DISCRETE TIME SYSTEMS AND Z-TRANSFORM

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Plan for the course

- Discrete time signals
  - Basic sequences and operations
  - Linear systems
  - Stability, causality, time invariance
- Linear time invariant (LTI) systems
  - Inpulse response and convolution
  - Parallel and cascade system combination
- Fourier transform of LTI systems
  - Definition and conditions for existence
- Z-transform
  - Definition and region of convergence (ROC)
  - Right, left-sided and finite duration sequences
  - ROC analysis
- Inverse z-transform
  - Definition and inspection method
  - Partial fraction expansion
  - Power series expansion
- Transform analysis of LTI systems
  - Linear constant coefficient difference equations
  - Stability and causality
  - Inverse systems
  - FIR and IIR systems

Literature

Alan V. Oppenheim - Ronald W. Schafer:
Discrete-Time Signal Processing
Pearson 2014
Second or Third Edition
ISBN 10: 1-292-02572-7
ISBN 13: 978-1-292-02572-8

Lecture format

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- Lecture (approx.1h45m including break) + exercise session
- Slides + large usage of blackboard
- No pre-scheduled breaks
  - Breaks are ”distributed” according to the complexity of the presented topics
- Please let me know if I erase the blackboard too quickly!

Today’s agenda

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- **Discrete time signals**
  - **Basic sequences and operations**
  - **Linear systems**
  - **Stability, causality, time invariance**
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  - Inpulse response and convolution
  - Parallel and cascade system combination
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  - FIR and IIR systems

Discrete time signals - Sequences

- Discrete signals can be represented as a sequence of numbers

$$x = \{x[n]\} \qquad -\infty < n < \infty$$

where n is an integer.

- In case such sequences arise from periodic sampling of an analog signal:

$$x[n] = x_a[nT_s] \qquad -\infty < n < \infty$$

where Ts is the *sampling interval* and $f_s = 1/T_s$ is the *sampling frequency*.

Discrete time signals - Sequences

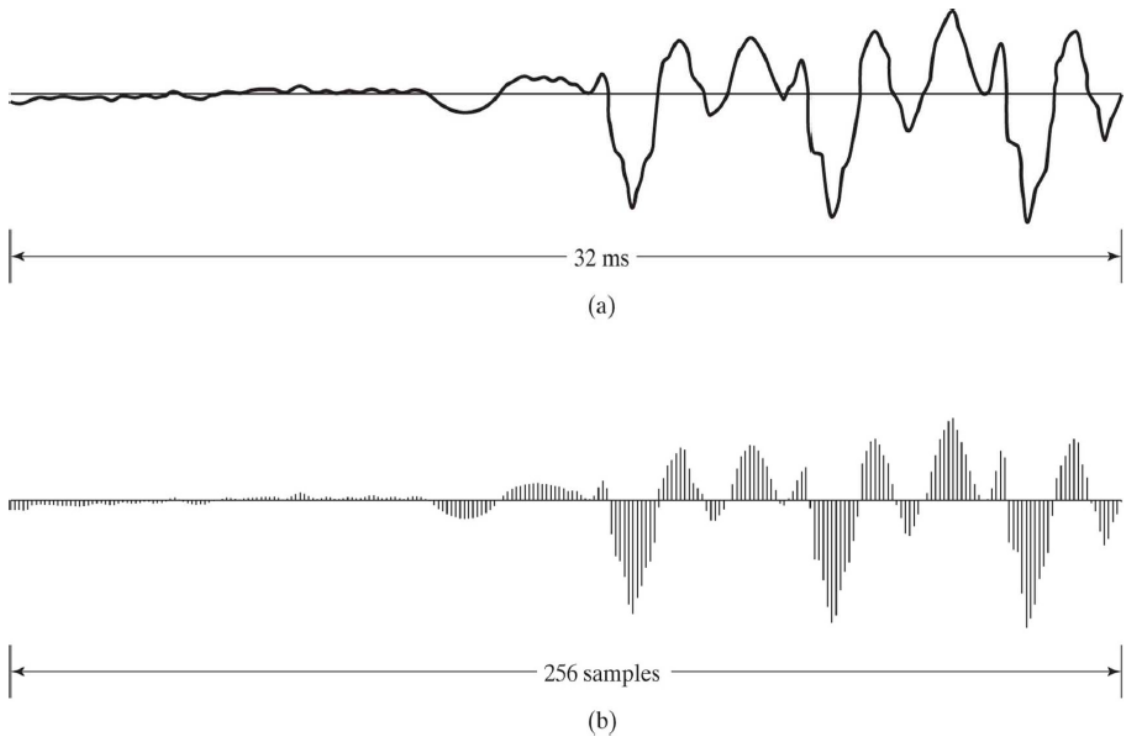


Fig 2 (13) (a) Segment of a continuous-time speech signal $x_a(t)$.
(b) Sequence of samples $x[n] = x_a(nT_s)$ obtained from the signal in part (a) with T_s $= 125\ \mu s$.

Discrete time signals - Sequences

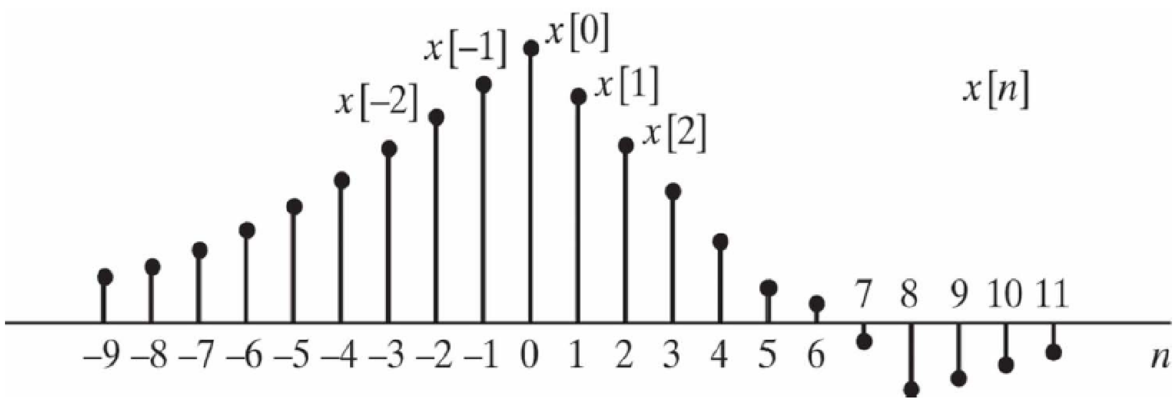


Fig 1 (13) Graphic representation of a discrete-time signal.

Basic sequences and operations

- y[n] is said to be a delayed (or shifted) version of the sequence x[n] if

  $y[n]=x[n-n_0]$, with n_0 integer

- The unit sample sequence is defined as

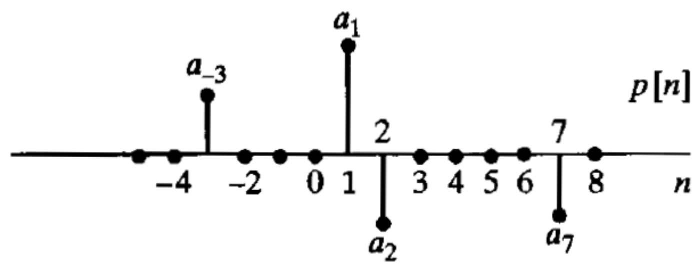
$$\delta[n] = \begin{cases} 0, & n \neq 0, \\ 1, & n = 0. \end{cases}$$

- An example of delayed unit sample sequence:

$$\delta[n-2] = \begin{cases} 0, & n \neq 2 \\ 1, & n = 2 \end{cases}$$

Basic sequences and operations

- An arbitrary sequence can be represented as a sum of scaled, delayed, impulses.



$$p[n] = a_{-3}\delta[n+3] + a_1\delta[n-1] + a_2\delta[n-2] + a_7\delta[n-7].$$

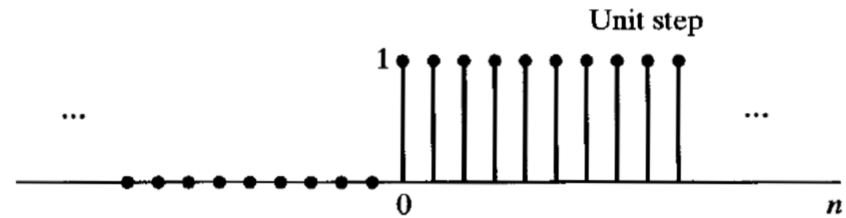
- More generally, any sequence can be expressed as:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

Basic sequences and operations

- The unit step sequence is defined as

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases}$$



- The unit step is related to the impulse by $u[n] = \sum_{k=-\infty}^{n} \delta[k];$ or $u[n] = \sum_{k=0}^{\infty} \delta[n-k].$

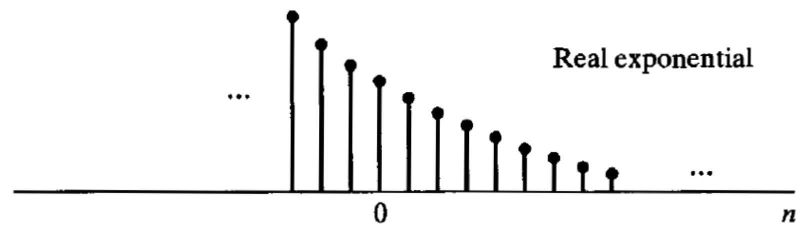
- Conversely, the impulse sequence can be expressed as the first backward difference of the unit step sequence:

$$\delta[n] = u[n] - u[n-1].$$

Basic sequences and operations

- The general form of an exponential sequence is

$$x[n] = A \cdot \alpha^n$$



- In which condition is the sequence decreasing with *n* ?

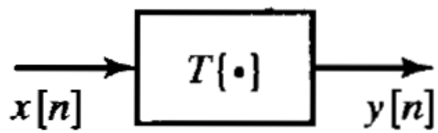
Discrete time systems

- Definition
- Properties
  - Linearity
  - Time invariance
  - Causality
  - Stability

Linear discrete time systems

- A discrete-time system is an operator that maps an input sequence x[n] to an output sequence y[n]

$$y[n] = T\{x[n]\}$$



- Examples of operators:

  - Delay $$y[n] = x[n - n_{delay}] \quad -\infty < n < \infty$$

  - Moving average $$y[n] = \frac{1}{M1 + M2 + 1} \sum_{k=-M1}^{M2} x[n-k]$$

  - FIR filter $$y[n] = \frac{1}{\sum_{k=0}^{M} b_k} \sum_{k=0}^{M} b_k \cdot x[n-k]$$

Linear discrete time systems

- The class of linear system is defined by the principle of superposition.

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n]$$
$$T\{a \cdot x[n]\} = a \cdot T\{x[n]\} = a \cdot y[n]$$

$$\Downarrow$$

$$T\{a \cdot x_1[n] + b \cdot x_2[n]\} = a \cdot T\{x_1[n]\} + b \cdot T\{x_2[n]\} = a \cdot y_1[n] + b \cdot y_2[n]$$

Linear discrete time systems

- The accumulator system

$$y[n] = \sum_{k=-\infty}^{n} x[k] \quad \Rightarrow \quad \text{Linear system}$$

- Consider the following

$$w[n] = \log_{10}(|x[n]|). \quad \Rightarrow \quad \text{Non-Linear system}$$

Time invariant discrete systems

- A time invariant system is a system for which a time delay/shift of the input sequence causes a corresponding shift in the output sequence.

$$x_1[n] = x[n - n_0] \Rightarrow y_1[n] = y[n - n_0].$$

- The accumulator is a time invariant system.
- Compressor is a non-time invariant system

$$y[n] = x[Mn], \qquad -\infty < n < \infty,$$

Causal discrete time systems

- A system is causal if, for every choice of n_0, the output sequence at the index $n=n_0$ depends only on the input sequence values for $n<=n_0$.

- Forward difference system

$$y[n] = x[n+1] - x[n].$$

→ Non- causal

- Backward difference system

$$y[n] = x[n] - x[n-1],$$

→ Causal

Stable discrete time systems

- A system is stable if and only if every bounded input sequence produces a bounded output sequence.

$$|x[n]| \le B_x < \infty, \qquad \text{for all } n. \quad \Rightarrow \quad |y[n]| \le B_y < \infty, \qquad \text{for all } n.$$

- Examples

$$y[n] = \sum_{k=-\infty}^{n} u[k] \quad \Rightarrow$$ Not stable

$$y[n] = x[n - n_d], \quad \Rightarrow$$ Stable

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k] \quad \Rightarrow$$ Stable

What about $y[n] = \log_{10}(|x[n]|)$?