

Transformation

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1. Model transformation(模型变换)
2. View/Camera transformation(视图变换)
3. Projection transformation(投影变换)
 - 3.1 Orthographic Projection(正交投影)
 - 3.2 Perspective Projection(透视投影)
 - 3.2.1 推理
 - 3.2.2 Fov and Aspect ratio

MVP分别代指三种变换 (Model, View, Projection)

1. Model transformation(模型变换)

2. View/Camera transformation(视图变换)

保证相机和物体的相对位置

3. Projection transformation(投影变换)

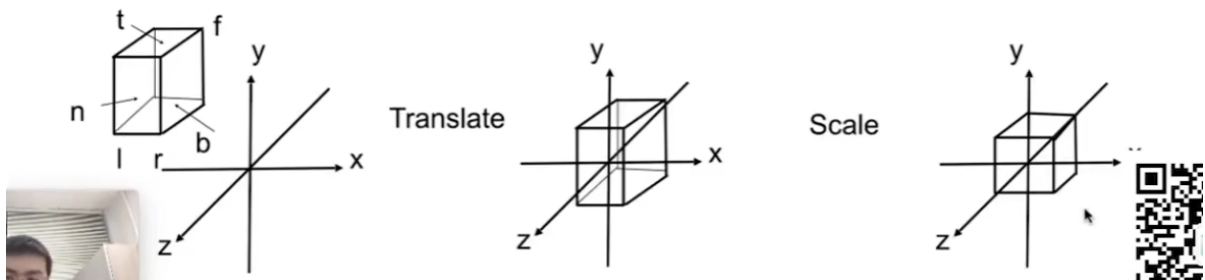
3.1 Orthographic Projection(正交投影)

3D to 2D Projection

Orthographic Projection

- Transformation matrix?
 - Translate (**center** to origin) **first**, then scale (length/width/height to **2**)

$$M_{ortho} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2}{t-b} & 0 & 0 \\ 0 & 0 & \frac{2}{n-f} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -\frac{r+l}{2} \\ 0 & 1 & 0 & -\frac{t+b}{2} \\ 0 & 0 & 1 & -\frac{n+f}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



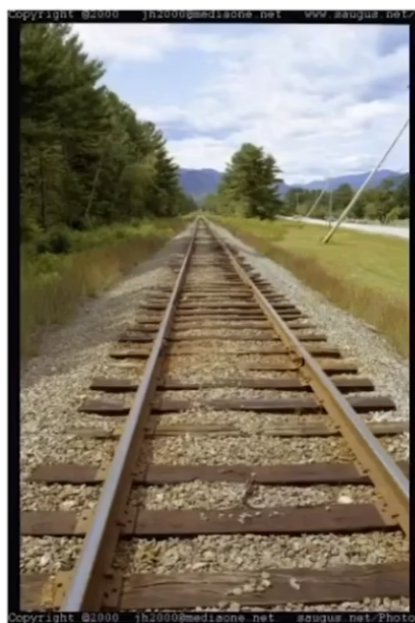
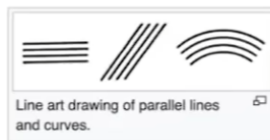
3.2 Perspective Projection(透视投影)

3.2.1 推理

- Euclid was wrong??!

In geometry, **parallel** lines are lines in a plane which do not meet; that is, two lines in a plane that do not intersect or touch each other at any point are said to be parallel. By extension, a line and a plane, or two planes, in three-dimensional Euclidean space that do not share a point are said to be parallel. However, two lines in three-dimensional space which do not meet must be in a common plane to be considered parallel; otherwise they are called **skew lines**. Parallel planes are planes in the same three-dimensional space that never meet.

Parallel lines are the subject of Euclid's parallel postulate.^[1] Parallelism is primarily a property of affine geometries and Euclidean geometry is a special instance of this type of geometry. In some other geometries, such as hyperbolic geometry, lines can have analogous properties that are referred to as parallelism.



[https://en.wikipedia.org/wiki/Parallel_\(geometry\)](https://en.wikipedia.org/wiki/Parallel_(geometry))

透视投影需要先将Frustum(截锥体)变成Cuboid(正方体), 再做正交投影

- How to do perspective projection

- First "squish" the frustum into a cuboid ($n \rightarrow n, f \rightarrow f$) ($M_{\text{persp} \rightarrow \text{ortho}}$)
- Do orthographic projection (M_{ortho} , already known!)

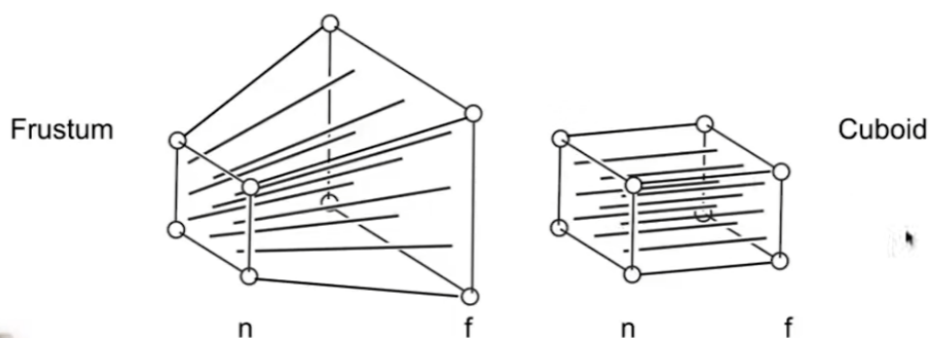


Fig. 7.13 from *Fundamentals of Computer Graphics, 4th Edition*

转换方法是使用相似三角形

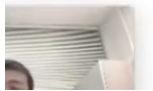
- In order to find a transformation

- Find the relationship between transformed points (x', y', z') and the original points (x, y, z)

$$y' = \frac{n}{z}y \quad x' = \frac{n}{z}x \text{ (similar to } y')$$

- In homogeneous coordinates,

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} nx/z \\ ny/z \\ \text{unknown} \\ 1 \end{pmatrix} \xrightarrow{\text{mult. by } z} \begin{pmatrix} nx \\ ny \\ \text{still unknown} \\ z \end{pmatrix}$$



可以得到除z轴外，透视投影矩阵所有的信息

- So the “squish” (persp to ortho) projection does this

$$M_{persp \rightarrow ortho}^{(4 \times 4)} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} nx \\ ny \\ \text{unknown} \\ z \end{pmatrix}$$

- Already good enough to figure out part of $M_{persp \rightarrow ortho}$

$$M_{persp \rightarrow ortho} = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ ? & ? & ? & ? \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{WHY?}$$





z轴满足 $z = n$ 或 $z = f$ 时没有任何变化，得到如下关系

- What do we have now?

$$(0 \quad 0 \quad A \quad B) \begin{pmatrix} x \\ y \\ n \\ 1 \end{pmatrix} = n^2 \quad \Rightarrow \quad An + B = n^2$$

- Any point's z on the far plane will not change



$$\begin{pmatrix} 0 \\ 0 \\ f \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 0 \\ f \\ 1 \end{pmatrix} == \begin{pmatrix} 0 \\ 0 \\ f^2 \\ f \end{pmatrix} \quad \Rightarrow \quad Af + B = f^2$$



- Solve for A and B


$$\begin{aligned} An + B &= n^2 \\ Af + B &= f^2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} A &= n + f \\ B &= -nf \end{aligned}$$

- Finally, every entry in $M_{\text{persp} \rightarrow \text{ortho}}$ is known!

- What's next?

- Do orthographic projection (M_{ortho}) to finish



$$M_{\text{persp}} = M_{\text{ortho}} M_{\text{persp} \rightarrow \text{ortho}}$$


得到:

$$M_{\text{persp} \rightarrow \text{ortho}} = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n + f & -nf \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

最终的透视投影矩阵为:

$$M_{\text{persp}} = M_{\text{ortho}} M_{\text{persp} \rightarrow \text{ortho}}$$

Tip:

经过 $M_{\text{persp} \rightarrow \text{ortho}}$ 后 n 与 f 平面之间的点会被**往 n 平面推**, 推导如下:

选取中间平面任意一点:

$$\begin{pmatrix} x \\ y \\ \frac{n+f}{2} \\ 1 \end{pmatrix}$$

左乘 $M_{persp \rightarrow ortho}$ 得到如下结果:

$$\begin{pmatrix} nx \\ ny \\ \frac{(n+f)^2}{2} - nf \\ \frac{n+f}{2} \end{pmatrix} == \begin{pmatrix} \frac{2nx}{n+f} \\ \frac{2ny}{n+f} \\ n + f - \frac{2nf}{n+f} \\ 1 \end{pmatrix}$$

其中新的Z轴坐标 $n + f - \frac{2nf}{n+f} == \frac{2n^2+2f^2}{2(n+f)}$, 而原Z轴坐标为 $\frac{n+f}{2} == \frac{n^2+f^2+2nf}{2(n+f)}$, 所以只需要判断 $2n^2 + 2f^2$ 和 $n^2 + f^2 + 2nf$ 大小即可。

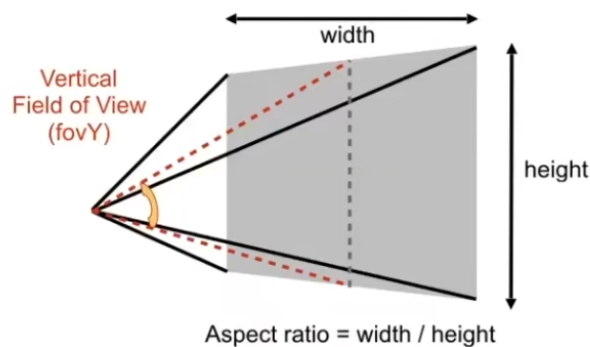
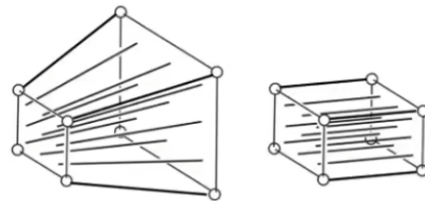
两边同减 $n^2 + f^2 + 2nf$ 得到 $(n - f)^2$ 和 0 , 所以新的Z轴更大, 而在右手坐标系中, $n > f$, 所以比之前更接近n平面

3.2.2 Fov and Aspect ratio

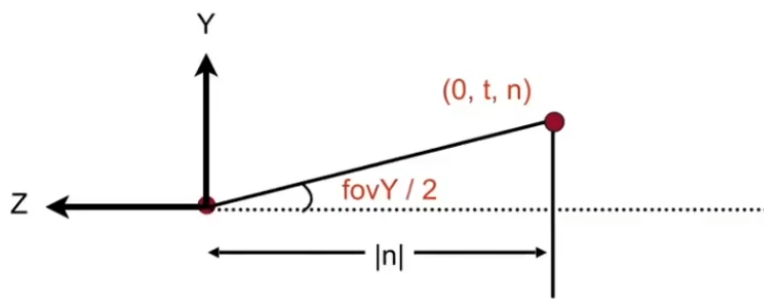
FovY(垂直视场角), Aspect ratio(宽高比)

- What's near plane's l, r, b, t then?

- If explicitly specified, good
- Sometimes people prefer: vertical **field-of-view** (fovY) and **aspect ratio** (assume symmetry i.e. l = -r, b = -t)



- How to convert from fovY and aspect to l, r, b, t?
 - Trivial



$$\tan \frac{fovY}{2} = \frac{t}{|n|}$$

$$aspect = \frac{r}{t}$$