Name:

EconS 424: Strategy and Game Theory Midterm #1 February 28th, 2020

1 A Cournot Game of Competing in Quantities w/ Fixed Costs

Consider two firms competing a la Cournot in a market with an inverse demand function of $p(Q) = a - b(Q)$ where $Q = q_i + q_j$ and total cost function of $TC_i(q_i) = F + c_i q_i$. Notice that each firm has the same fixed cost (F) but their marginal costs (c_i) are not equal to each other (i.e. $c_i \neq c_j$). This means these homogeneous product producing firms have asymmetric costs, and we can represent the Profit Maximization Problem (PMP_i) for firm i as:

$$\max_{q_i \geq 0} \ \pi_i = [a - b(q_i + q_j)]\, q_i - (F + c_i q_i)$$

1. (8 pts.) Find the Best Response Functions ($BRFs$) for each firm. How does the firm respond in their own quantities with respect to an increase in a, b, c_i, and q_j? More specifically, what are $\frac{\partial q_i(q_j)}{\partial a}, \frac{\partial q_i(q_j)}{\partial b}, \frac{\partial q_i(q_j)}{\partial c_i}, \frac{\partial q_i(q_j)}{\partial q_j}$, and what are their signs?

$$\max_{q_i \geq 0} \ \pi_i = [a - b(q_i + q_j)]\, q_i - (F + c_i q_i)$$

$$\frac{\partial \pi_i(q_i, q_j)}{\partial q_i} = a - 2bq_i - bq_j - c_i = 0$$

$$\implies \qquad BRF_i \equiv q_i(q_j) = \frac{(a - c_i)}{2b} - \frac{1}{2}q_j$$

And through symmetry we know

$$\implies \qquad BRF_j \equiv q_j(q_i) = \frac{(a - c_j)}{2b} - \frac{1}{2}q_i$$

Taking the comparative statics we get

$$\frac{\partial q_i(q_j)}{\partial a} = \frac{1}{2b} > 0 \quad , \quad \frac{\partial q_i(q_j)}{\partial b} = -\frac{(a - c_i)}{2b^2} < 0 \quad , \quad \frac{\partial q_i(q_j)}{\partial c_i} = -\frac{1}{2b} < 0 \quad , \quad \frac{\partial q_i(q_j)}{\partial q_j} = -\frac{1}{2} < 0$$

Where, intuitively we can see that as the demand curve shifts right, which increases our demand curve intercept (a), the firm responds with a higher quantity. As the slope of the demand curve (b) increases, which means it would become more negative, the firms responds is to decrease it's output. As the firm's own marginal cost (c_i) increases, it makes sense that the firm would have to respond by also decreasing it's own quantity produced. Lastly, when firm j increases their quantity produced, it is the best response of firm i to decrease production.

2. (7 pts.) Find the optimal equilibrium allocation for each firm when they are competing a la Cournot. That is, find q_i^* and q_j^*. How does firm i's equilibrium allocation change with respect to an increase in their own marginal costs (c_i) and their opponents marginal cost (c_j)? Which increase is larger in absolute magnitude?

   We yield the equilibrium allocation by simultaneously solving the BRFs s.t.

$$q_i(q_j) = \frac{(a-c_i)}{2b} - \frac{1}{2}\left[\frac{(a-c_j)}{2b} - \frac{1}{2}q_i\right]$$
$$4bq_i = 2(a-c_i) - (a-c_i) + bq_j$$
$$\implies \quad q_i^* = \frac{(a-2c_i+c_j)}{3b} \quad \textit{and} \quad q_j^* = \frac{(a-2c_j+c_i)}{3b}$$

by symmetry.

   Taking comparative statics of q_i^* s.t.

$$\frac{\partial q_i^*}{\partial c_i} = -\frac{2}{3b} < 0 \quad , \quad \frac{\partial q_i^*}{\partial c_j} = \frac{1}{3b} > 0$$

   we can see that an increase in firm i's own marginal costs (c_i) decreases their optimal equilibrium output, and an increase in firm j's marginal cost increases firm i's equilibrium output. This can be seen as a positive externality, and we can see that the absolute magnitude of the effect of firm i's own marginal cost is greater than the absolute effect of firm j's marginal cost.

3. (7 pts.) Now, consider that we have symmetric costs (i.e. $c_i = c_j = c$) in the competitive equilibrium and for all analyses from here on out. Find the competitive equilibrium quantities, prices, and profits. What happens to profit (π^*) as fixed costs increase?

   When costs are equivalent, we know quantities, price, and profits are all the same for every firm i. This implies we get the standard Cournot quantities of

$$q_i \equiv q_j \equiv q^* = \frac{(a-c)}{3b}$$

   and prices become

$$p^* = a - b\left(\frac{(a-c)}{3b} + \frac{(a-c)}{3b}\right)$$
$$= a - \frac{2(a-c)}{3}$$
$$= \frac{(a+2c)}{3}$$

   Using q^* and p^* we get π^* s.t.

$$\pi^* = p^*q^* - (F + cq^*)$$
$$= \left(\frac{(a+2c)}{3}\right)\left(\frac{(a-c)}{3b}\right) - \left(F + c\left(\frac{(a-c)}{3b}\right)\right)$$
$$= \left(\frac{(a+2c)}{3} - c\right)\left(\frac{(a-c)}{3b}\right) - F$$
$$= \frac{(a-c)}{3}\frac{(a-c)}{3b} - F$$
$$\implies \quad \pi^* = \frac{(a-c)^2}{9b} - F$$

Notice these profits are equivalent to the standard Cournot equilibrium with fixed costs are equal to 0 (i.e. $F = 0$). Taking the derivative with respect to F we get

$$\frac{\partial \pi^*}{\partial F} = -1 < 0$$

Which means that for every per-unit increase in fixed costs (F) we get get a per-unit decrease in profits.

4. (7 pts.) Now, assume that both firms are pooling resources and acting as cartel. Find equilibrium profits (π^C).

   We know that if two firms are operating in a cartel, they are agreeing to pool resources and operate as one entity in the market. This means we need to solve the monopolist maximization problem.

$$\max_{Q \geq 0} \ \pi_M = [a - b(Q)]\, Q - (F + cQ)$$

$$\frac{\partial \pi_M(Q)}{\partial Q} = a - 2bQ - c = 0$$

$$Q^M = \frac{(a-c)}{2b} \quad \Longrightarrow \quad q^C = \frac{1}{N}\frac{(a-c)}{2b} \quad = \quad \frac{1}{2}\frac{(a-c)}{2b} \quad = \quad \frac{(a-c)}{4b}$$

   Using monopolist quantities, we get monopoly profits

$$\pi^M = \left[a - b(Q^C)\right] Q^C - (F + cQ^C)$$

$$\Longrightarrow \pi^M = \frac{(a-c)^2}{4b} - F$$

   And then dividing by the number of firms that are participating in the cartel

$$\pi^C \equiv \frac{\pi^M}{N} = \frac{\pi^M}{2}$$

$$\Longrightarrow \frac{\pi^M}{2} = \frac{1}{2}\left(\frac{(a-c)^2}{4b} - F\right)$$

$$\pi^C = \frac{(a-c)^2}{8b} - \frac{F}{2}$$

5. (7 pts.) Now, consider that one of the firms in the cartel unilaterally deviates from the cartel to complete in quantities. Derive the profits from deviating (π_i^D) and the profits from not deviating (π_i^{ND}).

   The key here is to remember that when deviation occurs, the deviating firm sets their quantities at a level as if they were competing a la Cournot and leaves the other firm operating as a cartel. This implies that the deviating firm's profits (π^D) are

$$\pi_i^D = \left(a - b\left(\underbrace{\frac{(a-c)}{3b}} + \frac{(a-c)}{4b}\right)\right)\left(\underbrace{\frac{(a-c)}{3b}}\right) - c\left(\underbrace{\frac{(a-c)}{3b}}\right) - F$$

$$\Longrightarrow \pi_i^D = \frac{5(a-c)^2}{36b} - F$$

   Similarly we can find the profits for the firm that did not deviate (π_i^{ND}) s.t.

$$\pi_i^{ND} = \left(a - b\left(\underbrace{\frac{(a-c)}{4b}} + \frac{(a-c)}{3b}\right)\right)\left(\underbrace{\frac{(a-c)}{4b}}\right) - c\left(\underbrace{\frac{(a-c)}{4b}}\right) - F$$

$$\Longrightarrow \pi_i^{ND} = \frac{5(a-c)^2}{48b} - F$$

6. (7 pts.) (7 pts.) Plug these profits into a normal form game, and find all Pure Strategy Nash Equilibria ($psNE$).

P1 \ P2	Coordinate	Compete
Coordinate	$\underbrace{\frac{(a-c)^2}{8b} - \frac{F}{2}}_{\pi_1^C}, \underbrace{\frac{(a-c)^2}{8b} - \frac{F}{2}}_{\pi_2^C}$	$\underbrace{\frac{5(a-c)^2}{48b} - F}_{\pi_1^{ND}}, \underbrace{\frac{5(a-c)^2}{36b} - F}_{\underline{\pi_2^D}}$
Compete	$\underbrace{\frac{5(a-c)^2}{36b} - F}_{\underline{\pi_1^D}}, \underbrace{\frac{5(a-c)^2}{48b} - F}_{\pi_2^{ND}}$	$\underbrace{\frac{(a-c)^2}{9b} - F}_{\underline{\pi_1^*}}, \underbrace{\frac{(a-c)^2}{9b} - F}_{\underline{\pi_2^*}}$

Which is almost identical to the Cournot game, without fixed costs, we are familiar with. Thus the psNE of the game is

$$psNE = \left\{ \left(q_1^* = \frac{(a-c)}{3b} \; , \; q_2^* = \frac{(a-c)}{3b} \right) \right\}$$

7. (7 pts.) If we consider that a, b and c are fixed, what is the fixed cost needed to maintain the cartel? You should have a fixed cost greater than some fixed cost threshold (i.e. $F > \hat{F}$)

From class, we know that we need to find the condition s.t

$$\pi_i^C \equiv \frac{(a-c)^2}{8b} - \frac{F}{2} > \frac{5(a-c)^2}{36b} - F \equiv \pi_i^D$$

$$\implies F > \frac{5(a-c)^2}{18b} - \frac{(a-c)^2}{4b}$$

$$F > \frac{(a-c)^2}{36b} \equiv \hat{F}$$

Where we can interpret this as if fixed costs (F) is greater than $\hat{F}$ all players will sustain the cartel.

2 IDSDS, psNE and msNE

Consider the following simultaneous-move game played by player 1 (in rows) and player 2 (in columns).

		Player 2		
		x	y	z
	a	$2, 3$	$1, 4$	$3, 2$
Player 1	b	$5, 1$	$2, 3$	$1, 2$
	c	$3, 7$	$4, 6$	$5, 4$
	d	$4, 2$	$1, 3$	$6, 1$

1. (10 pts.) Which strategy pairs survive the application of iterative deletion of strictly dominated strategies (IDSDS)?

   - [*Hint*: You should be able to delete one strategy for player 2 and two strategies for player 1, leaving you with a 2 by 2 matrix. You will need to mix between two strategies to in order to use IDSDS to delete one of Player 1's strategies. You have three different combinations to choose from. Please report your resulting probability.]

   - For player 2 (column player), strategy z is strictly dominated by y. We can then delete column z, leaving us with the following reduced-form matrix.

		Player 2	
		x	y
	a	$2, 3$	$1, 4$
Player 1	b	$5, 1$	$2, 3$
	c	$3, 7$	$4, 6$
	d	$4, 2$	$1, 3$

For player 1 (row player), strategy a is strictly dominated by b. After deleting the row corresponding to a, we obtain

		Player 2	
		x	y
	b	$5, 1$	$2, 3$
Player 1	c	$3, 7$	$4, 6$
	d	$4, 2$	$1, 3$

At this point, we cannot delete any more strategies for players 1 or 2 if we restrict them to use pure strategies. However, if we allow player 1 to randomize between the strategies that provide the highest payoff, b and c. In particular, assigning a probability p to strategy b and the remaining probability $1-p$ to strategy c, player 1's expected payoff when player 2 chooses strategy x (in the left-hand column of the above matrix) is

$$5p + 3(1-p) = 2p + 3$$

which is larger than player 1's payoff from strategy d, 4, as long as $2p+3 > 4$, or solving for p, if $p > \frac{1}{2}$. Similarly, when player 2 chooses straegy y (in the right-hand column of the above matrix), player 1's expected payoff from randomizing between b and c becomes

$$2p + 4(1-p) = 4 - 2p$$

which is larger than player 1's payoff from strategy d, 1, as long as $4-2p > 1$, or solving for p, if $p < \frac{3}{2}$. This condition holds by assumption since probability p must be a number between 0 and 1. Therefore, any randomization between strategies b and c that assigns more than 50% probability on strategy b (that is, $p > 1/2$) yields a expected utility larger than the utility player 1 receives from strategy d. We can therefore claim that strategy d is strictly dominated, and delete the bottom row of the above matrix, leaving us with the followed reduced-form matrix.

		Player 2	
		x	y
Player 1	b	$5,1$	$2,3$
	c	$3,7$	$4,6$

At this point, we cannot delete any further strategies for players 1 or 2. Then, the strategy profiles surviving IDSDS are those in the four cells of the above matrix:

$$IDSDS = \{(b,x),(b,y),(c,x),(c,y)\}.$$

2. (5 pts.) Using your results from part (a), show that there is no Pure Strategy Nash Equilibrium (psNE) in this game.

- Using the strategy profiles that survived IDSDS, we can next underline best response payoffs, as depicted in the matrix below.

		Player 2	
		x	y
Player 1	b	$\underline{5},1$	$2,\underline{3}$
	c	$3,\underline{7}$	$\underline{4},6$

Since there is no cell where both players' payoffs are underlined, we can claim that there is no pure strategy Nash equilibrium in this game. There is, however, a mixed strategy Nash equilibrium, as we show in the next part of the exercise!

3. (10 pts.) Using your results from part (a), find a Mixed Strategy Nash Equilibrium (msNE) in this game.

- [You can use p to denote the probability with which player 1 randomizes, and q to denote the probability with which player 2 randomizes.]

- *Player 1.* If player 1 is randomizing, he must be indifferent between pure strategies b and c. His expected utility from choosing b (in the top row of the above matrix) is

$$EU_1(b) = 5q + 2(1-q) = 3q + 2$$

while his expected utility from selecting c (in the bottom row of the matrix) is

$$EU_1(c) = 5q + 4(1-q) = 4 - q.$$

Then, player 1 is indifferent between b and c if and only if $EU_1(b) = EU_1(c)$, which implies that

$$3q + 2 = 4 - q$$

and, after rearranging, $4q = 2$, or $q = \frac{1}{2}$.

- *Player 2.* If player 2 is randomizing, he must be indifferent between his pure strategies x and y. His expected utility from choosing x (in the left-hand column of the above matrix) is

$$EU_2(x) = 1p + 7(1-p) = 7 - 6p$$

while his expected utility from selecting y (in the right-hand column of the matrix) is

$$EU_2(y) = 3p + 6(1-p) = 6 - 3p.$$

Then, player 2 is indifferent between x and y if and only if $EU_2(x) = EU_2(y)$, which implies that

$$7 - 6p = 6 - 3p$$

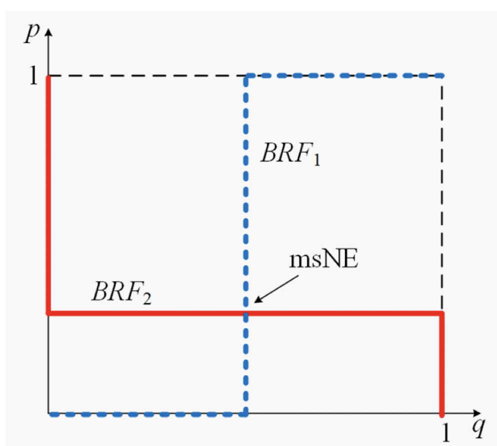
or, after rearranging, $1 = 3p$, or $p = \frac{1}{3}$.

- Therefore, the mixed strategy Nash equilibrium of the game is

$$\left\{ \left(\frac{1}{3}b, \frac{2}{3}c \right), \left(\frac{1}{2}x, \frac{1}{2}y \right) \right\}$$

where the first pair indicates player 1's randomization between b and c with probabilities 1/3 and 2/3 respectively, while the second pair represents player 2's randomization between x and y, each with 50% probability.

- *Graphical representation of the msNE.* The next figure depicts the best response functions for each player. (This was not required in the exam, but we include it here for completeness.) The best response functions only have a crossing point, which illustrates the mixed strategy Nash equilibrium of the game.



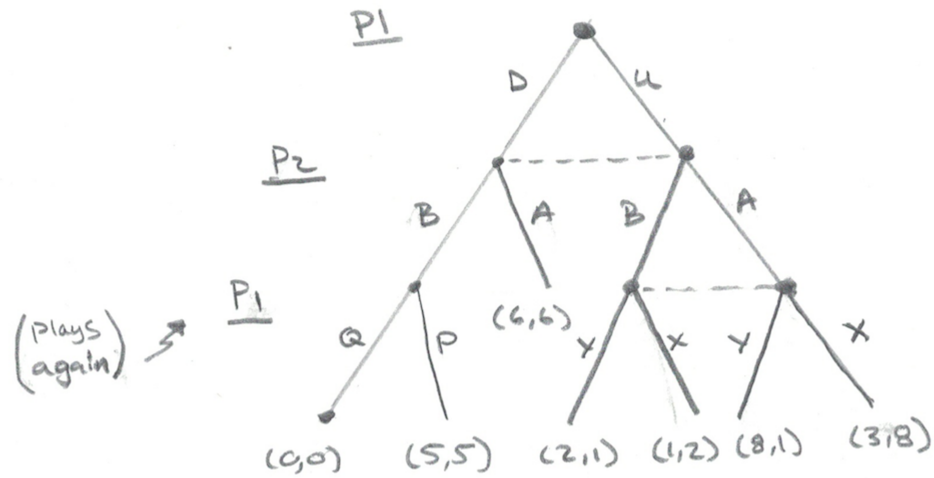
  - For player 1, note that when $q = 1$ (player 2 chooses x), his best response is to choose b, implying that he assigns full probability to b, that is, $p = 1$

at the top right-hand corner of the figure. When $q = 0$ (player 2 selects y), player 1's best response is to choose c, implying that he assigns no probability to b, that is, $p = 0$, at the bottom left-hand corner of the figure.

- For player 2, note that when $p = 1$ (player 1 chooses b), his best response is to choose y, implying that he assigns no probability to x, that is, $q = 0$ at the top left-hand corner of the figure. When $p = 0$ (player 1 selects c), player 2's best response is to choose x, implying that he assigns full probability to x, that is, $q = 1$, at the bottom right-hand corner of the figure.

3 Normal Form Game and spNE

1. (15 pts.) Consider the extensive form game below. Derive the normal form of this game and find all Pure Strategy Nash Equilibria (psNE).



Recall that we need to define the complete contingency plan for every player in the game, and notice that player 1 is the first and last mover. This means that he/she will have to act twice, and he/she needs to consider what happens when he/she chooses both D and U. This means that player 1 needs to choose D or U and then build his/her complete contingency plan for when he/she makes the next decision. The normal form game and the psNE can thus be represented as

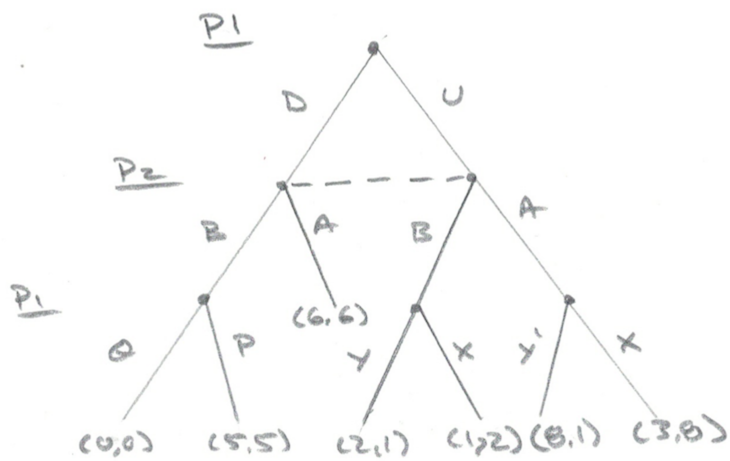
Normal Form

⇒

		P2 A	P2 B
P1	UXP	3,8	1,2
	UXQ	3,8	1,2
	UYP	8,1	2,1
	UYQ	8,1	2,1
	DXP	6,6	5,5
	DXQ	6,6	0,0
	DYP	6,6	5,5
	DYQ	6,6	0,0

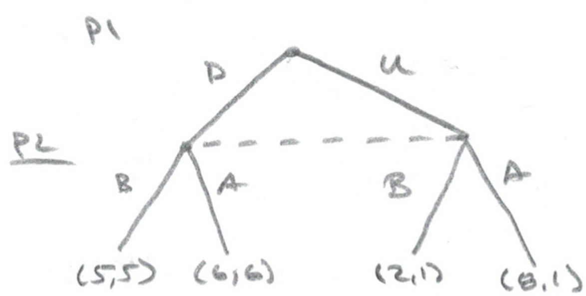
⇒ psNE = {(UYP, A), (UYQ, A)}

2. (10 pts.) Now, consider that we have removed the information set in the bottom right-hand portion of the extensive form game. Use backwards induction to find the Sub-game Perfect Nash Equilibrium (spNE).



Note that we have removed the information set in the bottom right portion of the game. If this information set was not removed, we would only have 2 proper sub-games. Now, with the removal of this information set, we have 5 proper sub-games (including the whole game in itself). With that said, we can use backward induction to reduce the extensive form game to

using Backward Induction, we get



With this reduced extensive game, we can now us our Nash Equilibrium concepts to solve for the unique psNE. With this psNE we can now completely define the Sub-game Perfect Nash Equilibrium (spNE) where we need to characterize the optimal decision for each player at each point they are called on to act. The spNE is defined as

$\Rightarrow$

		P2: A	P2: B
P1	D	$6, \underline{6}$	$\underline{5}, 5$
	u	$\underline{8}, \underline{1}$	$2, \underline{1}$

$\Rightarrow$ spNE $= \{(u, Y'/Y/P), (A)\}$