The z-transform

The z-transform of a sequence $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

where z is a complex variable, that can be generically expressed as

$$z = re^{jw}$$

For $r=1$,

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] \underbrace{e}_{z=e^{jw}}{}^{-jwn}$$

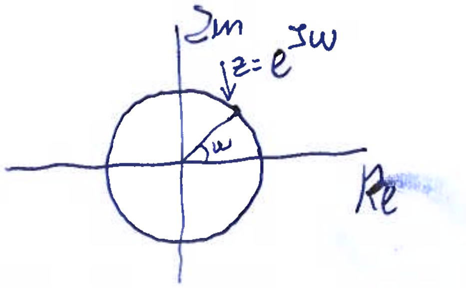
z-transform is the same as Fourier transform

More generally,

$$X(re^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] (re^{jw})^{-n} = \sum_{n=-\infty}^{+\infty} (x[n] r^{-n}) e^{-jwn}$$

This can be interpreted as the Fourier transform of the original sequence $x[n]$ and the exponential sequence r^{-n}.

Since z-transform is a function of a complex variable, it can be described and interpreted by using complex z-plane



The z-tranform evaluated on the unit circle corresponds to the Fourier transform.

The set of values of z for which the z-transform converges, is called the region of convergence (ROC)

all z for which $\rightarrow \sum_{n=-\infty}^{+\infty} |x[n]| \cdot |z|^{-n} < \infty \Rightarrow \sum_{n=-\infty}^{+\infty} |x[n] r^{-n}| < \infty$

z-transform may converge even if Fourier transform diverges

$$\sum_{n=-\infty}^{+\infty} |x[n] r^{-n}| < \infty \quad \text{even if} \quad \sum_{n=-\infty}^{+\infty} |x[n]| \rightarrow \infty$$

Example

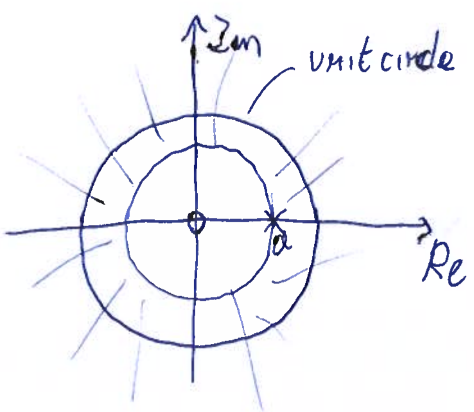
right-sided exponential sequence

$$x[n] = a^n u[n]$$

$$X(z) = \sum_{n=-\infty}^{+\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

This expression has the form $\sum_{n=0}^{\infty} q^n = \begin{cases} \frac{1}{1-q} & |q| < 1 \\ \infty & q \geq 1 \end{cases}$

$$\Rightarrow X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \quad \text{if } |az^{-1}| < 1 \Rightarrow |z| > |a|$$



zero in 0
pole in a

if $|a|<1$, ROC contains the unit circle $\rightarrow$ Fourier transform exists.

if $|a|>1$, Fourier transform does not exist

Left-sided exponential sequence

$$x[n] = -a^n u[-n-1] = \begin{cases} -a^n & n \leq -1 \\ 0 & n > -1 \end{cases}$$

$$X(z) = -\sum_{n=-\infty}^{+\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} =$$

Let us define $q = -n$

$$\Rightarrow n = -\infty \to q = +\infty$$
$$n = -1 \to q = 1$$

$$= -\sum_{q=1}^{\infty} (a^{-q} z^{q}) = -\sum_{q=1}^{\infty} (a^{-1} z)^q$$

Let us consider the following series

$$\sum_{q=0}^{\infty} (a^{-1} z)^q = 1 + \sum_{q=1}^{\infty} (a^{-1} z)^q$$

$$\Rightarrow \sum_{q=1}^{\infty} (a^{-1} z)^q = \sum_{q=0}^{\infty} (a^{-1} z)^q - 1$$

$$\Rightarrow -\sum_{q=1}^{\infty} (a^{-1} z)^q = 1 - \sum_{q=0}^{\infty} (a^{-1} z)^q$$

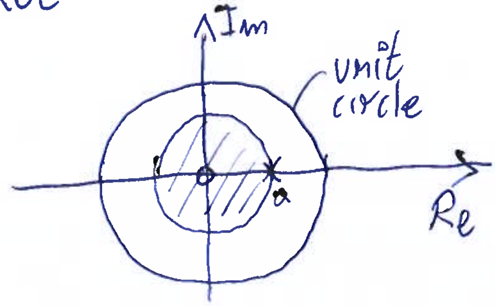
$$\sum_{q=0}^{\infty} r^q = \begin{cases} \frac{1}{1-r} & \text{if } |r|<1 \\ \infty & \text{if } |r| \geq 1 \end{cases}$$

$$\cancel{X(q) = \varphi}$$

$$X(z) = 1 - \sum_{q=0}^{\infty} (a^{-1} z)^q = 1 - \frac{1}{1 - a^{-1} z} = \frac{1}{1 - a z^{-1}} =$$

$$= \frac{z}{z-a} \quad \text{if } |a^{-1} z| < 1 \to \text{if } |z| < |a|$$

ROC



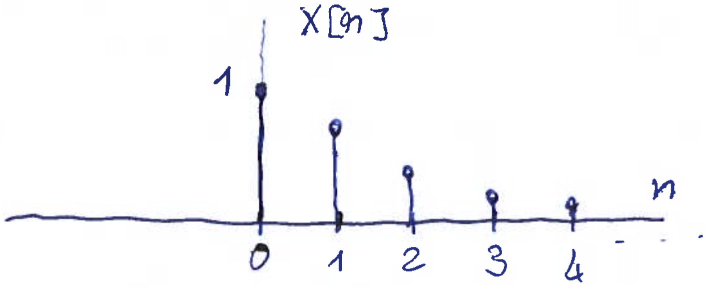
if $a < 1$, Fourier transform does not exist

if $a > 1$, Fourier transform exists.

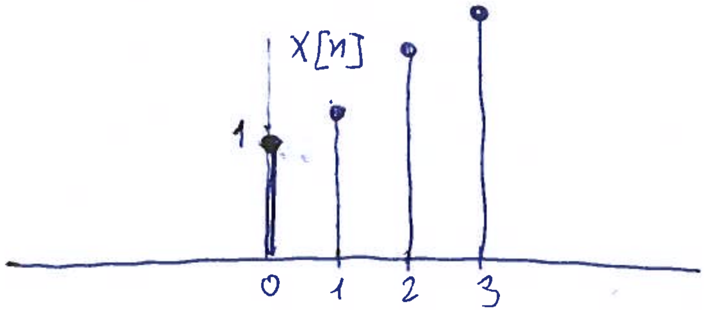
Right-sided exponential sequence

$$x[n] = a^n u[n]$$

$|a| < 1$



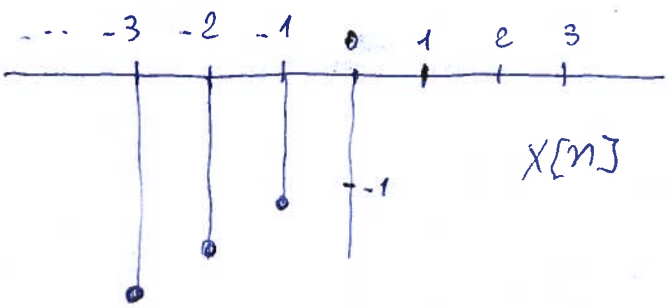
$|a| > 1$



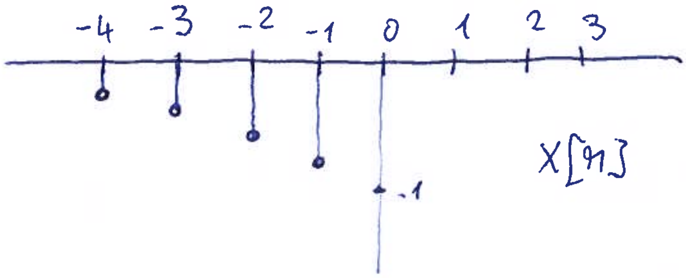
Left-sided exponential sequence

$$x[n] = -a^n u[-n-1]$$

$|a| < 1$



$|a| > 1$



z-transform of $x[n] = \begin{cases} a^n & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

finite length sequence

$$\sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n - \sum_{n=N}^{\infty} (az^{-1})^n =$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n - (az^{-1})^N \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n \left(1-(az^{-1})^N\right) =$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}}$$

$$= \frac{1-\frac{a^N}{z^N}}{1-\frac{a}{z}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

1 pole at $z=a$ → it is canceled by one of the zeros

$N-1$ order pole at 0

N zeros (roots of numerator polynomials)

$$z^k = a e^{j(2\pi k/N)}$$

In case az^{-1} is finite, i.e. $|a|<\infty$ and $z \neq 0$,

$\sum_{n=0}^{N-1} |az^{-1}|^n < \infty$ → ROC include the entire z-plane, with the exception of $z=0$.

z-transform of basic sequences

$$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & \text{elsewhere} \end{cases}$$

$$X[z] = \sum_{n=-\infty}^{+\infty} \delta[n] \cdot z^{-n} = 1 \cdot z^{-0} = 1$$

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$X[z] = \sum_{n=-\infty}^{+\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}} \quad \text{if } |z^{-1}| < 1 \rightarrow |z| > 1$$

$$\delta[n-m] = \begin{cases} 1 & n=m \\ 0 & \text{elsewhere} \end{cases}$$

$$X[z] = \sum_{n=-\infty}^{+\infty} \delta[n-m] z^{-m} = 1 \cdot z^{-m} = z^{-m}$$

z-transform of convolution

$$y[n]=\sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]$$

$$Y[z]=\sum_{n=-\infty}^{+\infty} y[n]\,z^{-n}=\sum_{n=-\infty}^{+\infty}\left\{\sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]\right\}z^{-n}=$$

$$=\sum_{k=-\infty}^{+\infty} x[k]\sum_{n=-\infty}^{+\infty} h[n-k]\,z^{-n}$$

$$m=n-k$$

$$Y[z]=\sum_{k=-\infty}^{+\infty} x[k]\left\{\sum_{m=-\infty}^{+\infty} h[m]\,z^{-m}\right\}z^{-k}=$$

$$=\sum_{k=-\infty}^{+\infty} x[k]\,z^{-k}\cdot\sum_{m=-\infty}^{+\infty} h[m]\,z^{-m}=X[z]\cdot H[z]$$

Delayed sequence

$$\tilde{x}[n]=x[n-n_0]$$

$$\tilde{X}[z]=\sum_{n=-\infty}^{+\infty} x[n-n_0]\,z^{-n}=$$

$$n-n_0=m$$

$$=\sum_{m=-\infty}^{+\infty} x[m]\,z^{-m}\,z^{-n_0}=X[z]\,z^{-n_0}$$