EconS 305: Intermediate Microeconomics w/o Calculus
Homework 2: Firm Behavior and Decision Making

Due: Friday, May 29th, 2020 at 5:00pm via Blackboard

- Please submit all homework solutions in the order the questions are presented and as **one .PDF**.

- Please **show all calculations** as these exercises are meant to refine your quantitative tool set. If I can not follow your calculations or it seems as you just "copy and pasted" answers from the internet, I will be deducting half the points from that solution.

1. **Cobb-Douglas Production Function with $\alpha + \beta = 1$ and a Fixed Cost**

   Consider the following profit maximization problem for the firm where the firm needs to decide the optimal amount of capital (K) and labor (L) that will maximize their profits. We model the firm's production output, using these inputs, with the classic Cobb-Douglas Production function of $Q = Af(K, L) = AK^{\alpha}L^{\beta}$. Note that A represents the level of total factor productivity where this can be interpreted differently depending on if A is greater than or less than ($\leq$ *or* $\geq$) 1. For more information about this production function please navigate to to this website (http://www2.hawaii.edu/∼fuleky/anatomy/anatomy.html). For simplicity, we assume that $\alpha + \beta = 1$ and $\alpha > 0$, which means that the firm does not specialize in just one good and that the production function yields constant returns to scale. Further note that to get total revenues ($TR(P, Q)$) of the firm, we multiply output (Q) by the price (P) they are receiving for each unit of output. Notice that this price is not a choice variable for the firm since they are being considered as price takers.

   Next, we introduce the firm's marginal costs where p_K is the price the firm has to pay to operate/maintain their capital, and p_L is the price (or wage rate) the firm needs to pay to use one unit of labor in production. We also introduce a fixed cost (F) in which the firm has to pay in order to enter the market and compete. We denote this as an "entry fee". Note that the total cost of the firm is $TC(K, L) = p_K K + p_L L + F$ where, in the field of economics, p_K and p_L are referred to as the firm's marginal costs of capital and labor, respectively.

   We represent the firm's Profit Maximization Problem (PMP) as Maximizing Total Revenues

$$\max_{K,L \geq 0} \; TR(K, L) = PAf(K, L)$$

$$\max_{K,L\geq 0} \; TR(K,L) = PAK^{\alpha}L^{(1-\alpha)}$$

subject to the firm's cost constraint of:

$$p_K K + p_L L + F = TC$$

CALCULUS PART:

Using constrained optimization techniques from calculus, we can set the problem up with a Lagrange multiplier s.t.

$$\mathcal{L}(K,L) = PAK^{\alpha}L^{1-\alpha} - \lambda[p_K K + p_L L + F - TC]$$

From here, we can take our derivatives and set them equal to zero

$$\frac{\partial \mathcal{L}(K,L)}{\partial K} = \alpha PAK^{(\alpha-1)}L^{1-\alpha} - \lambda p_K = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}(K,L)}{\partial L} = (1-\alpha)PAK^{\alpha}L^{(-\alpha)} - \lambda p_L = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}(K,q_2)}{\partial \lambda} = p_K K + p_L L + F - TC = 0 \tag{3}$$

where we now have three equations ((1),(2), and (3)), and two choice variables (K and L) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

(a) Find the firm's optimal allocation of inputs (K and L) to maximize its profits (i.e. find K^* and L^*). In economics, we also refer to this allocation as the firm's factor demands.

The optimal allocation of capital and labor respectively, is

$$K^*(p_K,\alpha,TC,F), L^*(p_L,\alpha,TC,F) = \left(\frac{\alpha}{p_K}(TC-F)\,,\; \frac{(1-\alpha)}{p_L}(TC-F) \right)$$

(b) What happens when the fixed costs for entry (F) are greater than the firm's total costs?

This would imply that the optimal allocation of capital and labor would be negative for the firm, and the firm would not enter the market.

(c) Find the firm's profits in equilibrium and simplify the equation. The firm's profits in equilibrium can be represented by:

$$\pi^*(K^*, L^*) = PA(K^*)^\alpha (L^*)^{(1-\alpha)} - p_K K^* - p_L L^* - F$$

$$\pi^* = PA\left(\frac{\alpha}{p_K}(TC-F)\right)^\alpha \left(\frac{(1-\alpha)}{p_L}(TC-F)\right)^{(1-\alpha)} - p_K\frac{\alpha}{p_K}(TC-F) - p_L\frac{(1-\alpha)}{p_L}(TC-F) - F$$

Simplifying

$$\pi^* = \left(PA\left(\frac{\alpha}{p_K}\right)^\alpha \left(\frac{(1-\alpha)}{p_L}\right)^{(1-\alpha)}\right)(TC-F) - TC$$

(d) Assuming that the firm has a total cost budget that cannot exceed $\overline{TC}$, find the condition on the entry fee (F) in which the firm will enter the market. To do this, set $0 < \pi^*$ and then solve for the entry fee (i.e. $F <$"the rest of the variables")

$$F < \frac{(\gamma - 1)}{\gamma}\overline{TC}$$

where

$$\gamma = \left(PA\left(\frac{\alpha}{p_K}\right)^\alpha \left(\frac{(1-\alpha)}{p_L}\right)^{(1-\alpha)}\right)$$

Intuitively, if the entry fee is less than $\frac{(\gamma-1)}{\gamma}\overline{TC}$, then the firm will find it profitable to enter since the entry fee is not too high.

2. **Cobb-Douglas with General Preferences without using the Lagrangian**

Consider the same setting we were operating in in Question 1, but now lets consider that $0 < \alpha + \beta \leq 1$. Notice that we cannot simply the problem as we did before (by replacing $\beta = 1 - \alpha$). Also notice that we are not including an entry fee for the firm.

We set up the firm's Profit Maximization Problem (PMP) as

$$\max_{K,L \geq 0} \pi(K,L) = PAf(K,L) - TC(K,L)$$

$$\max_{K,L \geq 0} \pi(K,L) = PAK^{\alpha}L^{\beta} - (p_K K + p_L L)$$

CALCULUS PART:

Notice that similar to Question 1, the firm's problem is a constrained optimization problem, but we are not using the Lagrangian method to solve it. This is because, as in a lot of cases, we do not need to consider a firm budget constraint because they do not have a total cost limit. If it is feasible for them to produce, they will as long as it is profitable.

$$\max_{K,L \geq 0} \pi(K,L) = PAK^{\alpha}L^{\beta} - p_K K - p_L L$$

From here, we can take our derivatives and set them equal to zero

$$\frac{\partial \pi(K,L)}{\partial K} = \alpha PAK^{(\alpha-1)}L^{\beta} - p_K = 0 \tag{4}$$

$$\frac{\partial \pi(K,L)}{\partial L} = \beta PAK^{\alpha}L^{(\beta-1)} - p_L = 0 \tag{5}$$

where we now have two equations ((4) and (5)), and two choice variables (K and L) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

(a) Find the firm's optimal allocation of inputs (K and L) to maximize its profit in equilibrium (i.e. find K^* and L^*).

The optimal allocation of capital and labor respectively, is

$$K^*(P, A, p_K, p_L, \alpha, \beta), L^*(P, A, p_K, p_L, \alpha, \beta)$$

$$= \left(\left[\frac{1}{PA} \left(\frac{p_L}{\beta}\right)^{\beta} \left(\frac{p_K}{\alpha}\right)^{(1-\beta)} \right]^{\frac{1}{\alpha+\beta-1}}, \frac{\beta}{\alpha}\frac{p_K}{p_L} \left[\frac{1}{PA} \left(\frac{p_L}{\beta}\right)^{\beta} \left(\frac{p_K}{\alpha}\right)^{(1-\beta)} \right]^{\frac{1}{\alpha+\beta-1}} \right)$$

(b) What happens when we invoke the restriction that $\alpha + \beta = 1$?

The optimal allocation of capital and labor become an "undefined" function in the power function of each demand. This implies that if the preference parameters sum to 1, then the production function will have constant returns to scale, and the optimal selection of capital and labor would be "infinity". In other words, there would be no constraint, and the firm would keep producing in order to maximize their profits. Note that the constraint of the function we have here is that the production function has diminishing returns to scale (i.e. $\alpha + \beta < 1$), so the optimal allocation is feasible.

(c) What is the Marginal Rate of Technical Substitution ($MRTS_{K,L}$)?

$$MRTS_{K,L} = \frac{MP_L}{MP_K} = \frac{\beta PAK^{\alpha}L^{(\beta-1)}}{\alpha PAK^{(\alpha-1)}L^{\beta}} = \frac{\beta K}{\alpha L}$$

Where intuitively, this is the rate in which you would have to substitute labor (L), to get one more unit of capital (K), at a given point on the isoquant curve.

3. **A Simple Cobb-Douglas, a Numerical Example**

Consider a similar setting to above, but we are going to simplify the model for an ease of calculation. We set $\alpha = \beta = \frac{1}{3}$, $A = 1$, and $F = 0$.

We set up the firm's Profit Maximization Problem (PMP) as

$$\max_{K,L\geq 0} \pi(K,L) \quad = Pf(K,L) - TC(K,L)$$

$$\max_{K,L\geq 0} \pi(K,L) \quad = PK^{\frac{1}{3}}L^{\frac{1}{3}} - (p_K K + p_L L)$$

CALCULUS PART:

$$\max_{K,L\geq 0} \pi(K,L) \quad = PK^{\frac{1}{3}}L^{\frac{1}{3}} - p_K K - p_L L$$

From here, we can take our derivatives and set them equal to zero

$$\frac{\partial \pi(K,L)}{\partial K} = \frac{1}{3}PK^{(-\frac{2}{3})}L^{\frac{1}{3}} - p_K = 0 \tag{6}$$

$$\frac{\partial \pi(K,L)}{\partial L} = \frac{1}{3}PK^{\frac{1}{3}}L^{(-\frac{2}{3})} - p_L = 0 \tag{7}$$

where we now have two equations ((6) and (7)), and two choice variables (K and L) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

(a) Find the firm's optimal allocation of inputs (K and L) to maximize its profit in equilibrium (i.e. find K^* and L^*).

The optimal allocation of capital and labor respectively, is

$$K^*(P, p_K, p_L), L^*(P, p_K, p_L) = \left(\frac{P^3}{27p_K^2 p_L}, \frac{P^3}{27p_K p_L^2} \right)$$

(b) What happens to each factor demand with an increase in it's own price (p_i) and the other factor's price (p_j)? Which one has a larger effect? What happens to output as the market price the firm is receiving (P) increase?

The factor demand for each good decreases with respect to their own price and the other good's factor price. Note that an increase in their own factor price decrease at a faster rate than if the other good's price increases. As market price increases, the firm's demand for each factor increases, and it increases at a faster rate than when they decreases due to an increases in the cost of the inputs..

(c) Please find the optimal supply (Q^*). Note that we can get this by plugging K^* and L^* back into the production function.

$$Q^* = (K^*)^{\frac{1}{3}}(L^*)^{\frac{1}{3}} = \left(\frac{P^3}{27p_K^2 p_L}\right)^{\frac{1}{3}} \left(\frac{P^3}{27p_K p_L^2}\right)^{\frac{1}{3}}$$

$$Q^* = \frac{P^2}{27p_K p_L}$$

(d) Please find the optimal profits for the firm (π^*). Note that we can get this by plugging K^*, L^*, and Q^* back into the profit function ($\pi^*(K^*, L^*, Q^*(K^*, L^*))$. To be clear, plug in the values you found as such

$$\pi^*(K^*, L^*, Q^*(K^*, L^*)) = PQ^* - p_K K^* - p_L L^*$$

$$\pi^*(K^*, L^*, Q^*(K^*, L^*)) = P\left(\frac{P^2}{27p_K p_L}\right) - p_K\left(\frac{P^3}{27p_K^2 p_L}\right) - p_L\left(\frac{P^3}{27p_K p_L^2}\right)$$

Simplifying

$$\pi^*(P, p_K, p_L) = \frac{P^3}{27p_K p_L}$$

4. **Cost Minimization Problem using a Lagrangian**

Consider the same setting we were operating in Question 1, but now lets change our analysis so that the firm is minimizing costs subject to the same Cobb-Douglas production function in Question 1. Intuitively, we can think of $\overline{Q}$ as the total maximum amount the market is willing to buy of the firm's good. This is the quantity demanded (Q_D) in which the firm will also supply (Q_S).

We consider the same conditions on the production function parameters where $\alpha + \beta = 1$ and $\alpha > 0$. Notice, similar to Question 1, we can simply the problem as we did before by replacing $\beta = 1 - \alpha$. Just as in Question 1, F is the entry fee to enter the market and that A is the level of total factor productivity.

We now set up the firm's Profit Maximization Problem (PMP) as a Cost Minimizing Problem (CMP).

$$\min_{K,L \geq 0} \quad C(K,L) = p_K K + p_L L + F$$

subject to the budget constraint of:

$$AK^{\alpha}L^{(1-\alpha)} = \overline{Q}$$

CALCULUS PART:
Using constrained optimization techniques from calculus, we can set the problem up with a Lagrange multiplier s.t.

$$\mathcal{L}(K, L : \lambda) = p_K K + p_L L + F - \lambda [AK^{\alpha}L^{1-\alpha} - \overline{Q}]$$

From here, we can take our derivatives and set them equal to zero

$$\frac{\partial \mathcal{L}(K,L;\lambda)}{\partial K} \quad = \quad p_K - \lambda [\alpha A K^{\alpha-1} L^{1-\alpha}] = 0 \tag{8}$$

$$\frac{\partial \mathcal{L}(K,L;\lambda)}{\partial q_2} \quad = \quad p_L - \lambda [(1-\alpha) A K^{\alpha} L^{(-\alpha)}] = 0 \tag{9}$$

$$\frac{\partial \mathcal{L}(q_1,q_2;\lambda)}{\partial \lambda} \quad = \quad AK^{\alpha}L^{1-\alpha} - \overline{Q} = 0 \tag{10}$$

where we now have three equations ((8),(9), and (10)), and two choice variables (K and L) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

(a) Find the firm's factor demands for K and L in equilibrium (i.e. find K^* and L^*).

$$K^*(p_K, p_L, A, \alpha, \overline{Q}), L^*(p_K, p_L, A, \alpha, \overline{Q}) = \left(\frac{\overline{Q}}{A} \left(\frac{\alpha}{(1-\alpha)} \frac{p_L}{p_K} \right)^{(1-\alpha)}, \frac{\overline{Q}}{A} \left(\frac{(1-\alpha)}{\alpha} \frac{p_K}{p_L} \right)^{\alpha} \right)$$

(b) What happens with an increases in $\overline{Q}$? Why does this happen?

As $\overline{Q}$ increases, both factor demands increase in order to minimize costs. This happens because since we are minimizing costs we are constrained by the production output the firm can produce, and what the firm produces can actually be interpreted as what the market demands (i.e. $Q_S = Q_D$ in equilibrium). This means that, as the quantity demanded goes up, so doesn't the quantity supplied, which can be represented by $\overline{Q}$.

(c) What is the Marginal Rate of Technical Substitution ($MRTS_{KL}$)?
Dividing the marginal product of capital (MP_K) by the marginal product of labor (MP_L) we get

$$
\begin{aligned}
MRTS_{KL} &= \frac{MP_K}{MP_L} \\
&= \frac{\alpha A K^{\alpha-1} L^{1-\alpha}}{(1-\alpha) A K^{\alpha} L^{-\alpha}} \\
&= \frac{\alpha L}{(1-\alpha)K}
\end{aligned}
$$

(d) What is the $MRTS_{KL}$ equal to in this problem? Does it make sense?

$$MRTS_{KL} \equiv \frac{\alpha L}{(1-\alpha)K} = \frac{p_K}{p_K}$$

It is equal to the price ratio of the inputs, and yes it makes sense as this is the necessary condition for most maximization problems.

5. **A Firm Cost Analysis**

Consider a firm with a total cost curve of $TC = 20Q^2 + 8Q + 90$ and marginal cost of $MC = 40Q + 8$. Without knowing the firm's production function, you are asked to conduct a cost analysis that will answer the following questions.

(a) What is the firm's fixed cost, variable cost, average total cost, and average variable cost?

i) $FC = 90$

ii) $VC = 20Q^2 + 8Q$

iii) $ATC = 20Q + 8 + 90\frac{1}{Q}$

iv) $AVC = 20Q + 8$

(b) Find the output level (Q) that minimizes average total cost.

Minimized ATC is at $Q = \frac{3}{\sqrt{2}}$

(c) Find the output level at which average variable cost is minimized.

Minimized AVC is at $Q = 0$

Question 1 (Math)

1a) $$\frac{\alpha \cancel{P} \cancel{A} K^{(\alpha-1)} L^{(1-\alpha)}}{(1-\alpha)\cancel{P}\cancel{A}\cancel{K^{\alpha}}\cancel{L^{-\alpha}}} = \frac{P_K}{P_L}$$

$$\Rightarrow \quad L = \frac{(1-\alpha)}{\alpha}\frac{P_K}{P_L}K \quad \Leftarrow (L(K))$$

$\Rightarrow$ Plugging into (B)

$$P_K K + \cancel{P_L}\left(\frac{(1-\alpha)}{\alpha}\frac{P_K}{\cancel{P_L}}K\right) + F = TC$$

$$\Rightarrow \quad P_K K\left(1 + \frac{(1-\alpha)}{\alpha}\right) + F = TC$$

$$\Rightarrow \quad K^* = \frac{\alpha}{P_K}(TC - F)$$

$\Rightarrow$ Plugging K^* into $L(K)$

$$L^* = \frac{(1-\alpha)}{\cancel{\alpha}}\frac{\cancel{P_K}}{P_L}\left(\frac{\cancel{\alpha}}{\cancel{P_K}}(TC-F)\right)$$

$$\Rightarrow \quad L^* = \frac{(1-\alpha)}{P_L}(TC-F)$$

1B) What happens if $F > TC$?

$\Rightarrow$ Negative output, Non feasible.

1c) Find, $\pi^* = PA(K^*)^{\alpha}(L^*)^{1-\alpha} - P_K K^* - P_L L^* - F$

$$\Rightarrow \pi^* = PA\left(\frac{\alpha}{P_K}(TC-F)\right)^{\alpha}\left(\frac{(1-\alpha)}{P_L}(TC-F)\right)^{1-\alpha} - P_K\left(\frac{\alpha}{P_K}(TC-F)\right) - P_L\left(\frac{(1-\alpha)}{P_L}(TC-F)\right) - F$$

$$\Rightarrow \quad = PA\left(\frac{\alpha}{P_K}\right)^{\alpha}\left(\frac{(1-\alpha)}{P_L}\right)^{1-\alpha}(TC-F) - \underbrace{\alpha(TC-F) - (1-\alpha)(TC-F)}_{= -(TC-F)} - F$$

$$\Rightarrow \pi^* = \left(PA\left(\frac{\alpha}{P_K}\right)^{\alpha}\left(\frac{(1-\alpha)}{P_L}\right)^{1-\alpha}\right)(TC-F) - TC$$

1D) Set $\pi^* = 0$, assume we have a TC budget ($\bar{TC}$), & solve $\underline{F}$.

$\Rightarrow$ <u>Need</u> $0 < \left(PA\left(\frac{\alpha}{P_K}\right)^{\alpha}\left(\frac{(1-\alpha)}{P_L}\right)^{(1-\alpha)}\right)(\bar{TC}-F)-\bar{TC}$

$\Rightarrow$ Let $\gamma = PA\left(\frac{\alpha}{P_K}\right)^{\alpha}\left(\frac{(1-\alpha)}{P_L}\right)^{(1-\alpha)}$

$\Rightarrow$ $0 < \gamma(\bar{TC}-F)-\bar{TC}$

$\Rightarrow$ $\bar{TC} < \gamma\bar{TC}-\gamma F$

$\Rightarrow$ $\gamma F < \bar{TC}(\gamma-1)$

$\Rightarrow$ $F < \frac{(\gamma-1)}{\gamma}\bar{TC}$ $\Rightarrow$ If Fixed Costs are less than than $\frac{(\gamma-1)}{\gamma}TC$, then the firm will find it profitable to enter since the entry fee (F) is not too high.

c)

Question #2 (Math)

2a) using equation (4) & (5) we have

(4) (5)

$$\Rightarrow \alpha P A K^{(\alpha-1)} L^{\beta} = P_K \quad \& \quad \beta P A K^{\alpha} L^{\beta-1} = P_L$$

where, we can use a trick (which is not necessary) o & multiplying (4) on both sides by "K" & (5) on both sides "L" we have

$$\Rightarrow \alpha P A K^{\alpha} L^{\beta} = P_K K \quad (4) \quad \& \quad \beta P A K^{\alpha} L^{\beta} = P_L L$$

$$\Rightarrow P A K^{\alpha} L^{\beta} = \frac{P_K K}{\alpha} \quad \& \quad P A K^{\alpha} L^{\beta} = \frac{P_L L}{\beta}$$

Combining since both $= P A K^{\alpha} L^{\beta}$

$$\Rightarrow \frac{P_K K}{\alpha} = \frac{P_L L}{\beta}$$

solving for L

L as a function of K

$$\Rightarrow L = \frac{\beta}{\alpha} \frac{P_K}{P_L} K$$

plugging back into (4)

$$\Rightarrow \alpha P A K^{(\alpha-1)} \left(\frac{\beta}{\alpha} \frac{P_K}{P_L} K \right)^{\beta} = P_K$$

And solving for K

$$\Rightarrow \alpha P A K^{(\alpha+\beta-1)} \left(\frac{\beta}{\alpha} \frac{P_K}{P_L} \right)^{\beta} = P_K$$

$$\Rightarrow K^{\alpha+\beta-1} = \frac{1}{\alpha P A} \left(\frac{\alpha}{\beta} \frac{P_L}{P_K} \right)^{\beta} P_K$$

$\Rightarrow$

Copying Over

$$\Rightarrow \quad K^{\alpha+\beta-1} = \frac{1}{\alpha PA}\left(\frac{\alpha}{\beta}\frac{P_L}{P_K}\right)^{\beta} P_K$$

$$\Rightarrow \quad K^{\alpha+\beta-1} = \frac{1}{PA}\left(\frac{P_L}{\beta}\right)^{\beta}\left(\frac{P_K}{\alpha}\right)^{1-\beta}$$

$$\Rightarrow \quad K^{*} = \left[\frac{1}{PA}\left(\frac{P_L}{\beta}\right)^{\beta}\left(\frac{P_K}{\alpha}\right)^{1-\beta}\right]^{\frac{1}{\alpha+\beta-1}} \quad (*)$$

Answer

pluggin $(*)$ into $L = \frac{\beta}{\alpha}\frac{P_K}{P_L}K$ from above

$$\Rightarrow \quad L^{*} = \frac{\beta}{\alpha}\frac{P_K}{P_L}\left[\frac{1}{PA}\left(\frac{P_L}{\beta}\right)^{\beta}\left(\frac{P_K}{\alpha}\right)^{1-\beta}\right]^{\frac{1}{\alpha+\beta-1}}$$

Part B: See Above

Part C: See Above

Question # 3 (Math)

a) Using (6) & (7)

$$\Rightarrow \quad \frac{1}{3} P K^{-2/3} L^{\frac{1}{3}} = P_K \qquad \& \qquad \frac{1}{3} P K^{1/3} L^{-2/3} = P_L$$

$\Rightarrow$ Lets solve this the typical way...

$\Rightarrow$ Solve for K in (6)

$$\Rightarrow \quad K^{2/3} P_K = \frac{1}{3} P L^{1/3}$$

$$\Rightarrow \quad K^{2/3} = \frac{P}{3P_K} L^{1/3}$$

$$\Rightarrow \quad K = \left(\frac{P}{3P_K}\right)^{3/2} L^{3/6} \equiv \left(\frac{P}{3P_K}\right)^{3/2} L^{1/2}$$

Plugging into (7)

$$\Rightarrow \quad L^{2/3} = \frac{1}{3P_L} P \left(\left(\frac{P}{3P_K}\right)^{3/2} L^{1/2} \right)^{1/3}$$

(taking everything to the third power)

$$L^2 = \left(\frac{P}{3P_L}\right)^3 \left(\frac{P}{3P_K}\right)^{3/2} L^{1/2}$$

$$\Rightarrow \quad L^{3/2} = \left(\frac{P}{3P_L}\right)^3 \left(\frac{P}{3P_K}\right)^{3/2}$$

$$\Rightarrow \quad L^* = \left[\left(\frac{P}{3P_L}\right)^3 \left(\frac{P}{3P_K}\right)^{3/2} \right]^{2/3}$$

$$= \left(\frac{P}{3P_L}\right)^2 \frac{P}{3P_K} \equiv \frac{P^3}{27 P_L^2 P_K}$$

$\Rightarrow$ Optimal Labor choice is

$$L^* = \frac{P^3}{27 P_L^2 P_K}$$

Plugging L^* back into equation 6 (the simplified one) $\Rightarrow$

$$\Rightarrow \quad K^* = \left(\frac{P}{3P_K}\right)^{3/2}\left(L^*\right)^{1/2}$$

$$\Rightarrow \quad K^* = \left(\frac{P}{3P_K}\right)^{3/2}\left(\frac{P^3}{27P_L^2P_K}\right)^{1/2}$$

$$K^* = \left(P^{3/2}P^{3/2}\right)\left(\frac{1}{3P_K}\right)^{3/2}\left(\frac{1}{27P_L^2P_K}\right)^{1/2}$$

$$= P^{(3/2+3/2)}\left(\frac{1}{3^{3/2+3/2}}\right)\left(\frac{1}{P_K^{3/2+1/2}}\right)\left(\frac{1}{P_L}\right)$$

$$= P^3\left(\frac{1}{3^3P_K^2P_L}\right)$$

$$= \frac{P^3}{27P_K^2P_L}$$

3b) see key

$$\Rightarrow \quad (K^*, L^*) = \left(\frac{P^3}{27P_K^2P_L}, \frac{P^3}{27P_L^2P_K}\right)$$

3c) Optimal Supply

$$\Rightarrow \quad Q^* = (K^*)^{1/3}(L^*)^{1/3} = \left(\frac{P^3}{27P_K^2P_L}\right)^{1/3}\left(\frac{P^3}{27P_KP_L^2}\right)^{1/3}$$

$$= \frac{P^2}{3^2P_KP_L} = \frac{P^2}{9P_KP_L}$$

3d) Optimal Profit

$$\Rightarrow \quad \pi^*(P, P_K, P_L) = PQ^* - P_KK^* - P_LL^*$$

$$= \frac{P^3}{9P_KP_L} - \frac{2P^3}{27P_KP_L}$$

$$= \left(\frac{3}{27} - \frac{2}{27}\right)\frac{P^3}{P_KP_L} = \frac{P^3}{27P_KP_L}$$

Question #4
(Math)

4a) using (8) & (9), & solving for λ

$$\Rightarrow \frac{P_K}{\alpha A K^{\alpha-1} L^{1-\alpha}} = \lambda \quad \& \quad \frac{P_L}{(1-\alpha) A K^{\alpha} L^{-\alpha}} = \lambda$$

Combining & Flipping $\Rightarrow$

$$\frac{\alpha \cancel{A} K^{\alpha-1} L^{1-\alpha}}{P_K} = \frac{(1-\alpha) \cancel{A} K^{\alpha} L^{-\alpha}}{P_L}$$

$\leftarrow$ same condition as Question 1

$$\Rightarrow \quad L = \frac{(1-\alpha)}{\alpha} \frac{P_K}{P_L} K \quad (*)$$

plugging into (10)

$$\Rightarrow \quad A K^{\alpha} \left(\frac{(1-\alpha)}{\alpha} \frac{P_K}{P_L} K \right)^{1-\alpha} = \bar{Q}$$

$$\Rightarrow \quad K^* = \frac{1}{A} \left(\frac{\alpha}{(1-\alpha)} \frac{P_L}{P_K} \right)^{1-\alpha} \bar{Q}$$

plugging into $(*)$

$$L^* = \frac{(1-\alpha)}{\alpha} \frac{P_K}{P_L} \left(\frac{1}{A} \left(\frac{\alpha}{(1-\alpha)} \frac{P_L}{P_K} \right)^{1-\alpha} \bar{Q} \right)$$

$$= \frac{1}{A} \left(\left(\frac{\alpha}{(1-\alpha)} \right)^{-1} \left(\frac{\alpha}{(1-\alpha)} \right)^{1-\alpha} \right) \left(\left(\frac{P_K}{P_L} \right)^{-1} \left(\frac{P_K}{P_L} \right)^{1-\alpha} \right) \bar{Q}$$

$$= \frac{1}{A} \left(\frac{\alpha}{(1-\alpha)} \frac{P_K}{P_L} \right)^{-\alpha} \bar{Q}$$

or

$$L^* = \frac{1}{A} \left(\frac{(1-\alpha)}{\alpha} \frac{P_L}{P_K} \right)^{\alpha} \bar{Q}$$

$\Rightarrow$ In summary,

$$K^*(P_K, P_L, A, \alpha, \bar{Q}), L^*(P_K, P_L, A, \alpha, \bar{Q}) = \left(\frac{1}{A} \left(\frac{\alpha}{(1-\alpha)} \frac{P_L}{P_K} \right)^{1-\alpha} \bar{Q} \ , \ \frac{1}{A} \left(\frac{(1-\alpha)}{\alpha} \frac{P_K}{P_L} \right)^{\alpha} \bar{Q} \right)$$

Question 5 (Math)

5a) i) A Fixed Cost is a cost that does not vary with output.

$\Rightarrow$ Set $Q = 0$ $\Rightarrow$ $TC = 20(0)^2 + 8(0) + 90$

$\Rightarrow$ $\boxed{FC = 90}$

ii)

$VC = TC - FC$

$\Rightarrow \underbrace{20Q^2 + 8Q + 90}_{TC} - \underbrace{90}_{FC}$

$\Rightarrow \boxed{VC = 20Q^2 + 8Q}$

iii) Average Total Cost (ATC)

$\Rightarrow \boxed{\frac{TC}{Q} = 20Q + 8 + 90/Q}$

iv) Average Variable Cost (AVC)

$\Rightarrow \frac{VC}{Q} = \frac{20Q^2 + 8Q}{Q} = \boxed{20Q + 8}$

5b) Minimum Average Total Cost (ATCmin) happens when

$ATC = MC$

$\Rightarrow 20Q + \cancel{8} + 90/Q = 40Q + \cancel{8}$

$90/Q = 20Q$

$\Rightarrow 90/20 = Q^2 \Rightarrow Q = \sqrt{9/2}$

$\boxed{= 3/\sqrt{2}}$

5c) Minimum AVC occures when

$AVC = MC$

$\Rightarrow 20Q + \cancel{8} = 40Q + \cancel{8}$

$\Rightarrow 20Q = 0 \Rightarrow$ @ $\boxed{Q = 0}$