PI PWM Temperature Controller

$mC\Delta T \ = \sum Q$

Energy supplied into the system:

$Q_{in} = Power * time = VIPwm_i T_s$; $V = Voltage;\ I \ = Current; Pwm_i = PWM\ perecentage;\ T_s\text{-} = Sampling\ Time$

$Q_{dissipative} = T_s UA(T_{out} - T_i)$; Heat loss or gain due to difference in surrounding temperature(T_{out})

Additional energy lost: $Q_{lost} = KT_s$

$mC\Delta T \ = \sum Q = Q_{in} - Q_{lost} + Q_{dissipative}$

$$mC(T_f - T_i) = VIT_sPwm_i - KT_s + T_sUA(T_{out} - T_i) \overset{Discretize}{\Longleftrightarrow} mC(T_n - T_{n-1}) = VIT_sPwm_{n-1} - KT_s + T_sUA(T_{out} - T_{n-1})$$

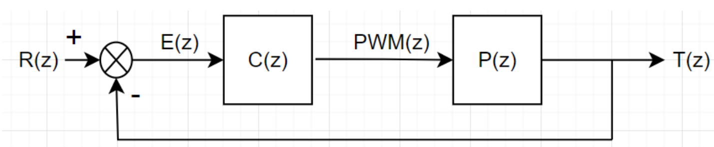
$$mC(T_n - T_{n-1}) = VIT_sPwm_{n-1} - KT_s + T_sUA(T_{out} - T_{n-1})$$
$$mCT_n - mCT_{n-1} = VIT_sPwm_{n-1} - KT_s + T_sUAT_{out} - T_sUAT_{n-1}$$
$$mCT_n \ = VIT_sPwm_{n-1} - KT_s + T_sUAT_{out} - T_sUAT_{n-1} + mCT_{n-1}$$
$$mCT_n \ = VIT_sPwm_{n-1} - KT_s + T_sUAT_{out} + (mC - T_sUA)T_{n-1}$$

$$T_n = \frac{VIT_sPwm_{n-1} - KT_s + T_sUAT_{out} + (mC - T_sUA)T_{n-1}}{mC} = \frac{VIT_s}{mC}Pwm_{n-1} - \frac{KT_s}{mC} + \frac{T_sUA}{mC}T_{out} + \frac{(mC - T_sUA)}{mC}T_{n-1};$$

$K_1 = \frac{VIT_s}{mC};\ K_2 = \frac{T_sUA}{mC};\ K_3 = 1 - \frac{T_sUA}{mC} = 1 - K_2;\ K_3 = \frac{KT_s}{mC}$

(1) $T_n = K_1Pwm_{n-1} - K_3 + K_2T_{out} + (1 - K_2)T_{n-1}$

Function (1) is non-linear; can be linearized

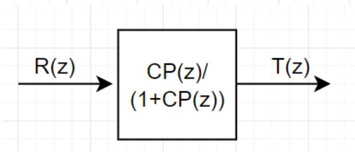


T = PCE; E= R- T

T = PC(R-T) = PCR - PCT = T

T(1+PC) = PCR

Transfer Function = Output/Input =T/R = H = PC/(1+PC)



The controller C(z) is a PI(Proportional Integral) controller. $C(z) = K_p + \frac{K_iT_s}{z-1} = \frac{K_p(z-1)+K_iT_s}{z-1}; Pwm = \ C\ E \ = \ C(R - T)$

$$C(z) = K_p + \frac{K_iT_s}{z-1} \ = \frac{K_p(z-1) + K_iT_s}{z-1}; Pwm(z) = \ C(z)E(z) = \ C(z)\big(R(z) - T(z)\big)$$

$$Pwm(z) = \left(\frac{K_p(z-1) + K_iT_s}{z-1}\right)\big(R(z) - T(z)\big) \Longrightarrow (z-1)Pwm(z) = \ \big(K_p(z-1) + K_iT_s\big)\big(R(z) - T(z)\big)$$

$$zPwm(z) - Pwm(z) = \ \big(K_pz \ - \ K_p + K_iT_s\big)\big(R(z) - T(z)\big) = K_pz(R(z) - T(z)) \ + (-\,K_p + K_iT_s)(R(z) - T(z))$$

$Multiply\ both\ sides\ by\ z^{-1} \Longrightarrow Pwm(z) - \ z^{-1}Pwm(z) = K_p(R(z) - T(z)) \ + \ z^{-1}(-\,K_p + K_iT_s)(R(z) - T(z))$

$X(z) \overset{Z}{\Leftrightarrow} x[n]; \qquad z^{-1}X(z) \overset{Z}{\Leftrightarrow} x[n-1];$

$$Pwm_n - \ Pwm_{n-1} = K_p(R_n - T_n) \ + \ (-\,K_p + K_iT_s)(R_{n-1} - T_{n-1})$$

$$(2)\ Pwm_n = K_p(R_n - T_n) \ + \ (-\,K_p + K_iT_s)(R_{n-1} - T_{n-1}) + Pwm_{n-1}$$

(2) equation is linear

we are interested in the mechanics of input reference R = constant => R = K*u[n]

(3) $R_n = \ \bar{R}u[n];$ $u[n] = 1, n >= 0$

Three variables (T,P,R) and three equations. Ready to linearize?

$(1) T_{n+1} = f(T_n, P_n, R_n) = K_1 Pwm_n - K_3 + K_2 T_{out} + (1-K_2)T_n$
$(2)\ Pwm_{n+1} = g(T_n, P_n, R_n) = K_p(R_{n+1} - T_{n+1}) + (-K_p + K_i T_s)(R_n - T_n) + Pwm_n$
$Pwm_{n+1} = g(T_n, P_n, R_n) = K_p(\bar{R} - (K_1 Pwm_n - K_3 + K_2 T_{out} + (1-K_2)T_n)) + (-K_p + K_i T_s)(\bar{R} - T_n) + Pwm_n$
$(3)\ R_{n+1} = h(T_n, P_n, R_n) = \bar{R}u[n]$

$X_{n+1} = \begin{bmatrix} T_{n+1} \\ Pwm_{n+1} \\ R_{n+1} \end{bmatrix}$, at steady state: $X_{n+1} = \begin{bmatrix} T_{n+1} \\ Pwm_{n+1} \\ R_{n+1} \end{bmatrix} = X_n = \bar{X} = \begin{bmatrix} f(T_n, Pwm_n, R_n) \\ g(T_n, Pwm_n, R_n) \\ h(T_n, Pwm_n, R_n) \end{bmatrix} = \begin{bmatrix} f(\bar{T}, \overline{Pwm}, \bar{R}) \\ g(\bar{T}, \overline{Pwm}, \bar{R}) \\ h(\bar{T}, \overline{Pwm}, \bar{R}) \end{bmatrix} = \begin{bmatrix} \bar{c} \\ \overline{Pwm}) \\ \bar{R} \end{bmatrix}$

$\begin{bmatrix} f(\bar{T}, \overline{Pwm}, \bar{R}) \\ g(\bar{T}, \overline{Pwm}, \bar{R}) \\ h(\bar{T}, \overline{Pwm}, \bar{R}) \end{bmatrix} = \begin{bmatrix} \bar{T} \\ \overline{Pwm}) \\ \bar{R} \end{bmatrix}$

$(1)\ T_{n+1} = T_n = \bar{T} = f(\bar{T}, \overline{Pwm}, \bar{R}) = K_1 \overline{Pwm} - K_3 + K_2 T_{out} + (1-K_2)\bar{T}$
$\bar{T} = K_1 \overline{Pwm} - K_3 + K_2 T_{out} + (1-K_2)\bar{T}$
$\bar{T} - (1-K_2)\bar{T} = K_1 \overline{Pwm} - K_3 + K_2 T_{out} = \bar{T}K_2 = K_1 \overline{Pwm} - K_3 + K_2 T_{out}$
$(4)\ \bar{T} = \frac{K_1}{K_2}\overline{Pwm_1} - \frac{K_3}{K_2} + T_{out}$

$(2)\ Pwm_{n+1} = Pwm_n = g(\bar{T}, \overline{Pwm}, \bar{R}) = \overline{Pwm}$
$\overline{Pwm} = K_p(\bar{R} - \bar{T}) + (-K_p + K_i T_s)(\bar{R} - \bar{T}) + \overline{Pwm}$
$\overline{Pwm} - \overline{Pwm} = 0 = K_p(\bar{R} - \bar{T}) + (-K_p + K_i T_s)(\bar{R} - \bar{T}) = (K_p + -K_p + K_i T_s)(\bar{R} - \bar{T}) = 0$
$(\bar{R} - \bar{T}) = 0 \Longrightarrow (5)\ \bar{R} = \bar{T}$
There error $\lim_{n\to\infty} E(t) = \bar{R} - \bar{T} = 0$ stabilizes to zero at steady state;
$(3)\ R_{n+1} = h(T_n, P_n, R_n) = \bar{R}u[n]$;
$\bar{R} = h(\bar{T}, \overline{Pwm}, \bar{R}) = \bar{R}u[n] = \bar{R}$
combine (4)+(5) ---> $\bar{T} = \bar{R}$, combine with (4) -> $\bar{T} = \bar{R} = \frac{K_1}{K_2}\overline{Pwm_1} - \frac{K_3}{K_2} + T_{out} \Longrightarrow (7)\ \frac{(\bar{R} - T_{out})K_2 + K_3}{K_1} = \overline{Pwm}$

the Jacobian $\nabla|_{\bar{T},\overline{Pwm},\bar{R}} = \left.\begin{bmatrix} \frac{\partial f}{\partial T} & \frac{\partial f}{\partial Pwm} & \frac{\partial f}{\partial R} \\ \frac{\partial g}{\partial T} & \frac{\partial g}{\partial Pwm} & \frac{\partial g}{\partial R} \\ \frac{\partial h}{\partial T} & \frac{\partial h}{\partial Pwm} & \frac{\partial h}{\partial R} \end{bmatrix}\right\|_{\bar{T},\overline{Pwm},\bar{R}} = \begin{bmatrix} 1-K_2 & K_1 & 0 \\ -K_p(1-K_2) - (-K_p + K_i T_s) & -K_p K_1 + 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$X_{n+1}' = \nabla|_{\bar{T},\overline{Pwm},\bar{R}} X_n' = \nabla|_{\bar{T},\overline{Pwm},\bar{R}} \begin{bmatrix} T_n \\ Pwm_n \\ R_n \end{bmatrix}$$

$T_{n+1} = (1-K_2)T_n + K_1 Pwm_n$
$Pwm_{n+1} = \left(-K_p(1-K_2) - (-K_p + K_i T_s)\right) T_n + (-K_p K_1 + 1)\ Pwm_n$
$K_4 = \left(-K_p(1-K_2) - (-K_p + K_i T_s)\right), K_5 = (-K_p K_1 + 1)$
$Pwm_{n+1} = K_4 T_n + K_5\ Pwm_n$

Linearized solution:
$T_{n+1} = (1-K_2)T_n + K_1 Pwm_n$
$Pwm_{n+1} = K_4 T_n + K_5\ Pwm_n$

$zT(z) = (1-K_2)T(z) + K_1 Pwm(z) \Leftrightarrow (8)\ \frac{T(z)}{Pwm(z)} = \frac{K_1}{z - 1 + K_2}$
$zPwm(z) = K_4 T(z) + K_5\ Pwm(z) \Leftrightarrow (9)\ \frac{T(z)}{Pwm(z)} = \frac{z - K_5}{K_4}$
characteristic equation:
(9)=(8) $\frac{z-K_5}{K_4} = \frac{K_1}{z-1+K_2} = (z - K_5)(z - 1 + K_2) = K_1 K_4 = z^2 + (-K_5 + K_2 - 1)z + -K_5(-1 + K_2) - K_1 K_4 = 0$
$b = (-K_5 + K_2 - 1), c = -K_5(-1 + K_2) - K_1 K_4$
$z^2 + bz + c = 0$

Simplified Transient Solution?

Lets simplify:

$$mC\Delta T \; = \sum Q = Qin + Qdissipative$$

$$Qdissipative = constant = K_1$$

$$mC(T_n - T_{n-1}) = VIT_sPwm_{n-1} - T_sK_1$$

$$T_n = \frac{VIT_s}{mC}Pwm_{n-1} - \frac{T_sK_1}{mC} + T_{n-1};\; K_2 = \frac{VIT_s}{mC};\; K_3 = \frac{T_sK_1}{mC}$$

(1) $T_n = K_2Pwm_{n-1} - K_3 + T_{n-1}$

(2) $Pwm_n = K_p(R_n - T_n) \; + \; (-K_p + K_iT_s)(R_{n-1} - T_{n-1}) + Pwm_{n-1}$

(3) $R_n = \; Ku[n]; u[n] = 1, n >= 0, u[n] = 0$ otherwise

at steady state $Error = \; (R_n - T_n) = 0;\; R_n = K, T_n = K, from\ (1)\ K_3 = K_2Pwm_{n-1}$

$$\overline{Pwm} = \frac{K_3}{K_2} = \frac{T_sK_1}{mC} * \frac{mC}{VIT_s} = \frac{T_sK_1}{VIT_s} = \frac{K_1}{VI}$$

$0 < \overline{Pwm} \leq 1, 0 < \frac{K_1}{VI} \leq 1, 0 < K_1 \leq VI$, but for transient state into steady state, VI cannot equal K1 so

$0 < K_1 < VI$

In stable steady state, Power lost cannot be greater than power put in.

at non-steady state step input K=Ref=75 Fahrenheit = 23.89 Celcius:

$$Pwm_n = K_p(K - T_n) \; + \; (-K_p + K_iT_s)(K - T_{n-1}) + Pwm_{n-1}$$

$$T_n = K_2Pwm_{n-1} - K_3 + T_{n-1}$$

```
Kp = 0.9097;
Ki = 0.1736;

Ts = 1;#seconds sample time
Ref = 23.8889;%reference temp in Celcius

V = 12; #Volts
I=2;     #amperes
m=110.6;#grams
C = .45;#Joules / g Celcius

K1 = 0.4*V*I;%pmax power = VI
K2 = V*I*Ts/(m*C);
K3 = Ts*K1/(m*C);

Tmax = 200; % seconds
t = [0:Ts:Tmax];

T(1) = 0;
Pwm(1) = 0;

for i = 2:length(t)
  T(i)=K2*Pwm(i-1)-K3+T(i-1);
  Pwm(i) = Kp*(Ref-T(i)) + (-Kp+Ki*Ts)*(Ref-T(i-1))+Pwm(i-1);
  if(Pwm(i)>1)
    Pwm(i)=1;
  elseif Pwm(i)<0
     Pwm(i)=0;
  end
endfor
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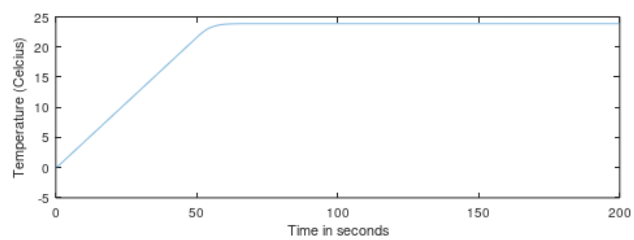

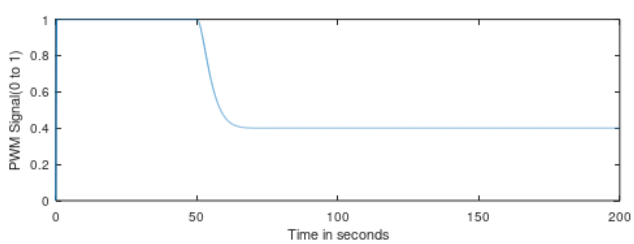

$$\overline{Pwm} = \frac{K_1}{VI} = \frac{0.4VI}{VI} = 0.4 = \overline{Pwm}$$

Kp = 0.9097

Ki = 0.1736