Module 5

1) $$\frac{1}{2}y[n-1] - \frac{9}{4}y[n] + y[n+1] = x[n]$$

By applying z-transform:

$$\frac{1}{2}Y(z)z^{-1} - \frac{9}{4}Y(z) + Y(z)z = X(z)$$

$$Y(z)\left(\frac{1}{2}z^{-1} - \frac{9}{4} + z\right) = X(z) \longrightarrow$$

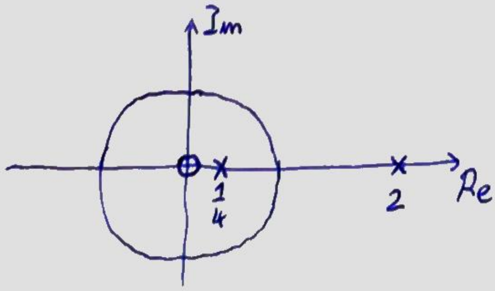
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{\frac{1}{2}z^{-1} - \frac{9}{4} + z} = \frac{z}{\frac{1}{2} - \frac{9}{4}z + z^2} =$$

$$= \left(\frac{z}{\left(z-\frac{1}{4}\right)(z-2)}\right) = \left(\frac{A_1}{z-\frac{1}{4}} + \frac{A_2}{z-2}\right)$$

$$A_1 = \left(z-\frac{1}{4}\right)X(z)\Big|_{z=\frac{1}{4}} = \frac{z}{z-2}\Big|_{z=\frac{1}{4}} = -\frac{4}{3}$$

$$A_2 = (z-2)\,X(z)\Big|_{z=2} = \frac{1}{z-\frac{1}{4}}\Big|_{z=2} = \frac{8}{7}$$

$$H(z) = \frac{-\frac{4}{3}}{z-\frac{1}{4}} + \frac{\frac{8}{7}}{z-2} = \frac{-\frac{1}{7}z^{-1}}{1-\frac{1}{4}z^{-1}} + \frac{\frac{8}{7}z^{-1}}{1-2z^{-1}}$$



if the system is stable

$\rightarrow \frac{1}{4} \leq |z| \leq 2$

$\longrightarrow$

$$h[n] = -\frac{1}{7}\left(\frac{1}{4}\right)^{n-1}u[n-1] - \frac{8}{7}(2)^{n-1}u[-n]$$

2) $$x[n] = \left(\frac{1}{2}\right)^n u[n] + (2)^n u[-n-1]$$

a) By inspection, the z-transform of the input above is

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}} \qquad \begin{matrix} |z| > \frac{1}{2} \\ |z| < 2 \end{matrix} \Rightarrow \frac{1}{2} < |z| < 2$$

$$y[n] = 6\left(\frac{1}{2}\right)^n u[n] - 6\left(\frac{3}{4}\right)^n u[n]$$

By inspection, the z-transform of the output signal is given by

$$Y(z) = \frac{6}{1-\frac{1}{2}z^{-1}} - \frac{6}{1-\frac{3}{4}z^{-1}} \qquad \begin{matrix} |z| > \frac{1}{2} \\ |z| > \frac{3}{4} \end{matrix} \rightarrow |z| > \frac{3}{4}$$

Therefore:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{6 - \frac{9}{2}z^{-1} - 6 + 3z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)} \cdot \frac{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)}{1-2z^{-1}-1+\frac{1}{2}z^{-1}} =$$

$$= \frac{-\frac{3}{2}z^{-1}}{\left(1-\frac{3}{4}z^{-1}\right)} \cdot \frac{\left(1-2z^{-1}\right)}{-\frac{3}{2}z^{-1}} = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}, \quad |z| > \frac{3}{4}$$

$\longrightarrow$

b) $H(z) = \frac{1}{1-\frac{3}{4}z^{-1}} - \frac{2}{1-\frac{3}{4}z^{-1}} \cdot z^{-1}$, $|z| > \frac{3}{4}$

By inspection:

$$h[n] = \left(\frac{3}{4}\right)^n u[n] - 2\left(\frac{3}{4}\right)^{n-1} u[n-1]$$

c) $H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}} \longrightarrow$

$$\rightarrow \left(1-\frac{3}{4}z^{-1}\right)Y(z) = \left(1-2z^{-1}\right)X(z)$$

$$Y(z) - \frac{3}{4}Y(z)z^{-1} = X(z) - 2X(z)z^{-1}$$

$$\Downarrow$$

$$y[n] - \frac{3}{4}y[n-1] = x[n] - 2x[n-1]$$

c) Since $|z| > \frac{3}{4}$, the ROC includes the unit circle

$\rightarrow$ stable system.

Since $h[n]$ does not depend on future values, the system is also causal.

3) $x[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1]$

By inspection, the z-transform of $x[n]$ is

$$X(z) = -\frac{1}{3}\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right) + \frac{4}{3}\left(\frac{1}{1-2z^{-1}}\right) \qquad \begin{matrix} |z| > \frac{1}{2} \\ |z| < 2 \end{matrix} \rightarrow \frac{1}{2} < |z| < 2$$

$$X(z) = \frac{-\frac{1}{3}(1-2z^{-1}) + \frac{4}{3}(1-\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} = \frac{1}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$

$$Y(z) = \frac{1-z^{-2}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$

$Y(z)$ has the same poles as $X(z) \rightarrow \frac{1}{2} < |z| < 2$

$$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-2} \Longrightarrow h[n] = \delta[n] - \delta[n-2]$$

4)

$$H(z) = \frac{1-3z^{-1}}{(1-z^{-1})(1-2z^{-1})} \qquad |z|>2$$

$$H_i(z) = \frac{(1-z^{-1})(1-2z^{-1})}{1-3z^{-1}} = \frac{1-3z^{-1}+2z^{-2}}{1-3z^{-1}}$$

$$\begin{array}{r|l} & -\frac{2}{3}z^{-1}+\frac{7}{9} \\ \hline -3z^{-1}+1 & 2z^{-2}-3z^{-1}+1 \\ & 2z^{-2}-\frac{2}{3}z^{-1} \\ \hline & // \quad -\frac{7}{3}z^{-1}+1 \\ & \quad -\frac{7}{3}z^{-1}+\frac{7}{9} \\ \hline & \quad // \quad \frac{2}{9} \end{array}$$

$$\rightarrow H_i(z) = \frac{7}{9} - \frac{2}{3}z^{-1} + \frac{2}{9}\cdot\frac{1}{1-3z^{-1}}$$

The possible choices for the region of convergence of $H_i(z)$ are $|z|>3$ and $|z|<3$. In both cases, the region of convergence overlaps with $|z|>2$.

If $|z|>3 \longrightarrow h_i[n] = \frac{7}{9}\delta[n] - \frac{2}{3}\delta[n-1] +$

$$+\frac{2}{9}(3)^n u[n]$$

if $|z|<3 \rightarrow h_i[n] = \frac{7}{9}\delta[n] - \frac{2}{3}\delta[n-1] - \frac{2}{9}(3)^n u[-n-1]$

5)
$$H(z) = \frac{21}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})(1-4z^{-1})}$$

Since the system is unstable, the ROC does not include the unit circle.
Possible ROCs are then the following

a) $|z|>2,\ |z|>4 \rightarrow |z|>4$

b) $2<|z|<4$

Let us express $H(z)$ in partial fractions form:

$$H(z) = \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-2z^{-1}} + \frac{A_3}{1-4z^{-1}}$$

$$A_1 = \left(1-\frac{1}{2}z^{-1}\right)H(z)\Big|_{z=\frac{1}{2}} = \frac{21}{(1-2z^{-1})(1-4z^{-1})}\Bigg|_{z=\frac{1}{2}} = -14$$

$$A_2 = (1-2z^{-1})H(z)\Big|_{z=2} = \frac{21}{(1-\frac{1}{2}z^{-1})(1-4z^{-1})}\Bigg|_{z=2} = -28$$

$$A_3 = (1-4z^{-1})H(z)\Big|_{z=4} = \frac{21}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}\Bigg|_{z=4} = 48$$

$$H(z) = -\frac{14}{1-\frac{1}{2}z^{-1}} - \frac{28}{1-2z^{-1}} + \frac{48}{1-4z^{-1}}$$

$\longrightarrow$

if $|z| > 4$

$$h[n] = -14\left(\frac{1}{2}\right)^n u[n] - 28(2)^n u[n] + 48(4)^n u[n]$$

if $2 < |z| < 4$

$$h[n] = -14\left(\frac{1}{2}\right)^n u[n] - 28(2)^n u[n] - 48(4)^n u[-n-1]$$