VEKTOR DI RUANG DIMENSI TIGA

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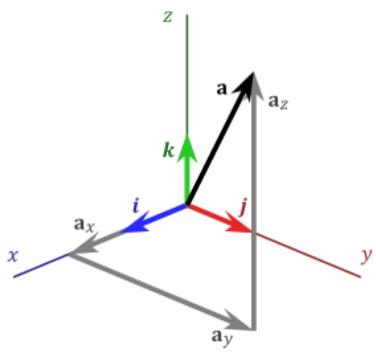
Vektor adalah besaran yang mempunyai panjang dan arah. Jika tanpa arah, maka disebut skalar.
Vektor dapat berada di ruang dimensi dua (R^2), dimensi tiga (R^3), atau dimensi n (R^n).
Mengapa tidak ada vektor di ruang dimensi satu/nol ?

VEKTOR SATUAN

Selanjutnya kita fokus ke vektor di ruang dimensi tiga.

$$\text{Sebarang vektor } \bar{a} \text{ di } R^3 \text{ dinyatakan sebagai : } \bar{a} = a_x i + a_y j + a_z k$$

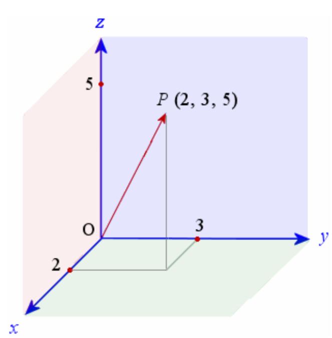
$$\begin{aligned} \text{dimana : } i &= (1, 0, 0) = \text{vektor satuan searah sumbu x} \\ j &= (0, 1, 0) = \text{vektor satuan searah sumbu y} \\ k &= (0, 0, 1) = \text{vektor satuan searah sumbu z} \end{aligned}$$



$$\text{Contoh : vektor } \overline{OP} = (2, 3, 5)$$

$\overline{OP}$ artinya titik pangkalnya di titik O (Origin) dan titik ujungnya di titik P.

$(2, 3, 5) = 2i + 3j + 5k$ artinya : 2 satuan kearah x, 3 satuan kearah y, dan 5 satuan kearah z.



PANJANG VEKTOR (NORM) YANG MELALUI TITIK ORIGIN

$$P = (p_x, p_y, p_z)$$

$$O = (0, 0, 0)$$

$$\overline{OP} = P - O = (p_x - 0, p_y - 0, p_z - 0)$$

$$\boxed{\left\| \overline{OP} \right\| = \sqrt{(p_x - 0)^2 + ((p_y - 0)^2 + (p_z - 0)^2}}$$

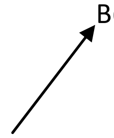
Contoh : Hitung panjang vektor $\overline{OP} = (2, 3, 5)$

Jawab :

$$\text{Berarti} : P = (2, 3, 5) \text{ dan } O = (0, 0, 0)$$

$$\left\| \overline{OP} \right\| = \sqrt{(2 - 0)^2 + (3 - 0)^2 + (5 - 0)^2} = \sqrt{4 + 9 + 25} = \sqrt{38}$$

PANJANG VEKTOR YANG TIDAK MELALUI TITIK ORIGIN = JARAK ANTARA DUA TITIK



$$\boxed{\left\| \overline{AB} \right\| = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

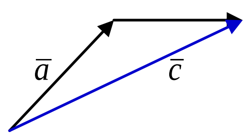
Contoh : Hitung jarak antara titik A(1, 2, 3) ke titik B(4, 6, 3)

Jawab : $\left\| \overline{AB} \right\| = d = \sqrt{(4 - 1)^2 + (6 - 2)^2 + (3 - 3)^2} = 5$

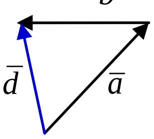
OPERASI PENJUMLAHAN/PENGURANGAN

SECARA GEOMETRI

a. $\overline{a} + \overline{b} = \overline{c}$



b. $\overline{a} - \overline{b} = \overline{d}$



SECARA ALJABAR

$$Misal : \overline{a} = (5,4,2) \; dan \; \overline{b} = (3,2,1), maka :$$

$$\overline{a} + \overline{b} = (5,4,2) + (3,2,1) = (8,6,3)$$

$$\overline{a} - \overline{b} = (5,4,2) - (3,2,1) = (2,2,1)$$

PERKALIAN TITIK (DOT PRODUCT)

Ada dua rumus, yaitu :

$$\text{Rumus 1} : \bar{u}.\bar{v} = (u_x)(v_x) + (u_y)(v_y) + (u_z)(v_z)$$

$$\text{Rumus 2} : \bar{u}.\bar{v} = \|\bar{u}\|\,\|\bar{v}\|\, Cos\,\theta$$

$$\text{dimana} : \theta = \text{sudut antara vektor } \bar{u} \text{ dengan vektor } \bar{v}.$$

Contoh soal :

Diketahui vektor : $\bar{u} = (2,-1,1)$ dan $\bar{v} = (1,1,2)$

Hitunglah : a. $(\bar{u} - \bar{v}).(2\bar{u} + \bar{v})$

b. Besarnya sudut antara $\bar{u}$ dan $\bar{v}$.

c. $3\bar{u} - 2\bar{v}$

Tentukan : d. Vektor satuan yang searah vektor $3\bar{u} - 2\bar{v}$.

JAWAB :

a. $(1,-2,-1).(5,-1,4) = (1)(5) + (-2)(-1) + (-1)(4) = 3$

b. $\bar{u}.\bar{v} = \|\bar{u}\|\,\|v\| Cos\,\theta$

$$\|\bar{u}\| = \sqrt{(2)^2 + (-1)^2 + (1)^2} = \sqrt{6}$$

$$\|\bar{v}\| = \sqrt{(1)^2 + (1)^2 + (2)^2} = \sqrt{6}$$

$$Cos\,\theta = \frac{\bar{u}.\bar{v}}{\|\bar{u}\|\,\|\bar{v}\|} = \frac{3}{(\sqrt{6})(\sqrt{6})} = \frac{3}{6} = \frac{1}{2}$$

$$\theta = arc\,Cos\left(\frac{1}{2}\right) = 60^\circ$$

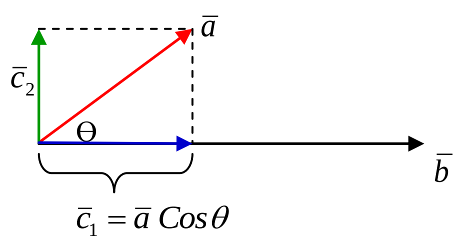
c. $3(2,-1,1) - 2(1,1,2) = (6,-3,3) - (2,2,4) = (4,-5,-1)$

d. $\|3\bar{u} - 2\bar{v}\| = \|4, -5, -1\| = \sqrt{16 + 25 + 1} = \sqrt{42}$

$$\text{Vektor satuan yg searah vektor } 3\bar{u} - 2\bar{v} = \frac{3\bar{u} - 2\bar{v}}{\|3\bar{u} - 2\bar{v}\|} = \frac{4}{\sqrt{42}} i - \frac{5}{\sqrt{42}} j - \frac{1}{\sqrt{42}} k$$

VEKTOR PROYEKSI

Proyeksi $\bar{a}$ pada $\bar{b}$: $\bar{c}_1$



$\text{Vektor proyeksi } \bar{a} \text{ pada } \bar{b} : \boxed{\bar{c}_1 = \dfrac{\bar{a}.\bar{b}}{\|\bar{b}\|^2}.\bar{b}}$

$\bar{c}_1 + \bar{c}_2 = \bar{a} \rightarrow \boxed{\bar{c}_2 = \bar{a} - \bar{c}_1 \text{ (vektor proyeksi } \bar{a} \text{ pada} \perp \bar{b})}$

Diketahui vektor : $\bar{a} = (1,1,2)$ dan $\bar{b} = (1,4,5)$

Tentukan :

a.Vektor proyeksi dari $\bar{a}$ pada $\bar{b}$

b.Vektor proyeksi dari $\bar{a}$ pada $\perp \bar{b}$ (disebut juga sebagai proyeksiortogonal $\bar{a}$ pada $\bar{b}$).

Jawab :

$$\text{a. } \bar{c}_1 = \frac{(1,1,2).(1,4,5)}{\left(\sqrt{1+16+25}\right)^2}(1,4,5) = \frac{1+4+10}{42}(1,4,5) = \left(\frac{15}{42}, \frac{60}{42}, \frac{75}{42}\right)$$

$$\text{b. } \bar{c}_2 = (1,1,2) - \left(\frac{15}{42}, \frac{60}{42}, \frac{75}{42}\right) = \left(\frac{27}{42}, \frac{-18}{42}, \frac{9}{42}\right)$$

PERKALIAN SILANG (CROSS PRODUCT)

Diketahui : $\bar{u} = (2,-1,1)$, $\bar{v} = (1,1,2)$ dan $\bar{w} = (1,2,1)$

Hitunglah : *a.* $\bar{u} \times \bar{v}$

b. $\bar{v} \times \bar{u}$

c. $(\bar{u} + \bar{v}) \times \bar{w}$

Jawab :

$$a. \bar{u} \times \bar{v} = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} j + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} k = -3i - 3j + 3k = (-3,-3,3)$$

$$b. \bar{v} \times \bar{u} = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} i - \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} j + \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} k = 3i + 3j - 3k = (3,3,-3)$$

Kesimpulan :

$\boxed{\bar{u} \times \bar{v} = -(\bar{v} \times \bar{u})}$

$$c. (\bar{u} + \bar{v}) \times \bar{w} = \{(2,-1,1) + (1,1,2)\} \times (1,2,1)$$

$$= (3,0,3) \times (1,2,1)$$

$$= \begin{vmatrix} i & j & k \\ 3 & 0 & 3 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} i - \begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix} j + \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} k = -6i - 0j + 6k = (-6,0,6)$$

Secara geometri :

$$\overline{u} \text{ x } \overline{v} = \|\overline{u}\| \|\overline{v}\| \text{Sin } \theta$$

$$\text{Catatan}: (\overline{u} \text{ x } \overline{v}) \text{ x } \overline{w} \neq \overline{u} \text{ x } (\overline{v} \text{ x } \overline{w})$$

SCALAR TRIPLE PRODUCT

Diketahui vektor: $\overline{u} = (3,-2,-5), \overline{v} = (1,4,-4)$, dan $\overline{w} = (0,3,2)$

Hitunglah : $\overline{u}.(\overline{v} \text{ x } \overline{w}),\ \overline{v}.(\overline{w} \text{ x } \overline{u}),\ \overline{w}.(\overline{u} \text{ x } \overline{v})$

Jawab :

$$\overline{u}.(\overline{v} \text{ x } \overline{w}) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix}$$

$$= \{(3)(4)(2) + (-2)(-4)(0) + (-5)(3)(1)\} - \{(-5)(4)(0) + (-2)(1)(2) + (3)(3)(-4)\}$$

$$= \{24 + 0 - 15\} - \{0 - 4 - 36\} = 49$$

$$\overline{v}.(\overline{w} \text{ x } \overline{u}) = \begin{vmatrix} 1 & 4 & -4 \\ 0 & 3 & 2 \\ 3 & -2 & -5 \end{vmatrix}$$

$$= \{(1)(3)(-5) + (4)(2)(3) + (-4)(-2)(0)\} - \{(-4)(3)(3) + (4)(0)(-5) + (1)(-2)(2)\}$$

$$= \{-15 + 24 + 0\} - \{-36 + 0 - 4\} = 49$$

$$\overline{w}.(\overline{u} \text{ x } \overline{v}) = \begin{vmatrix} 0 & 3 & 2 \\ 3 & -2 & -5 \\ 1 & 4 & -4 \end{vmatrix}$$

$$= \{(0)(-2)(-4) + (3)(-5)(1) + (2)(4)(3)\} - \{(2)(-2)(1) + (3)(3)(-4) + (0)(4)(-5)\}$$

$$= \{0 - 15 + 24\} - \{-4 - 36 + 0\} = 49$$

KESIMPULAN:

$$\boxed{\overline{u}.(\overline{v} \text{ x } \overline{w}) = \overline{v}.(\overline{w} \text{ x } \overline{u}) = \overline{w}.(\overline{u} \text{ x } \overline{v}) \rightarrow \text{SCALAR TRIPLE PRODUCT}}$$

SOAL UNTUK DICOBA SENDIRI :

1. Diberikan vektor $\overline{u} = (1,1,-1), \overline{v} = (6,7,-15)$ dan $\overline{w} = (-4,3,-1)$

   Hitunglah :

   a. $\overline{u} \text{ x } (\overline{v} + \overline{w})$

   b. $(\overline{u} - \overline{v}) \text{ x } \overline{w}$

   c. $\overline{u}.(\overline{v} \text{ x } \overline{w}),\ \overline{v}.(\overline{w} \text{ x } \overline{u}),\ \overline{w}.(\overline{u} \text{ x } \overline{v})$

2. Diberikan vektor $\overline{u} = 4i + 3j - 2k$ dan $\overline{v} 2i - j + 2k$

   Misalkan: $(\overline{u}.\overline{v})(\overline{u} \text{ x } \overline{v}) = ai + bj + ck$

   Tentukan nilai a, b dan c.

KOMBINASI LINIER

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Adalah cara untuk menyatakan suatu vektor sebagai jumlah kelipatan beberapa vektor lainnya dalam suatu ruang vektor.

Misal : $\overline{w} = k_1\overline{v}_1 + k_2\overline{v}_2 + k_3\overline{v}_3$

Dimana : $\overline{w}, \overline{v}_1, \overline{v}_2, \overline{v}_3 = \text{vektor}$

k_1, k_2, k_3 = skalar

Contoh :

Diketahui vektor-vektor : $\overline{u} = (1, 2, -1)$ dan $\overline{v} = (6, 4, 2)$ di R^3.

Tunjukkan apakah :

a. $\overline{w} = (9, 2, 7)$ merupakan kombinasi linier $\overline{u}$ dan $\overline{v}$?

b. $\overline{z} = (4, -1, 8)$ bukan merupakan kombinasi linier $\overline{u}$ dan $\overline{v}$?

Jawab :

$\overline{w} = k_1\overline{u} + k_2\overline{v}$

a.(9, 2, 7) = k_1 (1, 2, -1) + k_2 (6, 4, 2)

(9, 2, 7) = (k_1+6k_2, $2k_1+4k_2$, $-k_1+2k_2$)

1.$k_1+6k_2 = 9$

2.$2k_1+4k_2 = 2$

3.$-k_1+2k_2 = 7$

Pers.1 + pers.3 : $8k_2 = 16 \rightarrow k_2 = 2$ →substitusi pers.1 : $k_1+6(2) = 9 \rightarrow k_1 = 9 - 12 = -3$.

Cek ke pers. :

1.(-3)+6(2) = 9 →✓

2.2(-3)+4(2) = 2 →✓

3.-(-3)+2(2) = 7 →✓

Jadi : (9, 2, 7) = -3 (1, 2, -1) + 2 (6, 4, 2) →**adalah kombinasi linier.**

b.(4, -1, 8) = k_1 (1, 2, -1) + k_2 (6, 4, 2)

(4, -1, 8) = (k_1+6k_2, $2k_1+4k_2$, $-k_1+2k_2$)

1.$k_1+6k_2 = 4$

2.$2k_1+4k_2 = -1$

3.$-k_1+2k_2 = 8$

Pers.1 + pers.3 : $8k_2 = 12 \rightarrow k_2 = 3/2$ →substitusi pers.1 : $k_1+6(3/2) = 4 \rightarrow k_1 = 4 - 9 = -5$.

Cek ke pers. :

1.(-5)+6(3/2) = 4 →✓

2.2(-5)+4(3/2) = -4 → **X**

3.-(-5)+2(3/2) = 8 →✓

Jadi : (4, -1, 8) = -5 (1, 2, -1) + 3/2 (6, 4, 2) →**adalah bukan kombinasi linier.**

HIMPUNAN VEKTOR-VEKTOR YANG BEBAS LINIER/BERGANTUNG LINIER

Contoh1 :

Apakah vektor-vektor : $\overline{u} = (9,2,7), \overline{v} = (1,2,-1), \overline{w} = (6,4,2)$ bebas linier atau bergantung linier ?

Jawab :

Misal : $A = \begin{bmatrix} 9 & 2 & 7 \\ 1 & 2 & -1 \\ 6 & 4 & 2 \end{bmatrix}$ →maka det(A) = 0 →bergantung linier.

Contoh2 :

Tunjukkan bahwa vektor-vektor : $\overline{a} = (4, -1, 8),\ \overline{b} = (1, 2, -1),\ \overline{c} = (6, 4, 2)$ bebas linier atau bergantung linier.

Jawab :

Misal : $B = \begin{bmatrix} 4 & -1 & 8 \\ 1 & 2 & -1 \\ 6 & 4 & 2 \end{bmatrix}$ →maka det(B) = -24 →bebas linier.

SOAL UNTUK DICOBA SENDIRI :

1.Diketahui vektor-vektor : $\overline{a} = (3, 2, 1, -1), \overline{b} = (4, 3, 2, 1), \overline{c} = (18, 13, 8, 1)$.

Apakah $\overline{c}$ merupakan kombinasi linier dari $\overline{a}$ dan $\overline{b}$?

2.Diketahui vektor-vektor : $\overline{u} = (-2, 0, 4), \overline{v} = (3, -1, 6)$, dan $\overline{w} = (2, , -5, -5)$.

Apakah vektor-vektor tersebut bebas linier atau bergantung linier ?