The History of π

By Jacob Paltrowitz

The origin of much of mathematics is shapes. One shape that has been uniquely interesting to mathematicians for millennia is the circle. A circle is defined by the locus of points equidistant to a center. The circle seems like a natural shape to be interested in, as many phenomena in the world involve this shape. In fact, past mathematicians reasoned that circles were all similar. Therefore, there must be a constant ratio between the circumference and the diameter. We now know this ratio to be π.

The simplest way to approximate π is to construct a circle with a string, measure its circumference and diameter, and then divide. Following these steps, I concluded that $\pi \approx 3.35$. Please try this method at home, and see how close you can get. I suggest using a glass cup or some other circular object to aid with constructing the circle.

Past mathematicians needed more accurate approximations of the circumference of a circle. Many people made the astute observation that a circle's circumference approached the perimeter of inscribed regular polygons, as the polygons' number of sides goes to infinity. We can start by finding the perimeter of a regular hexagon inscribed in a circle with radius 1. If we split the hexagon into 6 equilateral triangles, we can see that the hexagon has perimeter 6. Now, we want to find the perimeter of larger regular polygons. The method used by Archimedes and Zu Chongzi, the two people who used this method to get good approximations very early on. The two mathematicians did their work completely separately; neither knew that the other existed, but they still got to the same information. They found a formula to find the perimeter of a regular polygon, directly from the perimeter of a polygon with half the number of sides.

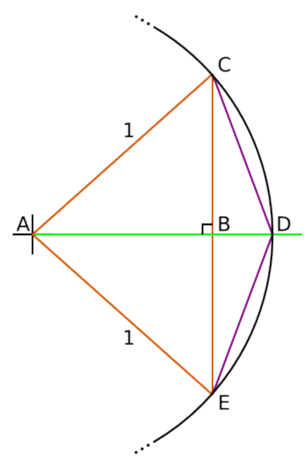

Figure 1: Diagram for the Formula

We shall consider a polygon with n sides, and a perimeter of $2nd$, where each side of the polygon has length $2d$. In our diagram, BC has length d. By the Pythagorean Theorem, $AB = \sqrt{1-d^2}$. Thus, $BD = 1 - \sqrt{1-d^2}$. Then,

$$CD = \sqrt{2 - 2\sqrt{1-d^2}}.$$

Using many laborious calculations, Zu Chongzi was able to calculate the perimeter of a 24,576 sided polygon. Not surprisingly, $24,576 = 6 \cdot 2^{12}$. Using these calculations, Chongzi was able to calculate π to 7 decimal places.

More recently, Isaac Newton used infinite summations and coordinate geometry to generate an approximation of π. He took the equation for a circle with center at $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$. This semicircle has the equation

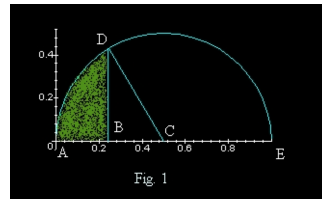

Figure 2: Newton's Diagram

$$y = \sqrt{\frac{1}{4} - (x - \frac{1}{2})^2}.$$

This can be rewritten as

$$y = \sqrt{x} \cdot \sqrt{1 - x}.$$

We can expand $\sqrt{1-x}$. $\sqrt{1-x}$ is equal to $(1-x)^{\frac{1}{2}}$, so it can be expanded using the binomial theorem. A part of the binomial theorem states that $(1+a)^n$ is equal to $1 + na + \binom{n}{2}a^2 + \binom{n}{3}a^3 + \cdots$. Now, we plug $n = \frac{1}{2}$ into the previous expansion, to get that

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \cdots$$

When evaluated, this can be simplified as

$$1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} \cdots$$

Newton then drew a line perpendicular to the x-axis at $(0, \frac{1}{4})$. If we look at the triangle formed by the center of the semicircle, the intersection of the perpendicular and the semicircle, and $(0, \frac{1}{4})$, we can see that it a 30-60-90 triangle. It is a right triangle with a hypotenuse of $\frac{1}{2}$ and one leg with length $\frac{1}{4}$. Newton then examined the area enclosed by the x-axis, the perpendicular, and the circle. This area is $\frac{1}{3}$ of a semicircle minus a triangle with area $\frac{\sqrt{3}}{32}$. Earlier in his mathematical career, Newton showed that the area bounded by a curve with the equation $y = ax^{\frac{m}{n}}$, the x-axis, and some perpendicular to the x-axis x units away from the origin was $\frac{an}{m+n}x^{\frac{m+n}{n}}$. This can be shown by applying the power rule, $\int x^n dx = \frac{x^{n+1}}{n+1}$. By this rule, $\int ax^{\frac{m}{n}} dx = \frac{ax^{\frac{m}{n}+1}}{\frac{m}{n}+1} = \frac{an}{m+n}x^{\frac{m+n}{n}}$. He also had concluded that the area under a curve formed by the summation of many terms was equal to the sum of the area under each of the curves. So the area bounded the curve $y = x^{\frac{1}{2}}\left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} \cdots\right)$, the x-axis, and the line $x = \frac{1}{4}$ can be found by inserting each term of the expansion of $\sqrt{x}\sqrt{1-x}$ into the previous formula. By doing this we get that the area we are try to find is equal to

$$\frac{2}{3}\left(\frac{1}{4}\right)^{\frac{3}{2}} - \frac{1}{2}\left(\frac{2}{5}\right)\left(\frac{1}{4}\right)^{\frac{5}{2}} - \frac{1}{8}\left(\frac{2}{7}\right)\left(\frac{1}{4}\right)^{\frac{7}{2}} - \frac{1}{16}\left(\frac{2}{9}\right)\left(\frac{1}{4}\right)^{\frac{9}{2}} - \ldots$$

When summed, this approaches 0.07677310678. Just summing the first four terms gets the area to be approximately 0.07678. So, we add the area of the 30-60-90 triangle, $\frac{\sqrt{3}}{32}$, to get 0.130899694527. So, the area of the entire circle is 6 times this value, getting .785398167099. We can then divide by the radius, $\frac{1}{2}$, squared in order to get that $\pi \approx 3.14159267$

A common usage of π is to measure angles. This usage of radians comes directly from the definition of π. We define a unit as the angle at which the radius of a circle equals its arc length. If we replicate this angle 2π times, we will get a full circle, as the circumference is 2π times the radius. Another way to find the area of a circle is to observe the limit of the areas of inscribed polygons. For an n-sided polygon inscribed in a circle, we can connect each vertex to the center creating n triangles. Each of these triangles has an area of $\frac{1}{2}\sin\alpha$ where $\alpha = \frac{2\pi}{n}$. So, the area of a circle is equal to the

$$\lim_{x\to\infty} \frac{1}{2} n \sin(\frac{2\pi}{n})$$

Using larger and larger values of n, we can reach increasingly accurate approximations for π.

A fun method to try at home is the Buffon Needle Problem (do not worry, no sharp objects are necessary for the problem). The setup of the problem involves parallel lines set 1 unit apart from each other. We then randomly place objects which I will refer to as needles, hence the name of the problem, resembling line segments with length 1. When we randomize the placement of the line segment, we can randomize the distance between the center of the needle and the nearest line, called d, as well as the acute angle formed by the extension of the needle with the parallel lines, called θ. We are going to count the fraction of needles that cross one of the parallel lines.

In order to cross a line, d has to be less than $\frac{1}{2}sin\theta$. d and θ are completely random selections, meaning any value within the bounds has an equal probability of being chosen. Since d is the distance to the closest line, $d \in [0, \frac{1}{2}]$. θ is a random acute angle, so $\theta \in [0, \frac{\pi}{2}]$. We can then graph $d < \frac{1}{2}\sin\theta$ within these bounds, where θ lies on the horizontal axis. In order to find the percentage of times where the needle crosses a line, we take

$$\frac{\int_0^{\frac{\pi}{2}} \frac{1}{2}\sin\theta d\theta}{\frac{\pi}{4}}$$

Since the derivative of sine is cosine, we can evaluate the integral portion to get $\frac{1}{2}$. So, as you throw needles, the percentage of them that cross lines should approach $\frac{2}{\pi}$. So, $\pi \approx 2$ times the reciprocal of the fraction of needles that cross.

More recently, people have used computers to generate many digits of pi. One simple and intuitive approach is to inscribe a unit circle in a square. This fraction, when multiplied by 4, will yield an approximation for π.

Some people have tried to claim a rational value for π. One famous example happened in Indiana in 1897. A man named Edward Goodwin claimed to prove that the ratio between the circumference and the diameter was exactly 3.2. The issue with his "proof," found under many layers of convoluted text was his assumption of the truth of this diagram. After examination, we can see that not only did Goodwin assume that $\pi = 3.2$, he also assumed that, by the Pythagorean theorem, $5\sqrt{2} = 7$. Despite these inaccuracies, Goodwin's false claims reached as far as state legislature. A bill

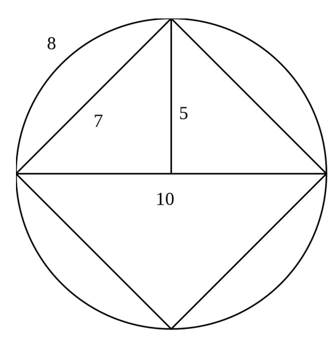

Figure 3: The Assumed Diagram

passed through the Indiana House of Representatives to grant the education system of Indiana free usage of this discovery. Thankfully, a Math Professor from Purdue University also happened to be at the State Legislature's chambers seeking funding so that he could show the error in Goodwin's proof. The Senate has indefinitely postponed this vote.

Citations

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