EconS 305: Intermediate Microeconomics w/o Calculus
Homework 2:
Firm Behavior and Decision Making

Due: Friday, May 29th, 2020 at 5:00pm via Blackboard

- Please submit all homework solutions in the order the questions are presented and as **one .PDF**.

- Please **show all calculations** as these exercises are meant to refine your quantitative tool set. If I can not follow your calculations or it seems as you just "copy and pasted" answers from the internet, I will be deducting half the points from that solution.

1. **Cobb-Douglas Production Function with $\alpha + \beta = 1$ and a Fixed Cost**

Consider the following profit maximization problem for the firm where the firm needs to decide the optimal amount of capital (K) and labor (L) that will maximize their profits. We model the firm's production output, using these inputs, with the classic Cobb-Douglas Production function of $Q = Af(K, L) = AK^\alpha L^\beta$. Note that A represents the level of total factor productivity where this can be interpreted differently depending on if A is greater than or less than $(\leq \; or \; \geq)$ 1. For more information about this production function please navigate to to this website (http://www2.hawaii.edu/~fuleky/anatomy/anatomy.html). For simplicity, we assume that $\alpha + \beta = 1$ and $\alpha > 0$, which means that the firm does not specialize in just one good and that the production function yields constant returns to scale. Further note that to get total revenues $(TR(P, Q))$ of the firm, we multiply output (Q) by the price (P) they are receiving for each unit of output. Notice that this price is not a choice variable for the firm since they are being considered as price takers.

Next, we introduce the firm's marginal costs where p_K is the price the firm has to pay to operate/maintain their capital, and p_L is the price (or wage rate) the firm needs to pay to use one unit of labor in production. We also introduce a fixed cost (F) in which the firm has to pay in order to enter the market and compete. We denote this as an "entry fee". Note that the total cost of the firm is $TC(K, L) = p_K K + p_L L + F$ where, in the field of economics, p_K and p_L are referred to as the firm's marginal costs of capital and labor, respectively.

We represent the firm's Profit Maximization Problem (PMP) as Maximizing Total Revenues

$$\max_{K,L\geq 0} \ TR(K,L) = PAf(K,L)$$

$$\max_{K,L\geq 0} \ TR(K,L) = PAK^{\alpha}L^{(1-\alpha)}$$

subject to the firm's cost constraint of:

$$p_K K + p_L L + F = TC$$

CALCULUS PART:
Using constrained optimization techniques from calculus, we can set the problem up with a Lagrange multiplier s.t.

$$\mathcal{L}(K,L) = PAK^{\alpha}L^{1-\alpha} - \lambda\left[p_K K + p_L L + F - TC\right]$$

From here, we can take our derivatives and set them equal to zero

$$\frac{\partial \mathcal{L}(K,L)}{\partial K} \quad = \quad \alpha PAK^{(\alpha-1)}L^{1-\alpha} - \lambda p_K = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}(K,L)}{\partial L} \quad = \quad (1-\alpha)PAK^{\alpha}L^{(-\alpha)} - \lambda p_L = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}(K,q_2)}{\partial \lambda} \quad = \quad p_K K + p_L L + F - TC = 0 \tag{3}$$

where we now have three equations ((1),(2), and (3)), and two choice variables (K and L) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

(a) Find the firm's optimal allocation of inputs (K and L) to maximize its profits (i.e. find K^* and L^*). In economics, we also refer to this allocation as the firm's factor demands.

(b) What happens when the fixed costs for entry (F) are greater than the firm's total costs?

(c) Find the firm's profits in equilibrium and simplify the equation. The firm's profits in equilibrium can be represented by:

$$\pi^*(K^*,L^*) = PA(K^*)^{\alpha}(L^*)^{(1-\alpha)} - p_K K^* - p_L L^* - F$$

(d) Assuming that the firm has a total cost budget that cannot exceed $\overline{TC}$, find the condition on the entry fee (F) in which the firm will enter the market. To do this, set $0 < \pi^*$ and then solve for the entry fee (i.e. $F <$ "the rest of the variables")

2. **Cobb-Douglas with General Preferences without using the Lagrangian**

Consider the same setting we were operating in in Question 1, but now lets consider that $0 < \alpha + \beta \leq 1$. Notice that we cannot simply the problem as we did before (by replacing $\beta = 1 - \alpha$). Also notice that we are not including an entry fee for the firm.

We set up the firm's Profit Maximization Problem (PMP) as

$$\max_{K,L \geq 0} \pi(K,L) = PAf(K,L) - TC(K,L)$$

$$\max_{K,L \geq 0} \pi(K,L) = PAK^{\alpha}L^{\beta} - (p_K K + p_L L)$$

CALCULUS PART:

Notice that similar to Question 1, the firm's problem is a constrained optimization problem, but we are not using the Lagrangian method to solve it. This is because, as in a lot of cases, we do not need to consider a firm budget constraint because they do not have a total cost limit. If it is feasible for them to produce, they will as long as it is profitable.

$$\max_{K,L \geq 0} \pi(K,L) = PAK^{\alpha}L^{\beta} - p_K K - p_L L$$

From here, we can take our derivatives and set them equal to zero

$$\frac{\partial \pi(K,L)}{\partial K} = \alpha PAK^{(\alpha-1)}L^{\beta} - p_K = 0 \tag{4}$$

$$\frac{\partial \pi(K,L)}{\partial L} = \beta PAK^{\alpha}L^{(\beta-1)} - p_L = 0 \tag{5}$$

where we now have two equations ((4) and (5)), and two choice variables (K and L) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

(a) Find the firm's optimal allocation of inputs (its factor demands) to maximize its profit in equilibrium (i.e. find K^* and L^*).

(b) What happens when we invoke the restriction that $\alpha + \beta = 1$?

(c) What is the Marginal Rate of Technical Substitution ($MRTS_{K,L}$)?

3. **A Simple Cobb-Douglas, a Numerical Example**

Consider a similar setting to above, but we are going to simplify the model for an ease of calculation. We set $\alpha = \beta = \frac{1}{3}$, $A = 1$, and $F = 0$.

We set up the firm's Profit Maximization Problem (PMP) as

$$\max_{K,L \geq 0} \pi(K,L) = Pf(K,L) - TC(K,L)$$

$$\max_{K,L \geq 0} \pi(K,L) = PK^{\frac{1}{3}}L^{\frac{1}{3}} - (p_K K + p_L L)$$

CALCULUS PART:

$$\max_{K,L \geq 0} \pi(K,L) = PK^{\frac{1}{3}}L^{\frac{1}{3}} - p_K K - p_L L$$

From here, we can take our derivatives and set them equal to zero

$$\frac{\partial \pi(K,L)}{\partial K} = \frac{1}{3}PK^{(-\frac{2}{3})}L^{\frac{1}{3}} - p_K = 0 \tag{6}$$

$$\frac{\partial \pi(K,L)}{\partial L} = \frac{1}{3}PK^{\frac{1}{3}}L^{(-\frac{2}{3})} - p_L = 0 \tag{7}$$

where we now have two equations ((6) and (7)), and two choice variables (K and L) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

(a) Find the firm's optimal allocation of inputs (its factor demands) to maximize its profit in equilibrium (i.e. find K^* and L^*).

(b) What happens to each factor demand with an increase in it's own price (p_i) and the other factor's price (p_j)? Which one has a larger effect? What happens to output as the market price the firm is receiving (P) increase?

(c) Please find the optimal supply (Q^*). Note that we can get this by plugging K^* and L^* back into the production function.

(d) Please find the optimal profits for the firm (π^*) and simplify. Note that we can get this by plugging K^*, L^*, and Q^* back into the profit function ($\pi^*(K^*, L^*, Q^*(K^*, L^*))$). To be clear, plug the values you found in part (a) into here

$$\pi^*(K^*, L^*, Q^*(K^*, L^*)) = PQ^* - p_K K^* - p_L L^*$$

4. **Cost Minimization Problem using a Lagrangian**

Consider the same setting we were operating in Question 1, but now lets change our analysis so that the firm is minimizing costs subject to the same Cobb-Douglas production function in Question 1. Intuitively, we can think of $\overline{Q}$ as the total maximum amount the market is willing to buy of the firm's good. This is the quantity demanded (Q_D) in which the firm will also supply (Q_S).

We consider the same conditions on the production function parameters where $\alpha + \beta = 1$ and $\alpha > 0$. Notice, similar to Question 1, we can simply the problem as we did before by replacing $\beta = 1 - \alpha$. Just as in Question 1, F is the entry fee to enter the market and that A is the level of total factor productivity.

We now set up the firm's Profit Maximization Problem (PMP) as a Cost Minimizing Problem (CMP).

$$\min_{K,L \geq 0} \quad C(K, L) = p_K K + p_L L + F$$

subject to the budget constraint of:

$$AK^\alpha L^{(1-\alpha)} = \overline{Q}$$

CALCULUS PART:
Using constrained optimization techniques from calculus, we can set the problem up with a Lagrange multiplier s.t.

$$\mathcal{L}(K, L : \lambda) = p_K K + p_L L + F - \lambda[AK^\alpha L^{1-\alpha} - \overline{Q}]$$

From here, we can take our derivatives and set them equal to zero

$$\frac{\partial \mathcal{L}(K, L; \lambda)}{\partial K} \quad = \quad p_K - \lambda[\alpha A K^{\alpha-1} L^{1-\alpha}] = 0 \tag{8}$$

$$\frac{\partial \mathcal{L}(K, L; \lambda)}{\partial q_2} \quad = \quad p_L - \lambda[(1-\alpha)AK^\alpha L^{(-\alpha)}] = 0 \tag{9}$$

$$\frac{\partial \mathcal{L}(q_1, q_2; \lambda)}{\partial \lambda} \quad = \quad AK^\alpha L^{1-\alpha} - \overline{Q} = 0 \tag{10}$$

where we now have three equations ((8),(9), and (10)), and two choice variables (K and L) to solve for.

CALCULUS PART FINISHED. YOUR CALCULATIONS START HERE.

(a) Find the firm's optimal allocation of inputs (its factor demands) to minimize its costs in equilibrium (i.e. find K^* and L^*).

(b) What happens with an increases in $\overline{Q}$? Why does this happen?

(c) What is the $MRTS_{KL}$ equal to in this problem? Does it make sense?

5. **A Firm Cost Analysis**

   Consider a firm with a total cost curve of $TC = 20Q^2 + 8Q + 90$ and marginal cost of $MC = 40Q + 8$. Without knowing the firm's production function, you are asked to conduct a cost analysis that will answer the following questions.

(a) What is the firm's fixed cost, variable cost, average total cost, and average variable cost?

(b) Find the output level (Q) that minimizes average total cost.

(c) Find the output level at which average variable cost is minimized.