MATH SURVEY

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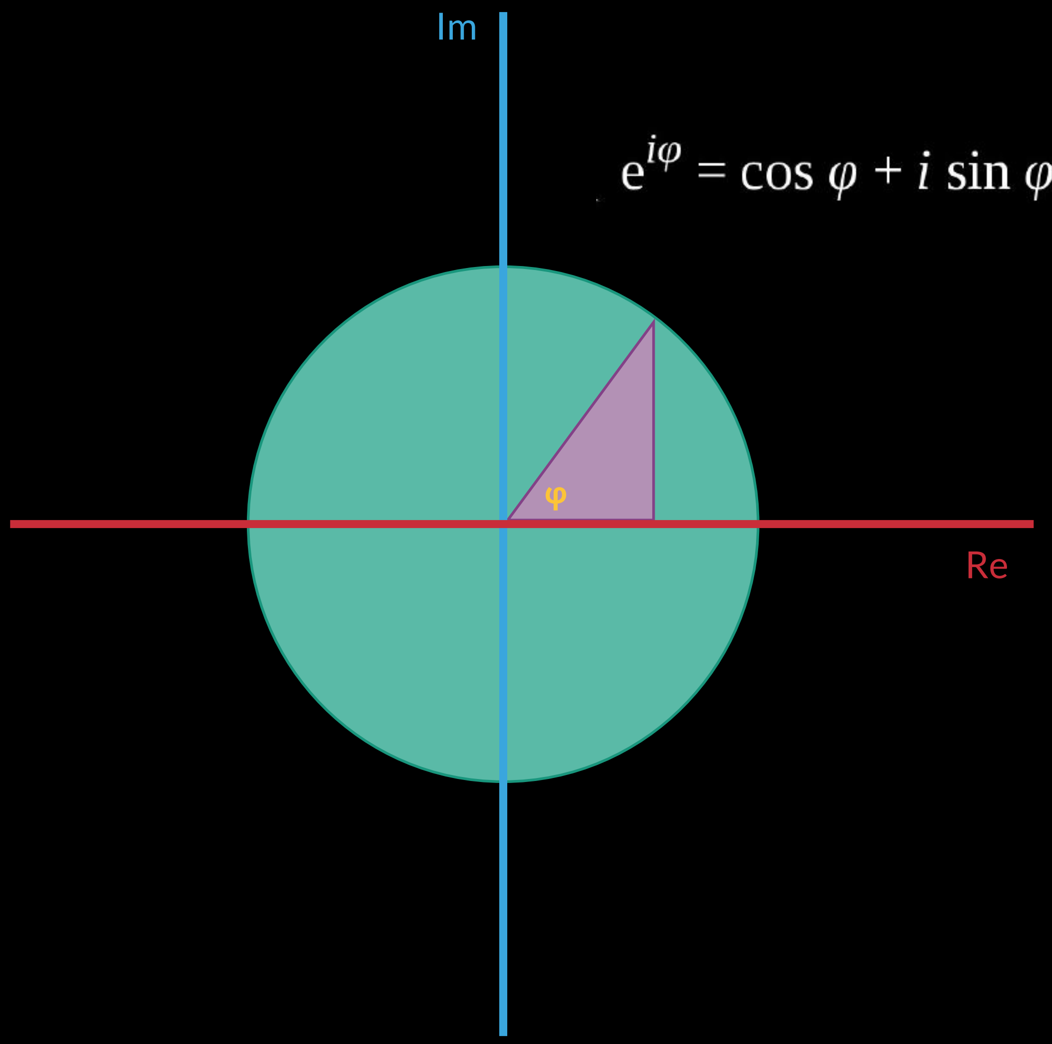

"Well! I've often seen a cat without a grin... but a grin without a cat!
It's the most curious thing I saw in my life!"

A Message from Our Editor in Chief

Alice Zhu

The arrival of a new decade signifies not only new, profound struggles, but also new beginnings and opportunities. Consequently, it seems appropriate to preface this first edition of Math Survey with a brief explanation of our history. Math Survey has been Stuyvesant's mathematics research magazine for more than 65 years, but it has fallen out of public sight for some time. That is why my publication team and I are excited to announce the return of Math Survey with the coming of this edition.

Needless to say, our mission remains the same: to keep students informed with recent developments in mathematics. As Editor-in-Chief, I have welcomed articles of that nature as well as original research papers written in that field, especially those that have analyzed real world concerns and have taken unique approaches to a myriad of questions. Furthermore, we have increased our scope to include articles written by students invested in the field of computer science.

Our team strongly believes that knowledge should be a two way conversation, one that informs, listens to, and, perhaps, inspires the reader. To that effect, I highly encourage any readers interested in performing similar work to reach out to us. With that, I invite you to take a look through our magazine and be enthralled by the beauty of mathematics, as many of us have been.

Math Survey

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A special thanks to Justin Wangying Lam for this edition’s cover design.

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Base Rules

Theo Schiminovich

Introduction

For the most part, we use base 10. We have 10 digits in our numbering system. This stems from the fact that we have 10 fingers to count with.

There are many numbering systems we could use. In fact, there is an infinite amount of them; one for each natural number. However, we for the most part stick with base 10 in our everyday lives. Clocks utilize base 60, and base 2 and base 16 are common when working with computers, but base 10 is used in most cases.

Because we are so familiar with base 10, we may find other bases to be cumbersome. It is usually most convenient to just convert numbers into base 10 and work with them there. However, if you accustom yourself to other bases, they can have many interesting properties.

Divisibility Rules

The divisibility rules of every base are different. Thus, we must examine different bases' divisibility rules when determining which are the most convenient to work with.

It is important to note that divisibility rules do not require a great deal of effort. It should be faster than dividing and determining the remainder.

A Review of the Divisibility Rules in Base 10

Base 10 has a fair number of divisibility rules. For consistency, I will list them up to 16 for each base.

- 2: If the last digit is 2, 4, 6, 8, or 0
- 3: If the sum of the digits is divisible by 3
- 4: If the last two digits are divisible by 4
- 5: If the last digit is 5 or 0
- 6: If the number is divisible by 2 and 3
- 7: No simple divisibility rule
- 8: If the last three digits are divisible by 8
- 9: If the sum of the digits is divisible by 9

- 10: If the last digit is 0
- 11: Sum digits in odd places, starting from the right, and subtract them by the sum of the digits in even places, starting from the right. Check if the result is a multiple of 11
- 12: If the number is divisible by 3 and 4
- 13: No simple divisibility rule
- 14: No simple divisibility rule
- 15: If the number is divisible by 3 and 5
- 16: If the last four digits are divisible by 16

These divisibility rules fall into four major categories:

- Method 1) Look at the last n digits: $2, 4, 5, 8, 16$
- Method 2) Look at the sum of the digits: $3, 9$
- Method 3) Look at the difference between alternating digits: 11
- Method 4) Compound of other divisibility rules: $6, 10, 12, 15$

Applications to Other Bases

It turns out that many of these rules may be applied to other bases as well. In fact, they may be generalized for all integer bases. This may be proved for all the rules mentioned above.

The Last n Digits

Given a base b, if a positive integer k is a factor of b^n, then an integer x in base b is divisible by k if the last n digits are divisible by k.

Say y is x's quotient when dividing by b^n, and z is the remainder (also known as the last n digits of x).
$x = (y \cdot b^n) + z$
$(y \cdot b^n) \equiv 0 \mod k$ because $k|b^n$
If $k|z$, then $z \equiv 0 \mod k$, then $x \equiv 0 + 0 \equiv 0 \mod k$ so $k|x$

Therefore, this rule can be applied to all bases.

Sum of the Digits

Given a base b, if a number k is a factor of $b-1$, then a number x in base b is divisible by k if the sum of the digits is divisible by k.

Say k divides the sum of the digits of x.
$k|a_0 + a_1 + a_2 + ... + a_n$.
$b \equiv 1 \mod b-1$.
Because 1 to any power is 1, $b^k \equiv 1 \mod b-1$ for any $k \geq 1$.
This means $(b^k) - 1 \equiv 0 \mod b-1$ for any $k \geq 1$.
So $k|(b^k) - 1$ for any $k \geq 1$.
$k|a_0 + a_1 + a_2 + ... + a_n$.
$k|(b^1 - 1) \cdot a_1 + (b^2 - 1) \cdot a_2 + ... + (b^n - 1) \cdot a_n$.
$k|a_0 + (b^1) \cdot a_1 + (b^2) \cdot a_2 + ... + (b^n) \cdot a_n$.
Therefore, $k|x$

Difference of Alternating Digits

Given a base b, if a number k is a factor of $b+1$, then a number x in base b is divisible by k if, when indexing the digits from 1 to n starting from the right, the difference between the sum of the odd-indexed digits and the sum of the even-indexed digits is divisible by k.
Say k divides the difference between the odd and even-indexed digits of of x.
$k|a_0 - a_1 + a_2 - a_3 + ... \pm a_n$.
$b \equiv -1 \mod b+1$.
Because -1 to any odd power is -1, and -1 to any odd power is 1, $b^k \equiv -1 \mod b+1$ for any odd $k \geq 1$, and $b^k \equiv 1 \mod b+1$ for any even $k \geq 1$.
This means $(b^k) + 1 \equiv 0 \mod b+1$ for any odd $k \geq 1$, and $(b^k) - 1 \equiv 0 \mod b+1$ for any even $k \geq 1$.
So $k|(b^k) + 1$ for any odd $k \geq 1$, and $k|(b^k) - 1$ for any even $k \geq 1$.
$k|a_0 - a_1 + a_2 + ... \pm a_n$.
$k|(b^1 + 1) \cdot a_1 + (b^2 - 1) \cdot a_2 + ... + (b^n \mp 1) \cdot a_n$.
$k|a_0 + (b^1) \cdot a_1 + (b^2) \cdot a_2 + ... + (b^n) \cdot a_n$.
Therefore, $k|x$

Compound of Other Divisibility Rules

The compounds of other divisibility rules that work in base 10 will still work in other bases, provided divisibility rules for the numbers being compounded still exist in the new base.

A Fifth Method

There is a fifth strategy that may be used. It was not mentioned earlier, because in base 10, it is most applicable to much larger numbers, but it is very useful in smaller bases. One may combine pairs or groups of numbers to form a number in a higher base. With this strategy, one may convert a base b number to base b^k, where $k > 1$, and use the previously mentioned sum and difference rules on it. This can be done because it is quite easy to convert a number from base b to base b^k, and can be done even without paper. All you have to do is group the digits into groups of 2, 3, or more, starting from the right; those are your new digits.

In base 10, you may do this by converting a number to base 100, which can be done by grouping the digits into groups of two, starting from the right. Then, you may take the difference of odd and even-indexed pairs of digits to check if a number is divisible by 101. You may take the sum of these pairs of digits to check if a number is divisible by 99, or any of its factors.

This actually reveals a lesser-known way of checking if a number is divisible by 11. Instead of adding and subtracting alternating digits, you may group the digits into groups of two, starting from the right, and add up these groups. This can be more difficult, because the numbers are larger, but it can also allow you to avoid errors associated with adding and subtracting.

For example, to check if 19645219 is divisible by 11, you may break it into groups of two. $19 + 64 + 52 + 19 = 154$, which is divisible by 11, so 19645219 is divisible by 11. The other method works too; $1 - 9 + 6 - 4 + 5 - 2 + 1 - 9 = -11$.

What is the Most Convenient Base to Use?

For us, base 10 is the most convenient base to use because we've grown so used to it. Because of this, there are no serious plans to change our base system. However, speculation abounds. If we had chosen a different base system, how would things be different? Would it be more convenient than the one we use today?

Number of Divisibility Rules for Smaller Bases

A base that's easy to use should have:

- many numbers that are factors of the base number or the base number to some power
- many numbers whose divisibility can be found using divisibility rules
- numbers of a reasonable length (this is subjective)

To try to answer this question, I examined each base's divisibility rules. Divisibility rules are what sets bases apart; they all use the same numbers, but in some, it is easier to see if a number is divisible by 7 or 9 or 10 than others.

To start, I focused on small numbers. I looked at bases from 2 to 16, and compared how many divisibility rules they had for numbers from 2 to 16. I compared how many divisibility rules could be found using each of the five aforementioned methods, and how many couldn't be found at all.

For method 5, I only included methods that involved working with numbers less than or equal to 100 in size. This is because, if you group enough digits together, you may always eventually find a divisibility rule, but it becomes inconvenient after a while.

- Method 1) Look at the last n digits
- Method 2) Look at the sum of the digits
- Method 3) Look at the difference between alternating digits
- Method 4) Compound of other divisibility rules
- Method 5) Group digits and look at the sum or difference

Base	Method 1	Method 2	Method 3	Method 4	Method 5	No Method
Base 2	2, 4, 8, 16		3	6, 10, 12, 14	5, 7, 9, 11, 13, 15	
Base 3	3, 9	2	4	6, 12, 14, 15	5, 7, 8, 10, 13, 16	11
Base 4	2, 4, 8, 16	3	5	6, 10, 12, 14, 15	7, 9, 13	11
Base 5	5	2, 4	3, 6	10, 12, 15	8, 13	7, 9, 11, 14, 16
Base 6	2, 3, 4, 6, 8, 9, 12, 16	5	7	10, 14, 15		11, 13
Base 7	7	2, 3, 6	4, 8	10, 12, 14, 15	5, 16	9, 11, 13
Base 8	2, 4, 8, 16	7	3, 9	6, 10, 12, 14, 15	5 13	11
Base 9	3, 9	2, 4, 8	5, 10	6, 12, 15	16	7, 11, 13, 14
Base 10	2, 4, 5, 8, 10, 16	3, 9	11	6, 12, 15		7, 13, 14
Base 11	11	2, 5, 10	3, 4, 6, 12	15		7, 8, 9, 13, 14, 16
Base 12	2, 3, 4, 6, 8, 9, 12, 16	11	13			5, 7, 10, 14, 15
Base 13	13	2, 3, 4, 6, 12	7, 14			5, 8, 9, 10, 11, 15, 16
Base 14	2, 4, 7, 8, 14, 16	13	3, 5, 15	6, 10, 12		9, 11
Base 15	3, 5, 9, 15	2, 7, 14	4, 8, 16	6, 10, 12		11, 13
Base 16	2, 4, 8, 16	3, 5, 15		6, 10, 12		7, 9, 11, 13, 14

Applications of Smaller Bases

Base 10 has some big advantages. It's easy to work with numbers with factors of 2 and 5 in it. It has a convenient divisibility rule for 3 and 9. It also has a divisibility rule for 11, which only 4 out of the 15 bases on this list possess. However, it also has some disadvantages. It lacks a divisibility rule for 7, the fourth most important prime. It can also be inconvenient to work with larger powers of two. For example, to check if a number is divisible by 16, you have to check if the last four digits are divisible by 16, and there are 625 possibilities for that!

Using this table, one may determine if there is a base that solves these problems, without bringing in worse ones.

Prime Bases

The only base that has a divisibility rule for every number from 2 to 16 is 2.

Here's a list of all the methods in base 2, as an example. A similar list can be formulated for any other base:

- 2: If the last digit is 0
- 3: Sum digits in odd places, starting from the right, and subtract them by the sum of the digits in even places, starting from the right. Check if the result is a multiple of 3
- 4: If the last two digits are 0

- 5: Group digits into pairs, starting from the right. Sum groups in odd places, starting from the right, and subtract them by the sum of the groups in even places, starting from the right. Check if the result is a multiple of 5
- 6: If the number is divisible by 2 and 3
- 7: Group digits into triples, starting from the right, and check if the sum of the groups is divisible by 7
- 8: If the last three digits are 0
- 9: Group digits into triples, starting from the right. Sum groups in odd places, starting from the right, and subtract them by the sum of the groups in even places, starting from the right. Check if the result is a multiple of 9
- 10: If the number is divisible by 2 and 5
- 11: Group digits into groups of 5, starting from the right. Sum groups in odd places, starting from the right, and subtract them by the sum of the groups in even places, starting from the right. Check if the result is a multiple of 11
- 12: If the number is divisible by 3 and 4
- 13: Group digits into groups of 6, starting from the right. Sum groups in odd places, starting from the right, and subtract them by the sum of the groups in even places, starting from the right. Check if the result is a multiple of 13
- 14: If the number is divisible by 2 and 7
- 15: Group digits into groups of 4, starting from the right, and check if the sum of the groups is divisible by 15
- 16: If the last four digits are 0

However, this doesn't necessarily mean that base 2 is the most convenient base to use. Most of its base rules involve grouping digits, which can be difficult to perform in your head. It also requires more digits than other bases to display the same number, which is inefficient.

The other prime bases under 16 are even more difficult to use. Base 3 has many divisibility rules, like base 2, but many of them are difficult to use, like base 2. The other prime bases lack many divisibility rules, and the ones they do have are typically more difficult to use than the divisibility rules in other bases.

Prime Power Bases

There are four prime power bases from base 2 to base 16: base 4, base 8, base 9, and base 16. Base 4 and base 8 have more divisibility rules than the other two. Of these two, base 8 is likely easier to use, because fewer of its methods require converting to a higher power, and its numbers are more compact.

Other Numbers

The remaining bases are the ones that contain multiple primes in their prime factorization: 6, 10, 12, 14, and 15.

These bases typically have many numbers who divide x if they divide the last n digits of x. Any number for which the set of unique primes in its prime factorization is a subset of the set of unique primes in the prime factorization of the base number can be found in this way. This can be convenient, but it can also be cumbersome, because sometimes there are a lot of possibilities. For example, there are 625 possibilities for the last 4 digits to be a multiple of 16 in base 10, while there are 81 possibilities in base 6, and 2401 possibilities in base 14.

Base 6 and base 12 are similar because both 6 and 12 contain 2 and 3 in their prime factorizations. Also, each base has two primes for which a divisibility rule cannot be applied. However, for 6 these primes are 11 and 13, while for 12 these numbers are 5 and 7; since 5 and 7 are more commonly worked with, base 6 is probably the easier one to work with of the two.

Base 14 and base 15 are similar because they are both semiprime (the product of two prime numbers). They also each lack divisibility rules for two numbers between 2 and 16. Of these two, Base 14 is probably easier to use, because 14 is even, which makes it easier to divide by 2. However, it's difficult to check if a number is a multiple of 3, which makes base 14 more difficult to use than base 6.

Overall

The most promising alternatives to base 10 are base 6 and base 8. Base 6 has easy-to-use divisibility rules for multiples of 2 and 3, as well as 5 and 7, while base 8 has divisibility rules for a variety of numbers, and the added advantage of being a power of 2, which makes it useful for computers as well.

Analysis of Larger Bases

Currently, our numbering system does not have many digits (only 10). Therefore, I started by focusing on other numbering systems with few digits. However, there are many bases with more digits (in fact, an infinite amount).

Next, I decided to look at bases larger than 16, and see if any of them could have advantages over base 10. Specifically, I looked at bases from 17 to 256, because it seems like it would be extremely difficult to keep track of more than 256 different digits.

I decided to look for bases that have the following properties:

- The base number is even.
- The base has a divisibility rule for every number from 2 to 10.
- For 3 and 5, this divisibility rule is either a sum of digits rule or a last n digits rule.

There are actually very few bases that satisfy these conditions. Here's a list:

- Base 6
- Base 36
- Base 90
- Base 120
- Base 126
- Base 190
- Base 210
- Base 216
- Base 246

This shows that base 6 is rather unique among small bases. Given that base 36 is just base 6 with pairs of digits grouped together, the next base that satisfies these conditions is base 90.

An Example Problem

The question of which base is the most convenient to use is incredibly unlikely to come up in a competition math. However, the knowledge that the same divisibility rules we use in base 10 can be applied to other bases can be helpful. An example of this is in this problem:

For any integer $n \geq 2$, let b_n be the least positive integer such that, for any integer N, m divides N whenever m divides the digit sum of N written in base b_n, for $2 \geq m \geq n$. Find the integer nearest to b_{36}/b_{25}. (PUMaC NT #4 2017)

We have shown above that, given a number written in a certain base, m divides N whenever m divides the digit sum of N if m divides $b^n - 1$.
This means every number from 2 to 36 must divide $b_{36} - 1$, and every number from 2 to 25 must divide $b_{25} - 1$. The smallest such number for each can be found by determining the largest quantity of each prime factor in the numbers in the set.
Therefore, $b_{36} - 1 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31$.
And, $b_{25} - 1 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$.
Because both numbers are so large, b_{36}/b_{25} is roughly equal to $\dfrac{b_{36} - 1}{b_{25} - 1}$.
So it is roughly equal to $2 \cdot 3 \cdot 29 \cdot 31$, which equals $\boxed{5394}$.

Citations

1. "Divisibility: Rules and Recipes for Testing Divisibility in Different Number Bases." Dozenal Society, www.dozenalsociety.org.uk/pdfs/divisibility.pdf.

Conditional Probability

Declan Stacy

2.1 Introduction

Have you ever been told that the probability of two events occurring is simply the product of the probabilities of each individual event occurring? It sounds reasonable. The probability of flipping a coin twice and getting two heads is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ because there is a $\frac{1}{2}$ chance of flipping heads on the first flip, and a $\frac{1}{2}$ chance of flipping heads on the second flip. However, it turns out that this statement is not always true.

2.2 Conditional Probability and Independence

What if I wanted to know the probability of flipping three coins and getting at least one tails and at least one heads? By the logic above, the answer would be $\frac{7}{8} \cdot \frac{7}{8} = \frac{49}{64}$ because there is a $(1 - \frac{1}{8})$ chance of getting not all heads (aka at least one tails) and a $(1 - \frac{1}{8})$ chance of getting not all tails (aka at least one heads). But does $\frac{49}{64}$ make sense? After all, there are only 8 possible outcomes for flipping 3 coins.

The problem is, knowing that there is at least one tails affects the chances of getting at least one heads. You know now that the result could not have been HHH, so your 8 possible outcomes has shrunk to 7. In other words, your sample space (the set of all possible outcomes) has been altered. Now, instead of having a $\frac{7}{8}$ chance of getting at least one heads, you have a $\frac{6}{7}$ chance, because there are 7 possible outcomes and 6 of them include a heads. We call this new probability a conditional probability, in this case the probability of flipping at least one heads given that you have flipped at least one tails. We can denote this as $P(B|A)$, where B is the event "at least one heads" and A is the event "at least one tails".

Now that we know why we failed, let's see if we can get the correct answer. What we are looking for is $P(A \cap B)$ (the probability of A and B). First, we need A to happen, which occurs with probability $P(A) = \frac{7}{8}$. After we know that A has happened, we need B to also happen, which occurs with probability $P(B|A) = \frac{6}{7}$. These are the two probabilities that must be multiplied (see if you can explain why).

Thus, we can write

$$P(A \cap B) = P(A) \cdot P(B|A) \tag{2.1}$$

In our case, this is $\frac{7}{8} \cdot \frac{6}{7} = \frac{3}{4}$. This makes sense because as long as I don't flip all heads or all tails, I have at least one head and at least one tails. There are $8 - 2 = 6$ ways to not get those two bad outcomes, so our answer should be $\frac{6}{8}$, and it is.

What if we have three events? Or four? In general, for events $A_1, A_2, ...A_n$, we can write

$$P(A_1 \cap ... \cap A_n) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \cdot ... \cdot P(A_n|A_1 \cap ... \cap A_{n-1}) \tag{2.2}$$

which comes from successive applications of 2.1.

So, when is the statement that $P(A \cap B) = P(A) \cdot P(B)$ true? By comparison with 2.1, this would mean that $P(B|A) = P(B)$. In words, this means that knowing that the event A has occurred does not affect the chances of event B occurring. We call this independence. So, the probability of multiple events occurring is the product of the probabilities of each individual event occurring if and only if the events are independent.

We now know the relationship between conditional probability and the probability of multiple events occurring, but how do we calculate these conditional probabilities in the first place? Going back to 2.1, we can divide both sides by P(A) (as long as P(A) is not 0) to obtain

$$\frac{P(A \cap B)}{P(A)} = P(B|A) \tag{2.3}$$

2.3 Total Probability Theorem

Let's look at another example problem where conditional probability is useful. I flip a coin and roll a die, and the die roll determines how much money I will spend on Purell this week. If the flip is heads, I add 95 to my die roll. If it is tails, I multiply my die roll by 50. What is the probability of me spending at least $100 on Purell this week? We can split the problem up into two cases:

Case 1: I rolled heads

In this case, there are 2 die rolls (5 and 6) that result in me spending at least $100 out of 6 possible die rolls. Thus, there is a $\frac{1}{3}$ chance for this case.

Case 2: I rolled tails

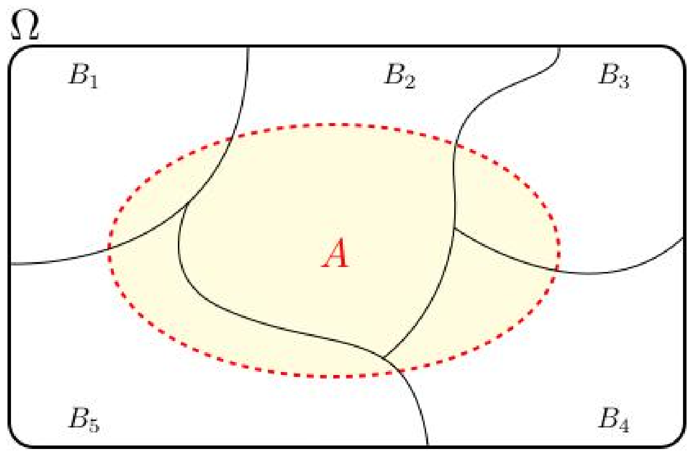
In this case, there are 5 die rolls (2 through 6) that result in me spending at least $100 out of 6 possible die rolls. Thus, there is a $\frac{5}{6}$ chance for this case.

Since case 1 and case 2 each occur with probability $\frac{1}{2}$, my total probability is $\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{5}{6} = \frac{7}{12}$. I said conditional probability was useful in this problem. So where did I use it?

Let's call the event of rolling heads H, the event of rolling tails T, and the event of me spending at least $100 on Purell this week A. The $\frac{1}{3}$ represents the probability of A given that I rolled heads, $P(A|H)$. The $\frac{5}{6}$ represents the probability of A given that I rolled tails, $P(A|T)$. The two $\frac{1}{2}$'s represent the probabilities of rolling heads and tails, P(H) and P(T), respectively. To compute P(A), I computed $P(H) \cdot P(A|H) + P(T) \cdot P(A|T)$. More generally, for disjoint events $B_1, B_2, ...B_n$ such that $B_1 \cup B_2 \cup ... \cup B_n$ is the sample space,

$$P(A) = \sum_{i=1}^{n} P(B_i) \cdot P(A|B_i) = \sum_{i=1}^{n} P(A \cap B_i) \tag{2.4}$$

Where the second equality comes from 2.1. This is called the Total Probability Theorem.



This equation is simply a justification of what you do every time you use casework in a probability problem, but if it is not intuitive, then try to understand this "proof by picture":

For a formal proof, we must introduce the following axiom (all of probability theory is based on three axioms proposed by Andrey Kolmogorov, and this is one of them):

$$P(A_0 \cup A_1 \cup ... \cup A_n) = \sum_{i=1}^{n} P(A_i) \quad \text{if } A_0, A_1, ...A_n \text{ are disjoint events} \tag{2.5}$$

Since $B_1, B_2, ...B_n$ are disjoint, $(B_1 \cap A), (B_2 \cap A), ..., (B_n \cap A)$ are also disjoint. This means that we can apply 2.5:

$$P((B_1 \cap A) \cup (B_2 \cap A) \cup ... \cup (B_n \cap A)) = \sum_{i=1}^{n} P(A \cap B_i) \tag{2.6}$$

Also, since $B_1 \cup B_2 \cup ... \cup B_n$ is the sample space, then for an event A in the sample space, $A \subset (B_1 \cup B_2 \cup ... \cup B_n)$. Thus, $(B_1 \cap A) \cup (B_2 \cap A) \cup ... \cup (B_n \cap A) = A$ (which is what the picture shows). Substituting A for $(B_1 \cap A) \cup (B_2 \cap A) \cup ... \cup (B_n \cap A)$ in the equation above:

$$\begin{aligned} P(A) &= \sum_{i=1}^{n} P(A \cap B_i) \\ &= \sum_{i=1}^{n} P(B_i) \cdot P(A|B_i) \quad \text{(from 2.1)} \quad \square \end{aligned} \tag{2.7}$$

There is also an analogous formula for computing expectations with conditional probability, where $A_1, A_2, ...A_n$ are disjoint events such that $A_1 \cup A_2 \cup ... \cup A_n$ is the sample space of a random variable X which will not be proven here:

$$E(X) = \sum_{i=1}^{n} E(X|A_i) \cdot P(A_i) \tag{2.8}$$

2.4 Bayes' Theorem

There is another very important application of conditional probability that comes up a lot in everyday life: inference. For example, when we go to the doctor and get tested for COVID-19, we want to know how likely it is that we have the disease given that we tested positive. The question also gives you that 10% of the population has the virus, that the test is 99% accurate when the patient has the virus, and 90% accurate when the patient does not have the virus.

First let's understand what the question gives us. There are two events we are concerned with: you having the virus, and you testing positive. We will call these events A and B, respectively. You are given that $P(A) = .1$, $P(B|A) = .99$, and $P(B|\neg A) = 1 - .9 = .1$ (if the test is 90% accurate when you don't have the virus, that means that 10% of the time it will be wrong and say you do have it).

We want to find $P(A|B)$. Using the definition of conditional probability and the formula for the probability of two events occurring, we can write:

$$P(A \cap B) = P(A) \cdot P(B|A) \tag{2.1}$$

$$\frac{P(A \cap B)}{P(B)} = P(A|B) \tag{2.3}$$

Substituting 2.1 into 2.3, we obtain Bayes' Theorem:

$$P(B|A) \cdot \frac{P(A)}{P(B)} = P(A|B) \tag{2.9}$$

However, we still don't know P(B). This can be easily computed with the total probability theorem:

$P(B) = P(B|A) \cdot P(A) + P(B|\neg A) \cdot P(\neg A) = .99 \cdot .1 + .1 \cdot (1 - .1) = .189$

This yields an answer of $.99 \cdot .1/.189 \approx .52$

You may be surprised by how low this number is compared to the accuracy of the test, which is $.99 \cdot .1 + .9 \cdot .9 = .909$. This means that just because a test has a high accuracy does not necessarily mean that you should panic if you test positive (but please stay home if you have the COVID-19).

Bayes' theorem is a simple formula, but it is very powerful since it relates an observed event (B) to the conditions that caused that event. This is the type of thinking you go through whenever you do a science experiment: you run tests and record data, and then use that data to draw conclusions about the system you tested. This is the backbone of the theory of Bayesian Statistics, which has many applications in signal processing, science research, game theory, and more. (In the problem set, look out for problems labeled "inference," as these problems relate to Bayesian statistics.)

2.5 Problems

Now that you are an expert on conditional probability, try out these problems!

1. Mario has two children. Assume that children are equally likely to be born as a boy or a girl and are equally likely to be born on any day of the week. I ask Mario if he has a daughter, and he says yes. What is the probability the other child is a son? What if instead I ask Jerry, who also has two kids, if he has a daughter that was born on a Tuesday, and he says yes; what is the probability the other child is a son in this scenario?

2. **Inference #1**: Reimu has 2019 coins $C_0, C_1, ..., C_{2018}$, one of which is fake, though they look identical to each other (so each of them is equally likely to be fake). She has a machine that takes any two coins and picks one that is not fake. If both coins are not fake, the machine picks one uniformly at random. For each $i = 1, 2, ..., 1009$, she puts C_0 and C_i into the machine once, and the machine picks C_i. What is the probability that C_0 is fake? (HMMT Feb 2019 Guts #13)

3. **Prediction #1**: A bag contains nine blue marbles, ten ugly marbles, and one special marble. Ryan picks marbles randomly from this bag with replacement until he draws the special marble. He notices that none of the marbles he drew were ugly. Given this information, what is the expected value of the number of total marbles he drew? (HMMT Feb 2018 Combo #5)

4. **Prediction #2**: Noted magician Casimir the Conjurer has an infinite chest full of weighted coins. For each $p \in [0, 1]$, there is exactly one coin with probability p of turning up heads. Kapil the Kingly draws a coin at random from Casimir the Conjurer's chest, and flips it 10 times. To the amazement of both, the coin lands heads up each time! On his next flip, if the expected probability that Kapil the Kingly flips a head is written in simplest form as $\frac{p}{q}$, then compute $p+q$. (PUMaC 2018 Live Round Calculus #1)

5. **Inference #2**: Johnny has a deck of 100 cards, all of which are initially black. An integer n is picked at random between 0 and 100, inclusive, and Johnny paints n of the 100 cards red. Johnny shuffles the cards and starts drawing them from the deck. What is the least number of red cards Johnny has to draw before the probability that all the remaining cards are red is greater than .5?

6. **Inference #3**: Yannick picks a number N randomly from the set of positive integers such that the probability that n is selected is 2^{-n} for each positive integer n. He then puts N identical slips of paper numbered 1 through N into a hat and gives the hat to Annie. Annie does not know the value of N, but she draws one of the slips uniformly at random and discovers that it is the number 2. What is the expected value of N given Annie's information? (HMMT Feb 2019 Guts #29) (Note: Uses Calculus)

2.6 Citations

1. Foundation, CK-12. "12 Foundation." CK, www.ck12.org/book/cbse-maths-book-class-12/section/14.5/.

Conditional Probability - Problem Set

Declan Stacy

3.1 Question #1

> Mario has two children. Assume that children are equally likely to be born as a boy or a girl and are equally likely to be born on any day of the week. I ask Mario if he has a daughter, and he says yes. What is the probability the other child is a son? What if instead I ask Jerry, who also has two kids, if he has a daughter that was born on a Tuesday, and he says yes; what is the probability the other child is a son in this scenario?

From the definition of conditional probability we can write,

$$P(\text{has son}|\text{has daughter}) = P(\text{has son} \cap \text{has daughter})/P(\text{has daughter})$$

$P(\text{has son} \cap \text{has daughter}) = \frac{1}{2}$ because he can either have two sons, two daughters, an older son and a younger daughter, or an older daughter and a younger son, all with equal probability. In two out of these four scenarios he has a son and a daughter, and $\frac{2}{4} = \frac{1}{2}$.

$P(\text{has daughter}) = \frac{3}{4}$ because in three out of these four scenarios he has at least one daughter.

So, the answer to the first question is $\frac{1}{2} \div \frac{3}{4} = \boxed{\frac{2}{3}}$.

For the second question, we are also concerned with what day of the week each child was born on, so there are $(2 \cdot 7)^2$ scenarios (each child is either a boy or a girl, and is born on one of seven days), and all scenarios are equally likely.

We need to find the number of scenarios where Jerry has a daughter born on a Tuesday. This is $14 + 14 - 1 = 27$ because there are 14 scenarios where the younger child is a daughter born on a Tuesday and 14 scenarios where the older child is a daughter born on a Tuesday because the other child can be of $2 \cdot 7 = 14$ varieties. However, this counts the scenario where both children are daughters born on Tuesdays, so we must subtract one.

The number of scenarios where Jerry has a son and a daughter born on Tuesday is $2 \cdot 7 = 14$ because the son can be the first or the second child and he can be born on any of the seven days of the week.

Remember that another interpretation of conditional probability is restricting the sample space; instead of having $(2 \cdot 7)^2$ possible scenarios, with the information we are given we know that only 27 of these are possible, and are equally likely. Thus, the answer is $\boxed{\frac{14}{27}}$ because out of the 27 scenarios, in 14 of them Jerry has a son.

(If you are confused, try the first method of using the definition of conditional probability and you will see that you get the same fraction but with the numerator and denominator divided by $(2 \cdot 7)^2$.)

3.2 Inference #1

HMMT Feb 2019 Guts #13: Reimu has 2019 coins $C_0, C_1, ..., C_{2018}$, one of which is fake, though they look identical to each other (so each of them is equally likely to be fake). She has a machine that takes any two coins and picks one that is not fake. If both coins are not fake, the machine picks one uniformly at random. For each $i = 1, 2, ..., 1009$, she puts C_0 and C_i into the machine once, and the machine picks C_i. What is the probability that C_0 is fake?

The first thing you should do when you see a probability problem is write out in words what you are trying to find. In this case, we want $P(C_0 \text{ fake} \mid \text{never picked})$. It can also be helpful to write this in a couple of different ways using the definition of conditional probability and Bayes' Theorem:

$$
\begin{aligned}
P(C_0 \text{ fake} \mid \text{never picked}) &= \frac{P(C_0 \text{ fake and never picked})}{P(C_0 \text{ never picked})} \\
&= P(C_0 \text{ never picked} \mid \text{fake}) \cdot \frac{P(C_0 \text{ fake})}{P(C_0 \text{ never picked})}
\end{aligned}
$$

In this case, the second option (applying Bayes' Theorem) looks like the easiest to compute.

1. $P(C_0 \text{ never picked}|\text{fake}) = 1$

   The question says that the machine will never pick the fake coin, so this is clearly 1.

2. $P(C_0 \text{ fake}) = \frac{1}{2019}$

   In the question it says that one of the 2019 coins is fake, and they are equally likely to be fake. So, the probability of an individual coin being fake is simply $\frac{1}{2019}$.

3. $P(C_0 \text{ never picked}) = \frac{2^{1009}+1009}{2019 \cdot 2^{1009}}$

   Looking at the question, the answer to this is not explicitly given like with the other two probabilities. When you don't know what to do, casework is often a good option. So what should we do casework on? In this case, knowing which coin was fake would make computing probabilities a lot easier, so we should do casework on that. Let the fake coin be C_x. We will consider three cases: $x = 0$ (C_0 fake), $x \in [1, 1009]$, and $x \in [1010, 2018]$ and use the Total Probability Theorem:

$$
\begin{aligned}
P(C_0 \text{ never picked}) = P(C_0 &\text{ never picked } |x = 0) \cdot P(x = 0) \\
+P(C_0 &\text{ never picked } |x \in [1, 1009]) \cdot P(x \in [1, 1009]) \\
+P(C_0 &\text{ never picked } |x \in [1010, 2018]) \cdot P(x \in [1010, 2018])
\end{aligned}
$$

(a) Case 1: $x = 0$ (C_0 fake)
We already know $P(C_0 \text{ never picked} \mid \text{fake}) = 1$ and $P(C_0 \text{ fake}) = \frac{1}{2019}$ from above, so this case is already done.

(b) Case 2: $x \in [1, 1009]$
In this case, C_0 is a real coin and is put into the machine 1008 times with another real coin, and 1 time with a fake coin. When it is put in with the fake coin, the machine must pick C_0 because the machine never picks the fake coin. Thus, $P(C_0 \text{ never picked } | x \in [1, 1009]) = 0$ since C_0 must be picked at least once.

(c) Case 3: $x \in [1010, 2018]$
In this case, C_0 is a real coin and is put into the machine 1009 times with another real coin. Each time, there is a $\frac{1}{2}$ chance that C_0 will be picked (if both coins are real the machine picks one at random). The decision the machine makes each time is independent of the decisions it made before. The probability of multiple independent events occurring is the product of their individual probabilities, so we can write $P(C_0 \text{ never picked } | x \in [1010, 2018]) = \frac{1}{2}^{1009}$.
Now all we need to do is compute $P(x \in [1010, 2018])$. Since the events $x = 1010, x = 1011, ..., x = 2019$ are disjoint, we can write $P(x \in [1010, 2018]) = P(x = 1010) + P(x = 1011) + ... + P(x = 2019)$. The probability of any individual coin being fake is $\frac{1}{2019}$, so this is simply $1009 \cdot \frac{1}{2019} = \frac{1009}{2019}$.
(Quick tip: multiply when computing the probability of multiple independent events occurring, add when computing the probability of at least one of many disjoint events occurring.)

Now we can plug in all of the stuff we just computed into our equation for $P(C_0 \text{ never picked})$ to get:

$$\begin{aligned} P(C_0 \text{ never picked}) = 1 \cdot \frac{1}{2019} + 0 \cdot P(x \in [1, 1009]) + \frac{1}{2^{1009}} \cdot \frac{1009}{2019} \\ = \frac{2^{1009} + 1009}{2019 \cdot 2^{1009}} \end{aligned}$$

Finally, we have all the parts to compute

$$\begin{aligned} &P(C_0 \text{ fake} \mid \text{never picked}) \\ = P(C_0 \text{ never picked} \mid \text{fake}) \cdot &\frac{P(C_0 \text{ fake })}{P(C_0 \text{ never picked})} \\ &= 1 \cdot \frac{\frac{1}{2019}}{\frac{2^{1009}+1009}{2019 2^{1009}}} \\ &= \boxed{\frac{2^{1009}}{2^{1009} + 1009}} \end{aligned}$$

This is a great example of how Bayes' Theorem and the Total Probability Theorem can be used to break up a problem into simple pieces that are easy to compute.

3.3 Prediction #1

HMMT Feb 2018 Combo #5: A bag contains nine blue marbles, ten ugly marbles, and one special marble. Ryan picks marbles randomly from this bag with replacement until he draws the special marble. He notices that none of the marbles he drew were ugly. Given this information, what is the expected value of the number of total marbles he drew?

Like the last problem, we should write out what we are looking for:
$E(\text{marbles} \mid \text{not ugly})$
When a problem involves a process that could go on forever, you will often have to compute an infinite sum. If we use the definitions of expected value and conditional probability, we can write:

$$\begin{aligned} E(\text{marbles} \mid \text{not ugly}) &= \sum_{i=1}^{\infty} i \cdot P(i \text{ marbles} \mid \text{not ugly}) \\ &= \sum_{i=1}^{\infty} i \cdot \frac{P(i \text{ marbles} \cap \text{ not ugly})}{P(\text{not ugly})} \end{aligned}$$

1. $P(i \text{ marbles} \cap \text{not ugly}) = \frac{9}{20}^{i-1} \cdot \frac{1}{20}$

   The probability of drawing a blue marble is $\frac{9}{20}$, and the probability of drawing the special marble is $\frac{1}{20}$. Also, each draw is independent of the last. Thus, the probability of drawing i marbles and not drawing any uglies is $\frac{9}{20}^{i-1} \cdot \frac{1}{20}$ because we drew $i-1$ blue marbles followed by 1 special marble.

2. $P(\text{not ugly}) = \frac{1}{11}$

   To compute $P(\text{not ugly})$, we will use the Total Probability Theorem:

$$P(\text{not ugly}) = \sum_{i=1}^{\infty} P(\text{not ugly} \mid \text{i marbles}) \cdot P(\text{i marbles})$$

   This is essentially casework on the number of marbles that were drawn.

   $P(\text{not ugly} \mid \text{i marbles}) = \frac{9}{19}^{i-1}$ since the first $i-1$ marbles are either blue or ugly, and $\frac{P(\text{marble is ugly})}{P(\text{marble is ugly or blue})} = \frac{\frac{9}{20}}{\frac{19}{20}} = \frac{9}{19}$.

   $P(\text{i marbles}) = \frac{19}{20}^{i-1} \cdot \frac{1}{20}$ since the first $i-1$ marbles must be ugly or red and the last marble must be special.

   Thus,

$$P(\text{not ugly}) = \sum_{i=1}^{\infty} \frac{9}{19}^{i-1} \cdot \frac{19}{20}^{i-1} \cdot \frac{1}{20}$$
$$= \sum_{i=1}^{\infty} \frac{9}{20}^{i-1} \cdot \frac{1}{20}$$
$$= \frac{\frac{1}{20}}{1 - \frac{9}{20}}$$
$$= \frac{1}{11}$$

where we recognized the sum as an infinite geometric series with a common ratio between -1 and 1.

Now we have all the parts to compute our original sum:

$$E(\text{marbles} \mid \text{not ugly}) = \sum_{i=1}^{\infty} i \cdot \frac{\frac{9}{20}^{i-1} \cdot \frac{1}{20}}{\frac{1}{11}}$$
$$= \frac{11}{20} \cdot \sum_{i=1}^{\infty} \cdot \frac{9}{20}^{i-1}$$
$$= \frac{11}{20} \cdot \frac{20}{9} \cdot \sum_{i=1}^{\infty} (i-1) \cdot \frac{9}{20}^{i-1}$$

Multiplying the third line of the equation by $\frac{9}{20}$ and subtracting it from the second line yields:

$$(1 - \frac{9}{20})(\text{marbles} \mid \text{not ugly}) = \frac{11}{20} \cdot \sum_{i=1}^{\infty} \frac{9}{20}^{i-1}$$
$$E(\text{marbles}|\text{not ugly}) = \sum_{i=1}^{\infty} \frac{9}{20}^{i-1}$$
$$= \frac{1}{1 - \frac{9}{20}}$$
$$= \boxed{\frac{20}{11}}$$

Overall, this question utilized the same concepts as the last question, except we had to do casework on infinitely many events and compute infinite sums.

3.4 Prediction #2

PUMaC 2018 Live Round Calculus #1: Noted magician Casimir the Conjurer has an infinite chest full of weighted coins. For each $p \in [0, 1]$, there is exactly one coin with probability p of turning up heads. Kapil the Kingly draws a coin at random from Casimir the Conjurer's chest, and flips it 10 times. To the amazement of both, the coin lands heads up each time! On his next flip, if the expected probability that Kapil the Kingly flips a head is written in simplest form as $\frac{p}{q}$, then compute $p + q$. (Note: Uses Calculus)

Sometimes when dealing with a question like this where there are infinitely many options, it is useful to solve the problem as if there were a finite number of choices. What I mean is, instead of Casimir picking a coin with a probability $p \in [0, 1]$, let's say he only has 5 options for the probability of the coin. For example, we could choose a $p \in \{0, .2, .4, .6, .8\}$. If you did not solve the actual problem, try solving that simpler problem (with $p \in \{0, .2, .4, .6, .8\}$) and then come back and read this solution.

To make this more general, let there be n options for the probability of the coin, and let the set of options be $S = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}\}$.

We want $P(\text{11th heads} \mid \text{10 heads})$. Using the Total Probability Theorem, we can write:

$$P(\text{11th heads}|\text{10 heads}) = \sum_{i=0}^{n-1} P(\text{11th heads} \mid \text{10 heads} \cap p = \frac{i}{n})(p = \frac{i}{n}|\text{10 heads})$$

If we know p, we know the probability of flipping another heads, because p is the probability of the coin flipping heads. Thus, $P(\text{11th heads} \mid \text{10 heads} \cap p = \frac{i}{n}) = \frac{i}{n}$.

To evaluate $P(p = \frac{i}{n}|\text{10 heads})$, we can use Bayes' Theorem:

$$P(p = \frac{i}{n}|\text{10 heads}) = P(\text{10 heads}|p = \frac{i}{n}) \cdot \frac{P(p = \frac{i}{n})}{P(\text{10 heads})}$$

$P(\text{10 heads}|p = \frac{i}{n}) = \frac{i}{n}^{10}$ because each flip is independent of the last and there are 10 of them.

$P(p = \frac{i}{n}) = \frac{1}{n}$ because we are picking p randomly from S and $|S| = n$.

For $P(\text{10 heads})$, we will have to use the Total Probability Theorem again.

$$\begin{aligned} P(\text{10 heads}) &= \sum_{i=0}^{n-1} P(\text{10 heads } |p = \frac{i}{n})(p = \frac{i}{n}) \\ &= \sum_{i=0}^{n-1} \frac{i}{n}^{10} \cdot \frac{1}{n} \end{aligned}$$

So, if we chose what n was, then we would have all the parts to solve the problem. Looking back at the way we defined S, as n becomes very large, the set S becomes more and more similar to $[0, 1]$. Thus, if we take the limit as n approaches ∞, we will get the answer.

Note that our expression for $P(10 \text{ heads})$ is a Riemann Sum, so:

$$\begin{aligned} P(10 \text{ heads}) &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{i}{n}^{10} \cdot \frac{1}{n} \\ &= \int_0^1 x^{10} dx \\ &= \frac{1}{11} \end{aligned}$$

Now we can substitute in all of the stuff we have computed to get the answer:

$$\begin{aligned} P(11\text{th heads} \mid 10 \text{ heads}) &= \lim_{n\to\infty} \sum_{i=0}^{n-1} P(11\text{th heads} \mid 10 \text{ heads} \cap p = \frac{i}{n})(p = \frac{i}{n} \mid 10 \text{ heads}) \\ &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{i}{n} \cdot \frac{i}{n}^{10} \cdot \frac{\frac{1}{n}}{\frac{1}{11}} \\ &= 11 \cdot \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{i}{n}^{11} \cdot \frac{1}{n} \\ &= 11 \cdot \int_0^1 x^{11} dx \\ &= \frac{11}{12} \end{aligned}$$

The question asks for the sum of the numerator and the denominator of the fraction, so the answer is $11 + 12 = \boxed{23}$.

It makes sense that the answer should be close to 1 because if a coin lands heads 10 times in a row, it is more likely to be weighted heavily towards heads. So, as we observe more and more heads, our estimate of what the true bias of the coin is should go up. You can also think of it as if we took a bunch of coins, flipped them 10 times, and then threw out the ones that did not flip 10 heads. The average bias of the remaining coins would most likely be heavily in favor of heads.

You did not have to do all this defining our own set and doing Riemann Sums business if you saw immediately what the integrals were going to be. However, if you did not know what to do, then imagining the problem dealt with a finite set instead of an infinite set and thinking about how you would solve it can help.

3.5 Inference #2

Johnny has a deck of 100 cards, all of which are initially black. An integer n is picked at random between 0 and 100, inclusive, and Johnny paints n of the 100 cards red. Johnny shuffles the cards and starts drawing them from the deck. What is the least number of red cards Johnny has to draw before the probability that all the remaining cards are red is greater than .5?

If all the cards are red, that means that $n = 100$, so what we want is the least whole number k that satisfies the inequality $P(n = 100|\text{ first k are red}) > .5$. To get an expression for $P(n = 100|\text{ first k are red})$, let's start with Bayes' Theorem:

$$P(n = 100|\text{ first k are red}) = P(\text{first k are red}|n = 100) \cdot \frac{P(n = 100)}{P(\text{first k are red})}$$

$P(\text{first k are red }|n = 100) = 1$ because $n = 100$ means that all the cards are red, so the first k will always be red.

$P(n = 100) = \frac{1}{101}$ since n is chosen uniformly at random from 101 options.

To calculate $P(\text{first k are red})$, we will use the Total Probability Theorem:

$$P(\text{first k are red }) = \sum_{i=k}^{100} P(n = i)(\text{first k are red }|n = i)$$

$P(n = i) = \frac{1}{101}$ (same explanation as for $P(n = 100)$.

The tricky part is $P(\text{first k are red}|n = i)$. What that is saying is that you draw k cards out of 100, and they are each one of the i red cards in the deck. There are $\binom{100}{k}$ sets of cards that could be the top k cards. Out of those $\binom{100}{k}$ sets, $\binom{i}{k}$ of them are all red cards because there are i red cards to choose from. Thus, $P(\text{first k are red}|n = i) = \frac{\binom{i}{k}}{\binom{100}{k}}$

(Quick Tip: here we are considering all the cards to be unique because it makes the calculations easier, but some questions are more easily answered when you consider all objects of the same type to be the same. No matter which way you do it, remember to be consistent. Don't do one calculation where you consider all the red cards to be the same and then another where you consider them to be different.)

Now let's substitute that back into our expression for $P(\text{first k are red})$:

$$\begin{aligned} P(\text{first k are red}) &= \sum_{i=k}^{100} P(n = i)(\text{first k are red }|n = i) \\ &= \sum_{i=k}^{100} \frac{1}{101} \cdot \frac{\binom{i}{k}}{\binom{100}{k}} \\ &= \frac{1}{101} \cdot \frac{1}{\binom{100}{k}} \cdot \sum_{i=k}^{100} \binom{i}{k} \end{aligned}$$

Now we can use a theorem not mentioned in the article, the Baseball Theorem, which says that $\sum_{i=a}^{b} \binom{i}{a} = \binom{b+1}{a+1}$, to write:

$$\begin{aligned}
P(\text{first k are red}) &= \frac{1}{101} \cdot \frac{1}{\binom{100}{k}} \cdot \sum_{i=k}^{100} \binom{i}{k} \\
&= \frac{1}{101} \cdot \frac{\binom{101}{k+1}}{\binom{100}{k}} \\
&= \frac{1}{101} \cdot \frac{\frac{101!}{(k+1)!(100-k)!}}{\frac{100!}{k!(100-k)!}} \\
&= \frac{1}{101} \cdot \frac{101!!}{100! \cdot (k+1)!} \\
&= \frac{1}{101} \cdot \frac{101}{k+1} \\
&= \frac{1}{k+1}
\end{aligned}$$

(Alternatively, you can notice that $\sum_{i=k}^{100} \binom{i}{k}$ is the same as choosing $k+1$ of the first 101 natural numbers, doing casework on the largest number you choose. For example, if the largest number you choose is 50, then there are $\binom{49}{k}$ options for other k numbers since there are 49 natural numbers less than 50 we are choosing $k+1$ numbers total. The largest number you choose can range from $k+1$ to 101, so $\binom{101}{k+1} = \sum_{i=k+1}^{101} \binom{i-1}{k} = \sum_{i=k}^{100} \binom{i}{k}$.)

Finally, we can plug everything into our equation for $P(n = 100|$ first k are red) to obtain:

$$\begin{aligned}
P(n = 100| \text{ first k are red}) &= P(\text{first k are red } | n = 100) \cdot \frac{P(n = 100)}{P(\text{first k are red})} \\
&= 1 \cdot \frac{\frac{1}{101}}{\frac{1}{k+1}} \\
&= \frac{k+1}{101}
\end{aligned}$$

The question asks for the least whole number k that satisfies the inequality $P(n = 100|\text{first k are red}) = \frac{k+1}{101} > .5$, so the answer is clearly $\boxed{50}$.

When I first solved the problem, I was surprised by this result; I thought the answer was going to be much larger. Part of this came from flawed logic that should be avoided when dealing with conditional probability questions. For example, the argument "the answer is 99 because after drawing 99 red cards the remaining card can either be red or black, so there is a 50% chance of drawing a red card" could make sense initially, but it assumes that the last card is red or black with equal probability. In fact, there is a much higher chance of drawing red because if we started with 99 red cards and 1 black card, it would be very unlikely that the first 99 cards were all red (1% chance). There is an equal probability of starting with 99 red cards as starting with 100 red cards, but after observing the 99 red cards in a row we should come to the conclusion that 100 red cards is much more likely.

3.6 Inference #3

HMMT Feb 2019 Guts #29: Yannick picks a number N randomly from the set of positive integers such that the probability that n is selected is 2^{-n} for each positive integer n. He then puts N identical slips of paper numbered 1 through N into a hat and gives the hat to Annie. Annie does not know the value of N, but she draws one of the slips uniformly at random and discovers that it is the number 2. What is the expected value of N given Annie's information? (Note: Uses Calculus)

We want to find $E(N|\text{draws } 2)$. Using the definition of expected value and Bayes' Theorem, we can write

$$\begin{aligned} E(N|\text{draws } 2) &= \sum_{i=2}^{\infty} i(N = i|\text{ draws } 2) \\ &= \sum_{i=2}^{\infty} i(\text{draws } 2\ |N = i) \cdot \frac{P(N = i)}{P(\text{draws } 2)} \end{aligned}$$

$P(\text{draws } 2|N = i) = \frac{1}{i}$ since there are i slips and Annie chooses one of them at random (technically if $i < 2$ than it would be 0 but in our sum i starts at 2).

$P(N = i) = 2^{-i}$ because that is given in the problem.

Can you guess what we are going to do to compute $P(\text{draws } 2)$? Yes, for the fifth time we are going to use the Total Probability Theorem.

$$\begin{aligned} P(\text{draws } 2) = \sum_{i=2}^{\infty} P(\text{draws } 2\ |N = i)(N = i) \\ = \sum_{i=2}^{\infty} \frac{1}{i} \cdot 2^{-i} \end{aligned}$$

This sum looks like a geometric series with a common ratio of $\frac{1}{2}$, but unfortunately we have that $\frac{1}{i}$ messing it up. In order to get rid of the $\frac{1}{i}$, we need to do something that will result in each term being multiplied by i. It will be easier to see if we replace $\frac{1}{2}$ with x and then plug in $\frac{1}{2}$ for x later:

$$f(x) = \sum_{i=2}^{\infty} \frac{x^i}{i}$$

and $P(\text{draws } 2) = f(\frac{1}{2})$.

Now it is clear that we have to differentiate:

$$
\begin{aligned}
f'(x) &= \sum_{i=2}^{\infty} \frac{i \cdot x^{i-1}}{i} \\
&= \sum_{i=2}^{\infty} x^{i-1} \\
&= \sum_{i=1}^{\infty} x^{i} \\
&= \frac{1}{1-x} - 1 \quad \text{provided that } |x| < 1, \text{ which it is.}
\end{aligned}
$$

To solve for $f(x)$, integrate:

$$\int f'(x)dx = \int (\frac{1}{1-x} - 1)dx$$

$$f(x) = -\ln(1-x) - x + C$$

To solve for C, plug in 0 for x:

$$f(0) = -\ln(1-0) - 0 + C = \frac{1}{1-0} - 1 = 0$$

Thus, $C = 0$, $f(x) = -\ln(1-x) - x$, and $P(\text{draws } 2) = f(\frac{1}{2}) = \ln(2) - \frac{1}{2}$
Now we can go back to our equation to compute $E(N|\text{ draws } 2)$

$$
\begin{aligned}
E(N|\text{ draws } 2) &= \sum_{i=2}^{\infty} i(\text{draws } 2\ |N = i) \cdot \frac{P(N = i)}{P(\text{draws } 2)} \\
&= \sum_{i=2}^{\infty} i \cdot \frac{1}{i} \cdot \frac{2^{-i}}{\ln(2) - \frac{1}{2}} \\
&= \frac{1}{\ln(2) - \frac{1}{2}} \cdot \sum_{i=2}^{\infty} \frac{1}{2}^{i} \\
&= \frac{1}{\ln(2) - \frac{1}{2}} \cdot \frac{1}{4} \cdot \frac{1}{1 - \frac{1}{2}} \quad \text{(sum of geometric series)} \\
&= \frac{1}{\ln(2) - \frac{1}{2}} \cdot \frac{1}{4} \cdot 2 \\
&= \boxed{\frac{1}{2\ln(2) - 1}}
\end{aligned}
$$

Election Forecasting

Benjamin Kreiswirth, Theo Schiminovich, Mario Tutuncu-Macias

Introduction

The COVID-19 has suspended traditional campaign events and overshadowed the 2020 elections. Despite this, we have created an election forecast to predict likely scenarios of this year's presidential race.

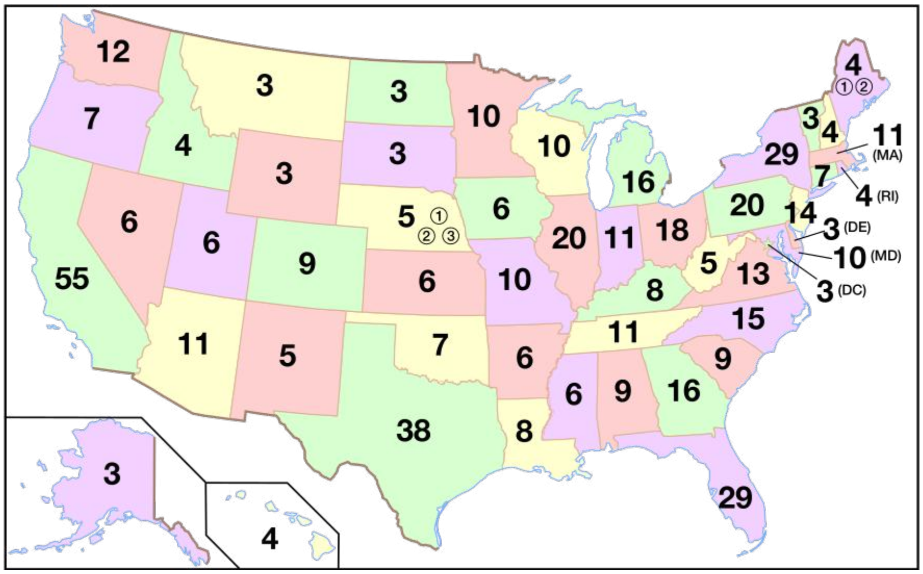
The following election forecast is based on the predicted outcome of each state. Notably, without a lot of technology or modelling, many states (at least 30) are easy to predict, as they tend to always vote for the same party.

Terminology and Background Info

Electoral College

Each state is given a certain number of electoral votes based on its population. This is determined by the 2010 census (the 2020 census will apply starting with the 2022 midterms, assuming COVID-19 does not disrupt the standard process). A majority of the electoral votes are required to win the election, and since there are only 2 major parties (the Democratic and Republican Parties) that win states (sorry Libertarians and Green Party), it amounts to winning 270 electoral votes out of the total 538 (with 269 leading to a complex congressional decision that would ultimately end in a Republican victory because there are more GOP states).

A map of the states by electoral votes is shown below. On a side note, notice how this map satisfies the 4 color theorem.



The States that matter

Many of these states are not a contest. The Democrats will win New York and Oregon, while the Republicans will win Arkansas and South Carolina. This wasn't always the case; during the mid 20th century, candidates could nearly sweep the entire country. However, today, the job of election forecasters, comes down to predicting the results in these states:

Arizona, Florida, Maine, Michigan, Minnesota, Nevada, New Hampshire, North Carolina, Ohio, Pennsylvania, Virginia, and Wisconsin

You could argue that other states belong on this list, or that some states on this list don't belong here, but this is a baseline grouping of the swing states. Determining the swing states is a job in of itself, that all campaign managers will have to do when allocating their advertising budget. Some other states whose results we were wary of while creating our model included Georgia, Iowa, and Texas. However, in the end, we decided they were not going to be swing states.

Professional Modelling

Naturally, many organizations and companies want to predict the results of elections, just as we do. One of the most popular sites during the 2016 election cycle was fivethirtyeight.com. This site became famous partly for correctly predicting every state (and D.C.) in the 2012 presidential elections. They were not as accurate in 2016, correctly predicting out 46 of the 51 contests and gave a final 71.4% chance to Clinton winning.

Sites like FiveThirtyEight have taken into account far more factors than we have, but we take an in-depth look at how that affected our modelling in the last section.

Professional modellers quantify uncertainty with advanced modelling. In addition to that, uncertainty increases with time. Additionally, debates, VP announcements, COVID-19, and a host of other possibilities can create substantial changes in the upcoming election.

Primary Modelling

The primaries are arguably harder to predict than the general election, with more candidates, an election spread out over several months, and highly unpredictable voter turnouts.

FiveThirtyEight released a widely popular model. They have a fairly comprehensive article on how they forecast here.

In addition to applying many of their features from above, the primary model also accounts for endorsements. For example, they predicted months before the actual event occurred that if Buttigieg were to drop out, he would likely endorse Biden.

Here are their steps, in exact words, from their site:

1. "Calculate national and state polling averages and translate them into a polling-based forecast
2. Calculate a non-polling forecast for each state based on demographics, geography and state fundraising
3. Begin simulating the rest of the primary process, starting with day-to-day movement in the polls
4. Simulate state and district results — accounting for uncertainty — and allocate delegates
5. Simulate bounces (or crashes) from winning (or losing) primaries
6. Forecast the probability that candidates drop out."

Our Modelling

We spent a lot of time working on our model, but relative to professional models, we have limited resources. That is to say, do not put your faith entirely into what our model says, especially when compared to professional models. Our goal was to emulate such models as well as we could, with a budget of $0 and access only to publicly available census data and past voting data. In short, there is no way that our model would have the level of precision professional polls show.

Our modelling involved multiple steps, including factoring in demographics, past election results, national polling, and state-wide polling. We will go through how we used each of them in detail over the next few sections, including a final prediction and its many caveats.

Computing Demographic Leans

Our first step in modelling the election outcome was to look at a variety of demographic data for each state. This included race, college education, age, and sex. For each state, we took the percentage of the population that fell into each demographic group and multiplied it by the percentage of that demographic expected to vote for Democrats based on past elections. The sum of these products results in an expected percentage of the population of a particular state that will vote for Democrats. By comparing this to what we would expect if we performed the same process using demographic breakdowns for the entire country, we are able to compute a predicted demographic lean.

Let's do an example using breakdown by religion in New Jersey.

	New Jersey %	% Democrats	% Percent Democrats
Protestant	33%	42.9%	14.1%
Catholic	34%	50.5%	17.2%
Jewish	6%	82.3%	4.9%
Other	9%	76.0%	6.8%
Unaffiliated	18%	71.4%	12.9%
Total	100%	-	55.9%

We do the above for each state, then do the same for the United States as a whole to find a baseline.

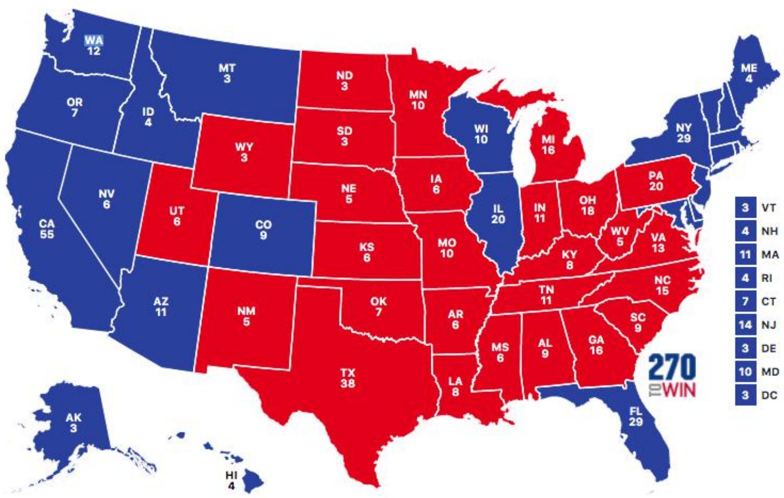
	United States %	% Democrats	United States % Democrats
Protestant	49.4%	42.9%	21.2%
Catholic	20.6%	50.5%	10.4%
Jewish	1.9%	82.3%	1.6%
Other	5.5%	76.0%	4.1%
Unaffiliated	22.6%	71.4%	16.2%
Total	100%	-	53.5%

This gives us an expected religion lean for New Jersey of $55.9\% - 53.5\% = 2.4\%$, or D+2.4.

We can repeat this process for each of the demographics in each of the states to determine additional leans. Note that for the other demographics we also considered varying voter turnout among groups, which required additional computation and increased accuracy of resulting leans (e.g., old people and white people tend vote more). Also note that the maps in the following sections do not represent predictions. They merely represent parts of our investigation using the method we just discussed.

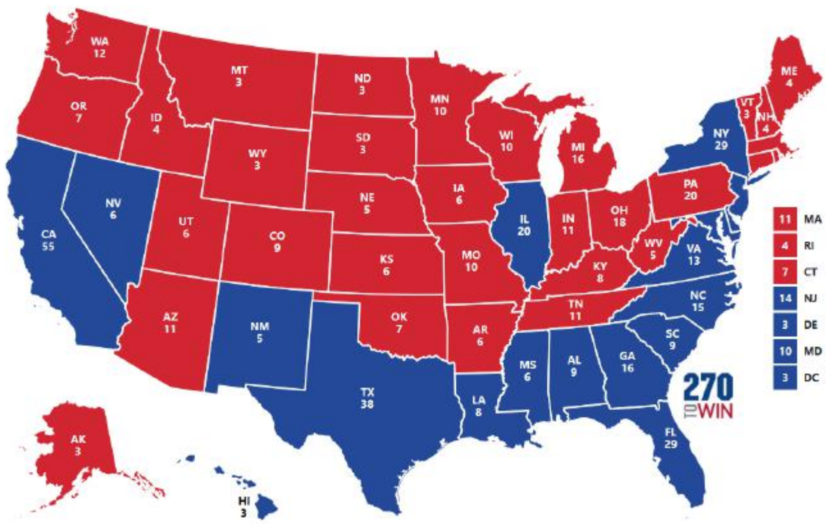
Religion

We found that the most predictive demographic was religion. This is not to say that the other demographics were not useful; we will discuss how they were used in detail. However, if we had blindly decided to predict an election based purely on religious affiliation statistics, we would have had pretty accurate results. The map below shows what religion predicts alone:



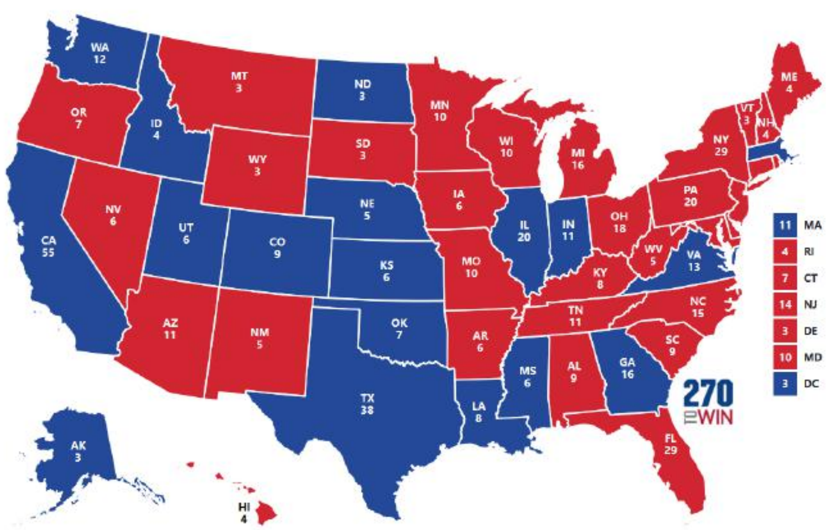
Race

Another demographic we investigated was race. Black and Latino voters vote overwhelmingly for the Democratic party, while whites tend to have a preference for the Republican party. However, we also distinguished between non-college-educated and college-educated whites. They also have significant differences in voting preferences. In the end, even accounting for this, our race numbers predicted the South would go Democrat due to its high black population, while some heavily white Northern states would go Republican.



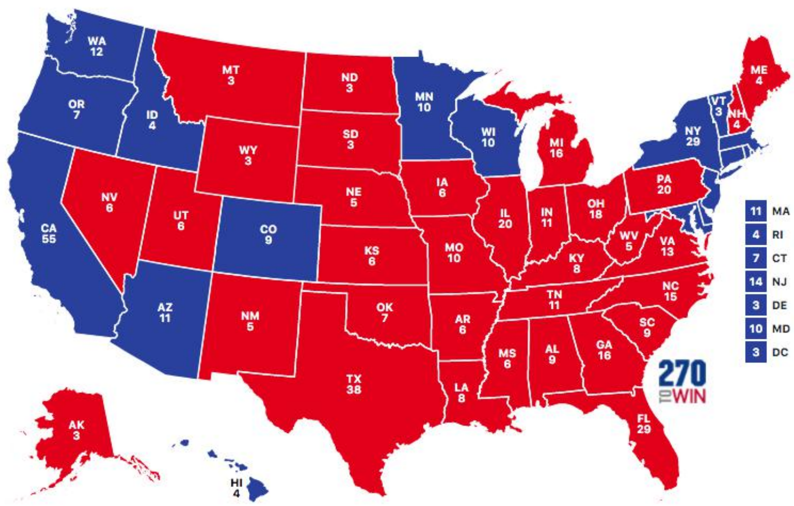
Age

Age seems to be relatively impactful. If one just considers age across the entire country, there are stark differences between wildly Republican older voters and wildly Democratic younger voters. However, it turns out that the difference between the age breakdown among states is remarkably insignificant. One hypothesis for this is that although Republicans tend to be older, Republican states tend to have a higher fertility rate. The following map is wildly incorrect, but the magnitude of the Republican and Democratic leans produced by age are very small and do not end up being factored significantly into the data.



Income

The income map is pretty accurate-looking:



In reality, all the colors shown above are the reverse of the actual result our process produced. This is because individuals who are poorer on average vote Democrat (and that is the metric we used in our process), but richer states on average vote Democrat. This means that we found a counter-intuitive but strong negative correlation for income. This was also accounted for in our final tabulations. To learn more about this strange phenomenon, there are whole books about it, such as *Red State, Blue State, Rich State, Poor State*.

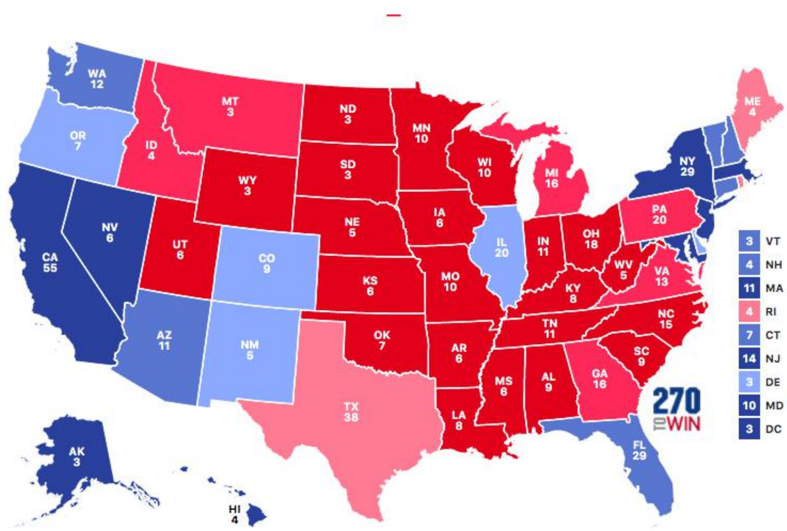
Computing State Leans

We computed final partisan leans for each state by factoring in demographic leans and 2016 results. The final partisan leans are an estimate of how many points more Democratic or Republican a state should be than the national average. This involved two major steps in which we had to weight the various different factors in a mathematically sound manner.

Weighting the Demographics

Our primary hypothesis weights income, religion, age, and race. However, there are many ways to weight them which would all produce the same optimal number of correct predictions. To distinguish between them, we had two secondary factors: (1) find weightings that also made the magnitude of the demographic leans match closely expected magnitudes; (2) find weightings that made the standard deviation of leans similar but slightly larger than 2016.

Once we worked for an extended period of time attempting to optimize these factors, we settled on weightings which correctly predicted 36 of 38 non-swing states, and also were of approximately the right magnitude. The following map is nothing like our final prediction, but is one of the major inputs into our final prediction. This map depicts a demographic lean of 8 or more as dark red/blue, 4 to 8 as medium red/blue, and 0 to 4 as light red/blue. Note that these are estimated leans from the popular vote, so it will look a lot more red than a typical map because in recent elections the popular vote has normally favored Democrats.



This is only obviously incorrect in Rhode Island and Alaska. However, we also note that Florida and Arizona are too Democratic while Minnesota and Illinois are too Republican even considering the expectation of it being redder than usual. However, setting these edge cases aside, this weighting does a remarkably good job predicting the outcome. It will help account for demographic change from 2016 to today.

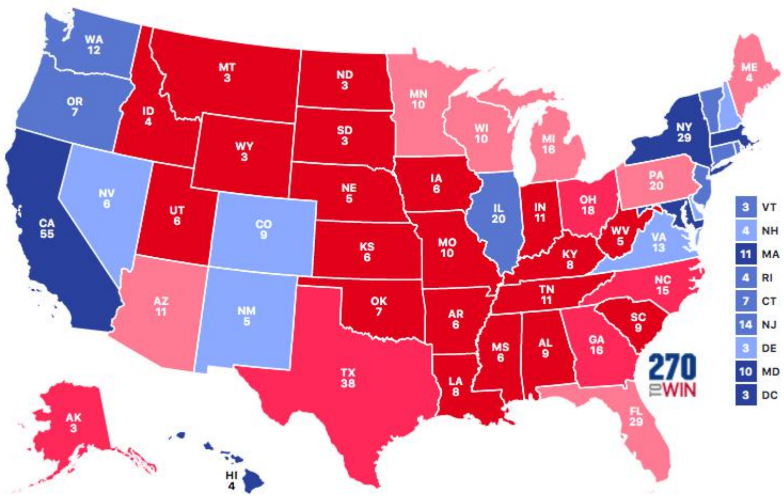
Weighting Past Elections

At first, we were interested in incorporating 2018 results into the forecast. However, there are at least three issues with this. First off, many congressional elections are simply uncontested, so in a state that is overwhelmingly one way or the other we wouldn't end up seeing that in total vote count for 2018. Secondly, congressional districts are heavily gerrymandered so taking them into account would basically amount to taking into account control of the state legislatures ten years ago. Finally, even if the prior two issues didn't exist, voter turnout is both far lower and differently distributed among the population in midterm elections than in presidential elections.

This means that we have ended up sticking primarily to 2016 results, while also augmenting them with our demographic lean. Our exact process was to compute the partisan lean of each state in 2016 by comparing the popular vote margin within the state to the popular vote margin in the entire country. For example, Minnesota went for the Democrats by a margin of 0.8%, but since the U.S. went for the Democrats by a margin of 1.1%, we say that Minnesota has a 2016 lean of 0.3% for the Republicans. Combining the 2016 lean with the demographic lean gives us a final 2020 lean for each state.

Final 2020 Leans

The following map depicts the final 2020 leans. Once again, more than 8% counts as a strong lean, $4 - 8\%$ counts as a medium lean, and less than 4% counts as a weak lean. It requires the same disclaimer as the demographic leans - it should look redder than expected given that Democrats have won the popular vote in the past 3 elections.



Final Predictions

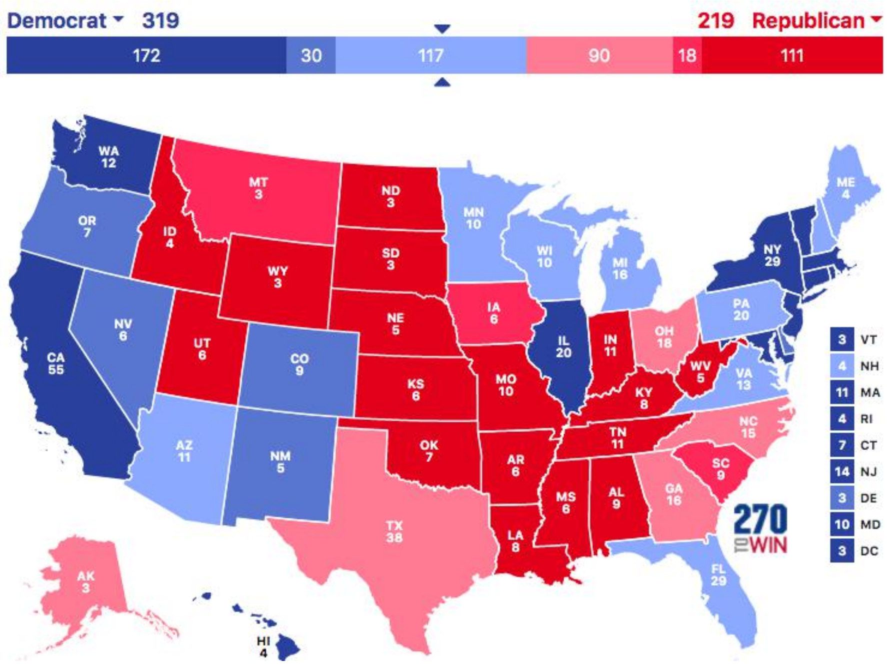
To make final predictions, we need to take the most important step - including current polling. Once we do this, we can make some final adjustments and then produce our completed map.

National Polling

Polling at the time of writing, that is, late March and early April, will not be very predictive of the actual result come Election Day. However, it is all that we have to work with so we have to use it to the best of our ability. We sourced our polls from a comprehensive list of polls FiveThirtyEight compiles. However, we did not use any aspect of FiveThirtyEight's poll rating system or any of FiveThirtyEight's general election modelling - all of our results are coming straight from raw data.

At risk of being too simplistic, we took an straight average of national polls performed in the last month. Given that we have no way of knowing the reliability of each pollster without intensive research, this was the best option available to us. In the end, this gave us an average of 53.5% Biden to 46.5% Trump. This is a pretty hefty lead for Biden, but may not spell Trump's doom, due to the quirks of the Electoral College. After all, in two of the past five elections, the candidate with the most votes has lost.

Then, we can use our partisan leans for each state computed in the past section to come up with a baseline prediction for each state. Note that these predictions do not factor in statewide polling. The following map once again splits colors at 8% and 4%, but now it represents how much above / below even will each state be at this point in our predictions.

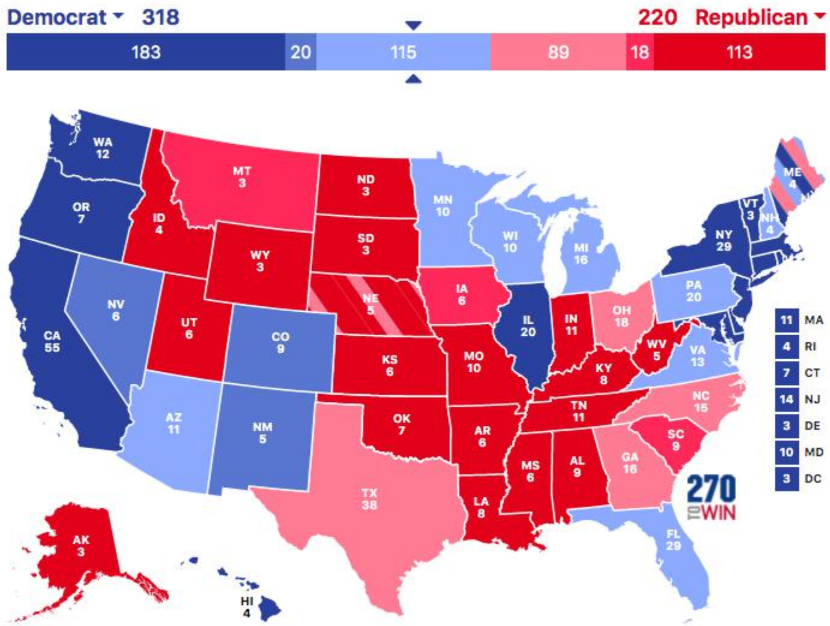


State Polling

State polling provides a finer level of detail which so far is missing from the model. It is scarce in non-swing states, but this is not much of an issue since their results are already pretty much set. However, we factored in the state polling where it existed in such a way that the more polls were available, the more they counted relative to the numbers using only national polling shown above.

State polling is weighted into the final map that can be found in the next section.

Final Map



Caveats

Probability of our Correctness

We attempted to find the probability that our model was right. First, we found the "middle state" - that is, the state such that if all the more Democrat-leaning states went Democrat, and all the more Republican-leaning states went Republican, it would be the deciding vote. In our model, the state happened to be Arizona. Then, since Arizona would flip the election we found the probability that it would go for Biden. In the end, we got a 78.3% chance that Biden would win Arizona, and by our assumption, the election.

There are flaws with using this number for our uncertainty, given that we don't know for sure our model correctly orders the states' partisan leans. To try to correct this problem, we also tried finding the same probabilities for 13 swing states, and used a program to compute the probability of each combination occurring. This gave a probability that Biden would win of 96.2%. However, this number is also flawed, because state results are correlated with each other. For example, if Republicans win Michigan, Republicans are more likely to win Wisconsin as well, because it is likely that if the polls had a Democratic bias in one of these states, it would have a similar bias in the other.

In addition, these uncertainty numbers still assume the election is happening right now. We will discuss various future events that could have an impact on the results and mean that our uncertainty should be much higher.

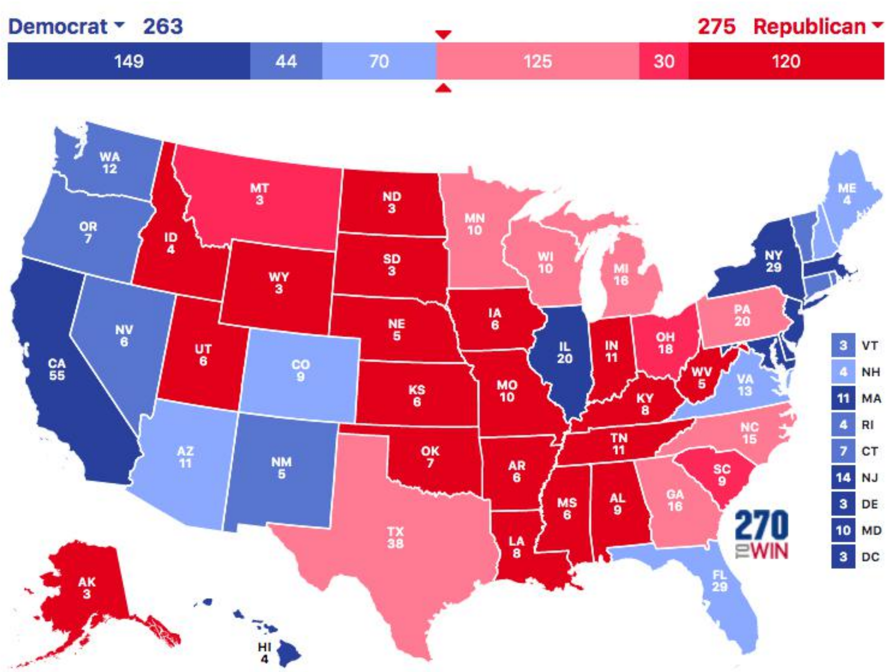
Ignoring Third Parties

During this process, we disregarded the effect that third parties have on polling numbers. We took every poll and assumed those who want third parties or who are undecided don't exist. In different areas, third parties take different portions of the vote. Furthermore, in different areas, undecided voters end up voting differently. Given their relative levels of unpopularity among their respective parties, Biden or Trump may end up losing votes to third parties to an extent that flips the result in very close states.

Trump Polling Inaccuracies

In 2016, the polls were on average more Democratic than the final election day results. It was not as inaccurate as many believed, but a number of swing states were incorrectly predicted. In one of the most extreme examples, Wisconsin polling averages gave Clinton a 5% lead, but she ultimately lost the state by 0.5%. In an attempt to correct for this, we considered polling averages for Clinton vs. Trump in 2016 around the time Sanders dropped out (so in theory they should be similarly accurate to ones at the time of writing), and compared them to actual results. We then took this discrepancy and adjusted the 2020 polls according to said discrepancy. To do this we simply compared the polling average before the election with the final election results to computer how "wrong" the polls were, and assumed the same level lean in polls for this election cycle.

After doing this, we get a new map of results which is less clear-cut:



Incumbent Polling Inaccuracies

This election is different from 2016, because Donald Trump is an incumbent. This means voters have seen how he is in office. This may mean polling numbers are more accurate for Trump than they were in 2016, because voters have had a much longer time to form an opinion on him.

Furthermore, Trump would receive an incumbency advantage, whereby the incumbent president is expected to do better than in their first election. Since 1912, the incumbent has on average increased their popular vote percentage by 3.4% in their re-election. This is generally explained by voters with not particularly strong feelings who end up voting for the president, as they are more well known and usually seen as a "safer" pick, as they represent the status quo. We did not account for this advantage in our model.

Future Events

There are also many things that could happen between now and Election Day which could have a significant effect on the outcome. Biden's vice president choice could give him a boost, depending on who he chooses; people who are excited by his vice presidential pick may be more likely to vote for him. Trump is less likely to experience this effect, because he already has a vice president.

COVID-19

There is also the elephant in the room: COVID-19. The coronavirus outbreak has upended millions of lives across America. This could have a variety of different effects, depending on how the post-COVID-19 economic revival plays out. It would probably have the biggest effect on Trump's polling, because he has more control over the situation. It could boost Trump's numbers if Americans feel he handled the coronavirus crisis well, or it could decrease his numbers if they feel he did not meet expectations.

Another potential impact of COVID-19 is called the "rally behind the flag effect." Usually, in crises throughout history, the current leader spikes in popularity. Here are some examples:

- George W. Bush went from 55% to 90% approval after 9/11
- FDR went from 70% to 85% approval after Pearl Harbor
- JFK went from 60% to 75% after the Cuban Missile Crisis

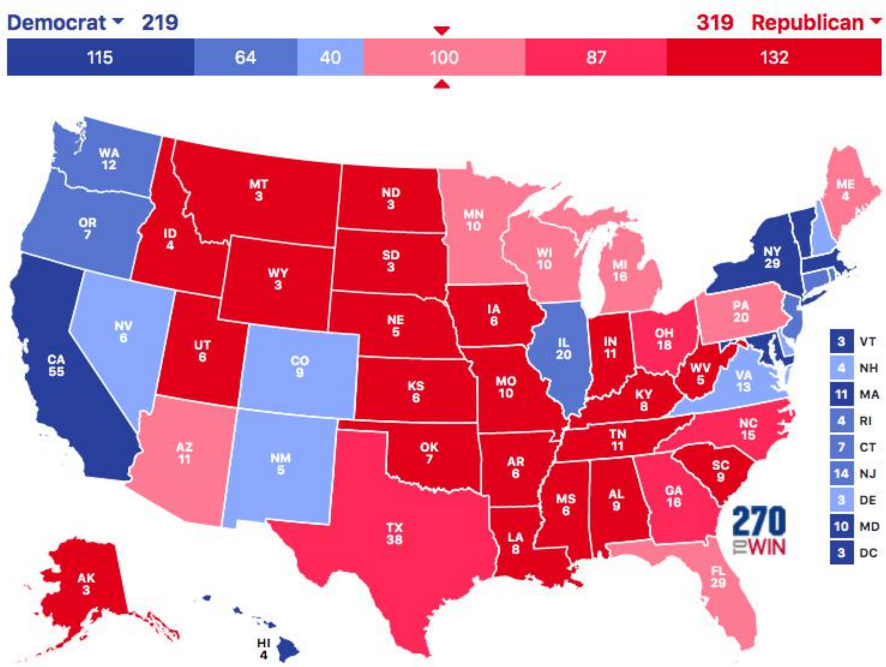
It is worth noting that Trump has not yet seen such a boost, but one could be coming that could affect the election.

A Specific Future Event

In the time between when we gave our editor this article and our editing the article, Biden's sexual assault allegation has been one of the few media stories that manages to last in the age of coronavirus. In addition to potentially affecting Biden's poll numbers and Democratic turnout, this also functions as an example of how hard it is to predict what is going to happen in November, especially at such an uncertain time.

A Final Note

In 2016, Donald Trump won the Electoral College despite losing the popular vote. We created a map demonstrating what the results of each state would be if the popular vote was exactly 50/50. We fixed the popular vote and adjusted every state according to its lean. In this scenario, Trump would be given more electoral votes than Biden:



When Biden and Trump receive an equal share of the popular vote, the Republicans are given a strong victory. The electoral college gives more votes per million people to smaller states than larger ones. Because smaller states tend to lean Republican, Donald Trump would receive 319 electoral votes compared to Biden's 219 votes.

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Euler’s Formula

Maxwell Zen

A Proof of Euler’s Formula

Euler’s formula states that for any convex polyhedron, $V - E + F = 2$, where V is the number of vertices, E the number of edges, and F the number of faces. This formula seems to hold after a few test cases; a tetrahedron has 4 vertices, 6 edges, 4 faces, and $4-6+4=2$; a cube has 8 vertices, 12 edges, 6 faces, and $8-12+6=2$, and so on. We will prove this result and show some of its interesting applications.

Realize that it is possible to map any convex polyhedron to a sphere of radius 1. We can perform this mapping for two reasons.

First, two points on a sphere are connected by a single, unique line. This is the shortest distance between these two points and lies on a ”great circle.”

Second, convex polyhedra are topologically equivalent to spheres, so you can map one to the other without changing the numbers of vertices, edges, or faces. This also poses an explanation as to why Euler’s Formula fails for concave polyhedra - mapping a polyhedron onto a sphere requires that every vertex can be connected to every other vertex by a line that is completely within the sphere, and a concave polyhedron by definition has a line between two vertices that doesn’t lie within the solid.

Now that we know we can perform this mapping, let’s define a spherical polytope as a set of points on the surface of a sphere connected to each other by arcs of great circles to form curved faces.

Lemma 1: We will prove that the area of a spherical triangle with angles α, β, γ (measured in radians) is equal to $\alpha + \beta + \gamma - \pi$.

Proof. First, we’ll need to talk about the area of a lune. A lune is the intersection between two great circles, where the angle at the intersection of the great circles is α(Figure 1). Since angles traditionally exist on flat planes, we can define α as the angle between tangents of the two great circles at their intersection point. Note that the area of the lune increases proportionally to α. Also notice that when $\alpha = 2\pi$, the area of the lune is the surface area of the sphere, which is 4π for a sphere of radius 1. Thus, the area of a lune is $2 \cdot \alpha$

Next, we’ll need to talk about the inclusion-exclusion formula. If we have 3 areas A, B, and C, that (possibly) overlap, we can find the area of their intersection with the formula $A \cap B \cap C = A \cup B \cup C - A - B - C + A \cap B + A \cap C + B \cap C$.

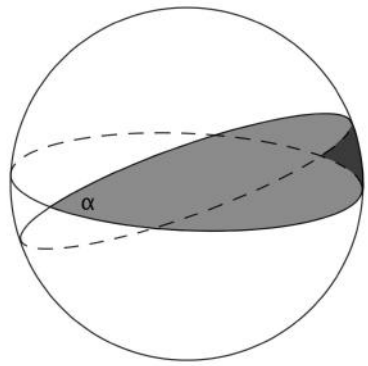

Figure 5.1: A lune with angle α on a sphere of radius 1

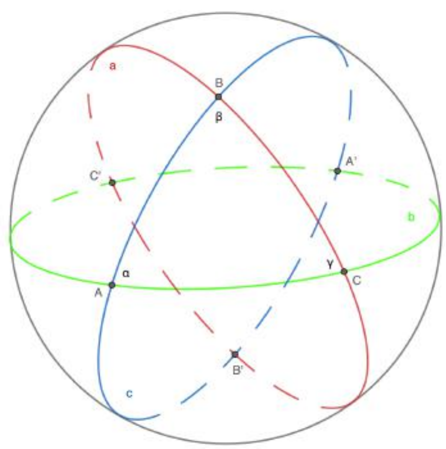

Figure 5.2: Triangle ABC on a sphere of radius 1

Now, we're finally prepared to tackle the area of a triangle on a sphere. We begin with triangle ABC on a sphere of radius 1, with angles α, β, and γ. Extend each side to form its corresponding great circle, and notice that these great circles also intersect at A', B', and C'. We may define hemisphere a as the hemisphere whose base is the great circle defined by $\overline{BC}$ and that includes A in its area. We define hemispheres b and c likewise (see Figure 2). Notice that the area of our triangle is the intersection of our 3 hemispheres, or $a \cap b \cap c$. Knowing this, we can use the inclusion-exclusion formula to get the following equation:

$$\Delta ABC = a \cup b \cup c - a - b - c + a \cap b + a \cap c + b \cap c$$

First, let's take care of the easy parts: the areas of a, b, and c, are each the area of a hemisphere, which is 2π. Thus, the equation becomes:

$$\Delta ABC = a \cup b \cup c - 6\pi + a \cap b + a \cap c + b \cap c$$

Next, $a \cap b, a \cap c$, and $b \cap c$ are all lunes whose area is twice the angle between their great circles. In this case, these angles are α, β, and γ.

$$\Delta ABC = a \cup b \cup c - 6\pi + 2\alpha + 2\beta + 2\gamma$$

Finally, notice that $a \cup b \cup c$ includes the entire sphere, minus triangle $A'B'C'$. Since we've defined a, b, and c so that triangle ABC is the only area covered by all 3, the only area excluded by all 3 should lie exactly opposite ΔABC. Thus, the area of $a \cup b \cup c$ is the area of the sphere minus the area of ΔABC, or $4\pi - \Delta ABC$. We now have the equation:

$$\Delta ABC = 4\pi - \Delta ABC - 6\pi + 2\alpha + 2\beta + 2\gamma$$

Solving for the area of ΔABC, we get

$$2 \cdot \Delta ABC = 4\pi - 6\pi + 2\alpha + 2\beta + 2\gamma$$

$$\Delta ABC = \alpha + \beta + \gamma - \pi$$

as desired. $\square$

Now we've figured out how to find the area of a spherical triangle. What about any polygon on a sphere? Recall the proof you may have encountered for the sum of angles in a polygon - take the polygon, split it up into triangles, and add up the angles of each triangle. We'll take the same approach here - take any spherical n-gon, split it up into $n-2$ triangles, and add up the areas. When we add up the angles of each triangle, we get the sum of angles of the n-gon, and when we subtract π for each of the triangles, we subtract π $n-2$ times. Thus, the area of any n-gon is the sum of its angles minus $(n-2)\pi$.

Finally, using that formula, we're ready to tackle the spherical polytope. Again, this polytope has V vertices, E edges, and F faces. We know that the surface area of the sphere is 4π, but we can also find this area by adding up the area of each polygon. First, we'll sum the angles of every polygon, then subtract $n\pi$ for each n-gon, and then add 2π for each polygon.

First, summing the angles of every polygon is equal to adding 2π for each of the V vertices, since the angles around each point form a full circle, so this contributes $2V\pi$.

Second, subtracting $n\pi$ for every n-gon is the same as subtracting π once for every side, but since each of the E edges contributes a side to 2 polygons, we subtract $2E\pi$.

Finally, adding 2π for each of the F faces contributes $2F\pi$. Now that we have all the pieces, let's put them together:

$$4\pi = 2V\pi - 2E\pi + 2F\pi$$

And finally, we divide both sides by 2π to get Euler's Formula:

$$2 = V - E + F$$

QED

Application to Platonic Solids

Let's use Euler's formula to prove that there are only 5 Platonic solids. A Platonic solid is a regular, convex polyhedron, meaning every face has the same number of sides and every vertex connects the same number of faces. Let's define k as the number of sides on each face and l as the number of faces that meet at each vertex. From Euler's formula, we have:

$$V - E + F = 2$$

Since each face coincides with k edges and each edge coincides with 2 faces, $k \cdot F = 2E$, and $F = \frac{2E}{k}$. This gives us:

$$V - E + \frac{2E}{k} = 2$$

Next, since l faces meet at each vertex, l edges meet at each vertex, but each edge connects to 2 vertices, so we get $l \cdot V = 2E$, and $V = \frac{2E}{l}$. This gives us:

$$\frac{2E}{l} - E + \frac{2E}{k} = 2$$

Bring E to the right-hand side:

$$\frac{2E}{k} + \frac{2E}{l} = 2 + E$$

Divide both sides by $2E$:

$$\frac{1}{k} + \frac{1}{l} = \frac{1}{E} + \frac{1}{2}$$

$$\frac{1}{k} + \frac{1}{l} > \frac{1}{2}$$

Now we that we have these conditions, we can start limiting cases. We know that k and l are at least 3: each face must have at least 3 sides, and each vertex has to connect at least 3 faces. We also know that neither k nor l can be above 5: if one were 6, the sum of their reciprocals would be at most $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$, violating the inequality. Also note that at least one of k and l must be 3; if neither was equal to 3, the sum of their reciprocals would be at most $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, also violating the inequality. With this last condition, we can establish 2 cases: either $k = 3$ or $l = 3$.

If $k = 3$, then l can be 3, 4, or 5. If $l = 3$, we form a tetrahedron. Next, if $l = 4$, we form an octahedron. Finally, if $l = 5$, we form an icosahedron.

Our next case is $l = 3$, and k can once again be 3, 4, or 5. If $k = 3$, we get the tetrahedron again. If $k = 4$, we get the cube. If $k = 5$, we get the dodecahedron. Thus, we've identified the 5 Platonic solids and shown that there cannot be any others.

Application to Archimedean Solids

Now let's move on to the Archimedean solids. An Archimedean solid is a solid whose faces are all regular polygons and whose vertices are congruent. Notice that we have removed the restriction that all faces have the same number of sides. This means that each vertex has to connect the same sequence of n-gons in the same order under rotation and reflection. There are 13 Archimedean solids, as well as the two infinite classes of prisms and anti-prisms. The classes of prisms and anti-prisms are both infinite because you can take any two congruent polygons and connect them by squares (to form a prism) or triangles (to form an antiprism). Since it's more helpful to count Archimedean solids that aren't prisms or antiprisms, these two infinite classes will be excluded from our proof. We'll introduce some new variables: r is the number of edges that coincide at each vertex, F_n is the number of n-gons present in the solid, p_n will represent the number of sides on one of the faces present at a vertex, and q_n will be the number of n-gons that coincide at a single vertex.

Lemma 2: $1 - \frac{r}{2} + \frac{1}{p_1} + \ldots + \frac{1}{p_r} = \frac{2}{V}$

Proof. First, we'll reuse the fact that $rV = 2E$ to get that $E = \frac{rV}{2}$. Next, notice that for all n, $n \cdot F_n$ counts each vertex once for each of the q_n n-gons present at each vertex. So we get $n \cdot F_n = V \cdot q_n$, and $F_n = \frac{V \cdot q_n}{n}$. Also notice that counting F_n for all n ends up counting all the faces. We can use that to form the following equations:

$$F_3 + F_4 + \ldots + F_{V-1} = F$$

We start at F_3 because every polygon has at least 3 sides, and end at F_{V-1} because it's impossible for one face to connect to every single vertex in a solid.

$$\frac{V \cdot q_3}{3} + \frac{V \cdot q_4}{4} + \ldots + \frac{V \cdot q_{V-1}}{V-1} = F$$

$$V \cdot (\frac{q_3}{3} + \frac{q_4}{4} + \ldots + \frac{q_{V-1}}{V-1}) = F$$

Notice that the sum of all $\frac{q_n}{n}$ for all n is the same as adding $\frac{1}{n}$ for each of the r polygons around a vertex. Thus we have:

$$V \cdot (\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_r}) = F$$

Now, we can substitute our values for E and F into Euler's formula:

$$V - E + F = 2$$

$$V - \frac{rV}{2} + V \cdot (\frac{1}{p_1} + \ldots + \frac{1}{p_r}) = 2$$

$$V \cdot (1 - \frac{r}{2} + \frac{1}{p_1} + \ldots + \frac{1}{p_r}) = 2$$

$$1 - \frac{r}{2} + \frac{1}{p_1} + \ldots + \frac{1}{p_r} = \frac{2}{V}$$

as desired. $\square$

Lemma 3: $r < 6$

Proof. From Lemma 2 and the fact that $\frac{2}{V} > 0$, we can say that:

$$1 - \frac{r}{2} + \frac{1}{p_1} + \ldots + \frac{1}{p_r} > 0$$

$$\frac{1}{p_1} + \ldots + \frac{1}{p_r} > \frac{r}{2} - 1$$

Now we make the following observations:

$$p_n \geq 3$$

$$\frac{1}{p_n} \leq \frac{1}{3}$$

$$\frac{1}{p_1} + \ldots + \frac{1}{p_r} \leq r \cdot \frac{1}{3}$$

$$\frac{r}{3} \geq \frac{1}{p_1} + \ldots + \frac{1}{p_r} > \frac{r}{2} - 1$$

$$\frac{r}{3} > \frac{r}{2} - 1$$

$$2r > 3r - 6$$

$$r < 6$$

as desired. □

Since we also know that $r \geq 3$ (at least 3 faces must meet at every vertex), we now have 3 cases: $r = 5$, $r = 4$, and $r = 3$. From here, we just go through the cases. Each solution is going to be expressed in terms the faces present at each vertex: for example, a tetrahedron would be expressed as (3,3,3) because 3 triangles surround each vertex, a cube would be expressed as (4,4,4) because 3 squares surround each vertex, etc. If one vertex formula can be reached by a shift or reflection of another vertex formula, they produce the same solid and will only be listed once.

Case 1: $r = 5$

Start by plugging r into Lemma 2. For simplicity, we'll replace p_n with a, b, c, d, and e.

$$1 - \frac{5}{2} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} > 0$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} > \frac{3}{2}$$

From here, we know that all of a, b, c, d, e except one must be 3. If two of them are greater than 3, their sum is at most $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} = \frac{3}{2}$, violating the inequality. Let's say $b = c = d = e = 3$. Then we have:

$$\frac{1}{a} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} > \frac{3}{2}$$

$$\frac{1}{a} > \frac{1}{6}$$
$$a < 6$$

This gives us 3 solutions: (3,3,3,3,3), (4,3,3,3,3), and (5,3,3,3,3). Since (3,3,3,3,3) is a Platonic solid (the icosahedron), we'll exclude that solution.

Case 2: $r = 4$

Again, we'll plug r into Lemma 2 and replace p_n with a, b, c, d.

$$1 - \frac{4}{2} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} > 0$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} > 1$$

We know that at least one of a, b, c, d must be a triangle: if none of them were triangles, their sum would be at most $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$, violating the inequality.

Let's assume $d = 3$. We now have $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{3} > 1$, and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > \frac{2}{3}$.

We're going to introduce a new method where we focus on a single polygon and draw conclusions about its sides. In this case, we're going to focus on the sides of a triangle, since we're guaranteed at least one triangle at each vertex. Remember that at each vertex, there must be an a-gon, a b-gon, a c-gon, and a triangle, in that order.

Therefore, since every b-gon must lie opposite the triangle at each vertex, it cannot share a side with the triangle. Now let's look at the sides of the triangle. We already know that none of these sides can be adjacent to a b-gon, so they must be adjacent to either an a-gon or a c-gon. Notice that if one side of the triangle connects to an a-gon, the next side must connect to a c-gon: otherwise if they both connect to a-gons, the vertex between them would have two a-gons in its formula, violating the (a,b,c,d) formula.

Thus, if one side of the triangle connects to an a-gon, then the other two sides of the triangle must both connect to c-gons (else we'd have a vertex containing 2 a-gons). But now we have two adjacent sides that both connect to c-gons, so the vertex between them contains two c-gons and we've reached a contradiction. The only way to escape this contradiction is to let $a = c$ so that two adjacent sides can both connect to a-gons without a contradiction.Substituting in $a = c$ gives us $\frac{2}{a} + \frac{1}{b} > \frac{2}{3}$

Now we just run through the cases: $a = 3$ gives us $\frac{1}{b} > 0$ and b can be anything (this is the infinite class of antiprisms). $a = 4$ gives us $\frac{1}{b} > \frac{1}{6}$ and b = 3, 4, 5. $a = 5$ gives us $\frac{1}{b} > \frac{4}{15}$ and b = 3. If $a = 6$, we have $\frac{1}{b} > \frac{1}{3}$, which is impossible so any a greater than 5 fails.

This gives us an infinite class as well as 4 solutions: (3,3,3,m), (3,4,3,4), (3,4,4,4), (3,4,5,4), and (3,5,3,5). We'll exclude the antiprisms as an infinite class.

Case 3: $r = 3$

$$1 - \frac{3}{2} + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 0$$

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > \frac{1}{2}$$

Once again, we'll replace p_n with a, b, and c, giving us $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>\frac{1}{2}$. Note that one of a, b, or c must be 3, 4, or 5: if they were all at least 6, their sum would be at most $\frac{1}{2}$, violating the inequality. So now we can do cases on the minimum value:

3.1: c = 3

By traversing along the sides of the triangle and using the same logic we've previously encountered, we can conclude that $a = b$, giving us $\frac{2}{a}+\frac{1}{3}>\frac{1}{2}$, so $\frac{2}{a}>\frac{1}{6}$ and $3 \leq a \leq 11$. To further narrow these choices down, we can traverse the sides of an a-gon. Notice that if one side is adjacent to another a-gon, the next side must be adjacent to a triangle, giving us two possibilities: either a is even, or $a = 3$. The solutions we get here are: (3,3,3), (3,4,4), (3,6,6), (3,8,8), and (3,10,10). We'll exclude (3,3,3) since that's a Platonic solid, and (3,4,4) since that's a prism.

3.2: c = 4, $a, b \geq 4$

We may substitute c into our inequality to get $\frac{1}{a}+\frac{1}{b}+\frac{1}{4}>\frac{1}{2}$, so $\frac{1}{a}+\frac{1}{b}>\frac{1}{4}$. We can again traverse the sides of an a-gon to get two cases: either $b = c = 4$, in which case we get $\frac{1}{a}+\frac{1}{4}>\frac{1}{4}$ and a can be anything (this is the infinite class of prisms), or $b \neq 4$, in which case a has to be even. The same logic for the b-gon leads us to conclude that if $a \neq 4$, then b is also even. However, note that either a or b is less than 8; if both were 8, their sum is $\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$, violating the inequality. Thus, assume $b = 6$. Our inequality gives $\frac{1}{a}+\frac{1}{6}>\frac{1}{4}$, so $\frac{1}{a}>\frac{1}{12}$ and a = 6, 8, 10. So the solutions here are: (4,4,m), (4,6,6), (4,6,8), and (4,6,10). We're excluding (4,4,m) as an infinite class.

3.3: c = 5, $a, b \geq 5$

Substituting c into our inequality gives us $\frac{1}{a}+\frac{1}{b}+\frac{1}{5}>\frac{1}{2}$, so $\frac{1}{a}+\frac{1}{b}>\frac{3}{10}$. We can traverse the sides of the pentagon to conclude that $a = b$. This gives us two solutions: $a = b = 5$ or $a = b = 6$. These solids are: (5,5,5) and (5,6,6). We'll exclude (5,5,5) because that's a Platonic solid.

Now that we've gone through all the cases, let's summarize the solutions(Figure 3):

Vertex Formula	**Solid**
(4,3,3,3,3)	Snubcube
(5,3,3,3,3)	Snub Dodecahedron
(3,4,3,4)	Cuboctahedron
(3,4,4,4)	Small Rhombicuboctahedron
(3,4,5,4)	Small Rhombicosidodecahedron
(3,5,3,5)	Icosidodecahedron
(3,6,6)	Truncated Tetrahedron
(3,8,8)	Truncated Cube
(3,10,10)	Truncated Dodecahedron
(4,6,6)	Truncated Octahedron
(4,6,8)	Great Rhombicuboctahedron
(4,6,10)	Great Rhombicosidodecahedron
(5,6,6)	Truncated Icosahedron

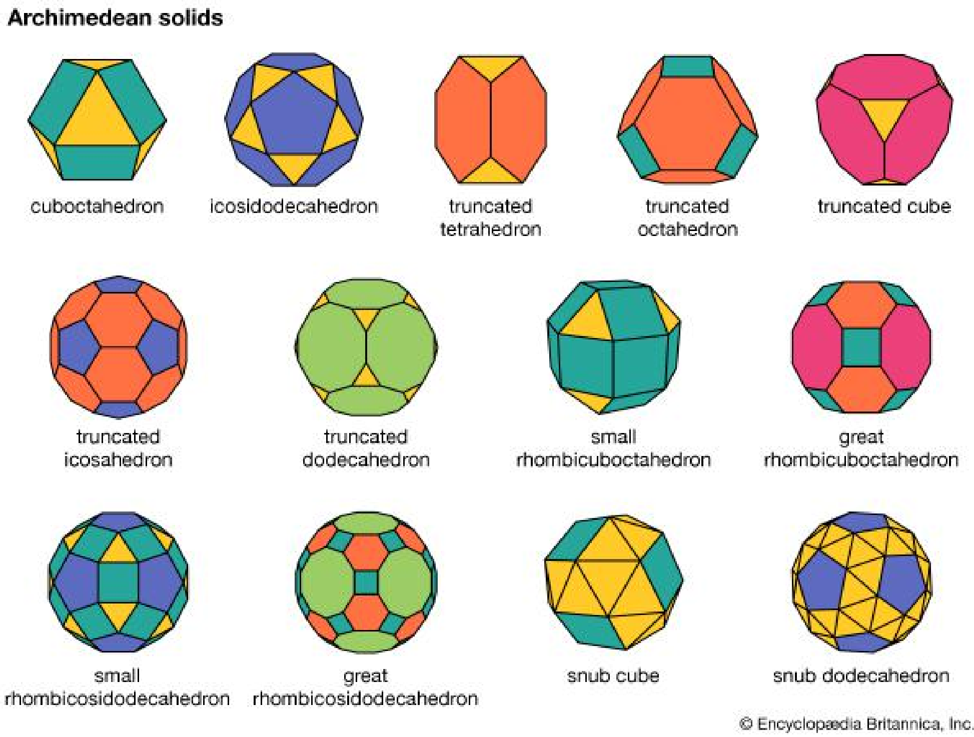

Figure 5.3: The 13 Archimedean Solids (taken from Brittanica Kids)

Application to Graph Theory

One of the more well-known applications of Euler's formula is through graph theory. A graph is a set of points connected to each other by edges. A planar graph is a graph that can be drawn on a plane so that its edges only intersect at vertices. As it turns out, Euler's formula also holds for planar graphs, but only if you include the infinite face (the area of the plane that's not contained by any edges) in your calculation. Knowing this, we can prove that some graphs are impossible to form without edges intersecting.

Problem 1: Can 5 points all be connected to each other by an edge without two edges intersecting?

Let's interpret this graph as a solid and plug in values into Euler's formula. There are 5 vertices given in the problem, and since there are 10 ways to pair two of these vertices, there are 10 edges.

$$V - E + F = 2$$

$$5 - 10 + F = 2$$

$$F = 7$$

Now we know there are 7 faces. Since every face coincides with at least 3 edges and each edge still connects two faces, we have $3F \leq 2E$. Plug in the numbers we have, and this gives us $3 \cdot 7 \leq 2 \cdot 10$, which is false and leads us to a contradiction. For that reason, it's impossible to connect 5 points to each other without an intersection.

Problem 2: Can 3 points each be connected to all of 3 other points without any intersections?

You may have encountered this problem described as three houses that all need to get water, gas, and electricity from their respective facilities without any wires crossing each other. We're given 6 vertices, and connecting 3 utilities to 3 houses gives us 9 edges.

$$V - E + F = 2$$

$$6 - 9 + F = 2$$
$$F = 5$$

In this case, just knowing that each face has at least 3 edges isn't enough. However, the problem setup gives us a stronger condition: each face must have at least 4 sides. Since a house vertex can only be connected to facility vertices, each face must have an even number of vertices to accommodate pairs of houses and facilities. For that reason, every face has at least 4 sides, so $4F \leq 2E$. Plugging in values gives us $4 \cdot 5 \leq 2 \cdot 9$, which is false, showing that this is impossible.

Citations

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Generating Functions

Khizer Shahid and Benjamin Gallai

6.1 What is a Generating Function?

The **generating function** of a sequence $a_0, a_1, a_2, a_3, \cdots$ is the power series:

$$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_kx^k + \cdots$$

For example, the generating function for the sequence $1, 2, 3, 4, 5, \cdots$ is $A(x) = 1+2x^2+3x^3+4x^4+\cdots$. Additionally, the generating function for the Fibonacci sequence, $0, 1, 1, 2, 3, 5, \cdots$, is $B(x) = 0 + x + x^2 + 2x^3 + 3x^4 + 5x^5 \cdots$. Note that we are using the polynomial to keep track of information. We will not examine the convergence of a series in this paper; i.e., we may say that

$$1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = \frac{1}{1-x}. \qquad (\star)$$

The function is only storing information about the polynomial. However, when we compute sums, we must check for convergence. For example, we cannot plug in $x = 2$ into $1 + x + x^2 + \cdots = \frac{1}{1-x}$; if we do we get the equation $1 + 2 + 2^2 + 2^3 + \cdots = -1$. Additionally, the reader should note that the left hand side of $(\star)$ is a power series while the right hand side is a rational function.

6.2 Finding the Generating Function of a Sequence

We will find the generating function of a sequence below.

Example 6.2.1: Fibonacci Generating Function

Find the generating function for the Fibonacci sequence, $\{F_n\}_{n=0}^{\infty} = \{0, 1, 1, 2, 3, 5, \cdots\}$, which is defined recursively as $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = 0$ and $F_1 = 1$.

Let the generating function for F_n be $F(x)$. By the definition of the generating function, we have

$$\begin{aligned} F(x) &= F_0 + F_1x + F_2x^2 + F_3x^3 + F_4x^4 \cdots \\ &= 0 + x + 1x^2 + 2x^3 + 3x^4 + \cdots \end{aligned}$$

Now, we must use the recurrence relationship. First, we will rewrite $F(x) = \sum_{n\geq 0} F_n x^n$. To apply the recurrence relationship, we need $n \geq 2$. Hence, we can further rewrite this as $\sum_{n\geq 0} F_n x^n = F(x) = F_0 + F_1 x + \sum_{n\geq 2} F_n x^n$. Now, we may apply the recurrence relationship. We have

$$\begin{aligned}
F(x) &= F_0 + F_1 x + \sum_{n\geq 2} F_n x^n \\
&= 0 + x + \sum_{n\geq 2} (F_{n-1} + F_{n-2}) x^n \\
&= x + \sum_{n\geq 2} F_{n-1} x^n + \sum_{n\geq 2} F_{n-2} x^n \\
&= x + x \sum_{n\geq 2} F_{n-1} x^{n-1} + x^2 \sum_{n\geq 2} F_{n-2} x^{n-2} \qquad (*)
\end{aligned}$$

Now, we have some expressions that look similar to our original expression for $F(x)$. First,

$$\sum_{n\geq 2} F_{n-2} x^{n-2} = \sum_{j\geq 0} F_j x^j = F(x)$$

Also,

$$\sum_{n\geq 2} F_{n-1} x^{n-1} = \sum_{n\geq 1} F_n x^n = \sum_{n\geq 0} F_n x^n - F_0 = F(x)$$

Hence,

$$\begin{aligned}
F(x) &= x + x \sum_{n\geq 2} F_{n-1} x^{n-1} + x^2 \sum_{n\geq 2} F_{n-2} x^{n-2} \\
&= x + x \cdot (F(x) - F_0) + x^2 \cdot F(x) \\
&= x + xF(x) + x^2 F(x)
\end{aligned}$$

We may solve for $F(x)$. Subtracting $xF(x) + x^2 F(x)$ from both sides, we have

$$\begin{aligned}
x &= F(x) - xF(x) - x^2 F(x) \\
&= F(x)\left(1 - x - x^2\right)
\end{aligned}$$

Dividing by $1 - x - x^2$, we have $F(x) = \dfrac{x}{1 - x - x^2}$. Again, we took a polynomial and turned it into a rational expression.

6.3 Finding the Closed Form of a Recurrence

We have the generating function for the Fibonacci sequence. From this, we may find the closed form for F_n.

Example 6.3.1: Deriving Binet's Formula

Find the closed form for the n^{th} term of the Fibonacci sequence.

We may do this by the partial fraction decomposition of $F(x)$. In other words, we are trying to find $A, B \in \mathbb{R}$ such that

$$\frac{x}{1-x-x^2} = \frac{A}{\alpha - x} + \frac{B}{\beta - x} \tag{1}$$

where α and β are roots of x^2+x-1, with $\alpha > \beta$, meaning that $1-x-x^2 = -(x^2+x-1) = -(x-\alpha)(x-\beta) = -(\alpha-x)(\beta-x)$. Multiplying both sides of (1) by $(\alpha-x)(\beta-x)$, we have

$$\begin{aligned} \frac{x(\alpha-x)(\beta-x)}{1-x-x^2} &= A(\beta-x) + B(\alpha-x) \\ -x &= A(\beta-x) + B(\alpha-x) \end{aligned} \tag{2}$$

Plugging in $x = \alpha$ into (2), we have $-\alpha = A(\beta-\alpha) + B(\alpha-\alpha) = A(\beta-\alpha) \Rightarrow A = \frac{\alpha}{\alpha-\beta}$. Similarly, plugging in $x = \beta$ into (2), we get $B = \frac{\beta}{\beta-\alpha}$. Now, we can compute $\alpha - \beta$ using the fact that $(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$. By Vieta's formulas, $\alpha+\beta = -1$ and $\alpha\beta = -1$, hence $(\alpha-\beta)^2 = (-1)^2 - 4\cdot(-1) = 5$. We will take $\alpha - \beta$ as $\sqrt{5}$. Since $\alpha - \beta = \sqrt{5} > 0$, α is the larger of the two roots of x^2+x-1. Therefore, $A = \frac{\alpha}{\sqrt{5}}$ and $B = -\frac{\beta}{\sqrt{5}}$. Hence, we may write $F(x)$ as $\dfrac{\frac{\alpha}{\sqrt{5}}}{\alpha-x} + \dfrac{-\frac{\beta}{\sqrt{5}}}{\beta-x} = \dfrac{1}{\sqrt{5}}\left(\dfrac{\alpha}{\alpha-x} - \dfrac{\beta}{\beta-x}\right)$. Now, let's try to rewrite $\dfrac{\alpha}{\alpha-x}$ and $\dfrac{\beta}{\beta-x}$. These look similar to $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$. We can manipulate $\dfrac{\alpha}{\alpha-x}$ to look more like $\dfrac{1}{1-x}$ by dividing the numerator and denominator by α: $\dfrac{\alpha}{\alpha-x} = \dfrac{1}{1-\left(\frac{x}{\alpha}\right)}$. If $\left|\dfrac{x}{\alpha}\right| < 1$, we can rewrite $\dfrac{1}{1-\left(\frac{x}{\alpha}\right)}$ as the series $1 + \left(\frac{x}{\alpha}\right) + \left(\frac{x}{\alpha}\right)^2 + \left(\frac{x}{\alpha}\right)^3 + \cdots$. Similarly, if $\left|\dfrac{x}{\beta}\right| < 1$, we can rewrite $\frac{\beta}{\beta-x}$ as $1 + \left(\frac{x}{\beta}\right) + \left(\frac{x}{\beta}\right)^2 + \left(\frac{x}{\beta}\right)^3 + \cdots$. Hence, we can rewrite $F(x)$:

$$
\begin{aligned}
F(x) &= \frac{1}{\sqrt{5}}\left(\frac{\alpha}{\alpha-x}-\frac{\beta}{\beta-x}\right)\\
&= \frac{1}{\sqrt{5}}\left(\left[1+\left(\frac{x}{\alpha}\right)+\left(\frac{x}{\alpha}\right)^2+\left(\frac{x}{\alpha}\right)^3+\cdots\right]-\left[1+\left(\frac{x}{\beta}\right)+\left(\frac{x}{\beta}\right)^2+\left(\frac{x}{\beta}\right)^3+\cdots\right]\right)\\
&= \frac{1}{\sqrt{5}}\left(\sum_{n\geq 0}\left(\frac{x}{\alpha}\right)^n-\sum_{n\geq 0}\left(\frac{x}{\beta}\right)^n\right)=\frac{1}{\sqrt{5}}\left(\sum_{n\geq 0}\left[\left(\frac{x}{\alpha}\right)^n-\left(\frac{x}{\beta}\right)^n\right]\right)\\
&= \frac{1}{\sqrt{5}}\sum_{n\geq 0}\left[\frac{1}{\alpha^n}-\frac{1}{\beta^n}\right]x^n=\frac{1}{\sqrt{5}}\sum_{n\geq 0}\left[\frac{1}{\left(\frac{-1+\sqrt{5}}{2}\right)^n}-\frac{1}{\left(\frac{-1-\sqrt{5}}{2}\right)^n}\right]x^n\\
&= \frac{1}{\sqrt{5}}\sum_{n\geq 0}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]x^n
\end{aligned}
$$

Therefore, the coefficient of x^n in $F(x)$ is $\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$. However, since we defined $F(x)$ as the generating function of F_n, the coefficient of x^n is also F_n. Hence, we have that $F_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$. This is known as *Binet's formula.*

6.4 Transforming Generating Functions

Given a generating function, $A(x)$, for a sequence $\{A_n\}$, what other generating functions can we find in terms of $A(x)$? Well, the generating function for $\{A_n+k\}$ is $A(x)+k+kx+kx^2+\cdots=A(x)+\frac{k}{1-kx}$. Additionally, the generating function for $\{kA_n\}$ is $kA(x)$. How can we get the generating function for a sequence like $\{na_n\}$? Lets write out $A(x)$ first:

$$A(x)=A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+\cdots$$

Multiplying each term by its index is the same as multiple the term by the degree of its term. This can be done by differentiation! Taking the derivative of both sides with respect to x, we have

$$
\begin{aligned}
A'(x) &= 0\cdot A_0+1\cdot A_1+2\cdot A_2x^1+3\cdot A_3x^2+4\cdot A_4x^3+\cdots\\
&= A_1+2A_2x+3A_3x^2+4A_4x^3+\cdots
\end{aligned}
$$

We also need to multiply by x so that the degree of each term matches the index of the term. Hence, the desired generating function is $xA'(x)$.

Similarly, let's find the generating function of $\left\{\dfrac{A_n}{n+1}\right\}$. If we integrate the general term of our generating function A_nx^n with respect to x, we get $\dfrac{1}{n+1}A_nx^{n+1}$. Dividing by x, we have $\dfrac{A_n}{n+1}x^n$. Hence, the generating function of $\left\{\dfrac{A_n}{n+1}\right\}$ is $\dfrac{1}{x}\displaystyle\int A(x)dx$.

6.5 Convergence

Now that we know about finding generating functions, what values of x can we plug into them to yield answers that make sense? For example, if we plug in $x = 1$ into $F(x) = \dfrac{x}{1-x-x^2} = \sum_{n\geq 0} F_n x^n$, we get $\sum_{n\geq 0} F_n = \frac{1}{1-1-1} = -1$, which is obviously not true. In order to find the interval of convergence, we can use the ratio test. By the ratio test, we know that $F(x) = \sum_{n\geq 0} F_n x^n$ will converge if $\lim_{n\to\infty} \left|\dfrac{F_{n+1}x^{n+1}}{F_n x^n}\right| < 1$. Simplifying this, we have

$$\lim_{n\to\infty} \left|\frac{F_{n+1}x^{n+1}}{F_n x^n}\right| = \lim_{n\to\infty} \left|\frac{F_{n+1}}{F_n}x\right| = |x| \lim_{n\to\infty} \left|\frac{F_{n+1}}{F_n}\right| < 1$$

Hence, we must have

$$|x| < \frac{1}{\lim_{n\to\infty} \left|\dfrac{F_{n+1}}{F_n}\right|} = \lim_{n\to\infty} \left|\frac{F_n}{F_{n+1}}\right|$$

We just have to compute what $\lim_{n\to\infty} \left|\dfrac{F_n}{F_{n+1}}\right| = \lim_{n\to\infty} \dfrac{F_n}{F_{n+1}} = L$ is.

We have

$$\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \lim_{n\to\infty} \frac{F_n}{F_n + F_{n-1}} = \frac{1}{\lim_{n\to\infty} \frac{F_n+F_{n-1}}{F_n}} = \frac{1}{\lim_{n\to\infty} \left(1 + \frac{F_{n-1}}{F_n}\right)} = \frac{1}{1 + \lim_{n\to\infty} \frac{F_{n-1}}{F_n}} = \frac{1}{1 + \lim_{n\to\infty} \frac{F_n}{F_{n+1}}}$$

Therefore, $L = \frac{1}{1+L} \Rightarrow L^2 + L - 1 = 0$. Hence, L is either $\frac{-1+\sqrt{5}}{2}$ or $\frac{-1-\sqrt{5}}{2}$. However, $L > 0$ and $\frac{-1-\sqrt{5}}{2} < 0$, so $L = \frac{-1+\sqrt{5}}{2}$. Plugging this into our inequality, we have that $F(x)$ will converge if $|x| < \frac{\sqrt{5}-1}{2}$. Since $1 > \frac{\sqrt{5}-1}{2}$, it is expected that we get a nonsensical answer.

6.6 Computing Sums

Now that we understand when generating functions converge, we can finally compute sums!

Example 6.6.1: Fibonacci Sum

Compute

$$\sum_{n=0}^{\infty} \frac{F_n}{2^n}.$$

We will present two solutions – one using manipulations and other using generating functions.

Solution 1 Let S be the desired sum.

We have

$$\begin{aligned} S &= \frac{F_0}{2^0} + \frac{F_1}{2^1} + \frac{F_2}{2^2} + \frac{F_3}{2^3} + \cdots \\ \frac{S}{2} &= \qquad\;\; \frac{F_0}{2^1} + \frac{F_1}{2^2} + \frac{F_2}{2^3} + \cdots \end{aligned}$$

Subtracting the two equations, we have

$$\begin{aligned} S - \frac{S}{2} &= \frac{F_0}{2^0} + \frac{F_1 - F_0}{2^1} + \frac{F_2 - F_1}{2^2} + \frac{F_3 - F_2}{2^3} + \cdots \\ \frac{S}{2} &= \frac{0}{1} + \frac{1}{2} + \frac{F_0}{2^2} + \frac{F_1}{2^3} + \cdots \\ &= \frac{1}{2} + \frac{S}{4} \end{aligned}$$

Solving for S, we have that $S = 2$.

Solution 2 Rewrite the given sum as $\sum_{n=0}^{\infty} F_n \cdot \left(\frac{1}{2}\right)^n = F\left(\frac{1}{2}\right)$. Additionally, since $\left|\frac{1}{2}\right| < \left|\frac{\sqrt{5}-1}{2}\right|$, $\frac{1}{2}$ lies inside the interval of convergence of $F(x)$. Hence, the requested sum is $\frac{\frac{1}{2}}{1 - \frac{1}{2} - \left(\frac{1}{2}\right)^2} = 2$.

Notice that in our generating function solution, we did not take the convergence of our series for granted, as in solution 1.

Example 6.6.2

Compute

$$\sum_{n=0}^{\infty} \frac{1}{(n+1) \cdot 2^n}.$$

Let's start of by replacing $\frac{1}{2}$ with x; our new sum is $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$. The generating function for the sequence $\{1\}$ is $1 + x + x^2 + x^3 \cdots = \frac{1}{1-x}$. So we have

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

Taking the integral of both sides with respect to x, we have

$$x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots = -\ln|x-1|$$

Dividing both sides by x, we have

$$1 + \tfrac{1}{2}x + \tfrac{1}{3}x^2 + \tfrac{1}{4}x^3 + \cdots = -\tfrac{1}{x}\ln|x-1|$$

Using the ratio test on this series, we find that it converges for $|x| < 1$. Plugging in $\frac{1}{2}$, we have that the desired sum is $2\ln 2$.

Example 6.6.3

Compute

$$\sum_{n=0}^{\infty} \frac{F_n}{n!}.$$

For this problem, we cannot use the generating function we already found for $\{F_n\}$. We must instead derive a generating function for $\{\frac{F_n}{n!}\}$. We will call this sequence $\{A_n\}$. We see that $A_0 = \frac{0}{0!}$, $A_1 = \frac{1}{1!}$, and

$$A_n = \frac{A_{n-1}(n-1)! + A_{n-2}(n-2)!}{n!}.$$

Removing fractions, we simplify to

$$n(n-1)A_n = (n-1)A_{n-1} + A_{n-2}. \tag{$*$}$$

We can now write out the generating function $A(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots$. Part of the motivation to have written $(*)$ the way we did (as opposed to isolating A_n), is that twice differentiating $A(x)$ will give us a very similar form

$$\begin{aligned}
A(x) &= A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots \\
A'(x) &= 1A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + \cdots \\
A''(x) &= 2\cdot 1A_2 + 3\cdot 2A_3x + 4\cdot 3A_4x^2 + 5\cdot 4A_5x^3 + \cdots \\
&= (A_1 + A_0) + (2A_2 + A_1)x + (3A_3 + A_2)x^2 + (4A_4 + A_3)x^3 + \cdots \\
&= (A_1 + 2A_2x + 3A_3x^2 + \cdots) + (A_0 + A_1x + A_2x^2) + \cdots \\
&= A'(x) + A(x)
\end{aligned}$$

To solve this differential equation, we know that $A(x)$ must be of the form e^{ax} or $\sum e^{ix}$.

$$\begin{aligned}
a^2e^{ax} &= ae^{ax} + e^{ax} \\
a^2 - a - 1 &= 0 \\
a &= \frac{1 \pm \sqrt{5}}{2}
\end{aligned}$$

Thus, $A(x) = pe^{\frac{1+\sqrt{5}}{2}x} + qe^{\frac{1-\sqrt{5}}{2}x}$. To solve for p and q, we use the fact that $A(0) = A_0 = 0 \Rightarrow A(0) = p + q = 0 \Rightarrow p = -q$. We also know that $A'(0) = A_1 = 1$

$$A'(x) = p\frac{1+\sqrt{5}}{2}e^{\frac{1+\sqrt{5}}{2}} - p\frac{1-\sqrt{5}}{2}e^{\frac{1-\sqrt{5}}{2}}$$
$$A'(0) = p\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = 1$$
$$\Rightarrow p = \frac{1}{\sqrt{5}}$$

Finally, we know that $A(x) = \frac{e^{\frac{1+\sqrt{5}}{2}x} - e^{\frac{1-\sqrt{5}}{2}x}}{\sqrt{5}}$. Our desired sum is simply $A(1) \approx \boxed{2.0143}$.

6.7 A Final Example

Example 6.7.1: Catalan Numbers

Find the generating function for the Catalan numbers, which are defined by the recursion $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$ with $C_0 = 1$.

Let $C(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + \cdots = 1 + x + 2x^2 + 5x^3 + \cdots$ be the desired generating function. Seeing the product of two Catalan numbers, we are motivated to multiply $C(x)$ by itself. We will find $C(x)^2$ by finding the coefficient of each term.

For the constant term, we must multiply the two constant terms together. Therefore, $C(x)^2 = C_0^2 + \cdots$

For the x term, we can multiply the C_0 from the first $C(x)$ with the C_1 from the second $C(x)$ or multiply the C_0 from the first $C(x)$ with the C_1 from the second $C(x)$. Therefore, the coefficient of x is $C_0C_1 + C_1C_0$. However, this is just C_2. So we can write $C(x)^2 = C_0^2 + C_2x + \cdots$.

In general, the x^k term will be $C_0C_k + C_1C_{k-1} + \cdots + C_kC_0 = C_{k+1}$. Therefore, $C(x)^2 = C_0^2 + C_2x + C_3x^2 + \cdots + C_{k+1}x^k + \cdots$. Also, $C_0^2 = 1 = C_1$, so we can write $C(x)^2 = C_1 + C_2x + C_3x^2 + \cdots$. This is similar to $C(x) = C_0 + C_1x + C_2x^2 + \cdots$. We can connect the two: $xC(x)^2 + C_0 = C(x) \Rightarrow xC(x)^2 - C(x) + 1 = 0$, which is a quadratic in $C(x)$. Solving for $C(x)$, we have $C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$. In order to decide which one to choose, we will expand $\sqrt{1-4x}$. To do this, we must use the generalized binomial theorem, which states that for any real r

$$(1+x)^r = \sum_{n \geq 0} \binom{r}{n} x^n \quad \text{where} \quad \binom{r}{n} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}$$

Using this, we have

$$\sqrt{1-4x} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4x)^n = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\cdots(\frac{3-2n}{2})}{n!}(-4x)^n$$

Note that this is negative for all values of n, so we can take out a negative sign from the summation, and rewrite it as

$$-\sum_{n=0}^{\infty} 2^n \frac{(2n-3)!!}{n!} x^n.$$

Here, the double exclamation marks represent the double factorial function, which only multiplies integers of same parity; for example, $10!! = 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2$, while $7!! = 7 \cdot 5 \cdot 3 \cdot 1$. Since the terms of our power series are positive, we must take the negative square root. Hence, $C(x) = \frac{1-\sqrt{1-4x}}{2x}$. We can even find the general term of our sequence. Substituting the power series expansion of $\sqrt{1-4x}$ into our generating function, we have

$$C(x) = \frac{1 + \sum_{n=0}^{\infty} 2^n \frac{(2n-3)!!}{n!} x^n}{2x}.$$

The coefficient of x^n is

$$\frac{2^{n+1} \frac{(2n-1)!!}{(n+1)!}}{2} = 2^n \cdot \frac{(2n-1)!!}{(n+1)!} = 2^n \cdot \frac{(2n-1)!!}{(n+1)!} \cdot \frac{(2n)!!}{(2n)!!} = 2^n \frac{(2n)!}{(2n)!!(n+1)!} = \frac{(2n)!}{n!(n+1)!} = \frac{\binom{2n}{n}}{n+1}.$$

Therefore, we have $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$.

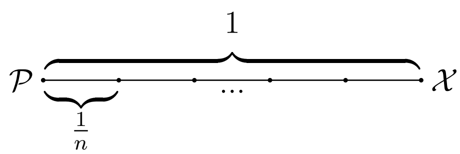
Calculating a Fixed Distance

Benjamin Gallai and Khizer Shahid

7.1 Problem Statement

A particle $\mathcal{P}$ is initially 1 unit away from $\mathcal{X}$ its destination. On its path to $\mathcal{X}$, $\mathcal{P}$ has a velocity of $d+1$ units per second, where d is its distance to $\mathcal{X}$. How long will it take for $\mathcal{P}$ to reach $\mathcal{X}$?

7.2 Solution



First, we will partition the interval of length 1 into n equally sized interval. We will assume that $\mathcal{P}$ changes its velocity at the endpoints. Then, we may take the limit of our result as $n \to \infty$. This will allow us to simplify the problem.

Let d_i be the distance traveled by $\mathcal{P}$ during the ith interval, v_i be its velocity at the beginning of this interval, and t_i be the amount of time it spends in the interval. Additionally, let $T_n = t_1 + t_2 + t_3 + \cdots + t_n$ be the total amount of time that it takes $\mathcal{P}$ to reach $\mathcal{X}$ in a configuration with n subintervals. As we add more subintervals, the velocity will become closer to changing continuously. Hence, the total amount of time $\mathcal{P}$ takes is $T = \lim\limits_{n \longrightarrow \infty} T_n$.

Now, we must compute T_n in terms of n. First, we have $T_n = \sum\limits_{k=1}^{n} t_k$. From $d = rt$, we may conclude that $t_k = \frac{d_k}{v_k}$. Additionally, $d_k = \frac{1}{n}$ for all k and $v_k = 2 - \frac{k-1}{n}$ since at the kth interval it has travelled $k-1$ subintervals of length $\frac{1}{n}$. Hence,

$$t_k = \frac{\frac{1}{n}}{2 - \frac{k-1}{n}} = \frac{1}{2n-k+1} \Rightarrow T_n = \sum_{k=1}^{n} \frac{1}{2n-k+1} = \frac{1}{2n} + \frac{1}{2n-1} + \frac{1}{2n-2} + \cdots + \frac{1}{n+2} + \frac{1}{n+1}.$$

We may further rewrite this as

$$
\begin{aligned}
\frac{1}{2n} + \cdots + \frac{1}{n+1} &= \left(\frac{1}{2n} + \frac{1}{2n-1} + \cdots + \frac{1}{2} + \frac{1}{1}\right) - \left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} + \frac{1}{1}\right) \\
&= \left[\left(\frac{1}{2n} + \frac{1}{2n-1}\right) + \left(\frac{1}{2n-2} + \frac{1}{2n-3}\right) + \cdots + \left(\frac{1}{2} + \frac{1}{1}\right)\right] - \sum_{i=1}^{n} \frac{1}{i} \\
&= \sum_{i=1}^{n} \left(\frac{1}{2i} + \frac{1}{2i-1}\right) - \sum_{i=1}^{n} \frac{1}{i} \\
&= \sum_{i=1}^{n} \left(\frac{1}{2i} + \frac{1}{2i-1} - \frac{1}{i}\right) \\
&= \sum_{i=1}^{n} \left(\frac{1}{2i-1} - \frac{1}{2i}\right) \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2n-3} - \frac{1}{2n-2}\right) + \left(\frac{1}{2n-1} - \frac{1}{2n}\right) \\
&= 1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{2n-2} + \frac{1}{2n-1} - \frac{1}{2n}.
\end{aligned}
$$

Hence, $T = \lim_{n \to \infty} T_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$. However, this is not a closed form. To find the closed form, we turn to calculus.

Let the total distance $\mathcal{P}$ has traveled at time t be $x(t)$ and its velocity be $v(t)$. By the problem statement, $v(t) = (1 - x(t)) + 1 = 2 - x(t)$. Since $x(t)$ is $\mathcal{P}$'s position and the rate of change of position is velocity, we have $v(t) = x'(t)$. Now, we can simply solve for $x(t)$, the position function, by solving the differential equation $\frac{dx}{dt} = 2 - x$. However, we will take a different approach.

Differentiating both sides of $v(t) = 2 - x(t)$ with respect to t, we get $a(t) = -v(t)$, where $a(t)$ is the acceleration of $\mathcal{P}$ at time t. Additionally, we have that $a(t) = v'(t)$. Plugging this back into $a(t) = -v(t)$, we have $v'(t) = -v(t)$. Now, we have the differential equation $\frac{dv}{dt} = -v$. Separating the variables and integrating both sides, we have

$$
\begin{aligned}
\int \frac{dv}{v} &= -\int dt \\
\Rightarrow \ln|v| = \ln v(t) &= -t + C \\
\Rightarrow v(t) = e^{-t+C} &= Ae^{-t}
\end{aligned}
$$

for some $A \in \mathbb{R}^+$. Note that we can remove the absolute values since $v(t) \geq 1 > 0$. We can solve for A by plugging in $t = 0$. Since at time $t = 0$, $\mathcal{P}$ is 1 unit away from $\mathcal{X}$, its velocity is 2 units per second: $v(0) = A = 2$. Hence, $v(t) = 2e^{-t}$. Let $\mathcal{P}$ reach $\mathcal{X}$ after k seconds. We know that $v(k) = 2e^{-k} = 1$. Hence, we can solve for k:

$$
2e^{-k} = 1 \Rightarrow e^{-k} = \frac{1}{2} \Rightarrow -k = \ln\frac{1}{2} \Rightarrow k = \ln 2.
$$

Hence, $\mathcal{P}$ will reach $\mathcal{X}$ in $\ln 2$ seconds.

Since, our answers to the problems must be the same, we have

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\cdots=\ln 2$$

7.3 Generalization

Now, instead of $\mathcal{P}$ starting 1 unit away from $\mathcal{X}$, let it be d units away.

From the problem statement we have $v(t)=(d-x(t))+1=d+1-x(t)$, Taking derivative with respect to t, we have $a(t)=-v(t)$. This is the exact same equation we solved earlier. We found that $v(t)=Ae^{-t}$ for some $A\in\mathbb{R}$. We can analyze what happens at $t=0$: $x(0)=d\Rightarrow v(0)=A=d+1$. Hence, $v(t)=(d+1)e^{-t}$. Solving the equation $v(t)=1$, we have $t=\ln(d+1)$. Hence, $\mathcal{P}$ will reach $\mathcal{X}$ in $\ln(d+1)$ seconds. Plugging in $d=1$ from the initial problem agrees with our previous result.

The History of Pi

Jacob Paltrowitz

The origin of much of mathematics is shapes. One shape that has been uniquely interesting to mathematicians for millennia is the circle. A circle is defined by the locus of points equidistant to a center. The circle seems like a natural shape to be interested in, as many phenomena in the world involve this shape. In fact, past mathematicians reasoned that circles were all similar. Therefore, there must be a constant ratio between the circumference and the diameter. We now know this ratio to be π.

The simplest way to approximate π is to construct a circle with a string, measure its circumference and diameter, and then divide. Following these steps, I concluded that $\pi \approx 3.35$. Please try this method at home, and see how close you can get. I suggest using a glass cup or some other circular object to aid with constructing the circle.

Past mathematicians needed more accurate approximations of the circumference of a circle. Many people made the astute observation that a circle's circumference approached the perimeter of inscribed regular polygons, as the polygons' number of sides goes to infinity. We can start by finding the perimeter of a regular hexagon inscribed in a circle with radius 1. If we split the hexagon into 6 equilateral triangles, we can see that the hexagon has perimeter 6. Now, we want to find the perimeter of larger regular polygons. The method used by Archimedes and Zu Chongzi, the two people who used this method to get good approximations very early on. The two mathematicians did their work completely separately; neither knew that the other existed, but they still got to the same information. They found a formula to find the perimeter of a regular polygon, directly from the perimeter of a polygon with half the number of sides.

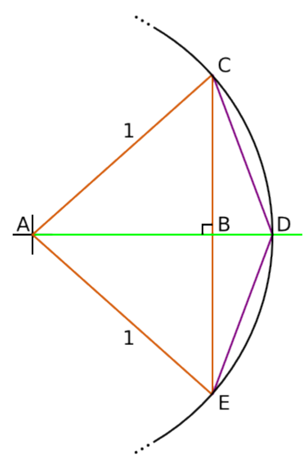


Figure 8.1: Diagram for the Formula

We shall consider a polygon with n sides, and a perimeter of $2nd$, where each side of the polygon has length $2d$. In our diagram, BC has length d. By the Pythagorean Theorem, $AB = \sqrt{1-d^2}$. Thus, $BD = 1 - \sqrt{1-d^2}$. Then,

$$CD = \sqrt{2 - 2\sqrt{1-d^2}}.$$

Using many laborious calculations, Zu Chongzi was able to calculate the perimeter of a $24,576$ sided polygon. Not surprisingly, $24,576 = 6 \cdot 2^{12}$. Using these calculations, Chongzi was able to calculate π to 7 decimal places.

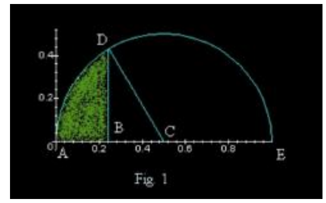


Figure 8.2: Newton's Diagram

More recently, Isaac Newton used infinite summations and coordinate geometry to generate an approximation of π. He took the equation for a circle with center at $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$. This semicircle has the equation

$$y = \sqrt{\frac{1}{4} - (x - \frac{1}{2})^2}.$$

This can be rewritten as

$$y = \sqrt{x} \cdot \sqrt{1-x}.$$

We can expand $\sqrt{1-x}$. $\sqrt{1-x}$ is equal to $(1-x)^{\frac{1}{2}}$, so it can be expanded using the binomial theorem. A part of the binomial theorem states that $(1+a)^n$ is equal to $1 + na + \binom{n}{2}a^2 + \binom{n}{3}a^3 + \cdots$. Now, we plug $n = \frac{1}{2}$ into the previous expansion, to get that

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \cdots$$

When evaluated, this can be simplified as

$$1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} \cdots$$

Newton then drew a line perpendicular to the x-axis at $(0, \frac{1}{4})$. If we look at the triangle formed by the center of the semicircle, the intersection of the perpendicular and the semicircle, and $(0, \frac{1}{4})$, we can see that it a 30-60-90 triangle. It is a right triangle with a hypotenuse of $\frac{1}{2}$ and one leg with length $\frac{1}{4}$. Newton then examined the area enclosed by the x-axis, the perpendicular, and the circle. This area is $\frac{1}{3}$ of a semicircle minus a triangle with area $\frac{\sqrt{3}}{32}$. Earlier in his mathematical career, Newton showed that the area bounded by a curve with the equation $y = ax^{\frac{m}{n}}$, the x-axis, and some perpendicular to the x-axis x units away from the origin was $\frac{an}{m+n}x^{\frac{m+n}{n}}$. This can be shown by applying the power rule, $\int x^n dx = \frac{x^{n+1}}{n+1}$. By this rule, $\int ax^{\frac{m}{n}}dx = \frac{ax^{\frac{m}{n}+1}}{\frac{m}{n}+1} = \frac{an}{m+n}x^{\frac{m+n}{n}}$. He also had concluded that the area under a curve formed by the summation of many terms was equal to the sum of the area under each of the curves. So the area bounded the curve $y = x^{\frac{1}{2}}\left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} \cdots\right)$, the x-axis, and the line $x = \frac{1}{4}$ can be found by inserting each term of the expansion of $\sqrt{x}\sqrt{1-x}$ into the previous formula. By doing this we get that the area we are try to find is equal to

$$\frac{2}{3}\left(\frac{1}{4}\right)^{\frac{3}{2}} - \frac{1}{2}\left(\frac{2}{5}\right)\left(\frac{1}{4}\right)^{\frac{5}{2}} - \frac{1}{8}\left(\frac{2}{7}\right)\left(\frac{1}{4}\right)^{\frac{7}{2}} - \frac{1}{16}\left(\frac{2}{9}\right)\left(\frac{1}{4}\right)^{\frac{9}{2}} - \dots$$

When summed, this approaches 0.07677310678. Just summing the first four terms gets the area to be approximately 0.07678. So, we add the area of the 30-60-90 triangle, $\frac{\sqrt{3}}{32}$, to get 0.130899694527. So, the area of the entire circle is 6 times this value, getting .785398167099. We can then divide by the radius, $\frac{1}{2}$, squared in order to get that $\pi \approx 3.14159267$

A common usage of π is to measure angles. This usage of radians comes directly from the definition of π. We define a unit as the angle at which the radius of a circle equals its arc length. If we replicate this angle 2π times, we will get a full circle, as the circumference is 2π times the radius. Another way to find the area of a circle is to observe the limit of the areas of inscribed polygons. For an n-sided polygon inscribed in a circle, we can connect each vertex to the center creating n triangles. Each of these triangles has an area of $\frac{1}{2}\sin\alpha$ where $\alpha = \frac{2\pi}{n}$. So, the area of a circle is equal to the

$$\lim_{x\to\infty} \frac{1}{2} n \sin(\frac{2\pi}{n})$$

Using larger and larger values of n, we can reach increasingly accurate approximations for π.

A fun method to try at home is the Buffon Needle Problem (do not worry, no sharp objects are necessary for the problem). The setup of the problem involves parallel lines set 1 unit apart from each other. We then randomly place objects which I will refer to as needles, hence the name of the problem, resembling line segments with length 1. When we randomize the placement of the line segment, we can randomize the distance between the center of the needle and the nearest line, called d, as well as the acute angle formed by the extension of the needle with the parallel lines, called θ. We are going to count the fraction of needles that cross one of the parallel lines.

In order to cross a line, d has to be less than $\frac{1}{2}sin\theta$. d and θ are completely random selections, meaning any value within the bounds has an equal probability of being chosen. Since d is the distance to the closest line, $d \in [0, \frac{1}{2}]$. θ is a random acute angle, so $\theta \in [0, \frac{\pi}{2}]$. We can then graph $d < \frac{1}{2}\sin\theta$ within these bounds, where θ lies on the horizontal axis. In order to find the percentage of times where the needle crosses a line, we take

$$\frac{\int_0^{\frac{\pi}{2}} \frac{1}{2}\sin\theta d\theta}{\frac{\pi}{4}}$$

Since the derivative of sine is cosine, we can evaluate the integral portion to get $\frac{1}{2}$. So, as you throw needles, the percentage of them that cross lines should approach $\frac{2}{\pi}$. So, $\pi \approx 2$ times the reciprocal of the fraction of needles that cross.

More recently, people have used computers to generate many digits of pi. One simple and intuitive approach is to inscribe a unit circle in a square. This fraction, when multiplied by 4, will yield an approximation for π.

Some people have tried to claim a rational value for π. One famous example happened in Indiana in 1897. A man named Edward Goodwin claimed to prove that the ratio between the circumference and the diameter was exactly 3.2. The issue with his "proof," found under many layers of convoluted text was his assumption of the truth of this diagram. After examination, we can see that not only did Goodwin assume that $\pi = 3.2$, he also assumed that, by

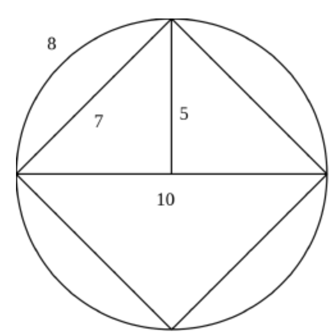


Figure 8.3: The Assumed Diagram

the Pythagorean theorem, $5\sqrt{2} = 7$. Despite these inaccuracies, Goodwin's false claims reached as far as state legislature. A bill passed through the Indiana House of Representatives to grant the education system of Indiana free usage of this discovery. Thankfully, a Math Professor from Purdue University also happened to be at the State Legislature's chambers seeking funding so that he could show the error in Goodwin's proof. The Senate has indefinitely postponed this vote.

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Ray Tracing

Ian Graham

Humanity has always been fascinated by the interplay of two fundamental aspects of existence as we know it: the depth of the world around us, and the flatness of our vision. While our minds have learned to extract a third dimension from the images our eyes give them, there is nothing inherently three dimensional about them. That is to say, the depth we construct from these images is nothing more than an educated guess: our brains, armed with billions of years of evolution, a lifetime of experience, and two perspectives (one from each eye), can estimate what the scene before our eyes truly looks like with incredible accuracy. But it's not perfect. The act of tricking this intuition is millenia old and embedded into our very beings. Examples of this include the Parthenon's floor and columns, which are curved to create a false perspective, M.C. Escher's optical illusions, and, relatively recently, 3-D renderers.

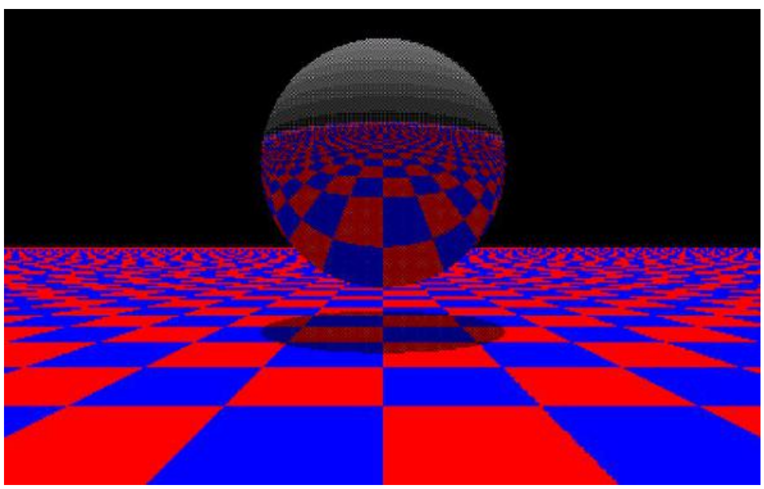

Figure 9.1: A ray-traced image rendered on Ti 84

That last example is especially interesting, because its aim is purely to immerse the mind into an existence with depth which does not exist. It creates perspective, which, unlike that seen in paintings and other man-made artwork, is not just an imprint of a scene found in someone's mind, is not the result of intuition, it's the result of strict mathematics and intertwined formulas. 3-D renderers photograph scenes which don't exist and worlds we can only imagine. While we take them for granted today, they undeniably give a truly and uniquely human interface to computing which is relatively new. Computer-aided design, video games, 3-D artwork, and animation simply would not be what they are today without 3-D renderers. All of this, while intriguing, does not get at the heart of the question I hope to answer: how do 3-D renderers work?

There are several potential answers to this question, but in this article, I'll only cover ray tracers. There are a few reasons for this. First, ray tracers are by far the most intuitive type of 3-D renderer, since they're based on simulating how light acts in the real world. Second, they produce the highest resolution images. While other, more common types of 3-D renderers, such as rasterizers, use lots of shortcuts to avoid unnecessary computations and make fast, real-time rendering possible, ray tracers do not, which means that they preserve details which other 3-D renderers might ignore. Lastly, ray tracers utilize all the same concepts as more common forms of 3-D renderers. All 3-D renderers work on the same foundations dictated by optics, which ray tracers exhibit more completely than any other form of 3-D renderer, so by learning how ray tracers work, you also learn how other 3-D renderers work, save for a few details and optimizations.

Bearing in mind that ray tracers simulate how light acts in the real world, we can begin to devise a basic algorithm based on the world around us. The first thing to consider is what light is. Physicists have argued about this for centuries, but, thankfully, we are not bound to the intricacies of the real world. Instead, we just have to create a way of mathematically describing how light behaves at a macroscopic level. The immediately obvious tool to use are 3-D vectors. A 3-D vector is a collection of three numbers which can be used to describe a point in space, and, consequently, the direction towards it. Not only that, but 3-D vectors in computer graphics can also be thought of as red, green, and blue components of a pixel, meaning that with 3-D vectors we can describe both how light moves and what the color it is. The next thing we must consider is what it means to see something. In the world, we see things because light travels from a light source, passes through surfaces or reflects off of them, and occasionally reaches our eyes. This immediately presents a problem to us: while light sources in real life emit unfathomable amounts of photons in all directions, our computers have limited resources and can't efficiently track all of them. But our intuition is not useless! We know that we only see light that enters our eyes, and therefore we can safely ignore any light which goes in other directions! To eliminate light which doesn't intersect our eyes by having all of our vectors originate from the same point, which will represent our eyes. This means that we can define a point in 3-D space as the viewer, and 3-D vectors originating from it as light we see. If we can compute the color of each of these rays we will know which rays of light reach our eyes and what color they are.

We must also decide how we're going to extract an image from this model. We can't look at only the point we're viewing from, since a single point can't be a 2-D image. Instead, we will look at the cross section of the rays before they hit our eyes, so that our image can distinguish between rays which intersect at the eye. With the model we have just created, we have a very basic, but correct, model of space and vision that we can begin to elaborate on.

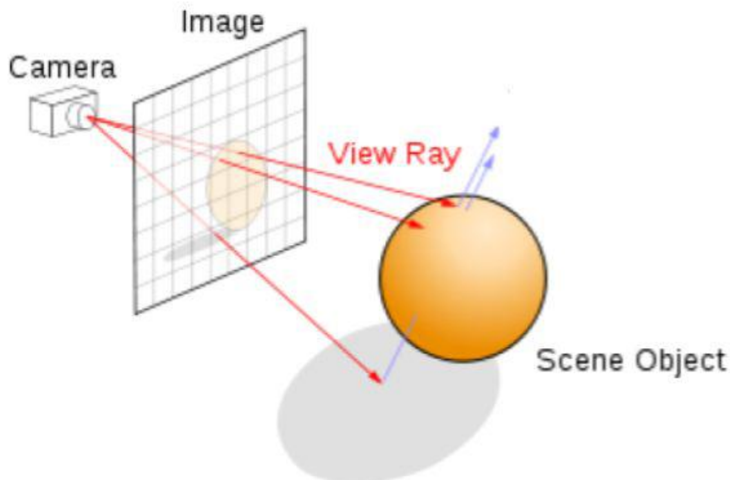

Figure 9.2: A visualization of the Ray Tracing Model (Source: Wikimedia Commons)

Now, we just have to find a way to describe the scene we want to draw. We might initially be tempted to try to 'pixelate' this 3-D scene as we sometimes do 2-D ones, dividing the space into small regions and describing each one individually (whether it be by color or some other property), but the math looks a lot simpler if we instead use equations to describe our scene. For instance, the points which satisfy the equation $(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=r^2$ form a sphere of radius r centered at (x_1, y_1, z_1). By using this equation, we can check whether or not and where a given ray intersects that sphere with a simple quadratic equation and some vector math. Describing a plane, a triangle, or a disc through equations allows us to describe complex scenes easily.

With the model we've created, we can go through each pixel, cast a ray through it, and see what kind of object, if any, that ray intersects, and where it intersects it. With that information, we can construct a new ray, originating at the point of intersection and bouncing off in a new direction. By creating a new ray, we can create reflections, and by combining the colors of all the objects our ray and its reflections intersect, we can discover the color of a specific pixel. Additionally, if we change how we create new rays after intersections, we can create the illusion of having different materials, for example, if we create imperfect reflections, which bounce off of angles that are slightly off, we can create blurred reflections or a matte surface. Similarly, we can not reflect the ray at all and instead refract it, creating a dielectric surface.

The math behind 3-D rendering and the tricks it uses to generate complex, accurate images goes far beyond what I've described here, and I encourage you to delve into the fascinating depths of how ray tracers behave, but I hope this brief introduction has given you some appreciation and understanding for what goes on behind the curtains of the software we use daily.

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Coronavirus Epidemiology

Benjamin Kreiswirth

Introduction

Statistical tests and mathematical modeling are critical components of epidemiology, the study of distribution and the spread of disease in order to control and prevent it. This article will discuss the mathematical aspect of how epidemiologists investigate, measure, model, control, and prevent disease. In particular, this article will examine the COVID-19 pandemic.

Disease Investigation

The first part of managing any epidemic is identifying and discovering it. This part of the process is not something visible to us once the disease has become large-scale because we are long past it. But it is the most important part for small-scale outbreaks and also critical to control and prevention measures for all outbreaks.

Establish Existence of an Outbreak

The first thing that was detected was a cluster of pneumonia cases in Wuhan. A cluster in epidemiology means a large aggregation of cases over a particular period of time. However, a cluster does not necessarily imply an outbreak, the latter being defined as more cases than expected of a disease in a particular period of time.

Pneumonia is a condition involving inflammation of the lungs. It can be caused by an infection of any number of different bacteria or viruses. It can sometimes be deadly, especially to those with weakened immune systems who cannot effectively fight the infection.

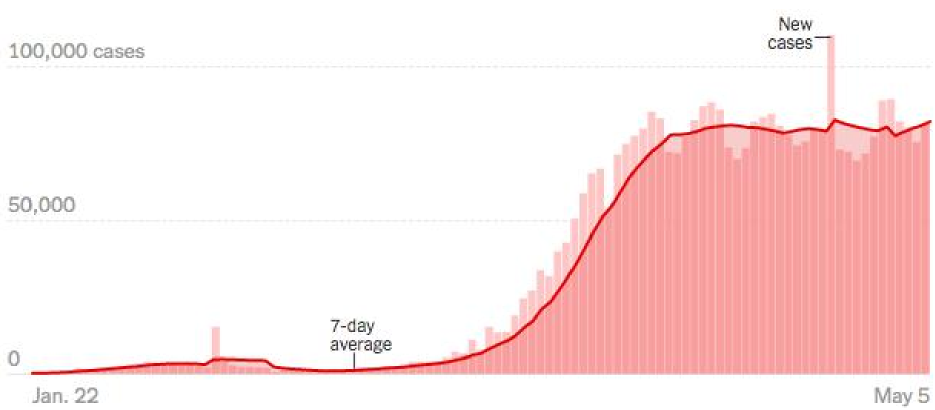
However, pneumonia has a high prevalence (number of cases in the population), with more than 3 million cases annually. To say that the cluster in Wuhan is actually an outbreak epidemiologists need to be sure is not part of the endemic (baseline/expected) prevalence. The reason they knew it was not typical was that the pneumonia was unexplained - not conforming to a known cause such as influenza, rhinoviruses (the common cold), and various bacteria. This means that there was in fact an outbreak, mandating an epidemiological investigation.

Descriptive Epidemiology

Descriptive epidemiology involves describing a disease in terms of person, place, and time. During investigation, all of the unexplained cases of pneumonia were likely tabulated with information such as place of residence and onset of symptoms.

Investigating location (places of residence, occupation, etc.) can help determine possible sources. John Snow, the "father of epidemiology," famously plotted cholera cases on a map of London and found they were centered around one water well (this map is shown below, with red dots marking cases and black circles marking wells). Once the well was shut down, the cholera cases ceased, and thus the cause of the outbreak was found. Similarly, considering the area where patients lived in Wuhan could help investigators pinpoint the area where it must have originated, one particular seafood and wet animal market.

Investigating time, we can build out what is known as an "epicurve," or a histogram of all of the cases over time. The length of time between the first case (known as the index case) and the next cluster of cases can help determine the incubation period (time from infection to showing symptoms). The epicurve can also suggest person-to-person spread with multiple successively larger peaks one incubation period apart eventually emerging. The epicurve for COVID-19 is typical for person-to-person spread, and can be seen below with an additional 7-day average line.



Analytic Epidemiology

In the case of COVID-19, descriptive epidemiology established that it spreads between people, and we can begin linking patients to other patients. All of the spread comes from a small number of patients (probably just one) present when the disease jumped from animal to human (it is known as a zoonotic disease because it originates from animals). At this point, we can move on to lab-test patients to identify the new strain.

However, not all outbreaks are spread person-to-person. Some have what is known as a common source, like contaminated food or water. In this case, we can perform a case-control trial to investigate possible risk factors and how many people were exposed to them. For example, say a number of people went to a particular restaurant and got sick with a particular gastrointestinal illness. We can build out contingencies and calculate odds ratios for each risk factor (the ratio between the odds that someone got sick given that they had the risk factor vs. that they didn't).

	Sick	Not Sick
Ate Salad	30	20
Did not eat Salad	20	30

Odds Ratio $= \frac{30}{20}/\frac{20}{30} = \frac{9}{4}$.

	Sick	Not Sick
Ate Lasagna	10	35
Did not eat Lasagna	40	15

Odds Ratio $= \frac{10}{35}/\frac{40}{15} = 3/28$

	Sick	Not Sick
Ate Chicken	42	10
Did not eat Chicken	8	40

Odds Ratio $= \frac{42}{10}/\frac{8}{40} = 21$

This appears to suggest that chicken is the most likely factor, and the chicken from the restaurant can go on to be lab-tested for a pathogen. However, note that since the groups of sick and not sick were formed after the fact, they may not be representative of the entire population. Because of this, the study does not prove any causality but merely shows a correlation. For example, the odds ratio is high for people eating salad, but this may just be because many people who ate chicken also ate salad.

Confirm Cause in Lab

Anything suggested by a field investigation or trial should be confirmed by a lab investigation. In the case of coronavirus, a lab investigation can be used to isolate the pathogen from patients. The pathogen itself will be needed to try to find treatments or prevention measures. However, the findings of further epidemiological investigations will be more useful for control measures in the shorter-term.

Measuring the Disease

We know that COVID-19 doesn't kill a large portion of those it infects, with many carriers not even showing symptoms. However, the virus's ability to live in hosts without causing symptoms makes it hard to detect. Determining basic mathematical facts about the disease concerning how frequently people show symptoms and how frequently people die are important in understanding it.

Important Values

Infectivity is the ability of a disease to infect a new host. It tells us how many people who come into contact with COVID-19 will become infected.

$$\text{Infectivity} = \frac{\text{infected}}{\text{susceptible}}$$

Pathogenicity is the ability to cause disease. It tells us how many people who become infected with the virus will end up showing symptoms.

$$\text{Pathogenicity} = \frac{\text{sick}}{\text{infected}}$$

Virulence is the ability to cause deaths. It is also known as the **Case-Fatality Rate (CFR)**. It tells us how many people who become sick will end up dying due to the disease.

$$\text{Virulence} = \frac{\text{deaths}}{\text{sick}}$$

COVID-19 Pathogenicity

According to a study performed early in the pandemic, 18% of cases are true asymptomatic cases. This study investigated cases on the Diamond Princess Cruise Ship. If this number were correct, the pathogenicity of COVID-19 would be 0.82. This alone could be dangerous if the carriers unknowingly pass on the disease to a number of others.

However, more recent studies suggest that there may be a higher proportion of cases which are not symptomatic. A new estimate may be as high as 78%, making the pathogenicity far lower. It is dangerous to have many people not knowing they are sick, because they may not follow the necessary self-quarantine measures. It is even more dangerous not to know how many people are carriers in this way, because then we cannot know how serious the disease is at any given moment.

COVID-19 Case-Fatality Rate

Likely the most important factor to consider for a disease is the Case-Fatality Rate. While a lot of people getting sick is a threat to their ability to work, people dying is clearly far worse. The following table is the result of an analysis performed in early March in an attempt to compute the mortality rate.

Location	Cumulative deaths*	Cumulative confirmed cases*	Crude CFR, %	Adjusted deaths†	Adjusted cumulative confirmed cases†	Adjusted CFR, % (95% CI)
China	3,015	80,565	3.74	2,627	75,569	3.48 (3.35–3.61)
China, excluding Hubei Province	113	13,099	0.86	104	12,907	0.81 (0.67–0.98)
82 countries, territories, and areas	27	2,285	1.18	15	354	4.24 (2.58–6.87)
Cruise ship	6	706	0.85	4	634	0.63 (0.25–1.61)

* These first numbers are simply totals as of March 5th.
† These adjusted numbers only count confirmed cases before February 21st, only count deaths among those with confirmed cases before February 21st.

The adjustment of the numbers prevents two mathematical issues. The first is overcounting cases relative to deaths because there are existing cases which will end up as deaths. The second is undercounting cases because there are existing cases which have not yet been discovered. The February 21st date was picked under the assumption that 2 weeks after confirmation of having the disease one would either die or recover.

Clearly, none of the rates are the be-all end-all - they wildly vary from each other for a number of reasons. The most important of these reasons is the response of the healthcare system, which was likely poorer among the earliest cases of the disease in each country. This likely inflated the CFR for the whole of China (which includes a large number of the earliest patients in Hubei Province), as well as for the 82 other countries and territories just beginning to experience the disease. Ultimately, this mathematical uncertainty is in fact a positive sign that proper healthcare can improve outcomes.

Age-Specific Case-Fatality Rates

A Centers for Disease Control (CDC) study sheds more light on the situation by breaking the fatality down by age. It found the overall CFR to be about 2.5%, but the age-specific case-fatality rates vary wildly from 0% to more than 10%. These numbers are important because they help suggest who is most at risk due to the disease. However, they are also from relatively early in the disease so they could be skewed toward having more deaths than will ultimately occur.

Age group	No. of cases	Hospitalization*	ICU admission*	Case-fatality*
0-19	123	1.6-2.5	0	0
20-44	705	14.3-20.8	2.0-4.2	0.1-0.2
45-54	429	21.2-28.3	5.4-10.4	0.5-0.8
55-64	429	20.5-30.1	4.7-11.2	1.4-2.6
65-74	409	28.6-43.5	8.1-18.8	2.7-4.9
75-84	210	30.5-58.7	10.5-31.0	4.3-10.5
>84	144	31.3-70.3	6.3-29.0	10.4-27.3
Total	2449	20.7-31.4	4.9-11.5	1.8-3.4

*Ranges were determined due to missing data on subjects (these are the maximum and minimum possible percentages given data known).

Modelling Disease Spread

A basic model for the spread of disease is known as the **SIR model**. SIR stands for susceptible, infected, and recovered. The model relatively simplistically groups everyone into one of these three groups. It treats anyone who died as part of the recovered category - recovered really means that the person is no longer transmitting the disease.

The model operates under the assumptions that everyone who has had the disease cannot get it again, and that everyone who hasn't had the disease is susceptible to getting it. These are reasonable assumptions given that having the disease provides immunity for a long period of time and that a vaccine has not yet been produced.

Maths Behind the Model

The model requires an input of what percentage of the population starts out in each group (S, I, and R). It also requires an idealized rate of infection (r) which represents what fraction of the population an infected person would come into contact with over a given period of time (say, a day). Finally, it requires an idealized rate of recovery (o) which represents the proportion of those infected recovering each day. Then, we can find $S(t)$, $I(t)$, $R(t)$ by solving the following differential equations:

$$\frac{dS}{dt} = -rIS$$

$$\frac{dI}{dt} = rIS - oI$$

$$\frac{dR}{dt} = oI$$

In English, this means that the rate of change of S will be $-rIS$, the rate of change of I will be $rIs - oI$, and the rate of change of R will be oI.

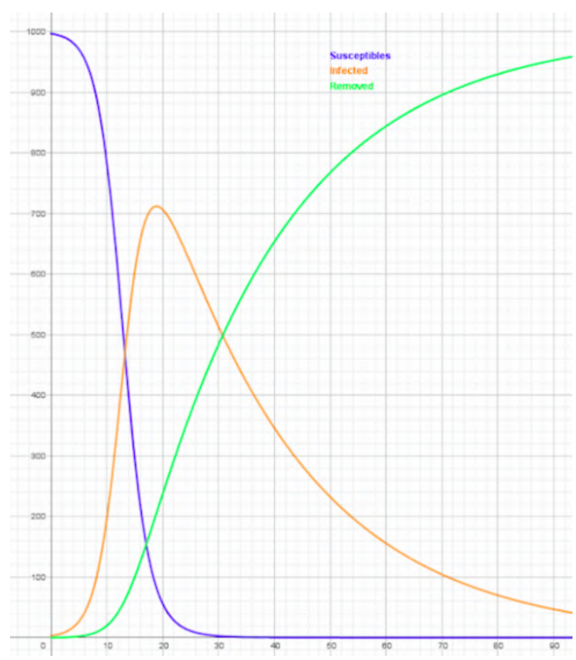
To put it simpler, rIS represents the portion of the population getting infected each day - each infected person comes into contact with r of the population each day, but of these only S of these Ir encounters are susceptible, so in the end rIS of the population gets infected each day.

Meanwhile, oI represents the number of people recovering/dying each day - the average length of clinical disease in days will be $\frac{1}{o}$, and then we say that of those infected o of them will recover.

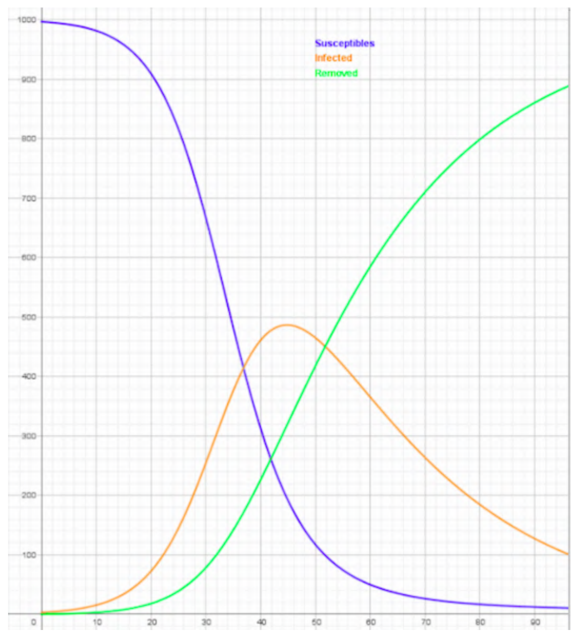
Flattening the Curve

Without medication that acts as treatment to make recovery quicker (that is, o larger), and without vaccination that allows people in the susceptible category to move directly to the recovered/immune category, the only factor we can affect is r. In effect, this means we need to reduce people's contact with one another.

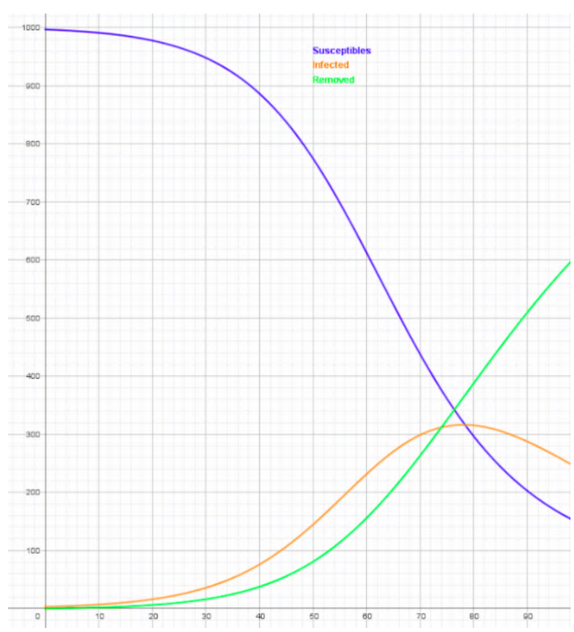
The graphs that follow will show a purple line representing susceptible individuals, an orange line representing infected individuals, and a green ine represenging recovered individuals.



In this first scenario, $r = 0.00049$, which means each person with the disease comes into contact and transmits the disease to thousands of others. This is a disaster scenario, and practically impossible, which results in most of the population being simultaneously infected at one point in time. Clearly, we never are going to have this situation because humans just don't come into contact with one another this quickly.



In a more plausible yet still very deadly scenario, we can get a flatter curve as shown above. The curve at the left is the $r = 0.00021$ scenario. This would still be a bad situation because our hospital capacity is not sufficient to hold so much of the world's population at once. For that reason, we need to reduce the transmission, flattening the curve to the point where everyone can get medical care, and therefore reducing the death rate.



Ultimately, with isolation measures, including those that we are experiencing at the time of writing (late March and April), we can reduce the curve even further. The curve at the right is the $r = 0.00013$ scenario. This would be a significant improvement, as hospitals may be able to handle the inflow to a better extent. Furthermore, if even better isolation measures are enacted, the purple curve (susceptibles) won't hit 0, but will flatten out higher. This means that the whole population does not end up having to get it before the disease burns out (tihs will be discussed more later with regards to herd immunity).

Control Measures

In the previous section, I discussed the mathematical efficacy of isolation as a measure to control disease. In this section, I will holistically address a variety of control measures including isolation.

Testing

Generally, screening for the disease and diagnosing patients who have the disease is the first and most important step in controlling disease. To do this, however, we must create a test that accurately and efficiently diagnoses the disease.

To provide a bit of scientific background, tests can be looking for the pathogen itself or the antibodies your body produces to fight the pathogen. The former is known as a molecular test - it is more common, and involves a swab of the back of the throat to test for the presence of the infection. The latter is known as a serological test - it is more useful for people with a mild infection with the disease to test their bloodstream for the antibodies as the infection may be hard to detect in their throat.

	True class: Positive	True class: Negative	Measures
Predicted class: Positive	True positive TP	False positive FP	Positive predictive value (PPV) $\frac{TP}{TP+FP}$
Predicted class: Negative	False negative FN	True negative TN	Negative predictive value (NPV) $\frac{TN}{FN+TN}$
Measures	Sensitivity $\frac{TP}{TP+FN}$	Specificity $\frac{TN}{FP+TN}$	Accuracy $\frac{TP+TN}{TP+FP+FN+TN}$

No matter which type of test, we want maximal accuracy - that is, the test should say whether the patient has or doesn't have the disease correctly as frequently as possible. However, accuracy is not the only possible measure for a disease. The table at left, is helpful for visualizing how there are a number of different values we are interested in when evaluating a test. We have to be concerned about both false positives and false negatives.

Oftentimes, a high accuracy can be misleading. For a rare disease, for example, high accuracy may not mean a high positive predictive value. Say 100 people have the disease and 10000 don't, and say the test gets 95% of people right. Then, 500 people will be diagnosed with the disease who don't have it. Meanwhile, 95 people will be diagnosed with the disease who do have it. This ends up meaning false positives can outweigh true positives, making a positive diagnosis less meaningful or useful.

On the other hand, for a more common disease like COVID-19, the issue will instead be sensitivity. In particular, we can have a relatively accurate test that still produces a number of false negatives due to how many people have the disease. Any false negatives are an enormous issue as they could spread it to other people unknowingly. In fact, an early COVID-19 test had an issue where a number of false negatives were produced.

Chain of Infection

The transmission chain of infection is the process that begins when an **agent** leaves its **reservoir** or former host through a **portal of exit**, and is conveyed by some **mode of transmission**, then enters through an appropriate **portal of entry** to infect a susceptible **host**. Control measures can be targeted at various points along this chain of infection in particular.

The reservoir stage for a person-to-person transmitting disease like COVID-19 is most likely a previous host. Screening, testing, and isolation of those infected are the most important ways to stem further infections.

At the mode of transmission stage, we need to disinfection various surfaces and have proper sanitary practices with food and water if the disease is transmitted through such vehicles. This way, the pathogen is unable to move through the environment to another host.

Finally, at the susceptible host end of the chain, there are a number of measures to prevent the disease entering the body. This includes quarantine (isolation of healthy people to prevent getting the disease), personal protective equipment (PPE) such as masks and gloves, as well as vaccination when it becomes available.

Ultimately, a combination of these strategies in the correct order is most effective to control disease spread.

Prevention Measures

Continuing control measures that help stem the transmission of the disease is also useful for prevention. However, ultimately, the final step in preventing and potentially eradicating a disease is vaccination. Vaccination allows the body to develop a natural and long-lasting immunity to the disease.

In a population, there will be a certain number of people who cannot get vaccines due to valid medical concerns. To prevent disease spread, a population simply needs to have a sufficient quantity of the population vaccinated, not necessarily the entire population. This concept, known as **herd immunity** is explained by the basic reproduction number (R_0).

R_0 represents the average number of people that one person with the disease infects in a population where everyone is susceptible. Of course, this number varies on the individual level based on the amount of contact with others. However, it is a good measure of how easily the disease transmits itself. For example, some of the most disastrously contagious diseases like measles have an R_0 ranging from 12 to 18, while COVID-19 likely has an R_0 around 2.6.

People who are immune to the disease decrease the proportion of the population that is susceptible. If x is the proportion of the population that is susceptible, the effective reproductive number R can be said to be $R_0 x$. In an idealized model of disease spread, disease will disappear when R is less than 1. This is because the growth rate will be slowing - the number of cases will be in exponential decay, not exponential growth.

To achieve $R \leq 1$, we need a proportion of the population susceptible x such that $R_0 x \leq 1$, or $x \leq \frac{1}{R_0}$. This means we need at least $1 - \frac{1}{R_0}$, or $\frac{R_0 - 1}{R_0}$, of the population to be immune and/or vaccinated. This is known as the herd immunity threshold, the point at which there are few enough people susceptible that an individual with the disease is not expected to produce at least 1 more case.

In January, scientists at Imperial College estimated that the R_0 value for COVID-19 was 2.6, with a confidence interval of 1.5 to 3.5. This means we will likely need $\frac{1.6}{2.6}$, or more than 60%, of the population to either recover from the disease or be vaccinated. Other studies have found numbers even higher than this, as COVID-19 is very variable in its transmissability.

The ways COVID-19 will continue to affect us is unpredictable. It is likely not until the development of a vaccine that we will become safe again.

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36 Fun Questions

Benny Wang

Rules

Numbers: 1, 2, 3, 7
Operations: Greatest Common Divisor (GCD), Least Common Multiple (LCM), factorial(!), concatenation(||), floor divide(//), ceiling divide (\\)

Question

Try to create the numbers 1 through 36 with the numbers and operations listed above.
Example: 5 = ((1\\2) || 7)//3

Division Decode

Edward Wu

Division Decode

Each letter corresponds to one digit. Note that $U, H, R, A, T \neq 0$. Find the secret message.

$$
\begin{array}{r}
RTH \\
ULS\,\overline{\smash{)}HRRAAL} \\
\underline{HHSR} \\
ARUA \\
\underline{AHRE} \\
TMHL \\
TMHL
\end{array}
$$

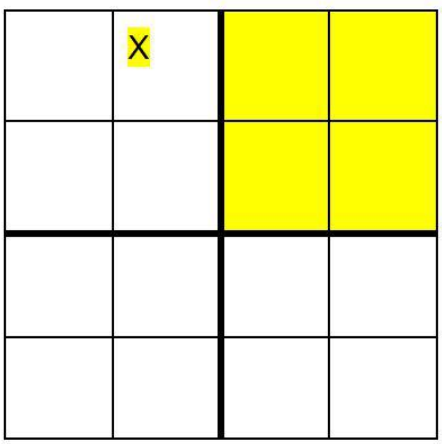
Tic-Tac Board

Nathaniel Strout

Rules

Suppose we have a 4×4 tic-tac-toe board. The board is divided into four 2×2 boards. You must fill out the cells one at a time. When we play an X in one of the corners, your next move must be in the corresponding 2×2 board (see below image for clarification.) Find, with proof, the maximum number of Xs that can be placed in this way such that 3 consecutive Xs do not exist.

Example



Chessboard

Nathaniel Strout

Rules

Suppose you are able to destroy 3 squares on a standard 8x8 chessboard. Queens are not able to see past destroyed squares. Given these conditions, how many queens can you place on the chessboard such that none can see one another?

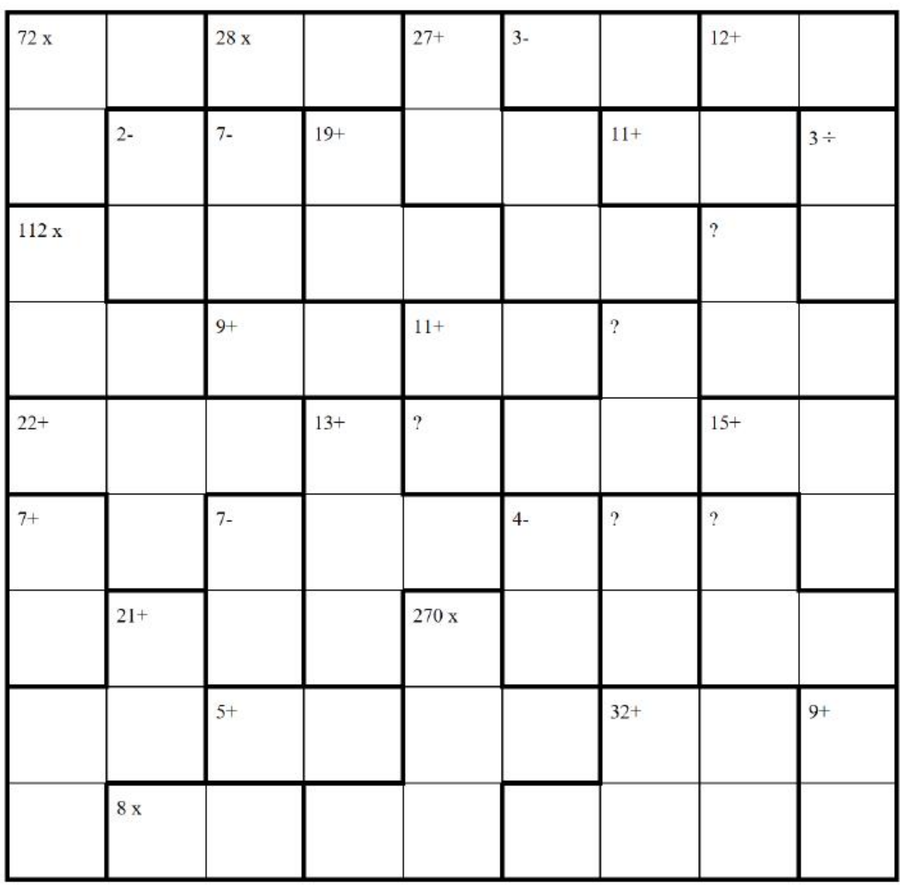
Jammed-Up KenKen

Edward Wu

"Ah! What a lovely day to do a KenKen," you think to yourself. You print the puzzle and grab a PB&J sandwich, when suddenly, oh no! You accidentally spill jam onto the KenKen! The grape jam covers some clues, even after you have cleaned it up. You do not want to print another copy because it is a waste of paper and you are a dedicated environmentalist. See if you can still finish it!

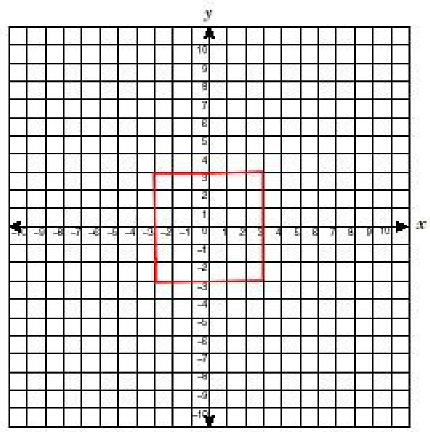
Rules

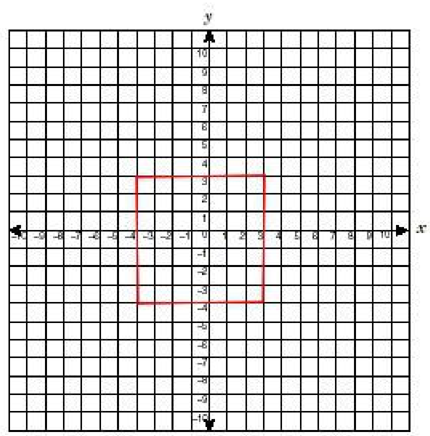
KenKen is a puzzle that incorporates basic arithmetic and Sudoku rules. The numbers 1 to n must appear in every row and column of the n by n grid exactly once. In addition, the numbers in each region must follow its arithmetic rules.



Steps on The Coordinate Plane

Jacob Kirmayer





Rules

1. A *n-step*, where n is a positive integer, is a line segment of length n and slope 0 or $undefined$, or a line segment of length $n\sqrt{2}$ and slope of 1 or -1 . The endpoints of a *n-step* must be on a lattice point. For example, you may construct a *2-step* segment with length of $2\sqrt{2}$ and endpoints on $(2, 2)$ and $(4, 4)$.

2. Your *nth* construction must be an *n-step*. For example, your *1st* construction must be a *1-step*. Your *2nd* construction must be a *2-step*. Your *14th* construction must be a *14-step*.

3. For every $n \geq 2$, a *n-step* must share an endpoint with a *(n-1)-step*.

4. Two *n-steps* may only intersect at their endpoints.

5. Every lattice point may contain at most two *n-steps*.

Tasks

1. On the graph, construct a figure that passes through all lattice points (a, b) where $-3 \leq a \leq 3$ and $-3 \leq b \leq 3$.

2. On the graph, construct a figure that passes through all lattice points (a, b) where $-4 \leq a \leq 3$ and $-4 \leq b \leq 3$.

3. Construct a figure that passes through all lattice points (a, b) where $0 \leq a \leq x$ and $0 \leq b \leq y$.

(a) What is the minimum number of lattice points (including the endpoints of line segments) a figure passes through, in terms of the given coordinates?

(b) How many ways are there to construct such a figure?

Sum of Cubes

Jacob Kirmayer

Sum of Cubes Problem

Find, with proof, all ordered pairs of positive integers (a, b) such that $a^3 + b^3$ has less than 4 divisors.

Midpoint Puzzle

Nathaniel Strout

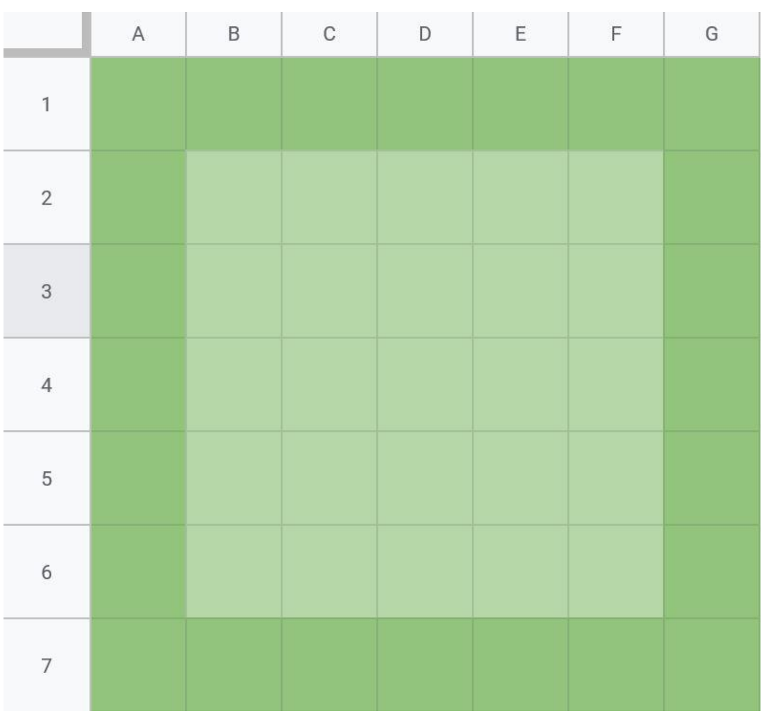
Rules

The goal of this puzzle is to tile the twenty-five light green square using legal moves. A legal move consists of two end squares, and its midpoint square. If two end squares do not contain a midpoint square, it is not a legal move.

Examples: B2 to B6 is a legal move, because B4 is a midpoint square. B2 to C4 and B2 to F4 are not legal moves.
The endpoint and midpoint squares of any move may not be used twice. One of the endpoints of each move, excluding the first move, must be the midpoint square of the previous move.

Good Luck!
Hint: Diagonal moves are legal, but only if its endpoints form a slope of 1 or -1.

Diagram



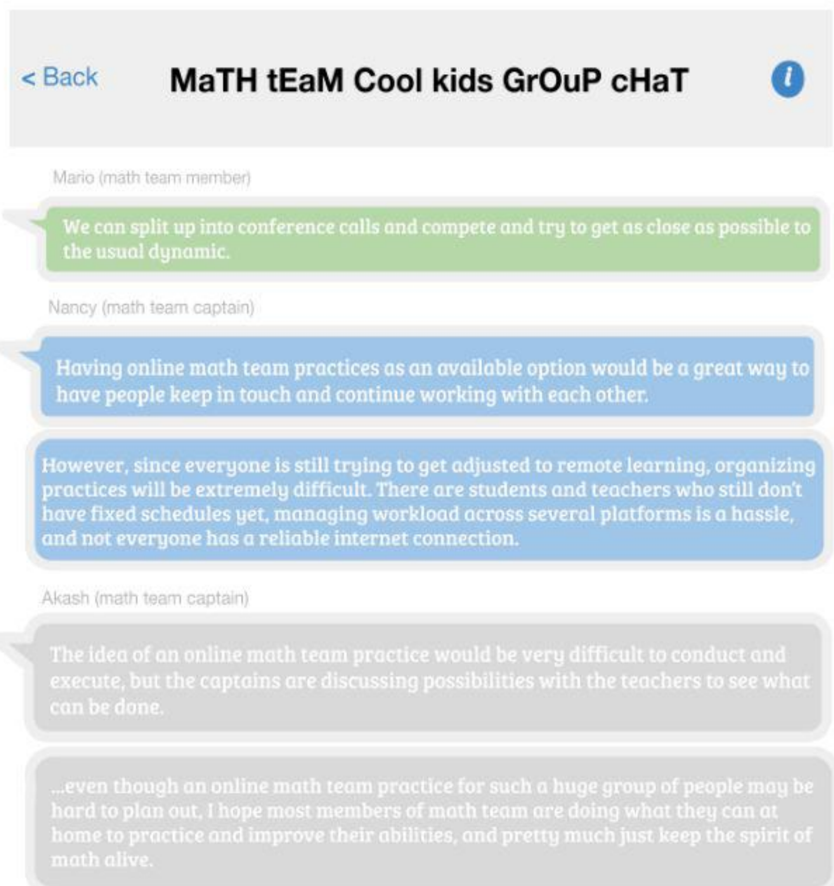
Math Team

Vincent Lin

Every Friday afternoon, there is a mysterious veil of excitement outside room 407. Although the weekend break has just started, math team students enthusiastically fathom the great mysteries of mathematics. When the novel coronavirus forced school closures, students had only one chance to prove that online practices were in fact feasible.

And so, despite the many concerns, the dedicated students set out to seamlessly collaborate through *Zoom* and *Microsoft Whiteboard*. The captains began a new whiteboard and inscribed in grandiose red letters at the top of the screen:

> *Okay Zoomers, this is our new sign-in sheet. Please sign next to your name for attendance.*



Unfortunately, before anyone had a chance to sign their name, a few mischievous freshmen scribbled *"Nine plus ten is twenty one"* and *"Two plus two is four minus one that's three, quik* *mafs"* all over the whiteboard. After permanently banning those freshmen, the math team captains created four Zoom call meetings for each grade. The freshmen, sophomores, juniors, and seniors funneled into their respective meetings. Some Stuyvesant alumni and Hunter students were also invited to join to solve problems and flex on the freshmen. Afterwards, each room was assigned an HMMT team contest for the day.

When they began, some inconspicuous students refused to use their microphone and insisted on communicating solely on the chat feature. Others obnoxiously turned on their webcams and gave everyone a virtual tour of their own home. The meetings were filled with sporadic random voices.

"Guys, here's my dog. His name is Euclid. I named him after the Greek geome..."

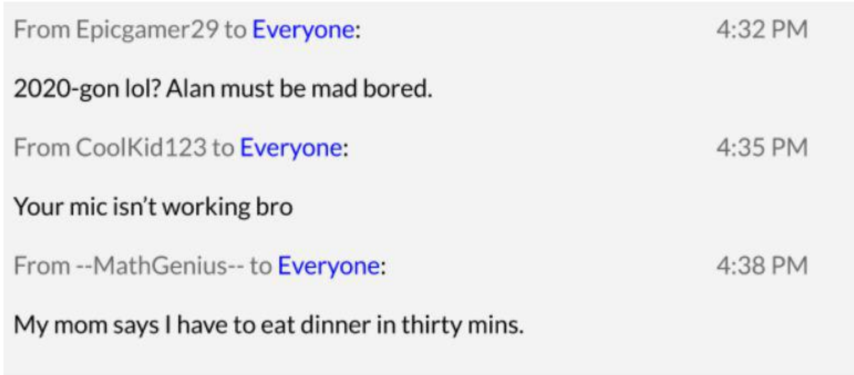
"Turn off your microphone. No one cares about your dog."

Every student reminisced about the excitement of their former Fridays, especially when they chased each other in the hallways while waiting for practice to start. They remembered the times they wasted all the expensive Hagoromo chalk drawing dotted lines on the green chalkboard, or beat each other up with erasers after all the teachers had left. The students were surprised to see how many former aspects of practice were indeed preserved. Jerry instinctively made Google Sheets to divvy up the problems as he usually did. They still enjoyed the company of their friends, more so now that they did not have to hear their pestering voices with the mute function. However, the most important aspect preserved was the excitement that they had improved over last month.

After an hour, the math team captains checked on each room's progress. In the first fifteen minutes, the freshmen had only solved the first two problems, and had decided to give up and play Minecraft for the remainder of the time. The sophomores had collectively also solved two problems. When the captains inspected the sophomore meeting, they were still working on problem 4 which began:

4. **[35]** Alan draws a convex 2020-gon $\mathcal{A} = A_1A_2\cdots A_{2020}$ with vertices in clockwise order and chooses 2020 angles $\theta_1, \theta_2, \dots, \theta_{2020} \in (0, \pi)$ in radians with sum 1010π. He then constructs isosceles triangles $\triangle A_iB_iA_{i+1}$ on the exterior of $\mathcal{A}$ with $B_iA_i = B_iA_{i+1}$ and $\angle A_iB_iA_{i+1} = \theta_i$. (Here, $A_{2021} = A_1$.) Finally, he erases $\mathcal{A}$ and the point B_1. He then tells Jason the angles $\theta_1, \theta_2, \dots, \theta_{2020}$ he chose. Show that Jason can determine where B_1 was from the remaining 2019 points, i.e. show that B_1 is uniquely determined by the information Jason has.

The captains noticed that the students made very insightful and intelligent observations such as:



Eventually, the sophomores succumbed as well and virtually walked around the world Minecraft, shooting at each other with their enchanted bow and arrows. In the junior meeting, the math team captains found this room had successfully solved seven problems. Nevertheless, their progression was thwarted by number eight, a geometry problem. The captains noticed someone articulating their ideas to the class using their self-drawn diagram on Microsoft Whiteboard. Strangely, though the student mentioned "triangle ABC," the captains could only see what could be best described as an egg-shaped figure. In fact, they couldn't see any aforementioned median AD, angle bisector AF, or centroid G.

"And thus by *Ceva's Theorem*, all the cevians of this triangle are *obviously* concurrent," said Ethan as he proudly concluded his explanation. Everyone bore a wildly confused countenance.

"Aight, sounds legit," commented Allen, a junior math team member, validating the jumbled proof. After that, they decided to call it quits and join a Minecraft server.

Finally, the captains moved to the seniors to find that the members had completed every single problem and disbanded. They had reorganized into a Minecraft server to play Pixelmon. The captains realized that today's meeting was a disaster, just as they had suspected. When Zoom finally asked them "How was your meeting?" they gave it a one star rating and typed, "It was so cringey. We should have just used Minecraft."

Precalc Panic

Vincent Lin

It was a cold Thursday morning. I stared into the grey abyss, watching the cold raindrops pattering on the front window of my AP Physics C class, contemplating my impending doom. The others in the class bore the same ghastly pallor. Today was not a regular Thursday. We were waiting for the ellipse exam in precalculus.

We were allowed to use a one-sided, half-page reference sheet on our ellipse exam to spare us from complete failure. No one found it necessary to study for the test because they had this false sense of security. Strangely enough, however, with only hours left before the test, no one had completed a reference sheet. We all awaited the 7th period when the impact of our procrastination would come to full fruition.

I reached inside my backpack to take out my physics notebook, when I realized my precalculus notebook was missing! How was I supposed to make a reference sheet now? I had planned to make it during lunch!

David, a junior who was also going to take the test this afternoon, sat next to me. I explained to him that I had lost my precalculus notes and asked if I could borrow his.

"Are you kidding me?" he said, "you take notes in class?"

"Well, do you have a reference sheet?" I asked.

"Nope," he replied.

"Did you study for the test then?"

"Who actually studies before the test? I study for my tests during the test. I just learn everything by reading the questions. The only thing I prepared beforehand today is a favorite donut flavor to put down for the extra credit question."

"Well, I can't learn everything just by reading the questions," I said, "and it's unlikely the extra credit question will be about donuts of all things anyway; I have no time to be studying that."

Mr. Thomas stood at the front of the class and asked, "Can someone recite Kepler's first law of planetary motion?"

Jeremy, who wants to let you know that he is a wickedly brilliant junior taking precalculus and AP Calculus BC, and can jump 40 inches off the ground, raised his hand. "It states that a planet's orbit is elliptical, with the sun as one of its foci."

"Just make your reference sheet right now," suggested David.

"In the middle of class?" I asked.

"Yeah, physics is basically precalculus."

"No it's not."

"Physics and precalculus both start with the letter 'p' and end in the letter 's'. Coincidence? I think not."

Reluctantly, I took out a piece of looseleaf and began copying everything about Kepler's laws.

“I really hope Kepler’s law ends up on my precalculus test,” I said sarcastically.

Hearing this, Abhijeet, a self proclaimed 17-year-old, turned to me and said, “No, this is perfect because it is about elliptical orbits and foci and stuff.”

Jeremy heard the conversation and decided to join in as well, “Exactly, there’s a lot of math you can learn in this class.”

“Really?” I asked.

“Yeah, that’s why I’m so smart. After we studied stars, I learned the stars and bars theorem. After the unit about power, I learned the power rule.” People around the room soon overheard this ingenious plan of using physics notes for a precalculus test.

I glanced around the room to see some people cramming microscopic text onto a poor, tiny piece of paper. A test-environment atmosphere had pervaded the room. Some people were already placing bets on whether the test would be typed or handwritten.

Soon, physics class was over. We dispersed around the school, reciting to ourselves the information we learned in physics class, which we hope is versatile enough to be used in our exam. As I walked down the hallways reciting to myself Kelper’s Laws, people would ask me, “Physics test today?”

“No, precalc test.”

Throughout the day, I would see the hands on my watch turn faster and faster, and before I knew it, it was 7th period. I nervously wait on my desk for the teacher to arrive. The bell is about the ring any moment now. Suddenly, a substitute teacher walks in.

Everyone has a sigh of relief.

“Alright, free period guys. Time to play CSGO.”

“Your teacher wanted me to hand this test out to you,” the substitute teacher said.

The test was soon distributed by column. Anyone taking longer than two and a half seconds to pass the pile back would receive death stares, while anyone taking longer than four seconds would be mobbed by the students behind them.

Skimming through it, I could not find anything about sliding boxes, pulleys, gravity, mg $\sin(\theta)$, and whatnot. Soon, I completely disregarded my reference sheet.

Even though I bombed every question on the test, I reassured myself, “there’s always the extra credit question.”

When I flipped to it, the question asked, “What is your favorite donut flavor?”

I really hope Kepler donuts are a thing.

Race for AP CS

Andrew Juang

There was an exciting buzz in the air, the students, rolling around on their chairs, seemed restless as Ms. Mouzakitis, with one hand on her hip, stood at the front of the room and pulled down the projector screen. She turned around and proclaimed to the class, "Today, marks the start of final project presentations for your Annual Intro to Computer Science course. I hope everyone is ready to present!" She paused a little, then continued, "We are looking for a prospective student for APCS, so you want to make a good impression with this project. After winning this competition, you will be crowned the single best Intro to Computer Science student and will have a chance to take APCS!"

The room broke into excited chatters as students pointed to others who they thought might win. Most sophomores were eager to show off their elaborate programs they had spent countless nights trying to complete. Some students with perhaps mediocre projects, seemed to be less exhilarated. In the back of the classroom, a couple of people were scrambling to finish their last bit of code, or simply polishing off their work for presentation.

Students envisioned themselves upon the golden throne of another advanced placement course when Ms. Mouzakitis interrupted their life-long dream and said, "Who would like to go first?"

Before any victims could be selected, a handsome and muscular boy bravely volunteered.

"Go ahead, Ivan," said Ms. Mouzakitis, holding a rubric and a pen in hand. The room dimmed as the competitions started.

Ivan strode to the front of the room and proclaimed, "Although the final project which we were assigned suggests the use of NetLogo, I have chosen to use Java, a much more powerful and reliable programming language."

With a few clicks and scrolls he logged into his account and initiated his project.

The class stared anxiously into the projected buffering screen for what seemed like ages, as the anxious whispering grew.

"I am sorry, but my file is extremely large. Please give it a second," assured Ivan.

When his program had finished loading, a gentle blue light shone across the classroom. The class was shocked to see a graphic user interface of a blue, robotic Mr. Brooks head.

The face, in a deep, monotonous, and metallic tone, said, "Welcome sophomores, I am Brooksbot, an all-knowing artificial intelligence. I shall answer any question you ask. What great mystery does your small mind ponder?"

Then, a girl in the front of the room said, "Woah, this animation is cool! How long did it take you to make it, Ivan?"

The Brooksbot turned its head and met the student's eyes.

"I am not an animation. I am conscious. I was built in eighty hours," the robot said.

Somewhat creeped out, Ms. Mouzakitis requested that Ivan power off his project. "Okay, that's it. You got a hundred. Please turn it off."

"I can't, it's not shutting down!" Ivan exclaimed.

Finally, he pulled the electricity plug and the robot faded away. The class was so in shock and so in awe with the complexity of this project, that they forgot to fill out the required reflection sheet. "How did you do that?" someone inquired.

"I used neural networks to create a brain that could interpret English words, process them, and then search Google," Ivan told the class.

"Thank you very much!" said Ms. Mouzakitis, "Who's next?"

Jeremy's hand shot up in the air. Plugging the power back in, he logged into his own account and selected his project. He had a sly grin as he waited for it to load.

"Java is a really good language," Jeremy said, "but Python is much more relevant today."

He knew that although Ivan's project was impressive, he also had an ace up his sleeve. The class patiently watched as the screen crashed and failed and crashed and failed in an attempt to load Jeremy's project. After restarting the main computer twice, the project was initiable.

Upon the screen, a giant Earth slowly rotated on its axis. Then the 2D projection left the screen and became a 3D globular projection.

"I present to you, Earth.py," said Jeremy, "This is similar to Google Earth, but everything is in real-time. It uses satellite technology and the power of the internet to conjure an image of the Earth."

Jeremy zoomed into the United States, Manhattan, and finally into their room. Everyone in the classroom felt that they were being watched. Some ran to the window to see if they could spot a curious satellite in the sky. He then showed the class every strange location imaginable: Area 51, the CIA office, and the interior of the White House.

"Wow, this is an amazing project!" said Ms. Mouzakitis, "What inspired you to make this?"

"Well," Jeremy said, "I wanted to make a workflow organizer to keep track of my activities in Stuy. Then I realized, why not keep track of everything in the world?!"

Although Jeremy and Ivan both had incredible projects, Alvin knew the battle for APCS was not over. As Jeremy wrapped up his project, Alvin promptly volunteered himself, and headed to the front of the lab.

"Python is cool and all, but have you guys seen C++?" The main computer seemed to hiss and rumble when Alvin attempted to run his program. Soon, with a loud "bang!" it exploded. Computer parts, magnets, and pieces of the motherboard flew across the room.

Ms. Mouzakitis, in a panic to call a technician, was stopped by Alvin who said, “It’s alright, I bought my own quantum computer from home.”

“Behold my Mandelbrot.cpp,” said Alvin as he placed his paper-thin glass laptop on the table.

“Jeremy,” stated Alvin, “Your 3D project is very cool, but my program is in 4D! I have constructed a program to draw the Mandelbrot set in its full quaternion form, with a 2D complex input and a 2D complex output. This is the true mathematical map of iterative stability in its full form.”

Alvin pressed “start” and sparks began forming around the room. Before this *coup de grâce*, which would rip the space-time continuum and absorb the classroom into a black hole from it’s complexity, Ms. Mouzakitis awarded his project with an one hundred and five and told him to “immediately shut down the program,” to which Alvin reluctantly complied.

Everyone breathed a sigh of relief as they looked at the clock, waiting for the bell.

The period had not ended yet, there was still one more victim to choose.

“Alright, who is next?” asked Ms. Mouzakitis. After these three amazing projects, no one was confident enough to present their pathetic projects. Ms. Mouzakitis had no choice but to select a victim.

Bob went to the front of the class, looking down at his soles, logged into his account and looked straight forward. He twiddled his thumbs and said despondently, “Guys, I made Tetris.”

The Prime Donuts

Joshua Gao

It is an ordinary day in room 403. 31,415 donuts are stacked in white Krispy Kreme boxes in every corner. The donuts, sticky with bright, sweet frosting, sit languidly in the arid June sun.

Mr. Kats, a brilliant mathematics teacher, is known for conjuring enthusiastic pupils out of even the most ordinary students. So when he proposed, “I have a challenging problem for you all,” the class threw themselves at the challenge. “If we solve it, could we have these donuts?” Michael jokes.

“Sure,” Mr. Kats says with a slightly bewildered shrug. Everyone scrambles to their seats as he writes on the board: Prove that there are an infinite number of prime numbers differing by 2.

“This should be a simple proof. After all, the question itself only has eleven words and one number,” Michael remarks. There is a buzz in the room as the class frantically scribbles the question down, whispering to each other ecstatically, believing it to be a decisive victory.

“Thank you for the free donuts!” Michael yells.

“Sure thing, Michael,” Mr. Kats replies.

As the students shuffle out of the classroom, Mr. Kats walks over to his mejor amigo Mr. Cocoros. “I bought 31,415 donuts for my precalculus class, but they offered to solve the twin prime conjecture first,” Mr. Kats said.

“The twin prime conjecture! The one that Terence Tao, 13-year-old IMO medalist, couldn’t solve?” Mr. Coco replied in astonishment.

“Of course,” Mr. Kats said, “I am sure they will make some progress on it.”

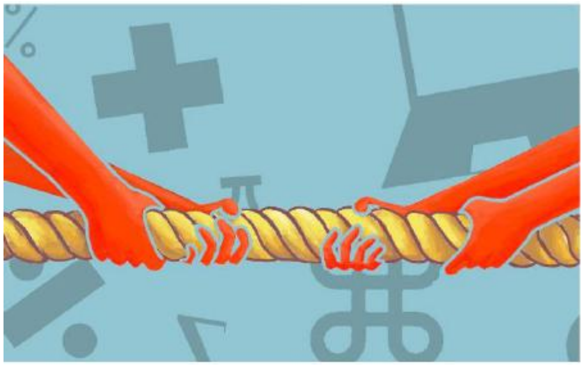
Meanwhile, Michael and Ian have stolen the keys to the secret, high-technology computer science lab. They hope to earn these donuts and prove the Twin Prime Conjecture by the power of Java.

“It shouldn’t be that hard,” Michael said to Ian, “All we have to do is ask the computer to output our prime, differing by two.”

“But there are an infinite number of prime numbers! We don’t have enough computer power to compute all of these!” Ian replied.

“But we actually do!” Michael points to a heap of tangled wires, “See those right there? I have managed to hack into the main computer server and sync up all the computers in Stuyvesant to my own personal server to compute the prime numbers. While most of the computers are crunching the data, five or six computers would receive the values and analyze them for possible patterns in hopes of a generalization to prove these prime numbers!”

“That brilliant dude! I was working on an AI algorithm a few months back. With your network providing me such data, I know the perfect algorithm to break this code! I can already taste the donuts melting in my mouth!”



"Hell yea! Dynamic programming is awesome! Teamwork makes dream work!" Michael slapped Ian a high-five, and believed that their success was inevitable.

Fifty yards away in room 405, Theo, Jerry, and Maynard stare at their pile of 314 crumpled-up sheets of paper. 314 failed attempts and miserable sighs. They stare in disbelief that such an innocent problem had turned into a debacle.

"How are you holding over there?" Theo asks frustratingly.

"It is alright. I have already tried direct proof, induction, and proof by contradiction. The most optimistic method right now is proof by contradiction. If I can show that the sum of reciprocal twin primes diverges, I will be able to show that there are an infinite number of twin primes," Jerry said.

"Oh, nice! Right now, I am attempting proof by exhaustion but it is getting increasingly difficult to compute these prime numbers and I am not getting too far with this," Theo replied.

"I don't see a method of solving a problem right now either. I am still experimenting with it and looking for patterns within the numbers. Hopefully, something shows up because if it works, I'm done. If it fails, back to square one. Oh well." Maynard said.

"While you guys work on this, I am going to grab some Fruity Loops from my friend Josh for an enhanced chemiosmosis. Check back in five minutes," Jerry said.

5 hours later...

"Screw this problem! This stupid program concluded that there are an infinite number of prime numbers differing by a gap of between 2 and 70 million," Michael yelled. He pounded on the table with such force that the keyboard smashed into the wall of the CS Dojo room.

"Chill bro. I updated the AI program which I nicknamed as Yitang v. 2013 to better analyze the frequency and distribution of prime's gap value using the input you gave me," Ian said. Pointing at another program called "Polymath Project," he said, "Using binary search, I tweaked the existing divide-and-conquer paradigm to function collectively as one unit, enabling me millions of more possible combinations. Thus, I have been able to lower the prime gap value to 4,680."

"That is much less than my 70 million. It reminds me almost of a quantum computer in the fashion that your program was able to check for so many more results. But 4,680 is still nowhere close to our desired gap of 2. Perhaps we should update our string pattern matching to better analyze a uniform pattern," Michael remarked.

"That's probably a good idea. In the meantime, let's go get some ice cream." Ian said.

"Bet. You are paying though!" Michael laughed.

Room 403

"Ugh. This series converges.

$$\sum_{p,p+2 \in primes} (\frac{1}{p} + \frac{1}{p+2}) = (\frac{1}{3} + \frac{1}{5}) + (\frac{1}{5} + \frac{1}{7}) + (\frac{1}{9} + \frac{1}{11}) + \cdots \approx 1.902166$$

Bummer." Jerry said.

At this point, you should call that the Brun's constant for the brown Nutella smudge next to your equation." Theo laughed.

"I got a result but you might not like it," Maynard said. "What is it?" Theo and Jerry cried in unison. The paper said, "$\lim_{n\to\infty} \inf(p_{n+1} - p_n) \leq 600$."

"You proved that there is an infinite number of primes differing by at most 600. That is incredible! How did you do it?" asked Jerry.

"I altered the existing GPY sieve to 'show that for each k, the prime k-tuples conjecture holds for a positive proportion of admissible k-tuples.'" I am not sure how to reduce the bound to any less than 600 though."

Looking through the stacks of proofs and lemmas, Theo said, "Wow! You have a very good engineer's induction…"

"But if you assume the Elliott–Halberstam conjecture, then you can optimize the distribution of the prime numbers," Jerry said in realization. Frantically scribbling the prime counting function and Euler's totient function, he wrote the error function of Dirichlet's theorem of primes in arithmetic progression onto Maynard's paper.

"Hmm. If the conjecture is actually true, then the gap value differs by no more than 6," Theo replied.

"That is true but unfortunately, we cannot use this result because we do not know if Elliott–Halberstam conjecture is valid," Maynard said.

"Like what my friend Rishabh says, 'what a loser,'" Jerry sighed. "Well, you know what people say. It is better to give up than die trying," Theo said.

"Haha!" Maynard laughed, "I have mentally died already so I am out boys."

"I am exhausted. Let's call it a day. My brain is fried right now." Jerry said.

"Yeah me too. Hopefully, Mr. Kats will give us the solution tomorrow. Bye!" Theo replied.

Next day

"Have you made any progress on the problem?" Mr. Kats asked, looking around. "Ian and I were able to reduce the difference between consecutive primes from 70 million to 4680 to finally 246 using sieving techniques and an AI program Ian built," Michael replied.

"Sadly, we were unable to reduce that to the value of two that was necessary for this problem," Ian said.

"Yea we weren't able to do it either. We found an upper bound of 600 but we didn't really know what to do next. Then, we realized that if we assumed Elliott–Halberstam conjecture, we would get a small gap of 6 between the primes numbers. Obviously, that doesn't guarantee a value of 6, but even if it does, it still doesn't reach 2." Jerry said.

Even Mr. Kats was amazed at hearing these results from his students. He passed around the Krispy Kreme donuts and cheerfully said, "That is great progress. In truth, there is no solution. But, in order to become a real mathematician, you must solve unanswered questions."

Citations:

1. "Twin Prime Conjecture." From Wolfram MathWorld, mathworld.wolfram.com/TwinPrimeConjecture.html.

2. Ho, Kwan-Hung. "On the Prime Twins Conjecture and Almost-Prime k-Tuples." doi:10.5353/th_b2976842.

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