Linear constant coefficient difference equations

An important class of LTI systems takes the following form

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{m=0}^{M} b_m\, x[n-m]$$

Example:
moving average system (causal system $\rightarrow M_1=0$)

$$y[n]-y[n-1]=\frac{1}{M_2+1}\left(x[n]-x[n-M_2-1]\right)$$

It is possible to analyze such systems via z-transform

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Rightarrow \quad H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{M} a_k z^{-k}}$$

This can be factorized as

$$H(z)=\left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

From $H(z)$ it is possible to derive the difference equation, and viceversa.

$\longrightarrow$

Whitout additional constraints/information, a linear constant coefficient difference equations for discrete time systems does not provide an <u>unique</u> specification of the output for a given input; i.e. for a given input sequence $x_s[n]$, there can be infinite $y[n]$ sequences fulfilling the condition

$$\sum_{k=0}^{N} a_k \, y[n-k] = \sum_{m=0}^{M} b_m \, x[n-m]$$

Additional conditions are then needed!

If a system is characterized by a linear constant coefficient difference equation and is specified to be linear, time invariant, and causal, the solution is unique.

For a given $H(z) = \dfrac{Y(z)}{X(z)}$, each possible choice of the region of convergence leads to a different impulse response, but they will all correspond to the same difference equation.

Example

$$H(z) = \frac{1+2z^{-1}+z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}} = \frac{Y(z)}{X(z)}$$

$$\rightarrow \left(1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}\right)Y(z) = \left(1+2z^{-1}+z^{-2}\right)X(z)$$

$$\rightarrow y[n]+\frac{1}{4}y[n-1]-\frac{3}{8}y[n-2] = x[n]+2x[n-1]+x[n-2]$$

How to evaluate the property of the system with $H(z)$?

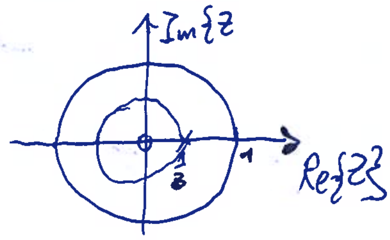
- A system is causal if $h[n]=0$ for $n<0$.
  This means, $h[n]$ is a right sided sequence
  $\rightarrow$ ROC of $H(z)$ must be outside the outermost pole
- System is stable $\rightarrow$ impulse response absolutely summable

$$\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$$

$\rightarrow$ ROC includes the unit circle

Example

$h[n] = \left(\frac{1}{3}\right)^n u[n] \quad \rightarrow \quad H(z) = \frac{1}{1-\frac{1}{3}z^{-1}} \qquad |z| > \frac{1}{3}$



ROC outside outermost pole $\rightarrow$ causal system
ROC includes unit circle $\rightarrow$ stable system

$h[n] = (2)^n u[n] \quad \longrightarrow \quad H(z) = \frac{1}{1-2z^{-1}} \qquad |z| > 2$

ROC outside outermost pole $\rightarrow$ causal system
ROC does not include unit circle $\rightarrow$ unstable

$h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] \quad \rightarrow \quad H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| < \frac{1}{2}$

ROC not outside outermost pole $\rightarrow$ non causal system

ROC does not include unit circle $\rightarrow$ unstable

Example

Consider the following difference equation

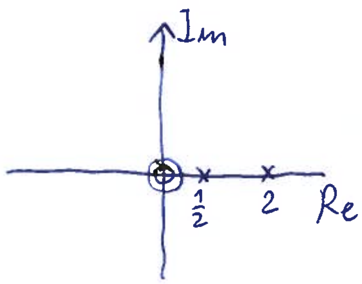
$$y[n] - \frac{5}{2} y[n-1] + y[n-2] = x[n]$$

By applying z-transform:

$$Y(z) - \frac{5}{2} z^{-1} Y(z) + z^{-2} Y(z) = X(z)$$

$$\rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{5}{2} z^{-1} + z^{-2}} = \frac{1}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 - 2z^{-1}\right)}$$

partial fraction



$$H(z) = \frac{A_1}{\left(1 - \frac{1}{2} z^{-1}\right)} + \frac{A_2}{\left(1 - 2z^{-1}\right)}$$

$$A_1 = \left(1 - \frac{1}{2} z^{-1}\right) H(z)\Big|_{z=\frac{1}{2}} = \frac{1}{1 - 2z^{-1}}\Big|_{z=\frac{1}{2}} = -\frac{1}{3}$$

$$A_2 = \left(1 - 2z^{-1}\right) H(z)\Big|_{z=2} = \frac{1}{1 - \frac{1}{2} z^{-1}}\Big|_{z=2} = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$$

$$H(z) = \frac{-\frac{1}{3}}{\left(1 - \frac{1}{2} z^{-1}\right)} + \frac{\frac{4}{3}}{\left(1 - 2z^{-1}\right)}$$

if $|z| > \frac{1}{2}$ and $|z| > 2 \rightarrow |z| > 2$, both sequences ~~have~~ are right sided

$$h[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{4}{3}(2)^n u[n]$$

ROC out of outermost pole $\rightarrow$ system is causal

ROC does not include the unit circle $\rightarrow$ unstable if $\frac{1}{2} < |z| < 2$, the sequence with pole in 2 is left sided, the sequence with pole in $\frac{1}{2}$ is right sided.

$$h[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1]$$

ROC is not out of outermost pole → non causal
ROC includes unit circle → stable system

if $|z| < \frac{1}{2}$, both sequences are left-sided

$$h[n] = \frac{1}{3}\left(\frac{1}{2}\right)^n u[-n-1] - \frac{4}{3}(2)^n u[-n-1]$$

ROC is not out of outermost pole → non causal
ROC does not include unit circle → unstable

Linear constant coefficient difference equation ⟷ impulse response $h(n)$

Linear constant coefficient difference equation ⟷ transfer function $H(z)$

impulse response $h(n)$ ⟷ transfer function $H(z)$

Let us suppose we have to implement in a digital device an accumulator.

The accumulator has impulse response

$$h[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n]$$

Let us calculate first the transfer function $H(z)$

$$H(z) = \frac{1}{1-z^{-1}} \qquad |z| > 1$$

causal system
unstable

Since $H(z) = \dfrac{Y(z)}{X(z)} \rightarrow \dfrac{Y(z)}{X(z)} = \dfrac{1}{1-z^{-1}}$

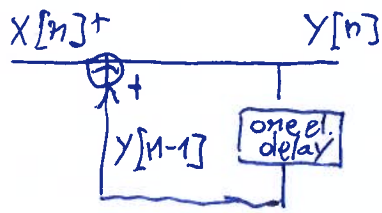
$$X(z) = (1-z^{-1})\,Y(z)$$

$$X(z) = Y(z) - Y(z)\,z^{-1}$$

$$\Big\downarrow z^{-1}$$

$$x[n] = y[n] - y[n-1]$$

$$y[n] = x[n] + y[n-1]$$



Inverse systems

$H_i(z)$ is the inverse system of $H(z)$ if

$$G(z) = H(z) \cdot H_i(z) = 1 \Rightarrow H_i(z) = \frac{1}{H(z)}$$

The time domain condition is

$$h[n] \otimes h_i[n] = \delta[n]$$

example

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

zeros at $z = c_k$
poles at $z = d_k$

$$\Downarrow$$

$$H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N}(1-d_k z^{-1})}{\prod_{k=1}^{M}(1-c_k z^{-1})}$$

zeros at $z = d_k$
poles at $z = c_k$

ROCs must overlap, such that $h[n] \otimes h_i[n] = \delta[n]$

If $H(z)$ is causal, its region of convergence is

$$|z| > \max_k |d_k|$$

Any appropriate ROC that overlaps with the condition above, is a valid ROC for $H_i(z)$.

Example

$$H(z) = \frac{1-0.5z^{-1}}{1-0.9z^{-1}} \qquad ROC: |z| > 0.9$$

$$\downarrow$$

$$H_i(z) = \frac{1-0.9z^{-1}}{1-0.5z^{-1}}$$

$H_i(z)$ has only 1 pole, $z=0.5$, and therefore 2 possible ROCs: $|z|<0.5$, or $|z|>0.5$.
The option $|z|>0.5$ overlaps with the ROC of $H(z)$.

$$H_i(z) = \frac{1}{1-0.5z^{-1}} - \frac{0.9z^{-1}}{1-0.5z^{-1}} =$$

$$= \frac{1}{1-0.5z^{-1}} - 0.9\left(\frac{1}{1-0.5z^{-1}} \cdot z^{-1}\right) =$$

$$= (0.5)^n u[n] - 0.9(0.5)^{n-1} \cdot u[n-1]$$

Example

$$H(z) = \frac{z^{-1} - 0{,}5}{1 - 0{,}9z^{-1}} \qquad |z| > 0.9$$

$$\rightarrow \quad H_i(z) = \frac{1 - 0{,}9z^{-1}}{z^{-1} - 0{,}5} = \frac{-1 + 0{,}9z^{-1}}{0{,}5 - z^{-1}} = \frac{-2 + 1{,}8z^{-1}}{1 - 2z^{-1}}$$

The possible regions of convergence are $|z| < 2$ and $|z| > 2$.
In this case, both ROCs overlap with $|z| > 0.9$.

$$H_i(z) = -2 \cdot \frac{1}{1 - 2z^{-1}} + 1.8 \cdot \frac{1}{1 - 2z^{-1}} z^{-1}$$

For $|z| < 2$

$$h_i[n] = 2(2)^n u[-n-1] - 1{,}8(2)^{n-1} u[-(n-1)-1] =$$
$$= 2(2)^n u[-n-1] - 1{,}8(2)^{n-1} u[-n]$$

For $|z| > 2$

$$h_i[n] = -2(2)^n u[n] + 1{,}8(2)^{n-1} u[n-1]$$

Impulse response for rational system functions

We saw that any rational system function with only poles of first order can be expressed as

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

where elements in the first summation only appears if $M > N$.

System causal $\rightarrow$ ROC outside outermost pole

$$\rightarrow \quad h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n]$$

If there is at least one term in the second sum

$\rightarrow$ $h[n]$ has not finite length $\rightarrow$

infinite impulse response (IIR) system.

If $H(z)$ has no poles (except for $z=0$), a partial fraction expansion is not possible

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} \rightarrow h[n] = \sum_{k=0}^{M} b_n \delta[n-k] =$$

$$= \begin{cases} b_n & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

The impulse response is finite in length

$\rightarrow$ finite impulse response system.

Example IIR filter

$h[n] = (a)^n u[n]$ → not finite length

$\downarrow$

$H(z) = \frac{1}{1-az^{-1}}$ $|z| > |a|$

causal system

if $|a| > 1$ → unstable

$|a| \leq 1$ → stable

Possible implementation via difference equation:

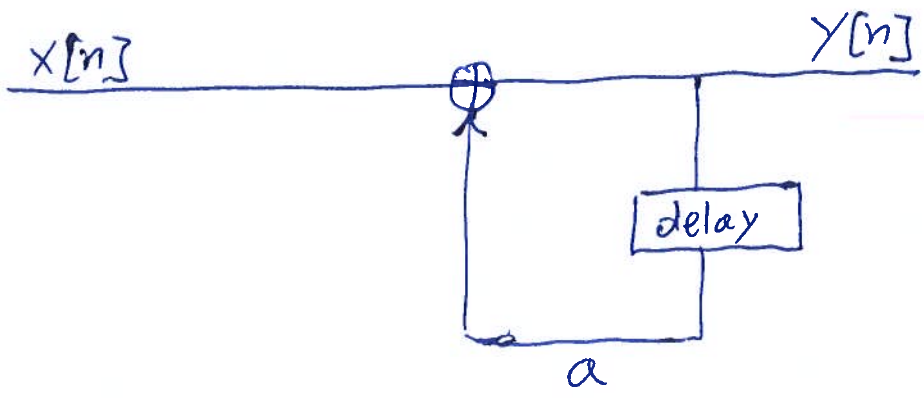
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-az^{-1}}$$

$$(1-az^{-1})Y(z) = X(z)$$

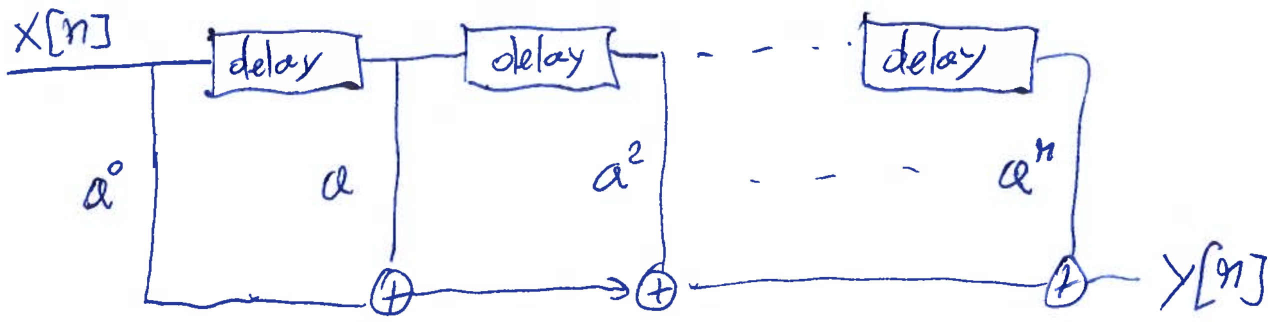
$$Y(z) - aY(z)z^{-1} = X(z)$$

$$\Big\downarrow Z^{-1}$$

$$y[n] - a\,y[n-1] = x[n] \rightarrow y[n] = x[n] + a\,y[n-1]$$



$$y[n] = \sum_{k=0}^{M} a^k x[n-k]$$



Example FIR filter

$$h[n] = (a)^n \qquad 0 \le n \le M$$

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n} = \sum_{n=0}^{M} a^n z^{-n} = \sum_{n=0}^{M} (a z^{-1})^n$$

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$$

$$\rightarrow H(z) = \frac{1-(az^{-1})^{M+1}}{1-az^{-1}} \qquad z \neq 0$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-(az^{-1})^{M+1}}{1-az^{-1}}$$

$$(1-az^{-1})Y(z) = (1-a^{M+1}z^{-M-1})X(z)$$

$$Y(z) - a Y(z) z^{-1} = X(z) - a^{M+1} X(z) z^{-M-1}$$

$$\Downarrow z^{-1}$$

$$y[n] - a y[n-1] = x[n] - a^{M+1} x[n-M-1]$$

$$y[n] = x[n] - a^{M+1} x[n-M-1] + a y[n-1]$$