Friday, May 28, 2021 11:16 PM

Numerical Algorithms

* What
* Contents
  - Solution of equation of single variable
  - Linear System of eq.
  - Matrix decomposition
  - Optimization

→ Analytical

→ Graphical

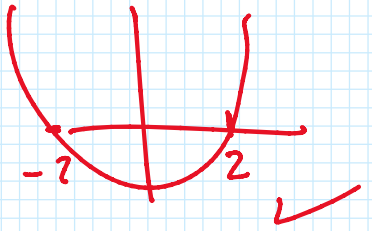
→ Numerical
initial guess
↓
Approximate solution
↺ Fast

$y = x^2 - 4 = 0$

$x^2 = 4$

$x = \pm\sqrt{4} = \pm 2$

1.9999 ...



$-\infty \leftarrow \ldots, -1, 2, \ldots \rightarrow \infty$

$-\infty < x < \infty$

Real Numbers $\mathbb{R}$

$0.4 + 0.8 == 1.2$?

$1.2 - 0.8 == 0.4$?

$1.2 - 0.4 == 0.8$?

$1.2 - 0.4 - 0.8 == 0$?

$0.1 + 0.1 == 0.2$?

$0.1 + 0.1 + 0.1 == 0.3$?

Saturday, May 29, 2021 7:49 AM

Decimal System

153 → 1 x 100 + 5 x 10 + 3 x 1

= 100 + 50 + 3

0 → 9

10^3	10^2	10^1	10^0
7	5	3	1

$1 \times 10^2 + 5 \times 10^1 + 3 \times 10^0$

$1000 + 500 + 30 + 1$

$= 153$

7531

Binary System

0, 1

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
0	0	0	0	0	1	1	0

$1 \times 2^2 + 1 \times 2^1$

$= 2 + 4 = 6$

Quiz

- $(0000\ 0000)_2 = 0$
- $(0000\ 0001)_2 = 1$
- $(0000\ 1111)_2 = 15$
- $(1000\ 0000)_2 = 128$
- $(1111\ 1111)_2 = 255$

$(29)_{10} = (\quad)_2$

$(231)_{10} = (\quad)_2$

29 = 16 + 8 + 4 + 1

00011101

Saturday, May 29, 2021 8:17 AM

10^{-1} →

Decimal

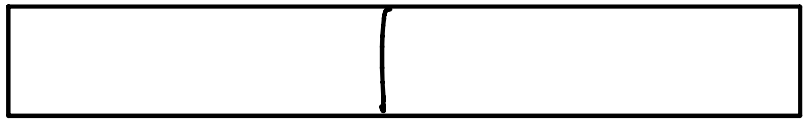
3.14

$3 \times 1 + 1 \times 0.1 + 4 \times 0.01$

$1 = 10^0$ $\quad 1 = 10^{-1}$ $\quad 1 = 10^{-2}$

1/10 1/100 1/1000 - -

$\cdots 10^3\ 10^2\ 10^1\ 10^0 \bullet 10^{-1}\ 10^{-2}\ 10^{-3}\ 10^{-4} \cdots$



Binary

1/2 1/4 1/8 1/16 — .

$\cdots 2^3\ 2^2\ 2^1\ 2^0 \bullet 2^{-1}\ 2^{-2}\ 2^{-3}\ 2^{-4} \cdots$

$2 + 1 + 1/2$

$= \boxed{3.5}$

Decimal Integer → Binary

Decimal real → Binary

$\boxed{1/3} \rightarrow 0.\overline{3333}\ldots$

$(1/10)_{10} \rightarrow 0.0001100110011\ldots$

Quiz

- $(0.1)_2 =$
- $(0.011)_2 =$
- $(0.101)_2 =$

→ $(0.5625)_{10} = (\quad)_2$

→ • $(0.1)_{10} = (\quad)_2$

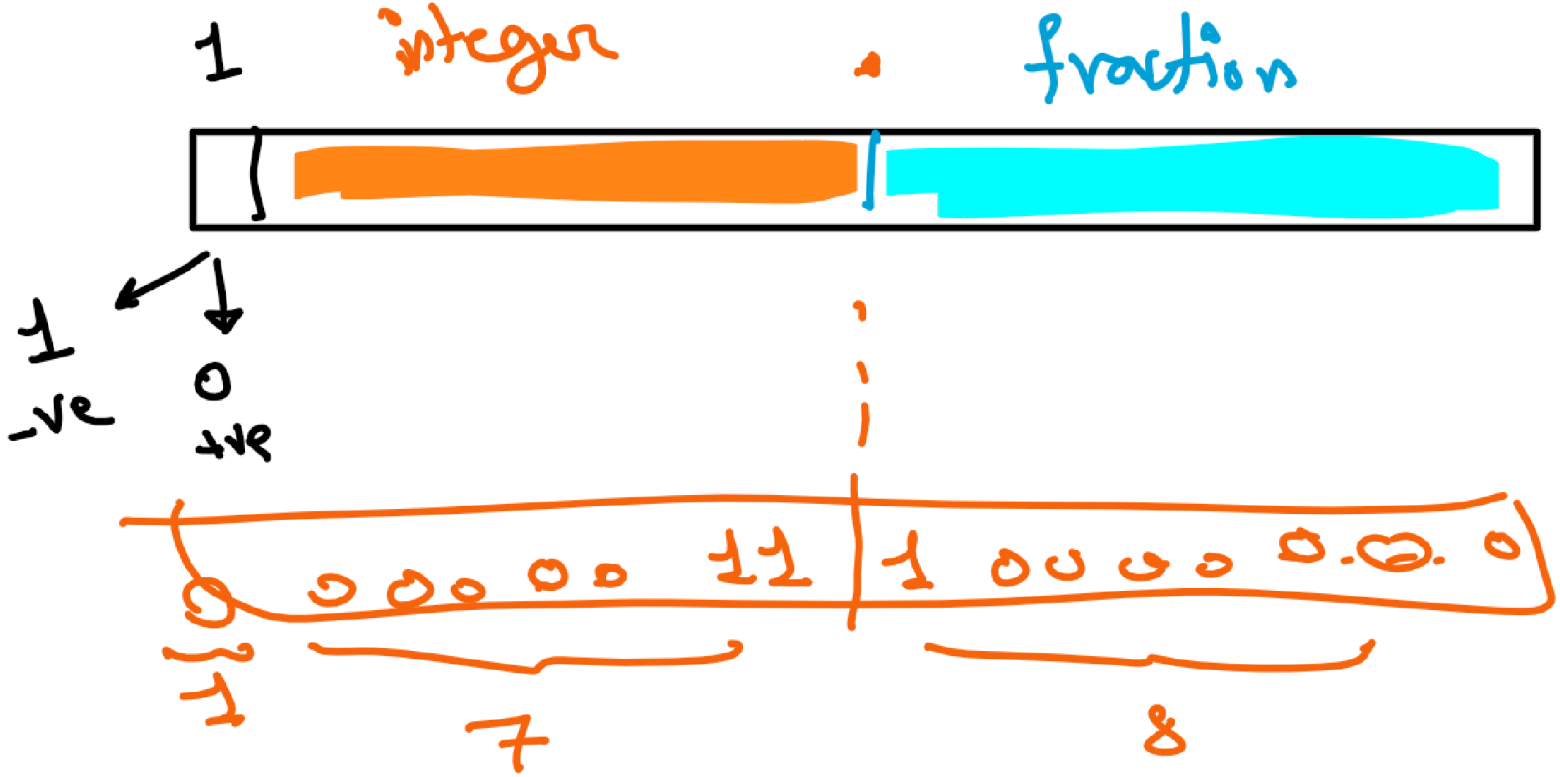
$0.5625 = 0.5 + 0.0625$ (1/16)

1/2

Saturday, May 29, 2021 8:47 AM

+0 → ⇐ −0

16-bit



Fixed

3.5

- limited & fixed representation

Point Representation

- Two distinct vals for $\emptyset$

Saturday, May 29, 2021 8:54 AM

Scientific Notation

365.25

$365.25 = 3.6525 \times 10^{2}$

$0.0036525 = 3.6525 \times 10^{-3}$

$R \in [1, 10)$

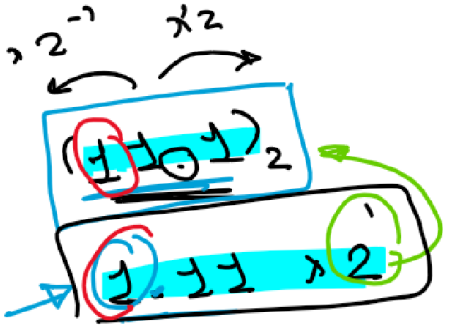
$X = \pm S \times 10^{E}$

$1 \leq S < 10$

Scientific Notation

$X = \pm S \times 2^{E}$

$1 \leq S < 2$



$S = 1.f$

$X = \pm (1 + f) \times 2^{E}$

① 1.011×2^{-3}

② 1.001001×2^{3}

③ 1.0×2^{0} ✓

Quiz

- $(0.001011)_2 =$
- $(1001.001)_2 =$
- $(1.0)_2 =$

Floating Point Representation

Saturday, May 29, 2021 9:12 AM

Floating Point

$$X = \pm (1 + f) \times 2^{E}$$

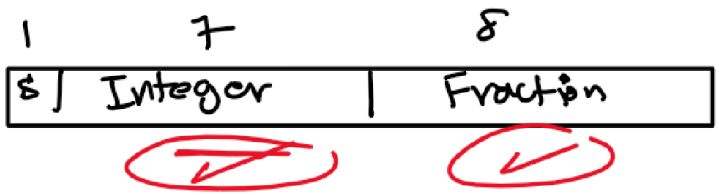
0 011 11001 → $+(1 + 0.11001) \times 2^{3}$

$= (1.011001) \times 2^{3}$

$= 1011.001$

* No fixed size for integers and fraction
* Exponent increases range

Fixed Point



-15 → +15

2^{5}

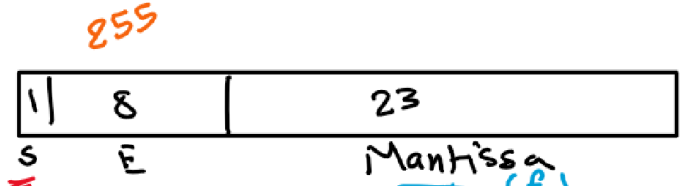
$2^{5} - 1$

$2^{-15} \rightarrow 2^{0} \rightarrow 2^{15}$

Saturday, May 29, 2021 9:22 AM

IEEE 745: Floating Point

Single Precision (32-bit)



255

00000

$X = (-1)^S * (1+f) * 2^{E_{10}-(127)_{10}}$

f: $\frac{128}{2}$

E: 1 → 254

0 255 special values

- Range

-126 → +127

Mantissa:

0 → $2^{23}-1$

0 → 255

2 → 254

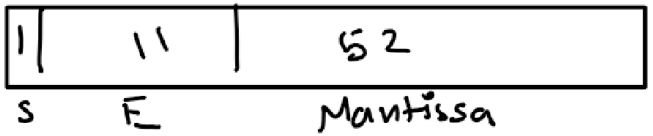
1 → 255

0 → 255

2^{E-127} Bias

0.0001

Double-Precision (64-bit)



Quiz

Single Precision

M

$(1 \quad 10000100 \quad 01100100\ldots0)_2$

$= (\quad)_{10}$

$X = (-1)^S * (1+M) \times 2^{E-127}$

132

128+4 = 127

$= -1 * (1+0.011001) \times 2^{5}$

$= -1.011001 \times 2^{5}$

32 16 8 4 2 1 1/2

$= -101100.1$

$= -(32+8+4).5 = \boxed{-44.5}$

$= -(32+8+4) \cdot 5 = \boxed{-44.5}$ ✓

44.5

$101100.1 \rightarrow 1.\boxed{011001} \times 2^{5}$

$5 + 127 = \boxed{132}$ (E)

Quiz

- Smallest positive number can be stored? Single Precision
- Largest " " " " " ?
- How to store zero? $E=0$, $M=0$ $\boxed{+0}$ $\boxed{-0}$
- How to store $+\infty$, $-\infty$?

$E = 255$, $M = 0$

$\frac{\infty}{\infty}$ = not defined $\quad \frac{0}{0}$ = not defined

Not a Number NaN

- $E = 255$ $M \neq 0$ $S = 0$

$(1+0) \times 2^{1-127} = \boxed{2^{-126}}$

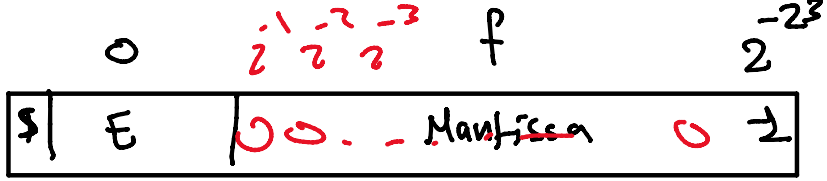
$0.\overline{111\cdots1} \approx 1$

$(1 + 0.\overline{111\cdots1}) \times 2^{254-127}$

$1 + 1$

$2 \times 2^{127} \approx \boxed{2^{128}}$

Single Precision



1001

Machine Epsilon ε

2^{-23}

$x_1 = (1+f) \times 2^0 = 1.0$

$x_2 = (1+0.000\ldots01) \times 2^0 = 1.0\text{----}01$

$x_1 = (1+f) \times 2^2 = (1.0) \times 2^2 = 100 = 4$

$x_2 = (1+0.00\ldots1) \times 2^2 = 1.000\ldots1 \times 2^2 = 100.00\ldots100$

$= 4.0 + 1 \times 2^{-21}$

$4+2$ (−22) X

Rounding

(−21) (−22) (−23)

ulp $2^{-21} \times 2^2 = 2^{-23} \times 2^2 = 2^{-21}$

$ulp = \varepsilon \times 2^E$

jump between two numbers

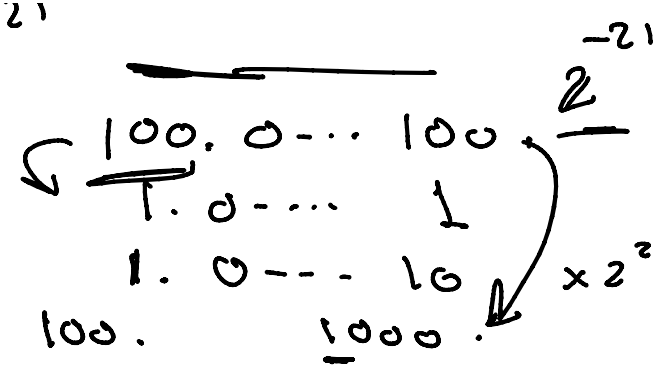
np.spacing(—)

Rounding Error

$x_2 = 4 + 2^{-21}$ —— 2^{-21}

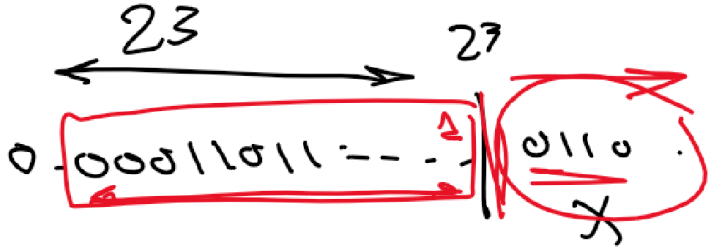
Rounding Error

$x_2 = 1 + 2^{-21}$



Saturday, May 29, 2021 12:35 PM

$(0.1)_{10}$ → infinite digits



Truncation Error

Rounding Error

Propagation Error

Cancellation Error

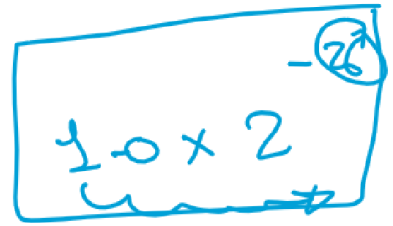
A - B

Absorption Error

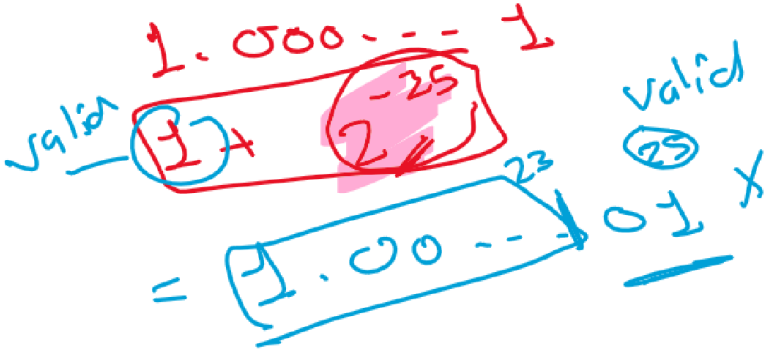
$x = 3.14159265358979$

$y = 3.14159265358682$

$x - y = 0$



1.0×2^0



Friday, May 28, 2021 11:16 PM

Root Finding Problem

$$2x - 4 = 0$$

$$f(x) = 2x - 4 = 0$$

2

$$2 \times 2 - 4 = 0$$

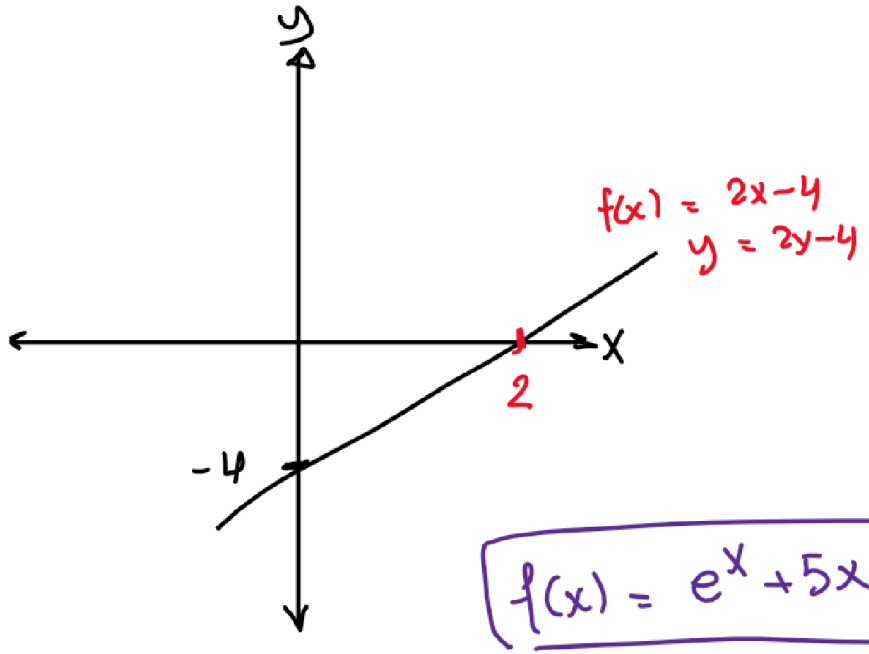
$$2x - 4 = 0$$

$$\frac{2x}{2} = \frac{4}{2}$$

$$\boxed{x = 2}$$

solution // root

$x = 2$ is a solution



$$f(x) = e^x + 5x - 2$$

Quiz

$$f(x) = 2x^2 + 6x + 4$$

roots?

$$x_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$(2x + 2)(x + 2) = 0$$

$$2x + 2 = 0$$

$$x = -1$$

$$x + 2 = 0$$

$$x = -2$$

Saturday, May 29, 2021 1:28 PM

Root Finding

- Bisection Method
- Fixed Point Method
- Newton's Method

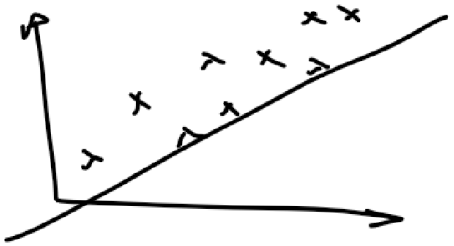
Numerical Algorithm :

1. Start initial guess
   root = ?
2. Iterations
   → improve root / guess
3. After N iterations
   guess ≈ actual root

min
max

ML ← optimization

$f(x) = mx + b$



Loss Function (m, b)

$$\frac{dL}{dm} = 0$$

Root Finding Problem

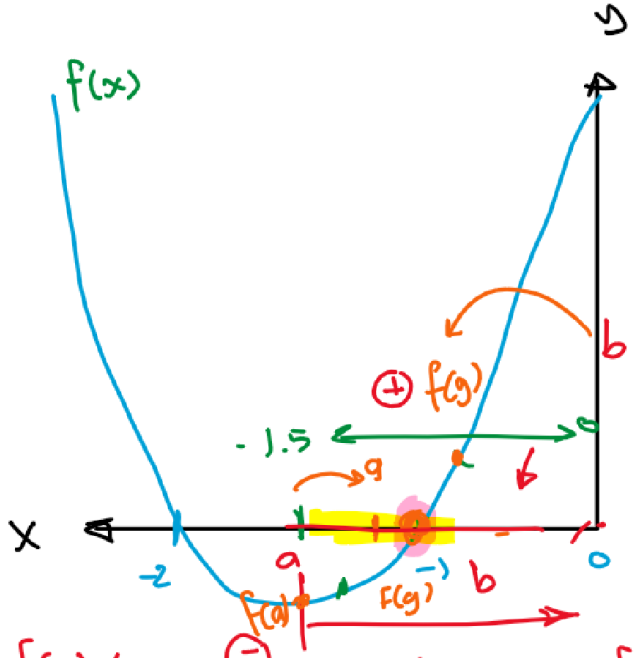
Saturday, May 29, 2021 1:51 PM

Bisection Method

$f(x) = 2x^2 + 6x + 4$

root

1. Interval
2. f(x)
3. tolerance ε



$|a-b|$ so small

$|a<b| < \varepsilon$

10^{-4}

$f(g)f(a) < 0$ → on different sides of x-axis

$f(g)f(a) > 0$

Algorithm

while $|a-b| > \varepsilon$

$g = \frac{a+b}{2}$

root on the left of g ? $f(a)f(g) < 0$

↳ $b = g$

root on the right of g ? else

↳ $a = g$

Saturday, May 29, 2021 2:24 PM

Fixed Point Method

$f(x) = e^x - 5x + 2 = 0$

$y = e^x - 5x + 2$ $\quad y = 0$

$e^x - 5x + 2 = 0$

$\ln e^x = \ln(5x - 2)$ → $g(x)$

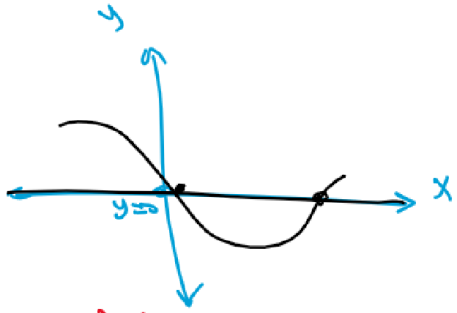
$x = \ln(5x-2)$

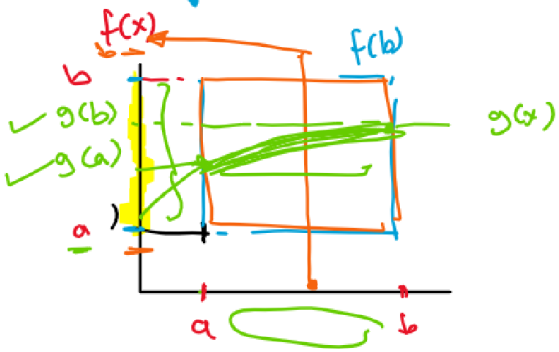
$y = x$ $\quad y = \ln(5x-2)$

$f(x) = 0$

$\Downarrow$

$g(x) = x$





$f(x) = e^x - 5x + 2$

$= e^x - x - 4x + 2 = 0$

$e^x - 4x + 2 = x$

$g(x)$

$e^x + 2 = 5x$

$\frac{e^x + 2}{5} = x$

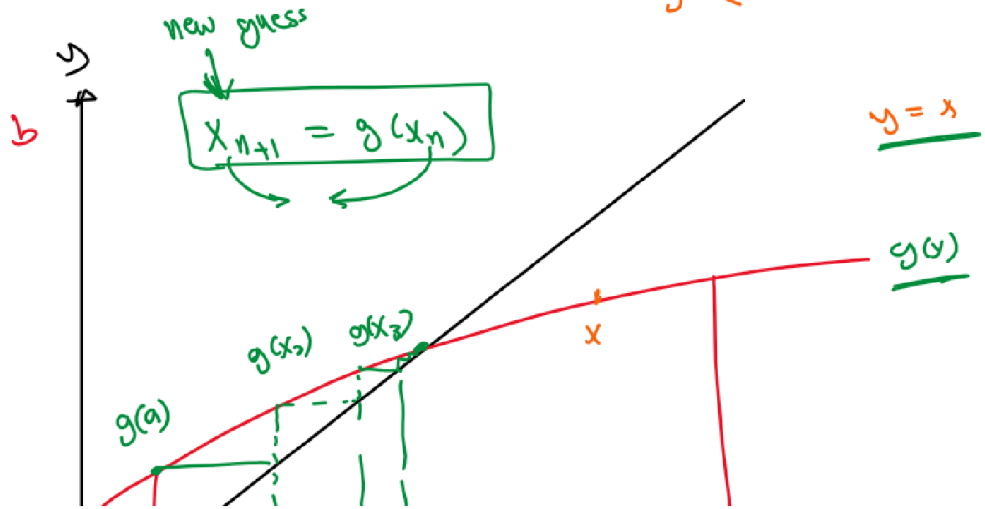
$g(x)$

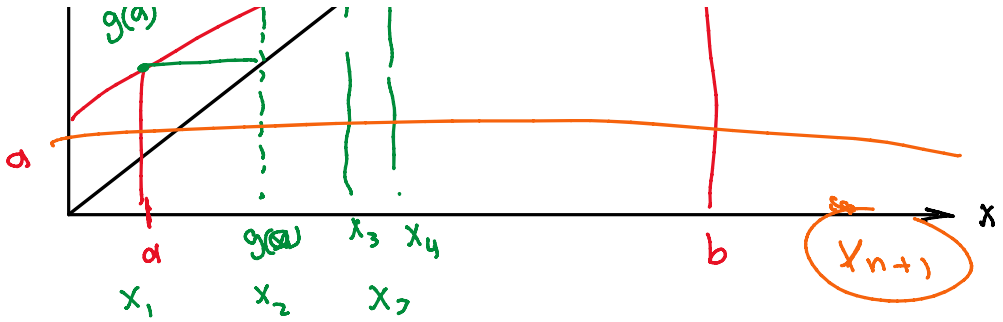
Conditions

$g(x)$ → continuous in $[a, b]$

→ $g(x) \in [a, b]$ $\forall x \in [a, b]$

$g(a \to b) \to a \to b$





Alg.

$p_n = a$

while $|p_{n+1} - p_n| > \varepsilon$

$p_{n+1} = g(p_n)$

Saturday, May 29, 2021 3:03 PM

Newton's Method

$$f'(x_1) = \tan\theta = \frac{f(x_1)}{\Delta x}$$

$$\Delta x = \frac{f(x_1)}{f'(x_1)}$$

$$x_1 - x_2 = \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$|x_{n+1} - x_n| < \varepsilon$$

Δx

1. $f(x)$
2. $\frac{df(x)}{dx}$, $f'(x)$
3. tolerance

Algorithm

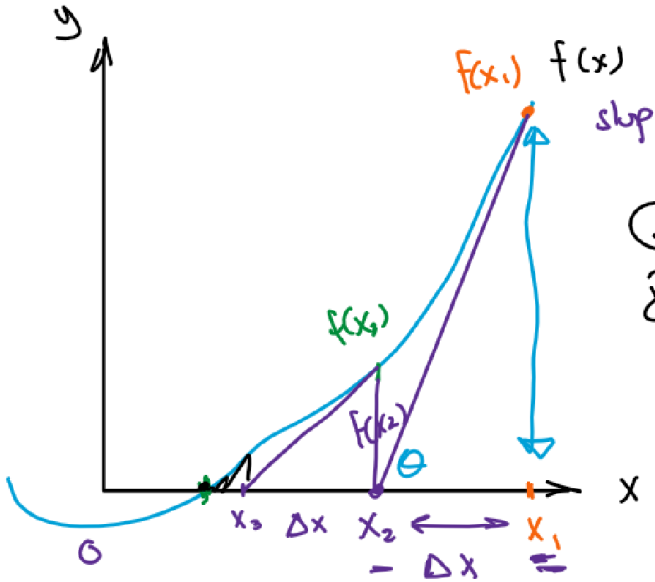
x_1 = guess

$x_2 = x_1 - f(x_1)/f'(x_1)$

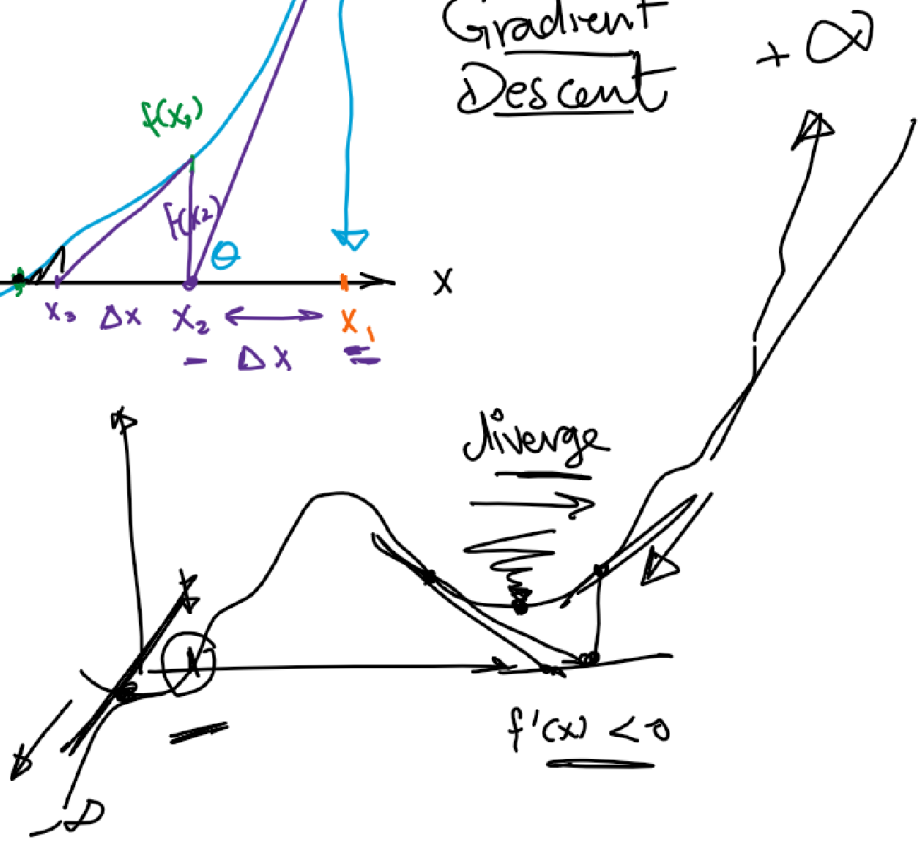
while $|x_2 - x_1| > \varepsilon$

$x_1 = x_2$

$x_2 = x_1 - f(x_1)/f'(x_1)$



Gradient Descent



Saturday, May 29, 2021 3:40 PM

HW ⓔ EP

① Implement 3 algorithms using

$$f(x) = 2x^2 + 6x + 4$$

How many iterations to reach the solution?

② Use different $g(x)$ for $f(x) = e^x - 5x + 2$ [Fixed Point]

other than $g(x) = \ln(5x - 2)$

③ Use Newton's method $p = 1$, what happens? why?

④ Apply Bisection Method on — $e^x - 5x + 2$

2.3 →