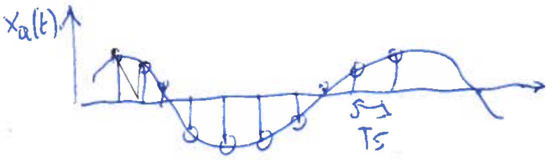
Discrete time signals

Discrete time signals can be represented as sequence of numbers

$$x = \{x[n]\} \quad -\infty < n < \infty$$

e.g. $x[-3] = 4,\ x[-2] = 3,\ x[-1] = 2.5,\ x[0] = 10,\ x[1] = 7, \ldots$

They can be generated by the sampling of an analog signal



$$x_n = x_a[n T_s]$$

sampling interval

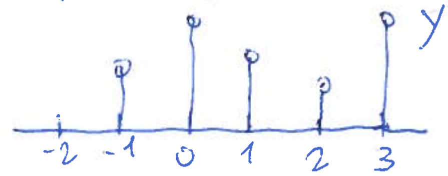
$f_s = \frac{1}{T_s}$, sampling frequency

Basic sequences

$y[n] = x[n - n_0]$ delayed version of $x[n]$

e.g.

$n_0 = 1$



$y[-1] = x[-1-1] = x[-2]$

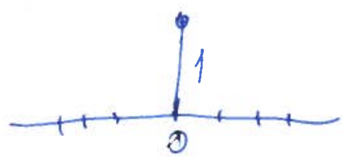
$y[0] = x[0-1] = x[-1]$

$\vdots$

$y[3] = x[3-1] = x[2]$

unit sample sequence

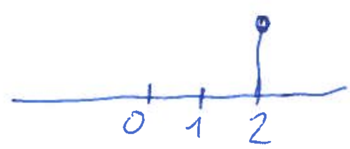
$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$



$$\delta[n - n_0] = \begin{cases} 0 & n - n_0 \neq 0 \rightarrow n \neq n_0 \\ 1 & n - n_0 = 0 \Rightarrow n = n_0 \end{cases}$$

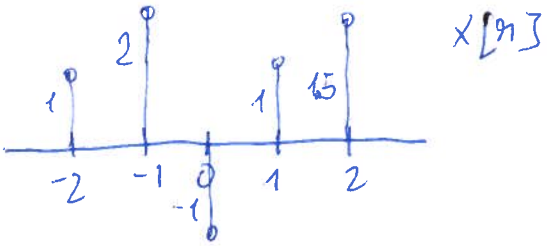
e.g. $n_0 = 2$

$\delta[n-2] \Rightarrow$



$\delta[n+2] \Rightarrow$

Why are unit sample sequences useful? An arbitrary sequence can be represented as a sum of scaled, delayed pulses

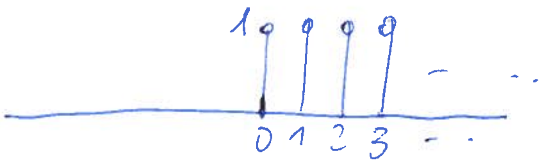


$$x[n] = 1\cdot\delta[n+2] + (2)\cdot\delta[n+1] + (-1)\,\delta[n] + (1)\,\delta[n-1] + (1.5)\cdot\delta[n-2]$$

In general, one can write

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]$$

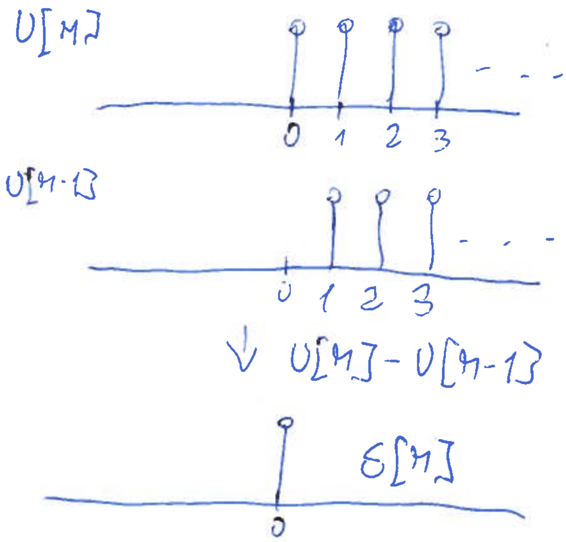
Unit step sequence



$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

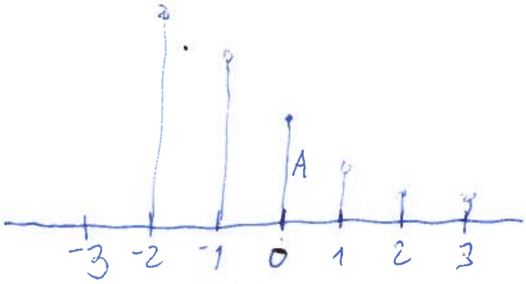
$$u[n] = \sum_{k=-\infty}^{n} \delta[k] \quad , \text{or} \quad \sum_{k=0}^{\infty} \delta[n-k]$$

$$\delta[n] = u[n] - u[n-1] \quad \Rightarrow$$



Exponential sequence

$$x[n] = A\alpha^{n}$$



The sequence decreases with n if $\alpha < 1$

Discrete time system

A discrete time system is an operator that maps an input sequence ~~to~~ $x[n]$ to an output sequence

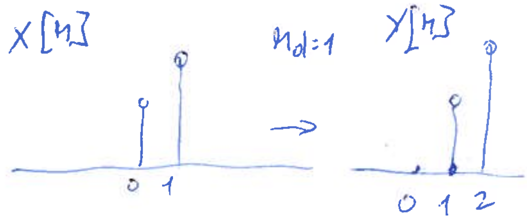
$$y[n] = T\{x[n]\}$$

$$x[n] \longrightarrow \boxed{T\{\cdot\}} \longrightarrow y[n]$$

examples

- delay operator

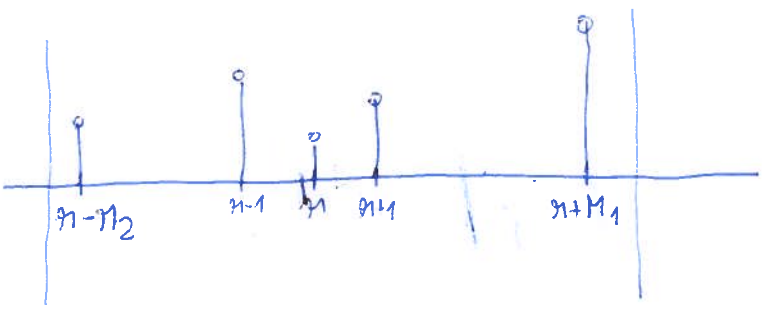
$$y[n] = x[n - n_d]$$



- moving average

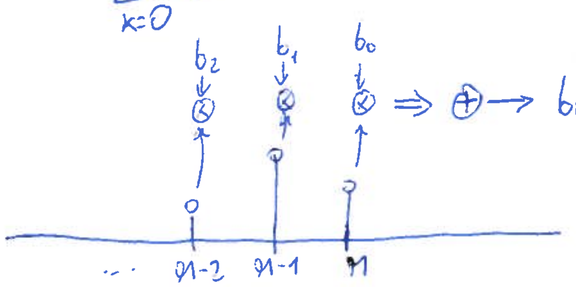
$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k] = \frac{1}{(M_1 + M_2 + 1)} \cdot \big(x[n+M_1] + x[n+M_1-1]$$

$$+ x[n+M_1-2] + \dots + x[n] + x[n-1] + \dots + x[n-M_2] \big)$$



- FIR filter

$$y[n] = \frac{1}{\sum_{k=0}^{M} b_k} \sum_{k=0}^{M} b_k x[n-k]$$



$\Rightarrow \oplus \rightarrow b_0 x[n] + b_1 \cancel{x}[n-1] + b_2 x[n-2]$

Linear system

When is it that a discrete time system is linear?

1) $T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} =$

$$= y_1[n] + y_2[n]$$

2) $T\{a\,x[n]\} = a\,T\{x[n]\}$

$$\Downarrow$$

$$T\{a x_1[n] + b x_2[n]\} = a\,T\{x_1[n]\} + b\,T\{x_2[n]\} = a\,y_1[n] + b\,y_2[n]$$

example: accumulator system $y[n] = \sum_{k=-\infty}^{n} x[k]$

Let us consider 2 sequences $x_1[n]$, $x_2[n]$

$$\Rightarrow y_1[n] = \sum_{k=-\infty}^{n} x_1[k], \quad y_2[n] = \sum_{k=-\infty}^{n} x_2[k]$$

Let us define $\tilde{x}[n] = a\,x_1[n] + b\,x_2[n]$

$$\Rightarrow \tilde{y}[n] = \sum_{k=-\infty}^{n} \tilde{x}[k] = \sum_{k=-\infty}^{n} (a x_1[k] + b x_2[k]) = \sum_{k=-\infty}^{n} a x_1[k] + \sum_{k=-\infty}^{n} b x_2[k] =$$

$$= a \sum_{k=-\infty}^{n} x_1[k] + b \sum_{k=-\infty}^{n} x_2[k] = a\,y_1[n] + b\,y_2[n]$$

V linear system

example

$$y[n] = \log_{10}(|x[n]|)$$

Let us define 2 sequences $x_1[n]$, $x_2[n]$

$$y_1[n] = \log_{10}(|x_1[n]|), \quad y_2[n] = \log_{10}(|x_2[n]|)$$

Let us define $\tilde{x}[n] = a x_1[n] + b x_2[n]$

$$\tilde{y}[n] = \log_{10}(|\tilde{x}[n]|) = \log_{10}(|a x_1[n] + b x_2[n]|) =$$

$$\neq a \log_{10}(|x_1[n]|) + b \log_{10}(|x_2[n]|)$$

non-linear system

Time invariant system

Shifting input sequence $\rightarrow$ shift in output sequence

$$x_1[n] = x[n-n_0] \longrightarrow y_1[n] = y[n-n_0]$$

example

accumulator $y[n] = \sum_{k=-\infty}^{n} x[k]$ is it time invariant?

$$y[n-n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$$

Let us check now the output of the system at the input sequence $x_1[n] = x[n-n_0]$

$$y_1[n] = \sum_{k=-\infty}^{n} x_1[k] = \sum_{k=-\infty}^{n} x[k-n_0] = \sum_{k_1=-\infty}^{n-n_0} x[k_1] = y[n-n_0]$$

$k_1 = k - n_0$

$\checkmark$ yes, it is time invariant!

example

Compressor system

$$y[n] = x[Mn]$$

it discards $(M-1)$ samples out of M

$$y[n-n_0] = x[M(n-n_0)] = x[Mn - Mn_0]$$

Let us check now the output of the system at the input sequence

$$x_1[n] = x[n-n_0]$$

$$y_1[n] = x_1[Mn] = x[Mn-n_0] \neq x[Mn - Mn_0]$$

non time-invariant system!

Causal system

for every choice of n_0, the output sequence at $n = n_0$ depends only on the input sequence values for $n \leq n_0$

Stable system

Every bounded input sequence produces a bounded output sequence

$$|x[n]| \leq B_x < \infty \;\; \forall n \Rightarrow |y[n]| \leq B_y < \infty \;\; \forall n$$

example: is accumulator stable? Let us consider as input $u[n]$

$$y[n] = \sum_{k=-\infty}^{n} u[k]$$

The input is bounded $u[n] \leq 1 \;\; \forall n$

$$y[n] = \sum_{k=-\infty}^{n} u[k] = \begin{cases} 0 & n < 0 \\ n+1 & n \geq 0 \end{cases}$$

There is no finite choice for B_y such that $(n+1) \leq B_y < \infty \;\; \forall n$

$\Rightarrow$ system unstable

example

$$y[n] = \log_{10}(|x[n]|)$$

is it stable?

For any values of the time index n such that $x[n]=0$

$\rightarrow y[n] = -\infty$

bounded input, unbounded output $\rightarrow$ unstable system