

DISCRETE TIME SYSTEMS AND Z-TRANSFORM

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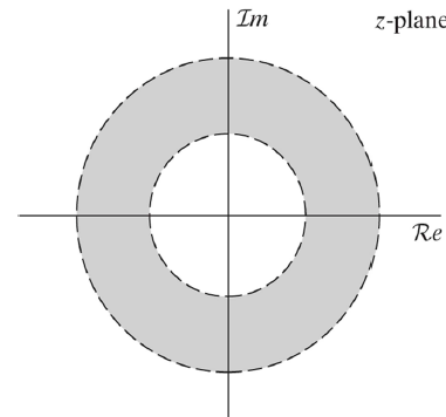
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What we have learned in module 3

Z- transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.$$

The set of values of z for which the z-transform converges is called the region of convergence (ROC).



We have calculated z-transform and ROC of

- Right sided exponential sequence
- Left Sided exponential sequence
- Sum of exponential sequences

We have studied the properties of the ROC, and common z-transform pairs.

The z-transform



TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$ z > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$ z < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$ z > a $
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$ z < a $
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z > a $
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z < a $
9. $\cos(\omega_0 n) u[n]$	$\frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$ z > 1$
10. $\sin(\omega_0 n) u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$ z > 1$
11. $r^n \cos(\omega_0 n) u[n]$	$\frac{1 - r \cos(\omega_0)z^{-1}}{1 - 2r \cos(\omega_0)z^{-1} + r^2 z^{-2}}$	$ z > r$
12. $r^n \sin(\omega_0 n) u[n]$	$\frac{r \sin(\omega_0)z^{-1}}{1 - 2r \cos(\omega_0)z^{-1} + r^2 z^{-2}}$	$ z > r$
13. $\begin{cases} a^n, & 0 \leq n \leq N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1 - a^N z^{-N}}{1 - az^{-1}}$	$ z > 0$

Today's agenda



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- Discrete time signals
 - Basic sequences and operations
 - Linear systems
 - Stability, causality, time invariance
- Linear time invariant (LTI) systems
 - Impulse response and convolution
 - Parallel and cascade system combination
- Fourier transform of LTI systems
 - Definition and conditions for existence
- Z-transform
 - Definition and region of convergence (ROC)
 - Right, left-sided and finite duration sequences
 - ROC analysis
- **Inverse z-transform**
 - **Definition and inspection method**
 - **Partial fraction expansion**
 - **Power series expansion**
- Transform analysis of LTI systems
 - Linear constant coefficient difference equations
 - Stability and causality
 - Inverse systems
 - FIR and IIR systems

Inverse z-transform

- Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n} \quad \begin{array}{l} z \in \mathcal{C} \\ ROC: \{z \mid X(z) < \infty\} \end{array}$$

- Inverse z-transform:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz,$$

where C is a closed circle around the origin which includes all the poles for $X(z)$.
In case C is the unit circle:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} (X(e^{i\omega}) \cdot e^{i\omega n}) d\omega$$

Inverse z-transform

- Z-transform

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Inverse z-transform

- How to calculate inverse z-transform?
 - Inspection method
 - Partial fraction expansion
 - Power series expansion

Inverse z-transform

- Less formal procedure than the Cauchy-Schwartz integral are preferable.
- **Inspection method:** "recognizing" certain transform pairs

$$x[n] = \underline{(a)^n \cdot u[n]} \quad -\infty < n < \infty \quad \xleftrightarrow{Z} \quad X(z)_h = \sum_{n=0}^{\infty} a^n \cdot z^{-n} = \frac{1}{1-a \cdot z^{-1}}; \quad \underline{|a| < |z|}$$

$$x[n] = \underline{-(a)^n \cdot u[-n-1]} \quad -\infty < n < \infty \quad \xleftrightarrow{Z} \quad X(z)_v = -\sum_{n=1}^{\infty} a^{-n} \cdot z^n = \frac{1}{1-a \cdot z^{-1}}; \quad \underline{|z| < |a|}$$

- Example

$$X(z) = \left(\frac{1}{1 - \frac{1}{2} z^{-1}} \right); \quad |z| > \frac{1}{2}$$



$$X[n] = \left(\frac{1}{2} \right)^n u[n]$$

Inverse z-transform

- Partial fraction expansion**

Sometimes inverse transforms cannot be found via simple inspection. However, it might be possible to obtain an alternative expression for $X(z)$ as a sum of simpler terms, each of which is tabulated.

$$X(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}} \quad \longrightarrow \quad X(z) = \frac{z^N \sum_{k=0}^M b_k z^{M-k}}{z^M \sum_{k=0}^N a_k z^{N-k}}$$

M zeros and N poles at non-zero locations in the z-plane

M-N poles at $z=0$ if $M > N$
or
N-M zeros at $z=0$ if $M < N$

$X(z)$ can be factorized as

$$X(z) = \frac{b_0 \prod_{k=1}^M (1 - c_k z^{-1})}{a_0 \prod_{k=1}^N (1 - d_k z^{-1})},$$

If $M < N$ and the poles are all first order, then

$$X(z) = \sum_{k=1}^N \frac{A_k}{1 - d_k z^{-1}}.$$

Coefficients can be found as $A_k = (1 - d_k z^{-1}) X(z) \big|_{z=d_k}.$

Inverse z-transform

- Example

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}, \quad |z| > \frac{1}{2}.$$



$$X(z) = \frac{A_1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1 - \frac{1}{2}z^{-1}\right)}.$$



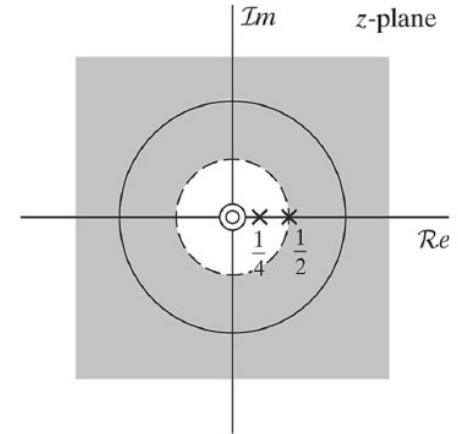
$$A_1 = \left(1 - \frac{1}{4}z^{-1}\right) X(z) \Big|_{z=1/4} = -1,$$

$$A_2 = \left(1 - \frac{1}{2}z^{-1}\right) X(z) \Big|_{z=1/2} = 2.$$

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)}.$$



$$x[n] = 2 \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n].$$



Inverse z-transform

- If $M \geq N$, the partial fraction expansion has the form

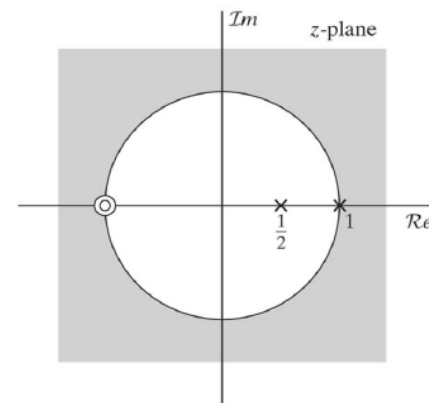
$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^N \frac{A_k}{1 - d_k z^{-1}}.$$

- The B_r coefficients can be found by long division of the numerator by the denominator, with the division process terminating when the remainder is of lower degree of the denominator.

Inverse z-transform

- Example

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{(1 + z^{-1})^2}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}, \quad |z| > 1.$$



$$X(z) = \underbrace{B_0}_{\text{circled}} + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}.$$

$$\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 \left[\frac{z^{-2} + 2z^{-1} + 1}{z^{-2} - 3z^{-1} + 2} \right] \xrightarrow{\text{blue arrow}} X(z) = 2 + \frac{-1 + 5z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}.$$

$$A_1 = \left[\left(\frac{-1 + 5z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})} \right) \left(1 - \frac{1}{2}z^{-1} \right) \right]_{z=1/2} = -9,$$

$$A_2 = \left[\left(\frac{-1 + 5z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})} \right) (1 - z^{-1}) \right]_{z=1} = 8.$$

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}.$$

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n].$$

Inverse z-transform

- The defining expression for the z-transform is a Laurent series where the sequence values $x[n]$ are the coefficients of z^{-n} .

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots, \end{aligned}$$

- Any value of the sequence can be then determined by finding the coefficient of the appropriate power of z^{-1}
- Example

$$X(z) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1} \quad \rightarrow \quad x[n] = \begin{cases} 1, & n = -2, \\ -\frac{1}{2}, & n = -1, \\ -1, & n = 0, \\ \frac{1}{2}, & n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1].$$

Inverse z-transform

- Transform by power series expansion (example 2)

$$X(z) = \log(1 + az^{-1}), \quad |z| > |a|.$$

$$\log(1 + x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots, \quad \Rightarrow \quad X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}.$$

$$x[n] = \begin{cases} (-1)^{n+1} \frac{a^n}{n}, & n \geq 1, \\ 0, & n \leq 0. \end{cases}$$