EconS 305: Intermediate Microeconomics w/o Calculus

Homework 5:
Repeated Games, Sequential Move Games and Product Differentiation

Due: Friday, June 19th, 2020 at 5:00pm via Blackboard

- Please submit all homework solutions in the order the questions are presented and as **one .PDF**.

- Please **show all calculations** as these exercises are meant to refine your quantitative tool set. If I can not follow your calculations or it seems as you just "copy and pasted" answers from the internet, I will be deducting half the points from that solution.

1. **A Cournot Game of Competing in Quantities w/ Fixed Costs - A Game Theory Extension with Sustainable Collusion**

   Note: *Please* ***re-do all calculations for this exercise****, even though Parts A through I are analogous to that of Homework 3 and 4, as it is important that you know how to derive the equilibrium results for this model. This is the base case model for many economic analyses, and it is important for Intermediate Microeconomic extensions.*

   Consider two firms competing a la Cournot in a market with an inverse demand function of $p(Q) = a - b(Q)$, where $Q = q_i + q_j$ and $a > c$, and a total cost function of $TC_i(q_i) = F + c_i q_i$. Notice that each firm has the same fixed cost (F) but their marginal costs (c_i) are not equal to each other (i.e. $c_i \neq c_j$). This means these homogeneous product producing firms have asymmetric costs, and we can represent the Profit Maximization Problem (PMP) for firm i as:

   CALCULUS PART:

$$\max_{q_i \geq 0} \ \pi_i = [a - b(q_i + q_j)]\, q_i - (F + c_i q_i)$$

$$\frac{\partial \pi_i(q_i, q_j)}{\partial q_i} = a - 2bq_i - bq_j - c_i = 0 \qquad (1)$$

And through symmetry we know that firm j's PMP is

$$\max_{q_j \geq 0} \ \pi_j = [a - b(q_i + q_j)]\, q_j - (F + c_j q_j)$$

$$\frac{\partial \pi_j(q_i, q_j)}{\partial q_j} = a - 2bq_j - bq_i - c_j = 0 \tag{2}$$

where we now have two equations ((1) and (2)), and two choice variables (q_i and q_j) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

(a) Find the optimal equilibrium allocation for each firm when they are competing a la Cournot. That is, find q_i^* and q_j^*. Note that this solution is analogous to the answer you derived in Question 3 of Homework 4.

We find the equilibrium allocation by simultaneously solving for q_i^* and q_j^* using each firm's BRFs derived from equations (1) and (2)

$$BRF_i \equiv q_i(q_j) = \frac{(a - c_i)}{2b} - \frac{1}{2}q_j \qquad BRF_j \equiv q_j(q_i) = \frac{(a - c_j)}{2b} - \frac{1}{2}q_i$$

$$q_i(q_j) = \frac{(a - c_i)}{2b} - \frac{1}{2}\left[\frac{(a - c_j)}{2b} - \frac{1}{2}q_i\right]$$

$$4bq_i = 2(a - c_i) - (a - c_i) + bq_i$$

$$\implies \quad q_i^* = \frac{(a - 2c_i + c_j)}{3b} \quad \textit{and} \quad q_j^* = \frac{(a - 2c_j + c_i)}{3b}$$

by symmetry.

(b) Now, consider that the firm's have symmetric costs (i.e. $c_i = c_j = c$) in the competitive equilibrium and for all analyses from here on out. Find the competitive equilibrium quantities (i.e. find q_i^* and q_j^*). Again, note that this solution is analogous to the answer you derived in Question 3 of Homework 4.

When costs are equivalent, we know quantities, price, and profits are all the same for every firm i. This implies we get the standard Cournot quantities of

$$q_i^* \equiv q_j^* = \frac{(a - c)}{3b}$$

Note that we can also label these quantities as $q_i^{Cournot}$ as they are the output of the Cournot Model.

(c) Find the equilibrium price (i.e. $p(Q^*) = a - b(Q^*)$). Again, note that this solution is analogous to the answer you derived in Question 3 of Homework 4.

$$
\begin{aligned}
p(Q^*) &= a - b\left(\frac{(a-c)}{3b} + \frac{(a-c)}{3b}\right) \\
&= a - \frac{2(a-c)}{3} \\
\implies p(Q^*) &= \frac{(a+2c)}{3}
\end{aligned}
$$

(d) Find the equilibrium profits (i.e. π^*). Again, note that this solution is analogous to the answer you derived in Question 3 of Homework 4.

Using q^* and p^* we get π^* s.t.

$$
\begin{aligned}
\pi^* &= p^* q^* - (F + cq^*) \\
&= \left(\frac{(a+2c)}{3}\right)\left(\frac{(a-c)}{3b}\right) - \left(F + c\left(\frac{(a-c)}{3b}\right)\right) \\
&= \left(\frac{(a+2c)}{3} - c\right)\left(\frac{(a-c)}{3b}\right) - F \\
&= \frac{(a-c)}{3}\frac{(a-c)}{3b} - F \\
\implies \quad \pi^{*\ Cournot} &= \frac{(a-c)^2}{9b} - F
\end{aligned}
$$

(e) Now, assume that both firms are pooling resources and acting as a cartel. Please re-write the equilibrium profits (π^{Cartel}) you found in Question 2 in Homework 4. The answer should be exactly the same answer you got in Question 2 of Homework 4.

$$
\pi^{Cartel} = \frac{(a-c)^2}{8b} - \frac{F}{2}
$$

(f) Now, consider that one of the firms in the cartel unilaterally deviates from the cartel to compete in quantities (i.e. sets their quantity at the competitive level instead of the cartel level). Derive the profits from deviating (π_i^{Dev}) and simplify the expression. Again, please note that this solution is analogous to the answer you derived in Question 3 of Homework 4. The key here is to remember that when deviation occurs, the deviating firm sets their quantities at a level as if they were competing a la Cournot and leaves the other firm operating as a cartel. This implies that the deviating firm's profits (π^D) become

$$
\pi_i^{Dev} = \left(a - b\left(q_i^{Cournot} + q_i^{Cartel}\right)\right)\left(q_i^{Cournot}\right) - c\left(q_i^{Cournot}\right) - F
$$

This implies that the deviating firm's profits (π_i^{Dev}) are

$$
\pi_i^{Dev} = \left(a - b\left(\underbrace{\frac{(a-c)}{3b}} + \frac{(a-c)}{4b}\right)\right)\left(\underbrace{\frac{(a-c)}{3b}}\right) - c\left(\underbrace{\frac{(a-c)}{3b}}\right) - F
$$

$$\implies \pi_i^{Dev} = \frac{5(a-c)^2}{36b} - F$$

(g) Similar to part f, we can find the profits of the firm that remains in the cartel while the other firm deviates. We will call these profits π_i^{NDev} (i.e. does not deviate) such that

$$\pi_i^{NDev} = \left(a - b\left(q_i^{Cournot} + q_i^{Cartel}\right)\right)\left(q_i^{Cartel}\right) - c\left(q_i^{Cartel}\right) - F$$

Use this formula to find the profits of the firm that is being deviated upon (i.e. π_i^{NDev}). Again, please note that this solution is analogous to the answer you derived in Question 3 of Homework 4.

Note that we can plug in our competing quantities.

$$\pi_i^{ND} = \left(a - b\left(\underbrace{\frac{(a-c)}{4b}} + \frac{(a-c)}{3b}\right)\right)\left(\underbrace{\frac{(a-c)}{4b}}\right) - c\left(\underbrace{\frac{(a-c)}{4b}}\right) - F$$

$$\implies \pi_i^{ND} = \frac{5(a-c)^2}{48b} - F$$

(h) Take the four different equilibrium profit functions you found (π^{Cartel}, $\pi^{Cournot}$, π^{Dev} and π^{NDev}), set all fixed costs equal to zero (i.e. $F = 0$), and compare them mathematically (i.e. from most profit gained to least profit gained). Once the comparison is done, plug these equations into a matrix following the matrix template given below. This is called a normal form game, and this allows us to determine the "best response" for each firm when trying to decide to participate in a cartel or compete in quantities. Again, please note that this solution is analogous to the answer you derived in Question 3 of Homework 4.

$\Rightarrow$

Firm j

		Cartel	Compete
Firm i	Cartel	$\pi_i^{Cartel}, \pi_j^{Cartel}$	$\pi_i^{NDev}, \pi_j^{Dev}$
	Compete	$\pi_i^{Dev}, \pi_j^{NDev}$	$\pi_i^{Cournot}, \pi_j^{Cournot}$

Plugging in the profits we found from above, we get a normal form matrix of plugging in values

$\Rightarrow$

Firm i \ Firm j	Cartel	Compete
Cartel	$\frac{(a-c)^2}{40b}$, $\frac{(a-c)^2}{40b}$	$\frac{(a-c)^2}{48b}$, $\frac{(a-c)^2}{36b}$
Compete	$\frac{(a-c)^2}{36b}$, $\frac{(a-c)^2}{48b}$	$\frac{(a-c)^2}{45b}$, $\frac{(a-c)^2}{45b}$

2i)

(i) Find the Pure Strategy Nash Equilibrium ($psNE$). Again, please note that this solution is analogous to the answer you derived in Question 3 of Homework 4.

$\Rightarrow$

Firm i \ Firm j	Cartel	Compete
Cartel	Cartel Cartel π_i , π_j	NDev Dev π_i , $\underline{\pi_j}$
Compete	Dev NDev $\underline{\pi_i}$, π_j	Cournot Cournot $\underline{\pi_i}$, $\underline{\pi_j}$

underlining all Best Responses

The the psNE of the game is

$$psNE = \left\{ \left(q_1^* = \frac{(a-c)}{3b} \;\; , \;\; q_2^* = \frac{(a-c)}{3b} \right) \right\}$$

Where the optimal choice for each firm is to choose to compete no matter what. Intuitively, this result shows that a firm's optimal move, when offered to participate cartel, is to deviate and break the cartel agreement. Thus, both firms will choose to compete in equilibrium.

(j) Now, consider that both firms are going to engage in an infinitely repeated game where they will need to simultaneously decide if they are going to compete or participate in a cartel at every time t. Please derive the Grim Trigger Strategy (GTS), using our profit notation (i.e. π^{Dev} and $\pi^{Cournot}$), where the firm will deviate at the time period $t = 0$, and then revert to the Nash Equilibrium payoff (i.e. $\pi^{Cournot}$) in all the time periods after that. In order to do this, recall that we need to consider the discounted stream of payoffs the firm will receive from $t = 0$ to $t = \infty$. Please refer to the Chapter 12 Game Theory lecture notes if you are having trouble with deriving the GTS payoff stream.

$$\begin{aligned} GTS &= \delta^0(\pi^{Dev}) + \delta^1(\pi^{Cournot}) + \delta^2(\pi^{Cournot}) + \delta^3(\pi^{Cournot}) + \dots \\ &= (1)(\pi^{Dev}) + \delta(\pi^{Cournot}) + \delta^2(\pi^{Cournot}) + \delta^3(\pi^{Cournot}) + \dots \\ &= \pi^{Dev} + \delta\pi^{Cournot}\left(\delta^0 + \delta^1 + \delta^2 + \dots\right) \end{aligned}$$

$$= \pi^{Dev} + \frac{\delta}{(1-\delta)}\pi^{Cournot}$$

(k) Now, please derive the stream of payoffs an individual firm will receive from participating in the cartel in the infinitely repeated game. Please use the general profit notation instead of the analytical notation.

$$\begin{aligned}
Cartel\ Payoffs &= \delta^0(\pi^{Cartel}) + \delta^1(\pi^{Cartel}) + \delta^2(\pi^{Cartel}) + \delta^3(\pi^{Cartel}) + \dots \\
&= (\pi^{Cartel})(\delta^0 + \delta^1 + \delta^2 + \delta^3 + \dots) \\
&= \pi^{Cartel}\frac{1}{(1-\delta)}
\end{aligned}$$

(l) Now, please find the condition on the firm's patience parameter (i.e. δ) in which will can guarantee that the firm's will both sustain the cartel agreement. Remember that the cartel agreement in the infinitely repeated game will be sustained if the discounted stream of payoffs from staying in the cartel are greater than the GTS payoffs.

The collusive agreement will be sustained if

$$Cartel\ Payoffs \quad \equiv \quad \pi^{Cartel}\frac{1}{(1-\delta)} > \pi^{Dev} + \frac{\delta}{(1-\delta)}\pi^{Cournot} \quad \equiv \quad GTS$$

$$\pi^{Cartel} > \pi^{Dev}(1-\delta) + \delta\pi^{Cournot}$$

$$\delta(\pi^{Dev} - \pi^{Cournot}) > \pi^{Dev} - \pi^{Cartel}$$

$$\delta > \frac{\pi^{Dev} - \pi^{Cartel}}{(\pi^{Dev} - \pi^{Cournot})} \quad \equiv \quad \overline{\delta}$$

(m) Please plug in the analytical solution to the general profit notation we have been using, and simplify. Simplifying should prove that $\overline{\delta} = \frac{1}{2}$ in this infinitely repeated game.

$$\delta > \frac{1}{2} \equiv \overline{\delta}$$

2. **A Bertrand Model with Product Differentiation**

Note: *I have given you the final answers to these problems, but I want you to* ***show all the work needed to derive these results****. Also, please make sure to interpret the results if the question asks you to interpret the results.*

Consider two firms competing in prices a la Bertrand selling differentiated goods. The demand function for every firm i is $q_i(p_i, p_j) = a - bp_i + p_j$ where $i,j \in \{1, 2\}$ and $i \neq j$. This means that the demand function for firm 1 and firm 2 are $q_1(p_1, p_2) = a - bp_1 + p_2$ and $p_2(q_1, q_2) = a - bp_2 + p_1$, respectively. In this model b represents the degree of product differentiation for every firm i. If $b > 1$ the products are considered to be differentiated (also known as heterogeneous), and if $b = 1$ the products are identical (also known as homogeneous). Each firm has the same constant marginal cost of c, and all firms' fixed costs are assumed to be equal to zero (i.e. $F = 0$). We formulate each firm's Profit Maximization Problem (PMP) as:

CALCULUS PART:

$$\max_{p_i \geq 0} \quad \pi_i = p_i \left[a - bp_i + p_j\right] - c\left[a - bp_i + p_j\right]$$

$$\frac{\partial \pi_i(p_i, p_j)}{\partial p_i} = a - 2bp_i + p_j + bc = 0 \tag{3}$$

And through symmetry we know that firm j's PMP is

$$\max_{p_j \geq 0} \quad \pi_j = p_j \left[a - bp_j + p_i\right] - c\left[a - bp_j + p_i\right]$$

$$\frac{\partial \pi_j(p_i, p_j)}{\partial p_j} = a - 2bp_j + p_i + bc = 0 \tag{4}$$

where we now have two equations ((3) and (4)), and two choice variables (p_i and p_j) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

(a) Please find the Best Response Functions for each firm (i.e. $BRF_i \equiv p_i(p_j)$ and $BRF_j \equiv p_j(p_i)$). How does firm i respond with it's own price (i.e. p_i) with an increase in a, b, c and p_j?

$$BRF_i \equiv p_i(p_j) = \frac{(a + bc)}{2b} + \frac{1}{2b}p_j$$
$$BRF_j \equiv p_j(p_i) = \frac{(a + bc)}{2b} + \frac{1}{2b}p_i$$

(b) Find the optimal equilibrium allocation for each firm when they are competing a la Bertrand with differentiated products. That is, find p_i^* and p_j^*, and please simplify. Are these prices the same?

$$(p_i^*, p_j^*) = \left(\frac{(a + cb)}{(2b - 1)}, \frac{(a + cb)}{(2b - 1)} \right)$$

(c) Find the optimal quantity demanded for each firm (i.e. $q_i^* = a - bp_i^* + p_j^*$ and $q_j^* = a - bp_j^* + p_i^*$).

$$(q_i^*, q_j^*) = \left(\frac{b(a - c(b-1))}{(2b-1)}, \frac{b(a - c(b-1))}{(2b-1)} \right)$$

(d) Find the equilibrium profits of each firm (i.e. π_i^* and π_j^*).

$$(\pi_i^*, \pi_j^*) = \left(\frac{b(a - c(b-1))^2}{(2b-1)^2}, \frac{b(a - c(b-1))^2}{(2b-1)^2} \right)$$

3. **A Cournot Model with Product Differentiation**

Consider two firms competing in quantities a la Cournot selling differentiated goods. The inverse demand function for every firm i is $p_i(q_i, q_j) = a - bq_i - dq_j$ where $i,j \in 1, 2$ and $i \neq j$. This means that the inverse demand functions for Firm 1 and Firm 2 are $p_1(q_1, q_2) = a - bq_1 - dq_2$ and $p_2(q_1, q_2) = a - bq_2 - dq_1$, respectively. In this model b and d represent the degree of product differentiation for every firm i and j. b is assumed to be greater than zero (i.e. $b > 0$), and for simplicity lets assume that d can take on any value between zero and b (i.e. $b \geq d \geq 0$). The products are considered to be differentiated (also known as heterogeneous goods) if $b \neq d$, and if $b = d$ the firms' products are identical (also known as homogeneous goods). Each firm has the same marginal cost of c, and all firm's fixed costs are assumed to be equal to zero (i.e. $F = 0$). We formulate each firm's Profit Maximization Problem (i.e. PMP) as:

CALCULUS PART:

$$\max_{q_i \geq 0} \quad \pi_i = [a - (bq_i + dq_j)]\, q_i - cq_i$$

$$\frac{\partial \pi_i(q_i, q_j)}{\partial q_i} = a - 2bq_i - dq_j - c = 0 \tag{5}$$

And through symmetry we know that firm j's PMP is

$$\max_{q_j \geq 0} \quad \pi_j = [a - (bq_j + dq_i)]\, q_j - cq_j$$

$$\frac{\partial \pi_i(q_i, q_j)}{\partial q_j} = a - 2bq_j - dq_i - c = 0 \tag{6}$$

where we now have two equations ((5) and (6)), and two choice variables (q_i and q_j) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

(a) Please find the Best Response Functions for each firm (i.e. $BRF_i \equiv q_i(q_j)$ and $BRF_j \equiv q_j(q_i)$). How does firm i respond with it's own quantity with an increase in a, b, c, d and q_j?

$$BRF_i \equiv q_i(q_j) = \frac{(a-c)}{2b} - \frac{d}{2b} q_j \qquad\qquad BRF_j \equiv q_j(q_i) = \frac{(a-c)}{2b} - \frac{d}{2b} q_i$$

Where, an increase in a (i.e. an outward shift in demand) increases q_i, an increase in b (i.e. the degree of product differentiation for firm i) increases q_i, an increase in c (i.e. the firm's marginal cost) decreases q_i, an increase in d (i.e. the degree of product differentiation for firm j) decreases q_i, and an increases in q_j decreases q_i.

(b) Find the optimal equilibrium allocation for each firm when they are competing a la Cournot with differentiated products. That is, find q_i^* and q_j^*. What happens to q_i^* as d increases towards b (i.e. $d \to b$)? What happens to q_i^* as d decreases towards zero (i.e. $d \to 0$)? What is the intuition behind this?

$$q_i(q_j) = \frac{(a-c)}{2b} - \frac{d}{2b}\left[\frac{(a-c)}{2b} - \frac{d}{2b}q_i\right]$$

$$\implies \quad q_i^* = \frac{(a-c)}{2b+d} \quad and \quad q_j^* = \frac{(a-c)}{2b+d}$$

by symmetry.

Where, as d moves closer to b (i.e. $d \to b$), the products become more similar (i.e. less differentiated), and both quantities move closer to the classic Cournot output (i.e. $q_i^* \to q_i^{Cournot}$). On the contrary, as d moves closer to 0 (i.e. $d \to 0$), the products become less similar (i.e. more differentiated), and both quantities move closer to the classic Monopoly output (i.e. $q_i^* \to q_i^{Monopoly}$). Intuitively, we can see that as the products become more differentiated, output for each product decreases because they are the only firm providing that good in that market. This means that have the incentive to act more as a monopolist than a duopolist.

(c) Find the optimal price for each firm (i.e. $p_i^* = a - bq_i^* - dq_j^*$ and $p_j^* = a - bq_j^* - dq_i^*$).

$$(p_i^*, p_j^*) = \left(\frac{a(2b+d) - (b+d)(a-c)}{2b+d}, \frac{a(2b+d) - (b+d)(a-c)}{2b+d}\right)$$

(d) Find the equilibrium profits of each firm (i.e. π_i^* and π_j^*).

$$(\pi_i^*, \pi_j^*) = \left(\frac{b(a-c)^2}{(2b+d)^2}, \frac{b(a-c)^2}{(2b+d)^2}\right)$$

(e) Now, lets assume that the products are identical (i.e. $b = d$). Plug b in for d in π_i^*, and simplify. What does this profit structure simplify to, and is it familiar? If so, what equilibrium profits are these profits analogous to?

$$\implies (\pi_i^{Cournot}, \pi_j^{Cournot}) = \left(\frac{(a-c)^2}{9b}, \frac{(a-c)^2}{9b}\right)$$

(f) Now, lets assume that the products are completely differentiated (i.e. $d = 0$). Plug 0 in for d in π_i^*, and simplify. What does this profit structure simplify to, and is it familiar? If so, what equilibrium profits are these profits analogous to?

$$\implies (\pi_i^{Monopoly}, \pi_j^{Monopoly}) = \left(\frac{(a-c)^2}{4b}, \frac{(a-c)^2}{4b}\right)$$

Where, the firms unambiguously have the incentive to try to differentiate their products as much as possible in order to capture monopoly profits.

4. **A Bertrand Model with Product Differentiation by Location**

Firms

Consider two firms competing a la Bertrand (i.e. a price competition) where each firm has constant marginal costs ($c > 0$). Each firm is located at the end of a "Linear City" interval where the interval is a line ranging from 0 to 1 inclusive (i.e. $[0, 1]$). In this setting, the product differentiation between each firm is its location.

Consumer Demand [*The Hotelling Demand Model*]

Consumers are distributed uniformly on this interval s.t. the distribution is a mass of consumers. We can imagine this distribution as being $X \sim UNIF(0, 1)$ where the expectation of X is $\frac{1}{2}$ (i.e. $\mathbb{E}(X) = \frac{1-0}{2} = \frac{1}{2}$). Let t be the per-unit transportation cost for the squared distance traveled (d^2). This implies the cost for each consumer to travel a certain amount of distance is td^2. This expression can represent a multitude of costs to the consumer. It can be the consumer's value of time, gasoline, or perhaps some learning curve when adopting the product. In this setting we will consider it to be the consumer's "traveling cost" to each firm. We can think of the transportation cost for each consumer x for Firm A to be tx^2 and for Firm B to be $t(1-x)^2$ since the consumer mass is distributed along the unit interval of X. Let s be the gross consumer surplus (i.e. its maximum willingness to pay for a good) where are going to assume that s is sufficiently high for all consumers to purchase a product. The utility that consumer i gets from consuming at most one unit good is

$$U_i = s - (p + td^2)$$

where p is the price of the good and td^2 is the travel cost. Notice that we can represent this utility for the consumer that purchases a good from Firm A and B respectively as

$$U_i^A = s - (p_A + tx^2) \qquad and \qquad U_i^B = s - (p_B + t(1-x)^2)$$

because we are assuming this consumers are evenly distributed on the same unit interval they will be traveling.

(a) Use these two derived utility functions to find the indifferent consumer (i.e. $\hat{x}$). This can be found by setting $U_i^A = U_i^B$ and solving for x. This x is $\hat{x}$ and is considered to be the demand that Firm A will capture.

In order to determine the quantity demanded for each firm, we need to find the consumer that is just indifferent between buying from Firm A and Firm B.

$\implies$ Let, $\hat{x}$ be the location of the indifferent consumer i s.t. the utility of consumer i is equal when buying from either firm s.t

$$U_i^A = U_i^B$$
$$s - p_A - t\hat{x}^2 = s - p_B - t(1-\hat{x})^2$$
$$p_A + t\hat{x}^2 = p_B + t(1-\hat{x})^2$$

$$
\begin{aligned}
p_A + t\hat{x}^2 &= p_B + t(1 - 2\hat{x} + \hat{x}^2) \\
p_A + t\hat{x}^2 &= p_B + t - 2t\hat{x} + t\hat{x}^2 \\
2t\hat{x} &= p_B - p_A + t \\
\hat{x} &= \frac{p_B - p_A}{2t} + \frac{1}{2} \quad \equiv \quad Q_A(p_A, p_B)
\end{aligned}
$$

where $Q_A(p_A, p_B)$ is the amount of quantity demanded Firm A realizes.

This implies that Firm B enjoys the demand of

$$
\begin{aligned}
Q_B(p_A, p_B) \quad \equiv \quad 1 - Q_A(p_A, p_B) \quad &\equiv \quad 1 - \hat{x} \\
= 1 - \left(\frac{p_B - p_A}{2t} + \frac{1}{2} \right) & \\
\implies \quad Q_B(p_A, p_B) \quad \equiv \quad \frac{p_A - p_B}{2t} + \frac{1}{2} &
\end{aligned}
$$

because we are working with a unit mass of consumers distributed across a unit interval from 0 to 1.

(b) Using the demands derived in Part (a) we can set up the the Profit Maximization Problems (PMPs) as:

CALCULUS PART:

Firm A's maximization problem is

$$
\max_{p_A \geq 0} \;\; \pi_A(p_A, p_B) = (p_A - c) \underbrace{Q_A(p_A, p_B)}_{\hat{x}}
$$

$$
\max_{p_A \geq 0} \;\; \pi_A(p_A, p_B) = (p_A - c)\, \hat{x}
$$

$$
\max_{p_A \geq 0} \;\; \pi_A(p_A, p_B) = (p_A - c) \left(\frac{p_B - p_A}{2t} + \frac{1}{2} \right)
$$

Taking the derivative with respect to p_A and setting equal to 0

$$
\frac{\partial \pi_A(p_A, p_B)}{\partial p_A} = \frac{p_B - p_A + t}{2t} - \frac{p_A}{2t} + \frac{c}{2t} = 0 \tag{7}
$$

Firm B's maximization problem is

$$
\max_{p_B \geq 0} \;\; \pi_B(p_A, p_B) = (p_B - c) \underbrace{Q_B(p_A, p_B)}_{(1 - \hat{x})}
$$

$$
\max_{p_B \geq 0} \;\; \pi_B(p_A, p_B) = (p_B - c)\, (1 - \hat{x})
$$

$$
\max_{p_B \geq 0} \;\; \pi_B(p_A, p_B) = (p_B - c) \left(\frac{p_A - p_B}{2t} + \frac{1}{2} \right)
$$

Taking the derivative with respect to p_B and setting equal to 0

$$\frac{\partial \pi_B(p_A, p_B)}{\partial p_B} = \frac{p_A - p_B + t}{2t} - \frac{p_B}{2t} + \frac{c}{2t} = 0 \tag{8}$$

where we now have two equations ((7) and (8)), and two choice variables (p_A and p_B) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

Using equations (7) and (8), please find the optimal price Firm A should set to capture their portion of the market (i.e. p_A^*), and the optimal price Firm B should set to capture their portion of the market (i.e. p_B^*). What happens when we set transportation costs to zero (i.e. $t = 0$)? What type of model does setting transportation costs equal to zero yield?

Solving for p_A gives the familiar Best Response Function of Firm A (BRF_A) in terms of the price of Firm B s.t.

$$p_A(p_B) = \frac{t+c}{2} + \frac{1}{2}p_B$$

And because this maximization problem is formulated using assumptions in which are symmetric (for example $c_A = c_B = c$) the Best Response Function of Firm B (BRF_B) is

$$p_B(p_A) = \frac{t+c}{2} + \frac{1}{2}p_A$$

This implies that we can solve for the optimal price set by Firm A as

$$p_A^* = \frac{t+c}{2} + \frac{1}{2}\left[\frac{t+c}{2} + \frac{1}{2}p_A^*\right]$$

$$\implies p_A^* = t + c$$

and through symmetry, the optimal price Firm B sets is

$$p_B^* = t + c$$

which is equivalent to the price of Firm A. Note that when transportation cost is set equal to zero ($t = 0$) the model reverts to the familiar Bertrand Duopoly model we have discussed in class where $p_A^* = p_B^* = c$.

(c) Find the optimal indifferent consumer (i.e. $\hat{x}^*$), which is also the optimal demand Firm A will receive, and find $(1 - \hat{x}^*)$, which is the optimal demand Firm B will receive.

To find the optimal indifferent consumer, the optimal prices from part (b) are plugged into the previously derived $\hat{x}$ s.t.

$$Q_A^*(p_A^*, p_B^*) \equiv \hat{x}^* = \left(\frac{(t+c)-(t+c)}{2t} + \frac{1}{2}\right)$$

$$\implies \hat{x}^* = \frac{1}{2}$$

$$\implies Q_B^*(p_A^*, p_B^*) \equiv (1 - \hat{x}^*) = \left(1 - \frac{1}{2}\right)$$

$$(1 - \hat{x}^*) = \frac{1}{2}$$

Which, can be seen as the optimal (or equilibrium) demand that Frim A and Firm B will realize, respectively.

(d) Find the equilibrium profits for Firm A and Firm B. What happens when we set $t = 0$? What does this condition resemble?

This implies that equilibrium profits for Firm A are

$$\pi_A^* = (p_A^* - c)(\hat{x}^*)$$

$$= ((t + c) - c)\left(\frac{1}{2}\right)$$

$$\implies \pi_A^* = \frac{t}{2}$$

and equivalently Firm B's equilibrium profits are

$$\pi_B^* = \frac{t}{2}$$

Where if transportation cost is set equal to zero ($t = 0$)

$$\pi_A^* = \pi_B^* = 0$$

which is the same result as the standard Bertrand Duopoly problem we have discussed in class. Also, note how profits and prices increase as the transportation cost increases for the consumer. This means that as the cost of traveling to a particular firm increase for the consumer, the firm's prices and profits also increase.

5. **A Three Firm Stackelberg Competition**

Consider a Cournout competition in quantities between three firms where Firm 1 is the first mover, Firm 2 is the second mover, and Firm 3 is the third, and final, mover. To be clear, Firm 1 sets their optimal quantity first, Firm 2 sets their optimal quantity second, and Firm 3 sets their optimal quantity last. The market inverse demand function the firms face is $p(Q) = a - b(q_1 + q_2 + q_3)$, and all firms have the same marginal cost of c. a is assumed to be strictly greater than c (i.e. $a > c$), and the marginal cost can be viewed as the price the firm pays for one additional unit of output. For simplicity, we assume the industry has no fixed costs of production (i.e. $F = 0$). We solve the sequential move game by backwards induction where Firm 3 maximizes their profits based on their best response to the output set by Firm 1 and Firm 2. This means we can specify the Profit Maximization Problem (PMP) for Firm 3 as:

CALCULUS PART:

$$\max_{q_3 \geq 0} \;\; \pi_3 = [a - b(q_1 + q_2 + q_3)]\, q_3 - cq_3$$

$$\frac{\partial \pi_3(q_1, q_2, q_3)}{\partial q_3} = a - 2bq_3 - bq_1 - bq_2 - c = 0$$

$$\implies BRF_3 \equiv q_3(q_1, q_2) = \frac{(a-c)}{2b} - \frac{1}{2}(q_1 + q_2) \tag{9}$$

Where, we can now use this anticipated Best Response Function for Firm 3 in the Profit Maximization Problem of Firm 2 such that:

$$\max_{q_2 \geq 0} \;\; \pi_2 = \left[a - b(q_1 + q_2 + \underbrace{q_3(q_1, q_2)}_{BRF_3}) \right] q_2 - cq_2$$

$$\max_{q_2 \geq 0} \;\; \pi_2 = \left[a - b\left(q_1 + q_2 + \left(\frac{(a-c)}{2b} - \frac{1}{2}(q_1 + q_2) \right) \right) \right] q_2 - cq_2$$

Which simplifies to:

$$\max_{q_2 \geq 0} \;\; \pi_2 = \left[\frac{(a-c)}{2} - \frac{b}{2}(q_1 + q_2) \right] q_2$$

Where we can now find the Best Response Function for Firm 2 (i.e. BRF_2) using calculus:

$$\frac{\partial \pi_2(q_1, q_2)}{\partial q_2} = \frac{(a-c)}{2} - \frac{b}{2} q_1 - bq_2 = 0$$

$$\implies BRF_2 \equiv q_2(q_1) = \frac{(a-c)}{2b} - \frac{1}{2} q_1 \tag{10}$$

Where, we can now use both of these anticipated Best Response Functions for Firm 2 and Firm 3 in the Profit Maximization Problem of Firm 1 such that:

$$\max_{q_1 \geq 0} \ \pi_1 = \left[a - b(q_1 + \underbrace{q_2(q_1)}_{BRF_2}) + \underbrace{q_3(q_1, \underbrace{q_2(q_1)}_{BRF_2})}_{BRF_3} \right] q_1 - cq_1$$

$$\max_{q_1 \geq 0} \ \pi_1 = \left[a - b \left(q_1 + \underbrace{\left(\frac{(a-c)}{2b} - \frac{1}{2} q_1 \right)}_{BRF_2} + \underbrace{\left(\frac{(a-c)}{2b} - \frac{1}{2} \left(q_1 + \underbrace{\left(\frac{(a-c)}{2b} - \frac{1}{2} q_1 \right)}_{BRF_2} \right) \right)}_{BRF_3} \right) \right] q_1 - cq_1$$

$$\max_{q_1 \geq 0} \ \pi_1 = \left[a - b \left(q_1 + \left(\frac{(a-c)}{2b} - \frac{1}{2} q_1 \right) + \left(\frac{(a-c)}{2b} - \frac{1}{2} \left(q_1 + \left(\frac{(a-c)}{2b} - \frac{1}{2} q_1 \right) \right) \right) \right) \right] q_1 - cq_1$$

Which simplifies to:

$$\max_{q_1 \geq 0} \ \pi_1 = \left[\frac{(a-c)}{4} - \frac{b}{4} q_1 \right] q_1$$

Where we can now find the optimal equilibrium quantity set by Firm 1 (i.e. q_1^*) using calculus:

$$\frac{\partial \pi_1(q_1)}{\partial q_1} = \frac{(a-c)}{4} - \frac{b}{2} q_1 = 0$$

$$\implies q_1^* = \frac{(a-c)}{2b} \tag{11}$$

where we now have the optimal equilibrium allocation for Firm 1 (equation (11)), and two Best Response Functions (equations (9) and (10)) for the firms that will move sequentially after Firm 1 has set their quantity as q_1^*.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

(a) Please find the equilibrium quantity each firm will set in order to maximize their profits in this sequential move game (i.e. write q_1^*, an then find q_2^* and q_3^*). Are the quantities decreasing or increasing relative to Firm 1's quantities? Why is this?

$$(q_1^*, q_2^*, q_3^*) = \left(\frac{(a-c)}{2b}, \frac{(a-c)}{4b}, \frac{(a-c)}{8b} \right)$$

Where, the quantities are decreasing with respect to Firm 1's quantity because of the effect of the first mover's advantage.

(b) Please find the equilibrium price each firm faces (i.e. find $p(Q^*)$).

$$p(Q^*) = \frac{(a + 7c)}{8}$$

(c) Please find the equilibrium profits of each individual firm (i.e. find π_1^*, π_2^* and π_3^*). Please rank them in the order of who is most profitable to the least profitable.

$$(\pi_1^*, \pi_2^*, \pi_3^*) = \left(\frac{(a - c)^2}{16b}, \frac{(a - c)^2}{32b}, \frac{(a - c)^2}{64b} \right)$$

Where,

$$\pi_1^* > \pi_2^* > \pi_3^*$$

Question #1 (Math)

1a) using equations (1) & (2) we get

$$BRF_i \equiv q_i(q_j) = \frac{(a-c_i)}{2b} - \frac{1}{2}q_j$$

&

$$BRF_j \equiv q_j(q_i) = \frac{(a-c_j)}{2b} - \frac{1}{2}q_i$$

Combining

$\Rightarrow$

$$q_i = \frac{(a-c_i)}{2b} - \frac{1}{2}\left[\frac{(a-c_j)}{2b} - \frac{1}{2}q_i\right]$$

$$\Rightarrow \quad q_i = \frac{(a-c_i)}{2b} - \frac{(a-c_j)}{4b} + \frac{1}{4}q_i$$

multiply by 4b

$$\Rightarrow \quad 4bq_i = 2(a-c_i) - (a-c_j) + bq_i$$

$$\Rightarrow \quad 3bq_i = a - 2c_i + c_j$$

$$\Rightarrow \quad q_i^* = \frac{(a-2c_i+c_j)}{3b}$$

also known as $q_i^{Cournot}$

$$\Rightarrow \quad q_j^* = \frac{(a-c_j)}{2b} - \frac{1}{2}\left[\frac{(a-2c_i+c_j)}{3b}\right]$$

$$= \frac{(a-c_j)}{2b} - \frac{(a-2c_i+c_j)}{6b}$$

$$= \frac{3(a-c_j) - (a-2c_i+c_j)}{6b}$$

$$= \frac{2a-4c_j+2c_i}{6b} = \frac{2(a-2c_j+c_i)}{6b}$$

$$= \frac{(a-2c_j+c_i)}{3b} \quad //$$

$\Rightarrow$

$$(q_i^*, q_j^*) = \left(\frac{(a-2c_i+c_j)}{3b}, \frac{(a-2c_j+c_i)}{3b}\right)$$

//

aka $(q_i^{Cournot}, q_j^{Cournot})$

$\Rightarrow$

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1b) Let $c_i = c_j = c$

$$\Rightarrow \quad (q_i^*, q_j^*) = \left(\frac{a-c}{3b}, \frac{a-c}{3b}\right)$$

1c)

$$P(Q^*) = \left(a - \cancel{b}\left(\frac{(a-c)}{3\cancel{b}} + \frac{(a-c)}{3\cancel{b}}\right)\right)$$

$$= \left(\frac{3a - 2(a-c)}{3}\right) = \frac{a+2c}{3} \quad //$$

1d)

$$\pi_i^* = \underbrace{\left(\left(\frac{a+2c}{3}\right) - c\right)}_{P(Q^*) - C} \underbrace{\frac{(a-c)}{3b}}_{q_i^*} - F$$

$$= \left(\frac{(a+2c-3c}{3}\right)\left(\frac{(a-c)}{3b}\right) - F$$

$$= \frac{(a-c)^2}{9b} - F$$

$\boxed{\text{Note, matches general form s.t. } \frac{(a-c)^2}{(N+1)^2 b} - F}$

where, $\pi_i^* = \pi_j^*$ due to symmetry

1e)

$$Q^* = \frac{(a-c)}{2b} \Rightarrow q_i^* = \frac{(a-c)}{4b} \equiv q^{cartel}$$

$$\Rightarrow P(Q^*) = \frac{(a+c)}{3} \qquad \Rightarrow \pi^{cartel} = \frac{\pi^{monopoly}}{N}$$

$$\Rightarrow \pi^{Monopoly} = \frac{(a-c)^2}{4b} - F$$

$$\Rightarrow \pi^{cartel} = \frac{(a-c)^2}{8b} - \frac{F}{2}$$

1f)

$$\pi_i^{Dev} = \left(a - b\left(\frac{(a-c)}{3b} + \frac{(a-c)}{4b}\right)\right)\frac{(a-c)}{3b} - c\frac{(a-c)}{3b} - F$$

$$\Rightarrow \left(\frac{12(a-c)}{12} - \frac{7(a-c)}{12}\right)\frac{(a-c)}{3b} - F$$

$$\Rightarrow \frac{5(a-c)^2}{36b} - F \equiv \pi_i^{Dev} \equiv \pi_j^{Dev}$$

$\Rightarrow$

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1g)

$$\pi_i^{NDev} = \left(a - b\left(\frac{(a-c)}{3b} + \frac{(a-c)}{4b}\right)\right)\frac{(a-c)}{4b} - c\,\frac{(a-c)}{4b} - F$$

$$\Rightarrow \quad = \left(\frac{12(a-c)}{12} - \frac{7(a-c)}{12}\right)\frac{(a-c)}{4b} - F$$

$$= \frac{5(a-c)}{48b} - F \equiv \pi_i^{NDev} \equiv \pi_j^{NDev}$$

1h)

Summarizing

$$\pi_i^N = \pi_i^{Cournot} = \frac{(a-c)^2}{9b} - F, \quad \pi_i^{Cartel} = \frac{(a-c)^2}{8b} - \frac{F}{2}$$

$$\pi_i^{Dev} = \frac{5(a-c)^2}{36b} - F, \quad \pi_i^{NDev} = \frac{5(a-c)^2}{48b} - F$$

Let $F = 0$ (i.e. no fixed costs)

$$\Rightarrow \quad \frac{5(a-c)^2}{36b} > \frac{(a-c)^2}{8b} > \frac{(a-c)^2}{9b} > \frac{5(a-c)^2}{48b}$$

where we can multiply by $1 = \frac{5}{5}$ & cancel

$$\Rightarrow \quad \underbrace{\frac{(a-c)^2}{36b}}_{\pi_i^{Dev}} > \underbrace{\frac{(a-c)^2}{40b}}_{\pi_i^{Cartel}} > \underbrace{\frac{(a-c)^2}{45b}}_{\pi_i^{Cournot}} > \underbrace{\frac{(a-c)^2}{48b}}_{\pi_i^{NDev}}$$

1 h (continued)

$\Rightarrow$

		Firm j: Cartel	Firm j: Compete
Firm i	Cartel	$\pi_i^{Cartel}, \pi_j^{Cartel}$	$\pi_i^{NDev}, \pi_j^{Dev}$
	Compete	$\pi_i^{Dev}, \pi_j^{NDev}$	$\pi_i^{Cournot}, \pi_j^{Cournot}$

plugging in values

$\Rightarrow$

		Firm j: Cartel	Firm j: Compete
Firm i	Cartel	$\frac{(a-c)^2}{40b}, \frac{(a-c)^2}{40b}$	$\frac{(a-c)^2}{48b}, \frac{(a-c)^2}{36b}$
	Compete	$\frac{(a-c)^2}{36b}, \frac{(a-c)^2}{48b}$	$\frac{(a-c)^2}{45b}, \frac{(a-c)^2}{45b}$

1 i)

where, doing our Best Response Analysis s.t.

Firm i: ① If Firm j plays Cartel, I play Compete ($\pi_i^{Dev} > \pi_i^{Cartel}$)

② If Firm j plays Compete, I play Compete ($\pi_i^{Cournot} > \pi_i^{NDev}$)

Firm j: ① If Firm i plays Cartel, I play Compete ($\pi_j^{Dev} > \pi_j^{Cartel}$)

② If Firm i plays Compete, I play Compete ($\pi_j^{Cournot} > \pi_j^{NDev}$)

$\Rightarrow$

		Firm j: Cartel	Firm j: Compete
Firm i	Cartel	$\pi_i^{Cartel}, \pi_j^{Cartel}$	$\pi_i^{NDev}, \underline{\pi_j^{Dev}}$
	Compete	$\underline{\pi_i^{Dev}}, \pi_j^{NDev}$	$\underline{\pi_i^{Cournot}}, \underline{\pi_j^{Cournot}}$

underlining all Best Responses

$\Rightarrow$ Nash Equilibrium $= (q_i^{*Cournot}, q_j^{*Cournot}) = \left\{\left(\frac{(a-c)}{3b}, \frac{(a-c)}{3b}\right)\right\}$

1j)

$$GTS = \delta^0(\pi^{Dev}) + \delta^1(\pi^{Cournot}) + \delta^2(\pi^{Cournot}) + \delta^3(\pi^{Cournot}) + \dots$$

$$\Rightarrow \quad = (\pi^{Dev}) + \delta\pi^{Cournot}(1 + \delta + \delta^2 + \delta^3 + \dots)$$

$$= \pi^{Dev} + \delta\pi^{Cournot}\underbrace{\left(\sum_{k=0}^{\infty} \delta^k\right)}$$

Geometric series where $\sum_{k=0}^{\infty} \delta^k = \frac{1}{(1-\delta)}$

$$= \pi^{Dev} + \frac{\delta}{(1-\delta)}\pi^{Cournot}$$

//

1k)

$$\text{Cartel Payoffs} = \delta^0\pi^{Cartel} + \delta^1\pi^{Cartel} + \delta^2\pi^{Cartel} + \delta^3\pi^{Cartel} + \dots$$

$$\Rightarrow \quad = \pi^{Cartel}\underbrace{(1 + \delta + \delta^2 + \delta^3 + \dots)}_{\text{Geometric Series}}$$

$$\Rightarrow \quad = \pi^{Cartel}\frac{1}{(1-\delta)}$$

1l) The Cartel can be sustained iff

$$\text{Cartel Payoffs} > GTS$$

$$\Rightarrow \quad \pi^{Cartel}\frac{1}{(1-\delta)} > \pi^{Dev} + \frac{\delta}{(1-\delta)}\pi^{Cournot}$$

$$\Rightarrow \quad \pi^{Cartel} > \pi^{Dev}(1-\delta) + \delta\pi^{Cournot}$$

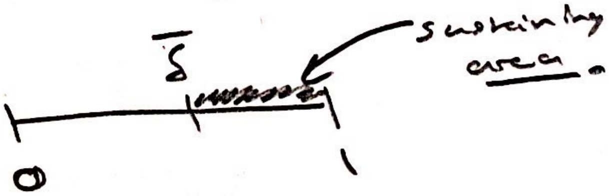
$$\Rightarrow \quad \delta(\pi^{Dev} - \pi^{Cournot}) > \pi^{Dev} - \pi^{Cartel}$$

$$\Rightarrow \quad \delta > \frac{\pi^{Dev} - \pi^{Cartel}}{\pi^{Dev} - \pi^{Cournot}} \equiv \bar{\delta}$$

Where IS

$\delta > \bar{\delta}$

Cartel Agreement will be sustained



$\Rightarrow$

1m)

$$\delta > \frac{\frac{5(a-c)^2}{36b} - \frac{(a-c)^2}{8b}}{\frac{5(a-c)^2}{36b} - \frac{(a-c)^2}{9b}}$$

$$\Rightarrow \quad \delta > \frac{\frac{5}{36} - \frac{1}{8}}{\frac{5}{36} - \frac{1}{9}} = \frac{1/72}{1/36} = \frac{1/\cancel{72}}{2/\cancel{72}}$$

$$\Rightarrow \quad \delta > \bar{\delta} = \frac{1}{2}$$

$\Rightarrow$ If $\delta > 1/2$, then collusion will be sustained. //

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2a) using equations (2) $ (4)

Question #2 (Math)

$$\Rightarrow \quad BRF_i = P_i(P_j) = \frac{(a+bc)}{2b} + \frac{1}{2b}P_j$$

$

$$BRF_j = P_j(P_i) = \frac{(a+bc)}{2b} + \frac{1}{2b}P_i$$

$\Rightarrow$ ① $a\uparrow \Longrightarrow P_i\uparrow$

② $b\uparrow \Longrightarrow P_i\downarrow$ since $\frac{a}{2b} + \frac{1}{2}c + \frac{1}{2b}P_j$ $\Rightarrow$ If $b\uparrow$ $P_i\downarrow$

③ $c\uparrow \Rightarrow P_i\uparrow$

④ $P_j\uparrow \Rightarrow P_i\uparrow$

2b) Solving for P_i^* first

$$\Longrightarrow \quad P_i = \frac{(a+cb)}{2b} + \frac{1}{2b}\left[\frac{(a+bc)}{2b} + \frac{1}{2b}P_i\right]$$

$$\Rightarrow \quad P_i = \frac{(a+cb)}{2b} + \frac{(a+bc)}{4b^2} + \frac{P_i}{4b^2}$$ multiply by $4b^2$

$$\Rightarrow \quad 4b^2P_i = 2b(a+cb) + (a+cb) + P_i$$

$$\Rightarrow \quad P_i(4b^2-1) = (2b+1)(a+cb)$$

$$\Rightarrow \quad P_i^* = \frac{(2b+1)(a+cb)}{\underbrace{(4b^2-1)}_{\text{Diff of squares} = ((2b)^2-(1)^2)}} = \frac{\cancel{(2b+1)}(a+cb)}{\cancel{(2b+1)}(2b-1)} = \frac{(a+cb)}{(2b-1)}$$

$$\Rightarrow \quad P_j^* = \frac{(a+bc)}{2b} + \frac{1}{2b}\left(\frac{a+cb}{2b-1}\right)$$

$$= \frac{(a+bc)(2b-1) + (a+cb)}{2b(2b-1)}$$

$$= \frac{(2b-\cancel{1}+\cancel{1})(a+bc)}{2b(2b-1)} = \frac{\cancel{2b}}{\cancel{2b}}\frac{(a+bc)}{(2b-1)}$$

$$\Rightarrow \quad (P_i^*, P_j^*) = \left(\frac{(a+cb)}{(2b-1)}, \frac{(a+cb)}{(2b-1)}\right)$$ ← prices are still the same !! :)

2c) $q_i^* = a - b\left(\frac{(a+cb)}{(2b-1)}\right) + \left(\frac{(a+cb)}{(2b-1)}\right)$

$\Rightarrow \quad q_i^* = \left(\frac{a(2b-1)}{(2b-1)} - \frac{b(a+cb)}{(2b-1)} + \frac{(a+cb)}{(2b-1)}\right)$

$= \left(\frac{a(2b-1)+(1-b)(a+cb)}{(2b-1)}\right)$

$= \left(\frac{\cancel{2}ab - \cancel{a} + \cancel{a} + cb - \cancel{ab} - cb^2}{(2b-1)}\right)$ or

$q_i^* = \frac{(ab+cb-cb^2)}{(2b-1)} = \frac{b(a-cb+c)}{(2b-1)} = \frac{b(a-c(b-1))}{(2b-1)}$

Through symmetry, we know $q_i^* = q_j^*$

$\Rightarrow \quad q_j^* = \frac{b(a+c(1-b))}{(2b-1)}$ when is $b=1$ $\Rightarrow q_i^* = a$

where, this is actually a corner solution where $p_i^* = c$

2d)

$\Rightarrow \pi_i^* = \left(p_i^* - c\right) q_i^*$

$= \left(\frac{(a+bc)}{(2b-1)} - c\right)\frac{b(a-c(b-1))}{(2b-1)}$

$= \left(\frac{a+bc-c(2b-1)}{(2b-1)}\right)\frac{b(a-c(b-1))}{(2b-1)}$

$= \left(\frac{a-cb+c}{(2b-1)}\right)\frac{b(a-c(b-1))}{(2b-1)} = \frac{b(a-c(b-1))^2}{(2b-1)^2}$

$\Rightarrow \pi_j^* = \frac{b(a-c(b-1))^2}{(2b-1)^2}$ by symmetry //

3a) using equations (5) $ (6)

Question #3 (math)

$\Rightarrow$ $BRF_i \equiv q_i(q_j) = \frac{(a-c)}{2b} - \frac{d}{2b}q_j$

$\Rightarrow$ $BRF_j \equiv q_j(q_i) = \frac{(a-c)}{2b} - \frac{d}{2b}q_i$

$\Rightarrow$ $a\uparrow \Rightarrow q_i \uparrow$

$b\uparrow \Rightarrow q_i \uparrow$ $\quad$ $ $q_j\uparrow \Rightarrow q_i \downarrow$

$c\uparrow \Rightarrow q_i \downarrow$

$d\uparrow \Rightarrow q_i \downarrow$

3b) starting w/ q_i

$\Rightarrow$ $q_i = \frac{(a-c)}{2b} - \frac{d}{2b}\left[\frac{(a-c)}{2b} - \frac{d}{2b}q_i\right]$

$\Rightarrow$ $q_i = \frac{(a-c)}{2b} - \frac{d(a-c)}{4b^2} + \frac{d^2}{4b^2}q_i$

$\Rightarrow$ $4b^2q_i = 2b(a-c) - d(a-c) + d^2q_i$

$\Rightarrow$ $q_i(4b^2-d^2) = (2b-d)(a-c)$

$\Rightarrow$ $q_i^* = \frac{(2b-d)(a-c)}{\underbrace{(4b^2-d^2)}_{\text{diff of squares}}} = \frac{(2b-d)(a-c)}{(2b-d)(2b+d)} = \frac{(a-c)}{(2b+d)}$

$\Rightarrow$ $q_j^* = \frac{(a-c)}{2b} - \frac{d}{2b}\left[\frac{(a-c)}{(2b+d)}\right]$

$= \frac{(a-c)(2b+d)}{2b(2b+d)} - \frac{d(a-c)}{2b(2b+d)}$

$= \frac{(a-c)(\cancel{2b})}{\cancel{2b}(2b+d)} = \frac{(a-c)}{(2b+d)}$

$\Rightarrow$ $(q_i^*, q_j^*) = \left(\frac{(a-c)}{(2b+d)}, \frac{(a-c)}{(2b+d)}\right)$

Intuition

$d \to b \Rightarrow q_i^* \to q_i^{\text{Cournot}}$

$d \to 0 \Rightarrow q_i^* \to q_i^{\text{Monopoly}}$

$\Rightarrow$ as products become more differentiated, they produce ∇ less

3c) $P_i^* = a - b\left(\frac{(a-c)}{(2b+d)}\right) - d\left(\frac{(a-c)}{(2b+d)}\right)$

$$= \left(\frac{a(2b+d) - (b+d)(a-c)}{(2b+d)}\right)$$

$$\Rightarrow \quad P_j^* = \left(\frac{a(2b+d) - (b+d)(a-c)}{(2b+d)}\right) \quad \text{by symmetry}$$

3d) $\pi_i^* = \left(\frac{a(2b+d) - (b+d)(a-c)}{(2b+d)} - c\right)\left(\frac{(a-c)}{(2b+d)}\right)$

$$\Rightarrow \quad = \left(\frac{(a-c)(2b+d) - (b+d)(a-c)}{(2b+d)}\right)\frac{(a-c)}{(2b+d)}$$

$$= \left(\frac{(a-c)(2b+\cancel{d}-b-\cancel{d})}{(2b+d)}\right)\frac{(a-c)}{(2b+d)}$$

$$\pi_i^* = \frac{b(a-c)}{(2b+d)}\frac{(a-c)}{(2b+d)} = \frac{b(a-c)^2}{(2b+d)^2}$$

$$\Rightarrow (\pi_i^*, \pi_j^*) = \left(\frac{b(a-c)^2}{(2b+d)^2}, \frac{b(a-c)^2}{(2b+d)^2}\right) \quad \text{by \underline{symmetry}}$$

3e) Let $d \to b$ s.t. $d = b$

$$\Rightarrow \pi_i^* = \frac{b(a-c)^2}{(2b+(b))^2}$$

plugging in

$$\Rightarrow \quad = \frac{b(a-c)^2}{(3b)^2} = \frac{\cancel{b}(a-c)^2}{9b^{\cancel{2}}} = \frac{(a-c)^2}{9b} \equiv \pi_i^{\text{Cournot}}$$

$\circ//$

3f) $\Rightarrow$ Let $d = 0$

$$\Rightarrow \pi_i^* = \frac{b(a-c)^2}{(2b+(0))^2} = \frac{b(a-c)^2}{(2b)^2} = \frac{(a-c)^2}{4b} \equiv \pi_i^{\text{Monopoly}}$$

$\circ//$

Question #4 (Math)

4a) See key

4b) using (7) & (8)

$$\Rightarrow \frac{\cancel{2t}P_A}{\cancel{2t}} = \frac{t+c}{2t} + \frac{1}{2t}P_B \qquad (7)$$

$$\Rightarrow BRF_A \equiv P_A(P_B) = \frac{t+c}{2} + \frac{1}{2}P_B$$

Combining

$$\Rightarrow \frac{\cancel{2t}P_B}{\cancel{2t}} = \frac{t+c}{2t} + \frac{1}{2t}P_B \qquad (8)$$

$$\Rightarrow BRF_B \equiv P_B(P_A) = \frac{t+c}{2} + \frac{1}{2}P_A$$

$$\Rightarrow P_A = \frac{t+c}{2} + \frac{1}{2}\left[\frac{t+c}{2} + \frac{1}{2}P_A\right]$$

$$\Rightarrow P_A = \frac{t+c}{2} + \frac{(t+c)}{4} + \frac{1}{4}P_A$$

$$\Rightarrow 4P_A = 2(t+c) + (t+c) + P_A$$

$$\Rightarrow P_A^* = \frac{3(t+c)}{3} = t+c$$

Same (P_A^*, P_B^*)

$$\Rightarrow P_B^* = \frac{(t+c)}{2} + \frac{1}{2}[t+c] = t+c$$

Note, If $t=0$, $\Rightarrow$ we get $P_A^* = P_B^* = c$

which is analagous to the bertrand model w/ transportation.

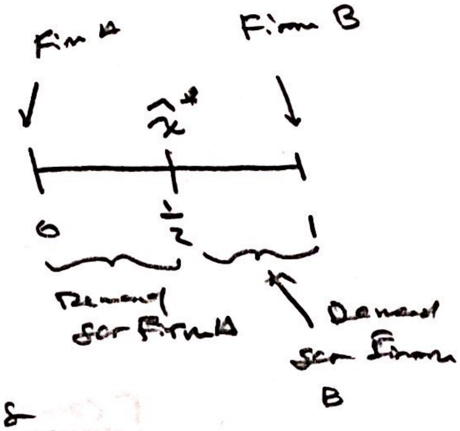
4c) To find $\hat{x}^* = Q_A^*(P_A^*, P_B^*)$ we plug in P_A^* & P_B^*

$$\Rightarrow \quad \hat{x}^* = \left(\frac{P_B^* - P_A^*}{2t} + \frac{1}{2}\right)$$

$$= \left(\frac{(t+c) - (t+c)}{2t} + \frac{1}{2}\right)$$

$$= \left(\frac{0}{2t} + \frac{1}{2}\right) = \frac{1}{2}$$

$\Rightarrow$



$$\Rightarrow \quad (1 - \hat{x}^*) = \left(1 - \frac{1}{2}\right) = \frac{1}{2}$$

$\Rightarrow$ They will split the $\underbrace{\text{consumers}}_{\text{(i.e. total demand)}}$ in half.

4d) $$\pi_A^* = (P_A^* - c)\hat{x}^* = (t + c - c)\left(\frac{1}{2}\right)$$

$$= \frac{t}{2}$$

Similarly

$$\pi_B^* = (P_B^* - c)(1 - \hat{x}^*) = (t + c - c)\left(\frac{1}{2}\right)$$

$$= \frac{t}{2}$$

$$\Rightarrow \quad (\pi_A^*, \pi_B^*) = \left(\frac{t}{2}, \frac{t}{2}\right)$$

@ $t = 0$, we get $\pi_i^* = 0$, which is the same as the Bertrand competition w/o product differentiation outcome. //

5A) using (9), (10), & (11)

Question #5 (Math)

$$\Rightarrow q_1^* = \frac{(a-c)}{2b} \qquad (11)$$

$$\Rightarrow BRF_2 \equiv q_2^*(q_1^*) = \frac{(a-c)}{2b} - \frac{1}{2}\left[\frac{(a-c)}{2b}\right]$$

$$= \frac{(a-c)}{2b} - \frac{(a-c)}{4b}$$

$$= \frac{2(a-c) - (a-c)}{4b}$$

$$= \frac{(a-c)}{4b}$$

$$\Rightarrow BRF_3 \equiv q_3^*(q_1^*, q_2^*) = \frac{(a-c)}{2b} - \frac{1}{2}\left[\frac{(a-c)}{2b} + \frac{(a-c)}{4b}\right]$$

$$= \frac{(a-c)}{2b} - \frac{1}{2}\left[\frac{3(a-c)}{4b}\right]$$

$$= \frac{(a-c)}{2b} - \frac{3(a-c)}{8b}$$

$$= \frac{4(a-c) - 3(a-c)}{8b} = \frac{(a-c)}{8b}$$

$$\Rightarrow (q_1^*, q_2^*, q_3^*) = \left(\frac{(a-c)}{2b}, \frac{(a-c)}{4b}, \frac{(a-c)}{8b}\right)$$

~~where~~, Quantities are decreasing due to the effect of the first movers advantage.

5b) $$p(Q^*) = \left(a - b\left(\frac{(a-c)}{2b} + \frac{(a-c)}{4b} + \frac{(a-c)}{8b}\right)\right)$$

$$= \left(a - \frac{7(a-c)}{8}\right) = \frac{a+7c}{8}$$

5c)

$$\pi_1^* = \left(\frac{(a-c)}{4} - \frac{\cancel{b}}{4} \frac{(a-c)}{2\cancel{b}} \right) \frac{(a-c)}{2b}$$

$$= \frac{(a-c)}{8} \frac{(a-c)}{2b} = \frac{(a-c)^2}{16b}$$

$$\pi_2^* = \left[\frac{(a-c)}{2} - \frac{\cancel{b}}{2} \left(\frac{(a-c)}{2\cancel{b}} + \frac{(a-c)}{4\cancel{b}} \right) \right] \frac{(a-c)}{4b}$$

$$= \left[\frac{(a-c)}{2} - \frac{3(a-c)}{8} \right] \frac{(a-c)}{4b}$$

$$= \frac{(a-c)}{8} \frac{(a-c)}{4b} = \frac{(a-c)^2}{32b}$$

$$\Rightarrow \pi_3^* = \left[(a-c) - \cancel{b} \left(\frac{(a-c)}{2\cancel{b}} + \frac{(a-c)}{4\cancel{b}} + \frac{(a-c)}{8\cancel{b}} \right) \right] \frac{(a-c)}{8b}$$

$$= \left[(a-c) - \frac{7(a-c)}{8} \right] \frac{(a-c)}{8b}$$

$$= \frac{(a-c)}{8} \frac{(a-c)}{8b} = \frac{(a-c)^2}{64b}$$

$$\Rightarrow \quad \pi_1^* > \pi_2^* > \pi_3^*$$

$$\Rightarrow \quad \frac{(a-c)^2}{16b} > \frac{(a-c)^2}{32b} > \frac{(a-c)^2}{64b}$$ //

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