Advanced Topics in Theoretical Physics

Module 2 Exam

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Monday 24^{th} November, 2025

Quantum battle of the sexes

(a)Pure strategy Nash equilibira

The pure strategy Nash equilibria of the classical Battle of the Sexes are

$$(O, O) \quad \text{and} \quad (A, A)$$

with payoffs (2,1) and (1,2).

(b) Quantum version of the game

The quantum version allows Alice and Bob choices to be entangled, creating correlations and superpositions and by applying local rotations to their qubits. After disentangling and measuring, the resulting probabilities determine the outcomes. This quantum approach enables better coordination and can lead to higher expected payoffs than the classical game. Assigning,

$$|0\rangle \equiv O \text{ (Opera)}, \quad |1\rangle \equiv A \text{ (Action)}. \tag{1}$$

The initial state of the two qubits is

$$|\psi_{\text{in}}\rangle = |00\rangle, = |0\rangle_A \otimes |0\rangle_B \tag{2}$$

with both players initially at Opera.The players qubits are then entangled using the operator

$$J(\gamma) = \exp\left(i\frac{\gamma}{2}\sigma_x \otimes \sigma_x\right), \quad 0 \leq \gamma \leq \frac{\pi}{2}. \tag{3}$$

Since $(\sigma_x \otimes \sigma_x)^2 = I \otimes I$, Using the general formula

$$e^{i\theta A} = \cos\theta\, I + i\sin\theta\, A, \quad \text{for } A^2 = I. \tag{4}$$

Applying this to $J(\gamma)$ gives

$$J(\gamma) = \cos\frac{\gamma}{2}\, I \otimes I + i\sin\frac{\gamma}{2}\, \sigma_x \otimes \sigma_x. \tag{5}$$

Applying it to the initial state:

$$|\psi_0\rangle = J(\gamma)|00\rangle = \cos\frac{\gamma}{2}|00\rangle + i\sin\frac{\gamma}{2}|11\rangle. \tag{6}$$

creating a superposition of both being at Opera and both being at Action.

Role of γ:

- $\gamma = 0$: no entanglement, the game reduces to the classical version.
- $0 < \gamma < \pi/2$: partial entanglement introduces quantum correlations, allowing new strategies.
- $\gamma = \pi/2$: maximal entanglement and fully correlated which can give rise to quantum equilibria that allows the players to coordinate their choices without direct communication and can outperform classical game.

Players can now apply local quantum strategies. Each player chooses a unitary operation from

$$U(\theta, \phi) = \begin{pmatrix} e^{i\phi}\cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & e^{-i\phi}\cos(\theta/2) \end{pmatrix}, \quad 0 \le \theta \le \pi,\ 0 \le \phi < 2\pi. \tag{7}$$

- Alice applies $U_A = U(\theta_A, \phi_A)$ to her qubit.
- Bob applies $U_B = U(\theta_B, \phi_B)$ to his qubit.

The state after local rotations is

$$|\psi_1\rangle = (U_A \otimes U_B)|\psi_0\rangle. \tag{8}$$

The disentangling operator $J^\dagger(\gamma)$ is applied:

$$|\psi_f\rangle = J^\dagger(\gamma)(U_A \otimes U_B)J(\gamma)|00\rangle. \tag{9}$$

Measurement is performed in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.These outcomes correspond to classical choices:

$$|00\rangle \to (O, O), \quad |01\rangle \to (O, A), \quad |10\rangle \to (A, O), \quad |11\rangle \to (A, A). \tag{10}$$

In the quantum game, each players rotation $U(\theta, \phi)$ is their own strategy. It changes the amplitude of the entangled state. When the final state is measured, the probabilities of classical outcomes are determined by these amplitudes. So, the choice of rotation controls which outcomes are more likely and enables players to achieve higher payoffs than classical strategies.

(c) Expected payoffs for maximally entangled case

Maximal entanglement Considering the maximally entangled case:

$$\gamma = \frac{\pi}{2} \quad \Rightarrow \quad J = \frac{1}{\sqrt{2}}\big(I \otimes I + i\,\sigma_x \otimes \sigma_x\big), \quad J^\dagger = \frac{1}{\sqrt{2}}\big(I \otimes I - i\,\sigma_x \otimes \sigma_x\big). \tag{11}$$

The initial state is

$$|\psi_{\text{in}}\rangle = |00\rangle. \tag{12}$$

Each player chooses from three strategies:

$$\begin{aligned} O = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \qquad \text{(classical Opera)} \\ A = i\sigma_x = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \qquad \text{(classical Action)} \\ Q = U(0,\pi/2) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, & \quad \text{(quantum move)}. \end{aligned} \tag{13}$$

The final state is

$$|\psi_f\rangle = J^\dagger\,(U_A \otimes U_B)\,J\,|00\rangle. \tag{14}$$

The probability of observing each classical outcome $|ij\rangle$ is

$$p_{ij} = |\langle ij|\psi_f\rangle|^2, \quad i,j \in \{0,1\}. \tag{15}$$

By choosing the right rotation parameters (θ_A, ϕ_A) and (θ_B, ϕ_B) the players can increase the probability of coordinated outcomes (O,O) or (A,A), potentially achieving higher expected payoffs than in any classical strategy.

Expected payoffs The payoff vectors (in the order $|00\rangle, |01\rangle, |10\rangle, |11\rangle$) are

$$\Pi_A = (2,0,0,1), \quad \Pi_B = (1,0,0,2). \tag{16}$$

Then the expected payoffs are

$$\$A = \sum_{i,j} p_{ij}\,\Pi_{A,ij}, \qquad \$B = \sum_{i,j} p_{ij}\,\Pi_{B,ij}. \tag{17}$$

For simplification,taking

$$|\phi\rangle := (U_A \otimes U_B)\,|\psi_0\rangle = \frac{1}{\sqrt{2}}\Big(U_A\,|0\rangle \otimes U_B\,|0\rangle \ + \ i\,U_A\,|1\rangle \otimes U_B\,|1\rangle\Big), \tag{18}$$

1. Alice: O, Bob: O

$$\begin{aligned} &U_A\,|0\rangle \otimes U_B\,|0\rangle = |0\rangle \otimes |0\rangle = |00\rangle\,, \\ &U_A\,|1\rangle \otimes U_B\,|1\rangle = |1\rangle \otimes |1\rangle = |11\rangle\,, \\ &|\phi\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + i\,|11\rangle). \end{aligned}$$

After applying $J^\dagger$ and simplifying one obtains

$$\boxed{|\psi_f\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.}$$

2. Alice: O, Bob: A

$$\begin{aligned}
U_A|0\rangle \otimes U_B|0\rangle &= |0\rangle \otimes i|1\rangle = i|01\rangle, \\
U_A|1\rangle \otimes U_B|1\rangle &= |1\rangle \otimes i|0\rangle = i|10\rangle, \\
|\phi\rangle &= \frac{i}{\sqrt{2}}(|01\rangle + i|10\rangle) = \frac{i}{\sqrt{2}}(|01\rangle - |10\rangle).
\end{aligned}$$

Applying $J^\dagger$ and simplifying yields

$$\boxed{|\psi_f\rangle = \frac{i\big(\,|01\rangle + |10\rangle\,\big)}{\sqrt{2}}.}$$

3. Alice: O, Bob: Q

$$\begin{aligned}
U_A|0\rangle \otimes U_B|0\rangle &= |0\rangle \otimes i|0\rangle = i|00\rangle, \\
U_A|1\rangle \otimes U_B|1\rangle &= |1\rangle \otimes (-i)|1\rangle = -i|11\rangle, \\
|\phi\rangle &= \frac{1}{\sqrt{2}}(i|00\rangle + i(-i)|11\rangle) = \frac{1}{\sqrt{2}}(i|00\rangle + |11\rangle).
\end{aligned}$$

After applying $J^\dagger$ and simplifying:

$$\boxed{|\psi_f\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.}$$

4. Alice: A, Bob: O

By symmetry with case 2 (swap players) one obtains

$$\boxed{|\psi_f\rangle = \frac{i\big(\,|01\rangle + |10\rangle\,\big)}{\sqrt{2}}.}$$

5. Alice: A, Bob: A

$$\begin{aligned}
U_A|0\rangle \otimes U_B|0\rangle &= i|1\rangle \otimes i|1\rangle = -|11\rangle, \\
U_A|1\rangle \otimes U_B|1\rangle &= i|0\rangle \otimes i|0\rangle = -|00\rangle, \\
|\phi\rangle &= -\frac{1}{\sqrt{2}}(|11\rangle + i|00\rangle).
\end{aligned}$$

After $J^\dagger$ and simplification:

$$\boxed{|\psi_f\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.}$$

6. Alice: A, Bob: Q

$$\begin{aligned}
U_A|0\rangle \otimes U_B|0\rangle &= i|1\rangle \otimes i|0\rangle = -|10\rangle, \\
U_A|1\rangle \otimes U_B|1\rangle &= i|0\rangle \otimes (-i)|1\rangle = -|01\rangle, \\
|\phi\rangle &= -\frac{1}{\sqrt{2}}(|10\rangle + i|01\rangle) = -\frac{1}{\sqrt{2}}(|10\rangle - |01\rangle).
\end{aligned}$$

Apply $J^\dagger$ and simplify:

$$\boxed{|\psi_f\rangle = \frac{i\big(|01\rangle + |10\rangle\big)}{\sqrt{2}}.}$$

7. Alice: Q, Bob: O

By symmetry with case 3 (swap players) one obtains

$$\boxed{|\psi_f\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.}$$

8. Alice: Q, Bob: A

By symmetry with case 6 (swap players) one obtains

$$\boxed{|\psi_f\rangle = \frac{i\big(|01\rangle + |10\rangle\big)}{\sqrt{2}}.}$$

9. Alice: Q, Bob: Q

$$\begin{aligned}
U_A|0\rangle \otimes U_B|0\rangle &= i|0\rangle \otimes i|0\rangle = -|00\rangle, \\
U_A|1\rangle \otimes U_B|1\rangle &= (-i)|1\rangle \otimes (-i)|1\rangle = -|11\rangle, \\
|\phi\rangle &= -\frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle) = -|\psi_0\rangle.
\end{aligned}$$

Applying $J^\dagger$ yields (up to a global phase)

$$\boxed{|\psi_f\rangle = |00\rangle.}$$

Final-state table

	O	A	Q
O	$\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}}$	$\frac{i(\lvert 01\rangle + \lvert 10\rangle)}{\sqrt{2}}$	$\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}}$
A	$\frac{i(\lvert 01\rangle + \lvert 10\rangle)}{\sqrt{2}}$	$\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}}$	$\frac{i(\lvert 01\rangle + \lvert 10\rangle)}{\sqrt{2}}$
Q	$\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}}$	$\frac{i(\lvert 01\rangle + \lvert 10\rangle)}{\sqrt{2}}$	$\lvert 00\rangle$

Alice O, Bob O

$$p_{00} = \tfrac{1}{2},\ p_{11} = \tfrac{1}{2}$$

$$\Pi_A = \tfrac{1}{2} \cdot 2 + \tfrac{1}{2} \cdot 1 = 1 + 0.5 = 1.5,$$
$$\$B = \tfrac{1}{2} \cdot 1 + \tfrac{1}{2} \cdot 2 = 0.5 + 1 = 1.5.$$

Alice O, Bob A

$$p_{01} = \tfrac{1}{2},\ p_{10} = \tfrac{1}{2}$$

$$\$A = 0 \cdot p_{00} + 0 \cdot p_{01} + 0 \cdot p_{10} + 1 \cdot p_{11} = 0,$$
$$\$B = 1 \cdot p_{00} + 0 \cdot p_{01} + 0 \cdot p_{10} + 2 \cdot p_{11} = 0.$$

Alice O, Bob Q

$$p_{00} = \tfrac{1}{2},\ p_{11} = \tfrac{1}{2}$$

$$\$A = \tfrac{1}{2} \cdot 2 + \tfrac{1}{2} \cdot 1 = 1.5,$$
$$\$B = \tfrac{1}{2} \cdot 1 + \tfrac{1}{2} \cdot 2 = 1.5.$$

Alice A, Bob O

$$p_{01} = \tfrac{1}{2},\ p_{10} = \tfrac{1}{2}$$

$$\$A = 0, \qquad \$B = 0.$$

Alice A, Bob A

$$p_{00} = \tfrac{1}{2},\ p_{11} = \tfrac{1}{2}$$

$$\$A = 1.5, \qquad \$B = 1.5.$$

Alice A, Bob Q

$$p_{01} = \tfrac{1}{2}, \; p_{10} = \tfrac{1}{2}$$

$$\$A = 0, \qquad \$B = 0.$$

Alice Q, Bob O

$$p_{00} = \tfrac{1}{2}, \; p_{11} = \tfrac{1}{2}$$

$$\$A = 1.5, \qquad \$B = 1.5.$$

Alice Q, Bob A

$$p_{01} = \tfrac{1}{2}, \; p_{10} = \tfrac{1}{2}$$

$$\$A = 0, \qquad \$B = 0.$$

Alice Q, Bob Q

$$p_{00} = 1$$

$$\$A = 1 \cdot 2 = 2, \qquad \$B = 1 \cdot 1 = 1.$$

Final 3×3 payoff matrix

With rows = Alice's choices (O, A, Q) and columns = Bob's choices (O, A, Q):

$$\boxed{\begin{bmatrix} (1.5,\ 1.5) & (0,\ 0) & (1.5,\ 1.5) \\ (0,\ 0) & (1.5,\ 1.5) & (0,\ 0) \\ (1.5,\ 1.5) & (0,\ 0) & (2,\ 1) \end{bmatrix}}$$

Each entry is $(\$A, \$B)$ for the corresponding strategy pair.

(d) Symmetric quantum strategy with maximal entanglement

We consider the maximally entangled case $\gamma = \pi/2$ and restrict the players' strategies to phase rotations:

$$U_A = U(0, \phi_A) = \begin{pmatrix} e^{i\phi_A} & 0 \\ 0 & e^{-i\phi_A} \end{pmatrix}, \quad U_B = U(0, \phi_B) = \begin{pmatrix} e^{i\phi_B} & 0 \\ 0 & e^{-i\phi_B} \end{pmatrix}. \tag{19}$$

The EWL final state is

$$|\psi_f\rangle = J^\dagger (U_A \otimes U_B) J |00\rangle, \tag{20}$$

where for maximal entanglement

$$J = \frac{1}{\sqrt{2}}(I \otimes I + i\sigma_x \otimes \sigma_x), \quad J^\dagger = \frac{1}{\sqrt{2}}(I \otimes I - i\sigma_x \otimes \sigma_x). \tag{21}$$

Acting on $|00\rangle$, we have

$$J|00\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + i|11\rangle\right), \quad (U_A \otimes U_B)J|00\rangle = \frac{1}{\sqrt{2}}\left(e^{i(\phi_A+\phi_B)}|00\rangle + ie^{-i(\phi_A+\phi_B)}|11\rangle\right). \tag{22}$$

Let $\alpha := \varphi_A + \varphi_B$.

$$|\phi\rangle := (U_A \otimes U_B)\,|\psi_0\rangle = \frac{1}{\sqrt{2}}\big(e^{i\alpha}\,|00\rangle + ie^{-i\alpha}\,|11\rangle\,\big). \tag{23}$$

$$\begin{aligned} |\psi_f\rangle &= \frac{1}{\sqrt{2}}\big((e^{i\alpha} - i \cdot ie^{-i\alpha})\,|00\rangle + (ie^{-i\alpha} - ie^{i\alpha})\,|11\rangle\,\big) \\ &= \frac{1}{\sqrt{2}}\big(2\cos\alpha\ |00\rangle + 2\sin\alpha\ |11\rangle\,\big) = \sqrt{2}\big(\cos\alpha\ |00\rangle + \sin\alpha\ |11\rangle\,\big). \end{aligned} \tag{24}$$

Hence

$$\boxed{|\psi_f\rangle = \cos\alpha\ |00\rangle + \sin\alpha\ |11\rangle\,.} \tag{25}$$

Therefore the measurement outcome probabilities are

$$p_{00} = \cos^2\alpha, \qquad p_{11} = \sin^2\alpha, \qquad p_{01} = p_{10} = 0. \tag{26}$$

A symmetric outcome with $p_{00} = p_{11} = \frac{1}{2}$ requires

$$\cos^2\alpha = \sin^2\alpha = \tfrac{1}{2} \quad \Longleftrightarrow \quad \cos 2\alpha = 0 \tag{27}$$

so the choices are

$$\alpha = \varphi_A + \varphi_B = \frac{\pi}{4}, \frac{\pi}{8}... \tag{28}$$

(infinitely many pairs)

Expected payoffs for that strategy pair

Using the payoff vectors (ordering $|00\rangle\,, |01\rangle\,, |10\rangle\,, |11\rangle$)

$$\Pi_A = (2, 0, 0, 1), \qquad \Pi_B = (1, 0, 0, 2),$$

the expected payoffs are

$$\$A = 2\,p_{00} + 0 \cdot p_{01} + 0 \cdot p_{10} + 1 \cdot p_{11} = 2\cos^2\alpha + \sin^2\alpha = 1 + \cos^2\alpha,$$

$$\$B = 1\,p_{00} + 0 \cdot p_{01} + 0 \cdot p_{10} + 2 \cdot p_{11} = \cos^2\alpha + 2\sin^2\alpha = 1 + \sin^2\alpha.$$

For the symmetric choice $\alpha = \pi/4$ (so $p_{00} = p_{11} = \frac{1}{2}$) we get

$$\boxed{\$A = 1 + \tfrac{1}{2} = 1.5, \qquad \$B = 1 + \tfrac{1}{2} = 1.5.} \tag{29}$$

Comparison with classical game strategy

- **Classical pure Nash equilibria:** The classical Battle-of-the-Sexes has two pure Nash equilibria
$$(O, O) \mapsto (2, 1), \qquad (A, A) \mapsto (1, 2),$$
each asymmetric (one player receives 2, the other 1). ONe player choice is always favoured more than other.

- **Classical mixed Nash equilibrium:** The mixed Nash yields expected payoffs $\$A_{\text{mix}} = \$B_{\text{mix}} = \frac{2}{3}$.

- **Quantum symmetric strategy:** The quantum strategy pair with $\varphi_A + \varphi_B = \pi/4$ yields equal payoffs $(1.5, 1.5)$. This outcome is

  1. symmetric (both players get the same expected payoff),
  2. better than the classical mixed Nash $\frac{3}{2} > \frac{2}{3}$.