Conic Sections

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What is a Cone?

Cones in General

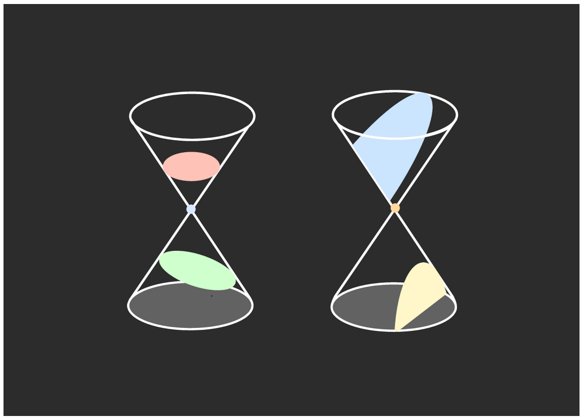


Figure 1: Image by Jennifer Sun

A cone is a figure in which a circular base tapers to a single point. Essentially, you pick a point P (the cone's vertex) and a circle ω (the cone's base) that lie on separate planes and draw the lines connecting P to every point on ω.

Right Cones

A right cone is a cone constructed using point P and circle ω (lying in separate planes) where if plane $\mathcal{T}$ contains circle ω, and if O is the center of ω, then $PO \perp P$.

The significance of a right cone, is that for any points X, Y on ω, $PX = PY$, always. The reason for this is $\angle POX = \angle POY = 90°$, and since $PO = PO$ and $OX = OY$ (since they are both radii of ω), the two triangles $\triangle PXO$ and $\triangle PYO$ are congruent.

This fact implies that we can actually generate a right cone using nothing but two lines! Lets say we construct two lines intersecting at point P, r and s. Rotate line r about s in 3-D space, and we will get some kind of a figure. To show this is a right cone, pick a point X on r. Since r is being rotated around s, the distance PX must remain constant, and the point X traces about a circle. Thus we have a right cone!

The interesting thing about this construction is that it produces not one, but two right cones. These cones are actually reflections of each other over P.

Spheres in Right Cones

Now imagine we want to fit a sphere inside a right cone. How would we do it? Can we even do it?

To return to our rotational idea of a cone, we can think of a sphere as a rotation of a circle about its diameter (to visualize this spin a coin on a table, its spinning shape appears to make the shape of a sphere!). Therefore, to get a sphere tangent to a right cone we simply make the axis of symmetry s contain the diameter of the circle, and the second line r tangent to the circle.

Lets say that X is the tangency point with r and the circle, then referring back to the section on right cones we realize that rotating X about s will yield a circle (since it is the base of a right cone).

The Two Tangents Lemma

The Two Tangents Lemma states that given a point P and points X, Y on circle ω such that PX and PY are tangent to ω, then $PX = PY$. (This can be proved fairly easily so it won't be proved here).

Now since a sphere is just a rotated circle, this lemma applies to spheres as well, and can be rephrased as such:

Given a point P and points X, Y on sphere Ω so that PX and PY are tangent to Ω then $PX = PY$.

What are the Conic Sections Anyway?

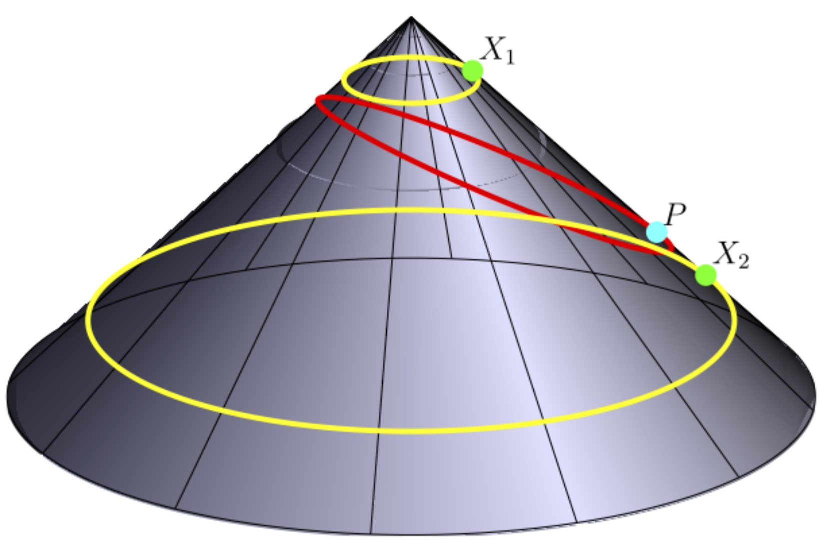
Now to get to the actual topic of the article, conic sections. The 3 conic sections are the circle, ellipse, parabola and hyperbola.

- *Circle:* A circle is a set of points equidistant from a single point.
- *Ellipse:* An ellipse is the set of points such that the sum of the distances from each point to two foci is a constant.
- *Parabola:* A parabola is the set of points equidistant from a point (the focus) and a line (the directrix).
- *Hyperbola:* A hyperbola is the set of points such that the positive difference between the distances from each point to two foci is a constant.

Ellipse

An ellipse is formed when a plane intersects a cone splitting it into two sections where one section has finite volume

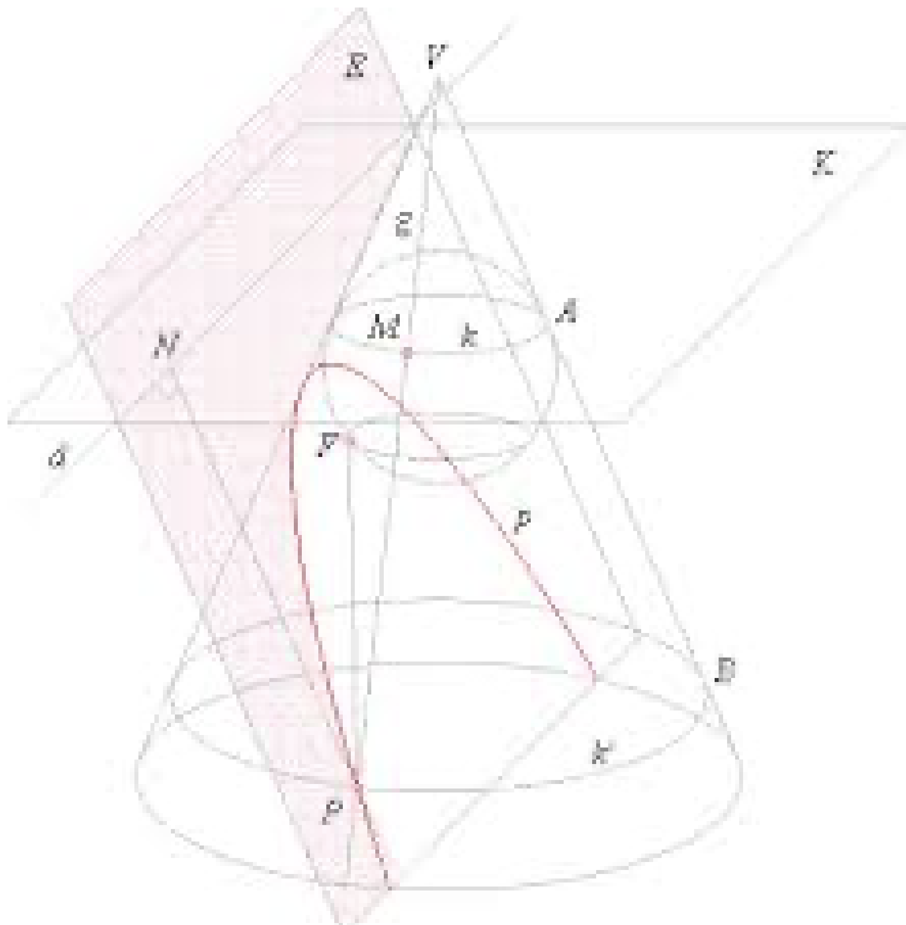
To prove that the figure described above is actually an ellipse, consider two distinct spheres Ω_1 and Ω_2 both inscribed in the cone and tangent to the plane at points F_1 and F_2 respectively. Now consider lines running along the cone from the vertex V intersecting Ω_1 and Ω_2 at X_1 and X_2, and with the plane at P. Since PX_1 is constant (by the two tangents lemma), and the length of PX_2 is constant (also by the two tangents), the length of X_1X_2 must also be constant since $X_1X_2 = PX_2 - PX_1$. Now lets say these lines running along the cone intersect the ellipse at point P, then $X_1X_2 = PX_1 + PX_2$. Since $PX_1 = PF_1$ and $PX_2 = PF_2$ (two tangents lemma), then $PF_1 + PF_2$ must equal X_1X_2, and must be a constant.



A diagram of the points P, X_1 and X_2 in a cone. Red is the ellipse, while yellow represents the ω's.

Note that since a circle is a type of ellipse, this also proves that a circle is also a conic section. (Try and visualize this!)

Parabola



A parabola is formed when a plane cuts a cone into two infinite sections. In other words, the plane was parallel to the line we rotated around the axis of symmetry to generate the cone.

Lets say the plane R is the plane generating the parabola, and lets say Ω is the sphere inscribed in the cone and tangent to R closest to the vertex of the cone. Lets also say that Ω intersects R at F, and it intersects the cone at circle ω. Let plane S contain ω, and let planes R and S intersect at line d.

Now consider the line radiating from the vertex V of the cone passing along the cone intersecting ω at Q and intersecting the parabola at P. Then $PF = PQ$ since PF and PQ are both tangent to Ω at points F and Q.

Now lets consider the circle δ on the cone so that the distance from the vertex to any point on δ equals VP. And let plane T contain δ. Then T must be parallel to S since the two are both perpendicular to the axis of symmetry of the cone. Because the two are parallel, if P' is the foot from P to d then there must exists points A and B on ω and δ so that $PP'AB$ is a parallelogram. Since $AB = VA - VB = VQ - VP$, $PP' = AB = PQ$ so we have proven the distance between the directrix d and focus F is equal for a point P on a parabola.

Hyperbola

The Hyperbola comes when a plane splits a cone into four different sections.

Try proving this conic section one on your own! It is much easier to prove than the parabola, and uses much of the same ideas as the ellipse and parabola. If you get stuck try abusing equivalent segments that come from the intersection of spheres and the cone!