

Horizontal Asymptotes

We discuss the circumstances that generate horizontal asymptotes and what they mean.

You can watch a lecture video on this here!

YouTube link: <https://www.youtube.com/watch?v=Y0D14W52N8Q>

Another key aspect of rational functions is their end term behavior. Often we are considering a ratio of functions to determine information about them for large (positive or negative) values of x , which is precisely what end term behavior is. Again this is done analytically using limits, but we can get a geometric understanding of what end term behavior means for rational functions, and how to determine what it is likely to be in some cases.

End term behavior for rational functions is essentially a measurement of their relative *growth rates* or rate at which the numerator and denominator functions increase in magnitude.

Definition 1 (Growth Rate). *The rate at which a mathematical function or expression increases as x gets larger. This is formalized with limits and derivatives in calculus.*

Generally speaking, whichever of the numerator or denominator function has the fastest growth rate “wins” and overpowers the other function in the ratio as x gets larger and larger.

If the numerator grows faster than the denominator, then the magnitude of the ratio goes to infinity. This means that the rational function $r(x)$ gets arbitrarily large in either the positive or negative direction for large enough x in either the positive or negative direction. If the denominator grows faster than the numerator however, then the rational function goes to zero as x gets larger; meaning that we have a horizontal asymptote at $y = 0$. Remember however that there is no reason that a function can’t cross a horizontal asymptote; in fact there are plenty of examples where a function will cross its own horizontal asymptote *infinitely many times*. Finally, in the very special case where the numerator and denominator functions both grow at the same relative speed, then there is a horizontal asymptote at the ratio of those growth rates. This is easiest to see in the very special case where the numerator and denominator functions are both polynomials.

Example 1. Find any horizontal asymptotes of the following function:

$$r(x) = \frac{3x^3 - 2x^2 + 1}{x^2 - 12}$$

To determine the horizontal asymptotes of $r(x)$ we need to consider which of the numerator or denominator functions “grows faster”, i.e. which one increases faster than the other as x keeps getting larger (positive or negative). Remember that we only care about the magnitude, not the sign for this part.

But, for polynomials we have already discussed how the largest power term eventually dominates the other terms; that is to say that the largest power has a larger growth rate than any term with a smaller power, regardless of coefficient. Since the numerator function is degree 3 and the denominator is degree 2, this means that the numerator function has higher growth rate and thus

Multiple Choice:

- (a) There is a horizontal asymptote at $y = 0$.
- (b) There are no horizontal asymptotes. ✓
- (c) There is a horizontal asymptote at infinity.
- (d) There is a horizontal asymptote at $y = 3$.

Example 2. Find any horizontal asymptotes of the following function:

$$r(x) = \frac{3x^3 - 2x^2 + 1}{x^4 - 2x^3 + x - 12}$$

Again, for polynomials largest power term eventually dominates the other terms; that is to say that the largest power has a larger growth rate than any term with a smaller power, regardless of coefficient. Since the numerator function is degree 3 and the denominator is degree 4, this means that the denominator function has higher growth rate and thus

Multiple Choice:

- (a) There is a horizontal asymptote at $y = 0$. ✓
- (b) There are no horizontal asymptotes.
- (c) There is a horizontal asymptote at infinity.
- (d) There is a horizontal asymptote at $y = -\frac{3}{2}$.

Example 3. Find any horizontal asymptotes of the following function:

$$r(x) = \frac{3x^3 - 2x^2 + 1}{-2x^3 + x - 12}$$

Again, for polynomials largest power term eventually dominates the other terms; that is to say that the largest power has a larger growth rate than any term with a smaller power, regardless of coefficient. Since the numerator function is degree 3 and the denominator is degree 3, this means that the denominator and numerator function have relatively similar growth rates, so...

Multiple Choice:

- (a) There is a horizontal asymptote at $y = 0$.
- (b) There are no horizontal asymptotes.
- (c) There is a horizontal asymptote at infinity.
- (d) There is a horizontal asymptote at $y = -\frac{3}{2}$. ✓