A Stochastic Model of Competing Binary Opinions in a Social Network of Individuals with Differing Levels of Involvement

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1 Introduction

Within a social network, it may appear common that two opposing opinions form among many individuals. As individuals interact with each other, transmission of stated opinions may occur causing an individual to change their opinion either in favor or in opposition to the opinion shared by a different individual. As opinions change within a network, there may be macroscopic and microscopic trends that can be observed and hopefully modelled. In particular, it is of interest to see how differing levels of involvement may effect how two competing opinions spread and organize the individuals of a social network. It is a hypothesis that people who are more involved or hold stronger beliefs with a specific opinion tend to organize their local network around others who share similar opinions. This hypothesis tends to be stated as 'two sides' not communicating with each other with the development of a lack of mutual understanding. It is another hypothesis that those who are move involved would have a higher frequency of transmission of opinions, thus investigation of this model would take that into account. Of principle interest in this report, though, is whether a proposed model can exhibit the 'silent' majority / 'vocal minority' behavior where most individuals are only at most leaning in opinion and not strongly opinionated.

2 A Tour of References

As it turns out, much work has already been done on attempting to model the evolution and spread of opinions within social networks. One paper uses interacting agents to model the evolution of opinion through the different types of discussions that agents may have. The article purports to "naturally" account for individual biases, but does not specifically seem to include agents of higher involvement or strategy [1]. Another article derives its model from mathematical physics, in particular, the ising model. In this model, nodes take on only two states which represent the two states of opinion [2]. Two other articles try to deliberate the strategies of opinion transmission in the social network. The first article discusses groups of "ordinary people" and "strong opinion leaders" and attempts to model the spread of opinions like the spread of a gas [3]. The other article divides opinion sharers into "opportunists" and "fanatics" with regards to how the agent's opinions can change [4].

3 Description of Model and Implementation

3.1 Individuals and Opinion Types

3.1.1 Mathematical Description

Let $I = \{I_1, ..., I_n\}$ be a set of nodes. Define the set of undirected graphs of I, by

$$G = \{\, f \mid f : I \times I \to \mathbb{Z}_2, f \text{ commutative}, f(I_k, I_k) = 0 \,\}.$$

Next, define the set of all opinion states as $S = \mathbb{Z}_5$ and define the set of all combinations (with repetition) of node states as

$$O = \{\, o \mid o : I \to S \,\}.$$

Given $o \in O$, define the statistics

$$\begin{aligned}
\#\text{Strong}_A &= \sum_{k=1}^{n} \mathbb{1}(o(I_k) = 0) \\
\#\text{Lean}_A &= \sum_{k=1}^{n} \mathbb{1}(o(I_k) = 1) \\
\#\text{Neutral} &= \sum_{k=1}^{n} \mathbb{1}(o(I_k) = 2) \\
\#\text{Lean}_B &= \sum_{k=1}^{n} \mathbb{1}(o(I_k) = 3) \\
\#\text{Strong}_B &= \sum_{k=1}^{n} \mathbb{1}(o(I_k) = 4) \\
\#\text{A} &= \#\text{Strong}_A + \#\text{Lean}_A \\
\#\text{B} &= \#\text{Strong}_B + \#\text{Lean}_B
\end{aligned}$$

3.1.2 Summary and Derivation

Each node I_k represents an individual in a social network. We can interpret every $g \in G$ as a graph in that if g evaluates to 0, then two nodes are not connected and if it evaluates to 1, then two nodes are connected. If two individuals are connected, then they share the ability to receive transmissions/communications from one another. Given a combination of opinion states $o \in O$, we can say that I_k contains the feature $o(I_k)$, where the states $0-4$ correspond to being of opinion Strong A, Leaning A, Neutral, Leaning B, and Strong B, respectively.

3.1.3 Implementation Details and Choices

For the purposes of simulation, we choose $g \in G$ to be of a type generated by the Barabasi-Albert model because we can choose m_o starting clusters and the resulting graph has the preferential-attachment property [5, p. 71]. There are criticisms abound for the Barabasi-Albert model, though, but we admit that any generated graph can be used in this model.

3.2 Markov Model of Transmissions and Opinion States

3.2.1 Mathematical Description

Given I, O, and $g \in G$ define the sample space $\Omega = O$ and define a stochastic process

$$X = \{\, X_t \,|\, t \in \mathbb{Z}, X_t : \Omega \to O, X_t \text{ bijective} \,\}.$$

We will later discuss how probabilities are assigned to events, but first we make the assumption that this stochastic process satisfies the Markov property, in that

$$\begin{aligned}&\mathbb{P}(X_t = o_t \in O | X_{t-1} = o_{t-1}, X_{t-2} = o_{t-2}, \ldots, x_{t-k} = o_{t-k})\\ &= \mathbb{P}(X_t = o_t \in O | X_{t-1} = o_{t-1}).\end{aligned}$$

Now, we assign probabilities inductively assuming the knowledge of an initial assignment of states $X_0 \in O$ through the following. First, denote the general transmission probabilities as $P_{\text{Transm}} : S \to [0, 1]$. For each $I_k \in I$, the probability of transmission is $P_{\text{Transm}} \circ o_0(I_k)$. Next, define the transition probability matrices A_s, A_l, N, B_l, B_s each of dimension 5×5 and where the sum of each row is 1. Also note that each of these matrices' elements are indexed starting at zero. Define a selector function for each matrix as $\text{Sel} = S \mapsto (A_s, A_l, N, B_l, B_s)$. For each $I_k, I_j \in I$ we define a single transition of I_k given I_j as $\hat{o}(I_k, I_j) \in S$ and define

$$\begin{aligned}&\mathbb{P}(\hat{o}(I_k, I_j) = a \qquad | X_0 = o_0, g(I_k, I_j) = 1) = P_{\text{Transm}} \circ o_0(I_j) \cdot \vec{e}^{\,T}_{o(I_k)} \text{Sel}(o_0(I_k)) \vec{e}_a,\\ &\mathbb{P}(\hat{o}(I_k, I_j) = o_0(I_k) | X_0 = o_0, g(I_k, I_j) = 0) = 1\end{aligned}$$

were $\vec{e}_j$ is a vector of all zeros except the j'th row is one. Now we define the delta single transition as $\Delta\hat{o}(I_k, I_j) = o_0(I_k) - \hat{o}(I_k, I_j)$. With all the previous defined, we can finally give the next state for each node as the following:

$$o_1(I_k) = \max\{0, \min\{5, \sum_{j=1}^{n} \Delta\hat{o}(I_k, I_j)\}\}.$$

We can continue inductively to define probabilities of each $X_t \in O$ occurring by replacing 0 and 1 with appropriate $t-1$ and t.

3.2.2 Summary and Derivation

At each time t is an associated configuration of node states $X_t = o_t \in O$. That is, at each time t each individual in the social network has an associated opinion. For a time step to $t+1$, each individual has a probability of sending out a transmission, given their opinion state. If an individual $I_k \in I$ only receives one transmission in total from all neighboring individuals, then they will transition states with a probability corresponding to the individual's opinion (the matrix), the other individual's opinion (the row), and the actual state we are checking to transition to (the column). This is of course if the other individual sent out a transmission and the other individual is a neighbor to I_k. If a particular individual receives multiple transmissions, though, then we need to sum all changes that would have occurred from repeated interactions with all neighboring nodes, while also keeping the sum in range of possible states, hence the max and min. An important observation is that if too many transmissions are sent in one time step, there is a possibility that an individual can change their opinion by a large amount. Thus, it may make sense to determine a time step small enough to keep transmission rates across all nodes small and prevent too many overlapping transmissions occurring. One major phenomenon excluded in the model is the ability for people to become 'unconnected.' We note that in coding simulations for the model, this option has the ability to be included, but it was observed that the output time series had a higher variance than without the assumption.

3.3 Proposed Model for Transition Probabilities

3.3.1 Remark

The important model is the one stated above, but in order for simulations to be made, it is important that the all the probabilities be filled out. Thus, we derive and propose some relationships to generate probabilities.

3.3.2 Mathematical Description

We first make the assumption that A_s and B_s, and A_l and B_l are similar in the sense that given each a_{ij} element in the first matrix, $b_{ij} = a_{4-i,4-j}$ in the second matrix. Also, the elements of N are similar to itself through the previous definition. Thus, if we define A_s, A_l, and N we have defined all the other matrices. We also assume that $P_{\text{Transm}}(0) = P_{\text{Transm}}(4)$ and $P_{\text{Transm}}(1) = P_{\text{Transm}}(3)$. Now define the following probabilities and probability changes:

name	variable	range
resistivity	r	$[0,1]$
resistivity bonus	r_b	$[0,1]$
receptiveness	e	$[0,1]$
backlash	b	$[0,1]$
backlash bonus	b_b	$[0,1]$
transmission frequency	f	$[0,1]$
frequency multiplier	f_m	$\mathbb{N}$
base non-change opinion	s_m	$[0,1]^3$
actual backlash	b_l	$[0,1]^3$

With these definitions, we can define the variables below the double bars as follows:

$$
\begin{aligned}
s_{m_1} &= \min\{1, r + 2r_b\} \\
s_{m_2} &= \min\{1, r + r_b\} \\
s_{m_3} &= r \\
b_{l_1} &= \min\{1, b + 2b_b\} \\
b_{l_2} &= \min\{1, b + b_b\} \\
b_{l_3} &= b
\end{aligned}
$$

Finally, we fully define

$$P_{\text{Transm}} : (0, 1, 2, 3, 4) \mapsto (\min\{1, f_m f\}, f, 0, f, \min\{1, f_m f\}).$$

$$A_s = \begin{bmatrix} s_{m_1} + (1-b)(1-s_{m_1}) & b(1-s_{m_1}) & 0 & 0 & 0 \\ s_{m_1} & (1-s_{m_1}) & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ s_{m_1} & (1-s_{m_1}) & 0 & 0 & 0 \\ s_{m_1} + b_{l_1}(1-s_{m_1}) & (1-b_{l_1})(1-s_{m_1}) & 0 & 0 & 0 \end{bmatrix}$$

$$A_l = \begin{bmatrix} e(1-b)(1-s_{m_2}) & s_{m_2} + (1-e)(1-b)(1-s_{m_2}) & b(1-s_{m_2}) & 0 & 0 \\ e(1-s_{m_2}) & s_{m_2} + (1-e)(1-s_{m_2}) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ (1-e)(1-s_{m_2}) & s_{m_2} & e(1-s_{m_2}) & 0 & 0 \\ b_{l_2}(1-s_{m_2}) & s_{m_2} + (1-e)(1-b_{l_2})(1-s_{m_2}) & e(1-b_{l_2})(1-s_{m_2}) & 0 & 0 \end{bmatrix}$$

$$N = \begin{bmatrix} 0 & e(1-b)(1-s_{m_3}) & s_{m_3} + (1-e)(1-b)(1-s_{m_3}) & b(1-s_{m_3}) & 0 \\ 0 & e(1-s_{m_3}) & s_{m_3} + (1-e)(1-s_{m_3}) & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & s_{m_3} + (1-e)(1-s_{m_3}) & e(1-s_{m_3}) & 0 \\ 0 & b(1-s_{m_3}) & s_{m_3} + (1-e)(1-b)(1-s_{m_3}) & e(1-b)(1-s_{m_3}) & 0 \end{bmatrix}$$

3.3.3 Summary and Derivation

To begin, we can interpret resistivity as proportion of transmissions which will not change an opinion in a time step across all node states. The resistivity bonus reflects the idea that a leaning or strongly opinionated person will be more resistant in changing their opinion in general. All other events occur under the purview that an individual is not resistant to changing their opinion, as in, $(1 - s_m)$. Receptiveness is the event that, upon receiving a transmission, the individual will 'like' the argument they hear and change their opinion in the direction of A or B. We note that receptiveness does not play a part in the strong transition matrices A_s and B_s because we assume that strongly opinionated people are not receptive to arguments and instead change their opinion if they are not being fully resistive. Backlash is the event that an individual receives a transmission from a strongly opinionated person and 'dislikes' the argument enough to change their opinion in the opposite direction. It is assumed that this backlash is especially worse from a strong opinion on the other side, hence the backlash bonus. Transmission frequency is how often an individual of an opinion type will send a transmission in a time step. The multiplier is how many times more likely a strongly opinionated individual will send a transmission. Finally, the 'similarity' argument between the matrices is to ensure that transitions based on the 'sameness' of opinion occur. To exemplify, in matrix B_l, the probability that a B-leaning individual, upon receiving a transmission from an A-leaning individual, will transition to neutral is $B_{l_{1,2}} = A_{l_{4-1,4-2}} = A_{l_{3,2}} = e(1 - s_{m_2})$ which is the probability that an A-leaning individual, upon receiving a transmission from a B-leaning individual, will transition to neutral.

3.3.4 Implementation Details and Choices

For purposes of simulation, we fix $r = 0.80$ so that individuals are mostly resistant to changing their opinions in a time step. We fix $f = 0.05$ so that transmissions are relatively infrequent in one time step so that we don't have the delta overloading issue as discussed in (3.2.2). We fix $f_m = 3$ on just a guess that a strongly opinionated person will be three times more likely to send a transmission. With these assumptions, we have reduced the "free" parameters to r_b, e, b, b_b. In our model, we also assumed that neutral individuals do not discuss the issue in a way that changes other opinions, or that they do not know of the issue. Thus, no transmissions are sent by neutral individuals.

4 Generated Data and Analysis

Much of the data is gathered through running Monte Carlo simulations of the model as built upon the implementation details in (3.1.3) and (3.3.3). From (3.1.3), we fixed $m_0 = 3$ (number of starting nodes) and $n = 100$ in the Barabasi-Albert model and used the following social network in all simulations (Figure 1).

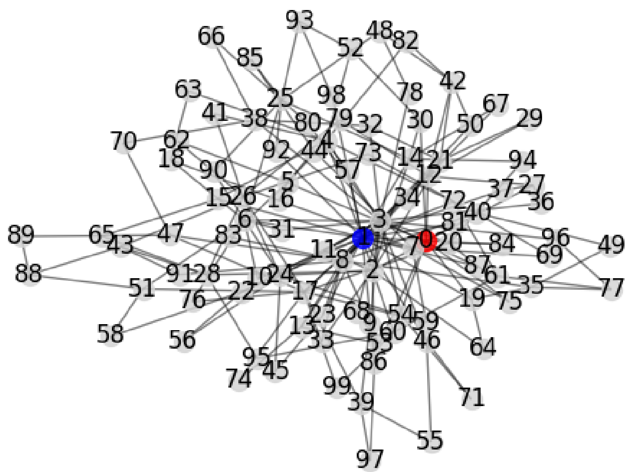

Figure 1: Social network used in all simulations. Starting states are all neutral (grey) except for two opposite strong leaning nodes (the red and blue nodes).

Taking into account the implementation details in (3.1.3), we run Monte Carlo simulations and generate time-step data. In the following is a time series gathered from a 'typical' run of 1000 time steps where b and e are chosen randomly uniformly in range $[0, 1]$ while b_b and r_b are chosen uniformly from $[0, (1-b)*2/3]$ and $[0, 0.2*2/3]$, respectively (extra bonuses are cutoff as seen in the definition for s_m and b_l) (Figure 2).

In these typical runs, we see in the time series that the count of all nodes seem to 'converge' to 'bands' and remain, perhaps in a brownian sense, in these bands. The size of these bands can be somewhat large, though, meaning that it is difficult to decipher visually which bands, on average, are higher or lower than one another (Figure 3).

Because of this issue, it makes sense that we would like to know when the node states have all reached a certain 'equilibrium' and the node counts fall within the bands. We would also like to compute the average of these bands once they reach equilibrium. If we look into the neutral count time series, we see that it falls rapidly at the beginning before reaching its equilibrium state. Thus, the neutral count time series should approximately hit a band when it reaches some sort of 'local' minimum for the first time. One idea to look for this occurrence was to peer at the moving averages of a time series to smooth it

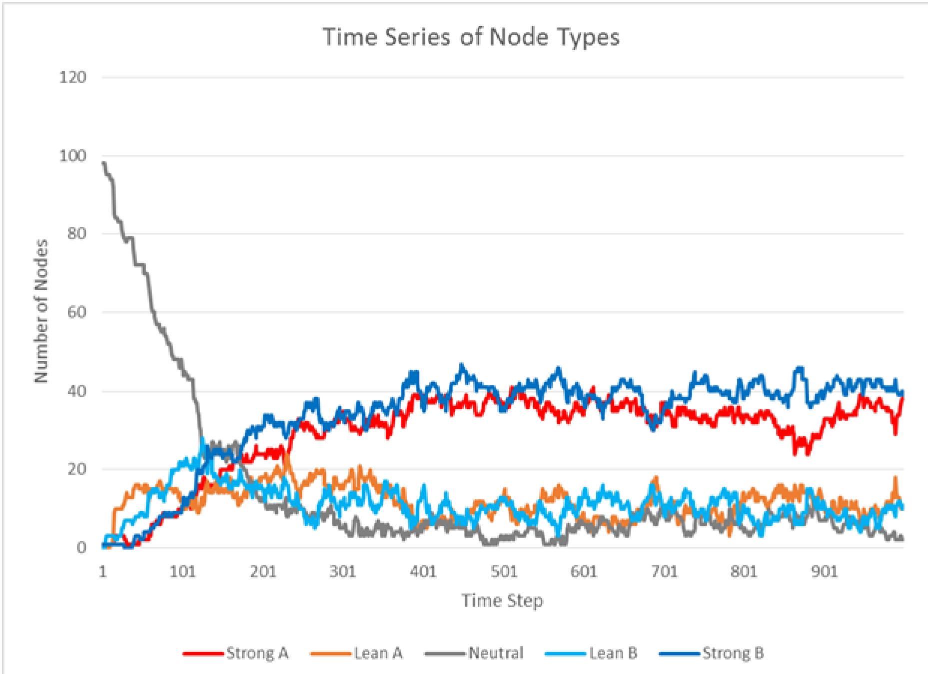

Figure 2: $(b, b_b, r_b, e) \approx (0.838, 0.088, 0.036, 0.256)$

out and then find the discrete derivative of the moving average. As it turns out, taking another moving average of the discrete derivatives gives us approximately when the time series converges. That is, when the moving average discrete derivative first hits zero (Figure 4). Because of the way the moving averages are computed, this first zero is chosen for $t > 50$. Given this information we define 'convergence' of all nodes to be when the 20-cumulative moving average of the discrete derivative of the 50-moving average of the neutral time series first hits 0 for $t > 50$.

In the ultimate goal of computing an 'average' of the band, we would like to remove the restriction of searching for the convergence point for each choice of parameters and each run and instead have a range of t where it should be likely, across all parameters and runs, that the nodes have reached equilibrium. To do so, we picked random parameters in the same manner for finding a 'typical run' (earlier discussed) and took 100 runs. The results show that the longest convergence of all these runs was $t = 521$. Thus, we say that it is likely, for $t \in [600, 1000]$ that the time series has reached 'convergence.' With this in mind, we compute the approximate average of each time series band as just the average across that range.

With these numerical definitions, we seek a 'silent-majority' situation where most nodes are not of the strong variety. That is, the sum of the averages

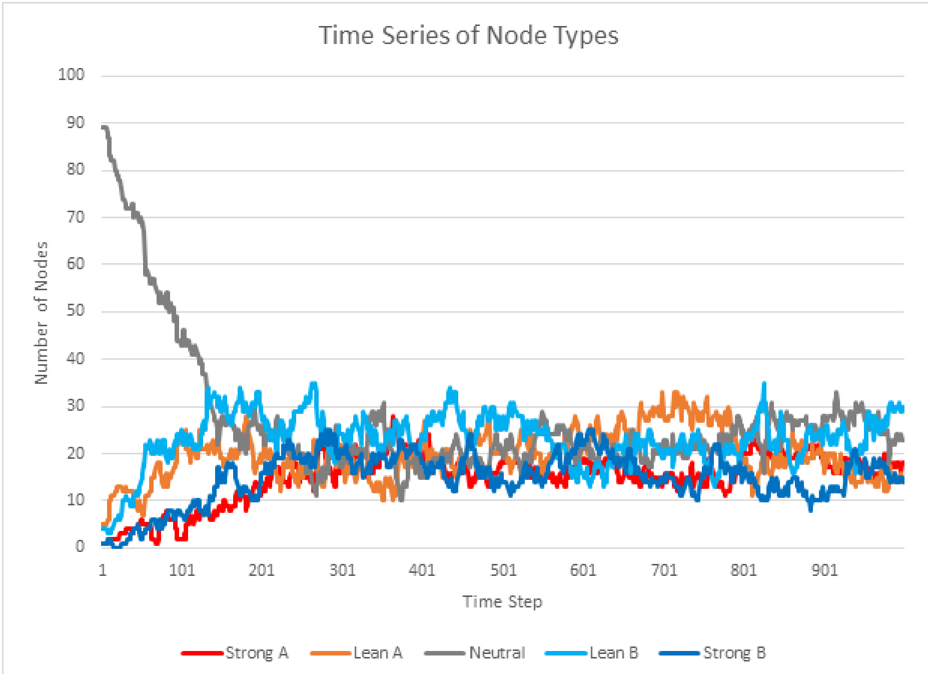

Figure 3: A time series where dominant bands are not as visually obvious. $(b, b_b, r_b, e) \approx (0.580, 0, 0, 0.573)$.

of the neutral and lean bands are higher than the sum of the strong opinion bands. The method to find this situation was to simply take 'typical' runs with randomly generated parameters until the above condition was satisfied. After many runs, the following parameters satisfied the above condition.

$$(b, b_b, r_b, e) \approx (0.580, 0.023, 0.007, 0.573).$$

Given how low the bonus parameters were, we instead check

$$(b, b_b, r_b, e) \approx (0.580, 0, 0, 0.573)$$

for reliability through taking 20 runs. From these runs the minimum, maximum, and average of the sum of lean A, lean B, and neutral bands was $65, 71$, and 68, respectively, indicating a high reliability that these parameters would give a scenario satisfying the above condition.

We note that, retroactively, Figures 3 and 4 actually employ this specific set of parameters. Taking a look at figure 3 also shows some interesting behavior in the convergence region as the neutral individuals at first convert to leaning individuals before some become strongly opinionated, as demonstrated by the strong series lagging in convergence. We also see that lean-A actually dips below both strong series around the 900-1000 time step region, though these lean-A individuals converted mostly to lean-B and not more strongly-opinionated individuals. In contrast to this specific set of parameters is the case illustrated in Figure 2, where Strong A and Strong B are clearly the dominant opinions.

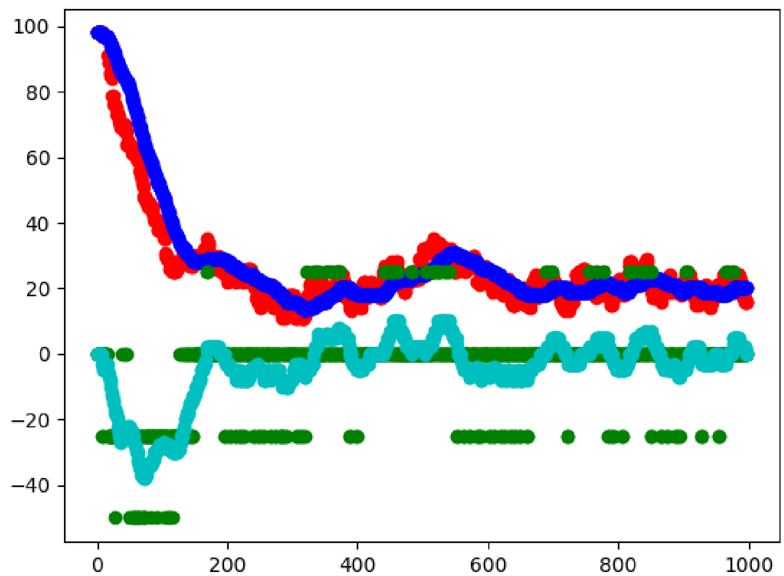

Figure 4: Red is the neutral time series. Blue is the 50-cumulative moving average of the neutral time series. Green is the discrete derivative of blue. Cyan is the 20-cumulative moving average of green. $(b, b_b, r_b, e) \approx (0.580, 0, 0, 0.573)$.

5 Conclusion

As seen from the data and analysis of the model, we found a set of parameters, namely $(b, b_b, r_b, e) \approx (0.580, 0, 0, 0.573)$ with which the model would reliably keep the neutral, A-leaning, and B-leaning counts above the strong-A and strong-B counts. Thus, a scenario where the 'silent majority' prevailed over the 'vocal minority' (and vocal they were with 3x the frequency!). In this search, we also developed numerical techniques to help analyze the model. For one, we described how to measure when the time series reaches a sort of 'equilibrium' or 'stationary' state among the nodes. We also assessed how to determine the approximate value of the bands after equilibrium is achieved. Ultimately, we wished to demonstrate that this theoretical model *might* have some efficacy. In the case of describing a desired behavior, we found it. Though, there are an overwhelming amount of choices that can be made. For one, the proposed transition probabilities model could be swapped out for a different one. In another way, not all parameter combinations could be 'swept' through because of the massive computation time it would take. Thus, the experiments show at least an inkling of promise for further study into the model. Additionally, to contrast with the Ising models, we were able to provide a divide between strong and lean opinions, which would be impossible to determine if there are only two states in total.

5.1 Personal Remark

It is of note that I do not have much experience with time series analysis. Therefore, I came up with some ideas of how to analyze the time series. Analysis of the model would be greatly enhanced if the Monte Carlo simulations were properly assessed using time series methods.

5.2 Immediate Enhancements

One observation not included in this report is the fact that if the resistivity is increased even further from 0.80, the variance of the bands could be reduced, somewhat.

5.3 Further Questions/Desired Behaviors to search for in the Model

- To what extent could an individual's involvement in transmitting an opinion actually help or hinder a cause?
- What is the actual process that 'camps' of localized networks organized by opinion form?
- Under the formation of 'camps' of opinion, do they stabilize with few new converts of strongly opinionated individuals joining, or do the two camps partition the global social network with little to no neutral or weakly leaning individuals left over?
- Is it possible under certain parameters for 'camps' to not form with free transmission of opinion between individuals of different beliefs?
- If we assume that a new divisive opinion is introduced to a social network through the addition of an involved individual, how quickly do the processes questioned earlier happen? (Note: This question is in part answered through our convergence definition).
- If we assume already a social structure organized around opinion, can a change in parameters undo this structure?
- Can parameters be estimated reasonably to 'fit' real world data in a statistical framework?

5.4 Personal Emphasis

I highly encourage anyone interested in further developing/gaining understanding of the model to run the code themselves and continue to 'play' around with it. Because of the massive number of 'choices' that can be made in the model, further methods may help append additional desired behaviors that the model could employ.

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