

# ***EconS 305: Intermediate Microeconomics w/o Calculus***

## ***Homework 5:***

### ***Repeated Games, Sequential Move Games and Product Differentiation***

***Due: Friday, June 19th, 2020 at 5:00pm via Blackboard***

- Please submit all homework solutions in the order the questions are presented and as **one .PDF**.

- Please **show all calculations** as these exercises are meant to refine your quantitative tool set. If I can not follow your calculations or it seems as you just “copy and pasted” answers from the internet, I will be deducting half the points from that solution.

#### **1. A Cournot Game of Competing in Quantities w/ Fixed Costs - A Game Theory Extension with Sustainable Collusion**

Note: Please **re-do all calculations for this exercise**, even though Parts A through I are analogous to that of Homework 3 and 4, as it is important that you know how to derive the equilibrium results for this model. This is the base case model for many economic analyses, and it is important for Intermediate Microeconomic extensions.

Consider two firms competing a la Cournot in a market with an inverse demand function of  $p(Q) = a - b(Q)$ , where  $Q = q_i + q_j$  and  $a > c$ , and a total cost function of  $TC_i(q_i) = F + c_i q_i$ . Notice that each firm has the same fixed cost ( $F$ ) but their marginal costs ( $c_i$ ) are not equal to each other (i.e.  $c_i \neq c_j$ ). This means these homogeneous product producing firms have asymmetric costs, and we can represent the Profit Maximization Problem (PMP) for firm  $i$  as:

#### **CALCULUS PART:**

$$\begin{aligned} \max_{q_i \geq 0} \pi_i &= [a - b(q_i + q_j)] q_i - (F + c_i q_i) \\ \frac{\partial \pi_i(q_i, q_j)}{\partial q_i} &= a - 2bq_i - bq_j - c_i = 0 \end{aligned} \tag{1}$$

And through symmetry we know that firm j's PMP is

$$\max_{q_j \geq 0} \pi_j = [a - b(q_i + q_j)] q_j - (F + c_j q_j)$$

$$\frac{\partial \pi_j(q_i, q_j)}{\partial q_j} = a - 2bq_j - bq_i - c_j = 0 \quad (2)$$

where we now have two equations ((1) and (2)), and two choice variables ( $q_i$  and  $q_j$ ) to solve for.

### **CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.**

- (a) Find the optimal equilibrium allocation for each firm when they are competing a la Cournot. That is, find  $q_i^*$  and  $q_j^*$ . Note that this solution is analogous to the answer you derived in Question 3 of Homework 4.
- (b) Now, consider that the firm's have symmetric costs (i.e.  $c_i = c_j = c$ ) in the competitive equilibrium and for all analyses from here on out. Find the competitive equilibrium quantities (i.e. find  $q_i^*$  and  $q_j^*$ ). Again, note that this solution is analogous to the answer you derived in Question 3 of Homework 4.
- (c) Find the equilibrium price (i.e.  $p(Q^*) = a - b(Q^*)$ ). Again, note that this solution is analogous to the answer you derived in Question 3 of Homework 4.
- (d) Find the equilibrium profits (i.e.  $\pi^*$ ). Again, note that this solution is analogous to the answer you derived in Question 3 of Homework 4.
- (e) Now, assume that both firms are pooling resources and acting as a cartel. Please re-write the equilibrium profits ( $\pi^{Cartel}$ ) you found in Question 2 in Homework 4. The answer should be exactly the same answer you got in Question 2 of Homework 4.
- (f) Now, consider that one of the firms in the cartel unilaterally deviates from the cartel to compete in quantities (i.e. sets their quantity at the competitive level instead of the cartel level). Derive the profits from deviating ( $\pi_i^{Dev}$ ) and simplify the expression. Again, please note that this solution is analogous to the answer you derived in Question 3 of Homework 4. The key here is to remember that when deviation occurs, the deviating firm sets their quantities at a level as if they were competing a la Cournot and leaves the other firm operating as a cartel. This implies that the deviating firm's profits ( $\pi^D$ ) become

$$\pi_i^{Dev} = (a - b(q_i^{Cournot} + q_i^{Cartel})) (q_i^{Cournot}) - c(q_i^{Cournot}) - F$$

- (g) Similar to part f, we can find the profits of the firm that remains in the cartel while the other firm deviates. We will call these profits  $\pi_i^{NDev}$  (i.e. does not deviate) such that

$$\pi_i^{NDev} = (a - b(q_i^{Cournot} + q_i^{Cartel})) (q_i^{Cartel}) - c(q_i^{Cartel}) - F$$

Use this formula to find the profits of the firm that is being deviated upon (i.e.  $\pi_i^{NDev}$ ). Again, please note that this solution is analogous to the answer you derived in Question 3 of Homework 4.

- (h) Take the four different equilibrium profit functions you found ( $\pi^{Cartel}$ ,  $\pi^{Cournot}$ ,  $\pi^{Dev}$  and  $\pi^{NDev}$ ), set all fixed costs equal to zero (i.e.  $F = 0$ ), and compare them mathematically (i.e. from most profit gained to least profit gained). Once the comparison is done, plug these equations into a matrix following the matrix template given below. This is called a normal form game, and this allows us to determine the “best response” for each firm when trying to decide to participate in a cartel or compete in quantities. Again, please note that this solution is analogous to the answer you derived in Question 3 of Homework 4.

$\Rightarrow$

		<u>Firm j</u>	
		Cartel	Compete
<u>Firm i</u>	Cartel	$\pi_i^{Cartel}, \pi_j^{Cartel}$	$\pi_i^{NDev}, \pi_j^{Dev}$
	Compete	$\pi_i^{Dev}, \pi_j^{NDev}$	$\pi_i^{Cournot}, \pi_j^{Cournot}$

- (i) Find the Pure Strategy Nash Equilibrium ( $psNE$ ). Again, please note that this solution is analogous to the answer you derived in Question 3 of Homework 4.
- (j) Now, consider that both firms are going to engage in an infinitely repeated game where they will need to simultaneously decide if they are going to compete or participate in a cartel at every time  $t$ . Please derive the Grim Trigger Strategy (GTS), using our profit notation (i.e.  $\pi^{Dev}$  and  $\pi^{Cournot}$ ), where the firm will deviate at the time period  $t = 0$ , and then revert to the Nash Equilibrium payoff (i.e.  $\pi^{Cournot}$ ) in all the time periods after that. In order to do this, recall that we need to consider the discounted stream of payoffs the firm will receive from  $t = 0$  to  $t = \infty$ . Please refer to the Chapter 12 Game Theory lecture notes if you are having trouble with deriving the GTS payoff stream.
- (k) Now, please derive the stream of payoffs an individual firm will receive from participating in the cartel in the infinitely repeated game. Please use the general profit notation instead of the analytical notation.
- (l) Now, please find the condition on the firm's patience parameter (i.e.  $\delta$ ) in which will can guarantee that the firm's will both sustain the cartel agreement. Remember that the cartel agreement in the infinitely repeated game will be sustained if the discounted stream of payoffs from staying in the cartel are greater than the GTS payoffs.
- (m) Please plug in the analytical solution to the general profit notation we have been using, and simplify. Simplifying should prove that  $\bar{\delta} = \frac{1}{2}$  in this infinitely repeated game.

## 2. A Bertrand Model with Product Differentiation

**Note:** *I have given you the final answers to these problems, but I want you to **show all the work needed to derive these results**. Also, please make sure to interpret the results if the question asks you to interpret the results.*

Consider two firms competing in prices a la Bertrand selling differentiated goods. The demand function for every firm  $i$  is  $q_i(p_i, p_j) = a - bp_i + p_j$  where  $i, j \in \{1, 2\}$  and  $i \neq j$ . This means that the demand function for firm 1 and firm 2 are  $q_1(p_1, p_2) = a - bp_1 + p_2$  and  $q_2(p_1, p_2) = a - bp_2 + p_1$ , respectively. In this model  $b$  represents the degree of product differentiation for every firm  $i$ . If  $b > 1$  the products are considered to be differentiated (also known as heterogeneous), and if  $b = 1$  the products are identical (also known as homogeneous). Each firm has the same constant marginal cost of  $c$ , and all firms' fixed costs are assumed to be equal to zero (i.e.  $F = 0$ ). We formulate each firm's Profit Maximization Problem (PMP) as:

### CALCULUS PART:

$$\begin{aligned} \max_{p_i \geq 0} \pi_i &= p_i [a - bp_i + p_j] - c [a - bp_i + p_j] \\ \frac{\partial \pi_i(p_i, p_j)}{\partial p_i} &= a - 2bp_i + p_j + bc = 0 \end{aligned} \quad (3)$$

And through symmetry we know that firm  $j$ 's PMP is

$$\begin{aligned} \max_{p_j \geq 0} \pi_j &= p_j [a - bp_j + p_i] - c [a - bp_j + p_i] \\ \frac{\partial \pi_j(p_i, p_j)}{\partial p_j} &= a - 2bp_j + p_i + bc = 0 \end{aligned} \quad (4)$$

where we now have two equations ((3) and (4)), and two choice variables ( $p_i$  and  $p_j$ ) to solve for.

### CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

- (a) Please find the Best Response Functions for each firm (i.e.  $BRF_i \equiv p_i(p_j)$  and  $BRF_j \equiv p_j(p_i)$ ). How does firm  $i$  respond with it's own price (i.e.  $p_i$ ) with an increase in  $a$ ,  $b$ ,  $c$  and  $p_j$ ?

$$\begin{aligned} BRF_i \equiv p_i(p_j) &= \frac{(a + bc)}{2b} + \frac{1}{2b}p_j \\ BRF_j \equiv p_j(p_i) &= \frac{(a + bc)}{2b} + \frac{1}{2b}p_i \end{aligned}$$

- (b) Find the optimal equilibrium allocation for each firm when they are competing a la Bertrand with differentiated products. That is, find  $p_i^*$  and  $p_j^*$ , and please simplify. Are these prices the same?

$$(p_i^*, p_j^*) = \left( \frac{(a + cb)}{(2b - 1)}, \frac{(a + cb)}{(2b - 1)} \right)$$

- (c) Find the optimal quantity demanded for each firm (i.e.  $q_i^* = a - bp_i^* + p_j^*$  and  $q_j^* = a - bp_j^* + p_i^*$ ).

$$(q_i^*, q_j^*) = \left( \frac{b(a - c(b - 1))}{(2b - 1)}, \frac{b(a - c(b - 1))}{(2b - 1)} \right)$$

- (d) Find the equilibrium profits of each firm (i.e.  $\pi_i^*$  and  $\pi_j^*$ ).

$$(\pi_i^*, \pi_j^*) = \left( \frac{b(a - c(b - 1))^2}{(2b - 1)^2}, \frac{b(a - c(b - 1))^2}{(2b - 1)^2} \right)$$

### 3. A Cournot Model with Product Differentiation

Consider two firms competing in quantities a la Cournot selling differentiated goods. The inverse demand function for every firm  $i$  is  $p_i(q_i, q_j) = a - bq_i - dq_j$  where  $i, j \in 1, 2$  and  $i \neq j$ . This means that the inverse demand functions for Firm 1 and Firm 2 are  $p_1(q_1, q_2) = a - bq_1 - dq_2$  and  $p_2(q_1, q_2) = a - bq_2 - dq_1$ , respectively. In this model  $b$  and  $d$  represent the degree of product differentiation for every firm  $i$  and  $j$ .  $b$  is assumed to be greater than zero (i.e.  $b > 0$ ), and for simplicity lets assume that  $d$  can take on any value between zero and  $b$  (i.e.  $b \geq d \geq 0$ ). The products are considered to be differentiated (also known as heterogeneous goods) if  $b \neq d$ , and if  $b = d$  the firms' products are identical (also known as homogeneous goods). Each firm has the same marginal cost of  $c$ , and all firm's fixed costs are assumed to be equal to zero (i.e.  $F = 0$ ). We formulate each firm's Profit Maximization Problem (i.e. PMP) as:

#### CALCULUS PART:

$$\begin{aligned} \max_{q_i \geq 0} \pi_i &= [a - (bq_i + dq_j)] q_i - cq_i \\ \frac{\partial \pi_i(q_i, q_j)}{\partial q_i} &= a - 2bq_i - dq_j - c = 0 \end{aligned} \quad (5)$$

And through symmetry we know that firm  $j$ 's PMP is

$$\begin{aligned} \max_{q_j \geq 0} \pi_j &= [a - (bq_j + dq_i)] q_j - cq_j \\ \frac{\partial \pi_i(q_i, q_j)}{\partial q_j} &= a - 2bq_j - dq_i - c = 0 \end{aligned} \quad (6)$$

where we now have two equations ((5) and (6)), and two choice variables ( $q_i$  and  $q_j$ ) to solve for.

#### CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

- Please find the Best Response Functions for each firm (i.e.  $BRF_i \equiv q_i(q_j)$  and  $BRF_j \equiv q_j(q_i)$ ). How does firm  $i$  respond with it's own quantity with an increase in  $a$ ,  $b$ ,  $c$ ,  $d$  and  $q_j$ ?
- Find the optimal equilibrium allocation for each firm when they are competing a la Cournot with differentiated products. That is, find  $q_i^*$  and  $q_j^*$ . What happens to  $q_i^*$  as  $d$  increases towards  $b$  (i.e.  $d \rightarrow b$ )? What happens to  $q_i^*$  as  $d$  decreases towards zero (i.e.  $d \rightarrow 0$ )? What is the intuition behind this?
- Find the optimal price for each firm (i.e.  $p_i^* = a - bq_i^* - dq_j^*$  and  $p_j^* = a - bq_j^* - dq_i^*$ ).
- Find the equilibrium profits of each firm (i.e.  $\pi_i^*$  and  $\pi_j^*$ ).
- Now, lets assume that the products are identical (i.e.  $b = d$ ). Plug  $b$  in for  $d$  in  $\pi_i^*$ , and simplify. What does this profit structure simplify to, and is it familiar? If so, what equilibrium profits are these profits analogous to?

- (f) Now, let's assume that the products are completely differentiated (i.e.  $d = 0$ ). Plug 0 in for  $d$  in  $\pi_i^*$ , and simplify. What does this profit structure simplify to, and is it familiar? If so, what equilibrium profits are these profits analogous to?

#### 4. A Bertrand Model with Product Differentiation by Location

##### Firms

Consider two firms competing a la Bertrand (i.e. a price competition) where each firm has constant marginal costs ( $c > 0$ ). Each firm is located at the end of a “Linear City” interval where the interval is a line ranging from 0 to 1 inclusive (i.e.  $[0, 1]$ ). In this setting, the product differentiation between each firm is its location.

##### Consumer Demand [ *The Hotelling Demand Model* ]

Consumers are distributed uniformly on this interval s.t. the distribution is a mass of consumers. We can imagine this distribution as being  $X \sim UNIF(0, 1)$  where the expectation of  $X$  is  $\frac{1}{2}$  ( i.e.  $\mathbb{E}(X) = \frac{1-0}{2} = \frac{1}{2}$  ). Let  $t$  be the per-unit transportation cost for the squared distance traveled ( $d^2$ ). This implies the cost for each consumer to travel a certain amount of distance is  $td^2$ . This expression can represent a multitude of costs to the consumer. It can be the consumer’s value of time, gasoline, or perhaps some learning curve when adopting the product. In this setting we will consider it to be the consumer’s “traveling cost” to each firm. We can think of the transportation cost for each consumer  $x$  for Firm A to be  $tx^2$  and for Firm B to be  $t(1-x)^2$  since the consumer mass is distributed along the unit interval of  $X$ . Let  $s$  be the gross consumer surplus (i.e. its maximum willingness to pay for a good) where we are going to assume that  $s$  is sufficiently high for all consumers to purchase a product. The utility that consumer  $i$  gets from consuming at most one unit good is

$$U_i = s - (p + td^2)$$

where  $p$  is the price of the good and  $td^2$  is the travel cost. Notice that we can represent this utility for the consumer that purchases a good from Firm A and B respectively as

$$U_i^A = s - (p_A + tx^2) \quad \text{and} \quad U_i^B = s - (p_B + t(1-x)^2)$$

because we are assuming these consumers are evenly distributed on the same unit interval they will be traveling.

- (a) Use these two derived utility functions to find the indifferent consumer (i.e.  $\hat{x}$ ). This can be found by setting  $U_i^A = U_i^B$  and solving for  $x$ . This  $x$  is  $\hat{x}$  and is considered to be the demand that Firm A will capture.

**Note:** I am *giving you the answer to Part A*. Please use these results in the analyses following Part A.

In order to determine the quantity demanded for each firm, we need to find the consumer that is just indifferent between buying from Firm A and Firm B.

$\implies$  Let,  $\hat{x}$  be the location of the indifferent consumer  $i$  s.t. the utility of consumer  $i$  is equal when buying from either firm s.t

$$U_i^A = U_i^B$$



$$s - p_A - t\hat{x}^2 = s - p_B - t(1 - \hat{x})^2$$

$$p_A + t\hat{x}^2 = p_B + t(1 - \hat{x})^2$$

$$p_A + t\hat{x}^2 = p_B + t(1 - 2\hat{x} + \hat{x}^2)$$

$$p_A + t\hat{x}^2 = p_B + t - 2t\hat{x} + t\hat{x}^2$$

$$2t\hat{x} = p_B - p_A + t$$

$$\hat{x} = \frac{p_B - p_A}{2t} + \frac{1}{2} \equiv Q_A(p_A, p_B)$$

where  $Q_A(p_A, p_B)$  is the amount of quantity demanded Firm A realizes.

This implies that Firm B enjoys the demand of

$$Q_B(p_A, p_B) \equiv 1 - Q_A(p_A, p_B) \equiv 1 - \hat{x}$$

$$= 1 - \left( \frac{p_B - p_A}{2t} + \frac{1}{2} \right)$$

$$\implies Q_B(p_A, p_B) \equiv \frac{p_A - p_B}{2t} + \frac{1}{2}$$

because we are working with a unit mass of consumers distributed across a unit interval from 0 to 1.

- (b) Using the demands derived in Part (a) we can set up the the Profit Maximization Problems (PMPs) as:

### CALCULUS PART:

Firm A's maximization problem is

$$\max_{p_A \geq 0} \pi_A(p_A, p_B) = (p_A - c) \underbrace{Q_A(p_A, p_B)}_{\hat{x}}$$

$$\max_{p_A \geq 0} \pi_A(p_A, p_B) = (p_A - c) \hat{x}$$

$$\max_{p_A \geq 0} \pi_A(p_A, p_B) = (p_A - c) \left( \frac{p_B - p_A}{2t} + \frac{1}{2} \right)$$

Taking the derivative with respect to  $p_A$  and setting equal to 0

$$\frac{\partial \pi_A(p_A, p_B)}{\partial p_A} = \frac{p_B - p_A + t}{2t} - \frac{p_A}{2t} + \frac{c}{2t} = 0 \quad (7)$$

Firm B's maximization problem is

$$\max_{p_B \geq 0} \pi_B(p_A, p_B) = (p_B - c) \underbrace{Q_B(p_A, p_B)}_{(1 - \hat{x})}$$

$$\begin{aligned}\max_{p_B \geq 0} \pi_B(p_A, p_B) &= (p_B - c)(1 - \hat{x}) \\ \max_{p_B \geq 0} \pi_B(p_A, p_B) &= (p_B - c) \left( \frac{p_A - p_B}{2t} + \frac{1}{2} \right)\end{aligned}$$

Taking the derivative with respect to  $p_B$  and setting equal to 0

$$\frac{\partial \pi_B(p_A, p_B)}{\partial p_B} = \frac{p_A - p_B + t}{2t} - \frac{p_B}{2t} + \frac{c}{2t} = 0 \quad (8)$$

where we now have two equations ((7) and (8)), and two choice variables ( $p_A$  and  $p_B$ ) to solve for.

### **CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.**

Using equations (7) and (8), please find the optimal price Firm A should set to capture their portion of the market (i.e.  $p_A^*$ ), and the optimal price Firm B should set to capture their portion of the market (i.e.  $p_B^*$ ). What happens when we set transportation costs to zero (i.e.  $t = 0$ )? What type of model does setting transportation costs equal to zero yield?

- (c) Find the optimal indifferent consumer (i.e.  $\hat{x}^*$ ), which is also the optimal demand Firm A will receive, and find  $(1 - \hat{x}^*)$ , which is the optimal demand Firm B will receive.
- (d) Find the equilibrium profits for Firm A and Firm B. What happens when we set  $t = 0$ ? What does this condition resemble?

## 5. A Three Firm Stackelberg Competition

Consider a Cournot competition in quantities between three firms where Firm 1 is the first mover, Firm 2 is the second mover, and Firm 3 is the third, and final, mover. To be clear, Firm 1 sets their optimal quantity first, Firm 2 sets their optimal quantity second, and Firm 3 sets their optimal quantity last. The market inverse demand function the firms face is  $p(Q) = a - b(q_1 + q_2 + q_3)$ , and all firms have the same marginal cost of  $c$ .  $a$  is assumed to be strictly greater than  $c$  (i.e.  $a > c$ ), and the marginal cost can be viewed as the price the firm pays for one additional unit of output. For simplicity, we assume the industry has no fixed costs of production (i.e.  $F = 0$ ). We solve the sequential move game by backwards induction where Firm 3 maximizes their profits based on their best response to the output set by Firm 1 and Firm 2. This means we can specify the Profit Maximization Problem (PMP) for Firm 3 as:

**CALCULUS PART:**

$$\begin{aligned} \max_{q_3 \geq 0} \pi_3 &= [a - b(q_1 + q_2 + q_3)] q_3 - cq_3 \\ \frac{\partial \pi_3(q_1, q_2, q_3)}{\partial q_3} &= a - 2bq_3 - bq_1 - bq_2 - c = 0 \\ \implies BRF_3 \equiv q_3(q_1, q_2) &= \frac{(a - c)}{2b} - \frac{1}{2}(q_1 + q_2) \end{aligned} \quad (9)$$

Where, we can now use this anticipated Best Response Function for Firm 3 in the Profit Maximization Problem of Firm 2 such that:

$$\begin{aligned} \max_{q_2 \geq 0} \pi_2 &= \left[ a - b(q_1 + q_2 + \underbrace{q_3(q_1, q_2)}_{BRF_3}) \right] q_2 - cq_2 \\ \max_{q_2 \geq 0} \pi_2 &= \left[ a - b \left( q_1 + q_2 + \left( \frac{(a - c)}{2b} - \frac{1}{2}(q_1 + q_2) \right) \right) \right] q_2 - cq_2 \end{aligned}$$

Which simplifies to:

$$\max_{q_2 \geq 0} \pi_2 = \left[ \frac{(a - c)}{2} - \frac{b}{2}(q_1 + q_2) \right] q_2$$

Where we can now find the Best Response Function for Firm 2 (i.e.  $BRF_2$ ) using calculus:

$$\begin{aligned} \frac{\partial \pi_2(q_1, q_2)}{\partial q_2} &= \frac{(a - c)}{2} - \frac{b}{2}q_1 - bq_2 = 0 \\ \implies BRF_2 \equiv q_2(q_1) &= \frac{(a - c)}{2b} - \frac{1}{2}q_1 \end{aligned} \quad (10)$$

Where, we can now use both of these anticipated Best Response Functions for Firm 2 and Firm 3 in the Profit Maximization Problem of Firm 1 such that:

$$\max_{q_1 \geq 0} \pi_1 = \left[ a - b \left( q_1 + \underbrace{q_2(q_1)}_{BRF_2} + \underbrace{q_3(q_1, q_2(q_1))}_{BRF_3} \right) \right] q_1 - cq_1$$

$$\max_{q_1 \geq 0} \pi_1 = \left[ a - b \left( q_1 + \underbrace{\left( \frac{(a-c)}{2b} - \frac{1}{2}q_1 \right)}_{BRF_2} + \underbrace{\left( \frac{(a-c)}{2b} - \frac{1}{2} \left( q_1 + \underbrace{\left( \frac{(a-c)}{2b} - \frac{1}{2}q_1 \right)}_{BRF_2} \right) \right)}_{BRF_3} \right) \right] q_1 - cq_1$$

$$\max_{q_1 \geq 0} \pi_1 = \left[ a - b \left( q_1 + \left( \frac{(a-c)}{2b} - \frac{1}{2}q_1 \right) + \left( \frac{(a-c)}{2b} - \frac{1}{2} \left( q_1 + \left( \frac{(a-c)}{2b} - \frac{1}{2}q_1 \right) \right) \right) \right) \right] q_1 - cq_1$$

Which simplifies to:

$$\max_{q_1 \geq 0} \pi_1 = \left[ \frac{(a-c)}{4} - \frac{b}{4}q_1 \right] q_1$$

Where we can now find the optimal equilibrium quantity set by Firm 1 (i.e.  $q_1^*$ ) using calculus:

$$\begin{aligned} \frac{\partial \pi_1(q_1)}{\partial q_1} &= \frac{(a-c)}{4} - \frac{b}{2}q_1 = 0 \\ \implies q_1^* &= \frac{(a-c)}{2b} \end{aligned} \tag{11}$$

where we now have the optimal equilibrium allocation for Firm 1 (equation (11)), and two Best Response Functions (equations (9) and (10)) for the firms that will move sequentially after Firm 1 has set their quantity as  $q_1^*$ .

### **CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.**

- Please find the equilibrium quantity each firm will set in order to maximize their profits in this sequential move game (i.e. write  $q_1^*$ , and then find  $q_2^*$  and  $q_3^*$ ). Are the quantities decreasing or increasing relative to Firm 1's quantities? Why is this?
- Please find the equilibrium price each firm faces (i.e. find  $p(Q^*)$ ).
- Please find the equilibrium profits of each individual firm (i.e. find  $\pi_1^*$ ,  $\pi_2^*$  and  $\pi_3^*$ ). Please rank them in the order of who is most profitable to the least profitable.