DISCRETE TIME SYSTEMS AND Z-TRANSFORM

Gilberto Berardinelli

Department of Electronic Systems, Aalborg University, Denmark

AALBORG UNIVERSITY
DENMARK

What we have learned in module 4

- Inverse z-transform

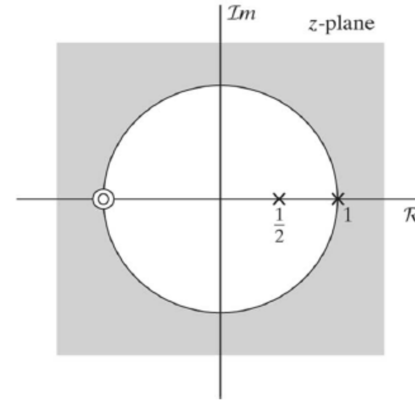
$$X(z) \Rightarrow x[n]$$

- How to calculate it?
  - Inspection method
  - Partial fraction expansion
  - Power series expansion

What we have learned in Module 4

- Example

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{(1 + z^{-1})^2}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}, \qquad |z| > 1.$$



$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}.$$

$$\begin{array}{r|l} & \quad 2 \\ \hline \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & z^{-2} + 2z^{-1} + 1 \\ & \underline{z^{-2} - 3z^{-1} + 2} \\ & \quad 5z^{-1} - 1 \end{array}$$

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}.$$

$$A_1 = \left[\left(2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}\right)\left(1 - \frac{1}{2}z^{-1}\right)\right]_{z=1/2} = -9,$$

$$A_2 = \left[\left(2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}\right)\left(1 - z^{-1}\right)\right]_{z=1} = 8.$$

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}.$$

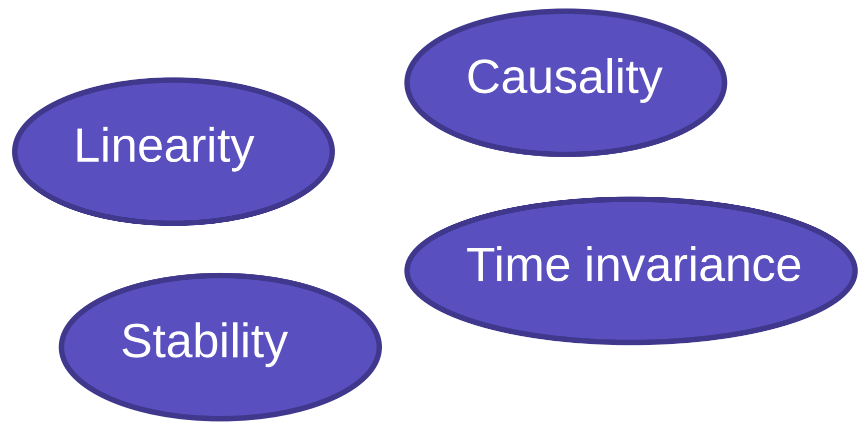
$$x[n] = 2\delta[n] - 9\left(\tfrac{1}{2}\right)^n u[n] + 8u[n].$$

What we have learned in module 1

- Linear discrete time system

$$
\begin{aligned}
T\{x_1[n] + x_2[n]\} &= T\{x_1[n]\} + T\{x_2[n]\} &&= y_1[n] + y_2[n] \\
T\{a \cdot x[n]\} &= a \cdot T\{x[n]\} &&= a \cdot y[n]
\end{aligned}
$$

- Other discrete system properties
  - Causality
  - Time invariance
  - Stability



Today's agenda

- Discrete time signals
  - Basic sequences and operations
  - Linear systems
  - Stability, causality, time invariance
- Linear time invariant (LTI) systems
  - Inpulse response and convolution
  - Parallel and cascade system combination
- Fourier transform of LTI systems
  - Definition and conditions for existence
- Z-transform
  - Definition and region of convergence (ROC)
  - Right, left-sided and finite duration sequences
  - ROC analysis
- Inverse z-transform
  - Definition and inspection method
  - Partial fraction expansion
  - Power series expansion
- **Transform analysis of LTI systems**
  - **Linear constant coefficient difference equations**
  - **Stability and causality**
  - **Inverse systems**
  - **FIR and IIR systems**

Stability and causality

- System is causal → h[n] must be a right handed sequence, and therefore the region of convergence of H(z) must be outside the outermost pole

- System is stable → impulse response absolutely summable

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty.$$

→ ROC of H[z] include the unit circle

Linear constant coefficient difference equations

- An important class of LTI systems takes the following form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m].$$

or $$y[n] = \frac{1}{a_0}\left(\sum_{m=0}^{M} b_m \cdot x[n-m] - \sum_{k=1}^{N} a_k \cdot y[n-k]\right)$$

- Applying z-tranform and linearity and time shift property, we obtain

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z),$$

$$\Downarrow$$

$$\left(\sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z).$$

Linear constant coefficient difference equations

$$\left(\sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z). \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$

- It is convenient to express the former in factored form:

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}.$$

- Important: a linear coefficient difference equation does not provide an <u>unique</u> specification of the output for a given input → for a given H(z), each choice of ROC leads to a different impulse response, but they will all correspond to the same difference equation.

Linear constant coefficient difference equations

Example

Consider the LTI system with input and output related through the difference equation

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n]. \tag{5.25}$$

From the previous discussions, $H(z)$ is given by

$$H(z) = \frac{1}{1 - \frac{5}{2}z^{-1} + z^{-2}} = \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - 2z^{-1}\right)}. \tag{5.26}$$

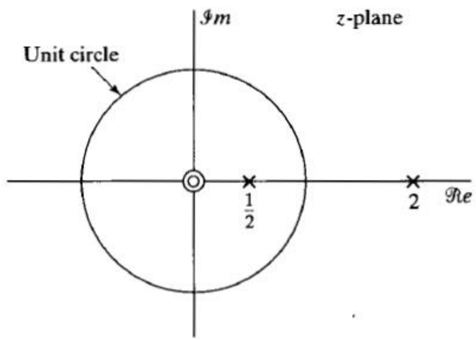


Figure 5.4 Pole–zero plot for Example 5.3.

The pole–zero plot for $H(z)$ is indicated in Figure 5.4. There are three possible choices for the ROC. If the system is assumed to be causal, then the ROC is outside the outermost pole, i.e., $|z| > 2$. In this case the system will not be stable, since the ROC does not include the unit circle. If we assume that the system is stable, then the ROC will be $\frac{1}{2} < |z| < 2$. For the third possible choice of ROC, $|z| < \frac{1}{2}$, the system will be neither stable nor causal.

Design of LTI systems

Linear constant coefficient difference equation

Impulse response
h(n)

Transfer function
H(z)

Implementation

Requirements & Analysis

Inverse systems

inverse

$$G(z) = H(z)H_i(z) = 1. \Rightarrow H_i(z) = \frac{1}{H(z)}.$$

- Equivalent time domain condition $$g[n] = h[n] * h_i[n] = \delta[n].$$
- Frequency response of an inverse system $$H_i(e^{j\omega}) = \frac{1}{H(e^{j\omega})};$$

Inverse systems

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}, \quad \Rightarrow \quad H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{k=1}^{M}(1 - c_k z^{-1})};$$

- The poles of Hi(z) are the zeros of H(z) and viceversa.
- ROCs of Hi(z) and H(z) must overlap.
- If H(z) is causal, its region of convergence is $|z| > \max_k |d_k|.$
- Any appropriate region of convergence that overlaps with the region specified above is a valid region of convergence for Hi(z).

Inverse systems

- Example

Let $H(z)$ be

$$H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}}$$

with ROC $|z| > 0.9$. Then $H_i(z)$ is

$$H_i(z) = \frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}}.$$

Since $H_i(z)$ has only one pole, there are only two possibilities for its ROC, and the only choice for the ROC of $H_i(z)$ that overlaps with $|z| > 0.9$ is $|z| > 0.5$. Therefore, the impulse response of the inverse system is

$$h_i[n] = (0.5)^n u[n] - 0.9(0.5)^{n-1} u[n-1].$$

In this case, the inverse system is both causal and stable.

Impulse response for rational system functions

- Any rational function of z^{-1} with only first order poles can be expressed as

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}},$$

where the elements in the first summation are only present if M>N.

- If the system is assumed to be causal, then the ROC is outside all the poles in the equation above, and it follows that

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n],$$

- If there is at least one term in the second sum, then h[n] will not be of finite length, i.e., it will not be zero outside of a finite interval → **infinite impulse response (IIR) system**

Impulse response for rational system functions

AALBORG UNIVERSITY
DENMARK

Example 5.6 A First-Order IIR System

Consider a causal system whose input and output satisfy the difference equation

$$y[n] - ay[n-1] = x[n]. \tag{5.36}$$

The system function is (by inspection)

$$H(z) = \frac{1}{1 - az^{-1}}. \tag{5.37}$$

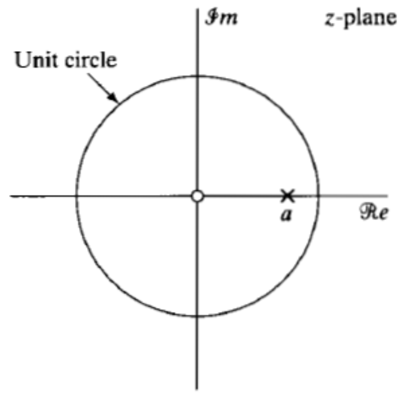


Figure 5.5 Pole–zero plot for Example 5.6.

Figure 5.5 shows the pole–zero plot of $H(z)$. The region of convergence is $|z| > |a|$, and the condition for stability is $|a| < 1$. The inverse z-transform of $H(z)$ is

$$h[n] = a^n u[n]. \tag{5.38}$$

Impulse response for rational system functions

- If H(z) has no poles except for z=0, a partial fraction expansion is not possible

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}.$$

$$\Downarrow$$

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_n, & 0 \le n \le M, \\ 0, & \text{otherwise.} \end{cases}$$

- In this case, the impulse response is finite in length, i.e. it is zero outside a finite interval → finite impulse response (FIR) system

Impulse response for rational system functions

- Example

Example 5.7 A Simple FIR System

Consider an impulse response that is a truncation of the impulse response of Example 5.6:

$$h[n] = \begin{cases} a^n, & 0 \le n \le M, \\ 0 & \text{otherwise.} \end{cases}$$

Then the system function is

$$H(z) = \sum_{n=0}^{M} a^n z^{-n} = \frac{1 - a^{M+1} z^{-M-1}}{1 - a z^{-1}}. \tag{5.42}$$

Since the zeros of the numerator are at

$$z_k = a e^{j2\pi k/(M+1)}, \qquad k = 0, 1, \ldots, M, \tag{5.43}$$

where a is assumed real and positive, the pole at $z = a$ is canceled by a zero. The pole–zero plot for the case $M = 7$ is shown in Figure 5.6.

Impulse response for rational system functions

- Example (continuation)

The difference equation satisfied by the input and output of the linear time-invariant system is the discrete convolution

$$y[n] = \sum_{k=0}^{M} a^k x[n-k]. \tag{5.44}$$

However, Eq. (5.42) suggests that the input and output also satisfy the difference equation

$$y[n] - ay[n-1] = x[n] - a^{M+1}x[n-M-1]. \tag{5.45}$$

These two equivalent difference equations result from the two equivalent forms of $H(z)$ in Eq. (5.42).

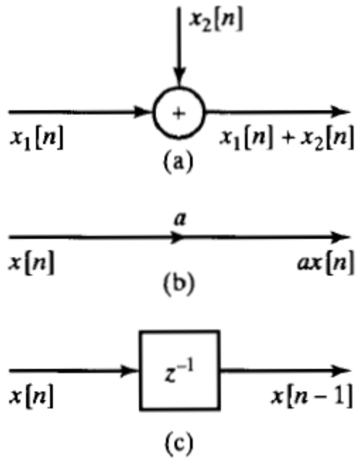
Block structures

- When linear time-invariant systems are implemented in hardware, the difference equation or the system function must be translated to an algorithm that can be implmented in the desired technology.

- The implementation of a linear time-invariant system by iteratively evaluating a recurrent formula obtained froma difference equation requires that delayed values of input, output and itnermediate sequences are available.
- Also, there is need for storage of past sequence values.
- We must also provide means for multiplication of the delayed sequence values by the coefficients, and means for adding the resulting products.

Block structures

- Block diagram symbols

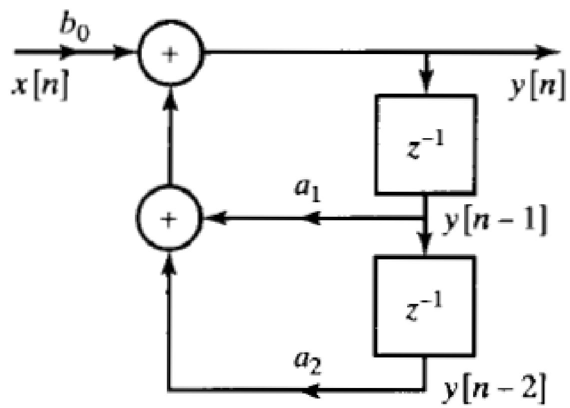


Block structures

- Example

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n]. \quad \Rightarrow \quad H(z) = \frac{b_0}{1 - a_1 z^{-1} - a_2 z^{-2}}.$$

Block diagram representation
of a difference equation

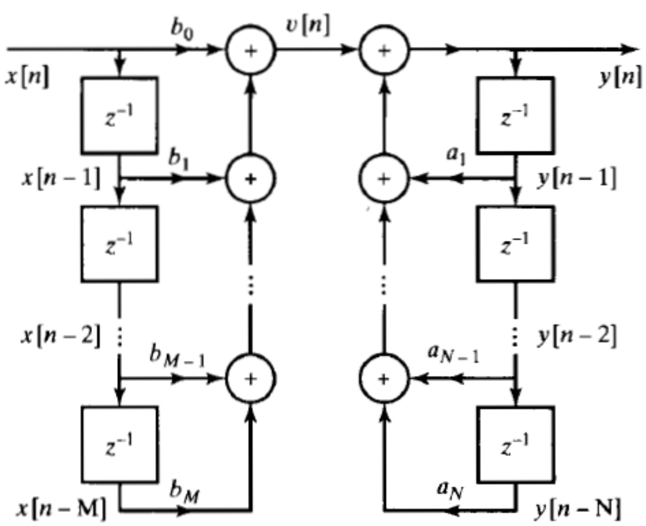


Block structures

- The example can be generalized to the form

$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \quad \Rightarrow \quad y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k].$$

- Block diagram



Direct form I implementation → direct realization of the difference equation

$$v[n] = \sum_{k=0}^{M} b_k x[n-k],$$

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + v[n].$$