

Base Rules

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Introduction

For the most part, we use base 10. We have 10 digits in our numbering system. This stems from the fact that we have 10 fingers to count with.

There are many numbering systems we could use. In fact, there is an infinite amount of them; one for each natural number. However, we for the most part stick with base 10 in our everyday lives. Clocks utilize base 60, and base 2 and base 16 are common when working with computers, but base 10 is used in most cases.

Because we are so familiar with base 10, we may find other bases to be cumbersome. It is usually most convenient to just convert numbers into base 10 and work with them there. However, if you accustom yourself to other bases, they can have many interesting properties.

Divisibility Rules

The divisibility rules of every base are different. Thus, we must examine different bases' divisibility rules when determining which are the most convenient to work with.

It is important to note that divisibility rules do not require a great deal of effort. It should be faster than dividing and determining the remainder.

A Review of the Divisibility Rules in Base 10

Base 10 has a fair number of divisibility rules. For consistency, I will list them up to 16 for each base.

- 2: If the last digit is 2, 4, 6, 8, or 0
- 3: If the sum of the digits is divisible by 3
- 4: If the last two digits are divisible by 4
- 5: If the last digit is 5 or 0
- 6: If the number is divisible by 2 and 3
- 7: No simple divisibility rule
- 8: If the last three digits are divisible by 8

- 9: If the sum of the digits is divisible by 9
- 10: If the last digit is 0
- 11: Sum digits in odd places, starting from the right, and subtract them by the sum of the digits in even places, starting from the right. Check if the result is a multiple of 11
- 12: If the number is divisible by 3 and 4
- 13: No simple divisibility rule
- 14: No simple divisibility rule
- 15: If the number is divisible by 3 and 5
- 16: If the last four digits are divisible by 16

These divisibility rules fall into four major categories:

- Method 1) Look at the last n digits: 2, 4, 5, 8, 16
- Method 2) Look at the sum of the digits: 3, 9
- Method 3) Look at the difference between alternating digits: 11
- Method 4) Compound of other divisibility rules: 6, 10, 12, 15

Applications to Other Bases

It turns out that many of these rules may be applied to other bases as well. In fact, they may be generalized for all integer bases. This may be proved for all the rules mentioned above.

The Last n Digits

Given a base b , if a positive integer k is a factor of b^n , then an integer x in base b is divisible by k if the last n digits are divisible by k .

Say y is x 's quotient when dividing by b^n , and z is the remainder (also known as the last n digits of x).

$$x = (y \cdot b^n) + z$$

$$(y \cdot b^n) \equiv 0 \pmod{k} \text{ because } k|b^n$$

$$\text{If } k|z, \text{ then } z \equiv 0 \pmod{k}, \text{ then } x \equiv 0 + 0 \equiv 0 \pmod{k} \text{ so } k|x$$

Therefore, this rule can be applied to all bases.

Sum of the Digits

Given a base b , if a number k is a factor of $b - 1$, then a number x in base b is divisible by k if the sum of the digits is divisible by k .

Say k divides the sum of the digits of x .

$$k | a_0 + a_1 + a_2 + \dots + a_n.$$

$$b \equiv 1 \pmod{b-1}.$$

Because 1 to any power is 1, $b^k \equiv 1 \pmod{b-1}$ for any $k \geq 1$.

This means $(b^k) - 1 \equiv 0 \pmod{b-1}$ for any $k \geq 1$.

So $k | (b^k) - 1$ for any $k \geq 1$.

$$k | a_0 + a_1 + a_2 + \dots + a_n.$$

$$k | (b^1 - 1) \cdot a_1 + (b^2 - 1) \cdot a_2 + \dots + (b^n - 1) \cdot a_n.$$

$$k | a_0 + (b^1) \cdot a_1 + (b^2) \cdot a_2 + \dots + (b^n) \cdot a_n.$$

Therefore, $k | x$

Difference of Alternating Digits

Given a base b , if a number k is a factor of $b + 1$, then a number x in base b is divisible by k if, when indexing the digits from 1 to n starting from the right, the difference between the sum of the odd-indexed digits and the sum of the even-indexed digits is divisible by k .

Say k divides the difference between the odd and even-indexed digits of x .

$$k | a_0 - a_1 + a_2 - a_3 + \dots \pm a_n.$$

$$b \equiv -1 \pmod{b+1}.$$

Because -1 to any odd power is -1 , and -1 to any even power is 1, $b^k \equiv -1 \pmod{b+1}$ for any odd $k \geq 1$, and $b^k \equiv 1 \pmod{b+1}$ for any even $k \geq 1$.

This means $(b^k) + 1 \equiv 0 \pmod{b+1}$ for any odd $k \geq 1$, and $(b^k) - 1 \equiv 0 \pmod{b+1}$ for any even $k \geq 1$.

So $k | (b^k) + 1$ for any odd $k \geq 1$, and $k | (b^k) - 1$ for any even $k \geq 1$.

$$k | a_0 - a_1 + a_2 - \dots \pm a_n.$$

$$k | (b^1 + 1) \cdot a_1 + (b^2 - 1) \cdot a_2 + \dots + (b^n \mp 1) \cdot a_n.$$

$$k | a_0 + (b^1) \cdot a_1 + (b^2) \cdot a_2 + \dots + (b^n) \cdot a_n.$$

Therefore, $k | x$

Compound of Other Divisibility Rules

The compounds of other divisibility rules that work in base 10 will still work in other bases, provided divisibility rules for the numbers being compounded still exist in the new base.

A Fifth Method

There is a fifth strategy that may be used. It was not mentioned earlier, because in base 10, it is most applicable to much larger numbers, but it is very useful in smaller bases. One may combine pairs or groups of numbers to form a number in a higher base. With this strategy, one may convert a base b number to base b^k , where $k > 1$, and use the previously mentioned sum and difference rules on it. This can be done because it is quite easy to convert a number

from base b to base b^k , and can be done even without paper. All you have to do is group the digits into groups of 2, 3, or more, starting from the right; those are your new digits.

In base 10, you may do this by converting a number to base 100, which can be done by grouping the digits into groups of two, starting from the right. Then, you may take the difference of odd and even-indexed pairs of digits to check if a number is divisible by 101. You may take the sum of these pairs of digits to check if a number is divisible by 99, or any of its factors.

This actually reveals a lesser-known way of checking if a number is divisible by 11. Instead of adding and subtracting alternating digits, you may group the digits into groups of two, starting from the right, and add up these groups. This can be more difficult, because the numbers are larger, but it can also allow you to avoid errors associated with adding and subtracting.

For example, to check if 19645219 is divisible by 11, you may break it into groups of two. $19 + 64 + 52 + 19 = 154$, which is divisible by 11, so 19645219 is divisible by 11. The other method works too; $1 - 9 + 6 - 4 + 5 - 2 + 1 - 9 = -11$.

What is the Most Convenient Base to Use?

For us, base 10 is the most convenient base to use because we've grown so used to it. Because of this, there are no serious plans to change our base system. However, speculation abounds. If we had chosen a different base system, how would things be different? Would it be more convenient than the one we use today?

Number of Divisibility Rules for Smaller Bases

A base that's easy to use should have:

- many numbers that are factors of the base number or the base number to some power
- many numbers whose divisibility can be found using divisibility rules
- numbers of a reasonable length (this is subjective)

To try to answer this question, I examined each base's divisibility rules. Divisibility rules are what sets bases apart; they all use the same numbers, but in some, it is easier to see if a number is divisible by 7 or 9 or 10 than others.

To start, I focused on small numbers. I looked at bases from 2 to 16, and compared how many divisibility rules they had for numbers from 2 to 16. I compared how many divisibility rules could be found using each of the five aforementioned methods, and how many couldn't be found at all.

For method 5, I only included methods that involved working with numbers less than or equal to 100 in size. This is because, if you group enough digits together, you may always eventually find a divisibility rule, but it becomes inconvenient after a while.

- Method 1) Look at the last n digits
- Method 2) Look at the sum of the digits

- Method 3) Look at the difference between alternating digits
- Method 4) Compound of other divisibility rules
- Method 5) Group digits and look at the sum or difference

Base	Method 1	Method 2	Method 3	Method 4	Method 5	No Method
Base 2	2, 4, 8, 16		3	6, 10, 12, 14	5, 7, 9, 11, 13, 15	
Base 3	3, 9	2	4	6, 12, 14, 15	5, 7, 8, 10, 13, 16	11
Base 4	2, 4, 8, 16	3	5	6, 10, 12, 14, 15	7, 9, 13	11
Base 5	5	2, 4	3, 6	10, 12, 15	8, 13	7, 9, 11, 14, 16
Base 6	2, 3, 4, 6, 8, 9, 12, 16	5	7	10, 14, 15		11, 13
Base 7	7	2, 3, 6	4, 8	10, 12, 14, 15	5, 16	9, 11, 13
Base 8	2, 4, 8, 16	7	3, 9	6, 10, 12, 14, 15	5 13	11
Base 9	3, 9	2, 4, 8	5, 10	6, 12, 15	16	7, 11, 13, 14
Base 10	2, 4, 5, 8, 10, 16	3, 9	11	6, 12, 15		7, 13, 14
Base 11	11	2, 5, 10	3, 4, 6, 12	15		7, 8, 9, 13, 14, 16
Base 12	2, 3, 4, 6, 8, 9, 12, 16	11	13			5, 7, 10, 14, 15
Base 13	13	2, 3, 4, 6, 12	7, 14			5, 8, 9, 10, 11, 15, 16
Base 14	2, 4, 7, 8, 14, 16	13	3, 5, 15	6, 10, 12		9, 11
Base 15	3, 5, 9, 15	2, 7, 14	4, 8, 16	6, 10, 12		11, 13
Base 16	2, 4, 8, 16	3, 5, 15		6, 10, 12		7, 9, 11, 13, 14

Applications of Smaller Bases

Base 10 has some big advantages. It's easy to work with numbers with factors of 2 and 5 in it. It has a convenient divisibility rule for 3 and 9. It also has a divisibility rule for 11, which only 4 out of the 15 bases on this list possess. However, it also has some disadvantages. It lacks a divisibility rule for 7, the fourth most important prime. It can also be inconvenient to work with larger powers of two. For example, to check if a number is divisible by 16, you have to check if the last four digits are divisible by 16, and there are 625 possibilities for that!

Using this table, one may determine if there is a base that solves these problems, without bringing in worse ones.

Prime Bases

The only base that has a divisibility rule for every number from 2 to 16 is 2.

Here's a list of all the methods in base 2, as an example. A similar list can be formulated for any other base:

- 2: If the last digit is 0
- 3: Sum digits in odd places, starting from the right, and subtract them by the sum of the digits in even places, starting from the right. Check if the result is a multiple of 3
- 4: If the last two digits are 0
- 5: Group digits into pairs, starting from the right. Sum groups in odd places, starting from the right, and subtract them by the sum of the groups in even places, starting from the right. Check if the result is a multiple of 5

- 6: If the number is divisible by 2 and 3
- 7: Group digits into triples, starting from the right, and check if the sum of the groups is divisible by 7
- 8: If the last three digits are 0
- 9: Group digits into triples, starting from the right. Sum groups in odd places, starting from the right, and subtract them by the sum of the groups in even places, starting from the right. Check if the result is a multiple of 9
- 10: If the number is divisible by 2 and 5
- 11: Group digits into groups of 5, starting from the right. Sum groups in odd places, starting from the right, and subtract them by the sum of the groups in even places, starting from the right. Check if the result is a multiple of 11
- 12: If the number is divisible by 3 and 4
- 13: Group digits into groups of 6, starting from the right. Sum groups in odd places, starting from the right, and subtract them by the sum of the groups in even places, starting from the right. Check if the result is a multiple of 13
- 14: If the number is divisible by 2 and 7
- 15: Group digits into groups of 4, starting from the right, and check if the sum of the groups is divisible by 15
- 16: If the last four digits are 0

However, this doesn't necessarily mean that base 2 is the most convenient base to use. Most of its base rules involve grouping digits, which can be difficult to perform in your head. It also requires more digits than other bases to display the same number, which is inefficient.

The other prime bases under 16 are even more difficult to use. Base 3 has many divisibility rules, like base 2, but many of them are difficult to use, like base 2. The other prime bases lack many divisibility rules, and the ones they do have are typically more difficult to use than the divisibility rules in other bases.

Prime Power Bases

There are four prime power bases from base 2 to base 16: base 4, base 8, base 9, and base 16. Base 4 and base 8 have more divisibility rules than the other two. Of these two, base 8 is likely easier to use, because fewer of its methods require converting to a higher power, and its numbers are more compact.

Other Numbers

The remaining bases are the ones that contain multiple primes in their prime factorization: 6, 10, 12, 14, and 15.

These bases typically have many numbers who divide x if they divide the last n digits of x . Any number for which the set of unique primes in its prime factorization is a subset of the set of unique primes in the prime factorization of the base number can be found in this way. This can be convenient, but it can also be cumbersome, because sometimes there are a lot of possibilities. For example, there are 625 possibilities for the last 4 digits to be a multiple of 16 in base 10, while there are 81 possibilities in base 6, and 2401 possibilities in base 14.

Base 6 and base 12 are similar because both 6 and 12 contain 2 and 3 in their prime factorizations. Also, each base has two primes for which a divisibility rule cannot be applied. However, for 6 these primes are 11 and 13, while for 12 these numbers are 5 and 7; since 5 and 7 are more commonly worked with, base 6 is probably the easier one to work with of the two.

Base 14 and base 15 are similar because they are both semiprime (the product of two prime numbers). They also each lack divisibility rules for two numbers between 2 and 16. Of these two, Base 14 is probably easier to use, because 14 is even, which makes it easier to divide by 2. However, it's difficult to check if a number is a multiple of 3, which makes base 14 more difficult to use than base 6.

Overall

The most promising alternatives to base 10 are base 6 and base 8. Base 6 has easy-to-use divisibility rules for multiples of 2 and 3, as well as 5 and 7, while base 8 has divisibility rules for a variety of numbers, and the added advantage of being a power of 2, which makes it useful for computers as well.

Analysis of Larger Bases

Currently, our numbering system does not have many digits (only 10). Therefore, I started by focusing on other numbering systems with few digits. However, there are many bases with more digits (in fact, an infinite amount).

Next, I decided to look at bases larger than 16, and see if any of them could have advantages over base 10. Specifically, I looked at bases from 17 to 256, because it seems like it would be extremely difficult to keep track of more than 256 different digits.

I decided to look for bases that have the following properties:

- The base number is even.
- The base has a divisibility rule for every number from 2 to 10.
- For 3 and 5, this divisibility rule is either a sum of digits rule or a last n digits rule.

There are actually very few bases that satisfy these conditions. Here's a list:

- Base 6
- Base 36
- Base 90
- Base 120
- Base 126
- Base 190
- Base 210
- Base 216
- Base 246

This shows that base 6 is rather unique among small bases. Given that base 36 is just base 6 with pairs of digits grouped together, the next base that satisfies these conditions is base 90.

An Example Problem

The question of which base is the most convenient to use is incredibly unlikely to come up in a competition math. However, the knowledge that the same divisibility rules we use in base 10 can be applied to other bases can be helpful. An example of this is in this problem:

For any integer $n \geq 2$, let b_n be the least positive integer such that, for any integer N , m divides N whenever m divides the digit sum of N written in base b_n , for $2 \leq m \leq n$. Find the integer nearest to b_{36}/b_{25} . (PUMaC NT #4 2017)

We have shown above that, given a number written in a certain base, m divides N whenever m divides the digit sum of N if m divides $b^n - 1$.

This means every number from 2 to 36 must divide $b_{36} - 1$, and every number from 2 to 25 must divide $b_{25} - 1$. The smallest such number for each can be found by determining the largest quantity of each prime factor in the numbers in the set.

Therefore, $b_{36} - 1 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31$.

And, $b_{25} - 1 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$.

Because both numbers are so large, b_{36}/b_{25} is roughly equal to $\frac{b_{36} - 1}{b_{25} - 1}$.

So it is roughly equal to $2 \cdot 3 \cdot 29 \cdot 31$, which equals 5394.

Citations

“Divisibility: Rules and Recipes for Testing Divisibility in Different Number Bases.” Dozenal Society, www.dozenalsociety.org.uk/pdfs/divisibility.pdf.