Inverse z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \longleftrightarrow x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

if C unit circle

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} (X(e^{i\omega}) \cdot e^{i\omega n}) \, d\omega$$

Less formal methods are preferable!

- Inspection method

  "recognizing" certain transform pairs

$$X(z) = \frac{1}{1-az^{-1}} \quad |a|<|z| \longrightarrow x[n] = (a)^n u[n]$$

$$X(z) = \frac{1}{1-az^{-1}} \quad |a|>|z| \longrightarrow x[n] = -(a)^n u[-n-1]$$

- Partial fraction expansion

  Let us assume $X(z)$ is expressed as a ratio of polynomials

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \Longrightarrow$$ let us multiply numerator and denominator by $z^N z^M$

$$\Rightarrow X(z) = \frac{z^N z^M \sum_{k=0}^{M} b_k z^{-k}}{z^N z^M \sum_{k=0}^{N} a_k z^{-k}} = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}}$$

$$X(z) = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}} =$$

- there are M zeros and N poles at non-zero locations
- if $N > M$, $N-M$ zeros at 0
- if $N < M$, $M-N$ poles at 0

$$X(z) = \frac{z^N (b_0 z^M + b_1 z^{M-1} + b_2 z^{M-2} + \dots + b_M)}{z^M (a_0 z^N + a_1 z^{N-1} + a_2 z^{N-2} + \dots + a_N)} =$$

Let us consider the original expression

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}} =$$

$$= \frac{b_0 \left(1 + \frac{b_1}{b_0} z^{-1} + \frac{b_2}{b_0} z^{-2} + \dots + \frac{b_M}{b_0} z^{-M}\right)}{a_0 \left(1 + \frac{a_1}{a_0} z^{-1} + \frac{a_2}{a_0} z^{-2} + \dots + \frac{a_N}{a_0} z^{-N}\right)} =$$

$$= \frac{b_0}{a_0} \frac{(1 + \bar{b}_1 z^{-1} + \bar{b}_2 z^{-2} + \dots + \bar{b}_M z^{-M})}{(1 + \bar{a}_1 z^{-1} + \bar{a}_2 z^{-2} + \dots + \bar{a}_N z^{-N})}$$

Polynomials at both numerator and denominator can be factorized to

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

Theorem of partial fraction decomposition

Let f and g be nonzero polynomials, and

$g = \prod_{i=1}^{K} p_i$, where p_i, $i = 1, \dots, K$ are irriducible polynomials.

There are then unique polynomials b and a_i such that

$$\frac{f}{g} = b + \sum_{i=1}^{K} \frac{a_i}{p_i}$$

If $\deg(f) < \deg(g) \rightarrow b = 0$

In our case, if $M < N$ and the poles are all first order, then $X(z)$ can be written as:

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}} = \frac{A_1}{1 - d_1 z^{-1}} + \frac{A_2}{1 - d_2 z^{-1}} + \dots + \frac{A_N}{1 - d_N z^{-1}}$$

How to calculate A_k?

Let us multiply both terms by $(1 - d_k z^{-1})$

$\longrightarrow$

$$(1-d_k z^{-1})X(z)=(1-d_k z^{-1})\left(\frac{A_1}{1-d_1 z^{-1}}+\dots+\frac{A_k}{1-d_k z^{-1}}+\dots+\frac{A_N}{1-d_N z^{-1}}\right)$$

$$(1-d_k z^{-1})X(z)=A_1\left(\frac{1-d_k z^{-1}}{1-d_1 z^{-1}}\right)+\dots+A_k+\dots+A_N\left(\frac{1-d_k z^{-1}}{1-d_N z^{-1}}\right)$$

Let us set $z=d_k$:

$$(1-d_k z^{-1})X(z)\Big|_{z=d_k}=A_1\left(\frac{1-d_k\cdot d_k^{-1}}{1-d_1 d_k^{-1}}\right)+\dots+A_k+\dots+A_N\left(\frac{1-d_k d_k^{-1}}{1-d_N d_k^{-1}}\right)=$$

$$=A_k$$

$$\Rightarrow\quad A_k=(1-d_k z^{-1})X(z)\Big|_{z=d_k}$$

Example

$$X(z)=\frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}\qquad |z|>\frac{1}{2}$$

$M=0,\ N=2$

$$X(z)=\frac{A_1}{\left(1-\frac{1}{4}z^{-1}\right)}+\frac{A_2}{\left(1-\frac{1}{2}z^{-1}\right)}$$

$$A_1=\left(1-\frac{1}{4}z^{-1}\right)\cdot X(z)\Big|_{z=\frac{1}{4}}=\left(1-\frac{1}{4}z^{-1}\right)\cdot\frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}\Bigg|_{z=\frac{1}{4}}=$$

$$=\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)}\Bigg|_{z=\frac{1}{4}}=-1$$

$$A_2 = \left(1 - \frac{1}{2} z^{-1}\right) X(z) \Big|_{z=\frac{1}{2}} = \frac{1}{\left(1 - \frac{1}{4} z^{-1}\right)} \Big|_{z=\frac{1}{2}} = 2$$

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4} z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2} z^{-1}\right)} \Rightarrow x[n] = 2\left(\frac{1}{2}\right)^n U[n] - \left(\frac{1}{4}\right)^n U[n]$$

In case $M \geq N$, then

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}$$

Example

$$X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} = \frac{(1-z^{-1})^2}{(1-\frac{1}{2}z^{-1})(1-z^{-1})} \qquad |z|>1$$

$M = N = 2$

$$X(z) = B_0 + \sum_{k=1}^{2} \frac{A_k}{1-d_k z^{-1}} = B_0 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}$$

How to calculate B_0? Via long division

$$\begin{array}{r|l} & \;2 \\ \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & z^{-2}+2z^{-1}+1 \\ & \underline{z^{-2}-3z^{-1}+2} \\ & \quad 5z^{-1}-1 \end{array}$$

$$X(z) = 2 + \frac{-1+5z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

$$A_1 = \left[\left(\frac{-1+5z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}\right)\left(1-\frac{1}{2}z^{-1}\right)\right]_{z=\frac{1}{2}} =$$

$$= \left[\frac{-1+5z^{-1}}{(1-z^{-1})}\right]_{z=\frac{1}{2}} = -9$$

$$A_2 = \left[\left(\frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)} \right) \left(1-z^{-1}\right) \right]_{z=1} = 8$$

Therefore

$$X(z) = 2 - \frac{9}{1-\frac{1}{2}z^{-1}} + \frac{8}{1-z^{-1}}$$

Since $|z|>1$,

$$\frac{1}{1-\frac{1}{2}z^{-1}} \Longrightarrow \left(\frac{1}{2}\right)^n u[n]$$

$$\frac{1}{1-z^{-1}} \Longrightarrow u[n]$$

Therefore

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$