Inverse Function - Analytic View

This section introduces the analytic viewpoint of invertability, as well as one-to-one functions.

Inverse Function - The Analytic View

The geometric view is insightful to understanding what the inverse *means*, but it doesn't really help us explicitly determine what the inverse of a function *is*. To do this, we use the analytic view.

Before we give a technique for explicitly obtaining an inverse, it is *very important* to know how to check if a function *actually is an inverse* analytically. This is because the process we have for obtaining an inverse can (and does) often fail, but it fails in a way that may not be clear without trying to verify if your result is a legitimate inverse. This means whenever you solve for an inverse of a function, you should *always* check to ensure it is an inverse according to the following definition.

Definition 1 (Inverse Function)**.** *A function $g(y)$ is an inverse to another function $f(x)$ if the following two compositions are true:*

$$f(g(y)) = y \text{ and } g(f(x)) = x$$

In other words, to show that g is the inverse function of f (ie $g(y) = f^{-1}(y)$), we must show that $f(g(y)) = y$ and $g(f(x)) = x$.

Example 1. *Consider the function $f(x) = x^3$ and $g(y) = \sqrt[3]{y}$. We wish to show that g is the inverse function of f.*

To do this we must show first that $f(g(y)) = y$;

$$f(g(y)) = (g(y))^3 = (\sqrt[3]{y})^3 = y\checkmark$$

Next we must show that $g(f(x)) = x$;

$$g(f(x)) = \sqrt[3]{f(x)} = \sqrt[3]{\boxed{x^3}} = x\checkmark$$

Thus, since we have shown that $f(g(y)) = y$ and $g(f(x)) = x$ we can conclude that $g(y) = f^{-1}(y)$, ie that g is the inverse function of f.

How to solve for inverse analytically

Remember from our geometric view, that the inverse function is the function that reverses the roles of x and y. In essence, the inverse function is switching the roles for the input and output variable. So to find a function that does this, we 'merely'[1] switch the independent and dependent variables, then solve for the independent variable again. Consider our previous example, but this time we will determine the inverse function.

Example 2. *Find the inverse function for the function $f(x) = x^3$.*

To find the inverse function we will first switch the input and output variable. Since there is no explicit output variable, we will assign one by setting $f(x) = y$, thus we switch the location of the x and y variables to go from $y = x^3$ to $\boxed{x} = \boxed{y}^3$.

Next we want to solve our new equality for y. To do this we need to cube root both sides, which gives:

$$\sqrt[3]{x} = \sqrt[3]{y^3} = y$$

So our proposed inverse function is $y = \sqrt[3]{x}$. Keep in mind this is only a proposed inverse until we prove it is an inverse by showing that $f(g(y)) = y$ and $g(f(x)) = x$ (which we did in the previous example). Once we have shown that it is indeed the inverse we can conclude that $f^{-1}(y) = \sqrt[3]{y}$ and we're done.

Problem 1 *In order to analytically solve for an inverse function you can...*

Learning outcomes:

[1] There's that 'merely' again, and yes this is the hard part

Multiple Choice:

(a) *Switch the x and y variable rolls, and solve for the new independent variable.* ✓

(b) *Change the x variable to another letter.*

(c) *Change the y variable to another letter.*

(d) *Use the horizontal line test to determine if an inverse exists.*