DISCRETE TIME SYSTEMS AND Z-TRANSFORM

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What we have learned in module 2

- The Discrete Time Fourier transform of a sequence x[n] is defined as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

ω is the radian frequency

- The Fourier transform determines how much of each frequency component is required to synthetize x[n].

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Inverse Fourier transform

- If x[n] is absolutely summable, then the Fourier transform exists.

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty.$$

Today’s agenda

- Discrete time signals
  - Basic sequences and operations
  - Linear systems
  - Stability, causality, time invariance
- Linear time invariant (LTI) systems
  - Inpulse response and convolution
  - Parallel and cascade system combination
- Fourier transform of LTI systems
  - Definition and conditions for existence
- **Z-transform**
  - **Definition and region of convergence (ROC)**
  - **Right, left-sided and finite duration sequences**
  - **ROC analysis**
- Inverse z-transform
  - Definition and inspection method
  - Partial fraction expansion
  - Power series expansion
- Transform analysis of LTI systems
  - Linear constant coefficient difference equations
  - Stability and causality
  - Inverse systems
  - FIR and IIR systems

The z-transform

- The z-transform of a sequence x[n] is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.$$

- The z-transform operator transforms the sequence x[n] into the function X[z], where z is a continuous complex variable.
- The z-transform reduces to the Fourier transform if $z = e^{j\omega}$
- More generally, the complex variable z can be expressed as $z = re^{j\omega}$.

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n},$$

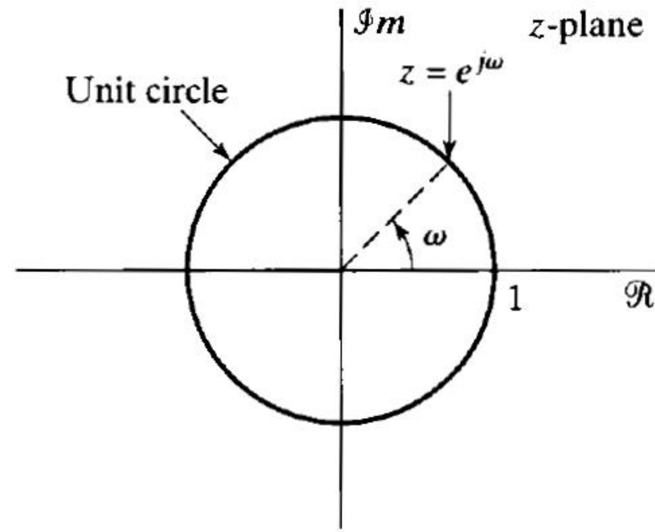
$$\Downarrow$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}.$$

The z-transform

- The z-transform evaluated in the unit circle corresponds to the Fourier transform.

- When it exists, the Fourier transform is simply X(z) with $z = e^{j\omega}$

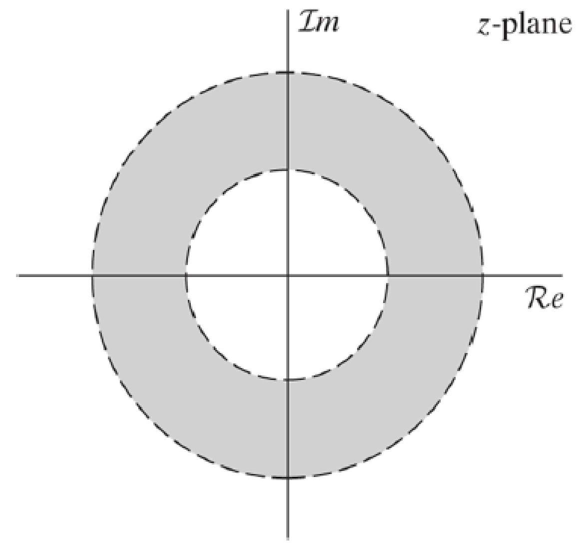


The z-transform

- The z-transform does not converge for all sequences or all values of *z.*
- The set of values of z for which the z-tranform converges is called the region of convergence (ROC).

$$\sum_{n=-\infty}^{\infty} |x[n]| \cdot |z|^{-n} < \infty$$

The ROC is a ring in the *z*-plane



The z-transform

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- There is possibility that the z-transform converges even if Fourier transform diverges

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot r^{-n} \cdot e^{-j\omega n}$$

$\sum_{n=-\infty}^{\infty} |x[n] \cdot r^{-n}|$ converges even if $\sum_{n=-\infty}^{\infty} |x[n]|$ diverges

The z-transform

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- The z-trasform is most useful when the infinite sum can be expressed in closed form.
- Among the most important and useful z-transforms, are those for which X(z) is a rational function inside the ROC, i.e.

$$X(z) = \frac{P(z)}{Q(z)},$$

where P(z) and Q(z) are polynomials in z.

- The values of z for which X(z)=0 are called the zeros of X(z), while the values of z for which Q(z)=0 are the poles of X(z).

The z-transform

- Right-sided exponential sequence

What is the ROC for the z-transform of $x[n] = \left(\frac{1}{2}\right)^n \cdot u[n]$, $-\infty < n < \infty$?

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$$

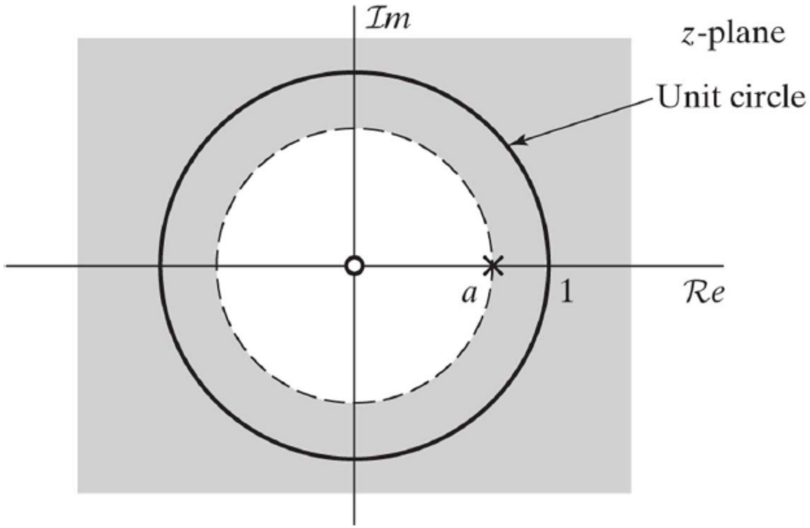
For convergence of $X(z)$, we require that

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty.$$

Thus, the ROC is the range of values of z for which $|az^{-1}| < 1$ or, equivalently, $|z| > |a|$. Inside the ROC, the infinite series converges to

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| > |a|. \tag{12}$$

The z-transform



$$x[n] = \left(\frac{1}{2}\right)^n \cdot u[n]$$

$$a = |a| = \left(\frac{1}{2}\right)$$

- For $|a| > 1$, the ROC does not contain the unit circle → Fourier transform does not exist

The z-transform

- Left-sided exponential sequence

What is the ROC for the z-transform of $x[n] = -\left(\frac{3}{2}\right)^n \cdot u[-n-1]$? $-\infty < n < \infty$?

$$x[n] = -a^n u[-n-1] = \begin{cases} -a^n & n \le -1 \\ 0 & n > -1. \end{cases}$$

Since the sequence is nonzero only for $n \le -1$, this is a *left-sided* sequence. The z-transform in this case is

$$\begin{aligned} X(z) &= -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} \\ &= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n. \end{aligned} \tag{15}$$

If $|a^{-1}z| < 1$ or, equivalently, $|z| < |a|$, the last sum in Eq. (15) converges, and using again the formula for the sum of terms in a geometric series,

$$X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| < |a|. \tag{16}$$

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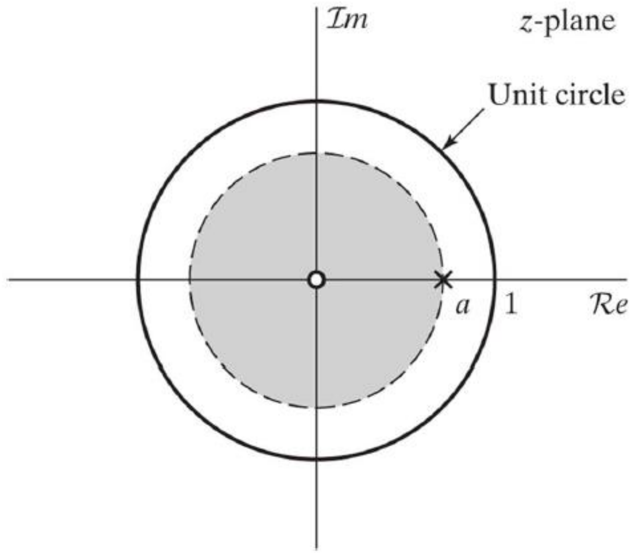
The z-transform



$$x[n] = -\left(\frac{3}{2}\right)^n \cdot u[-n-1]$$

$$a = \left(\frac{3}{2}\right)$$

- For $|a| < 1$, the sequence grows exponentially and therefore the Fourier transform does not exist.

The z-transform

- Sum of two exponential sequences

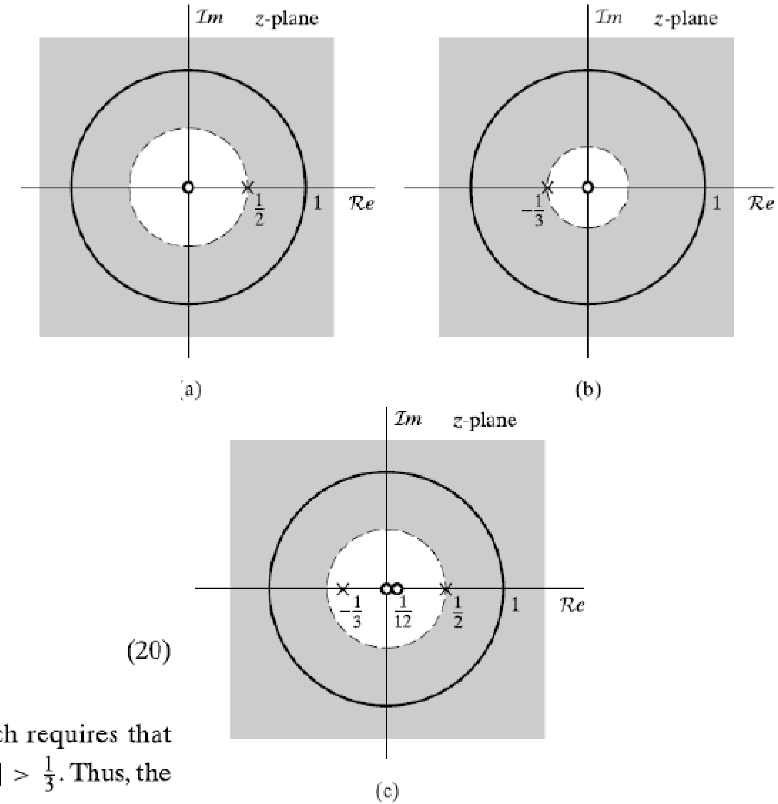
Consider a signal that is the sum of two real exponentials:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n].$$

The z-transform is

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \right\} z^{-n} \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right)^n u[n] z^{-n} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n \\
&= \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}} = \frac{2\left(1 - \frac{1}{12} z^{-1}\right)}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 + \frac{1}{3} z^{-1}\right)} \\
&= \frac{2z\left(z - \frac{1}{12}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)}.
\end{aligned} \tag{20}$$

For convergence of $X(z)$, both sums in Eq. (19) must converge, which requires that both $\left|\frac{1}{2} z^{-1}\right| < 1$ and $\left|\left(-\frac{1}{3}\right) z^{-1}\right| < 1$ or, equivalently, $|z| > \frac{1}{2}$ and $|z| > \frac{1}{3}$. Thus, the ROC is the region of overlap, $|z| > \frac{1}{2}$. The pole–zero plot and ROC for the z-transform of each of the individual terms and for the combined signal are shown in Figure 5.



The z-transform

- Two-sided exponential sequences

Consider the sequence

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]. \qquad (24)$$

Note that this sequence grows exponentially as $n \to -\infty$. Using the general result of Example 1 with $a = -\frac{1}{3}$, we obtain

$$\left(-\frac{1}{3}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{3},$$

and using the result of Example 2 with $a = \frac{1}{2}$ yields

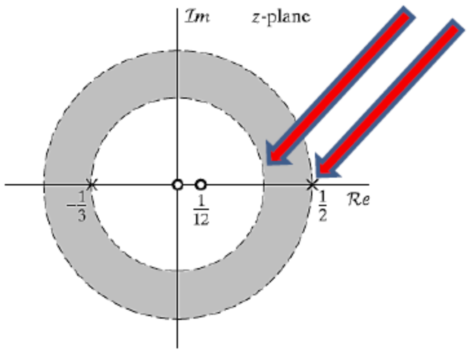
$$-\left(\frac{1}{2}\right)^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| < \frac{1}{2}.$$

Thus, by the linearity of the z-transform,

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad \frac{1}{3} < |z| \text{ and } |z| < \frac{1}{2},$$

$$= \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}.$$

Since the ROC does not contain the unit circle, the sequence in Eq. (24) does not have a Fourier transform.



The z-transform

- What is the z-transform of $x[n] = \begin{cases} 0{,}8^n, & for\ 0 \ll n \ll N-1 \\ 0, & ellers \end{cases}$??

$$\sum_{n=0}^{\infty}(az^{-1})^n - \sum_{n=N}^{\infty}(az^{-1})^n$$

$$\sum_{n=0}^{\infty}(az^{-1})^n - (az^{-1})^N \cdot \sum_{n=0}^{\infty}(az^{-1})^n$$

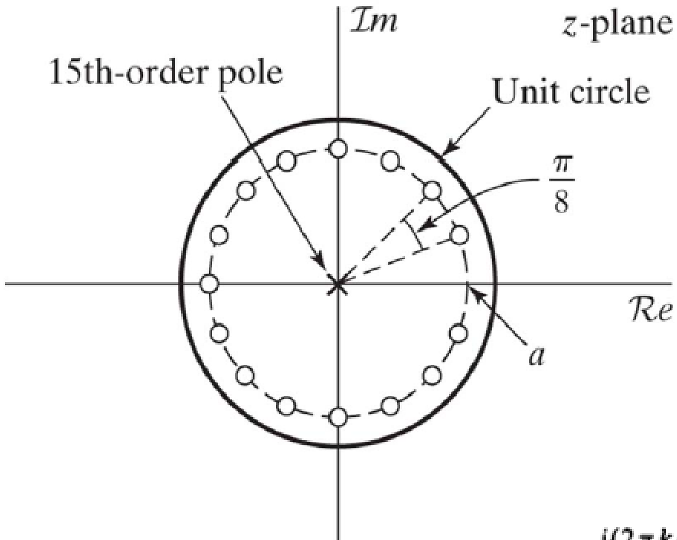
$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1}(az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}}\frac{z^N - a^N}{z-a}, \tag{3.23}$$

where we have used the general formula in Eq. (2.56) to sum the finite series. The ROC is determined by the set of values of z for which

$$\sum_{n=0}^{N-1}|az^{-1}|^n < \infty.$$

The z-transform



Pole–zero plot
N = 16 and a = 0,8

The ROC in this example consists of all values of z except z = 0.

$$z_k = ae^{j(2\pi k/N)}, \qquad k = 1, \ldots, N-1.$$

The z-transform

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TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m>0$) or ∞ (if $m<0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

The z-transform

PROPERTY 1: The ROC will either be of the form $0 \le r_R < |z|$, or $|z| < r_L \le \infty$, or, in general the annulus, i.e., $0 \le r_R < |z| < r_L \le \infty$.

PROPERTY 2: The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.

PROPERTY 3: The ROC cannot contain any poles.

PROPERTY 4: If $x[n]$ is a *finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \le n \le N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

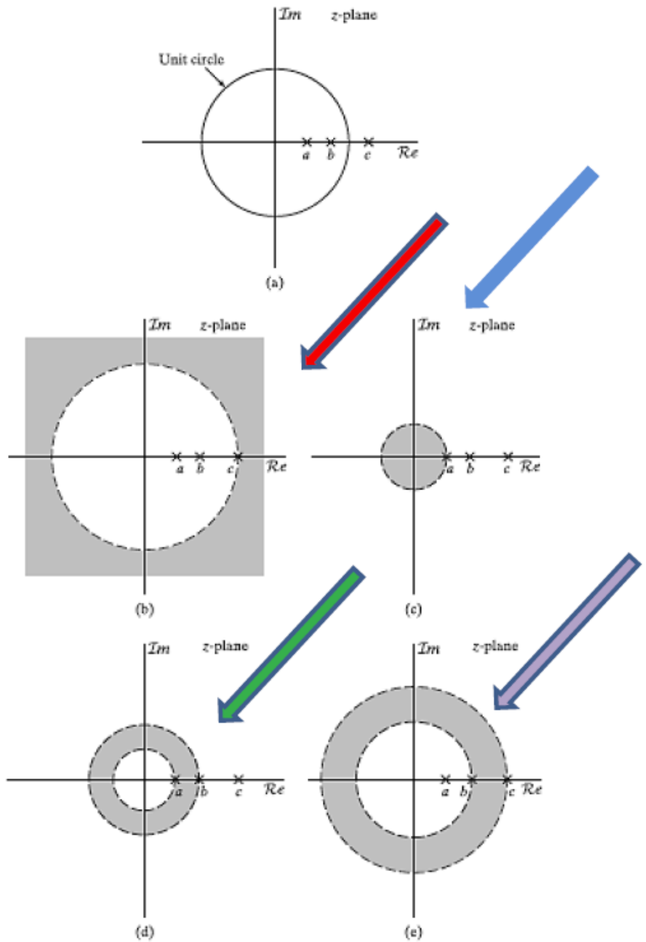
PROPERTY 5: If $x[n]$ is a *right-sided sequence*, i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

PROPERTY 6: If $x[n]$ is a *left-sided sequence*, i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

PROPERTY 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with Property 3, not containing any poles.

PROPERTY 8: The ROC must be a connected region.

The z-transform



Examples of four z-transforms with the same pole–zero locations, illustrating the different possibilities for the ROC, each of which corresponds to a different sequence:

(b) to a right-sided sequence,

(c) to a left-sided sequence,

(d) to a two-sided sequence, and

(e) to a two-sided sequence.

The z-transform

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TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n - n_0]$	$z^{-n_0} X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\| R_x$
4	3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

The z-transform

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Convolution in time ←→ multiplication in frequency domain

$$y[n] = \sum_{k=-\infty}^{\infty} x_1[k] \cdot x_2[n-k] \quad eller \quad y[n] = \sum_{k=-\infty}^{\infty} x_2[k] \cdot x_1[n-k] \quad for\ alle\ n$$

(Foldning er kommutativ)

$$y[n] = (x_1 * x_2)[n] = \sum_{k=-\infty}^{\infty} x_1[k] \cdot x_2[n-k]$$

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n] \cdot z^{-n} = \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k] \right\} z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]\, z^{-n}$$

Ved substitution: $m = n-k$,

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1[k] \left\{ \sum_{m=-\infty}^{\infty} x_2[m] \cdot z^{-m} \right\} z^{-k}$$

$$= X_1(z) \cdot X_2(z)$$

Altså:

$$x_1[n] * x_2[n] \overset{Z}{\longleftrightarrow} X_1(z) \cdot X_2(z)$$