Lineær algebra (LA)

anden del af kurset i beregningsteknik

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Delvist baseret på slides venligst udlånt af Hans Ebert

LINEÆR ALGEBRA

MM 3:	Fredag 17. marts 2023 kl. 08.15 i B2-104
Emner:	Lineær transformation Egenværdier, egenvektorer Karakteristisk ligning og karakteristisk determinant „Spor & determinant“ Egenrum, algebraisk og geometrisk multiplicitet Anvendelseseksempler: computergrafik, talrækker (Fibonaccis kaniner, det gyldne snit), kredsløbsanalyse og Markovmodeller
Læsning:	[EK] s. 129 – 134, 313 – 317, 322 - 338

Hvor mange løsninger?

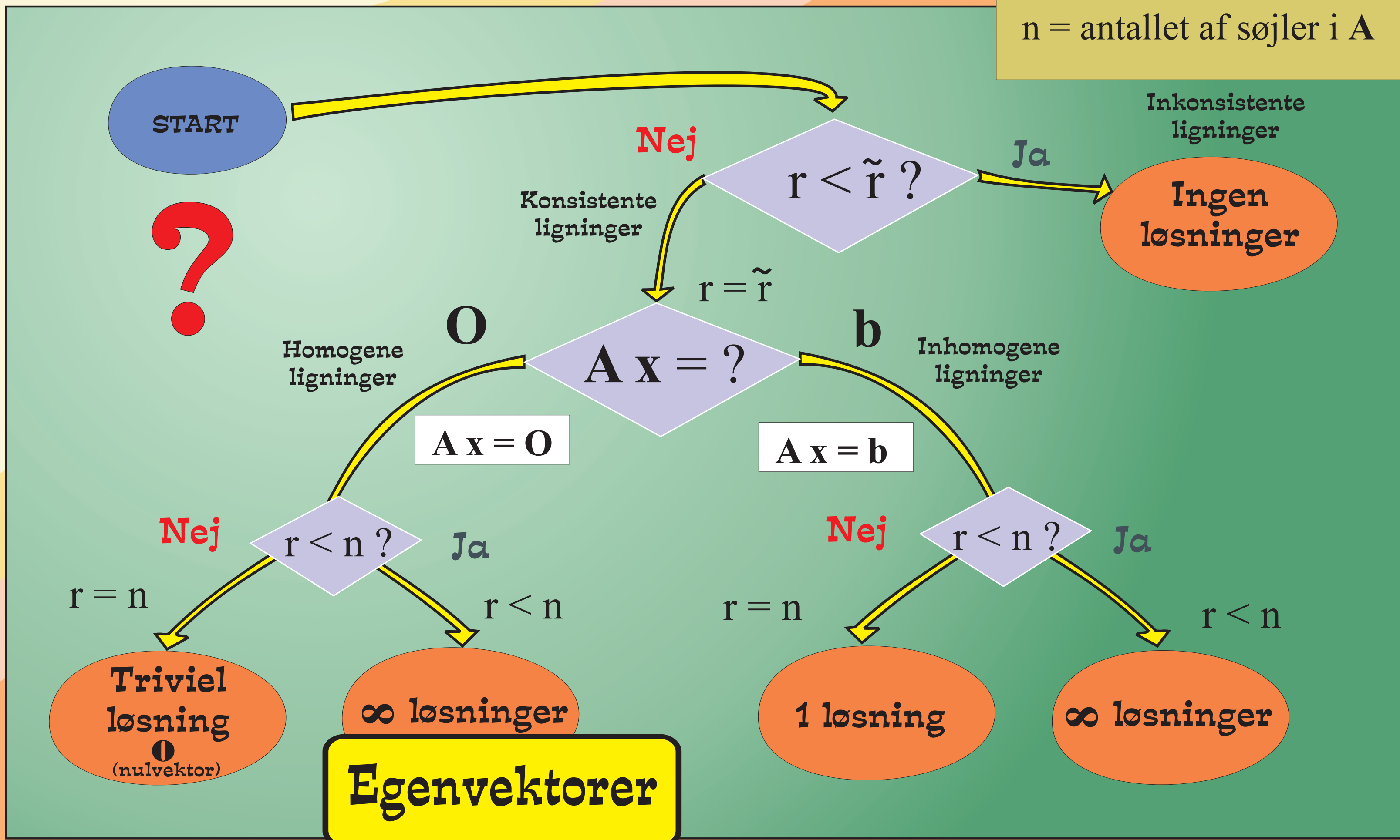
$r = \text{rang}(\mathbf{A})$, rangen af $\mathbf{A}$

$\tilde{r} = \text{rang}(\tilde{\mathbf{A}})$, rangen af totalmatrixen

$n =$ antallet af søjler i $\mathbf{A}$

Totalmatrix:

$\tilde{\mathbf{A}} = [\mathbf{A}|\mathbf{b}]$



Symmetrisk matrix

$$A^T = A$$

Eksempel

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

Skævsymmetrisk matrix

$$A^T = -A$$

$$\begin{bmatrix} 0 & -2 & 8 \\ 2 & 0 & 6 \\ -8 & -6 & 0 \end{bmatrix}$$

Orthogonal matrix

$$A^T = A^{-1}$$

$$\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

Eksempel

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \quad \theta = 30^\circ$$

$$\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

REGNEREGLER

Enhver kvadratisk matrix A kan skrives som:

$$A = R + S$$

hvor R er en symmetrisk matrix:

$$R = \frac{1}{2}(A + A^T)$$

hvor S er en skævsymmetrisk matrix:

$$S = \frac{1}{2}(A - A^T)$$

$$(A^2)^{-1} = (A^{-1})^2$$

$$(A^T)^{-1} = (A^{-1})^T$$

$$(A^{-1})^{-1} = A$$

$$(A^n)^T = (A^T)^n$$

$$\text{rang}(B^T A^T) = \text{rang}(AB)$$

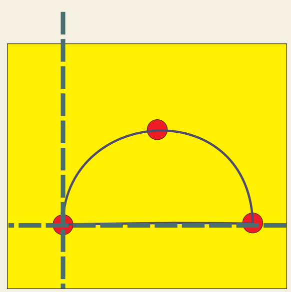
A^T har samme egenværdier som A

Transformation af billeder ved hjælp af matrix
Hvordan virker det?

Input-billede $\curvearrowright$ **A** $\curvearrowright$ **Output-billede**

A

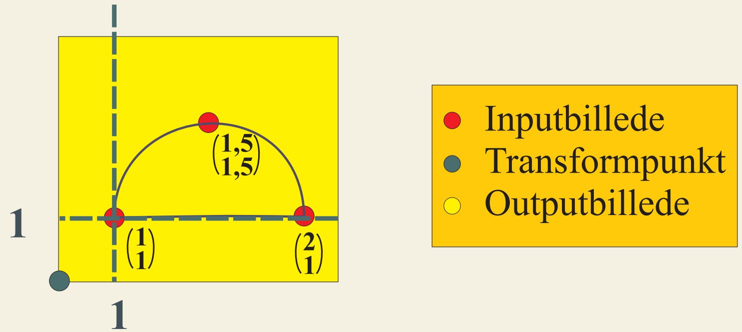
$$\begin{bmatrix} x \\ y \end{bmatrix} \curvearrowright A \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$



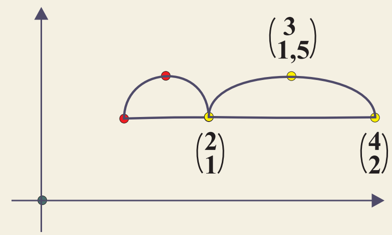
Gammel prik   Ny prik

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Transformation af billeder ved hjælp af matrix

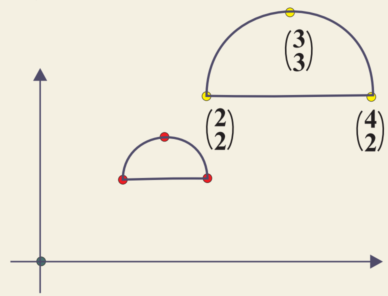


$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$



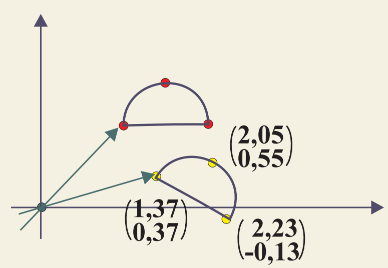
Stræk 2 gange i x-retningen

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$



Skaler 2 gange
(2 gange i x og 2 gange i y)

$$\frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}$$



Drej -30° (omkring trans-
formpunktet)

"Introduction to Video and Image Processing", T. Moeslund,
ISBN13: 9781447125020

9.1.5 Combining the Transformations

The four transformations can be combined in all kinds of different ways by multiplying the matrices in different orders, yielding a number of different transformations. One is shown in figure 9.1(f). Instead of defining the scale factors, the shearing factors and the rotation angle, it is common to merge these three transformation to one matrix. The combination of the four transformations is therefore defined as:

$$\begin{aligned} x' &= a_1 \cdot x + a_2 \cdot y + a_3 \\ y' &= b_1 \cdot x + b_2 \cdot y + b_3 \end{aligned} \Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a_3 \\ b_3 \end{bmatrix} \tag{9.8}$$

and this is the affine transformation. Below the relationships between equation 9.8 and the four above mentioned transformations are listed.

	a_1	a_2	a_3	b_1	b_2	b_3
Translation	1	0	Δ_x	0	1	Δ_y
Scaling	S_x	0	0	0	S_y	0
Rotation	$\cos\theta$	$-\sin\theta$	0	$\sin\theta$	$\cos\theta$	0
Shearing	1	B_x	0	B_y	1	0

Often *homogeneous coordinates* are used when implementing the transformation since they make further calculations faster. In homogeneous coordinates, the affine transformation becomes:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{9.9}$$

where $a_3 = \Delta x$ and $b_3 = \Delta y$.

FLASH MATRIXOPERATIONER

Matrix (flash.geom.Matrix)

```
Object
   |
   +-flash.geom.Matrix
```

```
public class Matrix
extends Object
```

The flash.geom.Matrix class represents a transformation matrix that determines how to map points from one coordinate space to another. By setting the properties of a Matrix object and applying it to a MovieClip or BitmapData object you can perform various graphical transformations on the object. These transformation functions include translation (*x* and *y* repositioning), rotation, scaling, and skewing.

Together these types of transformations are knows as *affine transformations*. Affine transformations preserve the straightness of lines while transforming, and parallel lines stay parallel.

To apply a transformation matrix to a movie clip, you create a flash.geom.Transform object, and set its Matrix property to the transformation matrix. Matrix objects are also used as parameters of some methods, such as the `draw()` method of the flash.display.BitmapData class.

A transformation matrix object is considered a 3 x 3 matrix with the following contents:

$$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ u & v & w \end{bmatrix}$$

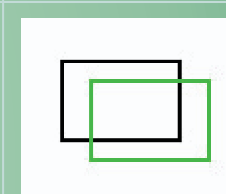
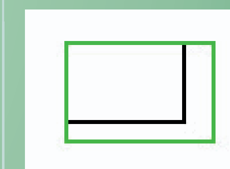
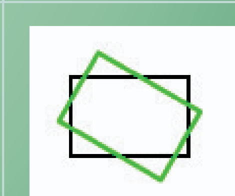
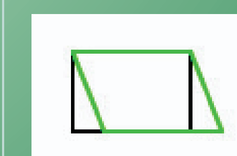
In traditional transformation matrixes the `u`, `v`, and `w` properties provide extra capabilities. The Matrix class can only operate in two-dimensional space so it always assumes that the property values `u` and `v` are 0.0, and that the property value `w` is 1.0. In other words the effective values of the matrix are as follows:

$$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

You can get and set the values of all six of the other properties in a Matrix object: `a`, `b`, `c`, `d`, `tx`, and `ty`.

FLASH MATRIXOPERATIONER

The Matrix class supports the four major types of transformation functions: translation, scaling, rotation, and skewing. There are specialized methods for three of these functions, as described in the following table.

Transformation	Method	Matrix values	Display result	Description
Translation (displacement)	`translate (tx, ty)`	$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Moves the image `tx` pixels to the right and `ty` pixels down.
Scaling	`scale(sx, sy)`	$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$		Resizes the image, multiplying the location of each pixel by `sx` on the *x* axis and `sy` on the *y* axis.
Rotation	`rotate(q)`	$\begin{bmatrix} \cos(q) & \sin(q) & 0 \\ -\sin(q) & \cos(q) & 0 \\ 0 & 0 & 1 \end{bmatrix}$		Rotates the image by an angle `q`, which is measured in radians
Skewing or shearing	None; must set the properties `b` and `c`.	$\begin{bmatrix} 0 & sk_y & 0 \\ sk_x & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		Progressively slides the image in a direction parallel to the *x* or *y* axis. The value `skx` acts as a multiplier controlling the sliding distance along the *x* axis; `sky` controls the sliding distance along the *y* axis.

Each transformation function alters the current matrix properties so that you can effectively combine multiple transformations. To do this, you call more than one transformation function before applying the matrix to its movie clip or bitmap target.

LINALG

Opskrift

1. **Dan den karakteristiske matrix:**

$$M = A - \lambda I$$

2. **Find den karakteristiske determinant**

$$\Delta M = |A - \lambda I|$$ polynomium / ligning

3. **Dette giver den karakteristiske ligning:**

$$a\lambda^{n} + b\lambda^{n-1} + c\lambda^{n-2} + \cdots = 0$$

4. **Løsningerne til den karakteristiske ligning giver spektret:** Spektral radius

$$S = \{\lambda_1, \lambda_2, \lambda_3, \cdots\}$$

5. **Dette indeholder mellem 1 og n egenværdier**

   Bemærk den algebraiske multiplicitet for λ_k (ordenen af roden).

6. **Indsæt hver λ_k i den karakteristiske matrix M og løs det homogene system (fx vha. Gaussisk elimination):**

$$A - \lambda I = 0$$

   Løsningerne er egenvektorerne.

7. **Bemærk den geometriske multiplicitet (antallet af lineært uafhængige egenvektorer for den givne λ_k)**