Linear time invariant systems

$y[n] = T\{x[n]\}$ general definition of a system

A general sequence $x[n]$ can be expressed as a linear combination of delayed and scaled unit pulses, i.e. $x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$

$$y[n] = T\left\{\sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]\right\} \underset{\substack{\text{linear}\\\text{system}}}{=} \sum_{k=-\infty}^{+\infty} x[k]\,T\{\delta[n-k]\}$$

Let us define $h_k[n] = T\{\delta[n-k]\}$, impulse response of linear system.

If the system is also time invariant, and $h[n]$ is the response to $\delta[n]$, then the response to $\delta[n-k]$ is $h[n-k]$

$$y[n] \underset{\substack{\text{linear,}\\\text{time invariant}}}{=} \sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]$$

This is called convolution sum, and is also expressed as

$$y[n] = x[n] * h[n]$$

Convolution is commutative

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{+\infty} h[k]\,x[n-k] = h[n] * x[n]$$

Commutative property of convolution

$$y[n] = x[n] \otimes h[n] = h[n] \otimes x[n]$$

Proof.

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k] =$$

$q = n - k \rightarrow$ if $k \rightarrow -\infty \Rightarrow q \rightarrow +\infty$

if $k \rightarrow +\infty \Rightarrow q \rightarrow -\infty$

$$= \sum_{q=-\infty}^{+\infty} x[n-q]\, h[q] = \sum_{q=-\infty}^{+\infty} h[q]\, x[n-q] = h[n] \otimes x[n]$$

The impulse response of accumulator system is

$$h[n] = \sum_{k=-\infty}^{n} \delta[k] \qquad\qquad y[n] = \sum_{k=-\infty}^{n} x[k]$$

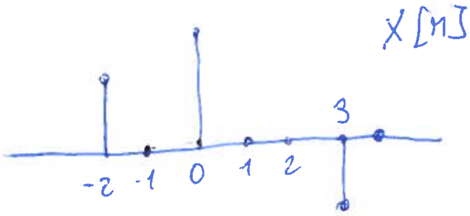
Demonstration

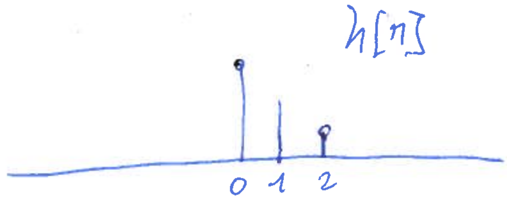
$$y[n] = x[n] \otimes h[n] = \sum_{k=-\infty}^{+\infty} x[k] \sum_{q=-\infty}^{n-k} \delta[q]$$

$$\sum_{q=-\infty}^{n-k} \delta[q] = \begin{cases} 1 & \text{if } n-k \geq 0 \rightarrow n \geq k \rightarrow k \leq n \\ 0 & \text{if } n-k < 0 \rightarrow n < k \rightarrow k > n \end{cases}$$

$$\rightarrow y[n] = \sum_{k=-\infty}^{n} x[k]$$ $\leftarrow$ definition seen in previous lecture

Example





$$x[n] = x_{-2}\,\delta[n+2] + x_0\,\delta[n] + x_3\,\delta[n-3]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, T\{\delta[n-k]\} = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

we can calculate the response to the individual samples of the input

$k = -2$

$$y_{-2}[n] = x_{-2}\, h[n+2]$$

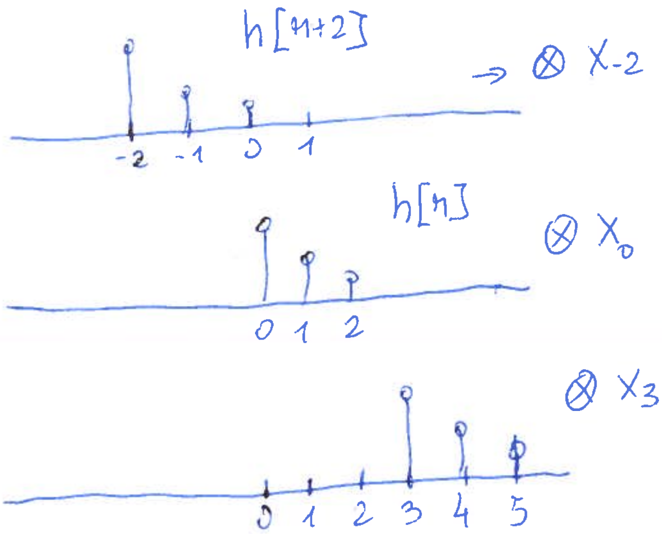
$k = 0$

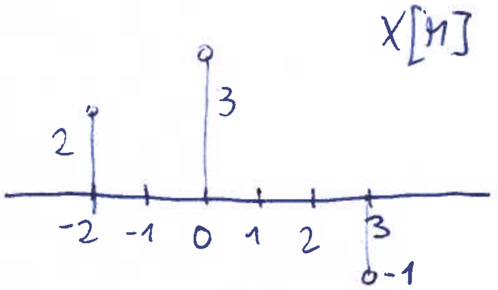
$$y_0[n] = x_0\, h[n]$$

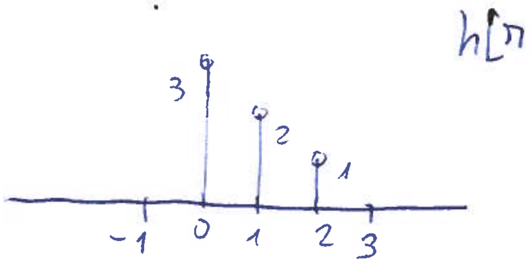
$k = 3$

$$y_3[n] = x_3\, h[n-3]$$

$$y[n] = y_{-2}[n] + y_0[n] + y_3[n] = x_{-2}\, h[n+2] + x_0\, h[n] + x_3\, h[n-3]$$







$$x[n] = 2\delta[n+2] + 3\delta[n] - \delta[n-3]$$

$$h[n] = 3\delta[n] + 2\delta[n-1] + \delta[n-2]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

$$= x[-2]\, h[n+2] + x[0]\, h[n] + x[3]\, h[n-3] =$$

$$= 2\, h[n+2] + 3\, h[n] - h[n-3]$$

$$h[n+2] = 3\delta[n+2] + 2\delta[n+1] + \delta[n]$$

$$h[n] = 3\delta[n] + 2\delta[n-1] + \delta[n-2]$$

$$h[n-3] = 3\delta[n-3] + 2\delta[n-4] + \delta[n-5]$$

$$y[n] = 2\left(3\delta[n+2] + 2\delta[n+1] + \delta[n]\right) +$$

$$+ 3\left(3\delta[n] + 2\delta[n-1] + \delta[n-2]\right) +$$

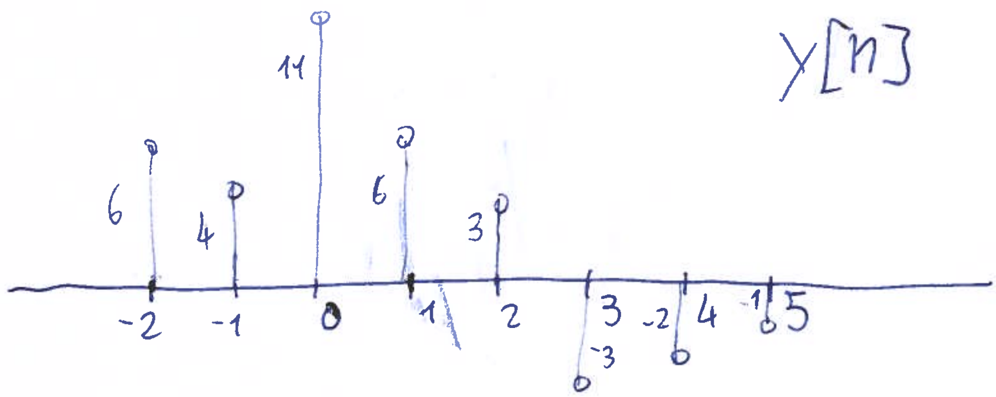
$$- \left(3\delta[n-3] + 2\delta[n-4] + \delta[n-5]\right) =$$

$$= 6\delta[n+2] + 4\delta[n+1] + 2\delta[n] + 9\delta[n] + 6\delta[n-1] +$$

$$+ 3\delta[n-2] - 3\delta[n-3] - 2\delta[n-4] - \delta[n-5] =$$

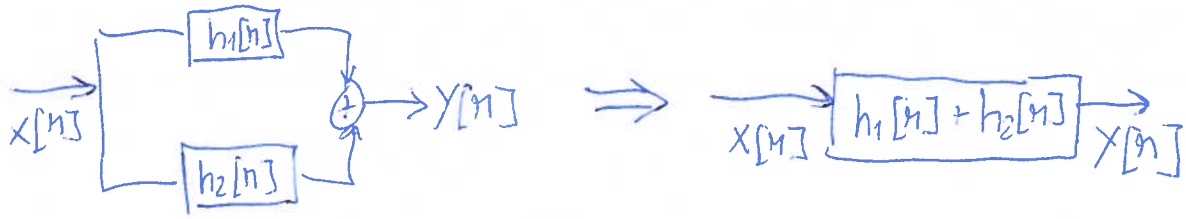
$$= 6\delta[n+2] + 4\delta[n+1] + 11\delta[n] + 6\delta[n-1] +$$

$$+ 3\delta[n-2] - 3\delta[n-3] - 2\delta[n-4] - \delta[n-5]$$

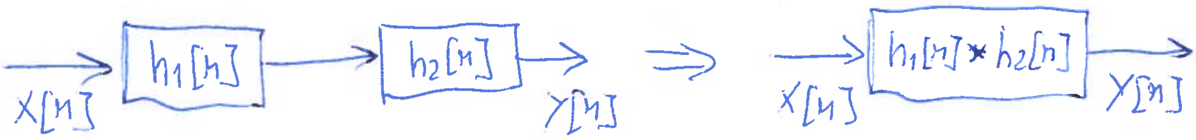


y[n]
11
6
4
6
3
-3
-2
-1
-2
-1
0
1
2
3
4
5

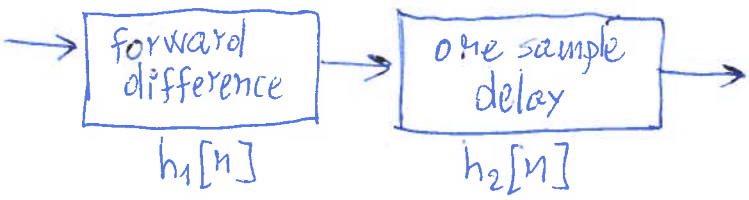
Parallel combination of LTI systems



Cascade combination of LTI systems



Example



$$h_1[n] = \delta[n+1] - \delta[n]$$

$$h_2[n] = \delta[n-1]$$

$$h_1[n] \otimes h_2[n] = (\delta[n+1] - \delta[n]) \otimes \delta[n-1] = \delta[n-1] \otimes (\delta[n+1] - \delta[n])$$

$$\delta[n-1] \otimes \delta[n+1] = \sum_{k=-\infty}^{+\infty} \delta[k-1]\,\delta[n+1-k]$$

$$\delta[k-1] = \begin{cases} 1 & k=1 \\ 0 & k \neq 1 \end{cases}$$

$$\delta[n+1-k]\Big|_{k=1} = \delta[n]$$

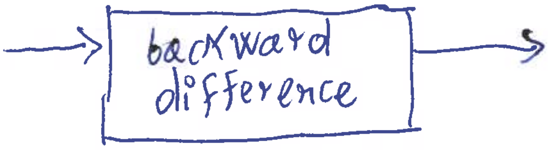
$$\Rightarrow \delta[n-1] \otimes \delta[n+1] = \delta[n]$$

$$\delta[n-1] \otimes \delta[n] = \sum_{k=-\infty}^{+\infty} \delta[k-1]\,\delta[n-k]$$

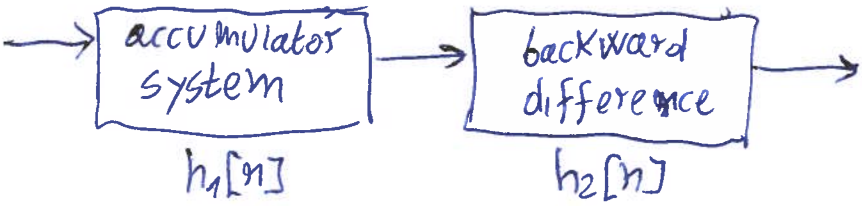
$$\delta[n-k]\Big|_{k=1} = \delta[n-1]$$

$$\delta[n-1] \otimes \delta[n] = \delta[n-1]$$

$$\Rightarrow \quad h_1[n] \otimes h_2[n] = \delta[n] - \delta[n-1]$$



example



$$h_1[n] = \sum_{k=-\infty}^{n} \delta[k] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \Rightarrow h_1[n] = u[n]$$

$$h_2[n] = \delta[n] - \delta[n-1]$$

$$h_1[n] \otimes h_2[n] = u[n] \otimes (\delta[n] - \delta[n-1])$$

$$u[n] \otimes \delta[n] = \sum_{k=-\infty}^{+\infty} u[k]\,\delta[n-k]$$

$$\delta[n-k] = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}$$

$$\Rightarrow \quad = u[n]$$

$$u[n] \otimes \delta[n-1] = u[n-1]$$

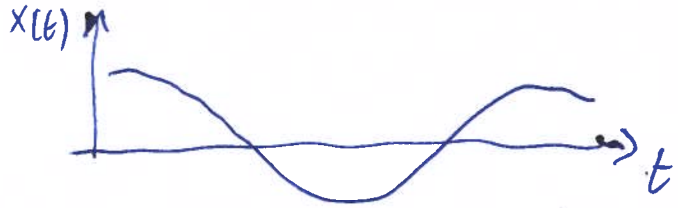
$\Rightarrow$

$h_1[n] \otimes h_2[n] = u[n] - u[n-1] = \delta[n]$

Backward difference is then the inverse system of the accumulator system

Fourier Transform of a continuous signal

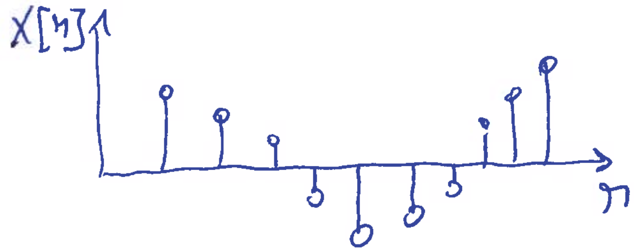
$x(t)$



$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

physichal interpretation: spectrum of a signal

What about the spectrum of a discrete signal, obtained by sampling $x(t)$?



It can be demonstrated that the spectrum of a discrete time signal can be calculated as

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

discrete time Fourier transform (DTFT)

A common notation for DTFT is $X(e^{j\omega})$

$X(e^{j\omega})$ is periodic of period 2π:

$$X\left(e^{j(\omega+2\pi)}\right) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j(\omega+2\pi)n} =$$

$$= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} \underbrace{e^{-j2\pi n}}_{1} = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} = X(e^{j\omega})$$

Since DTFT is 2π periodic, the inverse Fourier tranform can be calculated by integrating over $[-\pi, \pi]$:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

DTFT of impulse response

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n}$$

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\,e^{j\omega n}\,d\omega$$

What is the condition for existence of Fourier transform? Which signals can be represented as $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\,e^{j\omega n}\,d\omega$?

Fourier Transform exists if $\sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ series converges.

$$|X(e^{j\omega})| = \left|\sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}\right| \leq \sum_{n=-\infty}^{+\infty} |x[n]e^{-j\omega n}| =$$

$$= \sum_{n=-\infty}^{+\infty} |x[n]|\,\underbrace{|e^{-j\omega n}|}_{1} = \sum_{n=-\infty}^{+\infty} |x[n]| < \infty$$

Series $x[n]$ must be absolotely summable!

Example

$$x[n] = a^n u[n]$$

$$X(e^{jw}) = \sum_{n=0}^{\infty} a^n e^{-jwn} = \sum_{n=0}^{\infty} (a e^{-jw})^n$$

$$\left| \quad \sum_{n=0}^{\infty} z^n = \begin{cases} \infty & \text{if } z \geq 1 \\ \frac{1}{1-z} & \text{if } z < 1 \end{cases} \right.$$

$$= \frac{1}{1-ae^{-jw}} \quad \text{if } |ae^{-jw}| < 1 \rightarrow \text{if } |a| < 1$$

Fourier Transform exists if $|a| < 1$