Orbits

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1 Introduction

Orbits are the motion of extraterrestrial objects, such as planets, moons, and comets, around more massive objects, which can be other planets, stars, and even black holes. We can use mathematical models to predict the motion of objects in orbits. This article will primarily focus on the motion of exoplanets around stars to figure out their properties.

2 Orbit Types

Orbits can appear as many different shapes and sizes. However, in a simple two-body system (only two objects in orbit), orbits can have circular, elliptical, parabolic, and hyperbolic trajectories. Exoplanets orbit in elliptical orbits, while comets orbit in parabolic and hyperbolic trajectories. Comets have a much higher velocity than exoplanets (from conservation of momentum - comets are pieces of exoplanets or moons which have a lot of mass and little speed, so the comet must have a lot of speed for little mass). This reduces the impulse that the gravity has on them, making them able to escape the gravitational pull. For our main objective of exoplanets, elliptical orbits are going to be the main orbit type considered.

3 Ellipse Definition

An ellipse is basically a “flattened” circle, where two points on a circle that are connected to each other by the circle’s diameter are brought closer together, along with all other adjacent points, thus decreasing the diameter in that direction, creating the so-called “flattening” or “squishing” effect. The original circle is often referred to as the ‘auxiliary circle.’ The radius of the auxiliary circle is referred to as the semimajor axis, and the decreased radius is referred to as the semiminor axis.

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The equation of a circle with center at the origin in Cartesian form is:

$$x^2 + y^2 = r^2$$

Using the “flattening” property of an ellipse, we can divide the x and y coordinates by a and b, respectively (to scale both axes) to get the equation of the ellipse:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

More rigorously to the “flattening property”, the mathematical definition of an ellipse is the locus (set) of all points such that the sum of the distances between them and two foci is constant.

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The ellipse equation can also be represented in parametric form, with E as the parameter, which is:

$$x = a\cos(E)$$

$$y = b\sin(E)$$

4 Ellipse Properties

Instead of distance parameters, ellipses can also be characterized with angles. Besides E, being called the eccentric anomaly, ellipses can also be described by ν, called the true anomaly. The true anomaly is the angle formed from the distance from the focus to a point on the ellipse and the semimajor axis. Ellipses are also characterized by their eccentricity e, or a measure of how "squished" or "flattened" the ellipse is in the x direction compared to the y direction. Logically, it is a form of a ratio of a and b, which both describe the amount of squishing in the x direction and y direction, respectively. It is defined as:

$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

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5 Ellipse Constructions

Tangent angles on a star and its radius both influence the visible portion of an exoplanet's orbit and its orbital distance to the star. Two specific angles, β and $\alpha + \beta$, are the angles that will be important later on.

A construction that I will make from an ellipse follows: Create an ellipse and create a circle within it with center of one of the foci and having radius R_*, such that $R_* < a(1-e)$. Next, plot an arbitrary point such that it lies outside of the ellipse and draw a segment from the focus with the circle of radius R_* to that point. The angle that this segment creates with the axis defined by extending a is what I will call ϕ. Then, draw two parallel lines to that segment that are also tangent to the circle. These lines will intersect the ellipse a total of 4 times, but we only care about the intersections in the direction of the segment towards the arbitrary point, not π radians away in the opposite direction.

This construction is a model of two heavenly bodies, one being a star (circle with radius R_*) and another being an orbiting exoplanet. The inequality makes sure that the radius of the star doesn't block the trajectory of the orbiting exoplanet. The arbitrary point is the observer (like a satellite or Earth) and the points of intersection of the tangent lines represent the bounds in the exoplanet's orbit, whose elliptical arc from the two points of intersection is the distance that the observer sees when the exoplanet moves (transits) in front of the star. Angles β and $\alpha + \beta$ are two specific true anomaly (ν) angles that correspond to each intersection point, where β is the angle from the a-axis to the first intersection point and α is the separation angle between the two intersection points (so $\alpha+\beta$ is then the angle from the a-axis to the second intersection point). When the exoplanet does this, it blocks out some light (photon flux) emitted from the star that the observer was initially observing, and can be used to find out properties of the exoplanet. Fittingly, the method is called the exoplanet transit method.

Using the Pythagorean theorem, we can set the magnitude of the difference vector of $\vec{r_{\alpha+\beta}} - \vec{r_\beta}$ equal to $\sqrt{4R_*^2 + h^2}$ like so:

$$a^2(\cos(E_{\alpha+\beta}) - \cos(E_\beta)) + b^2(\sin(E_{\alpha+\beta}) - \sin(E_\beta)) = 4R_*^2 + h^2$$

Rearranging for h, we get:

$$h = \sqrt{a^2(\cos(E_{\alpha+\beta}) - \cos(E_\beta)) + b^2(\sin(E_{\alpha+\beta}) - \sin(E_\beta)) - 4R_*^2}$$

Since the figure is not a circle but rather an ellipse, r_β and $r_{\alpha+\beta}$ are pretty much always unequal (except when $\phi = 0$ or π), meaning we turn the inequality of r_β and $r_{\alpha+\beta}$ into an equality like so:

$$r_{\alpha+\beta} = r_\beta + h\sec\left(\frac{\alpha}{2}\right)$$

Rearranging this equation for α, we get:

$$\alpha = 2\sec^{-1}\left(\frac{r_{\alpha+\beta} - r_\beta}{h}\right)$$

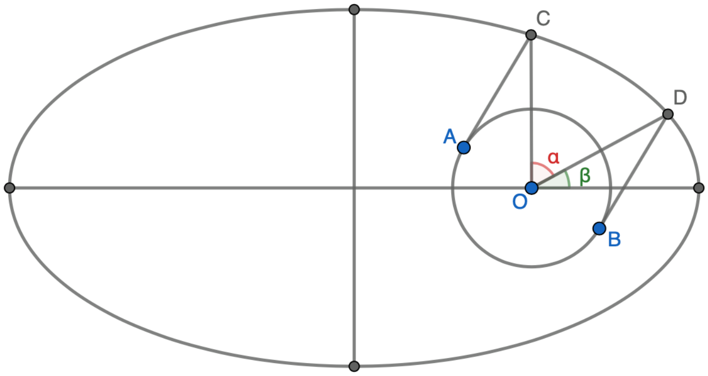

Figure 1: Elliptical orbit with two given distances from focus O (center of star) to two points on the ellipse, $OD = r_\beta$ and $OC = r_{\alpha+\beta}$, with corresponding ν angles β and $\alpha+\beta$. Parallel line segments BD and AC are also drawn from both r_β's and $r_{\alpha+\beta}$'s intersections with the ellipse tangent to circle O.

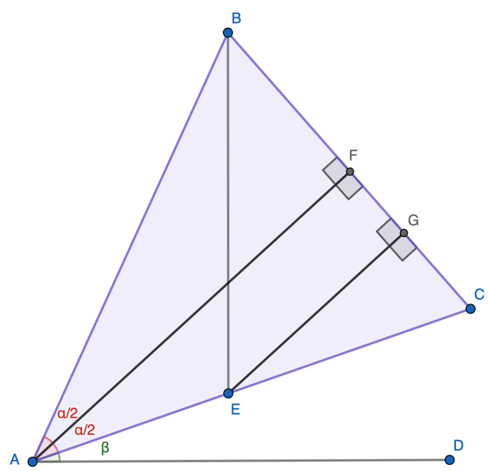

Figure 2: Zoomed in version of isosceles triangle formed with legs $AB = r_{\alpha+\beta}$ and $AE = r_\beta$ extended to equal the length of $AC = AB = r_{\alpha+\beta}$

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From the Pythagorean Theorem we get:

$$r = \sqrt{R_*^2 + p_i^2}$$

r here is the distance from the focus with the circle with radius R_* and p_i is the tangent distance from the circle to the intersection point on the ellipse. We can split up p_i as follows:

$$p_i = R_* \tan\left(\frac{\pi}{2} - \phi\right) + p_n$$
$$p_i = R_* \cot(\phi) + p_n$$

p_n here is the distance that is separated from $R_*\cot(\phi)$ by the a-axis to the tangent intersection point on the ellipse. We can determine an equation for p_n geometrically like so:

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Figure 3:

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Figure 4: Zoomed in version of Figure 3

$$p_n = \frac{b\sin(E)}{\sin(\phi)} = b\sin(E)\csc(\phi)$$

Substituting this equation for p_n into the equation for p_i, we get:

$$p_i = R_*\cot(\phi) + b\sin(E)\csc(\phi)$$

Squaring this equation to ultimately get p_i^2 to substitute into the equation of r we get:

$$p_i^2 = R_*^2\cot^2(\phi) + b^2\sin^2(E)\csc^2(\phi) + 2R_*b\sin(E)\cot(\phi)\csc(\phi)$$

Finally, substituting into the equation of R_* we get:

$$r = \sqrt{R_*^2 + R_*^2\cot^2(\phi) + b^2\sin^2(E)\csc^2(\phi) + 2R_*b\sin(E)\cot(\phi)\csc(\phi)}$$

Using the trigonometric identity $1 + \cot^2(\phi) = \csc^2(\phi)$ we can simplify like so:

$$r = \sqrt{R_*^2\csc^2(\phi) + b^2\sin^2(E)\csc^2(\phi) + 2R_*b\sin(E)\cot(\phi)\csc(\phi)}$$

We can see how r changes with respect to both ϕ and R_* by taking partial derivatives with respect to each like so:

$$\frac{\partial r}{\partial R_*} = \frac{1}{2} \cdot \frac{1}{\sqrt{r}} \cdot (2R_* \csc^2(\phi) + 2b\sin(E)\csc(\phi)\cot(\phi))$$

$$\frac{\partial r}{\partial R_*} = \frac{R_* \csc^2(\phi) + 2b\sin(E)\csc(\phi)\cot(\phi)}{\sqrt{r}}$$

Similarly, for ϕ:

$$\frac{\partial r}{\partial \phi} = \frac{1}{2} \cdot \frac{1}{\sqrt{r}} \cdot (-2R_*^2 \cot(\phi)\csc^2(\phi) - 2\cot(\phi)\csc^2(\phi)b^2\sin^2(E) - 2R_* b\sin(E)\csc(\phi)(\csc^2(\phi) + \cot^2(\phi)))$$

$$\frac{\partial r}{\partial \phi} = \frac{-R_*^2 \cot(\phi)\csc^2(\phi) - b^2\sin^2(E)\cot(\phi)\csc^2(\phi) - b\sin(E)\csc(\phi)(\csc^2(\phi) + cot^2(\phi))}{\sqrt{r}}$$

$$\int_{E_\beta}^{E_{\alpha+\beta}} \sqrt{a^2\cos^2(E) + b^2\sin^2(E)}\, dE$$

The expression above is the equation for the arc length (portion of the whole orbit) of the traversed exoplanet in its elliptical orbit. Mathematically speaking, the integral sums up all the infinitesimal arc portions for every infinitesimal change in E from angles E_β up to $E_{\alpha+\beta}$. We can differentiate this with R_* and ϕ to see how this arc length visible to the observer changes if we increase/decrease the radius of the star and if we change our location relative to the star.

First, to differentiate by R_*, we can use Leibniz's rule, like so:

$$\frac{d}{dR_*} \int_{E_\beta}^{E_{\alpha+\beta}} \sqrt{a^2\cos^2(E) + b^2\sin^2(E)}\, dE = \int_{E_\beta}^{E_{\alpha+\beta}} \frac{\partial}{\partial R_*} \sqrt{a^2\cos^2(E) + b^2\sin^2(E)}\, dE$$

Using what we have derived as r, we can determine the value of $\sqrt{a^2\cos^2(E) + b^2\sin^2(E)}$ like so:

$$\sqrt{a^2\cos^2(E) + b^2\sin^2(E)} = \sqrt{\sin^2(\phi)(r^2 - R_*\csc^2(\phi) - 2R_* b\sin(E)\cot(\phi)\csc(\phi)) + a^2\cos^2(E)}$$

$$= \sqrt{\left(1 - \frac{1}{1-e^2}\right)\left(r^2\sin^2(\phi) - 2R_* b\sin(E)\sin(\phi)\cot(\phi) - R_*^2\right)}$$

We can now substitute this new value into the integral like so:

$$\int_{E_\beta}^{E_{\alpha+\beta}} \frac{\partial}{\partial R_*} \sqrt{\left(1 - \frac{1}{1-e^2}\right)\left(r^2\sin^2(\phi) - 2R_* b\sin(E)\sin(\phi)\cot(\phi) - R_*^2\right)}\, dE$$

$$= \sqrt{\left(1 - \frac{1}{1-e^2}\right)} \int_{E_\beta}^{E_{\alpha+\beta}} \frac{\partial}{\partial R_*} \sqrt{\left(r^2\sin^2(\phi) - 2R_* b\sin(E)\sin(\phi)\cot(\phi) - R_*^2\right)}\, dE$$

We can do the entire process in a similar fashion to differentiate ϕ as we did with R_* like so:

$$\sqrt{\left(1 - \frac{1}{1-e^2}\right)} \int_{E_\beta}^{E_{\alpha+\beta}} \frac{\partial}{\partial \phi} \sqrt{\left(r^2\sin^2(\phi) - 2R_* b\sin(E)\sin(\phi)\cot(\phi) - R_*^2\right)}\, dE$$