History of Geometry

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The Elements **and the Parallel Postulate**

Roughly 2300 years ago, in Alexandria, Egypt, Euclid wrote *The Elements*, a series of theorems and proofs covering a variety of topics within geometry. The 48 proven theorems of *The Elements* were based on the 5 axioms, 23 definitions, and 5 common notions Euclid laid out at the beginning of the work. The definitions covered the vocabulary that was needed for understanding the rest of the work. The common notions were algebraic equivalences, like commutativity ($a + b = b + a$) and transitivity (if $a = b$ and $b = c$ then $a = c$). The 5 axioms are unproven statements that form the backbone of the logical system. Of the 5 axioms, four are reasonably straightforward, asserting that a line can be drawn between any two points, a line segment defines a single line, a circle is defined by its center and a radius, and all right angles are congruent. However, the fifth and final postulate is somewhat less obvious. It is most commonly translated to say that "if a straight line falling on two straight lines makes the interior angles on the same side less than the sum of two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles." In the diagram, this can be interpreted to say that if $\alpha + \beta < 180°$, then the two lines will intersect to the right of the diagram.

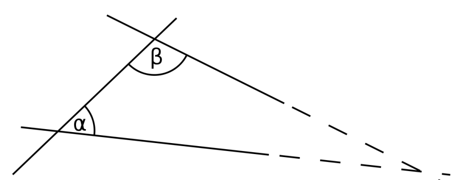

Figure 1: Diagram of the Parallel postulate

There are many statements that are equivalent to the parallel postulate. One very common such statement is the Playfair Axiom. The axiom states that given a line, l, and a point not on that line, p, there exists a unique line through the point, parallel to the line. We can demonstrate the equivalence by assuming that the Parallel Postulate is true, and using it to prove the Playfair Axiom, and vice versa. Assuming the Parallel Postulate, we can construct a unique parallel line by drawing a perpendicular to l through p, and then the line perpendicular to the perpendicular at p. This must be unique, as any other line would have to form an angle not equal to 90°, and thus would intersect l, by the parallel postulate. Next, assuming the Playfair Axiom, we can prove the Parallel postulate very similarly. Any line other than the one guaranteed by Playfair's Axiom would have to have a sum of angles less than 180°on one side of the perpendicular, and would have to intersect, as it is not a parallel line.

As it is considerably more complicated than the other axioms, the final postulate was subject to some scrutiny. In the 17th and 18th centuries, mathematicians began to question this axiom, often referred to as the fifth or parallel postulate. One of the most famous works on this topic, published in 1733 by a man named Girolamo Saccheri, was *Euclides ab omni nævo vindicatus*. Saccheri used a proof by contradiction to demonstrate the truth of the parallel postulate. Unfortunately, his writing relied on some unstated assumptions. Saccheri had a reputation of being a great logician and proof-writer, so these mistakes were uncharacteristic. So, it is speculated that Saccheri knew his mistakes, and understood the possibility of non-Euclidean Geometry, but did not want to be reprimanded by the Church, who took very strong opinions on maintaining the status quo, even in mathematics and science. His writings developed an idea of how a geometry where the parallel postulate was false could exist. Saccheri constructed a quadrilateral with two right angles at the base. For his contradiction, he assumed that the sum of the other two angles, called summit angles, was not 180°.

Hyperbolic Geometry

Saccheri's diagram leaves two possibilities for contradiction. Either, the sum of the two summit angles is greater than 180° or less than 180°. This geometry is formed by modifying the final postulate to say that there are at least two distinct lines through a point parallel to a separate line. From these two distinct lines, all of the lines between them are also parallel, so there are an infinite number of distinct parallel lines. This is the geometry on a saddle shaped, or negatively curved, plane (think like a Pringle), as this allows for lines to intersect in the ways dictated by the modified postulate. This geometry is called Hyperbolic Geometry (though it has no direct relation to the conic section) and the surface the geometry is performed on is called the hyperbolic plane.

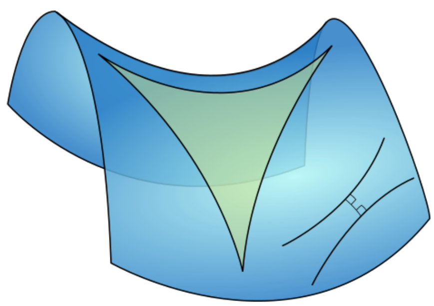

Figure 2: A diagram of the hyperbolic plane

Hyperbolic Geometry still satisfies the first four of Euclid's postulates: Two points define a line segment, a line segment defines a line, a circle is defined by a center and a radius, and all right angles are equal. The expression of these postulates might be slightly different than we expect, as circles and lines appear differently in hyperbolic geometry.

One extremely helpful model for Hyperbolic Geometry is the Poincaré Disk. This is a projection of the hyperbolic plane onto a unit circle. Lines are defined to be the circles orthogonal to, intersecting at right angles, the edge of the disk. Distances are shrunken as one approaches the edge of the disk, so the objects near the center of the projection appear larger than objects closer to the edge. points at the edge can be imagined to be points an infinite distance away from the center because of the continuously shrinking distances. We can verify that this model satisfies the fifth postulate of Hyperbolic Geometry as such: Given a line and a point not on the line, we can draw more than one parallel line as demonstrated below.

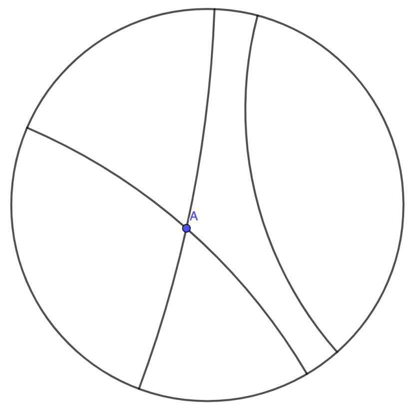

Figure 3: A demonstration of mutiple parallel lines through a single point on the Poincaré Disk

Triangles in hyperbolic geometry have some interesting properties. Like in Euclidean geometry, triangles are defined by three points. However, in hyperbolic geometry, the sum of the angles is always less than π, as can be extrapolated by dividing the Saccheri quadrilateral into two pieces. In the Saccheri quadrilateral, the sum of angles is by definition less than 2π. So, the sum of angles of a right triangle, after dividing the quadrilateral into 2 pieces, must be less than π. Triangles can also be formed using the points at infinity, or on the edge of the Poincaré Disk. In this case, the lines are said to be limiting parallel, as they never intersect, but get infinitely close to each other. A triangle with three ideal vertices would thus have an angle sum of 0. An example on the Poincaré Disk is shown below. In hyperbolic geometry, the area of a triangle is proportional to the defect, or the amount by which the sum is less than π. A very important result in Hyperbolic Geometry states that the area of a triangle, with angles A, B, and C, is equal to

$$\pi - A - B - C$$

This theorem provides some very nice corollaries, including an upper bound on the area of a triangle of π, which occurs with three limiting vertices.

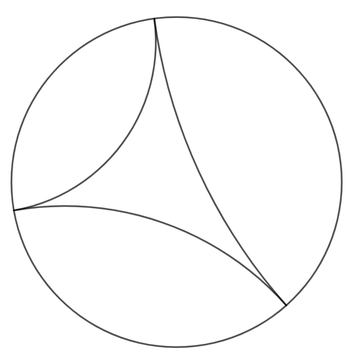

Figure 4: An example of a triangle with three limiting vertices

Spherical Geometry

Returning to Saccheri's quadrilaterals, there is a second, non-Euclidean case, as the sum of the summit angles can be greater than 180°. This is called spherical geometry. As the name implies, it is geometry on the surface of a sphere. So, since lines are the shortest distance between two points, in spherical geometry, lines are great circles, or the intersection of the sphere and a plane through the center of the sphere. A problem does arise when Euclid's postulates are verified. On a sphere, there are multiple lines through two antipodal, or opposite, points. In order to remedy this problem, points in spherical Geometry are conventionally defined to be the two antipodal points on a sphere. Now, two points, or sets of antipodal points, define a single line. When "points" are mentioned, assume a set of antipodal points is being referenced.

Any two lines, or great circles, must intersect. A great circle is the intersection of a plane with a sphere, going through the center. Thus the planes containing each of the great circles must intersect at the origin. Two planes cannot intersect at a point, they must intersect at a line. Where this line intersects the sphere is on both of the great circles, so the lines must intersect. So, there are no parallel lines in spherical geometry.

Spherical geometry has a special shape called a lune. It is formed by two lines. These two lines intersect at 1 antipodal set of points. We can calculate the area of a lune by considering the angle, α, between the two lines, measured in radians. The area of the lune is $\frac{\alpha}{2\pi}\left(4\pi r^2\right)$, as $\frac{\alpha}{2\pi}$ is the fraction of the sphere covered, and $4\pi r^2$ is the surface area of the sphere. This simplifies to 2α, as we can assume the use of the unit sphere.

Triangles can be formed in spherical geometry by choosing any three points. The three great circles, or lines, connecting those points create 8 triangles; we can choose which of the antipodal points to use. So, the three points do not define a unique triangle. From the 3 antipodal sets, we choose the points A, B, and C. We can let the angle at A in triangle ABC be a, and likewise for B and C. As per the previous paragraph, the area of the lune formed by A, its antipodal pair, and the lines AB and AC has area $2a$. If we consider the other lune formed by the same lines, it also has area $2a$. Similarly, the areas of the lunes with vertex at B have area $4b$, and likewise for C. These three lunes, when added together, cover the sphere plus two times the triangle plus two times triangle A'B'C', the antipodal triangle. If we let the area of the triangle be X,

$$4a + 4b + 4c = 4\pi + 4X$$

$$X = a + b + c - \pi$$

In this geometry, the sum of angles in a triangle is strictly greater than 180°,with an upper limit approaching 540°. We can see that the upper limit must be 540°, as a convex triangle cannot take up more than half of the sphere, so must have an area less than 2π. Using the area formula, $a + b + c$ must then be less than 3π.

Spherical Geometry is incredibly useful in navigation and sailing, as we live on a sphere. It was originally studied by the Greeks, after Ptolemy's discovery of the round nature of the Earth over two thousand years ago. This geometry was considered to be completely separate from the geometry done by Euclid, and thus non-contradictory. More recently, airlines use spherical geometry to determine flight paths, as ones that traverse great circles are the shortest and most efficient. This is why one would fly over the North Pole to get from San Francisco to Dubai.

Works Cited

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