

432 Class 18 Slides

github.com/THOMASELOVE/432-2018

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Setup

```
library(skimr)
library(arm)
library(rms)
library(boot)
library(MASS)
library(HSAUR)
library(broom)
library(tidyverse)
```

Today's Materials

Regression Models for Count Outcomes

- The polyps example
- Poisson Regression
- Overdispersion and Quasi-Poisson Regression
- Negative Binomial Regression

The polyps example

The polyps data frame within the HSAUR package describes the results of a placebo-controlled trial of a non-steroidal anti-inflammatory drug in the treatment of a condition called familial adenomatous polyposis (FAP).

```
head(HSAUR::polyps, 4)
```

	number	treat	age
1	63	placebo	20
2	2	drug	16
3	28	placebo	18
4	17	drug	22

Let's clean this up a little, and make it a tibble.

Cleaning the Polyps Data

```
pol432 <- HSAUR::polyps %>%  
  mutate(subject = 1:20) %>%  
  rename(polyps12m = number) %>%  
  select(subject, age, treat, polyps12m) %>%  
  tbl_df
```

Details

The tibble includes 20 observations (skim on next slide) on:

- `age` = the age of the patient at the start of the trial, in years
- `treat` = the patient's treatment arm
- `polyps12m` = (our outcome), the number of colonic polyps at 12 months for this patient

We want to understand how the number of colonic polyps at 12 months is related to both `treat` and `age`. Note that the actual trial was halted after a planned interim analysis suggested compelling evidence in favor of the drug over placebo. Data sources and the original NEJM reference (1993) may be found in `?HSAUR::polyps`

Skim of the pol432 data

Skim summary statistics



n obs: 20

n variables: 3

Variable type: factor

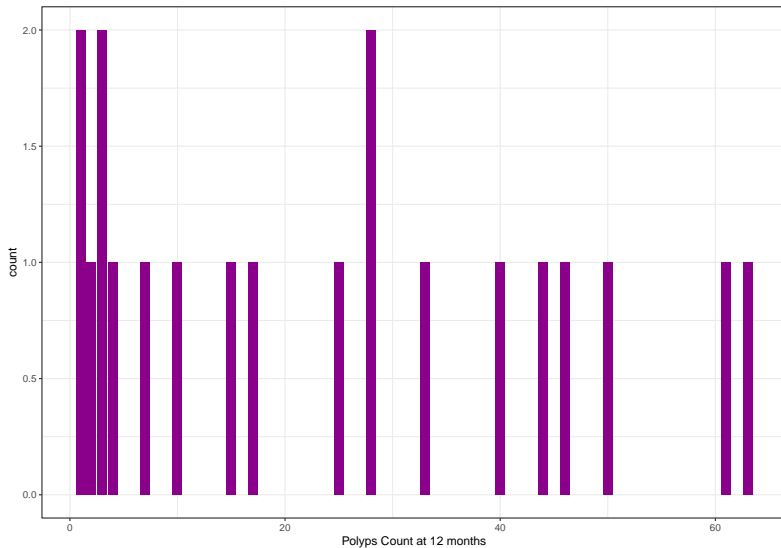
variable	missing	complete	n	n_unique	top_counts	ordered
treat	0	20	20	2	p1a: 11, dru: 9, NA: 0	FALSE

Variable type: numeric

variable	missing	complete	n	mean	sd	p0	p25	median	p75	p100	hist
age	0	20	20	25	9.05	13	19.75	22.5	27.75	50	
polyps12m	0	20	20	24.05	20.85	1	3.75	21	41	63	

Regression on a Count Outcome

Plot the Outcome



Why not model count data with a linear regression?

The data on polyps12m is count data. Why wouldn't we model this with linear regression?

- A count can only be positive. Linear regression would estimate some subjects as having negative counts.
- A count is unlikely to follow a Normal distribution. In fact, it's far more likely that the log of the counts will follow a Poisson distribution.

So, we'll try that. We'll run a generalized linear model with a log link function, ensuring that all of the predicted values will be positive, and using a Poisson error distribution. This is called **Poisson regression**.

Poisson regression may be appropriate when the dependent variable is a count of events. The events must be independent - the occurrence of one event must not make any other more or less likely.

Fit a Poisson Regression Model with glm

The default link function with the poisson family is the log.

```
mod_1 <- glm(polyps12m ~ treat + age,  
             data=pol432,  
             family=poisson())  
mod_1
```

Call: glm(formula = polyps12m ~ treat + age, family = poisson)

Coefficients:

(Intercept)	treatdrug	age
4.52902	-1.35908	-0.03883

Degrees of Freedom: 19 Total (i.e. Null); 17 Residual

Null Deviance: 378.7

Residual Deviance: 179.5 AIC: 273.9

Model Equation and confidence intervals

The model equation is:

$$\log(\text{polyps at 12 months}) = 4.53 - 1.36 \text{ treat} - 0.039 \text{ age}$$

```
> confint(mod_1)
```

Waiting for profiling to be done...

	2.5 %	97.5 %
(Intercept)	4.24184712	4.817700
treatdrug	-1.59549678	-1.133761
age	-0.05074339	-0.027393

Both treatdrug and age have CIs that exclude 0.

Model Summary (edited)

```
Call: glm(formula = polyps12m ~ treat + age,  
          family = poisson(), data = pol432)
```

	Estimate	Std. Error	z value	Pr(> z)	
(Intercept)	4.529024	0.146872	30.84	< 2e-16	***
treatdrug	-1.359083	0.117643	-11.55	< 2e-16	***
age	-0.038830	0.005955	-6.52	7.02e-11	***

(Dispersion parameter for poisson family taken to be 1)

Null deviance: 378.66 on 19 degrees of freedom
Residual deviance: 179.54 on 17 degrees of freedom
AIC: 273.88

Number of Fisher Scoring iterations: 5

There's a real problem here, but we'll get back to it.

Making Predictions

A subject of age 30 on the drug is predicted to have:

```
log(polyps at 12 months) = 4.53 - 1.36 treat - 0.039 age
log(polyps at 12 months) = 4.53 - 1.36 (1) - 0.039 (30)
log(polyps at 12 months) = 4.53 - 1.36 - 1.17 = 2, so...
polyps at 12 months = exp(2) = 7.4
```

so this subject is estimated by Model 1 to have 7.4 polyps.

Making the Prediction Automatically...

```
predict(mod_1, data.frame(treat = "drug", age = 30),  
        type = "response", se.fit = TRUE)
```

```
$fit
```

```
1
```

```
7.426338
```

```
$se.fit
```

```
1
```

```
0.8642734
```

```
$residual.scale
```

```
[1] 1
```

The `residual.scale` specifies the square root of the *dispersion* value used in computing the standard errors.

The Poisson Regression Model

The Poisson distribution is used to model a *count* outcome - that is, an outcome with possible values $(0, 1, 2, \dots)$. The model takes a somewhat familiar form to the models we've used for linear and logistic regression. If our outcome is y and our linear predictors X , then the model is:

$$y_i \sim \text{Poisson}(\theta_i)$$

The parameter θ must be positive, so it makes sense to fit a linear regression on the logarithm of this...

$$\theta_i = \exp(\beta_0 + \beta_1 X_1 + \dots \beta_k X_k)$$

Poisson Regression

The coefficients β can be exponentiated and treated as multiplicative effects. For example, if our model is for y_i = counts of polyps, with the regression equation:

$$y_i \sim \text{Poisson}(\exp(4.5 - 1.4(\text{treat} = \text{drug}) - 0.04 \text{ age}))$$

where $\text{treat} = \text{drug}$ is 1 if the treatment is drug, and 0 if the treatment is placebo, and age is in years, we can interpret the coefficients as follows...

Interpreting the Poisson Model

- The constant term, 4.5, gives us the intercept of the regression - the prediction if `treat = placebo` and `age = 0`. Since we have no one with age of zero, we try not to interpret this term.
- The coefficient of `treat = drug`, -1.4, tells us that the predictive difference between the drug and placebo treatments can be found by multiplying the polyps count by $\exp(-1.4) = 0.25$, yielding a reduction of 75%.
- The coefficient of `age`, -0.04, is the expected difference in count of polyps (on the log scale) for each additional year of age. Thus, the expected multiplicative *increase* is $e^{-0.04} = 0.96$, corresponding to a 4% negative difference in the count.

As with linear or logistic regression, each coefficient is interpreted as a comparison where one predictor changes by one unit, while the others remain constant.

Looking for Overdispersion

The notion of *overdispersion* arises here. When fitting generalized linear models with Poisson error distributions, the residual deviance and its degrees of freedom should be approximately equal if the model fits well.

If the residual deviance is far greater than the degrees of freedom, then overdispersion may well be a problem. In this case, the residual deviance is more than 10 times the size of the residual degrees of freedom, so that's a clear indication of overdispersion.

Residual deviance: 179.54 on 17 degrees of freedom

Poisson regression model requires that the outcome (here the polyps counts) be independent. A possible reason for the overdispersion we see here is that polyps likely do not occur independently of one another but may “cluster” together.

Dealing with Over-Dispersion

Poisson regressions do not supply an independent variance parameter σ , and as a result can be overdispersed, and usually are. Under the Poisson distribution, the variance equals the mean - so the standard deviation equals the square root of the mean. Gelman and Hill provide an overdispersion test in R for a Poisson model as follows...

```
yhat <- predict(mod_1, type = "response")
display(mod_1)
```

```
glm(formula = polyps12m ~ treat + age, family = poisson(), data = dat)
      coef.est coef.se
(Intercept)  4.53    0.15
treatdrug    -1.36    0.12
age          -0.04    0.01
---
n = 20, k = 3
residual deviance = 179.5, null deviance = 378.7 (difference = 199.2)
```

Overdispersion Test, Part 2

```
n <- 20; k <- 3
z <- (pol432$polyps12m - yhat) / sqrt(yhat)
cat("overdispersion ratio is ", sum(z^2) / (n - k), "\n")
```

overdispersion ratio is 10.72783

```
cat("p value of overdispersion test is ",
    pchisq(sum(z^2), df = n-k, lower.tail = FALSE), "\n")
```

p value of overdispersion test is 9.711657e-30

The sum of squared standardized residuals $\sum z^2 = 182.37$. The estimated overdispersion factor is $182.37/17 = 10.73$, and the p value here is 0 for all intents and purposes.

Conclusions from Overdispersion Test

This indicates that the probability is essentially zero that a random variable from a χ^2 distribution with $(n - k) = 17$ degrees of freedom would be as large as 182.37.

In summary, the polyps counts are overdispersed by a factor of more than 10, which is enormous (even a factor of 2 would be considered large) and also highly statistically significant. The basic correction for overdispersion is to multiply all regression standard errors by $\sqrt{10.73} = 3.28$.

Our main inferences are not too seriously affected by this adjustment, as it turns out, and we will see this in what's called an overdispersed Poisson model that we'll fit next.

Deal with Overdispersion via a Quasi-Likelihood Estimation Procedure

To deal with overdispersion, we could apply a quasi-likelihood estimation procedure.

```
mod_2 <- glm(polyps12m ~ treat + age,  
             data=pol432, family=quasipoisson())  
mod_2
```

```
Call:  glm(formula = polyps12m ~ treat + age, family = quasipoisson,  
          data = pol432)
```

Coefficients:

(Intercept)	treatdrug	age
4.52902	-1.35908	-0.03883

Degrees of Freedom: 19 Total (i.e. Null); 17 Residual

Summary of mod_2

```
Call: glm(formula = polyps12m ~ treat + age,  
          family = quasipoisson(), data = pol432)
```

	Estimate	Std. Error	t value	Pr(> t)	
(Intercept)	4.52902	0.48106	9.415	3.72e-08	***
treatdrug	-1.35908	0.38533	-3.527	0.00259	**
age	-0.03883	0.01951	-1.991	0.06284	.

(Dispersion parameter for quasipoisson family
taken to be 10.72805)

Null deviance: 378.66 on 19 degrees of freedom
Residual deviance: 179.54 on 17 degrees of freedom
AIC: NA

Number of Fisher Scoring iterations: 5

Comparing the Two Models

Estimate	Poisson Est (SE)	Quasi-Poisson Est (SE)
Intercept	4.53 (0.147)	4.53 (0.481)
treat	-1.36 (0.118)	-1.36 (0.385)
age	-0.039 (0.006)	-0.039 (0.020)

Poisson model and Quasi-Poisson model:

$$\log(\text{polyps at 12 months}) = 4.53 - 1.36 \text{ treat} - 0.039 \text{ age}$$

Confidence Intervals for `mod_2` coefficients

Waiting for profiling to be done...

	2.5 %	97.5 %
(Intercept)	3.59170079	5.481738085
treatdrug	-2.18466623	-0.653930343
age	-0.07969808	-0.003027068

The estimates in Model 2 are still statistically significant, but the standard errors for each coefficient are considerably larger when we account for overdispersion.

The Quasi-Poisson (Overdispersed Poisson) Regression

The quasipoisson and negative binomial (coming soon) models are very similar. We write the overdispersed “quasiPoisson” model as:

$$y_i \sim \text{overdispersed Poisson}(\mu_i = \exp(X_i\beta_i), \omega)$$

where ω is the overdispersion parameter, 10.73, in our case.

The Poisson model is just the overdispersed Poisson model with $\omega = 1$.

ANOVA for Poisson or Quasi-Poisson Regression

The ANOVA approach here (as with `glm`, generally) produces sequential tests, so in this case, we see whether `treat` by itself has a significant effect, and then whether `age`, given `treat` already in the model, has an impact. If we want to test the coefficients in another order, we need only to specify that order when we fit the model.

```
anova(mod_2, test = "Chisq")
```

Analysis of Deviance Table

Model: quasipoisson, link: log

Response: polyps12m

Terms added sequentially (first to last)

	Df	Deviance	Resid.	Df	Resid. Dev	Pr(>Chi)
NULL				19	378.66	
treat	1	150.101		18	228.56	0.0001836 ***
age	1	49.018		17	179.54	0.0325526 *

Signif. codes:

0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Making Predictions with Model 2

```
predict(mod_2, data.frame(treat = "drug", age = 30),  
        type = "response", se.fit = TRUE)
```

```
$fit
```

```
1
```

```
7.426338
```

```
$se.fit
```

```
1
```

```
2.830815
```

```
$residual.scale
```

```
[1] 3.27537
```

Fitting Poisson and Quasi-Poisson models using `Glm` from `rms`

The `Glm` function in the `rms` package can be used to fit these models.

Original Poisson Regression Model

Here's our original Poisson regression:

```
d <- datadist(pol432)
options(datadist = "d")
model1 <- Glm(polyps12m ~ treat + age, data=pol432,
              family=poisson(), x = T, y = T)
```


model1 from Glm

General Linear Model

```
Glm(formula = polyps12m ~ treat + age, family = poisson(), data =  
x = T, y = T)
```

Model Likelihood

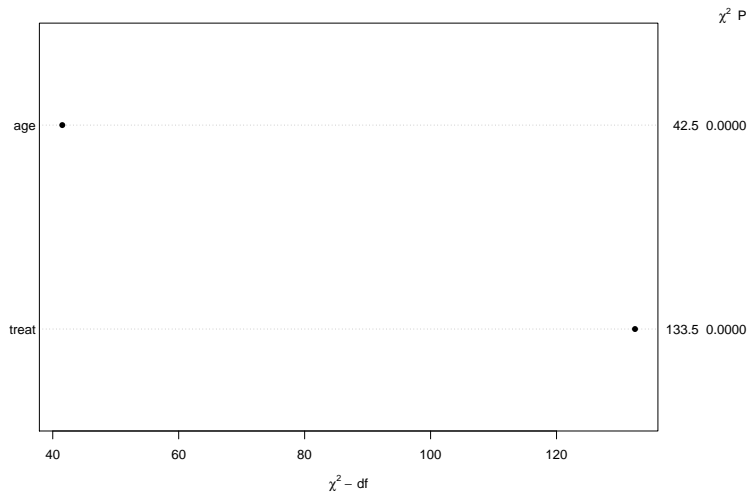
Ratio Test

Obs	20	LR chi2	199.12
Residual d.f.	17	d.f.	2
g	0.8324108	Pr(> chi2)	<0.0001

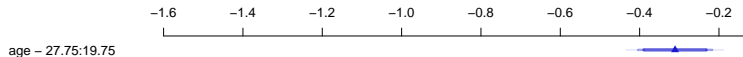
	Coef	S.E.	Wald Z	Pr(> Z)
Intercept	4.5290	0.1469	30.84	<0.0001
treat=drug	-1.3591	0.1176	-11.55	<0.0001
age	-0.0388	0.0060	-6.52	<0.0001

```
plot(anova(model1))
```

```
plot(anova(model1))
```



```
plot(summary(model1))
```



summary(model1)

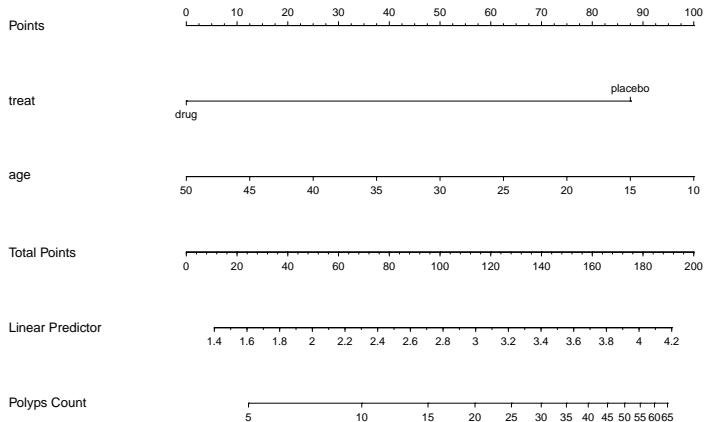
Effects

Response : polyps12m

Factor	Low	High	Diff.	Effect	S.E.
age	19.75	27.75	8	-0.31064	0.047642
treat - drug:placebo	1.00	2.00	NA	-1.35910	0.117640
Lower 0.95	Upper 0.95				
-0.41116	-0.21013				
-1.60730	-1.11090				

Nomogram

```
plot(nomogram(model1, fun = exp,  
             funlabel = "Polyps Count"))
```



Accounting for Overdispersion and Adding an Interaction Term

We can run an overdispersed model in `rms`, too. Just to mix things up a little, let's add an interaction term between `treat` and `age` and see if that improves our model at all.

```
d <- datadist(pol432)
options(datadist = "d")
model3 <- Glm(polyps12m ~ treat * age, data=pol432,
              family=quasipoisson(), x = T, y = T)
```

model3

General Linear Model

```
Glm(formula = polyps12m ~ treat * age, family = quasipoisson(  
  data = pol432, x = T, y = T)
```

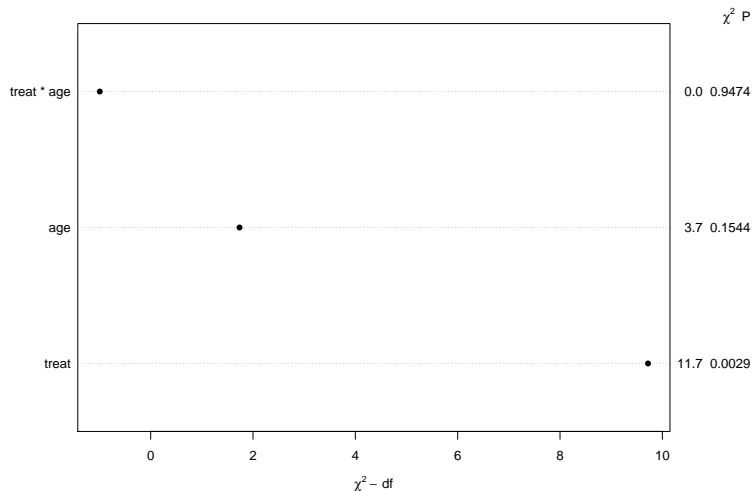
Model Likelihood

Ratio Test

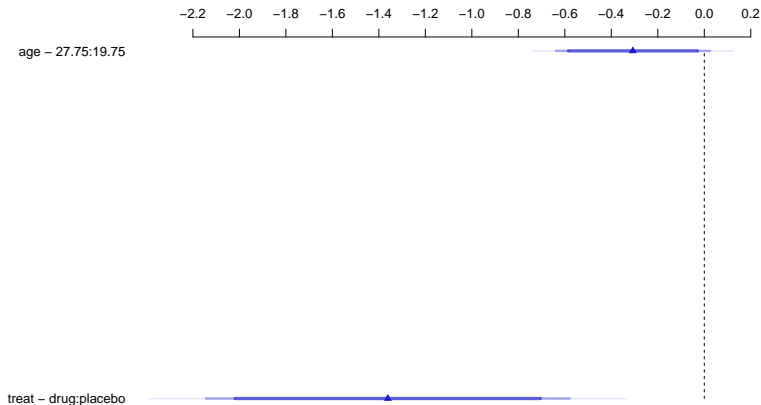
Obs	20	LR chi2	199.17
Residual d.f.	16	d.f.	3
g	0.8406447	Pr(> chi2)	<0.0001

	Coef	S.E.	Wald Z	Pr(> Z)
Intercept	4.5191	0.5173	8.74	<0.0001
treat=drug	-1.2573	1.5907	-0.79	0.4293
age	-0.0384	0.0211	-1.82	0.0683
treat=drug * age	-0.0046	0.0702	-0.07	0.9474

ANOVA: model3



Effects Summary Plot: model3



Adjusted to: treat=placebo age=22.5

Effects Summary Table: model3

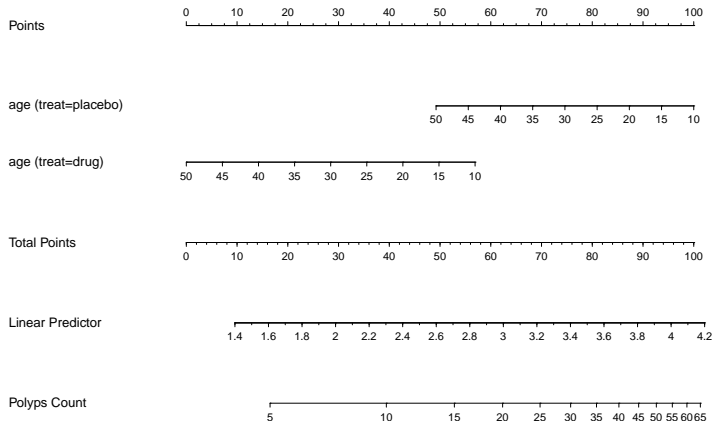
Effects Response : polyps12m

Factor	Low	High	Diff.	Effect	S.E.
age	19.75	27.75	8	-0.30722	0.16851
treat - drug:placebo	1.00	2.00	NA	-1.36140	0.39898
Lower 0.95	Upper 0.95				
-0.66446	0.050012				
-2.20720	-0.515650				

Adjusted to: treat=placebo age=22.5

Nomogram: model3

```
plot(nomogram(model3, fun = exp,  
              funlabel = "Polyps Count"))
```



Negative Binomial Regression

To fit a negative binomial regression model to predict the log(polyp counts), I'd use the `glm.nb` function from the MASS package, as follows...

```
mod_nb <- glm.nb(polyps12m ~ treat + age,  
                  data=pol432, link = log)
```

```
mod_nb
```

```
Call:  glm.nb(formula = polyps12m ~ treat + age, data = pol432,  
              init.theta = 1.719491)
```

Coefficients:

(Intercept)	treatdrug	age
4.52603	-1.36812	-0.03856

```
Degrees of Freedom: 19 Total (i.e. Null); 17 Residual
```

```
Null Deviance: 36.73
```

summary(mod_nb)

```
Call: glm.nb(formula = polyps12m ~ treat + age,  
             data = pol432, link = log, init.theta = 1.719491)
```

	Estimate	Std. Error	z value	Pr(> z)	
(Intercept)	4.52603	0.59466	7.611	2.72e-14	***
treatdrug	-1.36812	0.36903	-3.707	0.000209	***
age	-0.03856	0.02095	-1.840	0.065751	.

(Dispersion parameter for Negative Binomial(1.7195)
family taken to be 1)

Null deviance: 36.734 on 19 degrees of freedom
Residual deviance: 22.002 on 17 degrees of freedom
AIC: 164.88

Number of Fisher Scoring iterations: 1

Estimates and Standard Errors

Model	Poisson	Quasi-Poisson	Negative Binomial
Intercept	4.53 (0.147)	4.53 (0.481)	4.53 (0.594)
treat	-1.36 (0.118)	-1.36 (0.385)	-1.37 (0.369)
age	-0.039 (0.006)	-0.039 (0.020)	-0.039 (0.021)

Confidence Intervals for Coefficients

```
confint(mod_nb)
```

Waiting for profiling to be done...

	2.5 %	97.5 %
(Intercept)	3.3812139	5.716737794
treatdrug	-2.0960752	-0.633666100
age	-0.0776086	0.004319903

ANOVA for Negative Binomial Regression

The best way to run an ANOVA in this setting is to fit the model with and without the parameter you want to test.

```
mod_nb_notreat <- glm.nb(polyps12m ~ age,  
                          data=pol432, link = log)  
  
anova(mod_nb_notreat, mod_nb)
```

Likelihood ratio tests of Negative Binomial Models

Response: polyps12m

	Model	theta	Resid. df	2 x log-lik.	Test
1	age	1.042819	18	-166.818	
2	treat + age	1.719491	17	-156.880	1 vs 2
	df	LR stat.	Pr(Chi)		
1					
2	1	9.937965	0.001619045		

ANOVA for age in Negative Binomial model

```
mod_nb_noage <- glm.nb(polyps12m ~ treat,  
                        data=pol432, link = log)  
  
anova(mod_nb_noage, mod_nb)
```

Likelihood ratio tests of Negative Binomial Models

Response: polyps12m

	Model	theta	Resid. df	2 x log-lik.	Test
1	treat	1.467894	18	-159.8004	
2	treat + age	1.719491	17	-156.8800	1 vs 2
	df	LR stat.	Pr(Chi)		
1					
2	1	2.920354	0.08746867		

Making Predictions, with the Negative Binomial Model

```
predict(mod_nb, data.frame(treat = "drug", age = 30),  
        type = "response", se.fit = TRUE)
```

```
$fit
```

```
1
```

```
7.397731
```

```
$se.fit
```

```
1
```

```
2.310031
```

```
$residual.scale
```

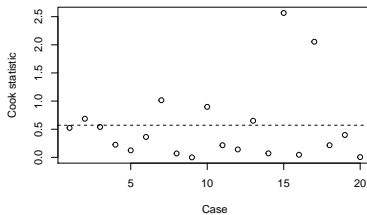
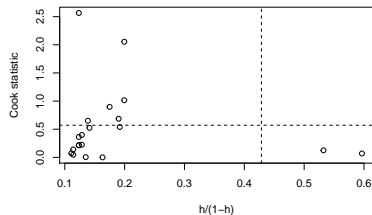
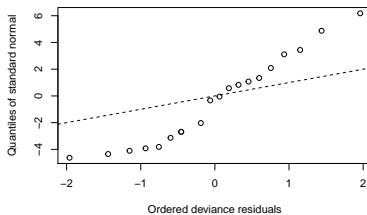
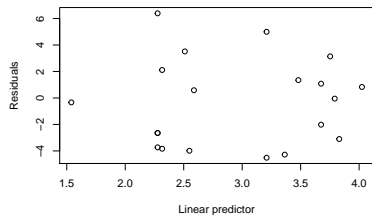
```
[1] 1
```

Diagnostic Plots for a Generalized Linear Model

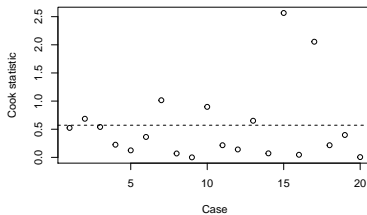
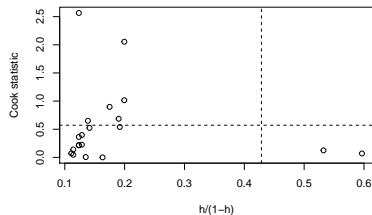
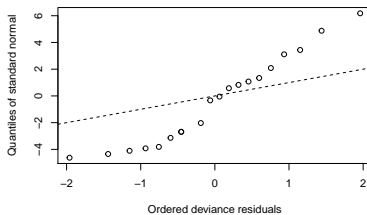
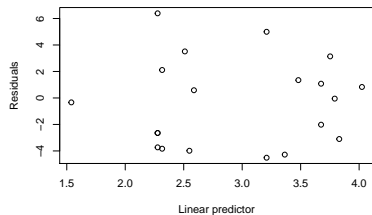
The `plots` function applied to one of these `glms` doesn't do anything that's too helpful. The `glm.diag.plots` function from the `boot` package makes a set of four plots:

- (Top, Left) Jackknife deviance residuals against fitted values. A *jackknife deviance* residual, also called a likelihood residual, is the change in deviance when this observation is omitted.
- (Top, Right) Normal Q-Q plot of standardized deviance residuals. (Dotted line shows expectation if those standardized residuals followed a Normal distribution, and these residuals generally should.)
- (Bottom, Left) Cook statistic vs. standardized leverage
 - $n = \#$ of observations, $p = \#$ of parameters estimated
 - Horizontal dotted line at $\frac{8}{n-2p}$. Points above line have high influence.
 - Vertical line is at $\frac{2p}{n-2p}$. Points to the right of the line have high leverage.
- (Bottom, Right) Index plot of Cook's statistic to identify observations with high influence.

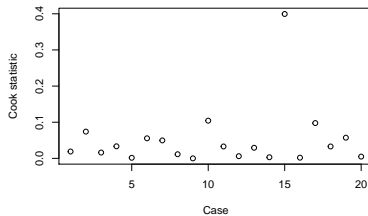
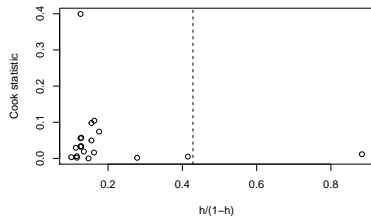
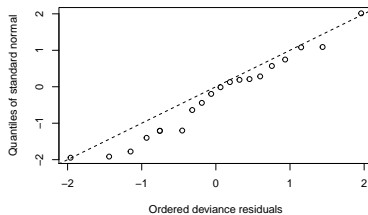
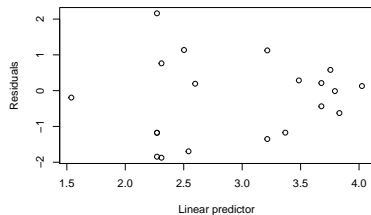
glm.diag.plots(mod_1)



Diagnostic Plots for `mod_2` are similar



Diagnostic Plots for mod_nb



Model Comparisons

Summary	Poisson	QuasiPoisson	Negative Binomial
log(counts) =	4.53 - 1.36	4.53 - 1.36	4.53 - 1.37
... treat	- 0.039	- 0.039	- 0.039
	age	age	age
treat effect ¹	1.36 (1.13, 1.60)	1.36 (0.65, 2.18)	1.37 (0.63, 2.10)
age effect	-0.04 (-0.05, -0.03)	-0.04 (-0.08, 0)	-0.04 (-0.08, 0)

¹Here we display the effect of being in the placebo group as compared to the drug group.

Model Comparisons

Summary	Poisson	Quasi-Poisson	Negative Binomial
Residual Deviance	179.5	179.5	22.0
Residual df	17	17	17
AIC	273.88	NA	164.9
Age 30, drug	7.43 (se = 0.86)	7.43 (2.83)	7.40 (2.31)
Deviance residuals	Not Normal	Not Normal	More Normal
Influential cases	15, 17	15, 17	maybe 15

Conclusion in this setting: It looks like the Negative Binomial model works best, of these options.

Next Time

The rest of Chapter 18

- Zero-Inflated Models
- Hurdle Models
- Tobit Models

plus a discussion of the Project 2 Instructions

PLEASE REMEMBER TO COMPLETE THE MINUTE PAPER!!