Module 4

a) $X(z) = (1+3z)(1+2z^{-1})(1-z^{-1}) =$

$$= (1+3z+2z^{-1}+6)(1-z^{-1}) = 1+3z+2z^{-1}+6-z^{-1}-3-2z^{-2}-6z^{-1} =$$

$$= +4+3z-5z^{-1}-2z^{-2}$$

$$\Downarrow$$

$$x[n] = 4\delta[n] + 3\delta[n+1] - 5\delta[n-1] - 2\delta[n-2]$$

b) $$X(z) = \frac{1-\frac{1}{2}z^{-1}}{1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} = \frac{1-\frac{1}{2}z^{-1}}{\left(1+\frac{1}{4}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)} =$$

$$= \frac{A_1}{1+\frac{1}{4}z^{-1}} + \frac{A_2}{1+\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$

$$A_1 = \left(1+\frac{1}{4}z^{-1}\right)X(z)\Big|_{z=-\frac{1}{4}} = \frac{\left(1-\frac{1}{2}z^{-1}\right)}{\left(1+\frac{1}{2}z^{-1}\right)}\Bigg|_{z=-\frac{1}{4}} = -3$$

$$A_2 = \left(1+\frac{1}{2}z^{-1}\right)X(z)\Big|_{z=-\frac{1}{2}} = \frac{\left(1-\frac{1}{2}z^{-1}\right)}{\left(1+\frac{1}{4}z^{-1}\right)}\Bigg|_{z=-\frac{1}{2}} = 4$$

$$X(z) = \frac{-3}{1+\frac{1}{4}z^{-1}} + \frac{4}{1+\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$

Since $|z| > \frac{1}{2}$, both sequences are right sided $\rightarrow$

$$x[n] = -3\left(-\frac{1}{4}\right)^n u[n] + 4\left(-\frac{1}{2}\right)^n u[n]$$

c) $$X(z) = \frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{4}z^{-2}} \qquad |z| > \frac{1}{2}$$

$$X(z) = \frac{1-\frac{1}{2}z^{-1}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{1}{1+\frac{1}{2}z^{-1}}$$

Since $|z| > \frac{1}{2}$, the sequence is right sided $\longrightarrow$

$$x[n] = \left(-\frac{1}{2}\right)^n u[n]$$

d) $$X(z) = \frac{1-az^{-1}}{z^{-1}-a} \qquad |z| > \left|\frac{1}{a}\right|$$

$$X(z) = \frac{1}{z^{-1}-a} - \frac{az^{-1}}{z^{-1}-a} = -\frac{1}{a}\frac{1}{1-\frac{1}{a}z^{-1}} - \frac{z^{-1}}{\frac{1}{a}z^{-1}-1} =$$

$$= -\frac{1}{a}\frac{1}{1-\frac{1}{a}z^{-1}} + \frac{1}{1-\frac{1}{a}z^{-1}}\cdot z^{-1}$$

Since $|z| > \left|\frac{1}{a}\right|$, both sequences are right-sided $\rightarrow$

$$x[n] = -\frac{1}{a}\left(\frac{1}{a}\right)^n u[n] + \left(\frac{1}{a}\right)^{n-1} u[n-1] =$$

$$= -\left(\frac{1}{a}\right)^{n+1} u[n] + \left(\frac{1}{a}\right)^{n-1} u[n-1]$$

e) $X(z) = \ln(1-4z) \qquad |z| < \frac{1}{4}$

We know that

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

We then have

$$\ln(1-4z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-4z)^n}{n} = \sum_{n=1}^{\infty} (-1) \frac{(-1)^n (-4)^n z^n}{n} =$$

$$= -\sum_{n=1}^{\infty} \frac{4^n}{n} z^n = -4z - \frac{4^2}{2} z^2 - \frac{4^3}{3} z^3 - \ldots\ldots$$

Therefore:

$$x[n] = -4\,\delta[n+1] - \frac{4^2}{2}\delta[n+2] - \frac{4^3}{3}\delta[n+3] + \ldots =$$

$$= -\frac{4^n}{n} u[-n-1]$$

f) $$X(z) = \frac{3 - 7z^{-1} + 5z^{-2}}{1 - \frac{5}{2}z^{-1} + z^{-2}} \qquad \frac{1}{2} < |z| < 2$$

$$X(z) = \frac{3 - 7z^{-1} + 5z^{-2}}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})}$$

Let us express $X(z)$ in the following form:

$$X(z) = B_0 + \frac{A_1}{1-2z^{-1}} + \frac{A_2}{1-\frac{1}{2}z^{-1}}$$

B_0 can be calculated via long division

$$\begin{array}{r|l} & 5 \\ \hline z^{-2} - \frac{5}{2}z^{-1} + 1 & 5z^{-2} - 7z^{-1} + 3 \\ & 5z^{-2} - \frac{25}{2}z^{-1} + 5 \\ \hline & // \quad \frac{11}{2}z^{-1} - 2 \end{array}$$

Therefore

$$X(z) = 5 + \frac{\frac{11}{2}z^{-1} - 2}{1 - \frac{5}{2}z^{-1} + z^{-2}} = 5 + \overbrace{\frac{\frac{11}{2}z^{-1} - 2}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})}}^{X_1(z)}$$

$$A_1 = (1-2z^{-1})X_1(z)\Big|_{z=2} = \frac{\frac{11}{2}z^{-1} - 2}{1 - \frac{1}{2}z^{-1}}\Bigg|_{z=2} = 1$$

$$A_2 = (1-\tfrac{1}{2}z^{-1})X_1(z)\Big|_{z=\frac{1}{2}} = \frac{\frac{11}{2}z^{-1} - 2}{1 - 2z^{-1}}\Bigg|_{z=\frac{1}{2}} = -3$$

Therefore we obtain

$$X(z) = 5 + \frac{1}{1-2z^{-1}} - \frac{3}{1-\frac{1}{2}z^{-1}}$$

Since $\frac{1}{2} < |z| < 2$, we obtain

$$x[n] = 5\delta[n] - 2^n u[-n-1] - 3\left(\frac{1}{2}\right)^n u[n]$$