Module 2

1) $x[n] = 2\delta[n+2] + 3\delta[n] + 2\delta[n-1] - \delta[n-4]$

$h[n] = \delta[n] + 3\delta[n-1] + 2\delta[n-2] + \delta[n-3]$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]$$

We can calculate the response to the individual samples of the input.

$k=-2 \quad x[-2]=2$

$$x[-2]\,h[n+2] = 2(\delta[n+2] + 3\delta[n+1] + 2\delta[n] + \delta[n-1]) =$$
$$= 2\delta[n+2] + 6\delta[n+1] + 4\delta[n] + 2\delta[n-1]$$

$k=0 \quad x[0]=3$

$$x[0]\,h[n] = 3(\delta[n] + 3\delta[n-1] + 2\delta[n-2] + \delta[n-3]) =$$
$$= 3\delta[n] + 9\delta[n-1] + 6\delta[n-2] + 3\delta[n-3]$$

$k=1 \quad x[1]=2$

$$x[1]\,h[n-1] = 2(\delta[n-1] + 3\delta[n-2] + 2\delta[n-3] + \delta[n-4]) =$$
$$= 2\delta[n-1] + 6\delta[n-2] + 4\delta[n-3] + 2\delta[n-4]$$

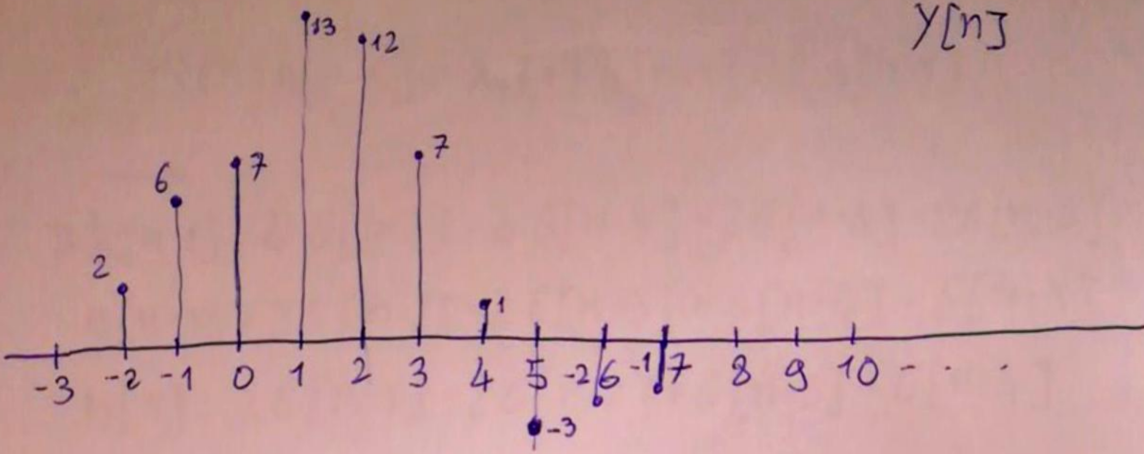
$k=4 \quad x[4]=-1$

$$x[4]\,h[n-4] = -(\delta[n-4] + 3\delta[n-5] + 2\delta[n-6] + \delta[n-7]) =$$
$$= -\delta[n-4] - 3\delta[n-5] - 2\delta[n-6] - \delta[n-7]$$

$$y[n] = x[-2]h[n+2] + x[0]h[n] + x[1]h[n-1] + x[4]h[n-4] =$$

$$= 2\delta[n+2] + 6\delta[n+1] + 7\delta[n] + 13\delta[n-1] + 12\delta[n-2] +$$

$$+ 7\delta[n-3] + \delta[n-4] - 3\delta[n-5] - 2\delta[n-6] - \delta[n-7]$$



2) If h represents an LTI system, then:

$$y_1[n] = \sum_{k=-\infty}^{+\infty} x_1[k]\, h[n-k] = 4\delta[n-2] - 4\delta[n-3] + 2\delta[n-4] - 2\delta[n-5]$$

$$\sum_{k=-\infty}^{+\infty} x_1[n]\, h[n-k] = x_1[1]\, h[n-1] = 2h[n-1]$$

$\longrightarrow$

$$2h[n-1] = 4\delta[n-2] - 4\delta[n-3] + 2\delta[n-4] - 2\delta[n-5]$$

$$h[n-1] = 2\delta[n-2] - 2\delta[n-3] + \delta[n-4] - \delta[n-5]$$

$$h[n] = 2\delta[n-1] - 2\delta[n-2] + \delta[n-3] - \delta[n-4]$$

We need to verify that

$$y_2[n] = \sum_{k=-\infty}^{+\infty} x_2[n]\, h[n-k] = x_2[-3]\, h[n+3] + x_2[-1]\, h[n+1] =$$

$$= -4\delta[n+2] + 4\delta[n+1] + \delta[n-2] - \delta[n-3]$$

- $x_2[-3]\, h[n+3] = -2h[n+3] = -2\cdot(2\delta[n+2] - 2\delta[n+1] + \delta[n] - \delta[n-1]) = -4\delta[n+2] + 4\delta[n+1] - 2\delta[n] + 2\delta[n-1]$

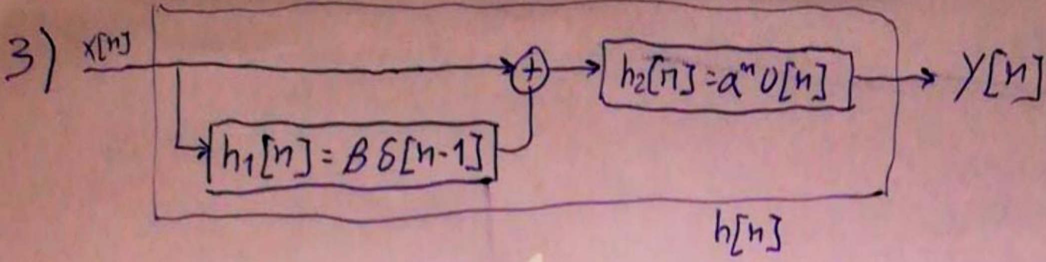
- $x_2[-1]\, h[n+1] = h[n+1] = 2\delta[n] - 2\delta[n-1] + \delta[n-2] - \delta[n-3]$

$$y_2[n] = -4\delta[n+2] + 4\delta[n+1] + \delta[n-2] - \delta[n-3] \quad \checkmark$$

h can then be an LTI system!

Since $h[n]$ does not depend on future values of n, the system is causal.

3)



$$y[n] = \left(x[n] + x[n] \otimes h_1[n]\right) \otimes h_2[n] =$$

$$= x[n] \otimes \left(\delta[n] + h_1[n]\right) \otimes h_2[n]$$

$$h[n] = \left(\delta[n] + h_1[n]\right) \otimes h_2[n] = h_2[n] + h_1[n] \otimes h_2[n] =$$

$$= a^n u[n] + \beta\,\delta[n-1] \otimes a^n u[n] = \cancel{\beta a^{n-1}}$$

$$= a^n u[n] + \beta a^{n-1} u[n-1]$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n] e^{-j\omega n} = \underbrace{\sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n}}_{(1)} +$$

$$+ \beta \underbrace{\sum_{n=-\infty}^{+\infty} a^{n-1} u[n-1] e^{-j\omega n}}_{(2)}$$

$$(1) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(a e^{-j\omega}\right)^n = \frac{1}{1 - a e^{-j\omega}}$$

$$\text{if } |a e^{-j\omega}| < 1 \rightarrow |a| < 1$$

$$(2)\ \beta \sum_{n=-\infty}^{+\infty} a^{n-1} u[n-1] e^{-j\omega n} \underset{\substack{n-1=q \\ n=q+1}}{=} \beta \sum_{q=-\infty}^{+\infty} a^q u[q] e^{-j\omega q} e^{-j\omega} =$$

$$= \beta e^{-j\omega} \sum_{q=0}^{\infty} a^q e^{-j\omega q} = \frac{\beta e^{-j\omega}}{1 - a e^{-j\omega}} \quad \text{if } |a| < 1$$

$\longrightarrow$

$$H(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}} + \frac{Be^{-j\omega}}{1-ae^{-j\omega}} = \frac{1+Be^{-j\omega}}{1-ae^{-j\omega}}$$

$|a|<1$

condition for existence of the Fourier transform

Since $h[n]=0$ for $n<0$, the system is causal.