Calculating a Fixed Distance

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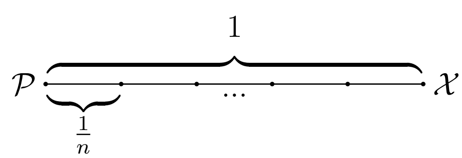
1 Problem Statement

A particle $\mathcal{P}$ is initially 1 unit away from $\mathcal{X}$ its destination. On its path to $\mathcal{X}$, $\mathcal{P}$ has a velocity of $d+1$ units per second, where d is its distance to $\mathcal{X}$. How long will it take for $\mathcal{P}$ to reach $\mathcal{X}$?

2 Solution



First, we will partition the interval of length 1 into n equally sized interval. We will assume that $\mathcal{P}$ changes its velocity at the endpoints. Then, we may take the limit of our result as $n \to \infty$. This will allow us to simplify the problem.

Let d_i be the distance traveled by $\mathcal{P}$ during the ith interval, v_i be its velocity at the beginning of this interval, and t_i be the amount of time it spends in the interval. Additionally, let $T_n = t_1 + t_2 + t_3 + \cdots + t_n$ be the total amount of time that it takes $\mathcal{P}$ to reach $\mathcal{X}$ in a configuration with n subintervals. As we add more subintervals, the velocity will become closer to changing continuously. Hence, the total amount of time $\mathcal{P}$ takes is $T = \lim_{n \to \infty} T_n$.

Now, we must compute T_n in terms of n. First, we have $T_n = \sum_{k=1}^{n} t_k$. From $d = rt$, we may conclude that $t_k = \frac{d_k}{v_k}$. Additionally, $d_k = \frac{1}{n}$ for all k and $v_k = 2 - \frac{k-1}{n}$ since at the kth interval it has travelled $k-1$ subintervals of length $\frac{1}{n}$. Hence,

$$t_k = \frac{\frac{1}{n}}{2-\frac{k-1}{n}} = \frac{1}{2n-k+1} \Rightarrow T_n = \sum_{k=1}^{n} \frac{1}{2n-k+1} = \frac{1}{2n}+\frac{1}{2n-1}+\frac{1}{2n-2}+\cdots+\frac{1}{n+2}+\frac{1}{n+1}.$$

We may further rewrite this as

$$\begin{aligned}
\frac{1}{2n}+\cdots+\frac{1}{n+1} &= \left(\frac{1}{2n}+\frac{1}{2n-1}+\cdots+\frac{1}{2}+\frac{1}{1}\right)-\left(\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{2}+\frac{1}{1}\right)\\
&= \left[\left(\frac{1}{2n}+\frac{1}{2n-1}\right)+\left(\frac{1}{2n-2}+\frac{1}{2n-3}\right)+\cdots+\left(\frac{1}{2}+\frac{1}{1}\right)\right]-\sum_{i=1}^{n}\frac{1}{i}\\
&= \sum_{i=1}^{n}\left(\frac{1}{2i}+\frac{1}{2i-1}\right)-\sum_{i=1}^{n}\frac{1}{i}\\
&= \sum_{i=1}^{n}\left(\frac{1}{2i}+\frac{1}{2i-1}-\frac{1}{i}\right)\\
&= \sum_{i=1}^{n}\left(\frac{1}{2i-1}-\frac{1}{2i}\right)\\
&= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{2n-3}-\frac{1}{2n-2}\right)+\left(\frac{1}{2n-1}-\frac{1}{2n}\right)\\
&= 1-\frac{1}{2}+\frac{1}{3}-\cdots-\frac{1}{2n-2}+\frac{1}{2n-1}-\frac{1}{2n}.
\end{aligned}$$

Hence, $T = \lim_{n\to\infty} T_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\cdots$. However, this is not a closed form. To find the closed form, we turn to calculus.

Let the total distance $\mathcal{P}$ has traveled at time t be $x(t)$ and its velocity be $v(t)$. By the problem statement, $v(t) = (1-x(t))+1 = 2-x(t)$. Since $x(t)$ is $\mathcal{P}$'s position and the rate of change of position is velocity, we have $v(t) = x'(t)$. Now, we can simply solve for $x(t)$, the position function, by solving the differential equation $\frac{dx}{dt} = 2-x$. However, we will take a different approach.

Differentiating both sides of $v(t) = 2-x(t)$ with respect to t, we get $a(t) = -v(t)$, where $a(t)$ is the acceleration of $\mathcal{P}$ at time t. Additionally, we have that $a(t) = v'(t)$. Plugging this back into $a(t) = -v(t)$, we have $v'(t) = -v(t)$. Now, we have the differential equation

$\frac{dv}{dt} = -v$. Separating the variables and integrating both sides, we have

$$\int \frac{dv}{v} = -\int dt$$
$$\Rightarrow \ln|v| = \ln v(t) = -t + C$$
$$\Rightarrow v(t) = e^{-t+C} = Ae^{-t}$$

for some $A \in \mathbb{R}^+$. Note that we can remove the absolute values since $v(t) \geq 1 > 0$. We can solve for A by plugging in $t = 0$. Since at time $t = 0$, $\mathcal{P}$ is 1 unit away from $\mathcal{X}$, its velocity is 2 units per second: $v(0) = A = 2$. Hence, $v(t) = 2e^{-t}$. Let $\mathcal{P}$ reach $\mathcal{X}$ after k seconds. We know that $v(k) = 2e^{-k} = 1$. Hence, we can solve for k:

$$2e^{-k} = 1 \Rightarrow e^{-k} = \frac{1}{2} \Rightarrow -k = \ln\frac{1}{2} \Rightarrow k = \ln 2.$$

Hence, $\mathcal{P}$ will reach $\mathcal{X}$ in $\ln 2$ seconds.

Since, our answers to the problems must be the same, we have

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots = \ln 2$$

3 Generalization

Now, instead of $\mathcal{P}$ starting 1 unit away from $\mathcal{X}$, let it be d units away.

From the problem statement we have $v(t) = (d - x(t)) + 1 = d + 1 - x(t)$, Taking derivative with respect to t, we have $a(t) = -v(t)$. This is the exact same equation we solved earlier. We found that $v(t) = Ae^{-t}$ for some $A \in \mathbb{R}$. We can analyze what happens at $t = 0$: $x(0) = d \Rightarrow v(0) = A = d + 1$. Hence, $v(t) = (d + 1)e^{-t}$. Solving the equation $v(t) = 1$, we have $t = \ln(d + 1)$. Hence, $\mathcal{P}$ will reach $\mathcal{X}$ in $\ln(d + 1)$ seconds. Plugging in $d = 1$ from the problem agrees with our previous result.