Math in Nature

Alex Zheng

1 Introduction

It's half-past twelve and you're still working on that geometry homework due tomorrow when a thought pops up: "What is the point of math? How is this gonna help me?" It's common knowledge that math can be utilized in different professions. Math can also be applied to different parts of daily life the quicker you're able to count change from your wallet, the less likely you're holding up a line behind you. However, in addition to the mundane day to day applications, math can also be observed within nature. Mathematicians have used math to explain nature for hundreds of years. Whether the hidden math in the orbits of celestial bodies or the seemingly random arrangement of flower petals, math is everywhere around us. It just takes an acute mind to notice and see the patterns.

2 Fibonacci Sequence

The Fibonacci Sequence is one of the most famous sequences in the world. The sequence first appeared in the book *Liber Abaci*, written by Italian mathematician Fibonacci (whose real name was Leonardo of Pisa). This sequence is commonly denoted as:

$$F_0 = 0, F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2}$$

for $n \geq 2$

Thus, the sequence goes: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$ where the sums of the two previous numbers generate the next number in the sequence. This may seem like a useless string of numbers, but it's the spiral that forms from this sequence that keeps popping up again and again in nature — Fibonacci's spiral. The spiral can be created on a 2D plane using the squares of the numbers in the Fibonacci sequence. Take two squares with lengths 1, add another square with lengths 2, then add another square with lengths 3. Continue this pattern while drawing a continuous arc from the corners of each square. This creates the famous Fibonacci spiral, which is constantly observed by mathematicians in nature.

From the pattern used to create the spiral, we notice an interesting pattern between the Fibonacci sequence and the squared version of the sequence. The product of two consecutive Fibonacci numbers is equal to the square of the first consecutive Fibonacci number plus the square of the previous Fibonacci numbers before it.

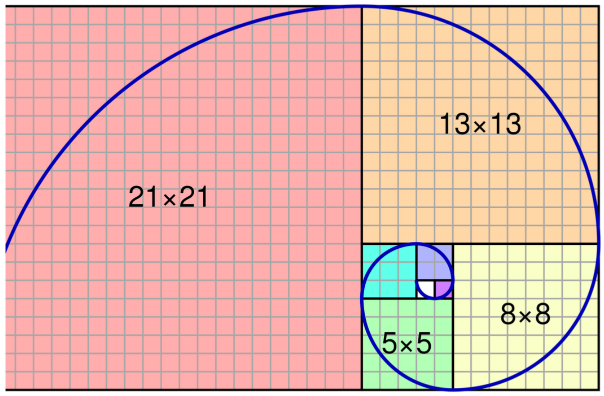

Figure 1: Drawing a Fibonacci spiral from the sequence

$$\text{Sequence: } 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$
$$\text{Squared: } 1, 1, 4, 9, 25, 64, 169, 441, 1156, 3025, \ldots$$
$$1 + 1 + 4 = 6 = 2 \cdot 3$$
$$1 + 1 + 4 + 9 = 15 = 3 \cdot 5$$
$$1 + 1 + 4 + 9 + 25 = 40 = 5 \cdot 8$$
$$1 + 1 + 4 + 9 + 25 + 64 = 104 = 8 \cdot 13$$

Using this information, we are able to form the conjecture:

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$$

Proof. We wish to prove that

$$F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}$$

We will proceed by induction.

Base Case: $n = 1$

$$F_1^2 = F_1 \cdot F_{1+1}$$
$$1^2 = 1 \cdot 1$$

Base case holds.

Induction Hypothesis: Assume that

$$F_1^2 + F_2^2 + F_3^2 + \cdots + F_k^2 = F_k F_{k+1}$$

Induction Step: Consider

$$F_1^2 + F_2^2 + F_3^2 + \cdots + F_k^2 + F_{k+1}^2$$

by the hypothesis,

$$= F_k F_{k+1} + F_{k+1}^2$$

$$= F_{k+1}(F_k + F_{k+1})$$

Using the Fibonacci sequence definition, we know that $F_k + F_{k+1} = F_{k+2}$ so we get

$$F_1^2 + F_2^2 + F_3^2 + \cdots + F_k^2 + F_{k+1}^2 = F_{k+1}F_{k+2}$$

as desired. $\square$

Interestingly, we can derive the second most famous irrational number using the Fibonacci sequence. When we reverse the sequence above and divide a Fibonacci number by the one before it, we're able to obtain the golden ratio. How close the number is to the actual Golden Ratio is dependent on how large the Fibonacci numbers are.

$$3/2 = 1.5$$

$$5/3 = 1.67$$

$$8/5 = 1.6$$

$$13/8 = 1.625$$

$$21/13 = 1.615$$

$$34/21 = 1.619$$

$$55/34 = 1.6176$$

$$89/55 = 1.61818$$

3 The Golden Ratio

The Golden Ratio is an irrational number commonly denoted with the Greek symbol phi (ϕ). It is also referred to as the divine proportion because the ratio ϕ is the most pleasing ratio to view. It's said that ancient Greek mathematicians first noticed the ratio but it was Euler who first described it as: "A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the lesses." Euclid said that an object is considered to be in the golden ratio when the sum of the two lengths over the longer side is equal to the longer side over the shorter side. This is written as:

$$\frac{a+b}{a} = \frac{a}{b} = \phi$$

Using Euler's definition of the Golden Ratio, a formula for the Golden Ratio was derived using the quadratic formula. The quadratic formula is used to derive a, which is then used to find the ratio between a and b.

$$b(a+b) = a^2$$

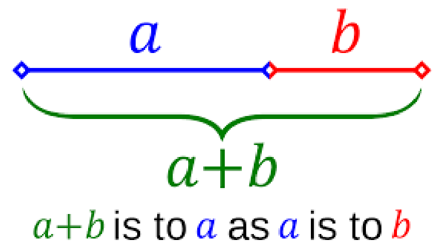

Figure 2: Euler's definition of the Golden Ratio

$$a^2 - ab - b^2$$
$$\frac{-b \pm \sqrt{(-b)^2 - 4(1)(-b)^2}}{2(1)} = a$$
$$\frac{1 \pm \sqrt{5}}{2} b = a$$
$$\phi = \frac{1 + \sqrt{5}}{2}$$

We know that the negative root not chosen because the ratio between lengths a and b isn't negative.

This is the most popular method for finding the Golden Ratio, but there are other methods to finding this value. Below is another more interesting way to write the Golden Ratio.

$$5^{0.5} \times 0.5 + 0.5$$

Since the Golden Ratio can be derived from the Fibonacci sequence, many things in nature can be observed in tandem with both of these numbers. The most prominent example are plants. The numbers of petals on a flower is usually a Fibonacci number. Lilies have 3 petals, buttercups have 5 and asters have 13. Sunflower seeds are both arranged in sets of Fibonacci numbers and also form multiple Fibonacci spirals. One of the most famous examples is the nautilus shell, which creates a near perfect Fibonacci spiral. This list goes on, from hurricane feeder bands to earthquake fractures and even the micro algae living in a pond. The Fibonacci sequence is found everywhere in nature.

There are, of course exceptions to the Fibonacci sequence and the Golden Ratio. In fact, there are more examples that don't follow this ratio than ones that follow this ratio. Therefore, it can't be precisely pinpointed why the Fibonacci number appears in nature; it's completely possible that this number is not as common in nature as mathematicians thought.

4 Fractals

Another branch of math constantly found in nature are fractals. Fractals are defined as a never ending, self-similar pattern. Zooming into a fractal yields the same result as when you look at the fractal zoomed out. Fractals can be generated by looping computer programs. In a nutshell, fractals optimize certain criterion like volume, surface area, or diffusion. This optimization is what causes fractals to appear again and again in nature. An example of fractals in nature is the creation of rivers. When rainwater flows down from an area of higher elevation, the water carves out river

Figure 3: Nautilus Shell

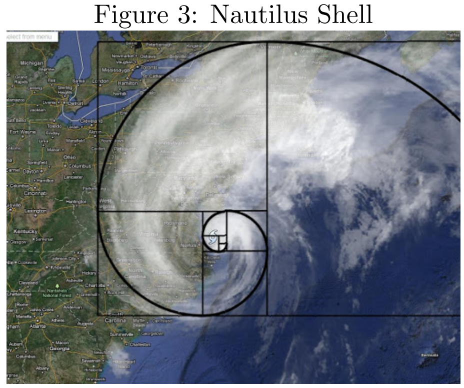

Figure 4: Hurricane Spiral

channels through hundreds if not thousands of years of erosion. River fractals form from this because the water flowing down is constantly "searching" for a way to make it down. The constant flow of water is a repetitive process which creates the fractals seen in the picture below.

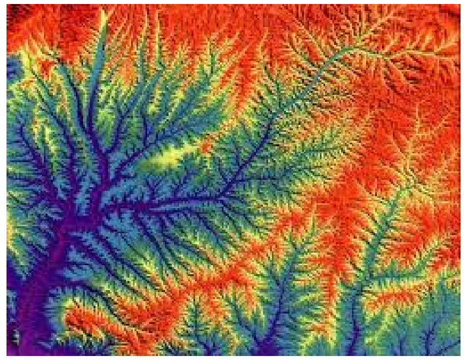

Figure 5: Elevation map of river fractals

Romanesco broccoli is an example of a natural fractal. The surface area of the plant is optimised due to these fractals. The more surface area a plant has correlates to more photosynthesis. More sunlight means more glucose production rate, which results in a better survival rate for the plant. Additionally, fractals are the simplest way for plants to create a structure using the least amount of DNA sequences (fractals repeat themselves). Natural fractals are a result of natural selection because fractals provide plants with the best chance of survival.

Fractals have some extremely fascinating properties, which anyone who has heard of the Coastline paradox may be familiar with. For a fractal's finite area, it has an infinite perimeter. Similarly, a coastline's length can vary drastically based on whether it's measured by intervals of 100 miles or 100 feet. A coastline that is infinite in length seems counter-intuitive, but it makes sense when you consider the existence of fractals along the coast. Every protruding border or cliff is able to increase the perimeter of the coastline.

Figure 6: Romanesco broccoli

5 Hexagons

One of the most prevalent shapes in nature is the hexagon. Why is this six-sided shape observed everywhere in nature? This actually comes down to some of the special geometric properties of hexagons. The most well known example of hexagons in nature is the beehive. As Charles Darwin puts it: “absolutely perfect in economising labour and wax.” Bees use hexagons because of how good hexagons are at tiling a 2D plane. Mathematically, hexagons are the most similar shape to circles that can still be tiled. In 1999, Thomas C. Hales proved in his ”The Honeycomb Conjecture” which stated that hexagons have the highest perimeter to area ratio out of all other shapes; hexagons were the most efficient way to tile an area. It requires the least amount of resources to tile a hexagon pattern. This is important for bees because it takes them 8oz of honey just to create 1oz of wax. Bees actually tile with circles and those circles harden into the ubiquitous hexagon shape because hexagons are the most energy-efficient shape. Architecturally, circles are the strongest shape for containing pressure. This is why tanks of compressed air or fire extinguishers are circular. This property of circles is passed to hexagons, allowing bees to use the hexagon holes as really effective storage. This is one of the many reasons bees chose to adopt the hexagon for their hives; hexagon are perfect for the bees to store pollen, nectar and larvae. This was probably also the product of natural selection in bees.

Figure 7: Beehive

When water freezes and ice forms, the water molecules form lattice shaped hexagons. Since H_2O is polar, the molecule isn't perfectly symmetrical. As seen in figure 6, the angle between two hydrogen atoms on a water molecule varies from 104.5° to 106.6°. Six water molecules is the perfect amount to create the hexagonal ring. The electrical force between adjacent water molecules roughly even out the angle difference. To our eyes, the angle appears to be 120°. This property of water causes ice and snowflakes to be hexagonal.

Hexagons can also be observed in plants. Plant cells are hexagonal. Like in beehives, hexagons are the toughest shape that uses the least resources. Plant cells absorb water and a strong shape is required to withstand the turgid pressure inside the cell.

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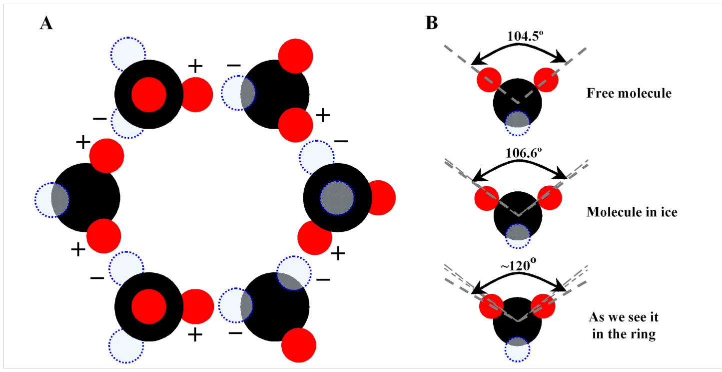

Figure 8: Formation of hexagons with water molecules

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