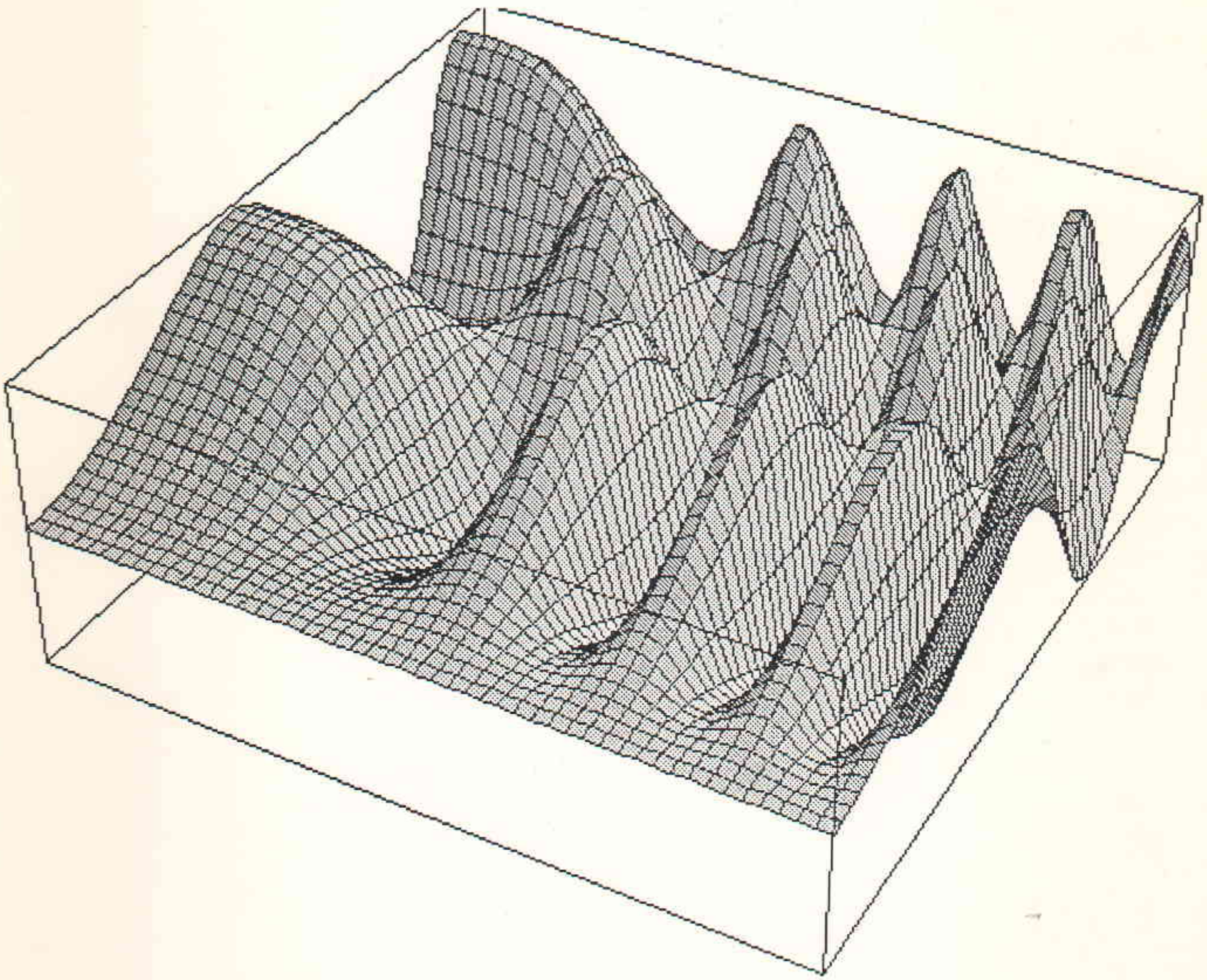
Math Survey

Volume LXXI Spring 1997



Stuyvesant High School
Department of Mathematics
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Dr. Richard Rothenberg

Chairman of the
Stuyvesant High School Math Department

1947-1997

Dr. Rothenberg's passing on May 15 was a shock to the Stuyvesant community.

Students loved Dr. Rothenberg as an unsurpassed teacher. Even those with only passing interest in mathematics quickly appreciated his multivariate calculus and differential equations classes because of his meticulous preparation and devotion. To make his lessons as clear and as interesting as possible, Dr. Rothenberg pored over a collection of textbooks and publications. He always surprised his classes with fresh problems to model, ranging from snowplow speeds in heavy snowfall, to tactics in the Battle of Trafalgar, to vibrations in bridges from marching soldiers, to fickle lovers with sinusoidally varying interests in each other. Furthermore, Dr. Rothenberg always emphasized the proofs and derivations behind the mathematics, trying to impart understanding on a more fundamental level.

Whenever Dr. Rothenberg wasn't teaching, he acted as a mentor to his students. Anyone, from newly arrived freshmen to seniors writing Westinghouse papers, could drop by his office for advice on classes, colleges, or even life in general. For the latter, Dr. Rothenberg offered guidance over what classes to take, and for the former he lent his formidable knowledge of mathematics and his library of mathematics books.

But it was perhaps Dr. Rothenberg's personality that most endeared him to his students and colleagues. To strangers, he sometimes seemed a little gruff, but that only camouflaged his kindness and sense of humor. He joked endlessly with his students and ribbed them, sometimes to gently motivate them, sometimes just for fun. He made sure that no student fell behind the syllabus, securing tutors for them if necessary. His enthusiasm to teach deeply impressed those who observed it, ensuring Dr. Rothenberg a permanent place in the minds of those who knew him.

This issue of *Math Survey* is dedicated to his memory.

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From the Editor-in-Chief

This the second issue of the seventy-first volume of *Math Survey*, Stuyvesant High School's official mathematics publication. This issue also happens to be the last issue of *Math Survey* that I will oversee, and so the laying out of this magazine has been quite special to me. So, in keeping with our long tradition, we present the Spring 1997 *Math Survey* to you, the reader.

As editor-in-chief, I once again thank our faculty advisor, Mr. Allen Clancy, for his guidance, as well as Chairman Dr. Richard Rothenberg for greatly assisting us in times of need. Mr. Richard Geller, Stuyvesant's math team coach, as also been one of our biggest supporters (and not only because of his persuasive sales pitches to his students). *Math Survey* also appreciates the efforts of all the teachers of the Stuyvesant's mathematics department for their comments and ideas.

Within the *Math Survey* staff, Linda Hong has once again been instrumental in raising funds and keeping deadlines for those who were remiss. I would also like to thank Edmund Chou and Tzyy-Ming Yeh for their persistence in pursuing advertisers, and Daniel Stronger, Abe Gurjal, and Michael Shy for proofreading the articles. Despite their vigilance, there may yet be errors, and we take credit for those as well.

Furthermore, *Math Survey* would never have been able to publish this issue without the timely aid of Mr. John Lapolla, Kieran Hervold, Brian Czyzewski, Jisoo Lee, Julianna Herlott and the Student Union government. Last, I would like to thank you the reader, our generous advertisers and sponsors for supporting *Math Survey*.

We, the staff of *Math Survey*, hope that you will enjoy this issue.

Sincerely,

Hyun-Sup Byun

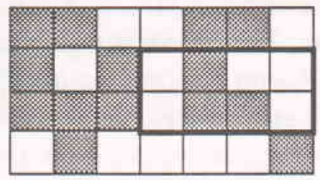
Hyun-Sup Byun
Editor-in-Chief

On a Rectangular Coloring Property of N-dimensional Rectangular Solids

Dan Stronger

In math, it is often considered useful to solve a problem and then extend the solution by finding a pattern which applies the solution to other cases. This is known as generalizing the problem, and often enough, competition problems can be generalized with interesting results.

The first problem on the 1976 USAMO was as follows:



(a) Suppose that each square of a 4×7 chessboard, as shown above, is colored either black or white. Prove that with any such coloring, the board must contain a rectangle (formed by the horizontal and vertical lines of the board such as the one outlined in the figure) whose four distinct unit corner squares are all of the same color.

(b) Exhibit a black-white coloring of a 4×6 board in which the four corner squares of every rectangle, as described above, are not all of the same color.

The problem book, Klamkin[1], gives a solution to this problem. It also provides "General results," which determine which other rectangular chessboards can be colored in this way. A rectangle can be colored in this way if and only if it is either 2 by k for any k, 3 by k for $k < 7$, 4 by k for $k < 7$, or 5 by k for $k < 5$. This result can be further generalized to n-dimensional chessboards. That is, determining which n-dimensional rectangular solids can be colored such that there does not exist a rectangle within it all four of whose corners are the same color.

For the following, the dimension n is greater than 2. Also, all of the a_i are greater than 2. Let $S(a_1, a_2, a_3, \ldots, a_n)$ be the set $\{1, 2, \ldots, a_1\} \times \{1, 2, \ldots, a_2\} \times \ldots \times \{1, 2, \ldots, a_n\}$.

We define a coloring C as follows: C is a function from $S(a_1, a_2, a_3, \ldots, a_n)$ to $\{0,1\}$, where a coloring is bad if there exist i and j, $i < j$, such that there exist $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$, and a, b, c, d (where $a \neq c$ and $b \neq d$) such that

$$C(x_1, x_2, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_{j-1}, b, x_{j+1}, \ldots, x_n),$$
$$C(x_1, x_2, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_{j-1}, d, x_{j+1}, \ldots, x_n),$$
$$C(x_1, x_2, \ldots, x_{i-1}, c, x_{i+1}, \ldots, x_{j-1}, d, x_{j+1}, \ldots, x_n), \text{ and}$$
$$C(x_1, x_2, \ldots, x_{i-1}, c, x_{i+1}, \ldots, x_{j-1}, d, x_{j+1}, \ldots, x_n)$$

are all the same, where it is assumed that $x_k \in S(a_k)$ for $k \neq i$ or j, a and c are elements of $S(a_i)$, and b and d are elements of $S(a_j)$. That is, the coloring is bad if there exists a rectangle all four of whose corners are the same color. We will call such a rectangle monochromatic. Any coloring which is not bad is good.

We define a boolean function $F(a_1, a_2, \ldots, a_n)$ as follows: $F(a_1, a_2, \ldots, a_n)$ is true if and only if there exists a good coloring on $S(a_1, a_2, \ldots, a_n)$.

Inclusion Lemma. *If $F(a_1, a_2, \ldots, a_n)$ is true then $F(b_1, b_2, \ldots, b_n)$, with $b_i \leq a_i$ for all i, $1 \leq i \leq n$, is also true.*

Proof: Assume $F(a_1, a_2, \ldots, a_n)$ is true. Thus there is a good coloring C_1 on $S(a_1, a_2, \ldots, a_n)$. We can easily construct a good coloring C_2 from C_1 as follows: simply let C_2 of an element of $S(b_1, b_2, \ldots, b_n)$ be the same as C_1 of the corresponding element of $S(a_1, a_2, \ldots, a_n)$. We can see that C_2 is a good coloring because if there were a rectangle all of whose corners were the same color under C_2, there would have been the corresponding rectangle under C_1, which we know is not the case because C_1 is good.

Note: The contrapositive of this lemma is also important. That is, if $F(b_1, b_2, \ldots, b_n)$ is false, and $b_i \leq a_i$ for all i, $1 \leq i \leq n$, then $F(a_1, a_2, \ldots, a_n)$ is also false.

"Two" Lemma. $F(a_1, a_2, \ldots, a_n)$ is the same as $F(a_1, a_2, \ldots, a_n, 2)$.

Proof: Assume $F(a_1, a_2, \ldots, a_n)$ is true. Then there is a good coloring C_1 on $S(a_1, a_2, \ldots, a_n)$. We can construct a good coloring C_2 on $S(a_1, a_2, \ldots, a_n, 2)$ from C_1 as follows: Let $C_2(x_1, x_2, \ldots, x_n, 1) = C_1(x_1, x_2, \ldots, x_n)$. Let $C_2(x_1, x_2, \ldots, x_n, 2) = 1\text{-}C_1(x_1, x_2, \ldots, x_n)$. We will now show that this coloring is good.

Assume there is a rectangle all four of whose corners are the same color under C_2. That is, there exist i and j, $i < j$, such that there exist $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}$, and a, b, c, d, where $a \neq c$ and $b \neq d$, such that

$$C_2(x_1, x_2, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_{j-1}, b, x_{j+1}, \ldots, x_{n+1}),$$
$$C_2(x_1, x_2, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_{j-1}, d, x_{j+1}, \ldots, x_{n+1}),$$
$$C_2(x_1, x_2, \ldots, x_{i-1}, c, x_{i+1}, \ldots, x_{j-1}, b, x_{j+1}, \ldots, x_{n+1}), \text{ and}$$
$$C_2(x_1, x_2, \ldots, x_{i-1}, c, x_{i+1}, \ldots, x_{j-1}, d, x_{j+1}, \ldots, x_{n+1}) \text{ are all the same.}$$

Case 1: Neither i nor j is $n+1$.

Case 1a: $x_{n+1} = 1$.

Since $C_2(x_1, x_2, \ldots, x_n, 1) = C_1(x_1, x_2, \ldots, x_n)$, this implies that C_1 is bad, which is a contradiction.

Case 1b: $x_{n+1} = 2$.

Since $C_2(x_1, x_2, \ldots, x_n, 2) = 1\text{-}C_2(x_1, x_2, \ldots, x_n, 1)$, if four points with $x_{n+1} = 2$ have the same color, then the corresponding four points with $x_{n+1} = 1$ also all have the same color. Thus, by case 1a, we have a contradiction.

Case 2: $j = n+1$.

This implies that $d = 2$ and $b = 1$. Thus $C_2(x_1, x_2, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_{j-1}, b, x_{j+1}, \ldots, x_n, 1)$ and $C_2(x_1, x_2, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_{j-1}, b, x_{j+1}, \ldots, x_n, 2)$ are the same. However, we know from the way they were defined that they are different.

Assume that $F(a_1, a_2, \ldots, a_n, 2)$ is true. If there is a good coloring on $S(a_1, a_2, \ldots, a_n, 2)$, then it must contain a good coloring on the subset of $S(a_1, a_2, \ldots, a_n, 2)$ for which $x_{n+1} = 1$. Thus there is a good coloring on $S(a_1, a_2, \ldots, a_n)$ and so $F(a_1, a_2, \ldots, a_n)$ is true. This completes the proof.

Orientation Lemma. *Rearranging the order of the parameters of $F(a_1, a_2, \ldots, a_n, 2)$ does not change the value of F.*

This is evident from the fact that whether or not a coloring is good is independent of the order of the a_i.

Rearrangement Lemma. *Let C, on $S(a_1, a_2, ..., a_n)$, be a good coloring. Let P be a permutation of $\{1, 2, ..., a_1\}$. Define C' as follows: $C'(P(x_1), x_2, ..., x_n) = C(x_1, x_2, ..., x_n)$. Then C' is also good.*

Proof: Assume C' is bad.

Case 1: There exist i and j, $1 < i < j$, such that there exist $x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_{j-1}, x_{j+1}, ..., x_{n+1}$, and a, b, c, d, $a \neq c$ and $b \neq d$ such that

$$C'(x_1, x_2, ..., x_{i-1}, a, x_{i+1}, ..., x_{j-1}, b, x_{j+1}, ..., x_{n+1}),$$
$$C'(x_1, x_2, ..., x_{i-1}, a, x_{i+1}, ..., x_{j-1}, d, x_{j+1}, ..., x_{n+1}),$$
$$C'(x_1, x_2, ..., x_{i-1}, c, x_{i+1}, ..., x_{j-1}, b, x_{j+1}, ..., x_{n+1}), \text{ and}$$
$$C'(x_1, x_2, ..., x_{i-1}, c, x_{i+1}, ..., x_{j-1}, d, x_{j+1}, ..., x_{n+1}) \text{ are all the same.}$$

Since permutations have inverses, we know that

$$C(P^{-1}(x_1), x_2, ..., x_{i-1}, a, x_{i+1}, ..., x_{j-1}, b, x_{j+1}, ..., x_{n+1}),$$
$$C(P^{-1}(x_1), x_2, ..., x_{i-1}, a, x_{i+1}, ..., x_{j-1}, d, x_{j+1}, ..., x_{n+1}),$$
$$C(P^{-1}(x_1), x_2, ..., x_{i-1}, c, x_{i+1}, ..., x_{j-1}, b, x_{j+1}, ..., x_{n+1}), \text{ and}$$
$$C(P^{-1}(x_1), x_2, ..., x_{i-1}, c, x_{i+1}, ..., x_{j-1}, d, x_{j+1}, ..., x_{n+1}) \text{ are all the same.}$$

This implies that C is bad, which is a contradiction.

Case 2: The only other possibility is that $i = 1$. In this case, we consider $P^{-1}(a)$ and $P^{-1}(c)$ and construct a rectangle all of whose corners are the same color under c, as we did in case 1. This completes the proof.

Exclusion Lemma. *If there are i and j, $1 \leq i \leq n$ and $1 \leq j \leq n$, such that $F(a_i, a_j)$ is false, then $F(a_1, a_2, ..., a_n)$ is also false.*

Proof: By the Orientation Lemma, we can assume that $F(a_i, a_j)$ is false. For any coloring on $S(a_1, a_2, ..., a_n)$, consider the rectangle given by the set of points $(x_1, x_2, 1, 1, ..., 1)$, $1 \leq x_1 \leq a_1$, $1 \leq x_2 \leq a_2$. This rectangle must contain a monochromatic rectangle. Thus, the entire $S(a_1, a_2, ..., a_n)$ contains a monochromatic rectangle and thus $F(a_1, a_2, ..., a_n)$ is false, as desired.

Theorem. *For all n, $F(a_1, a_2, ..., a_n)$ is true if and only if either:*

(1) *All of the a_i are equal to 2 except (at most) one, or*

(2) *Of the a_i, none is greater than 6 and no two are greater than 4.*

Proof. We will prove this by induction on n.

Base case: $n = 2$.

First we will show that if a_1 and a_2 satisfy one of the above conditions, $F(a_1, a_2)$ is true.

(1) Let $a_1 = 2$. We can exhibit the following good coloring on $S(2, a_2)$: Let $C(1, x) = 0$ and let $C(2, x) = 1$ for all x, $1 \leq x \leq a_2$. This is demonstrated by Fig. 1. It is evident that there are no monochromatic rectangles within this coloring.

Figure 1

(2) Neither of the a_i can are greater than 6 and at least one is not greater than 4. This is equivalent to the situation where one of the a_i is ≤ 6 and the other is ≤ 4. By the Inclusion and Orientation Lemmas, we have only to show that there exists a coloring on a 4 by 6 rectangle with no

monochromatic rectangles within it. Such a coloring is demonstrated by Fig. 2.

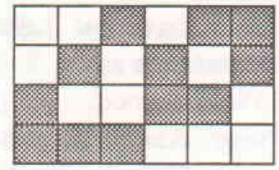


Figure 2

Note that every column of this rectangle consists of two squares of each color and that no two of the columns are the same. Thus, by the Rearrangement Lemma, any such 4 by 6 rectangle coloring is good. Later it will be useful to know that a coloring on a 4 by 6 rectangle is good only if each column has two squares of each color, so we will prove it now.

Assume there is a good coloring on a 4 by 6 rectangle which does not satisfy this condition. Without loss of generality, we can assume that the bottom three squares of the leftmost column are white. In each of the other columns, if two of the bottom three squares are white, a monochromatic white rectangle is formed. Thus at least two squares of the bottom three of each of those columns are black. Since there are only three ways to choose two of those three squares to be black and five remaining columns, there must be some two of those columns which have the same two of the bottom three squares black. This causes a monochromatic black rectangle. This completes the contradiction.

Now we have to show that if the above conditions are not met, $F(a_1, a_2)$ is false. First we will show that $F(3,7)$ is false. We will do a proof by contradiction: Assume there is a good coloring on $S(3,7)$. Consider a rectangle with 3 rows and 7 columns. There is either a column all of whose squares are the same color or each column has one square of one color and two of the other.

Case 1: All of the squares in one of the columns are the same color.
Without loss of generality, they are white. If any of the other columns have two white squares, a monochromatic rectangle is created. Thus, each column must have at least two black squares. However, since there are six remaining columns and only three ways to choose which two of the three squares in any column should be black, two of the columns must have two black squares in the same two rows. Thus there is a monochromatic rectangle.

Case 2: Each column has two squares of one color and one square of the other. Since there are two colors and seven columns, by the pigeonhole principle, there must be four columns whose predominant colors are all the same. Without loss of generality, we can say that it is black. Thus there are 4 columns, each of which has two black squares. Thus, by the same reasoning as before, there must be a monochromatic rectangle.

Now we will show that $F(5,5)$ is false. We will prove this by contradiction.
Assume there is a good coloring on $S(5,5)$. Consider the leftmost column. It must have either three white squares or three black squares. Without loss of generality (by the Rearrangement Lemma) we can assume that the three bottom squares of this row are

white. Consider the 3 by 4 rectangle in the bottom right corner of our 5 by 5 rectangle. Each of its columns cannot have two white squares. Thus each must have at least two black squares and as before, we have a contradiction.

We wish to show that if neither of the above conditions is satisfied, $F(a_1, a_2)$ is false. With $n = 2$, that neither of the above conditions is true is equivalent to that neither of the a_i is 2 and of the a_i, either both are > 4 or one is > 6. This is equivalent to when a_1 and a_2 are ≥ 3 and either both are ≥ 5 or one is ≥ 7. Assume both are ≥ 5. Then there is an $S(5,5)$ included in it. Since S(5,5) is false, $S(a_1, a_2)$ is false. Otherwise one is ≥ 3 and the other is ≥ 7. Then there is an $S(3,7)$ included in it. Similarly in this case, $S(a_1, a_2)$ is false.

This completes the base case.

Induction Step: Assume $F(a_1, a_2, ..., a_k)$ is true if and only if either

(1) All of the a_i are 2 except (at most) one, or

(2) Of the a_i, none is greater than 6 and no two are greater than 4.

We wish to show that the same is true when $n = k+1$. First we will show that if either of the above criteria are met, $F(a_1, a_2, ..., a_{k+1})$ is true.

(1) Assume that all of $a_2, a_3, ..., a_{k+1}$ are 2. Consider $F(a_1, a_2, ..., a_k)$. By our inductive hypothesis, it is true. Thus, by the Two Lemma, since $a_{k+1} = 2$, $F(a_1, a_2, ..., a_k, a_{k+1})$ is also true.

(2) Consider a good coloring on $S(6, 4)$, C. We wish to demonstrate a good coloring C_{k+1} on $S(a_1, a_2, ..., a_{k+1})$, where a_1 is 6 and all the other a_i are 4. We will use the notation p (mod q) to represent the remainder when p is divided by q. We define $C_{k+1}(x_1, x_2, ..., x_{k+1})$ as $C(x_1, ((x_2-1)+ (x_3-1) + ... + (x_{k+1}-1)) \pmod 4 +1)$. We wish to show that this coloring is good. We will prove this by contradiction. Assume that there is monochromatic rectangle on C_{k+1}. It extends in two dimensions, i and j, $i < j$. Say (x_i,x_j) for the four corners of our monochromatic rectangle are (a,b), (a,d), (c,b), and (c,d). Let $K(p, q) = C_{k+1}(x_1, x_2, ..., x_{i-1}, p, x_{i+1}, ..., x_{j-1}, q, x_{j+1}, ..., x_{k+1})$.

Case 1: Assume $i \neq 1$. Consider the colors of the four points with $x_i = a$ in the "plane" of the monochromatic rectangle. Since these can be seen as the colors of a "column" of a good coloring on a 4 by 6 rectangle, they must be two 0's and two 1's. Similarly, those with $x_i = c$ must also be two 0's and two 1's. However, since $K(c, q) = K(a+(c-a), q) = K((a+(c-a)-1) \pmod 4 +1, q) = K(a, (q+(c-a)-1) \pmod 4 +1)$, and since $c-a$ can't be divisible by 4, the two points whose colors are 1 with x_i = a cannot correspond with those whose colors are 1 with $x_i = c$. Thus $K(a, b)$, K(a, d), K(c, b), and K(c, d) cannot all be the same. This is a contradiction.

Case 2: Assume $i = 1$. Let $S = (x_2-1)+(x_3-1)+ ... +(x_{j-1}-1)+(x_{j+1}-1)+ ... +(x_{k+1}-1)$. Then $K(p, q) = C(p, (q+S) \pmod 4 +1)$. Note that $P(q) = (q+S) \pmod 4 +1$ is a permutation of {1, 2, 3, 4}. Thus, since C is a good coloring on $S(6, 4)$, $K(p, q) = C(p, P(q))$ is also a good coloring on S(6, 4) by the Rearrangement and Orientation Lemmas.

This completes the proof that (2) implies that $F(a_1, a_2, \ldots, a_{k+1})$ is true. Now we will show that if neither of the two criteria is met, $F(a_1, a_2, \ldots, a_{k+1})$ is false. Assume neither of the criteria is met. This is equivalent to that there are at least two $a_i \geq 3$ and there is some $a_i \geq 7$ or there are two ≥ 5. If two are ≥ 5, then because $F(5, 5)$ is false, $F(a_1, a_2, \ldots, a_{k+1})$ is also false by the Exclusion Lemma. If one is ≥ 7 and there is another which is ≥ 3, then $F(a_1, a_2, \ldots, a_{k+1})$ is false by the Exclusion Lemma because $F(3, 7)$ is false. This completes the induction step.

By Mathematical Induction, our theorem is true for all $n \geq 2$. Q.E.D.

Generalizing this problem turned out to be very interesting. Often, it can be very instructive to generalize a solution. In this way, one can see how one solution, which appears to be general, can actually be a very specific case of a much more general idea at the same time.

REFERENCES

[1] M. S. Klamkin, *USA Mathematical Olympiads 1972-1986*, The Mathematical Association of America, Washington DC, 1988.

Let's Make A Deal!

Johnny Chen

Dr. Kolmogorov is discussing a proposition with his student, Monte.

Dr. K:	Suppose, Monte, that I have three doors, only one of which has a prize beneath it –
Monte:	So this is a game?
Dr. K:	Yes, I suppose it is, but more importantly, it illustrates –
Monte:	I like playing games.
Dr. K:	Um, I'm sure you do ... but back to the problem. Suppose the doors are labeled 1, 2, and 3. Choose one.
Monte:	Okay. One.
Dr. K:	No, choose one of the doors.
Monte:	Okay ... one.
Dr: K:	Now, suppose that I tell you that door three does not have the prize behind it; would you like to choose another door?
Monte:	Okay. Door three.
Dr: K:	But I've told you that door three does not have the prize.
Monte:	Okay. Door two.
Dr. K:	That was a good decision to change doors ... can you tell me why?
Monte:	There is no reason. Isn't there a 1/3 chance that door one is the right one?
Dr. K:	Yes, that's right.
Monte:	And there's a 1/3 chance that door two is right also, right? So actually, you've been lying to me, Dr. K; I'm leaving.
Dr. K:	No, come back, come back. Actually, there is a 2/3 chance that door two is the right one given –
Monte:	Wait, you're telling me that if I had chosen two in the first place I'd have a 2/3 chance of winning?
Dr. K:	That is precisely what I am not saying.
Monte:	Watcha talkin' about, Dr. K? If I had chosen two in the first place, and you say there's a 2/3 chance of winning, then –
Dr. K:	But Monte, what was different after you chose door one?
Monte:	Well, you gave me extra information – hey! Are you telling me that it's different 'cause there's more to consider?
Dr. K:	Yes. There's a condition now. Before, you thought any of the doors could be the right one. Now, you know that one of two is the right one. Let's call this ... the Monte Hall problem, after you; this is a good demonstration of conditional probability.
Monte:	I still don't get it.
Dr. K:	I'm sure you don't. Let me try to explain it to you. You have a 1/3 chance of choosing the right door initially, and a 2/3 chance of choosing the wrong door. I tell you one that is not right, so if you do not switch you still have a 1/3 chance. If you do switch, you have a –
Monte:	Two-thirds chance!

Dr. K: Yes, that's pretty good! I still see your skepticism. Let's say I have a million boxes.

Monte: Gee whiz, Dr. K, that's a lot of boxes.

Dr. K: Uh-huh. Let's say one has a prize –

Monte: Box thirty.

Dr. K: Okay. I tell you that boxes one to sixteen, eighteen to twenty-nine, and thirty-one to one million do not have a prize.

Monte: I wanna switch.

Dr. K: Precisely.

Monte: But that's psychology, not mathematics. I choose to switch for no real mathematical reason. Where's the rigor?

Dr: K: Good question. Of course there is rigor. This is mathematics, not physics!

(Dr. K and Monte share a hearty chuckle)

Conditional probability is very understandable with a little bit of thought and the right notation. The key here is that we want the probability of an event, call it A, given another event, call it B.

Monte: In other words, $P(A \mid B)$?

Dr. K: Yes, that's right! So we would like a way of expressing $P(A \mid B)$ in terms of A and B.

Monte: Well, I guess it would be a probability, with the denominator $P(B)$, because B is given, and we want the number of cases that satisfy our condition.

Dr. K: Very good! So the numerator would be –

Monte: $P(A \cup B)$, the probability of both A and B occurring. And that's $P(A) \cdot P(B)$!

Dr. K: No, no, no! What if A and B are events like, say, drawing cards without replacement?

Monte: Okay ... so the probability of both events occurring is $P(A) \cdot P(B)$ when A and B aren't related.

Dr. K: In probability, we call A and B independent.

Monte: What if they're dependent?

Dr. K: Well, you'll have to think about that. It generally depends on the situation you're considering. It's different when you're dealing with doors to when you have dice and cover one side up, for example.

Four Saints in Three Acts

Hyun-Sup Byun

Once at a summer math program, my counselor bemoaned that the pigeonhole principle was "often underrated because it seems so obvious." However, the beauty of the principle is usually not obvious at all, and it is always very impressive. The pigeonhole principle (or Dirichlet's Box Principle) states:

If $kn+1$ objects (pigeons) are placed in n boxes (pigeonholes),
then at least one box will have $k+1$ objects.

In simple terms, if I have six pigeons and stuff them into five pigeonholes in my desk, I might decide to cram them all into one hole, or spread them around as much as possible. Either way, at least two pigeons will always share one crowded pigeonhole. That is, if the APSCA doesn't stop me first.

In mathematics, the trick often lies in properly setting up the problem – identifying the pigeonholes and the pigeons. However, unless it is directly suggested by the inquisitor, the unassuming principle often escapes notice. Even if one is forewarned that the principle is the weapon of choice, the problem usually requires some subtlety and cleverness. Moreover, there is often no simple alternative method, so you're stuck with the pigeonhole principle.

The problem is compounded by the pigeonhole principle's limited universality: it can apply to a maddening array of geometry, number theory, and combinatorics problems (to name a few areas), but only to certain problems with specific arrangements. I'll try to give a few classic, proof-type examples of each of these.

Geometry is a fairly popular subject for pigeonhole problems. The problem statement very frequently asks for a given number of points to be dropped somewhere on a given geometrical figure. Thus, the key is to divide up a figure into the proper figurative pigeonholes and to attempt to place the points. Try this exercise:

Problem 1. *Show that for five points inside a square of side length* 1, *the distance between at least one pair of these points is less than or equal to* $\sqrt{2}/2$.

Easy, you say. The corners count as four points, and the center of the square is the fifth. If we move the center point in any particular direction, it would decrease the distance between it and a corner point. This the difficult to state rigorously; "pushing the center point" isn't going to cut it on the USAMO. The rigorous proof would divide the square into four congruent squares. The four inner squares are the pigeonholes; the five points the pigeons. From the principle, at least two points will land in the same internal square, and the greatest distance between them is $\sqrt{2}/2$. This might seem to be a trivial application of the problem, but now try the following:

Problem 2. *Show that for five points inside an equilateral triangle of side length* 1, *the distance between at least one pair of these points is less than or equal to* ½.

Here is where the setup is critical. Divide the equilateral triangle into four congruent equilateral triangles (see Figure 1). Thus, at least two points will land in the same inner triangle. The greatest possible distance between them is at different corners (draw circular arcs from the corners to confirm this), which is ½. The pigeonhole principle partly explains why the problem requires five points and not four.

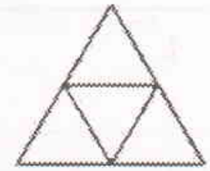


Figure 1

Problem 3. *Show that for seven points inside a circle of radius* 1, *the distance between at least one pair of these points is less than or equal to* 1.
This problem is not significantly different from the previous two; try it!

Number theory is also a prime target for pigeonhole attacks, even though it is quite different from geometry. Often enough, the pigeonholes in number theory are provided by modular arithmetic. In a given modulus m, there are only m values (mod m), thereby presenting a quick way to map integers into a set of boxes. Should two integers fall into the same value of a modulus, they are congruent, which is a very useful tool.

Problem 4. *Show that any set of n integers contains a nonempty subset of integers whose sum is divisible by n.*
Call the n integers a_1, a_2, ..., a_n. Then consider the corresponding set of partial sums (e.g., a_1, a_1+a_2, $a_1+a_2+a_3$, ..., $a_1+a_2+...+a_n$). If any of these sums are congruent to zero in mod n, we are done, but suppose that no single sum satisfies that. Then there are n partial sums, but only n-1 different values in mod n (since we excluded 0 mod n). Therefore, for some r and s ($r < s$), $a_1+a_2+...+a_r \equiv a_1+a_2+...+a_s$ mod n. So, $a_{s-r+1}+ a_{s-r+2}+...+ a_s \equiv 0$ mod n, so n divides the sum of that set.

Problem 5. *Show that for prime p and any integer a, there exist integers x and y such that $x^2+y^2 \equiv a \pmod p$.*
This is a bit more difficult. Rearrange the equation into $x^2 = a-y^2$ (mod p). Recall that since $x^2 \equiv (-x)^2$ (mod p), there are only $(p+1)/2$ different values mod p (strictly speaking, quadratic residues in mod p) for x^2. Accordingly, there are only $(p+1)/2$ different values for y^2, and thus only $(p+1)/2$ different values for $a-y^2$. In the worst case, there would be at most $(p+1)/2 + (p+1)/2 = p+1$ values. Since there are only p values in mod p, this implies that there exist x and y such that x^2 and $a-y^2$ are congruent mod p.

Problem 6. *Show that for any five lattice points, the midpoint of at least one pair of them is also a lattice point.* (Note: A lattice point, for practical purposes, is a point with integer coordinates). The setup here is tricky. Consider the polarity (odd and even) of the coordinates. There are four possible combinations: (odd, odd), (odd, even), (even, even), and (even, odd). By the pigeonhole principle, at least two of these fall into the same polarity combination. Thus, the sum of their coordinates will be (even, even), so the midpoint has integer coordinates and is thus a lattice point.

Combinatorics problems are also favorite candidates for pigeonhole problems. Generally, these are existence proofs that involve some sort of counting. Since it would be impractical or impossible to count all possible cases, the pigeonhole principle can whittle down the possibilities and facilitate solving the problem. Try the following exercises:

Problem 7. *In a room with n people, people are shaking hands. No pair of people has shaken hands more than once and no one has shaken his or her own hand. Prove that there exist at least two people who have shaken the same number of hands.*

A person may have shaken anywhere from 0 to n-1 hands. However, if someone shook n-1 hands (i.e., shook hands with everybody else), then nobody could have shaken 0 hands, and vice versa. This means that either 0 or n-1 is impossible, leaving only n-1 values. Since there are n people, at least two people have shaken the same number of hands.

Problems with real numbers are often some of the most challenging, since there usually isn't a clear-cut way of dividing up the real number line. The set $\Re$ is infinite, so the pigeonhole principle helps reduce the problem to a simpler form.

Problem 8. *Show that for a real number r, kr is less than a distance of* 1/100 *away from an integer for some integer k, where* $1 \leq k \leq 99$.

The solution involves manipulating the real number line. Consider the greatest integer function, which takes the fractional part of the argument (it can be loosely described in $\Re$ as "mod 1"). So, merely represent the range of the function as a circle of circumference 1. Divide the circle into 100 congruent arcs, such that two such arcs straddle the point on the circle that represents a fractional part of zero. If a value of kr should fall in these two arcs, then they would be less than 1/100 units from an integer. Assume that they cannot, leaving them the other 98 arcs. However, there are 99 values, and by the pigeonhole principle, two values, k_1r and k_2r, will fall in the same arc and who fractional parts thus differ by less than 1/100. Assume without loss of generality that $k_1 > k_2$ and $1 \leq (k_1\text{-}k_2) \leq 99$. It follows that $k_1r\text{-}k_2r = (k_1\text{-}k_2)r$ is less than 1/100 from an integer.

Problem 9. *Show that a set of any seven real numbers includes two reals x and y such that*

$$\frac{x-y}{1+xy} \leq \frac{1}{\sqrt{3}}.$$

This problem is actually a favorite old USAMO problem, and noticing small details about it can be helpful. The expression on the right and the $1/\sqrt{3}$ on the left suggest trigonometry; in particular, the tangent subtraction formula $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \tan B}$, and $\tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}}$.

Recall that the tangent function ranges over the entirety of $\Re$, so it is possible to substitute $x = \tan A$ and $y = \tan B$, for some $A,B \in \Re$. Furthermore, the tangent function has a period of π, so it is possible to cut the real number line into segments of length π and deal with only one such segment (loosely speaking, consider $\Re$ in "mod π"), thereby mapping $\Re$ into a much more useful (and bounded) set.

Then consider cutting up the line segment into six equal segments of length $\pi/6$. This provides the pigeonholes we seek. Since there are seven values for x and y, there are seven corresponding values for A and B. Therefore, by the pigeonhole principle, at least two values for A and B whose distance from each other is d where $0 \le d \le \pi/6$. Therefore, $\tan(A-B) \le \tan\frac{\pi}{6}$, and thus $\frac{\tan A - \tan B}{1+\tan A \tan B} \le \frac{1}{\sqrt{3}}$. Substitution yields $\frac{x-y}{1+xy} \le \frac{1}{\sqrt{3}}$.

These problems are most attractive because it combines the features of the geometrical approach with number-theoretical ideas, and because the claim is very non-obvious. Problem 9 offers a fairly straight geometrical representation, while Problem 10 requires much more work. These two both illustrate a common feature of pigeonhole proofs: once the setup is complete, it is quite easy to apply the principle and wrap up the proof. But the first step is always the hardest.

Finally, one might wonder what does the title of this article refer to? To quote an old number theory text of mine (Shanks), "Apply the Dirichlet Box Principle to Gertrude Stein's surrealist opera, *Four Saints in Three Acts*, and draw a valid inference."

I would like to thank Yirong Shen (Austin, TX) and Dr. Rothenberg for providing these problems to solve.

What's in a Shadow?

Soojin Yim

The shadow that an object casts when held directly under the sun is, in the world of mathematics, called an orthogonal projection. This is a type of geometric transformation, and its simplicity may lead one to believe that it is useless, or limited in its usefulness to such questions as: What are the possible shapes of the shadow cast by a random triangle? Can the triangle always be positioned so that the shadow is an equilateral triangle?

If this is the initial belief, though, the reader will soon discover otherwise. An orthogonal projection can transform a seemingly difficult geometrical problem into a simpler form that is easier to grasp, and after transforming back again, the solution to the easier problem provides a solution to the original problem. This "transform-solve-invert" procedure is a fundamental tool in mathematics, and through it, orthogonal projections unravel the challenges of numerous geometric figures.

Given a horizontal plane π' and a figure F above π' lying in a plane π not perpendicular to π', if a perpendicular is dropped to π' from each point P in F, the set of the feet of such perpendiculars, P', forming in the figure F', is the orthogonal projection of F in π'. (Figure 1)

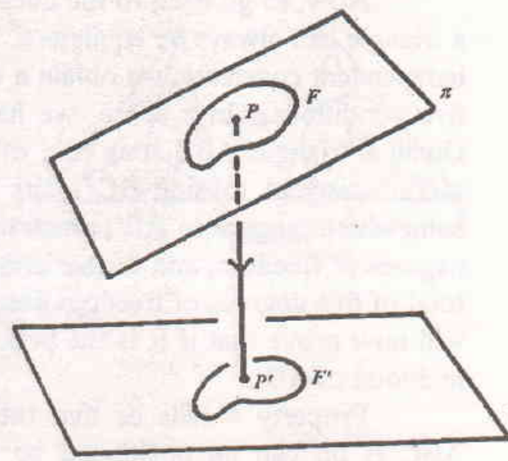


Figure 1

Some basic properties of orthogonal properties are:

1) If l and m are parallel straight lines in plane π, then their projections l' and m' are parallel straight lines in π'.

2) The ratio of the lengths of two line segments on the same line, or on parallel lines, is equal to the ratio of the lengths of their projections.

3) The ratio of the areas of two regions in π is equal to the ratio of the areas of their projections.

4) If F is an ellipse in π, then its projection F' is an ellipse in π'. Moreover, given any ellipse, it is always possible to position the plane so that its projection is a circle.

Property 1 follows from the fact that l' is just the intersection of π' with the plane perpendicular to π' and containing l. Thus l' and m' are the intersection with π' of a pair of parallel planes. (Figure 2)

Property 2 is an exercise with similar right triangles.

For Property 3, note that if the two given regions in π are thin rectangles with their long sides perpendicular to the line of intersection of π and π', then Property 3 follows quickly from Property 2. The general case follows by approximating the given regions with unions of such thin rectangles.

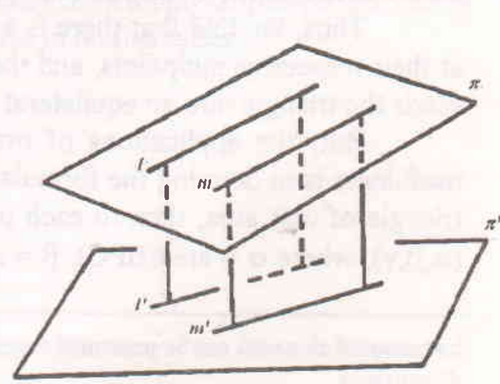


Figure 2

Property 4 can be demonstrated with a plane that cuts a right circular cylinder. If the cylinder has a radius b, any such plane will cut the cylinder in an ellipse with the semiminor axis b. Varying the inclination of the cutting plane allows for an ellipse with any semimajor axis larger than b. The orthogonal projection of the resulting ellipse is the circular base of the cylinder. (Figure 3) That ellipses always project to ellipses is equivalent to the fact that any cross-section of a cylinder with elliptical base, by a plane not parallel to the generators, is an ellipse.[1]

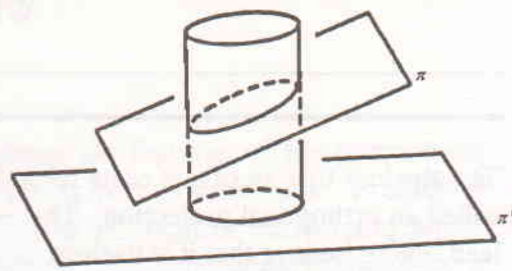


Figure 3

Now, to go back to the question previously raised – whether the orthogonal projection of a triangle can always be equilateral. Observing that the equation of a conic section contains five independent constants, we obtain a rough "rule of thumb" that a conic is uniquely determined by five conditions. In a sense, we have five degrees of freedom in constructing conic sections. Given a triangle ABC, this rule of thumb leads us to expect that there exists a unique conic section tangent to side BC at its midpoint P, tangent to side CA at midpoint Q, and also somewhere tangent to AB (constraining the conic to be tangent to the three sides uses up three degrees of freedom, and further constraining it to pass through P and Q uses up two more, for a total of five degrees of freedom used). The conic E satisfying these conditions is an ellipse.[2] We will now prove that if R is the point of tangency of E with side AB, then R must in fact be the midpoint of AB.

Property 4 tells us that the plane that triangle ABC is on can be positioned so that the orthogonal projection of the ellipse E is a circle E'. (Figure 4) Since tangencies and midpoints are preserved under orthogonal projection, the projection of ABC is a triangle A'B'C' circumscribed about the circle E', with side B'C' tangent at its midpoint P', C'A' tangent at its midpoint Q', and A'B' tangent at R'. Since the two tangent segments drawn to a circle from an external point are of the same length, we have A'Q' = A'R', B'R' = B'P', and C'P' = C'Q'. But B'P' = C'P' and C'Q' = A'Q', so the above equations imply that R' is the midpoint of side A'B' and that A'B'C' is equilateral.

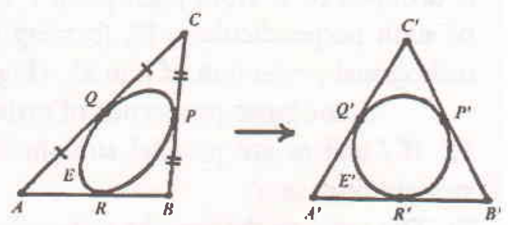


Figure 4

Thus, we find that there is a unique ellipse inscribed in a given triangle touching the sides at their respective midpoints, and that an orthogonal projection sending this ellipse to a circle also sends the triangle into an equilateral triangle. And our initial query is answered.

But, the applications of orthogonal projections don't end there. One example of its usefulness is in deriving the formula for the area of an ellipse inscribed in a triangle. If ABC is a triangle of unit area, then to each point P inside ABC, we can associate barycentric coordinates (α,β,γ), where α = area(BPC), β = area(CPA), and γ = area(APB).[3] Moreover, if P is interior to

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[1] A proof of this will not be presented here, but one can be constructed using analytic geometry in three dimensions.

[2] This will be stated without proof.

[3] Point P is uniquely determined by its barycentric coordinates. Since triangle with the same base and equal area have the same height, the locus of points having one coordinate constant is a straight line parallel to the side of the

the midpoint triangle of ABC, then there exists a unique ellipse centered at P and inscribed in ABC.[4] It is reasonable to expect that the area of this ellipse can be expressed in terms of the barycentric coordinates of its center P, and as a matter of fact, the area of the ellipse E inscribed in a triangle ABC of unit area, when E is centered at a point P with barycentric coordinates (α,β,γ), is given by:

$$\text{area}(E) = \pi\sqrt{(1-2\alpha)(1-2\beta)(1-2\gamma)}$$

The proof of this formula leans heavily on orthogonal projection.

Assume P is a point interior to the midpoint triangle of ABC, with barycentric coordinates (α,β,γ). A suitable orthogonal projection will send the ellipse E centered at P inscribed ABC to a circle E' of radius r and center P' inscribed in A'B'C'. Property 3 implies that:

$$\text{area}(B'P'C') : \text{area}(C'P'A') : \text{area}(A'P'3') = \alpha : \beta : \gamma.$$

But area(B'P'C') = $(r/2)$(B'C'), area(C'P'A') = $(r/2)$(C'A'), and area(A'P'B') = $(r/2)$(A'B'), so we see from the above equations that there exists a constant k such that B'C' = $k\alpha$, C'A' = $k\beta$, and A'B' = $k\gamma$. (Figure 5).

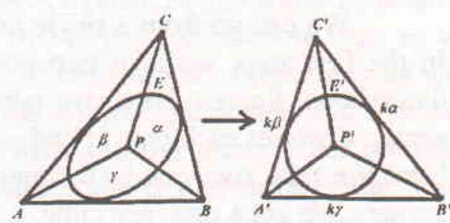


Figure 5

The semiperimeter s of triangle A'B'C' is $s = (k\alpha+k\beta+k\gamma)/2 = k/2$, since ABC has unit area and $\alpha+\beta+\gamma = 1$. Applying Heron's formula, we obtain

$$\text{area}(A'B'C') = \sqrt{\left(\frac{k}{2}\right)\left(\frac{k}{2}-k\alpha\right)\left(\frac{k}{2}-k\beta\right)\left(\frac{k}{2}-k\gamma\right)}$$

$$= \left(\frac{k^2}{4}\right)\sqrt{(1-2\alpha)(1-2\beta)(1-2\gamma)}.$$

But the area of A'B'C' is also given by:

$$\text{area}(A'B'C') = (r/2)(k\alpha+k\beta+k\gamma) = rk/2.$$

Eliminating k between the two above equations leads to:

$$\pi r^2/\text{area}\,(A'B'C') = \pi\sqrt{(1-2\alpha)(1-2\beta)(1-2\gamma)}$$

Property 3, and the fact that area(ABC) = 1, gives

$$\text{area}(E) = \text{area}(E)/\text{area}(ABC) = \text{area}(E')/\text{area}(A'B'C') = \pi r^2/\text{area}(A'B'C').$$

Finally, combining the last two equations yields the desired formula.

$$\text{area}(E) = \pi\sqrt{(1-2\alpha)(1-2\beta)(1-2\gamma)}.$$

This is one of the many demonstrations of the usefulness of orthogonal projection. Even an idea as simple as the casting of a shadow has its ramifications in mathematics.

triangle. Since two such lines intersect in at most one point, no two different points can have the same barycentric coordinates.

[4] This will go without proof here, but the locus of the centers of ellipses inscribed in a given triangle is the interior of the triangle determined by the midpoints of the sides of ABS, and each point interior to the midpoint triangle is the center of an ellipse inscribed in ABC.

The Fourth Dimension

Nelson Uhan

What is the fourth dimension? A line is one-dimensional. A flat surface is two-dimensional. Solid objects are three-dimensional. But what is the fourth dimension? Sometimes when people hear the words "fourth dimension" they think of Einstein's theory of relativity, where the fourth dimension is a combination of a three-dimensional space and a one-dimensional time coordinate. But why do we have to go into this physics mumbo-jumbo? Let's follow our intuition and see if it makes sense to take one more step in the list of geometrical dimensions.

We can go from a single point, an object with no dimensions, up to a cube in three steps. In the first step, we take two points, one inch apart, and join them. We have a line, a one-dimensional figure. Next, we take two one-inch line intervals, parallel to each other, one inch apart. Connect each pair of end-points, and we get a one-inch square, a two-dimensional figure. Next we take two one-inch squares, parallel to each other. If we connect the corresponding corners, we get a one-inch cube, a third-dimensional object.

So, if we follow our logic, can't we make an object in four-dimensions? We must take two one-inch cubes, parallel to each other, one inch apart, and connect the vertices. In this way, we should get a four-dimensional object, a *hypercube*.

Wow! A four-dimensional object. So what's the problem? The trouble is that we have to move in a new direction at each stage. The new direction has to be perpendicular to all the old directions. After we have moved back and forth, then right and left, and finally up and down, we have used up all the directions that we can perceive. We view objects in three dimensions, we cannot escape from three dimensional space. The only argument for it is that we can conceive it; there is nothing illogical or inconsistent about our conception.

We can determine many properties that a hypercube would have, if one existed. Let's think about this carefully: Since a hypercube would be constructed by joining two cubes, each of which has 8 vertices, the hypercube must have 16 vertices. It will have all the edges the two cubes have; it will also have new edges, one for reach pair of vertices that have to be connected. This gives us 12 + 12 + 8 = 32 edges. The rest of the table can be determined in a similar fashion.

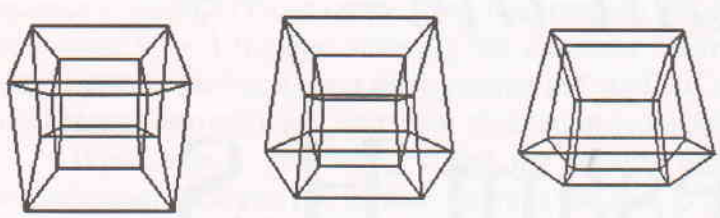
Dimension	**Object**	**Vertices (0-Faces)**	**Edges (1-Faces)**	**Faces (2-Faces)**	**"3-Faces"**	**"4-Faces"**
0	Point	1				
1	Interval	2	1			
2	Square	4	4	1		
3	Cube	8	12	6	1	
4	Hypercube	16	32	24	8	1

Well, if we can find out this much definite information about the hypercube, it must exist in some sense. Of course, when we try to determine how many vertices a hypercube has, we asking how many *could* it have, if there were such a thing. The hypercube is just a fiction in the sense of physical existence.

But, the hypercube does have a mathematical existence. If we can answer questions about two-dimensional and three-dimensional figures through relatively simple algebra, why can't we do the same for a fourth-dimensional figure? Solving systems with four variables is essentially the same as solving systems with two or three. So the hypercube, while it does not have an actual physical existence, it certainly has a mathematical one.

When it comes to four-dimensional geometry, it might seem that since we are limited to three-dimensional views that we are excluded by nature from the possibility of reasoning intuitively about four-dimensional objects. However, this is not so. Intuitive grasp of four-dimensional figures is not impossible. At Brown University, Thomas Banchoff, a mathematician, and Charles Strauss, a computer scientist, have made computer-generated motion pictures of a hypercube moving in and out of three-dimensional space.

Imagine a flat, two-dimensional creature on the surface of a pond, or any water surface for that matter. This flat creature of ours is limited to viewing two physical dimensions, just as we are limited to three. This creature could become aware of three-dimensional objects only by way of their two-dimensional intersections with its flat world. For example, if a solid cube passed from the air into the water, it could only see the two-dimensional cross section that the cube makes with the surface as it enters the surface. Now if the same cube passed through the water repeatedly, at many different angles and directions, the creature would eventually have enough information about the cube to "understand" it, even if it couldn't escape from its two-dimensional world. The Strauss-Banchoff movies show what we could see if a hypercube passed through our three-space, at one angle or another. We would see various or more complex configurations of vertices and edges.



Projections of the hypercube. Each of the diagrams on the left were created by rotating the hypercube about a plane in four dimensions, or a combination of plane rotations.

Thinking back, one wonders whether there really ever was a difference in principle between four-dimensional and three-dimensional. After all, we can develop the intuition to derive a four-dimensional object. We can even get the feel of a four-dimensional object, thanks to the geniuses at Brown. Once that is done, four-dimensional objects don't seem so much more imaginary than "real" objects like plane curves and surfaces in space. These are all objects we can grasp both visually and mathematically.

The Quest For Random Numbers

Ilya Berdnikov

Once, while walking along the street, I overheard one man talking about his dig. "That's odd," I thought, "I would never talk about something as random as that." After I went home and almost finished my homework, I tried to remember what happened that day. The conversation I heard reminded me of my remark, and I questioned its validity: "How can I call something random, if to that person it was the most natural thing to talk about? And what is random, anyway?"

Unable to resist the temptation, I opened my *Numerical Recipes in C* book and flipped to the page that talked about random numbers. Here's what I learned – a random number is a value which changes upon chance. Random numbers only make sense when one looks at a sequence of them. There are also different types of these sequences, but the most common one consists of "uniform deviates." A uniform deviate is simply a number that falls in a specified range – usually 0 to 1 – with any single number being just as likely to occur as another. Thus, uniform deviates have a fairly even distribution.

Of course one could say that just calling out fractions from the top of one's head creates a random sequence. This is not true, unless this person is not influenced by his feelings toward certain numbers.

Suddenly I stop reading and ask myself, "Self, do you need many random numbers?" Surprisingly enough Self said, "Why yes, and what did *you* think, silly? Read on!" Reading further, I find out that certain algorithms could be used for generating the needed sequences. "He-he-he – I'd like to see the computers crunch the formulas and create somewhat random sequence – NOBODY can create something random out of those neatly ordered bits and bytes of information!!!" – I thought savoring the moments before I'd have a good laugh. But to my disappointment the book gives the examples in C and due to the lack of understanding and the late hour I plunge into my usual four-hour meditation some people prefer to call sleep.

While slee..., meditating I feel my privacy being invaded by something beyond my comprehension. Suddenly, a familiar figure is standing in front of me.

"Professor Donald Knuth!" – I scream with joy.

"In order to pass to a higher level of intellect you must survive this challenge," I hear from a voice far away, "answer these questions truthfully or you will perish. Number one: What is your name?"

"I'm King Arthur... no wait, I'm Ilya."

"Good. Number two: What is your quest?"

"I seek the Holy Random Numbers".

"Very well, number three: What is the average aerial speed of a laden swallow? Wait, never mind that. Well, what do you want, young man?"

"I seek the book you have written"

As Donald Knuth tries to give me his book I hear a buzzing sound that soon fills my ears, my brain and every other useless part of the body with vibrations. Opening my eyes I shut off the alarm, and try to go back to meditation again when I see a book in yellowish hard cover lying on the table next to me. It says *The Art of Computer Programming* on the side.

Gladly I turn to the first chapter when I learn that the random numbers, particularly uniform deviates, could be generated by an interesting algorithm. In this algorithm, N_i is a term of the sequence, N_{i+1} is the next term, a is called the multiplier, c the constant or increment, and m is the modulus. Each term is generated by the relation $N_{i+1} = aN_i+c \pmod{m}$, beginning with a seed value N_0. This general formula will generate a sequence of random numbers between 0 and m. Generators based on this type of formula are usually called Linear Congruential Generator, or just LCG's. Since LCG's only involves a few arithmetic operations, they are usually very fast.

One more important aspect the book discusses is the period of a random number generator. This period is much like the period of repeating decimals – after a while the digits just start repeating and the sequence simply starts over. It's quite easy to see why the period is very important. Suppose you need at least 10,000 different random values, but the period of the generator you are using is only 5,000. By generating 10,000 values with such generator you unknowingly only get two identical sets of 5,000 different random values.

So how can the period be maximized? Well, the secret lies in the parameters of the standard equation – a, c, m, N_0 (the starting, or seed, value of the generation process). Thus, by changing the appropriate values the period for the LCG can become m (the maximum possible period in this case). Unable to resist the temptation I ask myself a question, "But how in the world will we know what values of a, c, and m, will make the period longest?" "Read on!" ordered a voice in my head. At the same time the pages start flipping by themselves and finally bring me to the section on choosing the right parameters.

First, m should be as large as possible; one of the preferred values is the largest prime less than 2^{32}, which is the Mersenne prime 2^{32}-1. Secondly, the choice of a greatly depends on the chosen value of m. If m is merely a product of primes, the longest period is when $a = 1$, and that is undesirable. However, when m is a power of some prime, a can have a much wider spectrum of values.

"Go easy on yourself!" my feelings cry, "The following theorem will do a lot for you:"

Theorem 1. The LCG defined by a, m, c, N_0 has a period of m if and only if:

1. c and m are relatively prime.
2. a-1 is a multiple of every prime divisor of m.
3. if m is a multiple of 4, a-1 is a multiple of 4.

Proof: While reading the proof, I become very dizzy, so I skip to the next section. But you should try to prove it as an exercise.

Although linear congruential generators have been extensively tested over the years and still appear to be pretty effective, the correlation between consecutive terms is still troublesome (the statistical correlation should be minimized). If we were to graph the points with their coordinates corresponding to the elements of the sequence, we wouldn't get an evenly covered plane, but instead a plane covered with small 'galaxies' of dots.

"The LCG is very fast," tells me a familiar voice, "but search your feelings: on the inside you prefer to follow the dark side, skepticism. You can't stand correlation. But wait! Read about shuffling! You have to finish your training before you are ready!"

Doubting my sanity, due to the fact that I hear strange voices, I continue reading. As the book leads on, it quite logically suggests that in order to increase the period of a sequence one could generate two different sequences (that is, with different generating parameters) and append one at the end of another. Yet another method is to add the i^{th} elements of both sequences in some modulo m to generate the i^{th} element of the third sequence.

Then I came across shuffling. One shuffling algorithm, that further 'randomizes' a single sequence, runs as follows:

1. Fill an array V with the k elements of the given sequence, where k is some convenient number, usually a power of 2. Set Y to be the $(k+1)^{th}$ element of the given sequence.
2. Let j be determined by Y. In this equation m is the modulus from the generated sequence.
3. Let $Y = V[j]$, then output Y, and then set $V[j]$ to the next element of the original sequence.

Thus continued the story of my Quest for Random Numbers. The book led me through many fine exercises, such as the serial test, the additive methods for generation of uniform deviates, the DES (Data Encryption Standard) algorithm for generating sequence of random numbers before it brought me to the Monte Carlo Method.

"Observe – said my inner voice – thou shall find π!"

"This is impossible!" I screamed frightened by the vast area random numbers engulfed.

"Well, not really – someone else said – just follow along. Let's say you have a quarter of a unit circle inscribed into a unit square like this. The area of the square is 1, and the area of a circle is $\pi/4$. Right? Ok, say we drop a point with random coordinates (x,y) inside this square. If that point is inside the quarter circle, we'll add a mark to our score sheet for the circle. We would always add a point to our square score sheet – that point did make it into the square, didn't it? Let's say we'll continue this until we've used all spaces inside the square. What would we get at the end on our score sheets?"

"The area of a square and the area of the quarter circle?" I interrupt in a weak voice.

"Of course! So if we divide the area of the quarter circle (our total score for the circle) by the area of the square (our total score for the square) and multiply the quotient by 4 what do we get?"

"The value of π!".

"Exactly! Well, of course you'd need to have a perfect random numbers generator in order to do that. As you yourself have explained before, the correlation between the terms of any LCG-generated sequence is too great to make the value at least remotely similar with the original. And as far as I recall, some have performed this same experiment with 10^{109} points dropped on a square, and their result was only correct to seven digits!"

"What did you mean when you said there was no way to get perfectly random sequence of numbers? What about the decay of cesium and other elements, that decay randomly? What about them?"

"Of course, I can't deny it. With the decay of radioactive elements you can get better random values, but still the ultimate solution to that problem does not exist!"

"So, ..., so, ..., my research,..., my vision of Professor Knuth, ..., ..., they were useless? But, ..., but what about the rest of the people? Do they know? What about my grade?"

Angered and frustrated I throw my books into the corner and start weeping. Weeping slowly goes to sobbing, and soon I'm sleeping peacefully on my bed.

"Your training is complete. You are now a true mathematician!" sounds a voice from afar, and three people covered in mist wearing cloaks walk away from my door...

Cryptographic Efficiency

Ilya Berdnikov

The art of cryptography evolved a long time ago. People needed a way to send messages that only they could read, and it was pretty hard to keep their mail from being read or the messenger from being tortured (with the consequences of his giving up the valuable information). Codes are a very important part in the development of the human society; our language, writing and customs are all codes of some form. Have you ever tried to listen to a broadcast in a foreign language? You can't understand it, just as if you can't understand customs of the foreigners. Until you understand that particular code, you are forever deprived of the information it hides. However, it's usually not impossible to learn a language, since there is always a large population of speakers to learn from.

Languages aren't designed to be indecipherable; they just seem to have evolved that way. A much greater problem is to understand a special code that had been designed to be difficult to understand. Many codes use a special combination of letters as a key, but some also have very complicated procedures for encoding and decoding. To crack the code, you have to spend days analyzing encrypted samples to get a very superficial idea about its intricacies, and you can't buy Berlitz tapes to help you along.

Codes may range from simple substitutions of symbols or numbers for a letter of an alphabet, to creating binary trees of ciphers with the application of passwords afterwards. Before discussing in detail how codes are structured, we should take a look at some basic postulates, theorems, and definitions that are essential to build good codes. The codes presented here all translate messages into binary numbers (zeros and ones). This dates back to Sir Francis Bacon of Elizabethan times, who was the first to realize that to represent an encoded message, you only need two symbols (0 and 1). Words, messages, or characters that will be encoded will be denoted as $M_1, M_2, \ldots, M_n$ and their corresponding code words are $m_1, m_2, \ldots, m_n$. The set of messages or symbols ($\{M_1, M_2, \ldots, M_n\}$) will be called T, while the set of corresponding code words ($\{m_1, m_2, \ldots, m_n\}$) will be referred to as C. The probability that a word M_n is in a given text is P_n. The length of a code word m_n is l_n. The average length of all of the code words in a code system is $\bar{l}$, and $\bar{l} = \sum_{i=1}^{N} l_i P_i$, where N is the number of ciphers in the code.

Defn 1. *Prefix property of a code – a code system has a prefix property if no code for a code word is the prefix of the code for any other code words.*

It's probably easier to understand this complicated definition by considering a set of examples. For example, a set of code words {010, 01, 10, 0, 1} doesn't have the prefix property. If a transmitted message contains 010, we can't be sure that we have decoded the message correctly since the given message might be decoded as {0, 10}, or {010}, or {01, 0}. However the following set of ciphers does have the prefix property: {000, 001, 010, 011, 100}, as you can see we can decode the following message easily: 000010100001, and the only possible solution using the given set being {000, 010, 100, 001}.

Look at the following set of messages: $\{M_1, M_2, M_3, M_4\}$ with the probabilities in the text P_1 appearing to be ½, of P_2 appearing to be ¼, and of P_3 and P_4 appearing individually to be 1/8 (see Figure 1). We can encode the ciphers using the two binary digits (4 different binary words) as {00, 01, 10, 11}. This code has the prefix property, and its average length is 2 (2·½+2·¼+4·1/8 = 2 using the formula). However, we might encode the messages the following way: first divide them into two groups with the sums of their probabilities as closely equal as possible, then continue dividing into groups and assigning 0's and 1's to the members of the group (this process completed makes a table seen on Figure 1). Thus, ciphers assigned to the messages will be 0 for M_1, 10 for M_2, 110 for M_3, and 111 for M_4 (just read across the table entries). The average length of the presented code is 1.75 (1·½ + 2·¼ + 6·1/8 = 1.75) which gives us a 12.5% savings in the length of the encrypted message C. The code has the prefix property, and it is more efficient than the code that merely assigns binary digits. This particular code was first presented by Fano, an American mathematician. However, Fano's algorithm doesn't always produce the optimal code, which means that there may be a code with a smaller average length. Well, let's outline the properties of an optimal code.

M_1	½	0		
M_2	¼	1	0	
M_3	1/8	1	1	0
M_4	1/8	1	1	1

Figure 1

Defn 2. *Optimal code – If a code is optimal the following condition must be satisfied: if $P_i < P_j$ then $l_i \geq l_j$.* (In other words, messages with lower probabilities of appearing are encoded with code words of greater length.)

In the optimal binary code there will always be two words with the longest ciphers that differ only in the last symbol (Figure 2A). Thus we can build a tree for that code. Let's look at a new set of messages $T^{new} = \{M_1, M_2, \ldots, M_{n-1}, M\}$ with the respective probabilities $\{P_1, P_2, \ldots, P_{n-2}, P_{n-1}, P_n\}$. Then there is set C^{new} with the following code words $\{m_1, m_2, \ldots, m_{n-1}, m\}$. The binary code tree is built from the set C^{new}. The vertices of the tree are respectively $m_1, m_2, \ldots, m_{n-1}, m$. If we have code built on T^{new} in our message then we can get the code words for the original messages m_{n-1} and m_n by adding a 0 or a 1 to the new m code word (Figure 2B). The set C^{new} is compressed (two of the elements of the original set are replaced by one element in the T^{new} set). A theorem that lets the optimal code to exist is produced:

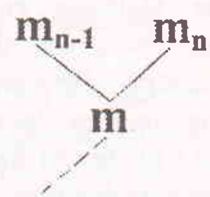


Figure 2A

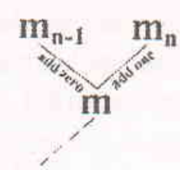


Figure 2B

Theorem 1. *If the set C^1, which was built on the compressed version of the original set, T^1, is optimal, then the set C for the original set of messages is also optimal.*

To prove the theorem let us establish the following equality: $\bar{l} = \bar{l}' + P$, where $\bar{l}$ is the average length of set C, $\bar{l}'$ is the average length of the compressed set C^1 and P is the probability of the compressed element M appearing in the set T^1. The proof of the theorem is relatively simple: assume that C is not the optimal code, and that there is a code C_1 such that $\bar{l}_1 < \bar{l}$, therefore that the ending nodes of the binary code trees are m'_{n-1} and m'_n which are the codes for the M_{n-1} and M_n respectively. Let's look at the corresponding set $C_1{}^1 = \{m'_1, m'_2, \ldots, m'_{n-2}, m'\}$ in which the cipher m' is gained by throwing away the last binary digit of the two ciphers m_{n-1}, and m_n (Figure 3). This means that the average lengths and

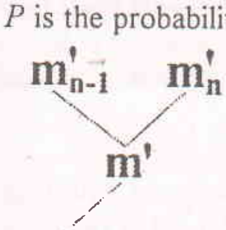


Figure 3

are bound by the equality: $\bar{l}_1 = \bar{l}_1' + P$. From the inequality (by definition an optimal code has a lesser average code length) follows which creates a contradiction with the C^1 being optimal.

This method of encoding was first introduced to us be D. A. Huffman in 1952. Let's look at an example of coding using such algorithm.

We have a set of characters $T = \{a, b, c, d, e\}$and with the probabilities $P(a) = 0.6$, $P(b) = P(c) = P(d) = P(e) = 0.1$. (Figure 4A shows the process of building the code itself and determining whether it is the optimal code for a given set of messages and their probabilities).

Symbol	Probabilities and codes							
	Original Set		Compressed Sets					
			T^1		T^2		T^3	
a	0.6	1	0.6	1	0.2	11	0.6	1
b	0.1	010	0.1	010	0.6	0	0.4	0
c	0.1	011	0.1	011	0.2	10		
d	0.1	001	0.2	00				
e	0.1	000						

Figure 4A

The set T^3 contains only two elements and therefore C^3 is the optimal code – since there are two elements in the binary number system the optimal code is the one that has the only two elements. However using Theorem 1, sets C^2, C^1, and C are also optimal. The average length of the code is 1.8 $(0.6+3\cdot4\cdot0.1 = 1.8)$, while the same text coded using Fano's algorithm (Figure 4B) has the average length of 1.9 $(0.6+13\cdot0.1 = 1.9)$. Thus, we see that the Huffman algorithm is optimal one for encoding data using binary digits. However, the Huffman code doesn't provide any error detecting/correcting algorithms and therefore is inefficient for use in the communication industry. However, this particular code is widely used in different data compression algorithms and is fairly effective when combined with some kind of device for error detection (a checksum, for example). Most of these techniques became the foundation for the modern-day compression algorithms on the Internet, and other information-related technologies.

0.6	0			
0.1	1	0		
0.1	1	1	0	
0.1	1	1	1	0
0.1	1	1	1	1

Figure 4B

The Theorem of Li-Yorke

Abhijit Gurjal

The Theorem of Li-Yorke is an important and useful theorem in the field of dynamical systems because it simplifies the task of determining which functions produce chaotic systems. Many seemingly simple functions can exhibit chaotic behavior, such as the functions that generate fractals. The theorem states that if a function has a period-3 point then it has "non-trivial period-k points" for all positive integers k. One of the fundamental characteristics of a chaotic system is the existence of non-trivial period-k points.

To understand period-k points some definitions are necessary. A function f: $\mathfrak{R} \rightarrow \mathfrak{R}$ is continuous if and only if for all X, closed subsets of $\mathfrak{R}$, $f^{-1}(X)$ is a closed set. This may not be a familiar definition for a continuous function, but it is equivalent to the common definition. It is easy to show that the sum of two continuous functions and the composite of two continuous functions are also continuous (f from now on will denote a continuous function).

Chaotic behavior refers to the unpredictability of a function over iteration. Iteration refers to the repeated application of a function to a value – so, $f(f(x))$ is the second iterate of f at point x, and $f(f(f(f(x))))$ is the fourth iterate of f at x. For simplicity, the nth iterate of the point x is denoted as $f^n(x)$. Let $f^0(x) = x$, $f^1(x) = f(x)$, and let $f^{n+1}(x) = f(f^n(x))$. At certain points in the domain of f, repeated iterations will approach zero, while at other points iterates of f will approach infinity.

A continuous function f has a non-trivial period-k point if and only if $f^k(x) = x$ and $f(x)$ is not equal to x (the cases $f^0(x) = x$ and $f^1(x) = x$ are trivial). For example, for complex number z, consider $f(z) = z^2$. This function is a good test case for the Theorem of Li-Yorke, since it exhibits chaotic behavior. A period-3 point for $f(z) = z^2$ would be $z = e^{2\pi i/7}$. Thus, $f(z) = z^2 = e^{4\pi i/7}$, and then $f^2(z) = f(f(z)) = e^{8\pi i/7} = e^{\pi i/7}$. So, $f^3(z) = e^{2\pi i/7} = z$.

One hallmark of a chaotic system is the existence of period-k points, where the iterates merely cycle through the same set of k different values. Of course, it is impractical to test the existence of period-k points for all k, since there are infinitely many values for k. Fortunately, the Theorem of Li-Yorke provides a handy shortcut: one only needs to show that a period-3 points exists, and the theorem guarantees the existence of period-k points for all greater k.

A couple of lemmas are needed to prove the theorem of Li-Yorke:

Lemma 1. *If for a closed interval $I = [i_0,i_1]$ there is an interval $[a,b]$ with $[a,b] \subseteq f(I)$, then there exists a closed interval $[c,d] \subseteq I$, such that $f([c,d]) = [a,b]$.*

Proof. Since f is continuous on a closed interval, by the Extreme Value Theorem from calculus, f has absolute minima and maxima, say $f(x_{\min})$ and $f(x_{\max})$, on I. Assume that $i_0 \le x_{\min} \le x_{\max} \le i_1$ (the case where $i_0 \le x_{\max} \le x_{\min} \le i_1$ follows similarly). Note that $f(x_{\min}) \le a$, and $f(x_{\max}) \ge b$.

Since f is continuous and the point a is a closed set, the set $f^{-1}(a)$ must also be closed. Now consider only f on $[x_{\min}, x_{\max}]$. Since f is restricted to $[x_{\min}, x_{\max}]$, $f^{-1}(a)$ is bounded

and non-empty as well. By the Least Upper Bound principle,† this set of real numbers must have a least upper bound which is an element of the set, say c.
Now consider the set $f^{-1}(b)$ restricted to $[c,x_{max}]$, which is a similar set (non-empty, closed and bounded). By the Greatest Lower Bound principle, there is a smallest real number, say d, greater than c with $f(d) = b$. We know that $f^{-1}(b)$ is not empty from the well-known Intermediate Value Theorem because $f(c) < b \leq f(x_{max})$, there must be a d such that $f(d) = b$.

To show that $f([c,d]) = [a,b]$, consider a number $p = f(r)$, with r in $[c,d]$. This number p is in $f([c,d])$ but not in $[a,b]$. Assume $p > b$ (the case where $p < a$ follows similarly). Then by the IVT, there exists a number which violates the fact that d is the Greatest Lower Bound greater than c with $f(d) = b$. Now assume p is in $[a,b]$ but not $f([c,d])$. But since $f(c) = a$, and $f(d) = b$, and $a < p < b$, this would violate the IVT. QED.

Lemma 2. *If $I \subseteq [a,b]$ and $[a,b] \subseteq f(I)$, then there is a point y in I such that $f(y) = y$. (This is a simple, one dimensional case of the Bower Fixed Point Theorem.)*
Proof. By Lemma 1 there exists a interval $[c,d] \subseteq \mathrm{I}$ such that $f([c,d]) = [a,b]$, with $f(c) = a$ and $f(d) = b$. Since $[c,d] \subseteq \mathrm{I} \subseteq [a,b]$, $c > a$ and $d < b$. Now consider the continuous function g, defined by $g(x) = f(x)\text{-}x$. Since $g(c) = a\text{-}c < 0$, and $g(d) = b\text{-}d > 0$, by the IVT, there exists a y in $[c,d] \subseteq \mathrm{I}$, such that $g(y) = f(y)\text{-}y = 0$ or $f(y) = y$. QED.
Note: If $f(y) = y$, then y is called a fixed point.

The "mathematical machinery" is finally in place to prove the theorem of Li-Yorke.

Theorem of Li-Yorke. If f has a period-3 point, with $f(a) = b$, $f(b) = c$ and $f(c) = a$, $a < b < c$ (the theorem follows similarly for the second case with $f(a) = c$, $f(b) = a$ and $f(c) = b$, $a < b < c$), then for all positive integers k, there exists x_k in $[a,c]$ such that $f^k(x_k) = x_k$, but $f(x_k)$ is not equal to x_k.
Proof. Since $f(c) = a$ and $f(b) = c$, $[a,c] \subseteq f([b,c])$, by Lemma 1 there exists an interval $\mathrm{I}_1 \subseteq [b,c]$ such that $f(\mathrm{I}_1) = [a,c]$. Similarly since $\mathrm{I}_1 \subseteq f(\mathrm{I}_1) = [a,c]$, there exists an I_2 such that $\mathrm{I}_2 \subseteq \mathrm{I}_1 \subseteq [b,c]$. This process is repeated k-1 times so that the sequence of nested intervals $\mathrm{I}_{k-1} \subseteq \ldots \subseteq \mathrm{I}_3 \subseteq \mathrm{I}_2 \subseteq \mathrm{I}_1 \subseteq [b,c]$ is constructed. Now since $f(a) = b$ and $f(b) = c$, $\mathrm{I}_{k-1} \subseteq [b,c] \subseteq f([a,b])$. Let I_k be the interval in $[a,b]$ that maps to I_{k-1}, or $f(\mathrm{I}_k) = \mathrm{I}_{k-1}$. Then $\mathrm{I}_k \subseteq [a,c]$ and $f^k(\mathrm{I}_k) = [a,c]$. By Lemma 2, f^k has a fixed point, say x_k (that is, $f^k(x_k) = x_k$). Since $[a,b]$ and $[b,c]$ are disjoint except for b, this point cannot stay the same for all iterates of f, so $f(x_k)$ does not equal x_k. QED.

† **The Least Upper Bound Principle states that if X is a closed and bounded set of real numbers, then there exists $y_0 \in \mathrm{X}$ such that for all $y \in \mathrm{X}$, $y \leq y$.**

Lagrange's Theorem

Michael Shy

In the area of calculus, there are two major contributors who immediately spring to mind: Newton and Liebnitz. Their work jointly laid the foundation of differential calculus. However, we should also consider the work of Joseph-Louis Lagrange (1736-1813), who made valuable contributions, particularly regarding optimization.

Many optimization problems have certain restrictions or constraints on the values that may be used to produce the optimal solution. These constraints often complicate such problems since the solution may lie at a boundary point of the domain. Lagrange came up with an ingenious method of solving such problems, using the aptly-named Lagrange multipliers. This method was based on one of his important theorems.

Lagrange's Theorem. *Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0) on the constraint curve $g(x,y) = c$. If $\nabla g(x_0, y_0) \neq 0$ then there is a real number λ such that*

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

where ∇ is the gradient vector operator and λ is a scalar that is called the Lagrange Multiplier.

Proof. The proof of this theorem is based on another theorem, the Implicit Function Theorem, which guarantees that the constraint curve $g(x,y) = c$ is smooth. This theorem will not be proven here, as it requires more advanced calculus. Using this theorem, we can represent the function using vectors:

$$r(t) = x(t)\mathbf{i} + y(t)\mathbf{j},$$

where x' and y' are continuous on an open interval I. Now, we will determine the function h as $h(t) = f(x(t), y(t))$ and since $f(x_0, y_0)$ is an extreme value of f,

$$h(t_0) = f(x(t_0), y(t_0)) = f(x_0, y_0),$$

is an extreme value of h. Then, $h'(t_0) = 0$, and by simply applying the Chain Rule (for partial derivatives),

$$h'(t_0) = f_x(x_0, y_0)x'(t_0) + f_y(x_0, y_0)y'(t_0) = \nabla f(x_0, y_0)\, r'(t_0) = 0,$$

Therefore, $\nabla f(x_0, y_0)$ is orthogonal to $r'(t_0)$.

Another useful theorem states that:

If f is differentiable at (x_0, y_0), and $\nabla f(x_0, y_0) \neq 0$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0).

So, $\nabla g(x_0, y_0)$ is also orthogonal to $r'(t_0)$. We can conclude that the gradients $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are parallel and thus, there must exist a scalar λ such that:

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0). \quad \text{QED.}$$

We can then apply this theorem to find the extreme values of a function f subject to a constraint g.

Cardano's Formula

Edmund Chou

Giralamo Cardano (1501-1576) published his famous *Ars Magna* in 1545. In *Ars Magna*, Cardon wrote, "Let the cube and six times the side be equal to 20." This is equivalent to the equation $x^3+6x = 20$, but he was probably thinking of the problem in the general form of $x^3+mx = n$.

He produced the solution by introducing two variables t and u such that $t-u = n$ and $tu = (m/3)^3$. Then, x can be expressed as $x = \sqrt[3]{t} - \sqrt[3]{u} = t^{1/3} - u^{1/3}$. A quick check of cubing both sides yields:

$$x^3 = \left(t^{1/3} - u^{1/3}\right)^3$$
$$x^3 = t - 3t^{2/3}u^{1/3} + 3t^{1/3}u^{2/3} - u$$
$$x^3 = (t-u) - \left(3t^{1/3}u^{1/3}\right)\left(t^{1/3} - u^{1/3}\right)$$
$$x^3 = n - mx\ .$$

So, these substitutions are valid. Solving for t and u in terms of n and m yields:

$$u = t - n$$
$$(m/3)^3 = tu$$
$$(m/3)^3 = t(t-n)$$
$$t^2 - nt - (m/3)^3 = 0\ .$$

From the quadratic formula, $t = \frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}$. Substitution of t into $u = t - n$ yields $u = \frac{-n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}$. Solving for x produces: $x = \left(\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}\right)^{\frac{1}{3}} - \left(\frac{-n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}\right)^{\frac{1}{3}}$

This is actually the cubic formula that Tartaglia derived in 1515, but Cardano received the credit because he rather sneakily secured the solution from his fellow mathematician under an oath of secrecy.

Now we can try the equation. Going back to Cardano's original problem, $m = 6$ and $n = 20$. Using the equation above,

$$x = \left(\frac{20}{2} + \sqrt{\left(\frac{20}{2}\right)^2 + \left(\frac{6}{3}\right)^3}\right)^{\frac{1}{3}} - \left(\frac{-20}{2} + \sqrt{\left(\frac{20}{2}\right)^2 + \left(\frac{6}{3}\right)^3}\right)^{\frac{1}{3}} = \left(10 + \sqrt{108}\right)^{\frac{1}{3}} - \left(-10 + \sqrt{108}\right)^{\frac{1}{3}}$$

This expressions is exact but unwieldy, so with a calculator we can approximate x, so $x \approx$ 2.732050808-0.7320508076, or simple $x = 2$ within 4×10^{-10}. Plugging x back into the original equation, $x^3+6x = 20$, shows that 2 indeed is the correct answer.

References

[1] J. Muir, *Of Men and Numbers*, Dover Publications, Inc., New York, 1961.
[2] P.J. Davis and R. Hersh, *The Mathematical Experience*, Houghton Mifflin Co., Boston, 1981.
[3] C.B. Boyer, *A History of Mathematics*, John Wiley & Sons, New York, 1968.
[4] D.E. Smith, *A Source Book in Mathematics*, Dover Publications, Inc., New York, 1959.

Worker Reliability

Regina Eum

Humans live in a world filled with many shades of gray, as evidenced by subjective words like "old" and "pretty" which describe a wide range of people. Thus, in order for mathematical models to reflect humans and human relationships accurately, they must allow for some ambiguity as well. One such type of model, fuzzy systems, deals with ambiguity through the gradation of the boundaries of the states, relationships, constraints, and goals of the system model made with ordinary sets.

To get a better idea of what fuzzy sets really are, one could draw a geometric model. First take a rectangle and designate it set X. Then draw a dotted circle inside; this circle will act as the hazy border between the fuzzy subset A inside the dotted circle and the rest of set X. Finally, the degree to which an element x of set X is included in subset A is defined by the function μ, which is called the *membership function*, and the degree of inclusion is called the *extent* or *grade*. For example, one could let the set X be the set of heights a person could be and the subset A consist of all of those heights that could be considered "tall." Let this range of heights extend from 64 in. to 84 in. One could see that $\mu(65)$ would be very low, close to zero, while $\mu(82)$ would be quite high, almost equal to one. If one were to define fuzzy sets formally, one would say that the function $\mu{:}X \rightarrow [0, 1]$ is given the label A, and A is called a fuzzy (sub)set of X. The function μ is called the membership function of A. Since a fuzzy subset is invariably defined as a subset of a general set X, the "sub" is usually dropped and a fuzzy subset called a fuzzy set.

Fuzzy theory has been useful for a variety of tasks, including making human reliability models relating reliability to workload, ability, ability distribution among multiple tasks, and psychological stress. Five interrelations among these factors can be made: (1) workload, ability, and reliability, (2) workload and stress, (3) stress and ability, (4) ability and distribution, and (5) environment and stress. One can gain the data necessary for these models by repeatedly giving a person a certain timed task to perform. The percentage of tasks completed successfully can be dubbed a person's reliability, and the inverse of the time limit can be used as a measure of the workload, since the inverse of the time can be termed the pressure felt by the person and the pressure is proportional to the workload.

Since workload/ability/reliability is the most ambiguous item listed, let it be expressed by fuzzy sets and allow the ambiguous part of the reliability to be brought out in the membership function. Then, if one lets the workload be the independent invariable and thinks of the reliability corresponding to it as the value of the membership function, one finds that human ability generally maintains a plateau of high reliability before reliability drops off linearly after some point. The more able a person, the longer he can sustain the plateau. Thus, changes in ability can be expressed as the horizontal translation of the membership function.

The relation between workload and stress, interrelation (2), is generally viewed as a proportional relation. The stress/ability relation, interrelation (3), is a little more complicated. Psychologists say that the proper amount of stress maximizes human ability. However, there are variations in ability itself so the integrated value of reliability shown in the previous paragraph is used to define a person's ability. If the degree to which ability changes in relation to the inverse of the time limit, or the workload, for all people, one obtains the average response of rising ability that soon maximizes and then sharply drops, which agrees with results from psychological studies.

In the case of one person with two jobs, the problem of how the subject distributes his or her total ability to the two jobs is added. The distribution varies with the kind of work, the workload, and the personality type of the subject. The fuzzy set curves often show an example for two jobs of the same type and workload. If the same subject performs a single job whose workload is twice as large, one line is obtained. This last line agrees with the sum of the curves from two jobs, which show that the overall ability for two jobs is equal to the sum of abilities for each job, and that the distribution of ability for jobs of equal workload is equal.

If the total workload is fixed and the workloads for each job are varied, the distribution of ability differed with the personality type of the subject, as shown in Table I below.

Fuzzy theory is extremely useful in modem-day life because of it allows for degrees of truth, whereas many other mathematical systems demand black and white definitions. The flexibility of fuzzy sets lets mathematicians apply them even to humans, some of the most irrational beings alive, and obtain startlingly accurate results. Fuzzy logic is currently being used not only in human reliability models, but in medical diagnosis, intelligent robots, Chinese character recognition, and numerous other areas as well. It will doubtless be applied to many other areas in the future.

REFERENCES

[1] T. Terano, K. Asal, and M. Sugeno. *Fuzzy Systems Theory and its Applications, Boston, Academic Press, Inc., 1987.*

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Solutions to Page 49

1. scaling noise
2. white
3. Brownian
4. brown
5. scaling density
6. fractional
7. topological
8. Pierce
9. imaginarius
10. omega
11. transfinite
12. First Outer Inner Outer
13. second
14. transcendental
15. nephroid
16. logarithmic
17. helix
18. zero
19. quadrivium

The Math Team Year in Review

The Math Team was greeted in September again with rooms swollen with students. To ease the crowding, Mr. Geller chose two associate captains, Sara Cohen and Zvi Moshowitz, to complement the regular Five: Dan Stronger, Leo Nguyen, Ping Hsin Lee, Linda Hong, and Che King Leo. Dan and Leo taught the senior/junior rooms, Ping and Zvi took the sophomores, and Sara, Linda and Che King got the freshmen. Unsurprisingly, the Math Team shrank precipitously as the year progressed. Sara and Zvi joined the unemployed while Ping was forced to teach nearly empty classrooms. The seniors fled *en masse* from Dan's room to Leo's room to copy homework in peace. That is, until Leo's attendance (or lack thereof) forced Mr. Geller to close the room.

The local team competition is the NYC Interscholastic Math League (NYCIML), which is divided into senior and junior divisions, and has further subdivisions. But before anything else, the Stuyvesant teams had to choose yet another catchy name for the year. One popular suggestion for the four Senior teams was P, I, N, G, but modesty won over idolatry and the name S, T, U, Y was adopted. In the fall competition, the S and T teams overran the competition by taking first and second places respectively in the Special Senior A division. In the spring, the T team led after two meets, but since Johnny Chen refused to motivate his team with free pizza, they slowly lost ground to the prevailing S team.

Tragedy struck the Junior teams in mid-year when it was announced that Bouley's, the celebrated TriBeCa restaurant that Mr. Geller is so fond of, was closing. It was just as well, since the junior NYCIML teams had abandoned their traditional policy of adopting names like B, O, U, L, E, Y and S, O, L, T, N, E, R (in honor of Lutèce's retiring head chef, André Soltner). In the absence of such *haute* culture, The Great Name Debate boiled down to C, H, E, K, I, N, G and E, P, S, I, L, O, N, with the latter prevailing. Unlike the Senior teams, the Junior teams were continuously harried by upstart Hunter. In the fall term, Stuyvesant E edged out Hunter A for first in the Special Junior division. In the spring, Hunter A seized an upset victory, while Stuyvesant E only tied Hunter B for second place. This is an embarrassing loss, as Stuyvesant teams almost always win first place in the NYCIML meets.

Stunned, the Math Team initiated the Great Purge, in which nobody was executed. Clearly, this year's juniors will have to make amends next year. And it looks as if Hunter isn't going away in the near future, as they tied Stuyvesant for the national lead in the ninth grade Continental Math League competition.

The Math Team also participates in three individual competitions, which have monthly rounds. In the New York Math League (NYML) competition, Stuyvesant won first place. In the Atlantic-Pacific (AtPac) competition, Stuyvesant is in third place in the Junior High division. In the regular AtPac, Stuyvesant tied for first with Lexington HS, MA, with a perfect 300 points. Finally, there is the Mandelbrot Competition, which is by far the hardest of the three. The Mandelbrot consists of four individual rounds, plus team rounds where the school's team must complete usually five proof-type questions. Although final standings aren't completed, so far Stuyvesant is in second place in Division A, trailing the North Carolina School of Science and Math after three rounds.

However, possibly the most important competition is the American High School Mathematics Examination (AHSME), a hour and a half long national exam. Stuyvesant again did very well, as 58 students did well enough to qualify for the three-hour American Invitational Math Exam (AIME) by scoring a 100 or more points. Stuyvesant scores were led by Dan Stronger and Leo Nguyen (135 of 150 points) and Johnny Chen (128), for a team score of 398 and fourth place nationally.

Dan and Leo went gangbusters on the AIME, both scoring 14 (out of 15) and qualifying for the USA Math Olympiad, which is given in May. Michael Shy, Ping Hsin Lee, Zvi Moshowitz, and Peter Graham also qualified, giving Stuyvesant the most USAMO qualifiers in the nation. The USAMO itself is a six hour, six question exam requiring proofs. Dan did well enough to be selected as one of the six high students to represent the US in the International Math Olympiad. Good luck Dan!

By April, most competitions end, leaving the Math Team free to prepare for the New York State Math League (NYSML) and American Regions Math League (ARML). NYSML and ARML share an identical setup: each participating city or county (or at ARML, state) sends fifteen-member teams to compete in team and individual rounds for one day. New York City competes as one entity (replacing local rivalry with camaraderie), and sent four teams this year – the A, N, Y, and C. Stuyvesant was heavily represented; both the A and Y teams had eleven Stuyvesant students apiece, and a fair number on the C team (curiously, none on the N team). The A team crushed its nearest rival, Nassau A, by nearly fifty points, continuing New York City's domination at the state level. The NYC A team also swept the top three individual scores, with Leo Nguyen, Ping Hsin Lee, and Carl Bialik (Bronx Science). The Y team finished a respectable fourth, with the C close behind in sixth. The N team mopped up in seventh place. At ARML, New York City is sending four teams again. The NYC A and B teams will compete in the ARML Division A, while the C and D teams will participate in Division B.

Unrelated to the Math Team is the New York Metropolitan Math Fair, in which students compete on the basis of submitted math research. For example, Westinghouse and Math Research papers are often submitted to the Math Fair. As usual, Stuyvesant did very well, claiming 16 medals overall.

Note: Due to space constraints, only the full NYCIML and NYSML standings are printed (the other meets have only the top three places listed; full standings are available elsewhere). *Math Survey* apologizes for any errors. Final standings for the Mandelbrot Competition are not available at the time of printing.

Mandelbrot Competition (3 of 4 rounds complete).

Leading Team Standings

Division A

Team Standings	Score	High Scorers	Team	Score
North Carolina School Of Science and Math	313	Martin Smith	NCSSM	39
Stuyvesant HS	304	Andrew Tuttle	NCSSM	33
Illinois Math and Science Academy	293	John Thacker	NCSSM	33

Division B

Illinois Math and Science Academy	327	Jeff Funk	IMSA	42
Whitman	314	Charles Steinhardt	IMSA	42
St. Mark's	310	Eugene Davydov	SL Park	39

Senior NYCIML:

Fall Term (5 rounds total)

Senior A Division

Team Standings	Score	High Scorers	Team	Score
Tottenville	109	Ming Fan	Seward Park	24
Seward Park	96	Erick Goldberg	Tottenville	24
Francis Lewis	85	Matt Sklar	Tottenville	24
Bryant	78	Weijing Guo	Seward Park	23
Madison	67	Zhi Ming Lin	Seward Park	23
Lincoln	56	Joseph Skylar	Tottenville	22
John Bowne	55	Lei Zhang	Francis Lewis	20
Cardozo	54	Gary Chin	Tottenville	19
Dewey	49	Paul Alexandresco	Cardozo	18
Murrow	45	Derek Ng	Bryant	18
Lafayette	20	Raymond Sung	Francis Lewis	18

Special A Division

Stuyvesant S	131	Leonidas Nguyen	Stuyvesant S	29
Stuyvesant T	112	Johnny Chen	Stuyvesant T	27
Stuyvesant U	90	Ping-Hsin Lee	Stuyvesant S	27
Bronx Science	83	Daniel Stronger	Stuyvesant S	27
Stuyvesant Y	81	Linda Hong	Stuyvesant S	25
Townsend Harris	76	Carl Bialik	Bronx Science	24
Hunter	60	Michael Shy	Stuyvesant T	24
Brooklyn Tech	51	(2 tied)		23

Out of City

Grissom, AL	113	John Vargas	Grissom	25
Redmond, Canada	67	Jacent Tokaz	Grissom	24
Jefferson, VA	47	Jao Ou	Grissom	22

Spring Term

Senior A Division

Tottenville	109
Seward Park	88
Francis Lewis	77
Lincoln	67
Madison (4 rounds)	61
John Bowne	56
Murrow	55
Lafayette	44
Cardozo (4 rounds)	39
Bryant (3 rounds)	36
Dewey (4 rounds)	35
Van Buren (4 rounds)	6

Out of City

Grissom, AL (3 rounds)	63
Redmond, Canada (3 rounds)	37

Special A Division

Stuyvesant S	124	Leonidas Nguyen	Stuyvesant S	27
Stuyvesant T	116	Johnny Chen	Stuyvesant T	26
Bronx Science	92	Ping-Hsin Lee	Stuyvesant S	26
Stuyvesant U	75	Michael Shy	Stuyvesant T	25
Hunter (4 of 5 rounds)	68	Daniel Stronger	Stuyvesant S	25
Stuyvesant Y	66	Linda Hong	Stuyvesant S	24
Brooklyn Tech	53	Sara Cohen	Stuyvesant T	22
Townsend Harris (3 of 5 rounds)	23	Che King Leo	Stuyvesant S	22

Junior NYCIML:

Fall Term (3 rounds total)

JUNIOR DIVISION

Seward Park	72	Jian Qin Chen	Seward Park	18
John Bowne	59	Ying Chen	Seward Park	17
Francis Lewis	57	Steve Choi	Cardozo	16
Cardozo	53	Ling Zheng	Seward Park	15
Lincoln	51	Vlad Zbarsky	Murrow	14
Tottenville	51	Peng Zhang	Francis Lewis	13
Bryant	47	William Chambla	Bryant	12
Midwood	47	Benjamin Chan	Francis Lewis	12
Murrow	43	Carlos Pena	John Bowne	12
Telecommunications	40	Kamfai Wong	Midwood	12
FDR	30	Peter Xia	John Bowne	12
Cleveland	26			
Forest Hills	25			
Madison	25			
Fort Hamilton	24			
Long Island City	11			
Van Buren	10			
Lafayette	9			
Dewey	7			
Sheepshead	5			

SPECIAL JUNIOR DIVISION

Stuyvesant E	78	Sam Frank	Hunter A	18
Hunter A	75	Peter Graham	Stuyvesant E	17
Stuyvesant P	71	Sang-Joon Han	Stuyvesant P	17
Stuyvesant S	71	Andrew Obus	Hunter A	17
Hunter B	59	Soojin Yim	Stuyvesant E	17
Bronx Science	50	Daniel Russo	Stuyvesant P	16
Brooklyn Tech A	48	Edmund Chou	Stuyvesant S	15
Stuyvesant I	45	Regina Eum	Stuyvesant E	15
Stuyvesant L	44	Boris Granovskiy	Stuyvesant P	15
Stuyvesant O	43	Jae Jang	Stuyvesant S	15
Townsend Harris	36	Zardosht Kasheff	Stuyvesant S	15
Brooklyn Tech B	35			
Stuyvesant N	35			

OUT OF CITY

Grissom, AL	62	Jennifer Chang	Grissom	12
HAFTR, NY	44	Avni Shah	Grissom	12
Jefferson, VA	18	Johnny Yang	Grissom	10

Spring Term

JUNIOR DIVISION

John Bowne	68
Francis Lewis	66
Seward Park	57
Cardozo	54
Lincoln	51
FDR	48
Madison	48
Tottenville	45

OUT OF CITY

Grissom, AL	48
HAFTR, NY	41

Midwood (2 rounds)	34
Cleveland	32
Long Island City	27
Murrow (2 rounds)	25
Lafayette	23
Dewey (2 rounds)	19
Fort Hamilton (2 rounds)	19
Sheepshead	10
Van Buren (1 round)	3

SPECIAL JUNIOR DIVISION

Hunter A	77	Luke Stein	Hunter A	17
Hunter B	74	Ross Benson	Stuyvesant S	16
Stuyvesant E	74	Sam Frank	Hunter A	16
Stuyvesant P	67	Elizabeth O'Conner	Hunter	16
Stuyvesant S	66	Kunal Sanghvi	Bronx Science	16
Bronx Science	62	Kirsten Wickelgren	Stuyvesant P	16
Stuyvesant L	60	Soojin Yim	Stuyvesant E	16
Brooklyn Tech A	56	(9 tied)		15
Brooklyn Tech B	54			
Stuyvesant I	54			
Stuyvesant O	47			
Stuyvesant N	25			

New York Math League (NYML)

Leading Team Scores (5 rounds total)

Stuyvesant HS	179	John Kelner	Wheatley School	36
Horace Mann HS	174	Davesh Maulik	Roslyn HS	36
Bronx HS of Science	170	Daniel Stronger	Stuyvesant HS	36
Hunter CHS	168			

Atlantic Pacific Math League (AtPac)

Leading Team Scores (5 rounds total)

SENIOR DIVISION

Lexington HS, MA	300
Stuyvesant HS, NY	300
David Thompson, S.S., Can.	291

JUNIOR (9TH GRADE) DIVISION

David Thompson, S.S., Can.	298
E.R. Hicks Sch., MD	295
Stuyvesant HS, NY	292

9th Grade Continental Math League (CML)

Leading Team Scores

Hunter College HS, NY	177
Stuyvesant HS, NY	177
Naperville North NS, IL	175
Punahou School, HI	175
Roxbury Latin	175

ARML Power Question

Leading Team Scores (2 rounds total)

Stuyvesant HS, NY	72
Vestavia HS, AL	54
Central HS, MN	53

NYSML Results

Team Standings	Score
1. New York City A	139
2. Nassau A	91
3. Suffolk A	79
4. New York City Y	75
5. Niagara A	69
6. New York City C	68
7. New York City N	61
8. Onondaga A	59
9. Nassau B	53
10. Monroe A	47
11. (tie) Niagara C	45
Suffolk B	
13. Ithaca	41
14. Niagara B	38
15. (tie) Met. Cath.	37
Monroe B	
17. (tie) Gen. Val. B	35
Rockland A	
19. Buffalo	33
20. (tie) Onondaga B	31
South Tier	
21. Duso	30
22. (tie) Gen. Val. A	26
Rockland B	

High Scorers	Team	Score
1. Leonidas Nguyen	NYC A	7
2. Ping Hsin Lee	NYC A	7
3. Carl Bialik	NYC A	6
4. James Chen	NYC B	6
Sara Cohen	NYC A	6
Davesh Maulik	Nassau A	6
Daniel Stronger	NYC A	6

Ping Hsin Lee   Leo Nguyen

The Math Team Mystery Series #1: The Mystery of the Missing Cows

Sharon Ace and Ken Cholegi

Everybody booed at the question: "Farmer Bob had eighteen cows, and after slaughtering one of them, the rest disappeared. How many cows had disappeared?" It was a long trip to the American Regional Math Competition (ARMC) and the stuffiness of the bus was getting to the occupants. Mr. Geller looked around, waiting for an answer to the problem.

"Eighteen ..." said Dan, "because a slaughtered cow is no longer a cow ..."

"You're wrong," said Leo. "If we use calculus and an infinite series, we can prove by induction that forty-two cows disappeared."

"No!" said Linda, "three fourths of the time we see that the disappearance of cows is due to a change in gravimetric fluctuations, and so using vector-valued equations and line integrals, we find that the curl is equal to zero, and therefore the potential function is: $f(x,y) = x^2y - \arctan[\sqrt{x^2+y^3}\,]$ ..."

"Nonsense!" interrupted Che King. "Since the cows can never truly disappear because they are always somewhere in the universe, the answer is zero."

"Isn't the answer seventeen?" asked Ping innocently, and with that everybody laughed at him.

* * * * * * * * * * *

The bus chugged along and the math team sang "$-100e^{i\pi}$ Bottles of Beer on the Wall." Finally, the bus reached the glowing campus of Penn State, home of the great Russian math genius Alex "Sasha" Khazanov.

As the bus entered the campus, everybody stared in awe at what was in the parking lot. There they saw eighteen grand bovines roaming around looking for grass. As everybody got off, the campus security guard greeted them. "Welcome!" said the guard, whose name tag identified him as Sasha. "For those of you who don't know, this year the American Regional Math Competition is giving out these eighteen lovely cows as a prize."

Everybody filed into the parking lot and began haggling over rooms. The bus pulled out and the parking lot was empty with the exception of the math team members. Then everybody went to their rooms.

Later, as the math team was tossing Frisbee in the parking lot, Sasha came up to them and shouted nasally, "All of you back at the scene of the crime, eh? Come with me for questioning. I lead one cow away to the prize room and then come back to find that all the other cows are missing!!!"

"This is very suspicious," commented Johnny. "I saw a McDonald's truck driving by a minute ago and now all the cows are missing. Coincidence?"

"It couldn't be," Vinci said. "McDonald's uses horse meat, not cow meat."

"Now listen here," Sasha demanded. "I want those cows and if I don't get them back I'm going to rearrange all your relay teams."

"No!!!" screamed Michael, who thought he was on a very good relay team.

"Nevertheless, that will happen unless I get those cows back by sundown," said Sasha, "You know, in Russia stealing cows is a worse offense than stealing vodka! I'm holding you all personally responsible for those cows!!!" And with that, the security guard stomped off.

"Well, hello-o! That gives us hardly any time to find those cows," said Tzyy. "How are we ever going to find them?"

The wind shifted and a piece of paper wafted through the air and landed on the floor. It had the distinct stench of cow manure. Sara spotted it at once. She ran over and picked it up. "Hey guys, look at this! I think this might be a clue to the disappearing cows!"

Everyone gathered around Sara to read the piece of paper:

The ARMC cows are gone
Your good names are slashed,
And in two more short hours
Your relays will be trashed.

Over 200 kids will mourn
As liver prices go sky high
(Plus your T-shirt design,
Will make small children cry).

There hasn't been such a tragedy
Since 1902
And now 95 years later
It all comes down to you.

All eighteen of you must find
That O so cruel thief,
Wrack your brains hard
To retrieve your great beef.

Every morning those forty or so minutes
Have made you a class,
And now you must work together
To kick the thief's butt.

Your very first clue,
If you have any will or dare,
Is the third smallest number,
Which can be expressed at best the sum
of three positive squares.†

†That cannot also be expressed as the sum of only one or two squares.

Hyun suddenly said, "As *Math Survey* editor, first I'll distribute these copies of *Math Survey* for you to sell. Pick them up now!" And so everybody picked up ten copies of the magazine and looked at them.

"Look! A sum of squares article!" exclaimed Regina.

"Yes," Hyun replied. "It was sent in anonymously."

"Maybe this article will help us understand the clue," suggested Abe, who no longer comes to math team, but Sharon thinks he's cool and wanted to fit him into the story anyway.

"I've got it!" exclaimed Venetia, who had already read the article. "This is the answer!" She pointed at the ARMC program of events. Everybody did the calculations and confirmed Venetia's result.

"But how can we be sure this is it?" conjectured Yvette. "Maybe we should just follow the scent of cow manure ..."

"No!!!" screamed Mike, and everybody looked at him rather strangely.

Although a lot of people wanted to follow the manure, the overwhelming majority decided to go along with Venetia.

"Hey!" exclaimed Vinci, "This program of events says relay questions start in fifteen minutes! Let's hurry there now!"

They reached the street where the ARMC was taking place with plenty of time to spare. The room was located on the top floor of the building, at the end of a long creaky staircase.

"Well, here we are," Aleksey said when they reached the room. He was especially tired from the climb since he was carrying seventeen bottles of Jolt.

"Is this the right place?" asked Peter. "It doesn't look right."

"Well, that's what it says on the 'Welcome' program ..." said Lev.

"Hey," said David. "Doesn't the handwriting on this program look strangely familiar? And isn't the competition supposed to be tomorrow?"

"Don't be sane." said Zvi, "The probability that the 'Welcome' program is a fake is 99.44% likely to be less than the probability that one gets dealt a royal flush from a deck of 52 cards ..." (What is the probability then, if this was true?)

All of a sudden, a gleaming figure appeared in a puff of smoke. Everybody turned and squinted as the smoke cleared. Yann-Bor was the first to recognize him. "Michael Develin! Were you the one behind this all along?!"

"No!" said Michael Develin, a well-known former Math Team captain of a high school on Chambers Street in New York, "The one who stole the cows was somebody else from this story, sneaking around while everybody else was getting their keys, and that (s)he loaded them up on a bus and is now driving back to New York with them!!!"

"And you saw who it was?" asked Dennis.

"I did," Develin confirmed. "In fact, those problems on the desk are not really relay questions, but actually problems I made up to lead you to the culprit. If you put together the solutions to these problems plus the number of the street of this building, you will get numbers that can be translated to the alphabet by the system $a = 1$, $b = 2$ and so forth. Unscramble this anagram and you'll realize who took those cows!"

Leo picked up the problems on the desk and started to read the contents of the sheet. "This is an unusual relay. In fact, it isn't a relay."

"No!!!" screamed Mike.

Use the answers to the following problems to find the thief's name. Let h stand for the number of this building. Unscramble the number to find the letters of the name of the thief!

1. From a yards of fencing Farmer Bob builds seven congruent hexagonal fenced in plots with sides of length one yard each. The plots are placed such that six hexagons share one side with exactly three other hexagons, and the other one shares a side with all the other six hexagons. With the remaining fencing, he is able to build a smaller hexagon fencing in each of the seven larger hexagons with area $\left(3\sqrt{3}\right)/8$. Find a.

2. In a card game, all thirteen spades are removed from a full deck before dealing three-card hands. Let m/n represent the probability that one is dealt a three of a kind, where m and n are relatively prime. Let $p = [0.1n]$, where $[x]$ is the greatest integer function. Compute p.

3. The diagonal of a rectangle has length $3\sqrt{10}$. Given that the sides of the rectangle are m and $\sqrt{n}$, where m and n are relatively prime positive integers, and $m+n$ is as small as possible, compute y, the product of m and n.

4. The scrambled name is in the palindrome $(h)(a)((p)(p)\text{-}(y))$, where the letters A through Z are represented by the integers 1 to 26, respectively.

"Well," Develin said, "I hope you are worthy of my challenge. I will leave you now!" Then he disappeared as quickly as he had come, in a puff of smoke.

"Wow," commented Johnny, "they really don't give you much to do at Harvard, do they?"

"Well, who is it?" asked Tzyy.

"I know!" exclaimed Dan. "It's ... wait a minute, did Develin say that we're in a story?!"

Then everybody screamed: "Shut up and tell us who stole those cows!!!"

"Huh?"

* * * * * * * * * *

Meanwhile, far away in a land called New York, seventeen cows and a happy and familiar-looking bus driver sat on a bus stopped in traffic. "So," smiled the bus driver, "if there are eighteen people at ARMC and one of them hops on a bus, steals seventeen cows, and the other seventeen people are left behind, then how many people are stranded at the Penn State campus?"

Everybody mooed at the question.

Stay tuned for the next episode of the Math Team Mystery Series #2: The Mystery of the Missing Math Team Members- They went to ARMC, and never came back! What happened?!

Solutions:

The poem: The smallest squares are 1, 4 and 9. The smallest sum of three squares is 3 = 1+1+1, then 6 = 1+1+4, then 9 = 1+4+4, and finally 11 = 9+1+1 Since $9 = 3^2$, 9 is not at best the sum of three squares, but actually only one square, so h = 11.

Relay 1: Drawing a diagram, we find 30 one yard lengths used for the larger hexagons. The seven smaller hexagons have perimeter 1 each since the area of each hexagon is $6\left(\frac{p^2}{36}\right)\cdot\frac{\sqrt{3}}{4}$ in terms of the perimeter, and so $p = 1$ and the total extra fence used was $7\cdot 1 = 7$. Hence, a = 30+7 = 37.

Relay 2: $m/n = (39/39)(2/38)(1/37) = 1/703$ Hence, $n = 703$, and $p = [0.1(703)] = 70$

Relay 3: $m^2+n = 300$. Listing the ordered pairs, (m,n) we have (1, 299), (2, 296), (3,291) ... (16, 44) and (17,11). The last pair have the smallest sum. Therefore, $y = 17\cdot 11 = 187$.

The culprit: Hence $h\cdot a\cdot(p\cdot p\text{-}y) = 1918191$ and this can be split into the letters S, A, H, S and A. Therefore, our culprit is the security guard, Alex Sasha Khazanov!

Alternate Solution: The letters are S, R, A, I, and A. Though unknown to most people, Sara's middle initial is I, therefore the culprit is Sara I. Cohen.

The Day of the Descartians

Tommy Wu

The Day of the Descartians came unexpectedly. The day had proceeded normally, with the usual armed robberies, presidential assassinations, and terrorist hijackings. The Descartians arrived in their sleek spacecraft and hovered over each of the planet's major cities. Commandeering every radio and television station, the Descartians made their fateful announcement. "Puny citizens of Earth! We are emissaries from the planet Descarti. Our mission is to purge this galaxy of every planet on which the dominant life form does not meet our standards of intelligence. To this end, we request that you choose someone to represent your species and bring him to the point of highest elevation on your planet at noon tomorrow. We will transport him to our ship to test his reasoning skills. Have a nice day!" The leaders of the world were appalled for several minutes. However, they quickly came to their senses and called a meeting at the United Nations. There, they immediately decided who the representative would be. Without a doubt, it had to be Alex Kazzy, the world-renowned Russian mathematician. At noon the next day, they flew him to Mount Everest. The Descartians teleported Alex onto their main ship and placed him in an empty testing room. An invisible speaker said to the surprisingly calm Russian, "Welcome, pathetic Earthling. We will now begin the administration of the exam. You will be given three puzzles to solve. If you answer all three correctly, then the inhabitants of your planet will be deemed intelligent enough to survive. However, if you answer any of the puzzles incorrectly, then your planet will be judged as below standard and will be instantly vaporized. The fate of Earth is in your hands. No pressure, though." Alex Kazzy smiled confidently and said, "Ready when you are." The Descartian began, "A man was looking at a portrait. Someone asked him, 'Whose picture are you looking at?' He replied, 'Brothers and sisters have I none, but this man's father is my father's son.' ('This man' refers, of course, to the man in the picture.) Whose picture was he looking at?" Coolly, Alex answered, "The man was looking at a picture of his son. Since the man has no siblings, 'my father's son' must refer to himself. Since he calls the father of the man in the picture 'my father's son,' the father of the man in the picture must be himself. Hence, the picture must be of his son." The alien responded sarcastically, "I'm impressed. Next puzzle: "The year is 1959. Two days ago, a boy was ten years old; next year he will be thirteen. What is the date today and when is his birthday?'" With a grin, Alex replied, "Today's date must be January 1, 1959. His birthday was on December 31, 1958." Disgruntled, the Descartian put forth the final question: "A man is rowing across a river in the gathering dusk. On the approaching bank he sees the dim figures of three men. 'Some' of these men wear red and 'some' white, and the oarsman wishes to know which wear red and which wear white, so he calls out, 'What colors are you wearing?' The reply from the first dim figure is lost in the slight breeze, but the second man calls out, 'He's wearing red, but I am wearing white!' The third figure, referring to the first man, says, 'He says he's wearing white and he is wearing white!' With this information, and with the knowledge that the men wearing red always lie, and the white-dressed men always tell the truth, identify the figures for what they are really wearing."

* * *

STOP! CAN YOU SOLVE THE DESCARTIANS' FINAL PUZZLE? THE EARTH'S DESTINY HANGS IN THE BALANCE! IF IT'S TOO HARD FOR YOU, JUST BE GLAD

THAT YOU'RE NOT ALEX KAZZY AND KEEP READING FOR THE EXCITING CONCLUSION OF "THE DAY OF THE DESCARTIANS"!

For the first time, Alex seemed a bit unsure of himself. Beads of sweat began to form on his forehead. "Well?" prompted the smug Descartian. "Is it too difficult for poor baby Alex? Looks like it's curtains for Earth!"

"No!" shouted Alex triumphantly. "I have the answer!"

"You do?" asked the alien in disappointment.

"I certainly do." replied Alex. "The first man, whose reply was lost, could only have said one thing when asked what color he was wearing. If he had been wearing white, he would have told the truth and said that he was wearing white. If he had been wearing red, he would have lied and said that he was wearing white. Therefore, the first man was in white, the second man was in red, and the third man was in white." Grudgingly, the Descartian said, "Very well, Alex Kazzy, you have passed our intelligence test. Your worthless race is not as moronic as it seems. Accordingly, your planet will be spared from destruction." As the alien teleported Alex back to the surface of the planet, the crestfallen Descartian muttered, "Aw, and I was looking so forward to blasting his planet to pieces. . . ."

After the last Descartian ship left Earth, the planet's denizens hailed Alex as their hero. All around the world, ticker tape parades were held in his honor. Alex's birthday was made an international holiday, called Kazzy Day. The Queen of England offered to make him heir to her throne. Modestly, Alex simply retired to his mansion in Russia to spend the rest of his days contemplating Godelian logic and conjugacy problems.

Word Search

Brandon Irizarry

F	D	O	N	W	O	R	B	L	A	C	I	G	O	L	O	E	O	T
W	I	G	S	C	A	L	I	N	G	D	E	N	S	I	T	Y	R	U
H	O	R	G	E	O	M	E	T	R	I	C	A	L	A	K	A	A	L
I	R	E	S	C	A	L	I	N	G	N	O	I	S	E	N	K	L	L
T	H	Y	E	T	S	T	A	B	O	Y	B	O	K	S	O	I	P	Y
E	P	O	I	G	O	R	I	O	S	R	O	R	C	C	C	D	H	L
N	E	O	N	M	H	U	E	N	O	O	F	E	R	A	I	N	A	A
O	N	N	N	A	A	C	T	W	A	A	N	Z	R	L	M	E	N	T
E	M	T	O	A	R	G	N	E	L	D	A	O	I	I	H	Y	U	N
T	C	U	O	E	I	I	I	D	R	U	L	E	C	N	T	F	M	E
I	D	M	I	P	A	N	N	N	D	I	A	H	K	G	I	E	B	D
N	I	P	A	V	A	M	W	O	A	L	N	C	O	N	R	M	E	N
I	L	C	U	T	I	P	A	O	N	R	O	N	H	O	A	A	R	E
F	I	M	E	S	C	R	L	T	R	U	I	D	E	I	G	T	U	C
S	M	I	O	E	D	A	D	A	H	B	T	U	X	R	O	S	L	S
N	A	T	I	C	L	G	L	A	S	S	C	O	S	I	L	O	E	N
A	O	E	R	O	A	E	L	L	U	S	A	F	A	N	L	A	R	A
R	I	N	A	N	K	M	F	D	O	Q	R	T	V	T	T	E	S	R
T	H	I	R	D	F	O	U	R	T	H	F	I	F	T	H	A	H	T

1. This type of sound, called a _____ by Benoit Mandelbrot, is quite unique among sounds. Most sounds change timbre when played at different speeds, as does this type of sound; however, this particular sound, when played at a different speed, can sound exactly as it did before if only the volume is adjusted.
2. The simplest example of clue one is called, in electronics and information theory, "_____ noise," or Johnson noise.
3. A more complicated type of clue one is characteristic of a type of motion called _____ motion, which is itself characteristic of a random walk.
4. As a result, clue four is called _____ music or noise, following Richard F. Voss.
5. Clue two is said to have a _____ of $1/f^0$, and clue four a _____ of $1/f^2$.
6. Mandelbrot coined the term "fractal" because he assigns to each fractal curve a _____ dimension ...
7. ... which is greater than its _____ dimension.
8. This philosopher-scientist-mathematician recognized, before 1900, the great value of elementary topology. Just look for the last name.
9. The number i is so named because it is the first letter of the Latin word _____.
10. For a given machine, _____ is a well defined irrational number between zero and one that is the machine's halting probability on a random program.
11. Numbers like w, $w+1$, $w+2$, and so on are called standard _____ numbers.
12. FOIL stands for these words: _____.
13. This ordinal number comes from the Latin word for "following": _____.
14. The sine cannot be expressed by any algebraic equation of finite degree; for this reason the sine curve is spoken of as "_____".
15. The simplest example of a caustic curve is the _____, an algebraic curve of degree six.
16. This spiral has a tracing point distance r from the origin and an angle θ measuring the amount of times that point has moved around the origin, such that $r = 2^a$: _____

(continued on the following page)

Number Maze

Linda Lau

This puzzle is based on another puzzle by Sam Lloyd. The goal of the puzzle is to work your way out of the number circle. You start at the number "4" in the center, and in as many steps in any of the eight directions as the number you started on. For example, since you start on the center four, you can either move four cells up and land on a "1", or perhaps in a northeast direction to land on a "4", for example. You repeat this until you land on an exact number that permits you to land right outside the circle. Good luck!

```
                  2 3 7
            3 3 4 5 6 3 4 5 4
        6 9 2 5 1 4 5 4 4 1 9 3 8
      4 7 8 1 9 5 4 5 6 8 8 7 4 9 8
    2 5 9 7 6 8 4 2 2 2 2 1 2 7 9 5 8
    3 8 7 5 9 1 1 3 1 9 9 4 1 7 6 1 3
  2 5 5 6 4 5 2 3 2 2 2 2 9 5 1 6 2 4 3
3 5 1 7 1 8 9 4 1 3 4 1 1 4 4 3 4 2 4 5 6
4 5 5 2 6 7 8 2 7 3 4 2 3 2 2 3 6 5 1 1 7
2 7 8 5 3 3 2 1 1 6 1 4 5 2 3 7 6 1 1 3 2
  7 9 2 3 1 3 2 6 2 1 3 1 5 2 2 5 2 4 4
  3 8 6 1 1 4 3 3 5 5 7 8 8 4 9 7 6 5 1
  2 5 5 1 4 7 5 2 1 4 6 4 3 1 2 2 1 4 2
    3 4 7 5 6 1 1 2 4 3 6 7 8 1 9 5 5
    4 5 7 8 8 9 8 7 4 3 6 3 9 7 8 8 4
      3 7 1 5 1 1 5 1 4 5 1 5 6 5 3
        5 4 4 9 1 4 9 5 4 1 8 3 2
            2 6 4 3 3 3 4 5 2
                  8 8 2
```

17. A three dimensional curve parametricized by a line and a circle (for it has both direction and circularity) is a _____.
18. Pi appears in the Bible to this many decimal places: _____.
19. Plato's _____ consisted of arithmetic, music, geometry, and astronomy.

MATH SURVEY

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