## Universidad Autónoma de Nuevo León

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# Materiales compuestos 2

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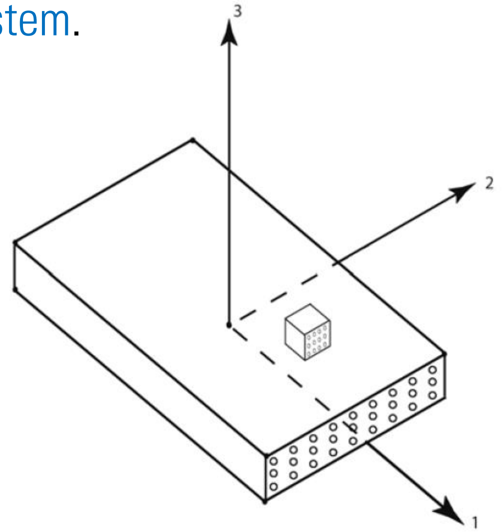
## Linear Elastic Stress-Strain Relations

Consider a single layer of fiber-reinforced composite material as shown in Fig. 1. In this layer, the 1-2-3 orthogonal coordinate system is used where the directions are taken as follows:

1. The 1-axis is aligned with the fiber direction.
2. The 2-axis is in the plane of the layer and perpendicular to the fibers.
3. The 3-axis is perpendicular to the plane of the layer and thus also perpendicular to the fibers.

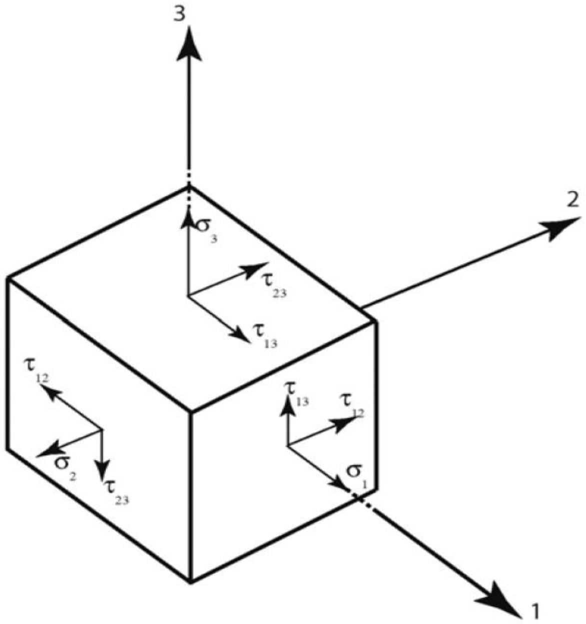
The 1-direction is also called the fiber direction, while the 2- and 3- directions are called the matrix directions or the transverse directions. This 1-2-3 coordinate system is called the principal material coordinate system. The stresses and strains in the layer (also called a lamina) will be referred to the principal material coordinate system.



# Linear Elastic Stress-Strain Relations

At this level of analysis, the strain or stress of an individual fiber or an element of matrix is not considered. The effect of the fiber reinforcement is smeared over the volume of the material. We assume that the two-material fiber-matrix system is replaced by a single homogeneous material. Obviously, this single material does not have the same properties in all directions. Such material with different properties in three mutually perpendicular directions is called an orthotropic material. Therefore, the layer (lamina) is considered to be orthotropic.

The stresses on a small infinitesimal element taken from the layer are illustrated in Fig. 2. There are three normal stresses $\sigma_1$, $\sigma_2$, and $\sigma_3$, and three shear stresses $\tau_{12}$, $\tau_{23}$, and $\tau_{13}$. These stresses are related to the strains $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$, $\gamma_{12}$, $\gamma_{23}$, and $\gamma_{13}$ as follows:



# Linear Elastic Stress-Strain Relations

In the following matrix equation (1), E1, E2, and E3 are the extensional moduli of elasticity along the 1, 2, and 3 directions, respectively. Also, vij (i, j = 1, 2, 3) are the different Poisson's ratios, while G12, G23, and G13 are the three shear moduli.

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}$$

This matrix can be written in a compact form as follows:

$$\{\varepsilon\} = [S]\{\sigma\}$$

where {ε} and {σ} represent the 6 $\times$ 1 strain and stress vectors, respectively, and [S] is called the compliance matrix. The elements of [S] are clearly obtained from the matrix, i.e. S11 = 1/E1, S12 = −v21/E2, . . . , S66 = 1/G12.

# Linear Elastic Stress-Strain Relations

The inverse of the compliance matrix [S] is called the stiffness matrix [C] given, in general, as follows:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

In compact form is written as follows:

$$\{\sigma\} = [C]\,\{\varepsilon\}$$

It is shown that both the compliance matrix and the stiffness matrix are symmetric, i.e. C21 = C12, C23 = C32, C13 = C31, and similarly for S21, S23, and S13.

# Linear Elastic Stress-Strain Relations

Therefore, the following expressions (5) can now be easily obtained:

$$C_{11} = \frac{1}{S}(S_{22}S_{33} - S_{23}S_{23})$$

$$C_{12} = \frac{1}{S}(S_{13}S_{23} - S_{12}S_{33})$$

$$C_{22} = \frac{1}{S}(S_{33}S_{11} - S_{13}S_{13})$$

$$C_{13} = \frac{1}{S}(S_{12}S_{23} - S_{13}S_{22})$$

$$C_{33} = \frac{1}{S}(S_{11}S_{22} - S_{12}S_{12})$$

$$C_{23} = \frac{1}{S}(S_{12}S_{13} - S_{23}S_{11})$$

$$C_{44} = \frac{1}{S_{44}}$$

$$C_{55} = \frac{1}{S_{55}}$$

$$C_{66} = \frac{1}{S_{66}}$$

$$S = S_{11}S_{22}S_{33} - S_{11}S_{23}S_{23} - S_{22}S_{13}S_{13} - S_{33}S_{12}S_{12} + 2S_{12}S_{23}S_{13}$$

## Linear Elastic Stress-Strain Relations

It should be noted that the material constants appearing in the compliance matrix in are not all independent. This is clear since the compliance matrix is symmetric. Therefore, we have the following equations relating the material constants (6):

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}$$
$$\frac{\nu_{13}}{E_1} = \frac{\nu_{31}}{E_3}$$
$$\frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3}$$

The above equations are called the reciprocity relations for the material constants. It should be noted that the reciprocity relations can be derived irrespective of the symmetry of the compliance matrix – in fact, we conclude that the compliance matrix is symmetric from using these relations. Thus, it is now clear that there are nine independent material constants for an orthotropic material.

## Linear Elastic Stress-Strain Relations

A material is called transversely isotropic if its behavior in the 2-direction is identical to its behavior in the 3-direction. For this case, E2 = E3, v12 = v13, and G12 = G13. In addition, we have the following relation:

$$G_{23} = \frac{E_2}{2(1+\nu_{23})}$$

It is clear that there are only five independent material constants (E1, E2, v12, v23, G12) for a transversely isotropic material.

A material is called isotropic if its behavior is the same in all three 1-2-3 directions. In this case, E1 = E2 = E3 = E, v12 = v23 = v13 = v, and G12 = G23 = G13 = G. In addition, we have the following relation:

$$G = \frac{E}{2(1+\nu)}$$

It is clear that there are only two independent material constants (E, v) for an isotropic material.

## Linear Elastic Stress-Strain Relations

Example 1.

For an orthotropic material, derive expressions for the elements of the stiffness matrix $C_{ij}$ directly in terms of the nine independent material constants.

Solution:

Substitute the elements of [S] from (1) into (5) along with using (6). This is illustrated in detail for C11 below. First evaluate the expression of S from (5) as follows:

$$
\begin{aligned}
S = {} & S_{11}S_{22}S_{33} - S_{11}S_{23}S_{23} - S_{22}S_{13}S_{13} - S_{33}S_{12}S_{12} + 2S_{12}S_{23}S_{13} \\
= {} & \frac{1}{E_1}\frac{1}{E_2}\frac{1}{E_3} - \frac{1}{E_1}\left(\frac{-\nu_{23}}{E_2}\right)\left(\frac{-\nu_{32}}{E_3}\right) \\
& - \frac{1}{E_2}\left(\frac{-\nu_{13}}{E_1}\right)\left(\frac{-\nu_{31}}{E_3}\right) - \frac{1}{E_3}\left(\frac{-\nu_{12}}{E_1}\right)\left(\frac{-\nu_{21}}{E_2}\right) \\
& + 2\left(\frac{-\nu_{12}}{E_1}\right)\left(\frac{-\nu_{23}}{E_2}\right)\left(\frac{-\nu_{31}}{E_3}\right) \\
= {} & \frac{1 - \nu_{23}\nu_{32} - \nu_{13}\nu_{31} - \nu_{12}\nu_{21} - 2\nu_{12}\nu_{23}\nu_{31}}{E_1E_2E_3} \\
= {} & \frac{1 - \nu_0}{E_1E_2E_3}
\end{aligned}
$$

$$\nu_0 = \nu_{23}\nu_{32} + \nu_{13}\nu_{31} + \nu_{12}\nu_{21} + 2\nu_{12}\nu_{23}\nu_{31}$$

$$
\begin{aligned}
C_{11} &= \frac{1}{S}(S_{22}S_{33} - S_{23}S_{23}) \\
&= \frac{E_1E_2E_3}{1 - \nu_0}\left[\frac{1}{E_2}\frac{1}{E_3} - \left(\frac{-\nu_{23}}{E_2}\right)\left(\frac{-\nu_{32}}{E_3}\right)\right] \\
&= \frac{(1 - \nu_{23}\nu_{32})\,E_1}{1 - \nu_0}
\end{aligned}
$$

## Linear Elastic Stress-Strain Relations

Similarly, the following expressions for the other elements of [C] can be derived:

$$C_{12} = \frac{(\nu_{21} + \nu_{31}\nu_{23})\,E_1}{1-\nu_0} = \frac{(\nu_{12} + \nu_{32}\nu_{13})\,E_2}{1-\nu_0}$$

$$C_{13} = \frac{(\nu_{31} + \nu_{21}\nu_{32})\,E_1}{1-\nu_0} = \frac{(\nu_{13} + \nu_{12}\nu_{23})\,E_3}{1-\nu_0}$$

$$C_{22} = \frac{(1 - \nu_{13}\nu_{31})\,E_2}{1-\nu_0}$$

$$C_{23} = \frac{(\nu_{32} + \nu_{12}\nu_{31})\,E_2}{1-\nu_0} = \frac{(\nu_{23} + \nu_{21}\nu_{13})\,E_3}{1-\nu_0}$$

$$C_{33} = \frac{(1 - \nu_{12}\nu_{21})\,E_3}{1-\nu_0}$$

$$C_{44} = G_{23}$$

$$C_{55} = G_{13}$$

$$C_{66} = G_{12}$$

## Linear Elastic Stress-Strain Relations

Example 2. Consider a 60-mm cube made of graphite-reinforced polymer composite material that is subjected to a tensile force of 100 kN perpendicular to the fiber direction, directed along the 2-direction. The cube is free to expand or contract. Use MATLAB to determine the changes in the 60-mm dimensions of the cube. The material constants for graphite-reinforced polymer composite material are given as follows:

$$V_f = 0.5$$

$$E_f = 303.83GPa \qquad E_m = 6.17GPa$$

$$\nu_{23} = 0.458, \qquad \nu_{12} = \nu_{13} = 0.248$$

$$G_{23} = 3.20 \text{ GPa}, \qquad G_{12} = G_{13} = 4.40 \text{ GPa}$$

Example 3. Repeat Example 2 if the cube is made of aluminum instead of graphite reinforced polymer composite material. The material constants for aluminum are E = 72.4 GPa and ν = 0.300. Use MATLAB:

```
d1 =

   -0.0069

>> d2 = epsilon(2)*60

d2 =

    0.0230

>> d3 = epsilon(3)*60

d3 =

   -0.0069
```

AF 5.

1.- Consider a 40-mm cube made of glass-reinforced polymer composite material that is subjected to a compressive force of 150 kN perpendicular to the fiber direction, directed along the 3-direction. The cube is free to expand or contract. Use MATLAB to determine the changes in the 40-mm dimensions of the cube. The material constants for glass-reinforced polymer composite material are given as follows: A=-0.0191, -0.1056, 0.2467

$$\begin{aligned} E_1 &= 50.0 \text{ GPa}, & E_2 &= E_3 = 15.20 \text{ GPa} \\ \nu_{23} &= 0.428, & \nu_{12} &= \nu_{13} = 0.254 \\ G_{23} &= 3.28 \text{ GPa}, & G_{12} &= G_{13} = 4.70 \text{ GPa} \end{aligned}$$

2.- Repeat Problem 2.7 if the cube is made of aluminum instead of glass-reinforced polymer composite material. The material constants for aluminum are E = 72.4GPa and ν = 0.300. Use MATLAB.

AF 5.

3.- When a fiber-reinforced composite material is heated or cooled, the material expands or contracts just like an isotropic material. This is deformation that takes place independently of any applied load. Let ΔT be the change in temperature and let α1, α2, and α3 be the coefficients of thermal expansion for the composite material in the 1, 2, and 3-directions, respectively. In this case, the stress-strain relation of and becomes as follows:

$$\begin{Bmatrix} \varepsilon_1 - \alpha_1 \Delta T \\ \varepsilon_2 - \alpha_2 \Delta T \\ \varepsilon_3 - \alpha_3 \Delta T \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}$$

Consider now the cube of graphite-reinforced polymer composite material of Example 2. Suppose the cube is heated 30ºC above some reference state. Given α1 = −0.01800 × 10−6/ ◦ C and α2 = α3 = 24.3 × 10−6/ ◦ C, use MATLAB to determine the changes in length of the cube in each one of the three directions.

$V_f = 0.5$

$E_f = 303.83 GPa$ $\quad E_m = 6.17 GPa$

$\nu_{23} = 0.458,$ $\quad \nu_{12} = \nu_{13} = 0.248$

$G_{23} = 3.20 \text{ GPa},$ $\quad G_{12} = G_{13} = 4.40 \text{ GPa}$

## Elastic Constants Based on Micromechanics

There are three different approaches that are used to determine the elastic constants for the composite material based on micromechanics. These three approaches are:

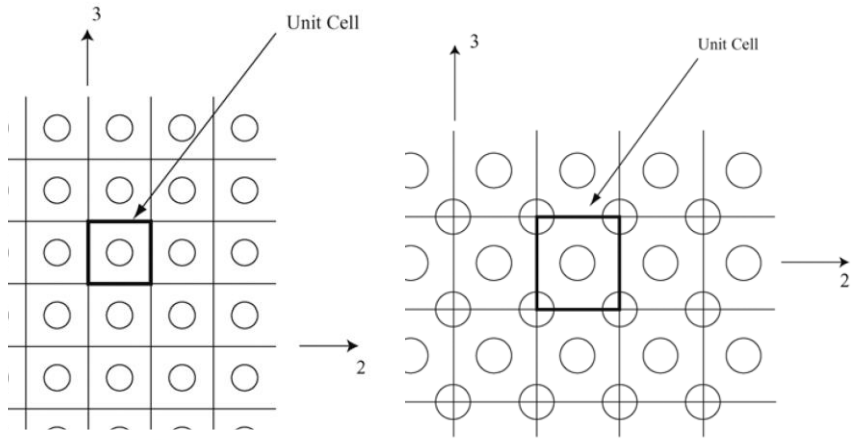
1. Using numerical models such as the finite element method.

2. Using models based on the theory of elasticity.

3. Using rule-of-mixtures models based on a strength-of-materials approach.

Consider a unit cell in either a square-packed array or a hexagonal-packed array. The ratio of the cross-sectional area of the fiber to the total cross-sectional area of the unit cell is called the fiber volume fraction and is denoted by Vf . The fiber volume fraction satisfies the relation 0 < Vf < 1 and is usually 0.5 or greater. Similarly, the matrix volume fraction Vm is the ratio of the cross-sectional area of the matrix to the total cross-sectional area of the unit cell. Note that Vm also satisfies 0 < Vm < 1. The following relation can be shown to exist between V f and Vm:

$$V^f + V^m = 1$$



# Elastic Constants Based on Micromechanics

In addition, the matrix material is assumed to be isotropic so that Em1 = Em2 = Em3 = Em and vm12 = vm. However, the fiber material is assumed to be only transversely isotropic such that Ef3 = Ef2 , vf13 = vf12, and vf23 = vf32 = vf .

Using the strength-of-materials approach and the simple rule of mixtures, we have the following relations for the elastic constants of the composite material. For Young's modulus in the 1-direction (also called the longitudinal stiffness), we have the following relation:

$$E_1 = E_1^f V^f + E^m V^m$$

For Poisson's ratio v12, we have the following relation:

$$\nu_{12} = \nu_{12}^f V^f + \nu^m V^m$$

For Young's modulus in the 2-direction (also called the transverse stiffness), we have the following relation:

$$\frac{1}{E_2} = \frac{V^f}{E_2^f} + \frac{V^m}{E^m}$$

For the shear modulus G12, we have the following relation:

$$\frac{1}{G_{12}} = \frac{V^f}{G_{12}^f} + \frac{V^m}{G^m}$$

For the coefficients of thermal expansion α1 and α2 we have the following relations:

$$\alpha_1 = \frac{\alpha_1^f E_1^f V^f + \alpha^m E^m V^m}{E_1^f V^f + E^m V^m}$$

$$\alpha_2 = \left[\alpha_2^f - \left(\frac{E^m}{E_1}\right)\nu_1^f(\alpha^m - \alpha_1^f)V^m\right]V^f + \left[\alpha^m + \left(\frac{E_1^f}{E_1}\right)\nu^m(\alpha^m - \alpha_1^f)V^f\right]V^m$$

However, we can use a simple rule-of-mixtures relation for α2 as follows:

$$\alpha_2 = \alpha_2^f V^f + \alpha^m V^m$$

While the simple rule-of-mixtures models used above give accurate results for E1 and v12, the results obtained for E2 and G12 do not agree well with finite element analysis and elasticity theory results. Therefore, we need to modify the simple rule-of-mixtures models shown above. For E2, we have the following modified rule-of-mixtures formula:

$$\frac{1}{E_2} = \frac{\frac{V^f}{E_2^f} + \frac{\eta V^m}{E^m}}{V^f + \eta V^m}$$

where η is the stress-partitioning factor (related to the stress σ2). This factor satisfies the relation 0 < η < 1 and is usually taken between 0.4 and 0.6.

## Elastic Constants Based on Micromechanics

Another alternative rule-of-mixtures formula for E2 is given by:

$$\frac{1}{E_2} = \frac{\eta^f V^f}{E_2^f} + \frac{\eta^m V^m}{E^m}$$

where the factors ηf and ηm are given by:

$$\eta^f = \frac{E_1^f V^f + \left[\left(1 - \nu_{12}^f \nu_{21}^f\right) E^m + \nu^m \nu_{21}^f E_1^f\right] V^m}{E_1^f V^f + E^m V^m}$$

$$\eta^m = \frac{\left[\left(1 - \nu^{m^2}\right) E_1^f - \left(1 - \nu^m \nu_{12}^f\right) E^m\right] V^f + E^m V^m}{E_1^f V^f + E^m V^m}$$

The above alternative model for E2 gives accurate results and is used whenever the modified rule-of-mixtures model of cannot be applied, i.e. when the factor η is not known.

# Elastic Constants Based on Micromechanics

The modified rule-of-mixtures model for G12 is given by the following formula:

$$\frac{1}{G_{12}} = \frac{\frac{V^f}{G_{12}^f} + \frac{\eta' V^m}{G^m}}{V^f + \eta' V^m}$$

where η′ is the shear stress-partitioning factor. Note that η′ satisfies the relation 0 < η′ < 1 but using η′ = 0.6 gives results that correlate with the elasticity solution.

Finally, the elasticity solution gives the following formula for G12:

$$G_{12} = G^m \left[ \frac{(G^m + G_{12}^f) - V^f (G^m - G_{12}^f)}{(G^m + G_{12}^f) + V^f (G^m - G_{12}^f)} \right]$$

# Elastic Constants Based on Micromechanics

Example 2. Consider a graphite-reinforced polymer composite lamina with the following material properties for the matrix and fibers:

$$
\begin{aligned}
E^m &= 4.62\,\mathrm{GPa}, & \nu^m &= 0.360\\
E_1^f &= 233\,\mathrm{GPa}, & \nu_{12}^f &= 0.200\\
E_2^f &= 23.1\,\mathrm{GPa}, & \nu_{23}^f &= 0.400\\
G_{12}^f &= 8.96\,\mathrm{GPa} & G_{23}^f &= 8.27\,\mathrm{GPa}
\end{aligned}
$$

Use MATLAB and the simple rule-of-mixtures formulas to calculate the values of the four elastic constants E1, v12, E2, and G12 for the lamina. Use Vf = 0.6. Plot in a single graph, E1 and E2 vs Vf.

AF 4.

1.- Consider a carbon/epoxy composite lamina with the following matrix and fiber material properties:

$$E_2^f = 14.8 \text{ GPa}, \quad E^m = 3.45 \text{ GPa}, \quad \nu^m = 0.36$$

Use MATLAB to calculate the transverse modulus E2 using the following three methods (use Vf = 0.65):
(a) the simple rule-of-mixtures formula.
(b) the modified rule-of-mixtures formula with η = 0.5.
(c) the alternative rule-of-mixtures formula. For this case, use Ef1 = 85.6GPa, vf12 = vf21 = 0.3.

2.- Consider the glass/epoxy composite lamina of Problem 1. Use MATLAB to plot a graph of the transverse modulus E2 versus the fiber volume fraction Vf for each one of the following cases. Use all values of Vf ranging from 0 to 1 (in increments of 0.1).
(a) the simple rule-of-mixtures formula.
(b) the modified rule-of-mixtures formula with η = 0.4.
(c) the modified rule-of-mixtures formula with η = 0.5.
(d) the modified rule-of-mixtures formula with η = 0.6.
(e) the alternative rule-of-mixtures formula with the values given in part (c) of Problem 1.
Make sure that all five graphs appear on the same plot.

AF 4.

3.- Consider a carbon/epoxy composite lamina with the following matrix and fiber material properties: Gf12 = 28.3 GPa, Gm = 1.27 GPa
Use MATLAB to calculate the shear modulus G12 using the following three methods (use V f = 0.55):
(a) the simple rule-of-mixtures formula.
(b) the modified rule-of-mixtures formula with η' = 0.6.
(c) the elasticity formula.

4.- Consider the glass/epoxy composite lamina of Problem 3. Use MATLAB to plot a graph of the shear modulus G12 versus the fiber volume fraction Vf for each one of the following cases. Use all values of Vf ranging from 0 to 1 (in increments of 0.1).
(a) the simple rule-of-mixtures formula.
(b) the modified rule-of-mixtures formula with η' = 0.6.
(c) the elasticity formula.
Make sure that all three graphs appear on the same plot.

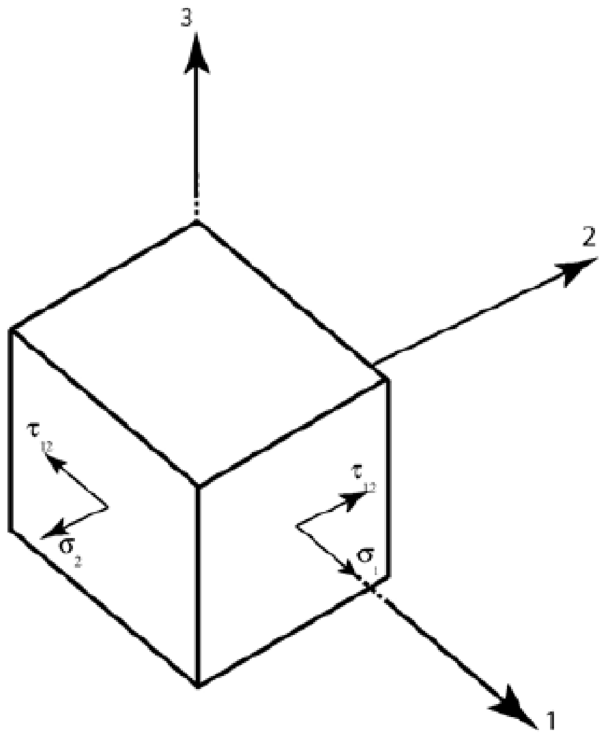
5.- Consider the graphite-reinforced polymer composite lamina of Example 2. Let the coefficients of thermal expansion for the matrix and fibers be given as follows:

$$\alpha^m = 41.4 \times 10^{-6}/\mathrm{K}$$
$$\alpha_1^f = -0.540 \times 10^{-6}/\mathrm{K}$$
$$\alpha_2^f = 10.10 \times 10^{-6}/\mathrm{K}$$

Use MATLAB to calculate α1 and α2 for the lamina. When calculating α2, use the two formulas.

## Plane stress

In the analysis of fiber-reinforced composite materials, the assumption of plane stress is usually used for each layer (lamina). This is mainly because fiber reinforced materials are utilized in beams, plates, cylinders, and other structural shapes which have at least one characteristic geometric dimension in an order of magnitude less than the other two dimensions. In this case, the stress components $\sigma_3$, $\tau_{23}$, and $\tau_{13}$ are set to zero with the assumption that the 1-2 plane of the principal material coordinate system is in the plane of the layer (lamina). Therefore, the stresses $\sigma_1$, $\sigma_2$, and $\tau_{12}$ lie in a plane, while the stresses $\sigma_3$, $\tau_{23}$, and $\tau_{13}$ are perpendicular to this plane and are zero (see Fig.).



## Plane stress

Using the assumption of plane stress, it is seen that the stress-strain relations of are greatly simplified. Setting σ3 = τ23 = τ13 = 0 in leads to the following:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ 0 \\ 0 \\ 0 \\ \tau_{12} \end{Bmatrix}$$

As a result of the plane stress assumption, we conclude that:

$$\gamma_{23} = 0$$
$$\gamma_{13} = 0$$
$$\varepsilon_3 = S_{13}\sigma_1 + S_{23}\sigma_2 \neq 0$$

Therefore (4.1) reduces to the following equation:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

The 3 × 3 matrix is called the reduced compliance matrix. The inverse of the reduced compliance matrix is the reduced stiffness matrix given as follows:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

Example 2. Consider a layer of graphite-reinforced composite material 200mm long, 100mm wide, and 0.200mm thick. The layer is subjected to an in-plane tensile force of 4 kN in the fiber direction which is perpendicular to the 100-mm width. Assume the layer to be in a state of plane stress and use the following elastic constants. Use MATLAB to determine the transverse strain ε1.A=0.0013

$$V_f = 0.5$$

$$E_f = 303.83GPa \qquad E_m = 6.17GPa$$

$$E_1 = 155.0 \text{ GPa}, \qquad E_2 = E_3 = 12.10 \text{ GPa}$$

$$\nu_{23} = 0.458, \qquad \nu_{12} = \nu_{13} = 0.248$$

$$G_{23} = 3.20 \text{ GPa}, \qquad G_{12} = G_{13} = 4.40 \text{ GPa}$$

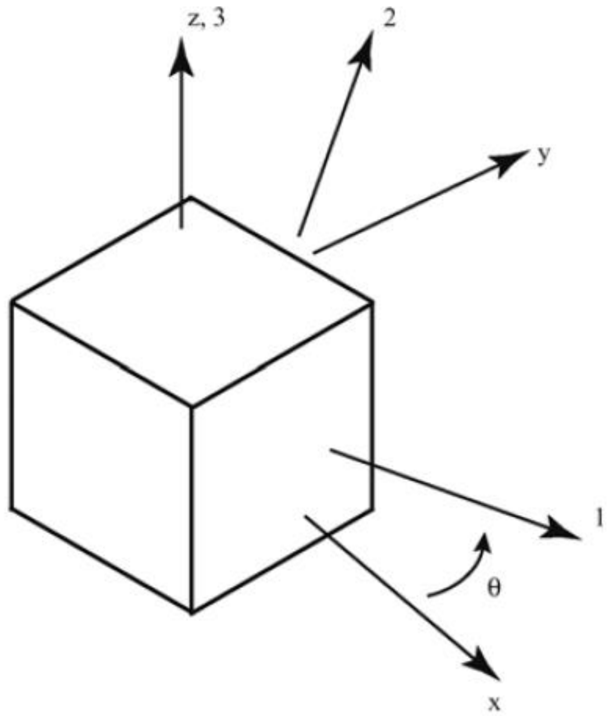
# Plane stress

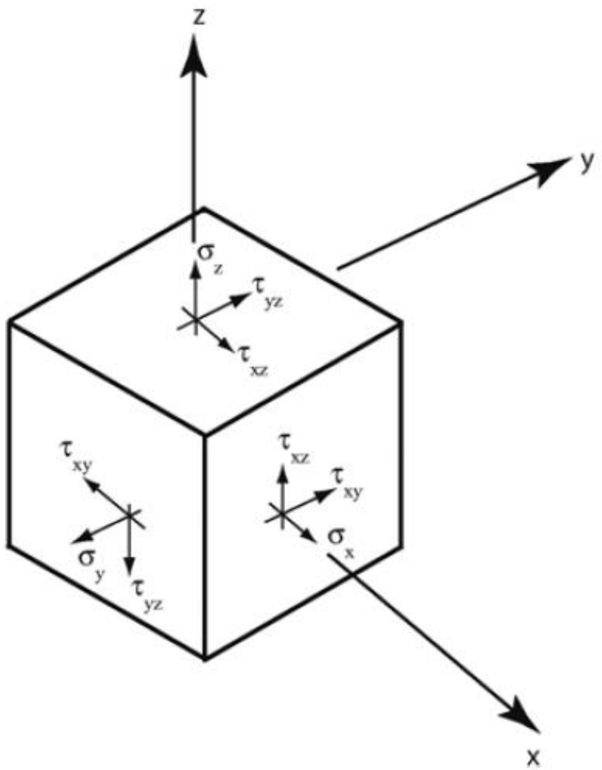
Example 3. Consider the layer of composite material of Example 2. Suppose that the layer is subjected to an in-plane compressive force of 2.5 kN in the 2-direction instead of the 4 kN force in the 1-direction. Use MATLAB to calculate the transverse strain $\varepsilon_3$ in this case.
A=0.0024

# Global Coordinate System

Consider an isolated infinitesimal element in the principal material coordinate system (1-2-3 system) that will be transformed into the x-y-z global coordinate system as shown in figure. The fibers are oriented at angle θ with respect to the +x axis of the global system. The fibers are parallel to the x-y plane and the 3 and z axes coincide. The orientation angle θ will be considered positive when the fibers rotate counterclockwise from the +x axis toward the +y axis.

The stresses on the small volume of element are now identified with respect to the x-y-z system. The six components of stress are now $\sigma_x$, $\sigma_y$, $\sigma_z$, $\tau_{yz}$, $\tau_{xz}$, and $\tau_{xy}$, while the six components of strain are $\varepsilon_x$, $\varepsilon_y$, $\varepsilon_z$, $\gamma_{yz}$, $\gamma_{xz}$, and $\gamma_{xy}$ (see figure.)



## Global Coordinate System

Note that in a plane stress state, it follows that the out-of-plane stress components in the x-y-z global coordinate system are zero, i.e. $\sigma_z = \tau_{yz} = \tau_{xz} = 0$.

The stress transformation relation is given as follows for the case of plane stress:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

where m = cosθ and n = sinθ. The above relation is written in compact form as follows:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [T] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

where [T] is the transformation matrix.

The inverse of the matrix [T] is $[T]^{-1}$ given as follows:

$$[T]^{-1} = \begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2 - n^2 \end{bmatrix}$$

where $[T]^{-1}$ is used in the following equation:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

## Global Coordinate System

Similar transformation relations hold for the strains as follows:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\gamma_{12} \end{Bmatrix} = [T] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\gamma_{12} \end{Bmatrix}$$

## Global Coordinate System

Similar to 1-2-3 coordinate system, in x-y-z is possible to convert from stresses to deformations or vice versa. Using:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

where the transformed reduced compliance matrix [ ¯ S] is given by:

$$[\bar{S}] = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} = [T]^{-1} \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} [T]$$

Similarly, we can derive the transformed reduced stiffness matrix [¯Q]:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

where [¯Q] is given by:

$$[\bar{Q}] = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} = [T]^{-1} \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} [T]$$

Example 3. Consider a plane element of size 40mm × 40mm made of graphite-reinforced polymer composite material whose elastic constants are given below. The element is subjected to a tensile stress $\sigma x$ = 200MPa in the x-direction. Use MATLAB to calculate the strains and the deformed dimensions of the element in the following two cases:

(a) the fibers are aligned along the x-axis.

(b) the fibers are inclined to the x-axis with an orientation angle $\theta = 30^\circ$ .

$$E_1 = 155.0 \text{ GPa}, \qquad E_2 = E_3 = 12.10 \text{ GPa}$$
$$\nu_{23} = 0.458, \qquad \nu_{12} = \nu_{13} = 0.248$$
$$G_{23} = 3.20 \text{ GPa}, \qquad G_{12} = G_{13} = 4.40 \text{ GPa}$$

a) First, calculate the Reduced Compliance matrix:

```
S =

    0.0065   -0.0016         0
   -0.0016    0.0826         0
         0         0    0.2273
```

Next,

the transformed reduced compliance matrix is calculated for part (a) with $\theta = 0^\circ$

```
S1 =

    0.0065   -0.0016         0
   -0.0016    0.0826         0
         0         0    0.2273
```

## Global Coordinate System

Next, the stress vector in the global coordinate system is setup in GPa as follows:

```
sigma =

    0.2000
         0
         0
```

Next, the strain vector is now calculated in the global coordinate system

```
epsilon =

    0.0013
   -0.0003
         0
```

Next, the change in the length in both the x- and y-direction is calculated next in mm as follows:

```
deltax =        deltay =        gammaxy =

    0.0516         -0.0128              0
```

The deformed dimensions are next calculated as follows:

```
dx =            dy =

   40.0516         39.9872
```

## Global Coordinate System

b)

```
dx =          dy =          gammaxy =

   40.7474       39.4438       -0.0111
```

AF 5.

4. Consider a plane element of size 50mm $\times$ 50mm made of glass-reinforced polymer composite material whose elastic constants are below. The element is subjected to a tensile stress σ = 100MPa in the x-direction. Use MATLAB to calculate the strains and the deformed dimensions of the element in the following three cases:

(a) the fibers are aligned along the x-axis.

(b) the fibers are inclined to the x-axis with an orientation angle θ = 45 ◦ .

(c) the fibers are inclined to the x-axis with an orientation angle θ = −45 ◦ .

$$
\begin{aligned}
E_1 &= 50.0 \text{ GPa}, & E_2 = E_3 &= 15.20 \text{ GPa} \\
\nu_{23} &= 0.428, & \nu_{12} = \nu_{13} &= 0.254 \\
G_{23} &= 3.28 \text{ GPa}, & G_{12} = G_{13} &= 4.70 \text{ GPa}
\end{aligned}
$$

## Elastic Constants Based on Global Coordinate System

The engineering properties or elastic constants was already introduced with respect to the lamina 1-2-3 coordinate system. Their evaluation was presented in based also on the 1-2-3 coordinate system. We can also define elastic constants with respect to the x-y-z global coordinate system. The elastic constants in the x-y-z coordinate system can be derived directly from their definitions, just as they were derived for the 1-2-3 coordinate system. The elastic constants based on the x-y-z global coordinate system are given as follows:

$$E_x = \frac{E_1}{m^4 + \left(\frac{E_1}{G_{12}} - 2\nu_{12}\right) n^2 m^2 + \frac{E_1}{E_2} n^4}$$

$$\nu_{xy} = \frac{\nu_{12}\left(n^4 + m^4\right) - \left(1 + \frac{E_1}{E_2} - \frac{E_1}{G_{12}}\right) n^2 m^2}{m^4 + \left(\frac{E_1}{G_{12}} - 2\nu_{12}\right) n^2 m^2 + \frac{E_1}{E_2} n^2}$$

$$E_y = \frac{E_2}{m^4 + \left(\frac{E_2}{G_{12}} - 2\nu_{21}\right) n^2 m^2 + \frac{E_2}{E_1} n^4}$$

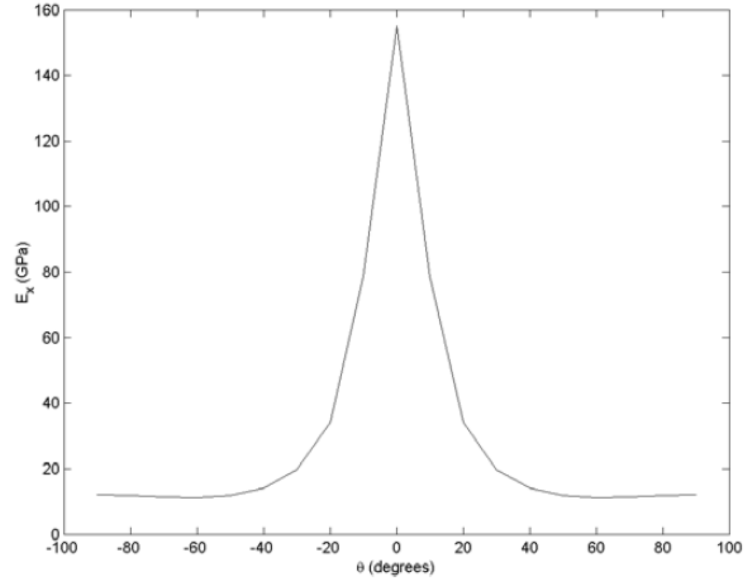
$$\nu_{yx} = \frac{\nu_{21}\left(n^4 + m^4\right) - \left(1 + \frac{E_2}{E_1} - \frac{E_2}{G_{12}}\right) n^2 m^2}{m^4 + \left(\frac{E_2}{G_{12}} - 2\nu_{21}\right) n^2 m^2 + \frac{E_2}{E_1} n^2}$$

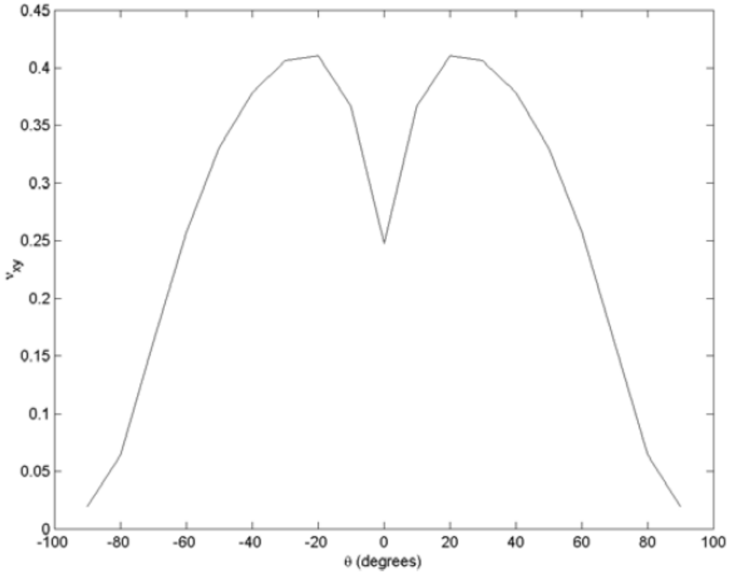
$$G_{xy} = \frac{G_{12}}{n^4 + m^4 + 2\left(\frac{2G_{12}}{E_1}\left(1 + 2\nu_{12}\right) + \frac{2G_{12}}{E_2} - 1\right) n^2 m^2}$$

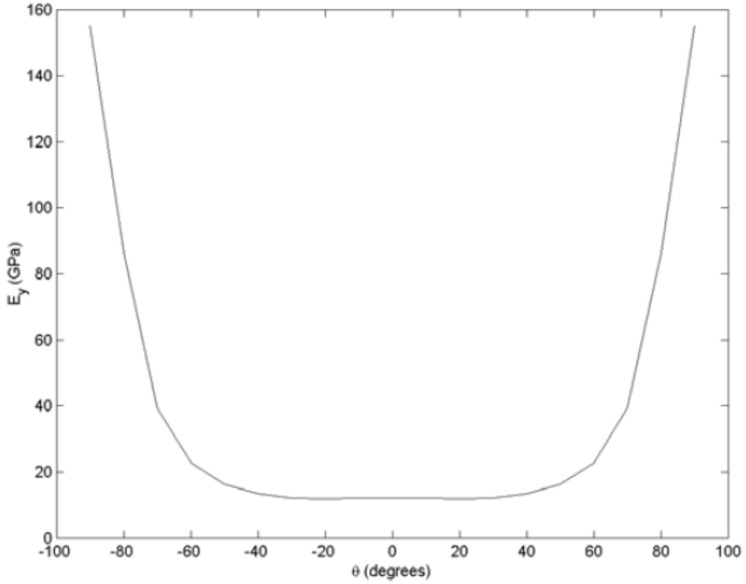
# Elastic Constants Based on Global Coordinate System

Example 2. Consider a graphite-reinforced polymer composite lamina with the below elastic constants. Use MATLAB to plot the values of the five elastic constants Ex, νxy, Ey, νyx, and Gxy as a function of the orientation angle θ in the rang −π/2 ≤ θ ≤ π/2.

$$
\begin{aligned}
E_1 &= 155.0 \text{ GPa}, & E_2 &= E_3 = 12.10 \text{ GPa} \\
\nu_{23} &= 0.458, & \nu_{12} &= \nu_{13} = 0.248 \\
G_{23} &= 3.20 \text{ GPa}, & G_{12} &= G_{13} = 4.40 \text{ GPa}
\end{aligned}
$$



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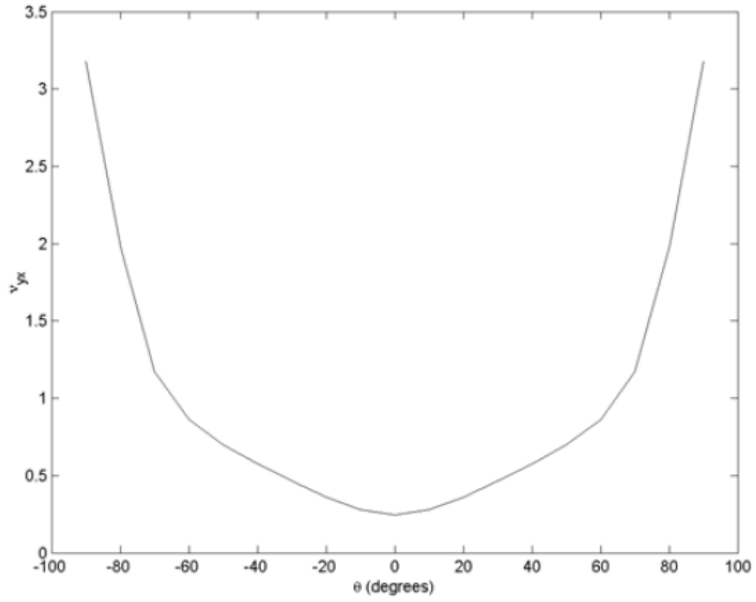
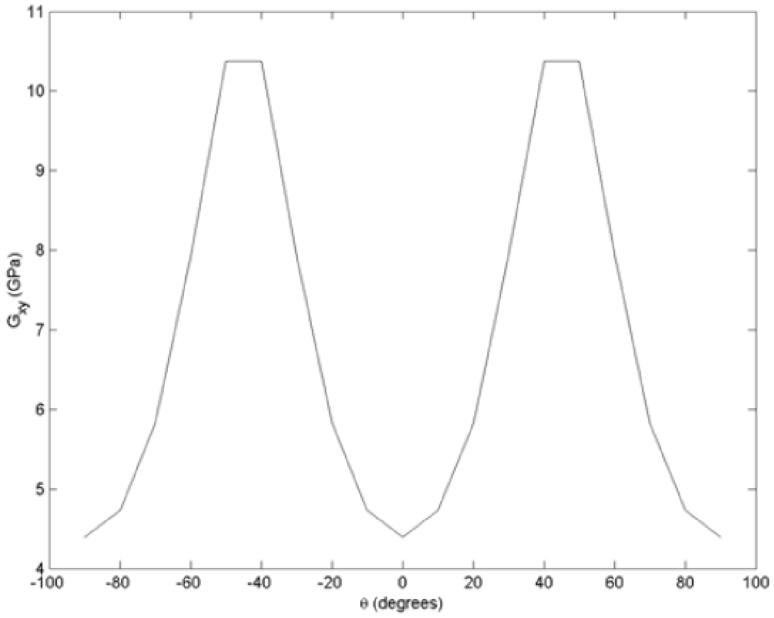
$$G_{23} = 3.20\ \text{GPa}, \qquad G_{12} = G_{13} = 4.40\ \text{GPa}$$



# Failure criteria

A successful design of a structure requires efficient and safe use of materials. Theories need to be developed to compare the state of stress in a material to failure criteria. It should be noted that failure criteria are only stated, and their application is validated by experiments.

For a laminate, the strength is related to the strength of each individual lamina. This allows for a simple and economical method for finding the strength of a laminate. Various theories have been developed for studying the failure of an angle lamina. The theories are generally based on the normal and shear strengths of a unidirectional lamina.

An isotropic material, such as steel, generally has two strength parameters: normal strength and shear strength. In some cases, such as concrete or gray cast iron, the normal strengths are different in the tension and compression. A simple failure theory for an isotropic material is based on finding the principal normal stresses and the maximum shear stresses. These maximum stresses, if greater than any of the corresponding ultimate strengths, indicate failure in the material.

Example 1. A cylindrical rod made of gray cast iron is subjected to a uniaxial tensile load, P. Given:

Cross-sectional area of rod = 2 in$^2$

Ultimate tensile strength = 25 ksi

Ultimate compressive strength = 95 ksi

Ultimate shear strength = 35 ksi

Modulus of elasticity = 10 Msi

$$\frac{P}{2} < 25 \times 10^3 \ \ or \ \ P < 50{,}000 \text{ lb},$$

## Failure criteria

However, in a lamina, the failure theories are not based on principal normal stresses and maximum shear stresses. Rather, they are based on the stresses in the material or local axes because a lamina is orthotropic and its properties are different at different angles, unlike an isotropic material.

In the case of a unidirectional lamina, there are two material axes: one parallel to the fibers and one perpendicular to the fibers. Thus, there are four normal strength parameters for a unidirectional lamina, one for tension and one for compression, in each of the two material axes directions. The fifth strength parameter is the shear strength of a unidirectional lamina.

$(\sigma_1^T)_{ult}$ = Ultimate longitudinal tensile strength (in direction 1),
$(\sigma_1^C)_{ult}$ = Ultimate longitudinal compressive strength (in direction 1),
$(\sigma_2^T)_{ult}$ = Ultimate transverse tensile strength (in direction 2),
$(\sigma_2^C)_{ult}$ = Ultimate transverse compressive strength (in direction 2), and
$(\tau_{12})_{ult}$ = Ultimate in-plane shear strength (in plane 12).

The failure criteria are based on first finding the stresses in the local axes and then using these five strength parameters of a unidirectional lamina to find whether a lamina has failed.

## Failure criteria – Maximum Stress

The stresses acting on a lamina are resolved into the normal and shear stresses in the local axes. Failure is predicted in a lamina, if any of the normal or shear stresses in the local axes of a lamina is equal to or exceeds the corresponding ultimate strengths of the unidirectional lamina.

Given the stresses or strains in the global axes of a lamina, one can find the stresses in the material axes. The lamina is considered to be failed if:

$$-(\sigma_1^C)_{ult} < \sigma_1 < (\sigma_1^T)_{ult}, \ or$$

$$-(\sigma_2^C)_{ult} < \sigma_2 < (\sigma_2^T)_{ult}, \ or$$

$$-(\tau_{12})_{ult} < \tau_{12} < (\tau_{12})_{ult}$$

is violated. Note that all five strength parameters are treated as positive numbers, and the normal stresses are positive if tensile and negative if compressive.

Each component of stress is compared with the corresponding strength; thus, each component of stress does not interact with the others.

# Failure criteria – Maximum Stress

Example 2. Find the maximum value of S > 0 if a stress of σx = 2S, σy = –3S, and τxy = 4S is applied to the 60° lamina of graphite/epoxy. Use maximum stress failure theory and the properties of a unidirectional graphite/epoxy lamina given as follow:

| Property | Symbol | Units | Glass/ epoxy | Boron/ epoxy | Graphite/ epoxy |
|---|---|---|---|---|---|
| Fiber volume fraction | $V_f$ | | 0.45 | 0.50 | 0.70 |
| Longitudinal elastic modulus | $E_1$ | GPa | 38.6 | 204 | 181 |
| Transverse elastic modulus | $E_2$ | GPa | 8.27 | 18.50 | 10.30 |
| Major Poisson's ratio | $\nu_{12}$ | | 0.26 | 0.23 | 0.28 |
| Shear modulus | $G_{12}$ | GPa | 4.14 | 5.59 | 7.17 |
| Ultimate longitudinal tensile strength | $(\sigma_1^T)_{ult}$ | MPa | 1062 | 1260 | 1500 |
| Ultimate longitudinal compressive strength | $(\sigma_1^C)_{ult}$ | MPa | 610 | 2500 | 1500 |
| Ultimate transverse tensile strength | $(\sigma_2^T)_{ult}$ | MPa | 31 | 61 | 40 |
| Ultimate transverse compressive strength | $(\sigma_2^C)_{ult}$ | MPa | 118 | 202 | 246 |
| Ultimate in-plane shear strength | $(\tau_{12})_{ult}$ | MPa | 72 | 67 | 68 |

Using T-equation the stresses in the local axes are:

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{bmatrix}=\begin{bmatrix}0.2500 & 0.7500 & 0.8660\\ 0.7500 & 0.2500 & -0.8660\\ -0.4330 & 0.4330 & -0.5000\end{bmatrix}\begin{bmatrix}2S\\ -3S\\ 4S\end{bmatrix}$$

$$=\begin{bmatrix}0.1714\times10^1\\ -0.2714\times10^1\\ -0.4165\times10^1\end{bmatrix}S.$$

Then, using the inequalities of the maximum stress failure theory:

$$-1500\times10^6 < 0.1714\times10^1 S < 1500\times10^6 \qquad -875.1\times10^6 < S < 875.1\times10^6$$

$$-246\times10^6 < -0.2714\times10^1 S < 40\times10^6 \qquad -14.73\times10^6 < S < 90.64\times10^6$$

$$-68\times10^6 < -0.4165\times10^1 S < 68\times10^6 \qquad -16.33\times10^6 < S < 16.33\times10^6.$$

All the inequality conditions (and S > 0) are satisfied if 0 < S < 16.33 MPa. The preceding inequalities also show that the angle lamina will fail in shear.

The maximum stress that can be applied before failure is:

$$\sigma_x = 32.66\,MPa,\ \sigma_y = -48.99\,MPa,\ \tau_{xy} = 65.32\,MPa.$$

Example 3. Consider a tube with a mean radius of 25 mm made of graphite-reinforced composite with a 10-layer wall with a stacking sequence of $[\pm 20/0_3]_s$. If we use the maximum stress failure criterion, what is the maximum allowable axial load? What layer or layers control failure?

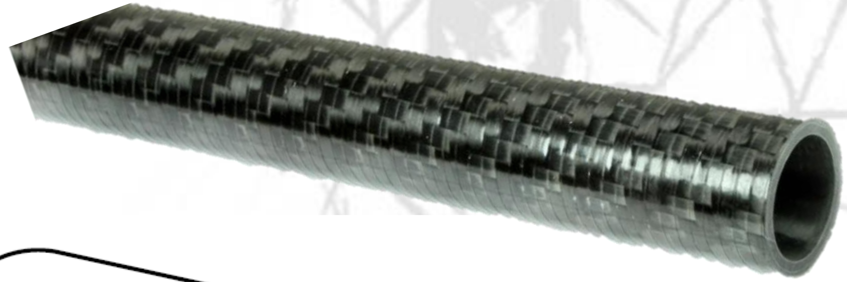
Assume that P is distributed uniformly around the circumference of the ends of the tube, and thus, in keeping with the definition of Nx as being a load per unit length of laminate, Nx is the load P divided by the circumference of the end; that is:

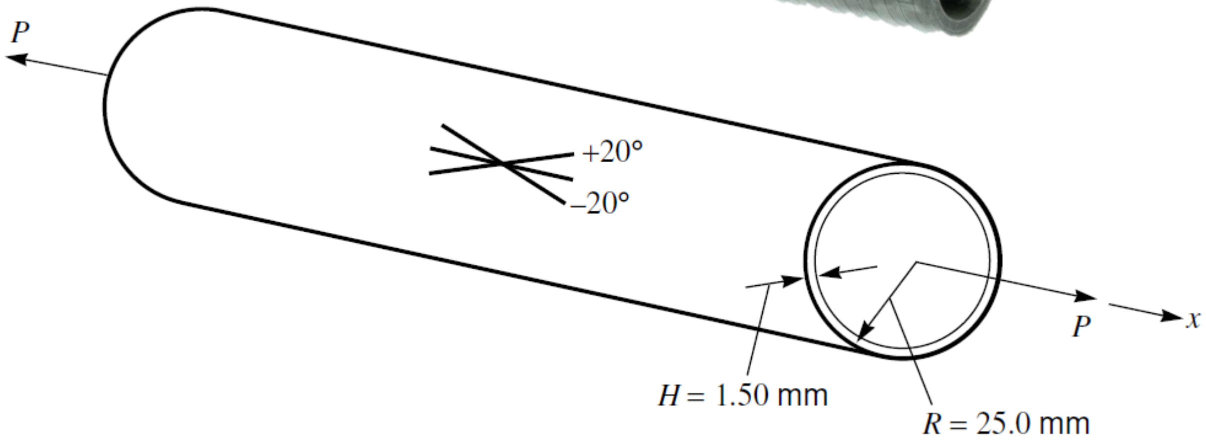
$$N_x = \frac{P}{2\pi R}$$

| | Graphite-reinforced |
|---|---|
| $\sigma_1^C$ | −1250 |
| $\sigma_1^T$ | 1500 |
| $\sigma_2^C$ | −200 |
| $\sigma_2^T$ | 50 |
| $\tau_{12}^F$ | 100 |

*TABLE 9.2. Principal material system stresses (Pa) in axially loaded tube for P = 1 N*

| Layer | $\sigma_1$ | $\sigma_2$ | $\tau_{12}$ |
|---|---|---|---|
| +20° | +3830 | −112.3 | −148.7 |
| −20° | +3830 | −112.3 | +148.7 |
| 0° | +4770 | −168.6 | 0 |





## Failure criteria – Tsai – Hill

This theory is based on the distortion energy failure theory of Von-Mises' distortional energy yield criterion for isotropic materials as applied to anisotropic materials. Distortion energy is actually a part of the total strain energy in a body. The strain energy in a body consists of two parts; one due to a change in volume and is called the dilation energy and the second is due to a change in shape and is called the distortion energy. It is assumed that failure in the material takes place only when the distortion energy is greater than the failure distortion energy of the material. Hill adopted the Von-Mises' distortional energy yield criterion to anisotropic materials. Then, Tsai adapted it to a unidirectional lamina. Based on the distortion energy theory, the proposed that a lamina has failed if:

$$\left[\frac{\sigma_1}{(\sigma_1^T)_{ult}}\right]^2 - \left[\frac{\sigma_1\sigma_2}{(\sigma_1^T)^2_{ult}}\right] + \left[\frac{\sigma_2}{(\sigma_2^T)_{ult}}\right]^2 + \left[\frac{\tau_{12}}{(\tau_{12})_{ult}}\right]^2 < 1.$$

Given the global stresses in a lamina, one can find the local stresses in a lamina and apply the preceding failure theory to determine whether the lamina has failed.

Example 4. Find the maximum value of $S > 0$ if a stress of $\sigma x = 2S$, $\sigma y = -3S$, and $\tau xy = 4S$ is applied to a 60° graphite/epoxy lamina. Use Tsai–Hill failure theory. Use the unidirectional graphite/epoxy lamina properties given in example 1.

From example 1: $\sigma_1 = 1.714\ S,\ \ \sigma_2 = -2.714\ S,\ \ \tau_{12} = -4.165\ S.$

Using the Tsai–Hill failure theory from Equation:

$$\left(\frac{1.714S}{1500\times10^6}\right)^2 - \left(\frac{1.714S}{1500\times10^6}\right)\left(\frac{-2.714S}{1500\times10^6}\right) + \left(\frac{-2.714S}{40\times10^6}\right)^2 + \left(\frac{-4.165S}{68\times10^6}\right)^2 < 1 \qquad S < 10.94\ MPa$$