DISCRETE TIME SYSTEMS AND Z-TRANSFORM

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What we have learned in module 1

- Discrete time signals by sampling an analog (time continuous) signal

$$x[n] = x_a[nT_s] \qquad -\infty < n < \infty$$

where Ts is the sampling interval and fs=1/Ts is the sampling frequency.

- Basic sequences

$$\delta[n] = \begin{cases} 0, & n \neq 0, \\ 1, & n = 0. \end{cases} \Rightarrow x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

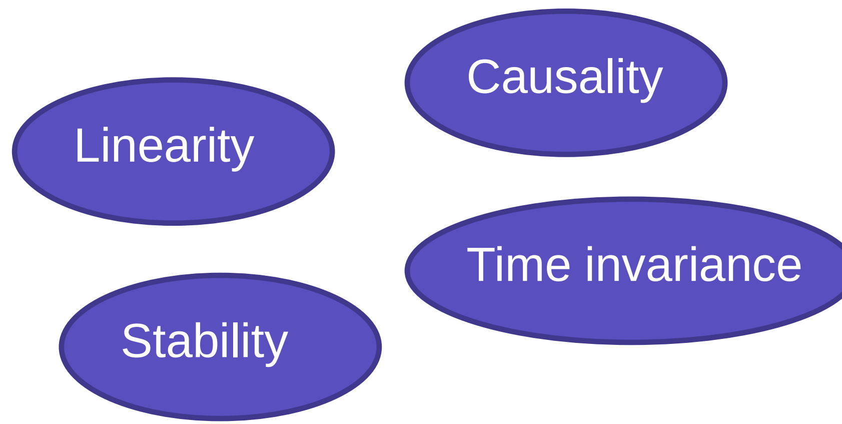
$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

What we have learned in module 1

- Linear discrete time system

$$
\begin{aligned}
T\{x_1[n] + x_2[n]\} &= T\{x_1[n]\} + T\{x_2[n]\} &&= y_1[n] + y_2[n] \\
T\{a \cdot x[n]\} &= a \cdot T\{x[n]\} &&= a \cdot y[n]
\end{aligned}
$$

- Other discrete system properties
  - Causality
  - Time invariance
  - Stability



- **What about systems that are both linear and time invariant?**

Today’s agenda

- Discrete time signals
  - Basic sequences and operations
  - Linear systems
  - Stability, causality, time invariance
- **Linear time invariant (LTI) systems**
  - **Impulse response and convolution**
  - **Parallel and cascade system combination**
- **Fourier transform of LTI systems**
  - **Definition and conditions for existence**
- Z-transform
  - Definition and region of convergence (ROC)
  - Right, left-sided and finite duration sequences
  - ROC analysis
- Inverse z-transform
  - Definition and inspection method
  - Partial fraction expansion
  - Power series expansion
- Transform analysis of LTI systems
  - Linear constant coefficient difference equations
  - Stability and causality
  - Inverse systems
  - FIR and IIR systems

Linear time invariant systems

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- Linear systems → principle of superposition
- General sequence can be expressed as a linear combination of delayed and scaled unit pulses → a linear system can be completely characterized by its impulse response.

$$y[n] = T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\} \Rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]h_k[n].$$

- If only linearity is imposed, $h_k[n]$ depends on both *k* and *n*.
- Time invariance: if h[n] is the response to delta[n], then the response to δ[n-k] is h[n-k].

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

convolution sum

$$y[n] = x[n] * h[n].$$

Linear time invariant systems

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- Convolution operation is commutative

$$x[n] * h[n] = h[n] * x[n].$$

- Convolution operation distributes over addition

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n].$$

Linear time invariant systems

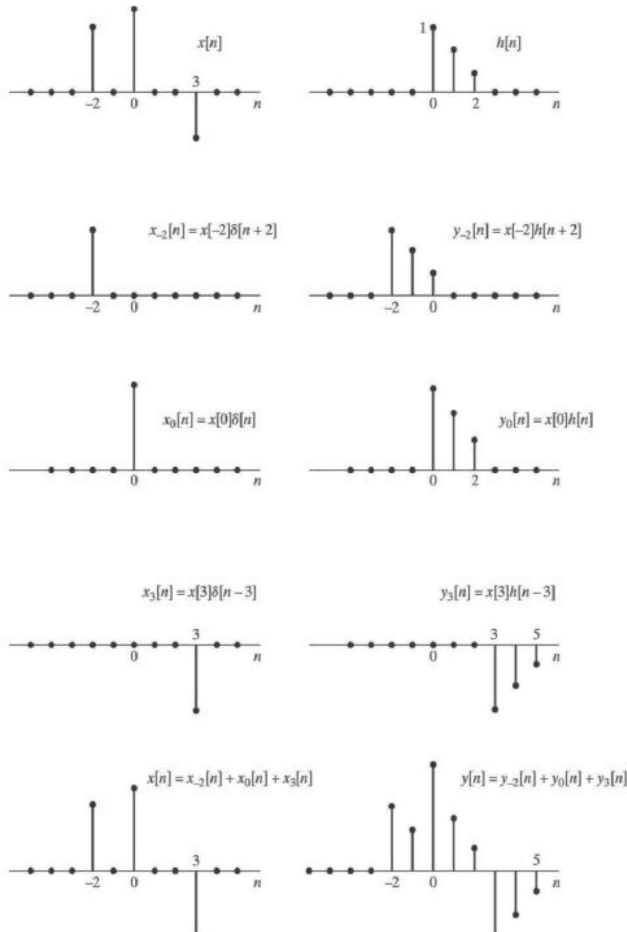


Fig 8 (27)

Representation of the output of an LTI system as the superposition of responses to individual samples of the input.

$$y[n] = x[n] * h[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \qquad \text{for all } n.$$

Linear time invariant systems

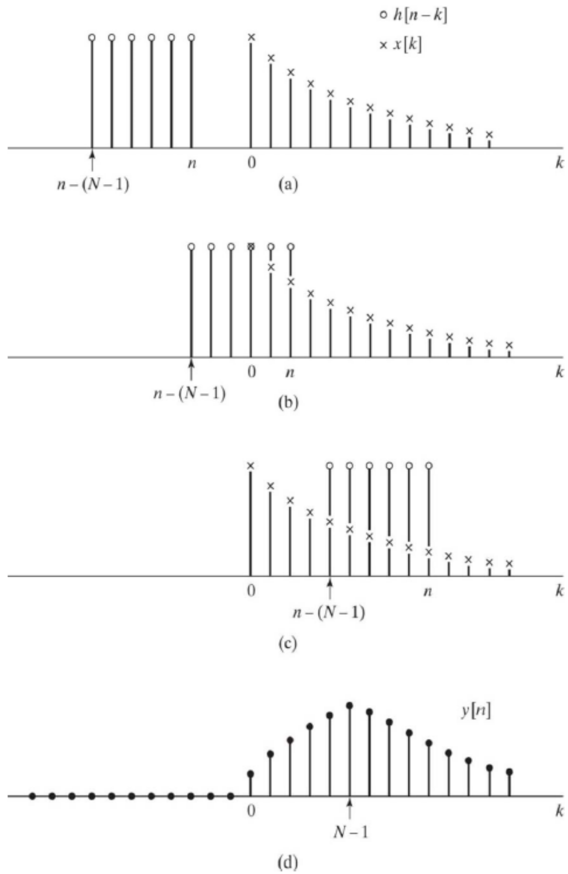


Fig 10 (30)

Sequence involved in computing a discrete convolution.

(a) – (c): The sequences $x[k]$ and $h[n-k]$ as a function of k for different values of n. (Only nonzero samples are shown.)

(d): Corresponding output sequence as a function of n.

h[n] = u[n] – u[n-N]

$$x[n] = \begin{cases} a^n, for\ n \geq 0 \\ 0, for\ n < 0 \end{cases}$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \qquad \text{for all } n.$$

Linear time invariant systems

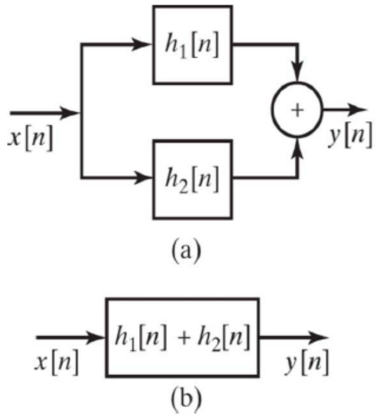


Fig 11 (33)

(a) Parallel combination of LTI systems
(b) An equivalent system

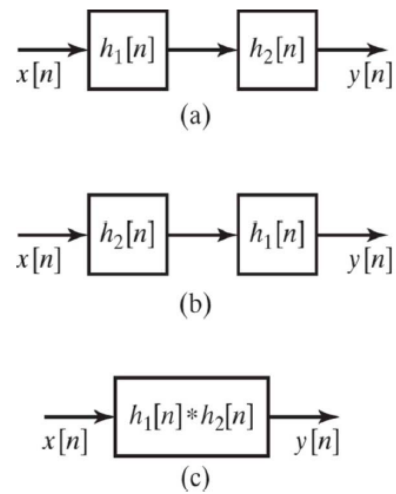


Fig 12 (33)

(a) Cascade combination of two LTI systems
(b) Equivalent cascade
(c) Single equivalent system.

Linear time invariant systems

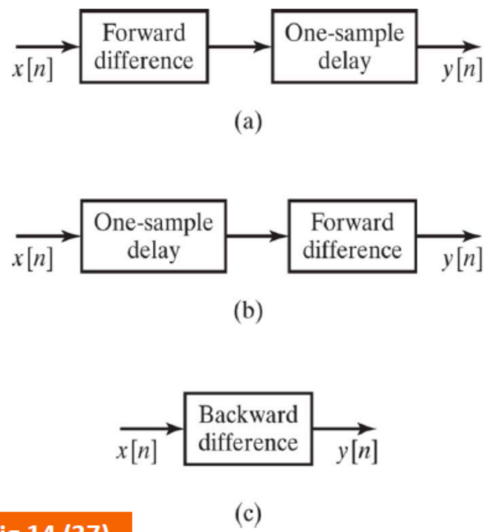
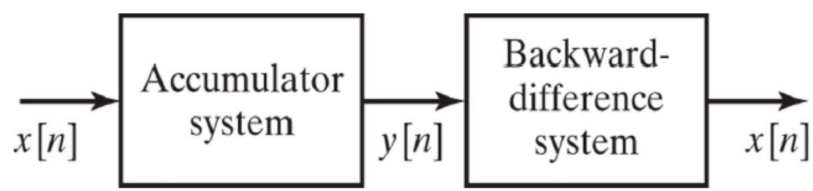


Fig 13 (36)

Equivalent systems found by using the commutative property of convolution.

Fig 14 (37)

An accumulator in cascade with a backward difference. Since the backward difference is the inverse system for the accumulator, the cascade combination is equivalent to the identity system.



Linear time invariant systems

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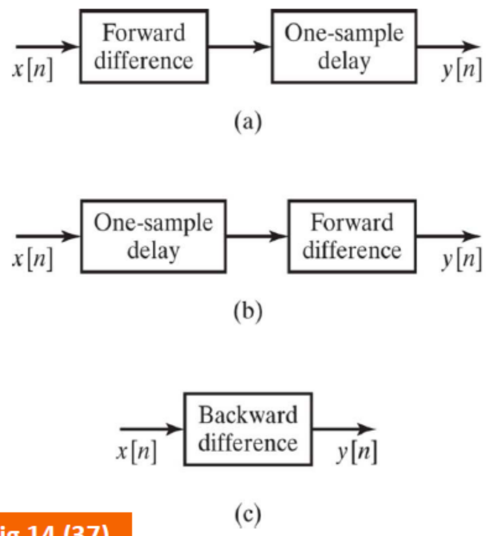

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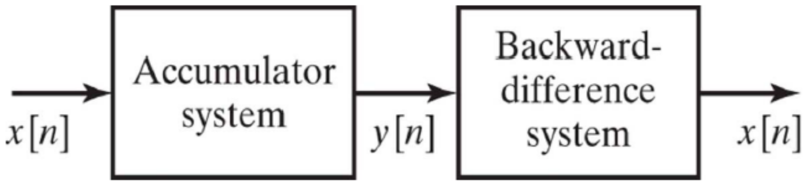


Fourier transform

- The Fourier transform of the function *x(t)* is defined as

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

- The inverse Fourier transform is defined as

$$x(t) = \int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega$$

Fourier transform

Physical interpretation: spectrum of a signal

$$\text{Time domain} \xrightarrow{\text{Fourier transform}} \text{Frequency domain}$$

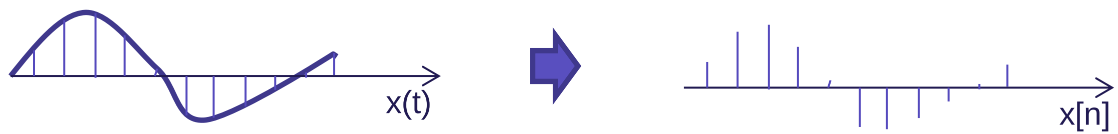
$$\text{Time domain} \xleftarrow{\text{Inverse Fourier transform}} \text{Frequency domain}$$

- The Fourier transform returns the amplitude and phase of sinusoidal signals at the different frequencies that compose the time domain signal.
- The Inverse Fourier transform returns the time domain signal which is composed of the sinusoidal signals at different amplitude and phases of the frequency domain representation.

Fourier transform

- What about the spectrum of a discrete signal, obtained by sampling x(t)?

time



frequency $X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$??

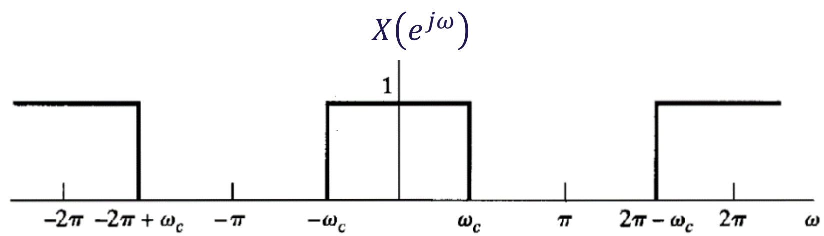
- It can be shown that the spectrum of a discrete time signal can be calculated as

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

Discrete time Fourier transform (DTFT)

Fourier transform

- A common notation for $X(\omega)$ is $X\left(e^{j\omega}\right)$.
- $X\left(e^{j\omega}\right)$ is periodic of period 2π → periodic spectrum



- Since $X\left(e^{j\omega}\right)$ is 2π – periodic, the inverse Fourier transform can be calculated by integrating over a single period, e.g. [- π,π].

$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega n}d\omega$$

Fourier transform

- The frequency response of a linear time invariant system is the Fourier transform of the impulse response,

$$\mathrm{H}(\omega) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n}$$

$$\Downarrow$$

$$\mathrm{h}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

Fourier transform

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- What is the condition of existence of the Fourier transform?

Determining the class of signals that can be represented by

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega,$$

is equivalent to considering the convergence of the infinite sum in $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$

$$|X(e^{j\omega})| < \infty \qquad \text{for all } \omega, \quad \Rightarrow \quad |X(e^{j\omega})| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right|$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]|\,|e^{-j\omega n}|$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty.$$

If x[n] is absolutely summable, than the Fourier transform exists.

Fourier transform

- Example: does the Fourier transform of the following sequence exist?

Let $x[n] = a^n u[n]$. The Fourier transform of this sequence is

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

$$= \frac{1}{1 - ae^{-j\omega}} \qquad \text{if } |ae^{-j\omega}| < 1 \quad \text{or} \quad |a| < 1.$$

Clearly, the condition $|a| < 1$ is the condition for the absolute summability of $x[n]$; i.e.,

$$\sum_{n=0}^{\infty} |a|^n = \frac{1}{1 - |a|} < \infty \qquad \text{if } |a| < 1. \tag{2.140}$$