

Fundamental Theorem of Algebra

This section covers one of the most important results in the last couple centuries in algebra; the so-called “Fundamental Theorem of Algebra.”

We will spend considerable time learning how to manipulate polynomials, typically in an effort to ascertain certain properties or values. It helps then, to know that our goal of factoring/manipulating the polynomials is actually possible. This is what the Fundamental Theorem of Algebra (and its corollaries) gives us, which we explore below.

YouTube link: https://www.youtube.com/watch?v=Qc-_Tp0ExWs

Fundamental Theorem of Algebra, aka Gauss makes everyone look bad.

In grade school, many of you likely learned some variant of a theorem that says any polynomial can be factored to be a product of smaller polynomials; specifically polynomials of degree one or two (depending on your math book/teacher they may have specified that they are polynomials of degree one, or so-called ‘linear’ polynomials). This is a paraphrased version of a *very* important and surprisingly recent theorem in mathematics called *The fundamental theorem of algebra* which is recorded formally here (don’t worry, explanation will follow).

Theorem 1 (Fundamental Theorem of Algebra). *Any polynomial $p(x)$ may be factored into a product of irreducible factors, where those factors are, at most, degree one in the complex numbers. That is to say for any $p(x)$ of the following form;*

$$p(x) = c_n x^n + c_{n-1} x^{n-1} + c_{n-2} x^{n-2} + \cdots + c_2 x^2 + c_1 x^1 + c_0 x^0$$

where each of $c_1, c_2, \dots, c_n \in \mathbb{R}$ (that is to say, each coefficient is a real number),¹ then we can factor $p(x)$ and rewrite it into the following form;

$$p(x) = (a_1 x - b_1)(a_2 x - b_2) \cdots (a_{n-1} x - b_{n-1})(a_n x - b_n)$$

Where a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n may be complex numbers.

In essence, what this theorem is saying, is that each polynomial can be factored down to the product of smaller polynomials, and that those polynomials can

Learning outcomes:

¹It’s actually not necessary for the coefficients to be real valued for this, but we will only consider this case.

be made to be at most degree one if we allow complex numbers. This has a corollary,² specifically;

Corollary 1. *Let $p(x)$ be a polynomial of degree n with the form:*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

Then the equation $p(x) = 0$ has at most n real solutions and exactly n complex solutions up to multiplicity³.

Let's see this corollary in action in an example.

Example 1 (Number of solutions of a polynomial). *Let's say we have the following polynomial;*

$$p(x) = x^4 + x^3 - 12x^2 + 13$$

and we want to determine how many x values satisfy the equation $p(x) = 13$. Currently our corollary doesn't quite work because it only applies when polynomials are equal to zero, but we can rewrite our current polynomial and manipulate it into that form. Specifically;

$$p(x) = x^4 + x^3 - 12x^2 + 13$$

$$13 = x^4 + x^3 - 12x^2 + 13$$

$$0 = \boxed{x^4 + x^3 - 12x^2}$$

In the last line above we have now manipulated our polynomial into a form where it equals zero, so our corollary tells us that there are at most 4 real solutions (ie at most 4 real values of x that satisfies this equality) and exactly 4 complex solutions. Remember that in both of these cases the "4" is up to multiplicity.

With a little manipulation we can see explicitly what values of x work;

$$\begin{aligned} 0 &= x^4 + x^3 - 12x^2 \\ &= x^2(x^2 + x - 12) \\ &= x^2(x - 3)(x + 4) \end{aligned}$$

Thus we see we have solutions of: $x = 0$ with multiplicity of 2, $x = \boxed{3}$ (the larger of the two remaining zeros) and $x = \boxed{-4}$ (the smaller of the two remaining zeros) both with multiplicity of 1. In this case, we do indeed have 4 solutions to the equation, but they are not unique as one of them has multiplicity of 2. That is to say, if we list the solutions from lowest (most negative) to largest (most positive) as: $\boxed{-4}, \boxed{0}, \boxed{0}, \boxed{3}$ we have four solutions as the corollary claimed.

²A corollary is like a 'theorem' except it follows immediately from the theorem it is a corollary of. You can think of a corollary to some theorem as being an immediate followup or consequence of the theorem.

³See the terminology subsection at the beginning of "explore polynomials" for a description of what it means to be "up to multiplicity" if you are unfamiliar

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Question 1 How many zeros (up to multiplicity) does the polynomial $p(x) = 3x^2 - 2x^3 + x$ have over the complex numbers?
There are zeros (up to multiplicity).
