Marbles

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Question

Johnny has a bag of marbles with n red marbles and n blue marbles. He keeps drawing marbles from the bag randomly without replacement, and stops when he has drawn an equal number of red and blue marbles. What is the probability that he goes through the entire bag (draws all $2n$ marbles) before stopping?

Solution

Answer: $\frac{1}{2n-1}$

Without loss of generality let the first marble that Johnny draws be blue, and assume that Johnny goes through the entire bag. Then the last marble Johnny draws must be red, as otherwise he would have went from having drawn more blue marbles than red marbles to having drawn more red marbles than blue marbles, so at some point he must have had an equal number of red and blue marbles and stopped drawing. Let B be a count of the number of blue marbles he has drawn and R be a count of the number of red marbles he has drawn, and every time Johnny draws a marble, plot (B, R) in the coordinate plane.

He starts at $(1, 0)$, and in order to go through the entire bag, he must reach $(n, n-1)$ at the end, and $R < B$ for all points (B, R) (since we can't have $B = R$ and $R < B$ at the start). The number of ways for this to happen is the number of up-right paths from $(1,0)$ to $(n, n-1)$ that lie weakly below the line $R = B - 1$, which is the $(n-1)$th Catalan number, $\frac{\binom{2n-2}{n-1}}{n}$.

Thus, there are $\frac{2*\binom{2n-2}{n-1}}{n}$ sequences of draws such that Johnny goes through the entire bag (remember we assumed the first marble was blue, but the first marble could also be red). There are $\binom{2n}{n}$ possible sequences of draws, so the answer is $\frac{2*\binom{2n-2}{n-1}}{(n*\binom{2n}{n})} = \frac{1}{2n-1}$.

References

"A Marble Falling into Water 03 by mimicry94 on DeviantArt." By mimicry94 on DeviantArt, www.deviantart.com/mimicry94/art/A-marble-falling-into-water-03-358339339.