{1}{2}\left[\frac{(a - c_j)}{2b} - \frac{1}{2} q_i\right]$$

$$4bq_i = 2(a - c_i) - (a - c_i) + bq_i$$

$$\implies \qquad q_i^* = \frac{(a - 2c_i + c_j)}{3b} \quad \textit{and} \quad q_j^* = \frac{(a - 2c_j + c_i)}{3b}$$

by symmetry.

(b) Now, consider that the firm's have symmetric costs (i.e. $c_i = c_j = c$) in the competitive equilibrium and for all analyses from here on out. Find the competitive equilibrium quantities (i.e. find q_i^* and q_j^*). Again, note that this solution is analogous to the answer you derived in Question 3 of Homework 3.

When costs are equivalent, we know quantities, price, and profits are all the same for every firm i. This implies we get the standard Cournot quantities of

$$q_i^* \equiv q_j^* = \frac{(a-c)}{3b}$$

Note that we can also label these quantities as $q_i^{Cournot}$ as they are the output of the Cournot Model.

(c) Find the equilibrium price (i.e. $p(Q^*) = a - b(Q^*)$). Again, note that this solution is analogous to the answer you derived in Question 3 of Homework 3.

$$\begin{aligned} p(Q^*) &= a - b\left(\frac{(a-c)}{3b} + \frac{(a-c)}{3b}\right) \\ &= a - \frac{2(a-c)}{3} \\ \implies p(Q^*) &= \frac{(a+2c)}{3} \end{aligned}$$

(d) Find the equilibrium profits (i.e. π^*). Again, note that this solution is analogous to the answer you derived in Question 3 of Homework 3.

Using q^* and p^* we get π^* s.t.

$$\begin{aligned} \pi^* &= p^* q^* - (F + cq^*) \\ &= \left(\frac{(a+2c)}{3}\right)\left(\frac{(a-c)}{3b}\right) - \left(F + c\left(\frac{(a-c)}{3b}\right)\right) \\ &= \left(\frac{(a+2c)}{3} - c\right)\left(\frac{(a-c)}{3b}\right) - F \\ &= \frac{(a-c)}{3}\frac{(a-c)}{3b} - F \\ \implies \quad \pi^{*\ Cournot} &= \frac{(a-c)^2}{9b} - F \end{aligned}$$

(e) Now, assume that both firms are pooling resources and acting as cartel. Please re-write the equilibrium profits (π^{Cartel}) you found in Question #2. The answer should be the same answer you got in Question #2.

$$\pi^{Cartel} = \frac{(a-c)^2}{8b} - \frac{F}{2}$$

(f) Now, consider that one of the firms in the cartel unilaterally deviates from the cartel to compete in quantities (i.e. sets their quantity at the competitive level instead of the cartel level). Derive the profits from deviating (π_i^{Dev}) and simplify the expression. The key here is to remember that when deviation occurs, the deviating firm sets their quantities at a level as if they were competing a la Cournot and leaves the other firm operating as a cartel. This implies that the deviating firm's profits (π^D) become

$$\pi_i^{Dev} = \left(a - b\left(q_i^{Cournot} + q_i^{Cartel}\right)\right)\left(q_i^{Cournot}\right) - c\left(q_i^{Cournot}\right) - F$$

This implies that the deviating firm's profits (π_i^{Dev}) are

$$\pi_i^{Dev} = \left(a - b\left(\underbrace{\frac{(a-c)}{3b}} + \frac{(a-c)}{4b}\right)\right)\left(\underbrace{\frac{(a-c)}{3b}}\right) - c\left(\underbrace{\frac{(a-c)}{3b}}\right) - F$$

$$\implies \pi_i^{Dev} = \frac{5(a-c)^2}{36b} - F$$

(g) Similar to part f, we can find the profits of the firm that remains in the cartel while the other firm deviates. We will call these profits π_i^{NDev} (i.e. does not deviate) such that

$$\pi_i^{NDev} = \left(a - b\left(q_i^{Cournot} + q_i^{Cartel}\right)\right)\left(q_i^{Cartel}\right) - c\left(q_i^{Cartel}\right) - F$$

Use this formula to find the profits of the firm that is being deviated upon (i.e. π_i^{NDev}).

Note that we can plug in our competing quantities.

$$\pi_i^{ND} = \left(a - b\left(\underbrace{\frac{(a-c)}{4b}} + \frac{(a-c)}{3b}\right)\right)\left(\underbrace{\frac{(a-c)}{4b}}\right) - c\left(\underbrace{\frac{(a-c)}{4b}}\right) - F$$

$$\implies \pi_i^{ND} = \frac{5(a-c)^2}{48b} - F$$

(h) Take the four different equilibrium profit functions you found (π^{Cartel}, $\pi^{Cournot}$, π^{Dev} and π^{NDev}), set all fixed costs equal to zero (i.e. $F = 0$), and compare them mathematically (i.e. from most profit gained to least profit gained). Once the comparison is done, plug these equations into a matrix following the matrix template given below. This is called a normal form game, and this allows us to determine the "best response" for each firm when trying to decide to participate in a cartel or compete in quantities.

$\Rightarrow$

		Firm j	
		Cartel	Compete
Firm i	Cartel	$\pi_i^{Cartel}, \pi_j^{Cartel}$	$\pi_i^{NDev}, \pi_j^{Dev}$
	Compete	$\pi_i^{Dev}, \pi_j^{NDev}$	$\pi_i^{Cournot}, \pi_j^{Cournot}$

Plugging in the profits we found from above, we get a normal form matrix of

plugging in values

$\Rightarrow$

		Firm j	
		Cartel	Compete
Firm i	Cartel	$\frac{(a-c)^2}{40b}, \frac{(a-c)^2}{40b}$	$\frac{(a-c)^2}{48b}, \frac{(a-c)^2}{36b}$
	Compete	$\frac{(a-c)^2}{36b}, \frac{(a-c)^2}{48b}$	$\frac{(a-c)^2}{45b}, \frac{(a-c)^2}{45b}$

2i)

(i) Find the Pure Strategy Nash Equilibrium ($psNE$).

$\Rightarrow$

		Firm j	
		Cartel	Compete
Firm i	Cartel	$\pi_i^{Cartel}, \pi_j^{Cartel}$	$\pi_i^{NDev}, \underline{\pi_j^{Dev}}$
	Compete	$\underline{\pi_i^{Dev}}, \pi_j^{NDev}$	$\underline{\pi_i^{Cournot}}, \underline{\pi_j^{Cournot}}$

underlining all Best Responses

The the psNE of the game is

$$psNE = \left\{ \left(q_1^* = \frac{(a-c)}{3b} \;\; , \;\; q_2^* = \frac{(a-c)}{3b} \right) \right\}$$

Where the optimal choice for each firm is to choose to compete no matter what. Intuitively, this result shows that a firm's optimal move, when offered to participate cartel, is to deviate and break the cartel agreement. Thus, both firms will choose to compete in equilibrium.

4. **Bertrand Price Competition - A Game Theory Extension**

Consider two firms competing a la Bertrand in a market with demand $Q = 1-p$. For simplicity, both firms face no marginal costs, i.e., $c = 0$. Recall the rules of Bertrand competition: if firm i sets a lower price than firm j, $p_i < p_j$, firm i captures the entire market (Q) and makes profits of

$$\pi_i^* = p_i Q = p_i \underbrace{(1 - p_i)}_{\text{Q}}$$

while firm j gets zero sales, and thus, zero profits (i.e. $\pi_j^* = 0$). If both firms set the same price, $p_i = p_j$, then both firms obtain half of the above profits, $\frac{1}{2}p_i(1 - p_i)$.

(a) Find the competitive equilibrium price for every firm (i.e. p_i^*, and the equilibrium profits for every firm (i.e. π_i^*). Recall that in a one shot game, firms will undercut each other until they set a price equal to their marginal cost (i.e. $p_i^* = c$).

As discussed in class, both firms have incentives to undercut each other's price until they both converge on a common price that coincides with their marginal cost, $c = 0$. Such a profile of equilibrium prices $p_i^B = p_j^B = c = 0$ yields zero profits, $\pi_i^B = 0$, for every firm i where the superscript B denotes Bertrand competition.

(b) Now assume that the firms form a cartel. This implies that the firms would jointly act as a monopolist and split the monopoly profits evenly. We set the monopolist's problem up as

$$\max_{p \geq 0} \ \ \pi = p \underbrace{(1 - p)}_{\text{Q}} = p - p^2$$

CALCULUS PART:

where from here, we can take our derivatives and set them equal to zero

$$\frac{\partial \pi(p)}{\partial p} \quad = \quad 1 - 2p = 0 \tag{6}$$

where we now have one equation ((6)), and one choice variable (p) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

What is the price that every firm should set in order to maximize the profits of the cartel? Notice that the firm will not divide this price by the number of firms in the market as this would yield lower profits. Once the optimal price is found, find monopoly profits and the individual profits that every firm makes in the cartel. Please label them as π^{Cartel}.

Solving for p yields a profit-maximizing price for the monopoly of

$$p^{Monopoly} = p^{Cartel} = \frac{1}{2}$$

where if the price was set higher or lower, they would yield lower total monopoly profits. To see this, you can graph the profit function and see that it is an inverted parabola.
Given the above price, monopoly profits become

$$\pi^{Monopoly} = p^{Cartel}(1 - p^{Cartel}) = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}$$

$$\implies \pi_i^{Cartel} = \frac{\pi^{Monopoly}}{N}$$

$$\pi_i^{Cartel} = \frac{1}{4} * \frac{1}{2} = \frac{1}{8}$$

implying that every firm i earns half of that (i.e. $\pi_i^{Cartel} = \frac{1}{8}$). Note that the cartel profit is higher than what firms make when competing in prices found in part (a).

(c) Now assume that a firm tries to under-cut the other firm's price (*say, by a value of* ϵ) in an attempt to capture the whole market. We can denote this price deviation as

$$p^{Dev} = p^{Cartel} - \epsilon = \frac{1}{2} - \epsilon$$

What are the firm's profits if they deviate from the cartel agreement, and what are the firm's profits if they are the one that sticks to the cartel agreement? Please label them as π^{Dev} and π^{NDev}, respectively.

Notice that in Bertrand competition that if a firm chooses to deviate from a cartel price (p^{Cartel}) they would just undercut the price of their cartel partner (i.e. now becoming a competitor) in order to capture the whole market. What this means is that the firm would infinitesimally slightly under-cut their opponents price by a value of ϵ and then make monopoly profits less their ϵ

$$p^{Dev} = p^{Cartel} - \epsilon = \frac{1}{2} - \epsilon.$$

Where deviation profits will become

$$\pi^{Dev} = p^{Dev}(1 - p^{Dev}) = \left(\frac{1}{2} - \epsilon\right)\left(1 - \left(\frac{1}{2} - \epsilon\right)\right) = \frac{1}{4} - \epsilon^2$$

and the profits from being deviated against will be

$$\pi^{NDev} = p\underbrace{(1 - p)}_{\text{Q=0}} = 0$$

(d) Using the profits found in parts a through c (i.e. $\pi^{Bertrand}, \pi^{Cartel}, \pi^{Dev}$ and π^{NDev}), please construct the same $2x2$ normal form matrix you did in Question 3. For reference, the matrix is

$\Longrightarrow$

		Firm j	
		Cartel	Compete
Firm i	Cartel	$\pi_i^{Cartel}, \pi_j^{Cartel}$	$\pi_i^{NDev}, \pi_j^{Dev}$
	Compete	$\pi_i^{Dev}, \pi_j^{NDev}$	$\pi_i^{Cournot}, \pi_j^{Cournot}$

where you can substitute $\pi^{Bertrand}$ for $\pi^{Cournot}$.

(e) Find the condition on ϵ that will insure that the cartel stays intact.

Where, for there to exist a profitable deviation we need that

$$\pi^{Cartel} \equiv \frac{1}{8} \geq \frac{1}{4} - \epsilon^2 \equiv \pi^{Dev}$$

$$\epsilon \geq \frac{1}{2\sqrt{2}} \equiv \bar{\epsilon}$$

Where if ϵ is greater than $\bar{\epsilon}$, firms would not deviate from the cartel agreement. Intuitively, this would mean that the deviation would yield less profits than if they were to stick with the cartel.

5. **An Increase in the Elasticity of Demand Decreases the Monopolist's Markup Price for that Consumer**

(a) Using the general definition of Marginal Revenue (i.e. $MR = p + \frac{\Delta p}{\Delta q} p$), please derive the Lerner Index. Please show all calculations. What does the Lerner Index tell us about the monopolist's price?

Please see answer attached below in the Math .PDF.

(b) Using the Lerner Index you derived in part a, please derive the Inverse Elasticity Pricing Rule (IEPR). Please show all calculations.

Please see answer attached below in the Math .PDF.

(c) Assuming that the elasticity of demand for consumer 1 is greater than the elasticity of demand for consumer 2 (i.e. $\epsilon_1^D > \epsilon_2^D$), please prove that the markup price for consumer 1 is less than the markup price for consumer 2. Please do not prove this trivially, and make sure you show all steps.

Hint: Start by assuming that the $p(MC, \epsilon_2^D) > p(MC, \epsilon_1^D)$.

Please see answer attached below in the Math .PDF.

1a)

Using equation (1) & (2)

Question #1 (Math)

(1) $\Rightarrow \quad q_1 = \frac{1}{2} q_2$

plugging into (2)

$$\Rightarrow \quad q_2 = \frac{(a-c)}{2b} + \frac{1}{2} q_1$$

$$\Rightarrow \quad q_2 = \frac{(a-c)}{2b} + \frac{1}{2}\left[\frac{1}{2} q_2\right]$$

$$\Rightarrow \quad q_2 = \frac{(a-c)}{2b} + \frac{1}{4} q_2$$

$$\Rightarrow \quad 4q_2 = \frac{4(a-c)}{2b} + q_2$$

$$\Rightarrow \quad q_2^* = \frac{4(a-c)}{6 \cdot 2b} = \frac{2(a-c)}{3b}$$

$$\Rightarrow \quad q_1^* = \frac{1}{\cancel{2}}\left[\frac{\cancel{2}(a-c)}{3b}\right] = \frac{(a-c)}{3b}$$ ← Familiar?

1b) Prices

$$\Rightarrow \quad P(q_1^*) = a - \cancel{b}\left[\frac{(a-c)}{3\cancel{b}}\right] = \frac{2a+c}{3}$$

$$P(q_2^*) = a - \cancel{b}\left[\frac{2(a-c)}{3\cancel{b}}\right] = \frac{3a-2a+2c}{3} = \frac{a+2c}{3}$$

$\Rightarrow$ Is $a > c$

$$P(q_1^*) > P(q_2^*)$$

$$\frac{\cancel{2}a+\cancel{c}}{\cancel{3}} > \frac{\cancel{a}+2c}{\cancel{3}}$$

$\Rightarrow \quad a > c$ ✓ by assumption.

$\Rightarrow$

Scanned with CamScanner

1c) πs

①

$$\pi_1 = \left(\frac{2a+c}{3}\right)\left(\frac{(a-c)}{3b}\right) - (c)\left(\frac{(a-c)}{3b}\right)$$

Subtracting out

$$= \left(\frac{2a+c-3c}{3}\right)\left(\frac{(a-c)}{3b}\right)$$

Duopoly

$$= \frac{2(a-c)}{3}\frac{(a-c)}{3b} = \frac{2(a-c)^2}{9b} = 2\pi^c$$

②

$$\pi_2 = \underbrace{\left(\frac{a+2c}{3} - c\right)}_{p(q_2^*)-c}\left(\underbrace{\frac{2(a-c)}{3b}}_{q_2^*} - \underbrace{\frac{(a-c)}{3b}}_{q_1^*}\right)$$

$$= \left(\frac{a-c}{3}\right)\left(\frac{(a-c)}{3b}\right) = \frac{(a-c)^2}{9b} = \pi^{*c}$$

Profit in typical Cournot duopoly

$$\Rightarrow \pi^* = \pi_1^* + \pi_2^*$$

$$= \frac{2(a-c)^2}{9b} + \frac{(a-c)^2}{9b} = \frac{3(a-c)^2}{9b} = \frac{(a-c)^2}{3b} = \pi^{PD}$$

1d) Using monopoly π's from HW2 wkset,

$$\pi^M < \pi^{PD}$$

$$\Rightarrow \frac{(a-c)^2}{4b} < \frac{(a-c)^2}{3b}$$

$\Rightarrow$ Firms have incentive to price discriminate.

2a)

Question #2 (Math)

using (3)

$$\Rightarrow \quad 2bQ = a - c$$

$$\Rightarrow \quad Q^* = \frac{(a-c)}{2b}$$

$$\Rightarrow \quad q_i^{\text{Cartel}} = \frac{(a-c)}{2b}\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad q_i^{\text{Cartel}} = \frac{(a-c)}{4b}$$

2b)

$$P(Q^*) = (a - b(Q^*))$$

$$= a - \cancel{b}\left(\frac{(a-c)}{2\cancel{b}}\right)$$

$$= \frac{2a - a + c}{2} = \frac{a+c}{2}$$ Same as the Monopoly

2c)

$$\pi^* = P(Q^*)Q^* - cQ^* - F$$

$$\Rightarrow \quad \left(\left(\frac{a+c}{2}\right) - c\right)\left(\frac{(a-c)}{2b}\right) - F$$

$$\Rightarrow \quad \left(\frac{(a-c)}{2}\right)\left(\frac{(a-c)}{2b}\right) - F = \frac{(a-c)^2}{4b} - F = \pi^{\text{Monoply}}$$

2d)

$$\Rightarrow \quad \pi^{\text{Cartel}} = \frac{\pi^{\text{Monopoly}}}{N}$$

$$\Rightarrow \quad \pi^{\text{Cartel}} = \left(\frac{(a-c)^2}{4b} - F\right)/2$$ where $N = 2$

$$\Rightarrow \quad = \frac{(a-c)^2}{8b} - \frac{F}{2}$$

3a)
Using (4) & (5)

Question #3
(MATH)

$\Rightarrow$ (4) $\quad 2bq_i = a - c_i - bq_j$

$$\Rightarrow BRF_i = q_i(q_j) = \frac{(a-c_i)}{2b} - \frac{1}{2}q_j$$

Combining

(5) $\quad 2bq_j = a - c_j - bq_i$

$$\Rightarrow BRF_j = q_j(q_i) = \frac{(a-c_j)}{2b} - \frac{1}{2}q_i$$

$$\Rightarrow q_i(q_j) = \frac{(a-c_i)}{2b} - \frac{1}{2}\left[\frac{(a-c_j)}{2b} - \frac{1}{2}q_i\right]$$

everything is in terms of q_i

$$\Rightarrow q_i = \frac{(a-c_i)}{2b} - \frac{(a-c_j)}{4b} + \frac{1}{4}q_i$$

x4 both sides

$$\Rightarrow 4bq_i = 2(a-c_i) - (a-c_j) + b\cancel{q}_i$$

$$\Rightarrow 3bq_i = (2a-a) - 2c_i + c_j$$

$$\Rightarrow q_i^* = \frac{(a-2c_i+c_j)}{3b}$$

Plug into other BRF_j $\Rightarrow$

$$q_j(q_i^*) = \frac{(a-c_j)}{2b} - \frac{1}{2}\left[\frac{(a-2c_i+c_j)}{3b}\right]$$

$$q_j^* = \frac{(a-c_j)}{2b} - \frac{(a-2c_i)+c_j)}{6b}$$

$$= \frac{3(a-c_j) - (a-2c_i+c_j)}{6b}$$

$$= \frac{(2a - 3c_j + 2c_i - c_j)}{6b}$$

*Also, Cournot q_i, q_j Cournot by notation.

$$\Rightarrow q_j^* = \frac{(2a - 4c_j + 2c_i)}{6b} = \frac{2(a-2c_j+c_i)}{6b} = \frac{(a-2c_j+c_i)}{3b}$$

$$\Rightarrow (q_i^*, q_j^*) = \left(\frac{(a-2c_i+c_j)}{3b}, \frac{(a-2c_j+c_i)}{3b}\right)$$

Scanned with CamScanner

3b) $\Rightarrow (q_i^*, q_j^*)$ become $\frac{(a-2c+c)}{3b}$ since $c_i = c_j = c$

$$\Rightarrow \left(q_i^{\text{cournot}}, q_j^{\text{cournot}}\right) = \left(\frac{a-c}{3b}, \frac{a-c}{3b}\right) \equiv q_i^*$$

3c)

$$P(Q^*) = a - b\left(\frac{(a-c)}{3b} + \frac{(a-c)}{3b}\right)$$

$$= a - \frac{2(a-c)}{3}$$

$$= \frac{(3a-2a+c)}{3} = \frac{a+c}{3}$$

⊛ Note, not a function of b.

3d)

aka q_i^{cournot}

$$\Rightarrow \pi_i^* = p^*(Q^*)q_i^* - (F + cq_i^*)$$

$$\Rightarrow \quad = \left(\frac{a+c}{3}\right)\left(\frac{(a-c)}{3b}\right) - \left(F + c\left(\frac{(a-c)}{3b}\right)\right)$$

$$= \underbrace{\left(\frac{a+c}{3} - c\right)}\left(\frac{(a-c)}{3b}\right) - F$$

$$= \frac{(a-c)}{3}\frac{(a-c)}{3b} - F$$

$$\pi^{*\text{cournot}} = \frac{(a-c)^2}{9b} - F$$

Note the similarity w/

$$\boxed{\begin{array}{l}\text{* Imperfect Competition} \\ \pi = \frac{(a-c)^2}{(N+1)^2 b} - F\end{array}}$$

3e) From Q2

$$\pi^{\text{cartel}} = \frac{(a-c)^2}{8b} - \frac{F}{2}$$

(f) $\Rightarrow$

3f)

$$\pi_i^{Dev} = (a - b(q_i^{Cournot} + q_i^{cartel}))(q_i^{Cournot}) - c(q_i^{Cournot}) - F$$

$$= \left(a - b\left(\frac{(a-c)}{3b} + \frac{(a-c)}{4b}\right)\right)\left(\frac{(a-c)}{3b}\right) - c\left(\frac{(a-c)}{3b}\right) - F$$

$$\Rightarrow \;= \left(\frac{12a - 4(a-c) - 3(a-c) - 12c}{12}\right)\frac{(a-c)}{3b} - F$$

$$= \frac{12(a-c) - 7(a-c)}{12}\;\frac{(a-c)}{3b} - F$$

$$= \frac{5(a-c)}{36\,b} - F$$

3g) Similarly, assume you act as cartel while they compete

$$\Rightarrow \pi_i^{NDev} = (\underbrace{a - b\left(\frac{(a-c)}{3b} + \frac{(a-c)}{4b}\right)}) \left(\frac{(a-c)}{4b}\right) - c\left(\frac{(a-c)}{4b}\right) - F$$

$$= \frac{5(a-c)}{12}\;\frac{(a-c)}{4b} - F$$

$$= \frac{5(a-c)^2}{48\,b} - F$$

3h) <u>Summarizing</u>

$$\Rightarrow \pi_i^{*} = \pi_i^{Cournot} = \frac{(a-c)^2}{9b} - F \;,\quad \pi_i^{Cartel} = \frac{(a-c)^2}{8b} - \frac{F}{2}$$

$$\pi_i^{Dev} = \frac{5(a-c)^2}{36b} - F \;,\quad \pi_i^{ND} = \frac{5(a-c)}{48b} - F$$

Shutting down F (ie set F=0),

$\Rightarrow$ Simplifying, & comparing

Interesting

$$\underbrace{\frac{(a-c)^2}{36b}}_{\pi_i^{Dev}} > \underbrace{\frac{(a-c)^2}{40b}}_{\pi_i^{Cartel}} > \underbrace{\frac{(a-c)^2}{45b}}_{\substack{\pi_i^{Cournot} \\ = \pi_i^{*} = \pi_i^{Compete}}} > \underbrace{\frac{(a-c)^2}{48b}}_{\pi_i^{NDev}}$$

3h (continued)

$\Rightarrow$

Firm j

Firm i	Cartel	Compete
Cartel	$\pi_i^{Cartel}, \pi_j^{Cartel}$	$\pi_i^{NDev}, \pi_j^{Dev}$
Compete	$\pi_i^{Dev}, \pi_j^{NDev}$	$\pi_i^{Cournot}, \pi_j^{Cournot}$

plugging in values

$\Rightarrow$

Firm j

Firm i	Cartel	Compete
Cartel	$\frac{(a-c)^2}{40b}, \frac{(a-c)^2}{40b}$	$\frac{(a-c)^2}{48b}, \frac{(a-c)^2}{36b}$
Compete	$\frac{(a-c)^2}{36b}, \frac{(a-c)^2}{48b}$	$\frac{(a-c)^2}{45b}, \frac{(a-c)^2}{45b}$

3i)

where, doing our Best Response Analysis s.t.

Firm i: ① If Firm j plays Cartel, I play Compete ($\pi_i^{Dev} > \pi_i^{Cartel}$)

② If Firm j plays Compete, I play Compete ($\pi_i^{Cournot} > \pi_i^{NDev}$)

Firm j: ① If Firm i plays Cartel, I play Compete ($\pi_j^{Dev} > \pi_j^{Cartel}$)

② If Firm i plays Compete, I play Compete ($\pi_j^{Cournot} > \pi_j^{NDev}$)

$\Rightarrow$

Firm j

Firm i	Cartel	Compete
Cartel	$\pi_i^{Cartel}, \pi_j^{Cartel}$	$\pi_i^{NDev}, \underline{\pi_j^{Dev}}$
Compete	$\underline{\pi_i^{Dev}}, \pi_j^{NDev}$	$\underline{\pi_i^{Cournot}}, \underline{\pi_j^{Cournot}}$

underlining all Best Responses

$\Rightarrow$ Nash Equilibrium $= (q_i^{*Cournot}, q_j^{*Cournot}) = \left\{\left(\frac{(a-c)}{3b}, \frac{(a-c)}{3b}\right)\right\}$

Scanned with CamScanner

Question #4
(Math)

4a) Bertrand Competitive Equilibrium

$\Rightarrow \quad p^* = c \quad$ where $\quad c = 0$

$\Rightarrow \quad p^* = 0$

$\Rightarrow \quad \pi^* = 0$

4b) using equation (6)

$\Rightarrow \quad 1 - 2p = 0$

$\Rightarrow \quad p^* = \frac{1}{2} \quad = p^m = p^c$

$\Rightarrow \quad \pi^{Monopoly} = p^c(1 - p^c)$

$= \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}$

$\Rightarrow \quad \pi_i^{Cartel} = \frac{\pi^{Monoply}}{N} = \frac{1}{4} * \frac{1}{2} \quad$ where $N = 2$

$= \frac{1}{8}$

4c)

$p^{Dev} = p^{Cartel} - \varepsilon = \frac{1}{2} - \varepsilon$

$\Rightarrow \quad \pi^{Dev} = p^D(1 - p^D)$

$= \left(\frac{1}{2} - \varepsilon\right)\left(1 - \left(\frac{1}{2} - \varepsilon\right)\right)$

$= \left(\frac{1}{2} - \varepsilon\right)\left(\frac{1}{2} + \varepsilon\right)$ ← Difference of squares

$= \frac{1}{4} - \varepsilon^2$

$\pi^{NDev} = 0 \quad$ since $\quad Q^{NDev} = 0$

$\Rightarrow$

3d)

$\Rightarrow$

	Cartel	Compete
Cartel	$\frac{1}{8}, \frac{1}{8}$	$0, \frac{1}{4}-\varepsilon^2$
Compete	$\frac{1}{4}-\varepsilon^2, 0$	$0, 0$

$\Rightarrow$ We want $\pi^{Cartel} > \pi^{Dev}$ to sustain collusion

$$\Rightarrow \quad \frac{1}{8} > \frac{1}{4} - \varepsilon^2$$

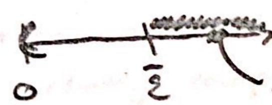
$$\Rightarrow \quad \varepsilon^2 > \frac{1}{8}$$

$$\varepsilon > \sqrt{1/8}$$

$$\Rightarrow \quad \varepsilon > \frac{1}{\sqrt{4}\sqrt{2}}$$

$$\varepsilon > \frac{1}{2\sqrt{2}} \equiv \bar{\varepsilon}$$

$\Rightarrow$ IS $\varepsilon > \bar{\varepsilon}$ cartel will be sustained.



IS in here, price decrease is too intense.

Question #5 (math)

5a) Knowing, $MR = p + \frac{\Delta P}{\Delta q} q$ & that $MR = MC$ is the monopolist's profit maximization condition

$$\Rightarrow \quad MR = MC$$

$$\Rightarrow \quad p + \frac{\Delta P}{\Delta Q} q = MC$$

multiply by 1

$$\Rightarrow \quad p + \frac{\Delta P}{\Delta Q} q \left(\frac{p}{p}\right) = MC$$

$$\Rightarrow \quad P + \frac{\Delta P}{\Delta q} \frac{q}{p} P = MC$$

since $\frac{\Delta P}{\Delta q} \frac{q}{p} = \frac{\Delta P / p}{\Delta q / q} = \frac{1}{\varepsilon_D}$

$$\Rightarrow \quad P + \frac{\Delta P / p}{\Delta q / q} P = MC$$

$$\Rightarrow \quad P + \frac{1}{\varepsilon_D} P = MC$$

$$\Rightarrow \quad P - MC = -\frac{1}{\varepsilon_D} P$$

$$\Rightarrow \quad \frac{(P - MC)}{P} = -\frac{1}{\varepsilon_D}$$

It tells us the % mark-up in price relative to the perfectly competitive equilibrium

5b) solve for p

$$\Rightarrow \quad \frac{(p - MC)}{p} = -\frac{1}{\varepsilon_D}$$

$$\Rightarrow \quad p + \frac{1}{\varepsilon_D} p = MC$$

$$\Rightarrow \quad p = \frac{\varepsilon_D}{(1 + \varepsilon_D)} MC \equiv \underline{IERP}$$

Scanned with CamScanner

5c) Assuming $\varepsilon_1^D > \varepsilon_2^D$, we start our proof w/ what we are trying to prove such that

$$P_2 > P_1$$

$$\Rightarrow \quad P(MC, \varepsilon_2^D) > P(MC, \varepsilon_1^D)$$

invoking IEPR

$$\Rightarrow \quad \frac{\varepsilon_D^2}{(1+\varepsilon_D^2)}\cancel{MC} > \frac{\varepsilon_D^1}{(1+\varepsilon_D^1)}\cancel{MC}$$

$$\Rightarrow \quad \frac{(1+\varepsilon_D^1)}{\varepsilon_D^1} > \frac{(1+\varepsilon_D^2)}{\varepsilon_D^2}$$

$$\Rightarrow \quad \frac{1}{\varepsilon_D^1} + \cancel{1} > \frac{1}{\varepsilon_D^2} + \cancel{1}$$

$$\Rightarrow \quad \frac{1}{\varepsilon_D^1} > \frac{1}{\varepsilon_D^2}$$

$$\Rightarrow \quad \varepsilon_D^2 > \varepsilon_D^1$$

where If $\varepsilon_D^i < 0$ Then the sign flips and the result holds true when we take the absolute value

$$\Rightarrow \quad |\varepsilon_D^2| < |\varepsilon_D^1|$$