Harmonic Geometry

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Harmonic, or projective, geometry is a special way of looking at ratios in a geometric figure. Many difficult problems can be solved with only the two most important constructions. The first construction is the harmonic bundle on a line. The second construction is the harmonic quadrilateral on a circle.

The Harmonic Product

There is no point in talking about harmonic geometry without first understanding the harmonic product (also called the cross-ratio). This is a value you can find given four points A, B, X, and Y on a line or on a circle. It is denoted as $(A, B; X, Y)$ and defined as

$$(A, B; X, Y) = \frac{XA \cdot YB}{XB \cdot YA}$$

The four points A, B, X, and Y are called a harmonic bundle if this harmonic product is -1.

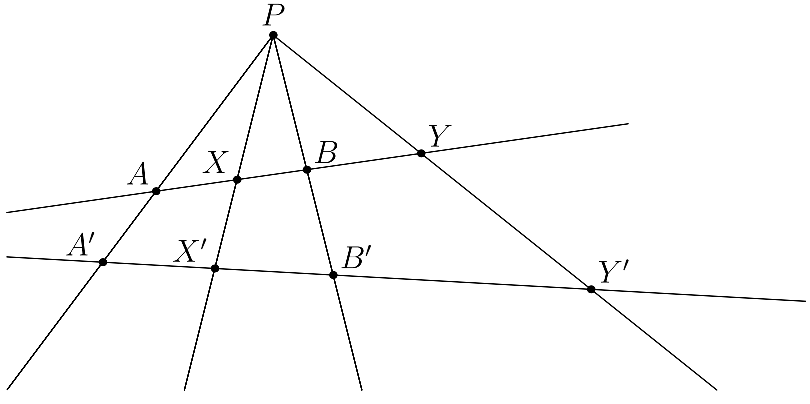
Directed/Signed Lengths

Directed lengths are usually used in harmonic/projective geometry. Given collinear points A, B, C, we have that $AB \cdot AC$ is positive if B is between A and C (i.e. lies on segment $\overline{AC}$) and negative otherwise.

We also define the harmonic product/cross-ratio $(A, B; X, Y)$ to be positive if segments $\overline{AB}$ and $\overline{XY}$ do not intersect, and negative if they do.

Line to Line Projections

This is a basic construction of harmonic geometry that is necessary to solve many other projective questions. This construction allows you to project four points from one line to four points on another line without changing its harmonic product. This construction is also called a pencil tip sometimes (it makes sense; just look at it).



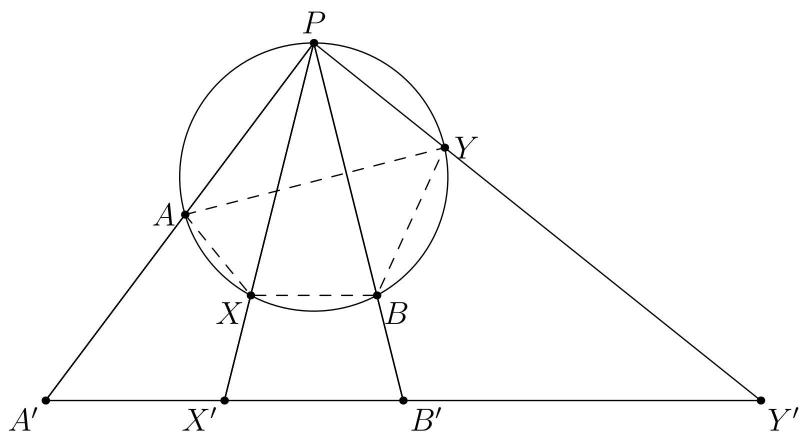
$(A, B; X, Y) = (A', B'; X', Y')$

This kind of transformation is extremely important and versatile because you can use any point to project through (in this case P).

Exercise: Prove $(A, B; X, Y) = (A', B'; X', Y')$.
Hint: Use the sine area formula to find the ratios of $\sin\angle APX$, $\sin\angle XPB$, $\sin\angle BPY$, and $\sin\angle YPA$.

Line to Circle Projections

A similar projection can be done with a line onto points of a circle, instead of points on another line. This also does not change harmonic product and can be used to develop harmonic quadrilaterals (which I will get into in the next section).



$$(A, B; X, Y) = (A', B'; X', Y')$$

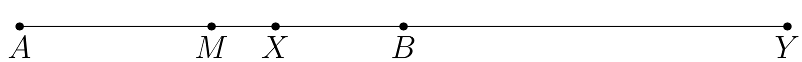
This transformation is less versatile than the previous, especially because of the condition that P must be on the circle, but it is still useful and can solve some problems, one of which I will give later.

Exercise: Prove $(A, B; X, Y) = (A', B'; X', Y')$ again for this case.
Hint: Use the last exercise's ratio of sines again, and calculate $(A, B; X, Y)$ in terms of the same sines using the Law of Sines.

Harmonic Bundles

Harmonic bundles are sets of four points that have a harmonic product of -1. These harmonic bundles have many special properties that can be exploited.

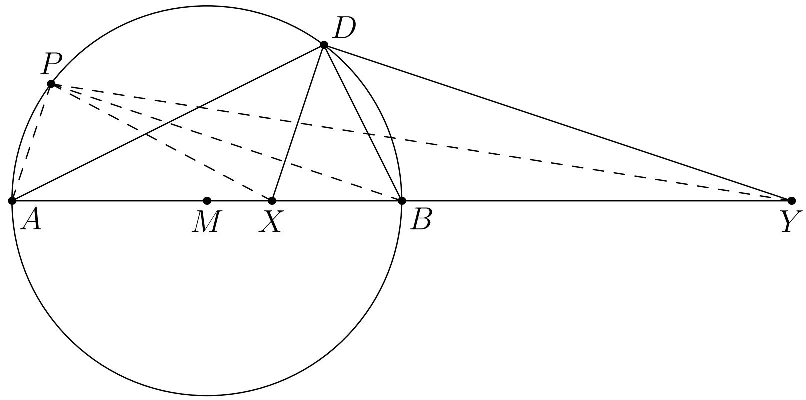
1. $(A, B; X, Y)$ is harmonic if and only if $MX \cdot MY = MA^2$ where M is the midpoint of $\overline{AB}$.



   Exercise: Prove this result, namely $(A, B; X, Y) = -1 \iff MX \cdot MY = MA^2$.

   Hint: Put A, B, X, Y, and M on a number line and try solving for the harmonic product by simplifying $MX \cdot MY = MA^2$.

2. The following is the configuration of the Apollonian Circle.



Let $(A, B; X, Y)$ be a harmonic bundle. Let D be a point on the circle with diameter $\overline{AB}$. We will show that as D moves along the circle (which is called the Apollonian Circle of $\overline{XY}$) that DB is an angle bisector of $\angle XDY$.

From our previous exercises, we know that

$$|(A, B; X, Y)| = \frac{(\sin \angle ADX)(\sin \angle BDY)}{(\sin \angle XDB)(\sin \angle YDA)}$$

We know that $\angle BDA$ is right because it subtends a semicircle. Calling $\angle XDB$, α and $\angle BDY$, β, we can start substituting some values.

$$|(A, B; X, Y)| = \frac{(\sin(90 - \alpha))(\sin \beta)}{(\sin \alpha)(\sin(90 - \beta))}$$

Using some basic trig identities we get:

$$|(A, B; X, Y)| = \frac{(\cos \alpha)(\sin \beta)}{(\sin \alpha)(\cos \beta)}$$

Even more algebra:

$$|(A, B; X, Y)| = \frac{\tan \beta}{\tan \alpha}$$

Since $(A, B; X, Y)$ is a harmonic bundle, we have:

$$1 = \frac{\tan \beta}{\tan \alpha}$$

$$\tan \alpha = \tan \beta$$

Since α and β are both between 0 and 90, for $\tan \alpha$ to be $\tan \beta$, α must be β. Since $\alpha = \beta$, DB must be an angle bisector of $\angle XDY$, as desired. As one more example, I've included P, which is another arbitrary point on the Apollonian circle.

One fascinating property of the Apollonian circle is that the ratio $\frac{PX}{PY}$ is fixed for any P on the circle. This is a direct consequence of the angle bisector theorem and our above result.

Point at Infinity

There is an important kind of harmonic bundle to take note of. Imagine three of our points, which can be called A, X, and B, are in a line, and one is the midpoint of the other two. Suppose we want to complete this harmonic bundle. Where would the last point be? This question seems to be sort of unsolvable.

$$(A, B; X, Y) = \frac{XA \cdot YB}{XB \cdot YA}$$

If X is the midpoint of A and B, then $XA = XB$ This now gives us:

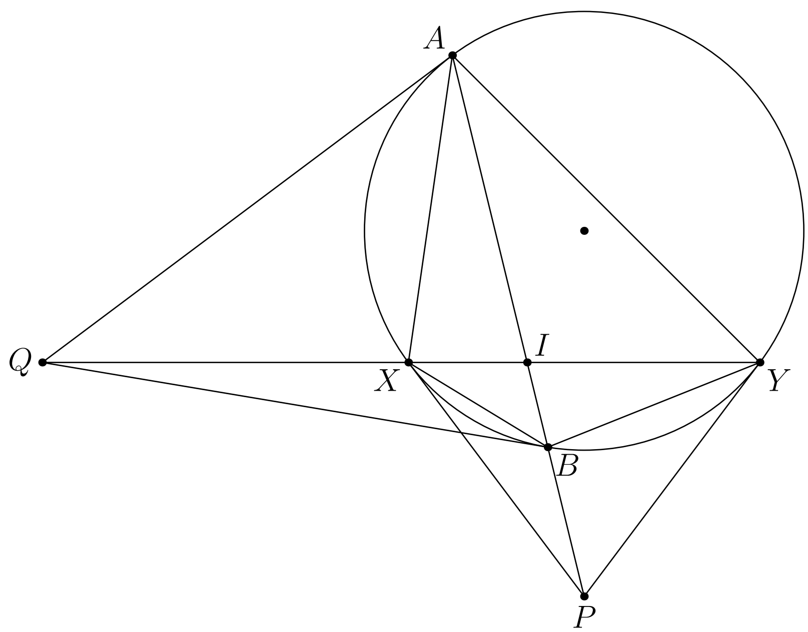
$$(A, B; X, Y) = -\frac{YB}{YA}$$

If we want this to be a harmonic bundle, we need $\frac{YB}{YA}$ to be equal to 1. This means YB and YA are equal in length but once we consider direction, it becomes obvious Y cannot be the midpoint of A and B. This means that Y must lie at infinity on either side of the line AB. This means we have a valid harmonic bundle, where X is the midpoint of AB and Y lies on the point at infinity on line AB. This is useful in some niche situations, mostly when you have parallel lines. We denote the point at infinity as P_∞.

Harmonic Quadrilaterals

The harmonic quadrilateral is a harmonic bundle where the four points make a cyclic quadrilateral. This quadrilateral has many properties as well.

Given a cyclic quad $AXBY$ with diagonals intersecting at I, construct points P, Q along AB and XY respectively such that PX, PY, QA, QB are tangents to $AXBY$ at their respective points.



- The line AB is concurrent with the tangents to the circle at X and Y and the line XY is concurrent with the tangents to the circle at A and B if and only if $AXBY$ is a harmonic quad: $(A, B; X, Y) = -1$
- $\triangle BXP \sim \triangle XAP$. The same is true for other triangles of the same type like $\triangle BYP \sim \triangle YAP$.
- XY is the Y symmedian (the reflection of the median over the angle bisector) for $\triangle AYB$ and the X symmedian for $\triangle AXB$. Similarly, AB is the A symmedian for $\triangle XAY$ and the B symmedian for $\triangle XBY$. I don't want to include too much on symmedians, since the focus is projections, but symmedians and its relation to harmonic geometry is very interesting, and I encourage you to look into it.
- When projecting a point through itself, the line through which you project is the tangent. For example, if you wanted to project $AXBY$ through X onto AP, to project X through X, follow the rule of projecting through the tangent. So to project X, you would project X through XP (the tangent to X) onto AP which goes to P. After projecting A, Y, and B, we get that A, I, B, and P form a harmonic bundle. Similarly, Q, X, I, and Y also form a harmonic bundle.

Some Problems

Here are some problems that can be done with the concepts above.

1. (AIME) $\triangle ABC$ is inscribed in circle ω. Points P and Q are on side AB with $AP < AQ$. Rays CP and CQ meet ω again at S and T (other than C), respectively. If $AP = 4$, $PQ = 3$, $QB = 6$, $BT = 5$, and $AS = 7$, then find ST.

   Note: Answer is not necessarily an integer.

2. (AIME) The incircle ω of $\triangle ABC$ is tangent to BC at X. Let $Y \neq X$ be the other intersection of AX with ω. Points P and Q lie on AB and AC, respectively, so that PQ is tangent to ω at Y. Assume that $AP = 3$, $PB = 4$, and $AC = 8$. Find AQ.

   Note: Answer is not necessarily an integer.

3. (NEMO) Let $ABCD$ be a convex quadrilateral, with $AB = 6$, $BC = 9$, $CD = 15$, and $DA = 10$. Point P is chosen on the line AC so that $PB = PD$. Given that $AC = 11$, find PB.

4. (Math-M-Addicts) Suppose A is a point outside of circle ω. The two tangents to ω passing through A touch this circle at points B and C. Line l passing through A intersects ω at points D and E. Chord BM of ω is parallel to DE. Prove that line MC bisects DE.

Hints:

1. Use circle to line projections.

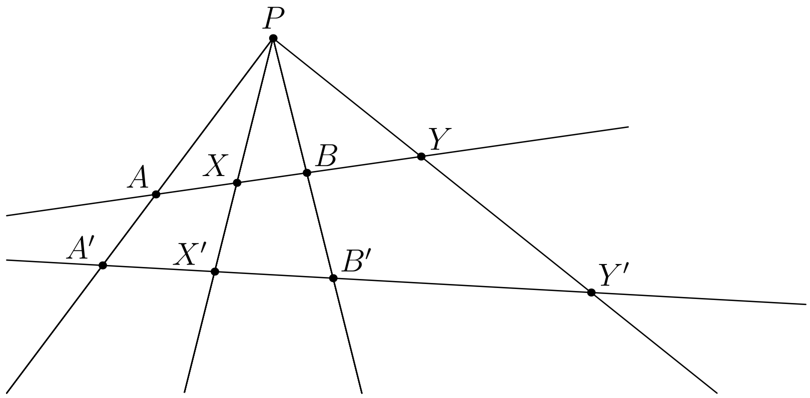
2. I will not give a hint for this one since there are many, many steps, but see what you can figure out with the information you are given and projecting a bunch of things. This is where the versatility of harmonic geometry comes in.

3. Use the fact that $ABCD$ is harmonic and use some of the similar triangles I proposed in the "Harmonic Quadrilaterals" section.

4. Use the concept I said would be useful when you have parallel lines and combine that with some harmonic quadrilateral concepts.

Solutions

First Exercise



Let us look at the ratios of the areas of triangles $\triangle PAX$, $\triangle PXB$, $\triangle PBY$, and $\triangle PYA$. Using the sine area formula, we get that

$$\frac{[PAX]}{[PXB]} = (\frac{1}{2}(PA)(PX)\sin\angle APX) \div (\frac{1}{2}(PX)(PB)\sin\angle XPB)$$

This simplifies to $\frac{PA\sin\angle APX}{PB\sin\angle XPB}$. We also know that we can write this ratio of areas as the ratio of their bases, since they share a height. This gives us

$$\frac{AX}{XB} = \frac{(PA)\sin\angle APX}{(PB)\sin\angle XPB}$$

We can use a similar argument on triangles $\triangle PBY$ and $\triangle PYA$. This gives us that

$$\frac{YB}{YA} = \frac{(PB)\sin\angle BPY}{(PA)\sin\angle YPA}$$

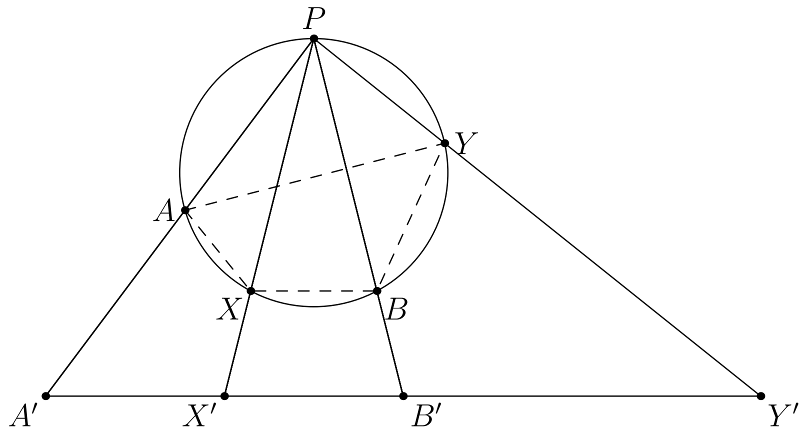
Multiplying these two equations results in

$$|(A, B; X, Y)| = \frac{(\sin\angle APX)(\sin\angle BPY)}{(\sin\angle XPB)(\sin\angle YPA)}$$

Since this means that the harmonic mean is fully dependent on angles (it can be solved for in terms of only angles not using any lengths), we know this will work for another set of points A', B', X', and Y'.

Second Exercise

If we are able to conclude that the new harmonic mean is also equal to this ratio of sines, we will have deduced that the harmonic mean is the same.



Here, we can use the Law of Sines to find our ratios. We know, by Law of Sines, that

$$\frac{\sin \angle APX}{AX} = \frac{\sin \angle PAX}{PX}$$

and that

$$\frac{\sin \angle XPB}{XB} = \frac{\sin \angle PBX}{PX}$$

We also know that $\sin \angle PAX = \sin \angle PBX$ because $\angle PAX$ and $\angle PBX$ subtend opposite arcs, meaning they are supplementary, meaning their sines are the same. Dividing and then moving a few things around gives us that

$$\frac{AX}{XB} = \frac{\sin \angle APX}{\sin \angle XPB}$$

A similar argument with Law of Sines on triangles $\triangle PYB$ and $\triangle PYA$ using the common side of PY gives that

$$\frac{BY}{YA} = \frac{\sin \angle BPY}{\sin \angle YPA}$$

Multiplying gives us that

$$|(A, B; X, Y)| = \frac{(\sin \angle APX)(\sin \angle BPY)}{(\sin \angle XPB)(\sin \angle YPA)}$$

which is the same result as when using a line. This means that the harmonic mean for any circle is the same as the harmonic mean for the line, as long as the circle passes through P.

Ending Problems

Problem 1

Imagine the rays from C outwards to A, P, Q, and B. This is similar to our pencil configuration that we used for line to line and line to circle projections. We have a circle in this configuration, so it will likely be a line to circle projection. Now that our circle has been established (circumcircle of $\triangle ABC$), we should look for the line. We only have one line that intersects this pencil configuration, being AB. Note that our line can be any line whatsoever, so it sharing points A and B with the circle is not a problem at all. This gives us

$$(A, Q; P, B) = (A, T; S; B)$$

We can evaluate $(A, Q; P, B)$:

$$(A, Q; P, B) = \frac{4 \cdot 6}{3 \cdot 13} = \frac{8}{13}$$

This means that

$$\frac{SA \cdot BT}{ST \cdot BA} = \frac{8}{13}$$

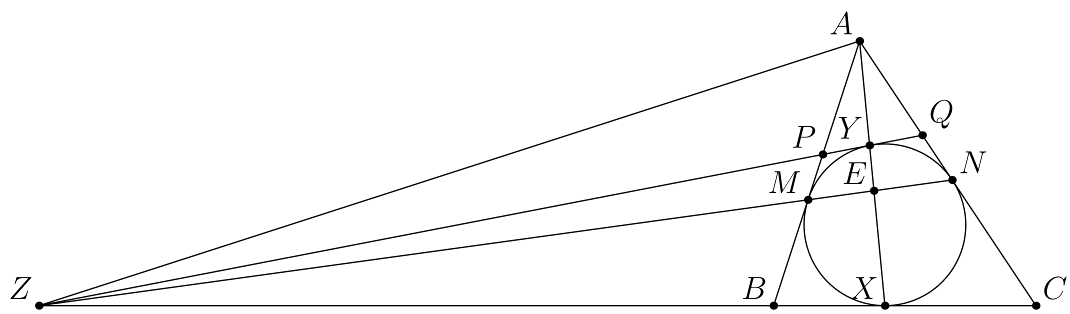
Substituting, we get

$$\frac{7 \cdot 5}{ST \cdot 13} = \frac{8}{13}$$

Finally, solving for the desired ST gives us that $ST = \frac{35}{8}$.

Problem 2

We have that the line XY passes through A, which is the intersection point of the tangents of the meeting points of the incircle and AB and AC. This means that X, Y, and those meeting points (lets call them M on AB and N on AC) form a harmonic quadrilateral. Call the intersection of XY and MN, E. We know that A, Y, E, and X form a harmonic bundle, either by projecting through M or N, or just remembering this as a construction, and using it as a rule. Either way, we know we can extend MN so that it intersects with the tangents of X and Y. Call this point Z.



If we connect Z to our harmonic bundle of A, Y, E, and X and extend, we can see two new harmonic bundles: One formed by A, P, M, and B, and another formed by A, Q, N, and C. Calling the length AM, x, we can solve for these harmonic products in terms of x, set them equal to 1, solve for x and then use this same value of x to represent AM, since the lengths of tangents are equal.

$$|(A, M; P, B)| = \frac{3}{x-3} \cdot \frac{7-x}{7} = 1$$

Solving gives that $x = \frac{21}{5}$ Now setting our desired $AQ = y$, we can solve for $|(A, N; Q, C)|$.

$$|(A, N; Q, C)| = \frac{y}{x-y} \cdot \frac{8-y}{8} = 1$$

Substituting x and solving for y reveals $AQ = \frac{168}{59}$.

Problem 3

Immediately from the side lengths, we get that $ABCD$ is a harmonic quadrilateral. This means that when we extend AC, it intersects the tangents of B and D. We know that the point P must lie on AC and be an equal distance from B and D. Since we know AC passes through the intersection of the tangents, and that the intersection of tangents are equal lengths from both B and D, we know that P must be this special point that lies on the intersection of tangents. From there we have similar triangles to solve.

$$\frac{PA}{PB} = \frac{PB}{PC} = \frac{AB}{BC} = \frac{2}{3}$$

$$PC = \frac{9}{4}AB$$

$$PC = PA + 11$$

Solving gives us that $PA = \frac{44}{5}$ and $PC = \frac{99}{5}$. We are asked for PB which is their geometric mean: $PB = \frac{66}{5}$

Problem 4

In this problem, we have that the two tangents to ω from A hit ω at B and C. Then we have another chord DE that passes through A. This means that $BECD$ is a harmonic quadrilateral. Now we project through M onto DE. D and E stay in the same place, while C goes to K, because K is the intersection of DE and M. Now the projection of B is strange. BM is parallel to DE, so it never hits. Another way of thinking of this is that B projects onto the point at infinity, call it X. Now that we have that $(E, D; K, X)$ is a harmonic bundle, and that X is the point at infinity on the line DE. This means that K must be the midpoint of DE, which is exactly what we wanted to prove.