EconS 305: Intermediate Microeconomics w/o Calculus

Homework 1:
Consumer Preference and Decision Making

Due: Friday, May 22nd, 2020 at 5:00pm via Blackboard

- Please submit all homework solutions in the order the questions are presented and as **one .PDF**.

- Please redo all the [**EXAMPLE**] solutions for Question 1, and submit them with your homework.

- Please **show all calculations** as these exercises are meant to refine your quantitative tool set. If I can not follow your calculations or it seems as you just "copy and pasted" answers from the internet, I will be deducting half the points from that solution.

1. **Cobb-Douglas with** $\alpha + \beta = 1$

   Consider the following utility maximization problem of the consumer where the consumer prefers to consumer some amount of rice q_1 and beans q_2. We model the consumer's preferences for these goods with the classic Cobb-Douglas Utility function $U(q_1, q_2) = q_1^{\alpha} q_2^{\beta}$. For more information on this utility function please navigate to to this website (http://www2.hawaii.edu/∼fuleky/anatomy/anatomy.html). For simplicity, we assume that $\alpha + \beta = 1$ which means that the consumer prefers more of each good, and that each additional amount of good yields the same utility as the the previous amount of good. We refer to this in economics as a consumer having constant returns to scale in terms of consumption. This also means that we can represent the utility function as $U(q_1, q_2) = q_1^{\alpha} q_2^{1-\alpha}$ since $\beta = 1 - \alpha$.

   Next, we employ a linear budget constraint on the consumer $p_1 q_1 + p_2 q_2 = M$ where p_1 is the price the consumer pays to consumer one unit of rice, and p_2 is the price the consumer pays to consumer one unit of beans. M is the consumer's entire budget for rice and beans, and we assume the consumer is willing to spend their entire budget for rice and beans, on rice and beans.

   We represent the consumer's Utility Maximization Problem (UMP) as

$$\max_{q_1, q_2 \geq 0} \; U(q_1, q_2) = q_1^{\alpha} q_2^{1-\alpha}$$

subject to the budget constraint of:

$$p_1 q_1 + p_2 q_2 = M$$

CALCULUS PART:
Using constrained optimization techniques from calculus, we can set the problem up with a Lagrange multiplier s.t.

$$\mathcal{L}(q_1, q_2) = q_1^{\alpha} q_2^{1-\alpha} - \lambda \left[p_1 q_1 + p_2 q_2 - M \right]$$

From here, we can take our derivatives and set them equal to zero

$$\frac{\partial \mathcal{L}(q_1, q_2)}{\partial q_1} = \alpha q_1^{(\alpha-1)} q_2^{1-\alpha} - \lambda p_1 = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}(q_1, q_2)}{\partial q_2} = (1-\alpha) q_1^{\alpha} q_2^{(-\alpha)} - \lambda p_2 = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}(q_1, q_2)}{\partial \lambda} = p_1 q_1 + p_2 q_2 - M = 0 \tag{3}$$

where we now have three equations ((1),(2), and (3)), and two choice variables (q_1 and q_2) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

(a) Find the consumer's demand for q_1 and q_2 in equilibrium (i.e. find q_1^* and q_2^*).

Solving for λ in equations (1) and (2), and setting them equal to each other yield a result of

$$\frac{\alpha q_1^{(\alpha-1)} q_2^{1-\alpha}}{p_1} = \frac{(1-\alpha) q_1^{\alpha} q_2^{(-\alpha)}}{p_2}$$

Canceling on both sides

$$\implies \frac{\alpha q_1^{\cancel{\alpha}-1} q_2^{1-\cancel{\alpha}}}{p_1} = \frac{(1-\alpha)\cancel{q_1^{\alpha}}\cancel{q_2^{(-\alpha)}}}{p_2}$$

Where we get one choice variable in terms of the other choice variable

$$q_2 = \frac{(1-\alpha)}{\alpha} \frac{p_1}{p_2} q_1 \tag{4}$$

Using this equation, we plug it into equation (3) and solve for the only choice variable we have (q_1) such that

$$p_1 q_1 + p_2 \left(\frac{(1-\alpha)}{\alpha} \frac{p_1}{p_2} q_1 \right) = M$$

$$p_1q_1 + \cancel{p_2}\left(\frac{(1-\alpha)}{\alpha}\frac{p_1}{\cancel{p_2}}q_1\right) = M$$

$$p_1q_1\left(1+\frac{(1-\alpha)}{\alpha}\right) = M$$

$$p_1q_1\left(\frac{\cancel{\alpha}-(1-\cancel{\alpha})}{\alpha}\right) = M$$

$$p_1q_1\left(\frac{1}{\alpha}\right) = M$$

$$\implies \quad q_1^*(p_1, M, \alpha) = \frac{\alpha}{p_1}M \tag{5}$$

which, intuitively, is the consumer's demand function for rice. Notice that the the consumer's demand function for rice is a function in terms of variables in which the consumer is not choosing (i.e. p_1, M, α). Plugging this function back into equation (4), we get

$$q_2^* = \frac{(1-\alpha)}{\alpha}\frac{p_1}{p_2}q_1^*$$

$$q_2^* = \frac{(1-\alpha)}{\cancel{\alpha}}\frac{\cancel{p_1}}{p_2}\left(\frac{\cancel{\alpha}}{\cancel{p_1}}M\right)$$

$$q_2^*(p_2, M, \alpha) = \frac{(1-\alpha)}{p_2}M \tag{6}$$

which is the consumers demand function for beans.

In summary, the quantity demanded for rice (q_1^*) and beans (q_2^*), respectively, is

$$(q_1^*(p_1, M, \alpha), q_2^*(p_2, M, \alpha)) = \left(\frac{\alpha}{p_1}M\ ,\ \frac{(1-\alpha)}{p_2}M\right) \tag{7}$$

(b) What happens to each demand with an increase in it's own price (p_i), the other good's price (p_j), the consumer's budget (M), and the consumer's preference for rice (α)?

i) The consumer's quantity demanded for rice and beans both decrease with respect to an increase in their own price. This should be a property of your demand functions as it follows the law of demand.

ii) Both demand functions are not directly affected by the price of the other good. With that said, notice how if we were to plug in the budget constraint for M that we would have a positive relationship as the other price increases. This would follow the same intuition that we would expect from our theory about the consumer.

iii) Both demand functions increase as the budget of the consumer increases. This makes sense as an increase in wealth would mean that the consumer could consume more rice and beans.

iv) Notice that as α increases, the consumer's demand for rice increases and their demand for beans decreases. This is because of the condition we set on α such that $\alpha + \beta = 1$. This implies that, from a preference stand-point, these goods can be substituted to some degree for the consumer. This is a neat concept because economists can then build models that estimate these preference parameters to determine equilibrium conditions.

(c) Find the consumers utility in terms of their optimal demand (i.e. find $U(q_1^*, q_2^*)$). In economics, we refer to this as the indirect utility function as it is in terms of exogenous parameters (factors other than the choice variables).

$$U(q_1^*, q_2^*) = \left(\frac{\alpha}{p_1}M\right)^{\alpha}\left(\frac{(1-\alpha)}{p_2}M\right)^{1-\alpha}$$

$$U(q_1^*, q_2^*) = \left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{(1-\alpha)}{p_2}\right)^{1-\alpha} M$$

(d) Find the own price elasticity of demand for rice using $\frac{\partial q_1^*}{\partial p_1} = -\frac{\alpha}{p_1^2}M$.

Using the definition of the own price elasticity of demand s.t.

$$\varepsilon_{q_1^*,p_1} = \frac{\Delta q_1^*}{\Delta p_1}\frac{p_1}{q_1^*} = \frac{\partial q_1^*}{\partial p_1}\frac{p_1}{q_1^*}$$

And substituting in our results

$$\varepsilon_{q_1^*,p_1} = -\frac{\alpha M}{p_1^2}\frac{p_1}{\left(\frac{\alpha M}{p_1}\right)}$$

$$\varepsilon_{q_1^*,p_1} = -\frac{\alpha M}{p_1^2}\frac{p_1^2}{\alpha M} = -1$$

Where we can interpret this as a "unit elastic" demand where a 1% increase in good 1's price, yields a 1% decrease in the quantity demanded for good 1 (rice). Also note that it is the standard in economics to take the absolute value of the elasticity so that we can characterize the type of elasticity it is (i.e. it is unit elastic since $|\varepsilon_{q_1^*,p_1}| = 1$).

(e) Find the cross price elasticity of demand for rice using $\frac{\partial q_1^*}{\partial p_2} = 0$.

Using the definition of the cross price elasticity of demand s.t.

$$\varepsilon_{q_1^*,p_2} = \frac{\Delta q_1^*}{\Delta p_2}\frac{p_2}{q_1^*} = \frac{\partial q_1^*}{\partial p_2}\frac{p_2}{q_1^*}$$

And substituting in our results

$$\varepsilon_{q_1^*,p_1} = 0\frac{p_2}{\left(\frac{\alpha M}{p_2}\right)}$$

$$\varepsilon_{q_1^*,p_1} = 0$$

Where we can interpret this as a "perfectly inelastic" demand with respect to the other good's price where a 1% increase in good 2's price, yields a 0% change in the quantity demanded for good 1 (rice). Essentially, a price change in beans has no effect on the consumer's quantity demanded for rice. Also note that it is the standard in economics to take the absolute value of the elasticity so that we can characterize the type of elasticity it is (i.e. it is perfectly inelastic since $|\varepsilon_{q_1^*,p_2}| = 0$).

2. **Cobb-Douglas with General Preferences**

Consider the same setting we were operating in in Question 1, but now lets consider that $0 < \alpha + \beta \leq 1$. Notice that we cannot simply the problem as we did before (by replacing $\beta = 1 - \alpha$). Also note that adding this condition to the preference parameters implies that the consumer has decreasing returns to scale in consumption.

We set up the consumer's Utility Maximization Problem (UMP) as

$$\max_{q_1, q_2 \geq 0} \; U(q_1, q_2) = q_1^{\alpha} q_2^{\beta}$$

subject to the budget constraint of:

$$p_1 q_1 + p_2 q_2 = M$$

CALCULUS PART:
Using constrained optimization techniques from calculus, we can set the problem up with a Lagrange multiplier s.t.

$$\mathcal{L}(q_1, q_2 : \lambda) = q_1^{\alpha} q_2^{\beta} - \lambda \left[p_1 q_1 + p_2 q_2 - M \right]$$

From here, we can take our derivatives and set them equal to zero

$$\frac{\partial \mathcal{L}(q_1, q_2; \lambda)}{\partial q_1} = \alpha q_1^{(\alpha-1)} q_2^{\beta} - \lambda p_1 = 0 \tag{8}$$

$$\frac{\partial \mathcal{L}(q_1, q_2; \lambda)}{\partial q_2} = \beta q_1^{\alpha} q_2^{(\beta-1)} - \lambda p_2 = 0 \tag{9}$$

$$\frac{\partial \mathcal{L}(q_1, q_2; \lambda)}{\partial \lambda} = p_1 q_1 + p_2 q_2 - M = 0 \tag{10}$$

where we now have three equations ((8),(9), and (10)), and two choice variables (q_1 and q_2) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

(a) Find the consumer's demand for q_1 and q_2 in equilibrium (i.e. find q_1^* and q_2^*).

The quantity demanded for rice (q_1^*) and beans (q_2^*), respectively, is

$$(q_1^*(p_1, M, \alpha, \beta), q_2^*(p_2, M, \alpha, \beta)) = \left(\frac{\alpha}{(\alpha + \beta)} \frac{M}{p_1} \, , \, \frac{\beta}{(\alpha + \beta)} \frac{M}{p_2} \right)$$

(b) What are the similarities and differences between the demands you found in Question 1?

Note that the preference parameters are now relative to the sum of the preference parameters. Before, we knew that $\alpha + \beta = 1$, and this condition showed up in the denominator but because it was $= 1$ we did not see it.

(c) What happens to each demand with an increase in it's own price (p_i), the other good's price (p_j), the consumer's budget (M), and the consumer's preference for rice (α)?

i) The consumer's quantity demanded for rice and beans both decrease with respect to an increase in their own price. This should be a property of your demand functions as it follows the law of demand.

ii) Both demand functions are not directly affected by the price of the other good. With that said, notice how if we were to plug in the budget constraint for M that we would have a positive relationship as the other price increases. This would follow the same intuition that we would expect from our theory about the consumer.

iii) Both demand functions increase as the budget of the consumer increases. This makes sense as an increase in wealth would mean that the consumer could consume more rice and beans.

iv) Notice that as α increases, the consumer's demand for rice increases, and their demand for beans decreases. Alternatively, Notice that as β increase, the consumer's demand for beans increases, and their demand for rice decreases. We can see this if we keep the other parameter, the one that is not of interest, constant. This is because of the condition we set on α and β, which is $\alpha + \beta \leq 1$. This implies that, from a preference stand-point, these goods can be substituted to some degree for the consumer, but it is now relative to the sum total of the relationship between α and β.

(d) Find the consumers utility in terms of their optimal demands (i.e. find $U(q_1^*, q_2^*)$).

$$U(q_1^*, q_2^*) = \left(\frac{\alpha}{(\alpha+\beta)}\frac{M}{p_1}\right)^{\alpha}\left(\frac{\beta}{(\alpha+\beta)}\frac{M}{p_2}\right)^{\beta}$$

$$U(q_1^*, q_2^*) = \left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\left(\frac{M}{(\alpha+\beta)}\right)^{\alpha+\beta}$$

(e) Find the own price elasticity of demand for rice using $\frac{\partial q_1^*}{\partial p_1} = -\frac{\alpha}{(\alpha+\beta)p_1^2}M$.

Using the definition of the own price elasticity of demand s.t.

$$\varepsilon_{q_1^*, p_1} = \frac{\Delta q_1^*}{\Delta p_1}\frac{p_1}{q_1^*} = \frac{\partial q_1^*}{\partial p_1}\frac{p_1}{q_1^*}$$

And substituting in our results

$$\varepsilon_{q_1^*,p_1} = -\frac{\alpha M}{(\alpha+\beta)p_1^2}\frac{p_1}{\left(\frac{\alpha M}{(\alpha+\beta)p_1}\right)}$$

$$\varepsilon_{q_1^*,p_1} = -\frac{\alpha M}{(\alpha+\beta)p_1^2}\frac{(\alpha+\beta)p_1^2}{\alpha M} = -1$$

Where we can interpret this as a "unit elastic" demand where a 1% increase in good 1's price, yields a 1% decrease in the quantity demanded for good 1 (rice). Also note that it is the standard in economics to take the absolute value of the elasticity so that we can characterize the type of elasticity it is (i.e. it is unit elastic since $|\varepsilon_{q_1^*,p_1}| = 1$).

(f) Find the cross price elasticity of demand for rice using $\frac{\partial q_1^*}{\partial p_2} = 0$.

Using the definition of the cross price elasticity of demand s.t.

$$\varepsilon_{q_1^*,p_2} = \frac{\Delta q_1^*}{\Delta p_2}\frac{p_2}{q_1^*} = \frac{\partial q_1^*}{\partial p_2}\frac{p_2}{q_1^*}$$

And substituting in our results

$$\varepsilon_{q_1^*,p_1} = 0\frac{p_2}{\left(\frac{\alpha M}{(\alpha+\beta)p_2}\right)}$$

$$\varepsilon_{q_1^*,p_1} = 0$$

Where we can interpret this as a "perfectly inelastic" demand with respect to the other good's price where a 1% increase in good 2's price, yields a 0% change in the quantity demanded for good 1 (rice). Essentially, a price change in beans has no effect on the consumer's quantity demanded for rice. Also note that it is the standard in economics to take the absolute value of the elasticity so that we can characterize the type of elasticity it is (i.e. it is perfectly inelastic since $|\varepsilon_{q_1^*,p_2}| = 0$).

(g) Find the income elasticity of demand for rice using $\frac{\partial q_1^*}{\partial M} = \frac{\alpha}{(\alpha+\beta)p_1}$.

Using the definition of the income elasticity of demand s.t.

$$\varepsilon_{q_1^*,M} = \frac{\Delta q_1^*}{\Delta M}\frac{M}{q_1^*} = \frac{\partial q_1^*}{\partial M}\frac{M}{q_1^*}$$

And substituting in our results

$$\varepsilon_{q_1^*,M} = \frac{\alpha}{(\alpha+\beta)p_1}\frac{M}{\left(\frac{\alpha M}{(\alpha+\beta)p_1}\right)}$$

$$\varepsilon_{q_1^*,p_1} = 1$$

3. **Log Utility with an Ad Valorem Tax**

Consider a similar setting to Question 2, but now we have a new utility function. This utility function is called the "Log" utility function, and is one of the more simpler "production" functions to work with. We are still considering that $0 < \alpha + \beta \leq 1$, and now we are introducing an "Ad Valorem Tax" $(1 + \tau)$ to the price of rice (good 1). Notice that the tax is some "added value" to the price of rice, and that this it is applied to the price of each and every unit sold. Also notice how it enters into the budget constraint of the consumer. This is the price the consumer pays, and this is also the price the theoretical producer will get.

We modify the Cobb-Douglas and set up the consumer's Utility Maximization Problem (UMP) as

$$U(q_1, q_2) = \log(q_1^{\alpha} q_2^{\beta})$$
$$U(q_1, q_2) = \log(q_1^{\alpha}) + \log(q_2^{\beta})$$

$$\max_{q_1, q_2 \geq 0} \; U(q_1, q_2) = \alpha \log(q_1) + \beta \log(q_2)$$

subject to the budget constraint of:

$$p_1(1 + \tau)q_1 + p_2 q_2 = M$$

CALCULUS PART:
Using constrained optimization techniques from calculus, we can set the problem up with a Lagrange multiplier s.t.

$$\mathcal{L}(q_1, q_2 : \lambda) = \alpha \log(q_1) + \beta \log(q_2) - \lambda [p_1(1 + \tau)q_1 + p_2 q_2 - M]$$

From here, we can take our derivatives and set them equal to zero

$$\frac{\partial \mathcal{L}(q_1, q_2; \lambda)}{\partial q_1} = \frac{\alpha}{q_1} - \lambda p_1(1 + \tau) = 0 \tag{11}$$

$$\frac{\partial \mathcal{L}(q_1, q_2; \lambda)}{\partial q_2} = \frac{\beta}{q_2} - \lambda p_2 = 0 \tag{12}$$

$$\frac{\partial \mathcal{L}(q_1, q_2; \lambda)}{\partial \lambda} = p_1(1 + \tau)q_1 + p_2 q_2 - M = 0 \tag{13}$$

where we now have three equations ((11),(12), and (13)), and two choice variables (q_1 and q_2) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

(a) Find the consumer's demand for q_1 and q_2 in equilibrium (i.e. find q_1^* and q_2^*).

The quantity demanded for rice (q_1^*) and beans (q_2^*), respectively, is

$$(q_1^*(p_1, M, \alpha, \beta, \tau), q_2^*(p_2, M, \alpha, \beta)) = \left(\frac{\alpha}{(\alpha+\beta)} \frac{M}{p_1} \frac{1}{(1+\tau)}, \frac{\beta}{(\alpha+\beta)} \frac{M}{p_2} \right)$$

(b) What are the similarities and differences between the demands you found in Question 2?

Everything is the same except for addition of the tax in the demand for rice. notice how the tax does not change anything about the demand for beans.

(c) Find the consumers utility in terms of their optimal demand (i.e. find $U(q_1^*, q_2^*)$).

$$U(q_1^*, q_2^*) = \alpha \log \left(\frac{\alpha}{(\alpha+\beta)} \frac{M}{p_1} \frac{1}{(1+\tau)} \right) + \beta \log \left(\frac{\beta}{(\alpha+\beta)} \frac{M}{p_2} \right)$$

(d) Find the own price elasticity of demand for rice using $\frac{\partial q_1^*}{\partial p_1} = -\frac{\alpha}{(\alpha+\beta)p_1^2} \frac{M}{(1+\tau)}$.

Using the definition of the own price elasticity of demand s.t.

$$\varepsilon_{q_1^*, p_1} = \frac{\Delta q_1^*}{\Delta p_1} \frac{p_1}{q_1^*} = \frac{\partial q_1^*}{\partial p_1} \frac{p_1}{q_1^*}$$

And substituting in our results

$$\varepsilon_{q_1^*, p_1} = -\frac{\alpha}{(\alpha+\beta)p_1^2} \frac{M}{(1+\tau)} \frac{p_1}{\left(\frac{\alpha M}{(\alpha+\beta)p_1(1+\tau)} \right)}$$

$$\varepsilon_{q_1^*, p_1} = -\frac{\alpha M}{(\alpha+\beta)p_1^2(1+\tau)} \frac{(\alpha+\beta)p_1^2(1+\tau)}{\alpha M} = -1$$

(e) Find the Marginal Rate of Substitution (MRS) using the equations from above.

$$MRS_{1,2} = -\frac{\Delta q_1}{\Delta q_2} = \frac{MU_{q_2}}{MU_{q_1}} = \frac{\frac{\partial U(q_1, q_2)}{\partial q_2}}{\frac{\partial U(q_1, q_2)}{\partial q_1}}$$

Where we can use the first part of the derivatives from the Lagrangian such that

$$MRS_{1,2} = \frac{\frac{\partial U(q_1, q_2)}{\partial q_2}}{\frac{\partial U(q_1, q_2)}{\partial q_1}} = \frac{\frac{\beta}{q_2}}{\frac{\alpha}{q_1}}$$

$$= \frac{\beta q_1}{\alpha q_2}$$

Which means that the rate in which the goods can be substituted is determined by the ratio of the preference parameters and your current bundle you are looking to deviate from. We can also apply a negative sign to this ratio, but the interpretation of the MRS would be the same.

4. **Stone-Geary with a Unit Tax**

Consider the same setting we were operating in in Question 2, but now lets consider the Stone-Geary utility function with the preference parameter condition of $\alpha + \beta = 1$. Notice, similar to Question 1, we can simply the problem as we did before (by replacing $\beta = 1 - \alpha$. Also consider that a regulator implements a unit tax (τ) on each unit of rice consumed, and that $\tau > 1$. Notice how this tax enters into the budget constraint, and directly taxes the quantity consumed of good 1.

We set up the consumer's Utility Maximization Problem (UMP) as

$$\max_{q_1, q_2 \geq 0} \; U(q_1, q_2) = (q_1 - \gamma_1)^{\alpha}(q_2 - \gamma_2)^{(1-\alpha)}$$

subject to the budget constraint of:

$$p_1 q_1 \tau + p_2 q_2 = M$$

CALCULUS PART:
Using constrained optimization techniques from calculus, we can set the problem up with a Lagrange multiplier s.t.

$$\mathcal{L}(q_1, q_2 : \lambda) = (q_1 - \gamma_1)^{\alpha}(q_2 - \gamma_2)^{(1-\alpha)} - \lambda\left[p_1 q_1 \tau + p_2 q_2 - M\right]$$

From here, we can take our derivatives and set them equal to zero

$$\frac{\partial \mathcal{L}(q_1, q_2; \lambda)}{\partial q_1} = \alpha(q_1 - \gamma_1)^{(\alpha-1)}(q_2 - \gamma_2)^{(1-\alpha)} - \lambda p_1 \tau = 0 \tag{14}$$

$$\frac{\partial \mathcal{L}(q_1, q_2; \lambda)}{\partial q_2} = (1-\alpha)(q_1 - \gamma_1)^{\alpha}(q_2 - \gamma_2)^{(-\alpha)} - \lambda p_2 = 0 \tag{15}$$

$$\frac{\partial \mathcal{L}(q_1, q_2; \lambda)}{\partial \lambda} = p_1 q_1 \tau + p_2 q_2 - M = 0 \tag{16}$$

where we now have three equations ((14),(15), and (16)), and two choice variables (q_1 and q_2) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

(a) Find the consumer's demand for q_1 and q_2 in equilibrium (i.e. find q_1^* and q_2^*).

$$(q_1^*(p_1, M, \alpha, \beta, \tau), q_2^*(p_2, M, \alpha, \beta)) = \left(\gamma_1(1-\alpha) + \frac{\alpha(M - \gamma_2 p_2)}{p_1 \tau} \; , \; \gamma_2 \alpha + \frac{(1-\alpha)(M - \gamma_1 p_1 \tau)}{p_2} \right)$$

(b) Find the consumers utility in terms of their optimal demand (i.e. find $U(q_1^*, q_2^*)$). How does the consumer's indirect utility change with an increase in the unit tax?

$$U(q_1^*, q_2^*) = \left(\frac{\alpha(M - \gamma_2 p_2)}{p_1 \tau} - \gamma_1 \alpha\right)^{\alpha} \left(\frac{(1-\alpha)(M - \gamma_1 p_1 \tau)}{p_2} - \gamma_2(1-\alpha)\right)^{(1-\alpha)}$$

As the tax increases, the consumers indirect utility goes down.

5. **Three Factor Cobb-Douglas Utility Function**

Consider that we are looking to model a consumer that has preference over 3 different goods (rice, beans and corn). We are using the same Cobb-Douglas production function, but we are assuming there are three goods with three different preference parameters. In order to maintain the appropriate functional form to model a consumer's utility, we impose the restriction that $0 < \alpha + \beta + \gamma \leq 1$.

We set up the consumer's Utility Maximization Problem (UMP) as

$$\max_{q_1, q_2, q_3 \geq 0} \quad U(q_1, q_2, q_3) = q_1^{\alpha} q_2^{\beta} q_3^{\gamma}$$

subject to the budget constraint of:

$$p_1 q_1 + p_2 q_2 + p_3 q_3 = M$$

CALCULUS PART:
Using constrained optimization techniques from calculus, we can set the problem up with a Lagrange multiplier s.t.

$$\mathcal{L}(q_1, q_2, q_3 : \lambda) = q_1^{\alpha} q_2^{\beta} q_3^{\gamma} - \lambda \left[p_1 q_1 + p_2 q_2 + p_3 q_3 - M \right]$$

From here, we can take our derivatives and set them equal to zero

$$\frac{\partial \mathcal{L}(q_1, q_2, q_3; \lambda)}{\partial q_1} = \alpha q_1^{(\alpha-1)} q_2^{\beta} q_3^{\gamma} - \lambda p_1 = 0 \tag{17}$$

$$\frac{\partial \mathcal{L}(q_1, q_2, q_3; \lambda)}{\partial q_2} = \beta q_1^{\alpha} q_2^{(\beta-1)} q_3^{\gamma} - \lambda p_2 = 0 \tag{18}$$

$$\frac{\partial \mathcal{L}(q_1, q_2, q_3; \lambda)}{\partial q_3} = \gamma q_1^{\alpha} q_2^{\beta} q_2^{(\gamma-1)} - \lambda p_3 = 0 \tag{19}$$

$$\frac{\partial \mathcal{L}(q_1, q_2, q_3; \lambda)}{\partial \lambda} = p_1 q_1 + p_2 q_2 + p_3 q_3 - M = 0 \tag{20}$$

where we now have four equations ((17),(18),(19) and (20)), and three choice variables (q_1, q_2 and q_3) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

(a) Find the consumer's demand for q_1, q_2, and q_3 in equilibrium (i.e. find q_1^*, q_2^* and q_3^*).

$$(q_1^*, q_2^*, q_3^*) = \left(\frac{\alpha}{(\alpha + \beta + \gamma)} \frac{M}{p_1}, \frac{\beta}{(\alpha + \beta + \gamma)} \frac{M}{p_2}, \frac{\gamma}{(\alpha + \beta + \gamma)} \frac{M}{p_3} \right)$$

2a)

Question #2 (Math)

Using equations (8) & (9)

$\Rightarrow$ Solve for λ & set equal

$$\Rightarrow \quad \frac{\alpha q_1^{(\alpha-1)} \cancel{q_2^{\beta}}}{P_1} = \frac{\beta \cancel{q_1^{\alpha}} q_2^{(\beta-1)}}{P_2} \equiv \lambda$$

$$\Rightarrow \quad q_2 = \frac{\beta}{\alpha} \frac{P_1}{P_2} q_1 \qquad \text{where } q_2(q_1) \quad \circledast$$

$\Rightarrow$ Plug into (10)

$$\Rightarrow \quad P_1 q_1 + \cancel{P_2}\left(\frac{\beta}{\alpha} \cdot \frac{P_1}{\cancel{P_2}} q_1\right) = M$$

$$\Rightarrow \quad P_1 q_1 + \frac{\beta}{\alpha} P_1 q_1 = M$$

$$\Rightarrow \quad P_1 q_1 \left(1 + \frac{\beta}{\alpha}\right) = M$$

$$\Rightarrow \quad q_1^* = \frac{\alpha}{(\alpha+\beta)} \frac{M}{P_1}$$

plug into $\circledast$

$$q_2^* = \frac{\beta}{\cancel{\alpha}} \frac{\cancel{P_1}}{P_2}\left(\frac{\cancel{\alpha}}{(\alpha+\beta)} \frac{M}{\cancel{P_1}}\right)$$

$$= \frac{\beta}{(\alpha+\beta)} \frac{M}{P_2}$$

$$\Rightarrow \quad q_1^*, q_2^* = \left(\frac{\alpha}{(\alpha+\beta)} \frac{M}{P_1}, \frac{\beta}{(\alpha+\beta)} \frac{M}{P_2}\right)$$

$\circ//$

3a) using equation (11) & (12) we can solve for λ s.t.

Question #3 (Math)

$$\underset{(11)}{\Rightarrow \frac{\alpha}{q_1} = \lambda P_1(1+\tau)} \quad \& \quad \underset{(12)}{\frac{\beta}{q_2} = \lambda P_2}$$

$$\Rightarrow \frac{\alpha}{q_1 P_1(1+\tau)} = \lambda \quad \& \quad \frac{\beta}{q_2 P_2} = \lambda$$

Combining

$$\Rightarrow \frac{\alpha}{q_1 P_1(1+\tau)} = \frac{\beta}{q_2 P_2} \equiv \lambda$$

$$\Rightarrow q_2 = \frac{\beta}{\alpha}\frac{P_1(1+\tau)}{P_2} q_1 \quad (*)$$

where q_2 is is a function of q_1 s.t. $q_2(q_1)$

plugging into (13)

$$\Rightarrow P_1(1+\tau)q_1 + \cancel{P_2}\left(\frac{\beta}{\alpha}\frac{P_1(1+\tau)}{\cancel{P_2}} q_1\right) = M$$

$$\Rightarrow P_1(1+\tau)q_1 + \frac{\beta}{\alpha}P_1(1+\tau)q_1 = M$$

$$\Rightarrow P_1(1+\tau)q_1\left(1 + \frac{\beta}{\alpha}\right) = M$$

$$\Rightarrow P_1(1+\tau)q_1 = \frac{\alpha}{(\alpha+\beta)} M$$

$$\Rightarrow q_1^* = \frac{\alpha}{(\alpha+\beta)}\frac{M}{P_1(1+\tau)}$$

(q_1^*, q_2^*)

plugging into (*)

$$\Rightarrow q_2^* = \frac{\beta}{\cancel{\alpha}}\frac{\cancel{P_1(1+\tau)}}{P_2}\left(\frac{\cancel{\alpha}}{(\alpha+\beta)}\frac{M}{\cancel{P_1(1+\tau)}}\right) = \frac{\beta}{(\alpha+\beta)}\frac{M}{P_2} \quad //$$

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4a) using equations (14) & (15)

Question #4 (Math)

$\Rightarrow$ (14) = (15)

$$\frac{\alpha(q_1-\gamma_1)^{\alpha-1}(q_2-\gamma_2)^{1-\alpha}}{p_1\tau} = \frac{(1-\alpha)\cancel{(q_1-\gamma_1)^{\alpha}}\,\cancel{(q_2-\gamma_2)^{-\alpha}}}{p_2} \equiv \lambda$$

(both equal to λ)

$\Rightarrow$ solving for $q_2(q_1)$

$$(q_2-\gamma_2) = \frac{(1-\alpha)}{\alpha}\frac{p_1\tau}{p_2}(q_1-\gamma_1)$$

$$\Rightarrow \quad q_2 = \gamma_2 + \frac{(1-\alpha)}{\alpha}\frac{p_1\tau}{p_2}(q_1-\gamma_1) \qquad \circledast$$

plugging into BC (16), & solve for q_1^*

$$\Rightarrow \quad p_1 q_1\tau + p_2\left(\gamma_2 + \frac{(1-\alpha)}{\alpha}\frac{p_1}{p_2}\tau(q_1-\gamma_1)\right) = M$$

$$p_1 q_1\tau + p_1\tau(q_1-\gamma_1)\frac{(1-\alpha)}{\alpha} + p_2\gamma_2 = M$$

$$p_1 q_1\tau + p_1\tau\frac{(1-\alpha)}{\alpha} - p_1\tau\gamma_1\frac{(1-\alpha)}{\alpha} + p_2\gamma_2 = M$$

$$p_1 q_1\tau\left(1+\frac{(1-\alpha)}{\alpha}\right) - p_1\tau\gamma_1\frac{(1-\alpha)}{\alpha} + p_2\gamma_2 = M$$

$$p_1 q_1\tau\left(\frac{1}{\alpha}\right) - p_1\tau\gamma_1\frac{(1-\alpha)}{\alpha} + p_2\gamma_2 = M$$

$$\Rightarrow \quad \frac{p_1 q_1\tau}{\alpha} = p_1\tau\gamma_1\frac{(1-\alpha)}{\alpha} - p_2\gamma_2 + M$$

$$\Rightarrow \quad q_1^* = \gamma_1(1-\alpha) + \frac{\alpha(M-p_2\gamma_2)}{p_1\tau}$$

plug back into $\circledast$ to get q_2^*

$\Rightarrow$

$\Rightarrow$ from $\oplus$

$$q_2 = \gamma_2 + \frac{(1-\alpha)}{\alpha}\,\frac{P_1\tau}{P_2}\left(\gamma_1(1-\alpha) + \frac{\alpha(M-P_2\gamma_2)}{P_1\tau} - \gamma_1\right)$$

$$q_2 = \gamma_2 + \frac{(1-\alpha)}{\alpha}\,\frac{P_1\tau}{P_2}\left(\frac{\alpha(M-P_2\gamma_2)}{P_1\tau} - \alpha\gamma_1\right)$$

$$q_2 = \gamma_2 + \frac{(1-\alpha)(M-P_2\gamma_2)}{P_2} - \frac{\cancel{\alpha}\,\gamma_1(1-\alpha)P_1\tau}{\cancel{\alpha}\,P_2}$$

$$\Rightarrow \quad q_2 = \gamma_2 + \frac{(1-\alpha)(M-P_2\gamma_2-\gamma_1P_1\tau)}{P_2}$$

$$q_2 = \gamma_2 - (1-\alpha)\gamma_2 + \frac{(1-\alpha)(M-\gamma_1P_1\tau)}{P_2}$$

$$\Rightarrow \quad q_2^* = \alpha\gamma_2 + \frac{(1-\alpha)(M-\gamma_1P_1\tau)}{P_2} \quad .//$$

5a) using (17) & (18) Question #5 (Math)

$$\Rightarrow \quad \frac{\alpha q_1^{(\alpha-1)} \cancel{q_2^{\beta}} \cancel{q_3^{\gamma}}}{p_1} = \frac{\beta \cancel{q_1^{\alpha}} q_2^{\beta-1} \cancel{q_3^{\gamma}}}{p_2} \equiv \lambda$$

$$\Rightarrow \quad q_2 = \frac{\beta}{\alpha}\frac{p_1}{p_2} q_1 \qquad (*1)$$

& using (17) & (19)

$$\frac{\alpha q_1^{\alpha-1} \cancel{q_2^{\beta}} \cancel{q_3^{\gamma}}}{p_1} = \frac{\gamma \cancel{q_1^{\alpha}} \cancel{q_2^{\beta}} q_3^{\gamma-1}}{p_2}$$

$$\Rightarrow \quad q_3 = \frac{\gamma}{\alpha}\frac{p_1}{p_3} q_1 \qquad (*2)$$

plugging into BC (20)

$$\Rightarrow \quad p_1 q_1 + \cancel{p_2}\left(\frac{\beta}{\alpha}\frac{p_1}{\cancel{p_2}} q_1\right) + \cancel{p_3}\left(\frac{\gamma}{\alpha}\frac{p_1}{\cancel{p_3}} q_1\right) = M$$

$$\Rightarrow \quad p_1 q_1 \left(1 + \frac{\beta}{\alpha} + \frac{\gamma}{\alpha}\right) = M$$

$$\Rightarrow \quad q_1^* = \frac{\alpha}{(\alpha+\beta+\gamma)}\frac{M}{p_1}$$

plugging into (*1)

$$q_2 = \frac{\beta}{\cancel{\alpha}}\frac{\cancel{p_1}}{p_2}\left(\frac{\cancel{\alpha}}{(\alpha+\beta+\gamma)}\frac{M}{\cancel{p_1}}\right) \Rightarrow q_2^* = \frac{\beta}{(\alpha+\beta+\gamma)}\frac{M}{p_2}$$

& plugging q_1^* into (*2)

$$q_3 = \frac{\gamma}{\cancel{\alpha}}\frac{\cancel{p_1}}{p_2}\left(\frac{\cancel{\alpha}}{(\alpha+\beta+\gamma)}\frac{M}{\cancel{p_1}}\right) \Rightarrow q_3^* = \frac{\gamma}{(\alpha+\beta+\gamma)}\frac{M}{p_3}$$

//