Polynomial Long Division

In this section we explore how to factor a polynomial out of another polynomial using polynomial long division

Many of the sections remaining in this topic are methods to find *roots* or *zeros* of a polynomial, but not how to *factor* the polynomial. One may wonder then how it is that they are still under the general heading of factoring if they don't actually factor the polynomial. This is where the techniques for 'Polynomial Division' come in.

YouTube link: https://www.youtube.com/watch?v=f99CoPU8ROM

Polynomial Long Division

The most general, intuitive, and arguably useful, type of polynomial division is *polynomial long division*. The key idea to learning and remembering this technique is that it is *exactly* the same process as you *first* learned for regular long division with numbers, albeit with some more complex notation and a few more potential pitfalls. It has probably been a little while since you last used long division in its most formal way though, so we will give a quick rehash here. This may seem silly, but it is *extremely* helpful to review the formal method of long division with numbers, so you can compare it to polynomial long division, so don't skip this part thinking that you already know how to do long division, trust me on this! (**Note:** You can click the image to make it larger!)

Formal Long Division Revisited

You likely haven't had to really do long division in quite some time. Even if you have written out the dreaded long division symbols to compute something, you probably did a mental shorthand version that, although perfectly accurate and fine with numbers, is a bit misleading for our purposes of polynomial long division.

We will begin by observing a few 'obvious' facts in the long division example to the right that will be incredibly important for the polynomial version. Notice that the number being divided, 6504 represents (in words) 6 groups of one thousand, 5 groups of one hundred, *no* groups of ten, and 4 groups of one. But, we don't omit the 'groups of ten' (because that's not how our number system works) instead we write a 0 to represent that we have no groups of that size present. This is akin to writing the polynomial $p(x) = x^2 - 4$ as $p(x) = x^2 + 0x - 4$. Unlike our number system we typically just omit the 'missing' parts of a polynomial, but this is actually a *big mistake* when trying to do polynomial long division, it's *very important* to make sure to include *every* term of the polynomial (up to the leading term), even if that means adding in a term with a zero coefficient.

$$
\begin{array}{r}
232 \\
28\,\overline{)6504} \\
\underline{56} \\
90 \\
\underline{84} \\
64 \\
\underline{56} \\
8
\end{array}
$$

Next consider the first step; putting 28 into 65. In practice, one typically accomplishes this by figuring out what multiple of 28 is closest to 65 without going over, and then writes that under the 65 and puts the multiplier at the top... ie the thought process is something along the lines of '56 is the closest under 65', you write the 56 under the 65, and then think 'that's 2 times 28' so you write the 2 on the line above. In the formal sense however this is actually backwards. What you are *suppose* to do is divide 65 by 28 to get '2 with a remainder of 7', write the 2 on the top *then* multiply 28 by 2 to get 56 and write it underneath the 65.

In practice there is no difference, and it is often easier doing it the 'wrong way'. However, in polynomial division it is *absolutely necessary* to do it the 'right way'. That is to say, to divide the number, write the 'whole' piece on the top of the line, and then multiply to get the value you will need to subtract and place it below the dividend.

This may seem silly, but polynomial long division works *exactly* like the ***formal*** version of long division. Unfortunately, the *formal* version of long division isn't really the version that people tend to use, which is why the review is helpful; more as a way to make sure we are thinking about the 'correct' version of long division and not using mental shortcuts. People often try to do polynomial long division using the mental short-cuts they've developed for long division of numbers. Unfortunately that almost always goes horribly horribly wrong with polynomials.

Not to beat a dead horse, but polynomial division follows the exact same process as the formal long division above. Thus we will go through an example and point out the parallels; starting with writing out the divisor (thing we are dividing by) and dividend (the thing we are dividing) in the same spots and in the same way, and then go through the same steps (conceptually speaking).

(**Note:** You can click the image to make it larger!)

Learning outcomes:

Example 5: Use polynomial long division to divide $x^3 + x^2 - x - 1$ by $x - 1$.

Solution and/or Discussion:
We will write the same kind of setup as we did before, so since the divisor is $x - 1$ and the dividend is $x^3 + x^2 - x - 1$ we will write:

$$x - 1\overline{)x^3 + x^2 - x - 1}$$

Next, as before, we will try to divide the first term by the divisor to see what we want to put on the line above. The key here though is that *we only care about the leading term of the divisor and the part of the dividend we are dividing into.* Thus we only look at the x in $x - 1$ and the x^3 in the $x^3 + x^2$. Thus dividing x^3 by x we get a 'multiplier' of x^2 (this is analogous to the 2 that we had in the long division example when we divided 65 by 28). Thus we write the x^2 at the top, which gives us;

$$\begin{array}{r|rrrr} & x^2 & & & \\ \hline x-1) & x^3 & +x^2 & -x & -1 \end{array}$$

Next we *multiply the divisor by the multiplier factor (x^2) to find what we will subtract by.* This is analogous to the previous example of multiplying the 28 by the 2 to get 56, then subtracting the 56 from the 65. Doing this we now have the following (notice the distribution of the negative sign):

$$\begin{array}{r|rrrr} & x^2 & & & \\ \hline x-1) & x^3 & +x^2 & -x & -1 \\ & -x^3 & +x^2 & & \end{array}$$

Finally; we compute the subtraction and then drop the next term down to get the next piece of the dividend that we want to divide into. Thus we have completed the first full iteration of polynomial long division, getting:

$$\begin{array}{r|rrrr} & x^2 & & & \\ \hline x-1) & x^3 & +x^2 & -x & -1 \\ & -x^3 & +x^2 & & \\ \hline & & 2x^2 & -x & \end{array}$$

Thus, if we then continue this until we reach the last possible term we have the following:

$$\begin{array}{r|rrrr} & x^2 & +2x & +1 & \\ \hline x-1) & x^3 & +x^2 & -x & -1 \\ & -x^3 & +x^2 & & \\ \hline & & 2x^2 & -x & \\ & & -2x^2 & +2x & \\ \hline & & & x & -1 \\ & & & -x & +1 \\ \hline & & & & 0 \end{array}$$

This then says that: $\frac{x^3 + x^2 - x - 1}{x - 1} = x^2 + 2x + 1$ with *no* remainder. But another way to write this (after multiplying both sides of the previous equation by $(x - 1)$) is:

$$x^3 + x^2 - x - 1 = (x - 1)(x^2 + 2x + 1)$$

Which is the factored form we were looking for.

Non-zero Remainders

Let's look at the example above but divide by $x - 2$ instead of $x - 1$. Then we get: (**Note:** You can click the image to make it larger!)

$$\begin{array}{r|rrrr} & x^2 & +3x & +5 & \\ \hline x-2) & x^3 & +x^2 & -x & -1 \\ & -x^3 & +2x^2 & & \\ \hline & & 3x^2 & -x & \\ & & -3x^2 & +6x & \\ \hline & & & 5x & -1 \\ & & & -5x & +10 \\ \hline & & & & 9 \end{array}$$

Just like with regular long division, if you have a remainder you have some choices on how to express that remainder. Let's say you divide 45 by 7; you'd get 6 with a remainder of 3. You could proceed to find a decimal version, but sevenths tend to have terrible decimal expansion. Instead we could write it as $\frac{45}{7} = 6 + \frac{3}{7}$, or we could leave it as $\frac{45}{7}$. This is exactly what we do with polynomials. We could leave it in its original fraction form, or we could write:

$$\frac{x^3 + x^2 - x - 1}{(x - 2)} = x^2 + 3x + 5 + \frac{9}{x - 2}$$

There is one other form that is often helpful, as we often want to replace the original polynomial with a "factored form", rather than dividing out the divisor piece. So we can instead write:

$$x^3 + x^2 - x - 1 = (x - 2)\left(x^2 + 3x + 5 + \frac{9}{x - 2}\right)$$

Which form we want depends on what we are doing. In either case however we can see one of the 'factors' is no longer a polynomial. Once that happens a lot of our tools vanish which is why we tend to only record the result of our division if it has a zero remainder; not because the long division failed (it worked just fine), but rather because the result is annoying (like dealing with the "sevenths" in our number example, rather than getting a whole number.)