An introduction to complex numbers

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1 Introduction: The Imaginary unit i and the anatomy of a complex number

Image by Rebecca Bao

The most common definition of the imaginary unit i is $\sqrt{-1} = i$ (the principal square root is being used here). There is also the definition $i^2 = -1$ (the one that I somewhat prefer). Note that with square roots, or any even root for that matter, of negative real values, you have to be careful when it comes to things that involve the following product:

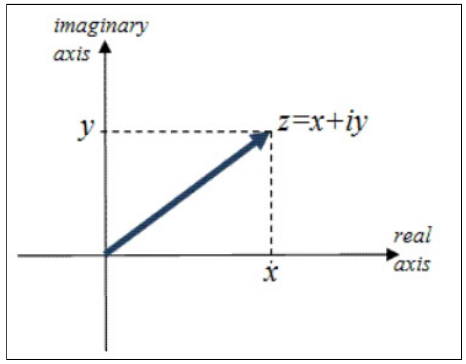
$$\sqrt{-1} * \sqrt{-1} = \sqrt{1} = 1$$

This is obviously not true; $\sqrt{-1} * \sqrt{-1}$ should be -1. Here's the restriction: when taking the product of two square roots of negative real numbers, one should first convert each factor to a complex value before multiplying. This conversion looks like this: $\sqrt{-89} = i\sqrt{89}$

Complex numbers have a real part and a complex part. For example, the complex value $z = a+bi$ has a real part of a and an imaginary part of b. You can write this out as $Re(z) = a$, $Im(z) = b$. If the complex number has a real part 0, the number is pure imaginary. A number such as 7 is a sort of trivial complex number, with an imaginary part of 0. You can graph complex numbers on the complex plane, which has a real and an imaginary axis. This $a + bi$ form is also known as the rectangular form of a complex number.

In terms of number systems, $N \subset Z \subset Q \subset R$; the complex numbers are a superset of the real numbers. They are unique and have real world applications, as there are things complex numbers do that real numbers are limited in. The set of complex numbers is similar to R^2, the set of ordered real pairs (you can think of each complex number as an ordered pair of it's real and imaginary part). However, a basic advantage to complex numbers over R^2 is that taking the product of complex numbers is simple and results in another complex number; trying to take the product of two ordered real pairs is more complicated .

The complex number $x + iy$, with real component x and imaginary component y. A vector can also be used to describe a complex number.

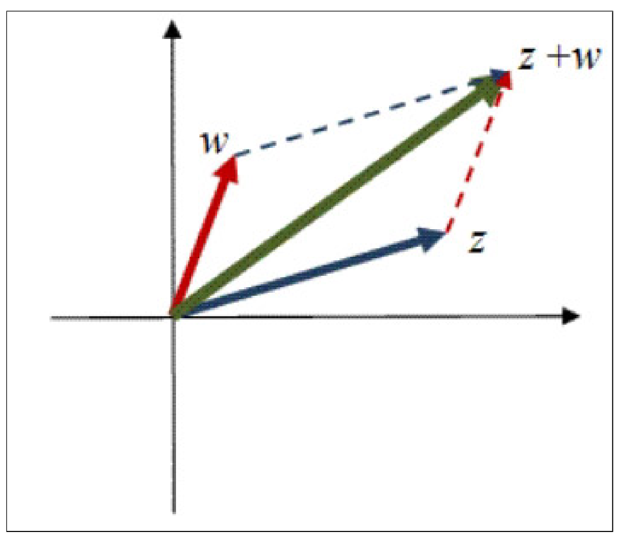


Real world applications of complex numbers arise in fluid dynamics in 2-dimensions, or in engineering to represent rotations in a plane, as the complex numbers provide such an elegant representation of the 2-dimensional system.

1.1 Basic Arithmetic: Sum, modulus and conjugate of complex numbers

Taking the sum of complex numbers is simple: you just add their respective real and imaginary components. For example: $(43 + 7i) + (12 - 34i) = (43 + 12) + i(7 - 34) = 65 - 27i$. In other words, $z_1 + z_2 = [Re(z_1) + Re(z_2)] + i[Im(z_1) + Im(z_2)]$. You just treat the "$i$" as any other regular algebraic variable and add like terms. A more intuitive way to explain this is by using the vector of the complex number. When adding two complex numbers, you're taking the sum of two vectors: you translate the base of one vector to the tip of the other.

z and w are complex numbers. Their respective vectors are added to get the sum of the complex numbers.



The absolute value/radius/modulus of a complex number, is how a number lies from the origin in the complex plane. Using the distance formula, we find that the modulus r of complex number $z = a + bi$ is $r = \sqrt{a^2 + b^2}$. Also, $|z_1 - z_2|$, the absolute value of two complex numbers, is how far apart two complex numbers lie from each other on the complex plane. An equation such $|z - (3 + 4i)| = 3$ would represent the set of all complex numbers that are 3 units away from $3 + 4i$. The solution set would form a circle if graphed.

The conjugate of complex number $z = a+bi$, which is represented as $\overline{z}$, is equal to $a-bi$. In other words, it is the complex number reflected over the real axis. It has the property where if you take a product of a complex number and its conjugate, you always get the modulus squared. In fact, this is how you get the formula for the sum of squares:

$$z * \overline{z} = (a+bi)(a-bi) = a^2 + b^2 + abi - abi = a^2 + b^2$$

There are many properties of complex conjugates; here are a few of them:

$$\overline{cz} = c * \overline{z} \text{ where } c \in R$$

$$\overline{\overline{z}} = z$$

$$\overline{z_1} + \overline{z_2} = \overline{z_1 + z_2}$$
$$\overline{z_1 z_2} = \overline{(a_1 + b_1 i)(a_2 + b_2 i)} = \overline{a_1 a_2 - b_1 b_2 + i(a_1 b_2 + a_2 b_1)} =$$
$$a_1 a_2 - b_1 b_2 - i a_1 b_2 - i a_2 b_1 = (a_1 - b_1 i)(a_2 - b_2 i) = \overline{z_1} * \overline{z_2}$$
$$\overline{z}^n = \overline{z} * \overline{z} * \overline{z} * \ldots\ldots * \overline{z} = \overline{z^n}$$

Here is an example of a well-known theorem involving polynomials that utilizes the properties of conjugates:

Complex Conjugate Theorem: Given that polynomial $P(x)$, if $a+bi$ is a root of the polynomial, then $a-bi$ must also be a root.

$$\text{Given: } P(z) = 0 = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$
$$\text{Show: } P(\overline{z}) = 0$$

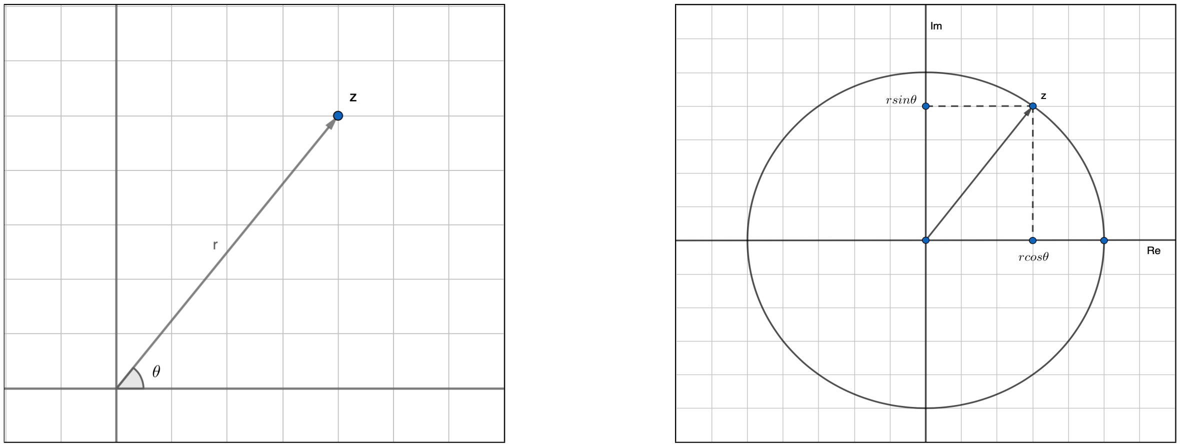
$$P(\overline{z}) = a_n(\overline{z})^n + a_{n-1}(\overline{z})^{n-1} + \ldots + a_1 \overline{z} + a_0 = a_n \overline{z^n} + a_{n-1} \overline{z^{n-1}} + \ldots + a\overline{z} + a_0$$

We can combine the coefficients under the conjugate sign because we were given that they are real:

$$P(\overline{z}) = \overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \ldots + \overline{a_1 z} + \overline{a_0} = \overline{a_n z^n + a_{n-1} z^{n-1} + \ldots a_1 z + a_0} = \overline{0} = 0$$

2 The two polar forms of complex numbers

So far, we have been considering complex numbers in the form $a+bi$, with it's real component a and imaginary component b. There is another way to represent complex numbers that is a lot more useful in certain situations. This uses the modulus of the complex number and its argument the terminal angle formed with the right x-axis and the vector that goes to the complex number. In the following diagram, r is the modulus and θ is the argument. In other words $|z| = r, arg(z) = \theta$. We can convert between the rectangular and polar form using these equations: $\tan^{-1}(\frac{b}{a}) = \theta$ and $r = \sqrt{a^2 + b^2}$



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On the diagram on the right; notice how there is a circle drawn with radius r that goes through the complex number z. Let us assume that $r = 1$; this would create the complex unit circle. Such as with unit circles on $R^2, Re(z) = x = \cos\theta$ and $Im(z) = y = \sin\theta$. If the unit circle were to be dilated by a scale factor of r, then each x and y coordinates would also be scaled by constant of proportionality r. Knowing this, $Re(z) = r\cos\theta$ and $Im(z) = r\sin\theta$. This is how we arrive at the first polar form of complex numbers:

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) \text{ which is sometimes written as } z = rcis\theta.$$

$$^*cis(\theta) = \cos(\theta) + i\sin(\theta)$$

In the polar form, there are two things needed to define a complex number: the radius and the argument. Something to note is that you can technically always add or subtract 2π to the argument and have it be the same complex number. Similarly, you can always add π to the argument and negate r and you'll still end up with the same number (however, we'll just assume $r \geq 0$ and $\theta \in [0, 2\pi)$ for the rest of this paper unless indicated otherwise). There is also the *principal* angle of a complex number, which is the interval $(-\pi, \pi]$.

There is a second similar polar form (it also involves the radius and argument). To find it, we need to use the Taylor series for e^x, $\sin(x)$, and $\cos(x)$. Remember that for differentiable function f about $x = a$. the Taylor Series expansion is the following equation:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \text{ where } f^{(n)} \text{ is the nth order derivative of } f$$

Luckily e^x, $\sin(x)$ and $\cos(x)$ are simple to differentiate and have repeating patterns. We will center all of them at $x = 0$

n	$\frac{d^n}{dx^n}(e^x)$	$atx = 0$	$\frac{d^n}{dx^n}(\cos x$	$atx = 0$	$\frac{d^n}{dx^n}(\sin x)$	$atx = 0$
0	e^x	1	$\cos x$	1	$\sin x$	0
1	e^x	1	$-\sin x$	0	$\cos x$	1
2	e^x	1	$-\cos x$	-1	$-\sin x$	0
3	e^x	1	$\sin x$	0	$-\cos x$	-1
4	e^x	1	$\cos x$	1	$\sin x$	0
5	e^x	1	$-\sin x$	0	$\cos x$	1
6	e^x	1	$-\cos x$	-1	$-\sin x$	0

$$e^x = \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + ...$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + ...$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + ...$$

$$e^{i\theta} = \frac{i\theta}{1} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + ... =$$

$$1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ... + i(x - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + ...) = \cos\theta + i\sin\theta$$

$$r(\cos\theta + i\sin\theta) = re^{i\theta}$$

This is the exponential polar form of complex numbers. If $r = 1$ and $\theta = \pi$, it results in $e^{\pi i} = \cos\pi + i\sin\pi = -1$: Euler's famous identity. It describes a complex number with a radius of 1 and an argument of π, which geometrically speaking would end up at the location $-1 + 0i$

2.1 The elegant geometric explanation of the product / real powers/ roots of complex numbers (binomial approach vs polar approach)

Multiplication:

When multiplying several complex numbers, using the rectangular $a + bi$ form is quite cumbersome. Even when raising a complex number to a real power, it can get quite difficult to deal with. In certain contest problems where the rectangular form is easier to work with than the polar form, sometimes treating the rectangular form $a + bi$ as a binomial and binomial expanding it is useful.

$$(a + bi)^n = \sum_{k=1}^{n} \binom{n}{k} a^{n-k}(bi)^k = \sum_{k=1}^{n} \binom{n}{k} a^{n-k}b^k i^k$$

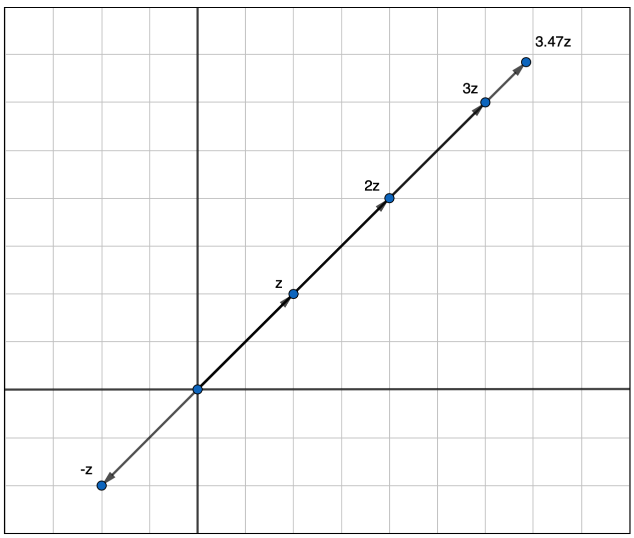
Raising i to natural number powers have a pattern so this is not super hard to do (if the numbers are small and you remember some binomial coefficients). You could split this sum apart to cases where k=1,5,9,13... , $k = 2, 6, 10, 14...$, $k = 3, 7, 11, 15...$ and $k = 4, 8, 12, 16...$ to deal with the respective real and imaginary parts more neatly, but I find this to be of little use as I have not found any contest problem where it's necessary to use binomial expansion with very large powers in this way (the only place where I found this useful is in the formulas for $\cos(n\theta)$ and $\sin(n\theta)$.

$$(3 - 2i)^4 = 3^4 + 4(3)^3(-2i) + 6(3)^2(-2i)^2 + 4(3)(-2i)^3 + (-2i)^4 =$$
$$81 - 216i - 216 + 96i + 16 = -119 - 120i$$

This method is rather limited: it's easy to make a mistake (especially with the i's), it's arithmetically cumbersome and it generally works well only with the Gausian integers (the set of complex numbers with real and imaginary parts that are integers).

Using the polar approach to multiplication, powers and roots is a lot more elegant. Let's start off with something simple: multiplying a complex number by a real scale factor (let's call it a scalar). The vector representing the complex number z would just get scaled by a factor of that scalar. A negative scalar would just reverse the direction of the vector representing that complex number.

When multiplying two complex number in the polar form, observe the following property:



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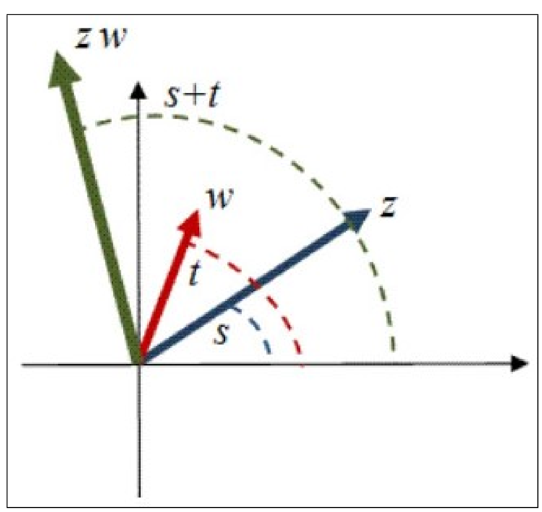
$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 (e)^{i(\theta_1+\theta_2)} = r_1 r_2 (cis(\theta_1 + \theta_2)$$

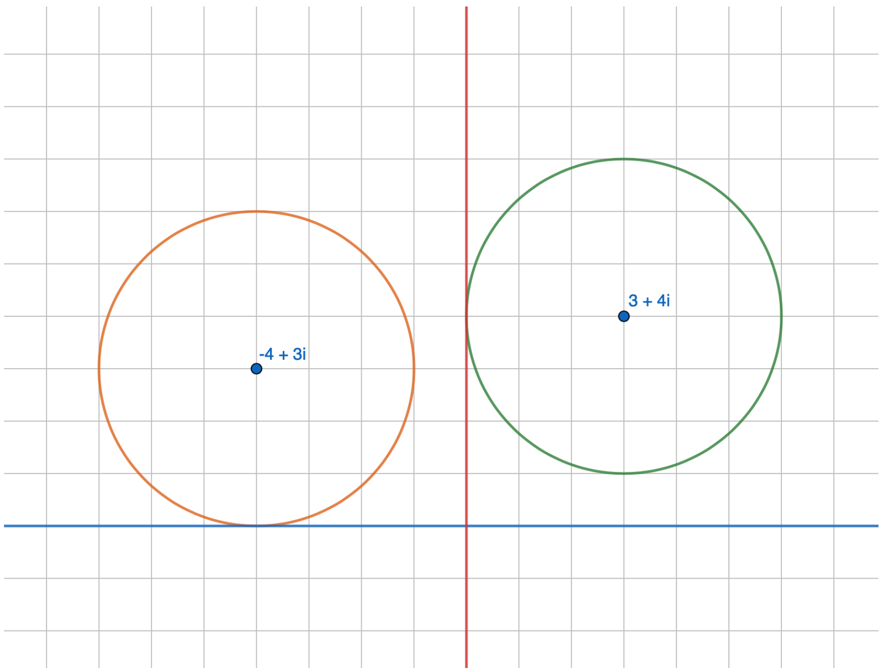
When you multiply two complex numbers, we multiply the radii and add the arguments to get the resulting complex number.

The leftmost complex number vector represents the product of the two other complex number vectors. You add the angle measures and multiply the arguments to get the product.



For this reason, multiplying any number in the plane by i would result in a counter-clockwise rotation of 90 degrees. Multiplying any number in the plane by -1 would result in a rotation of 180 degrees. Multiplying z by 2.23($cis(23.4^o)$) has the effect of rotating z by 23.4 degrees about the origin and multiplying the radius of z by a factor of 2.23. This concept allows complex multiplication to represent transformations of points / figures.

Remember $|z - (3 + 4i)| = 3$ from section 1.1? If we set complex number $y = z \cdot cis\frac{\pi}{2}$ and graph the solution set for y, then we get the solution set for z rotated $\frac{\pi}{2}$ radians counterclockwise about the origin. We can write the equation with y as $|\frac{y}{cis(\frac{\pi}{2})} - (3+4i)| = 3$. Here is a diagram describing the transformation:



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You can also divide complex numbers by dividing the radii and subtracting the arguments:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} cis(\theta_1 - \theta_2)$$

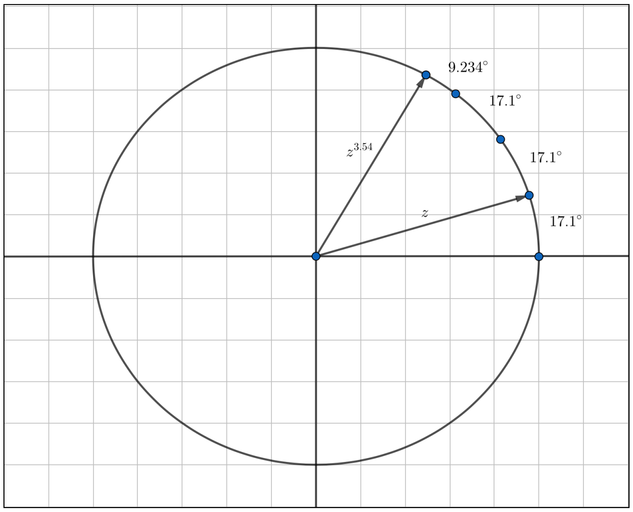
Real powers of complex numbers

Once we understand how multiplication works, real powers of complex numbers isn't super hard to understand:

$$|n| \geq 1$$

$$z^n = (re^{i\theta})^n = r^n e^{i(n\theta)} = r^n(cis(n\theta))$$

All you do is raise the radius to the nth power and multiply the argument by n. This makes sense geometrically because if you do repeated multiplication, you will have to dilate the radius n times and you will have to rotate the argument n times. On the unit circle where $r = 1$, this can be easily visualized. On the complex unit circle, if you raise some number with an argument of 17.1 degrees ($z = cis(17.1^o)$) to the power of 3.54 , the resulting number $z^{3.54}$ would be $cis(17.1^o \cdot 3.54) = cis(60.534^o) \approx 0.4919 + 0.8706i$



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Complex Numbers

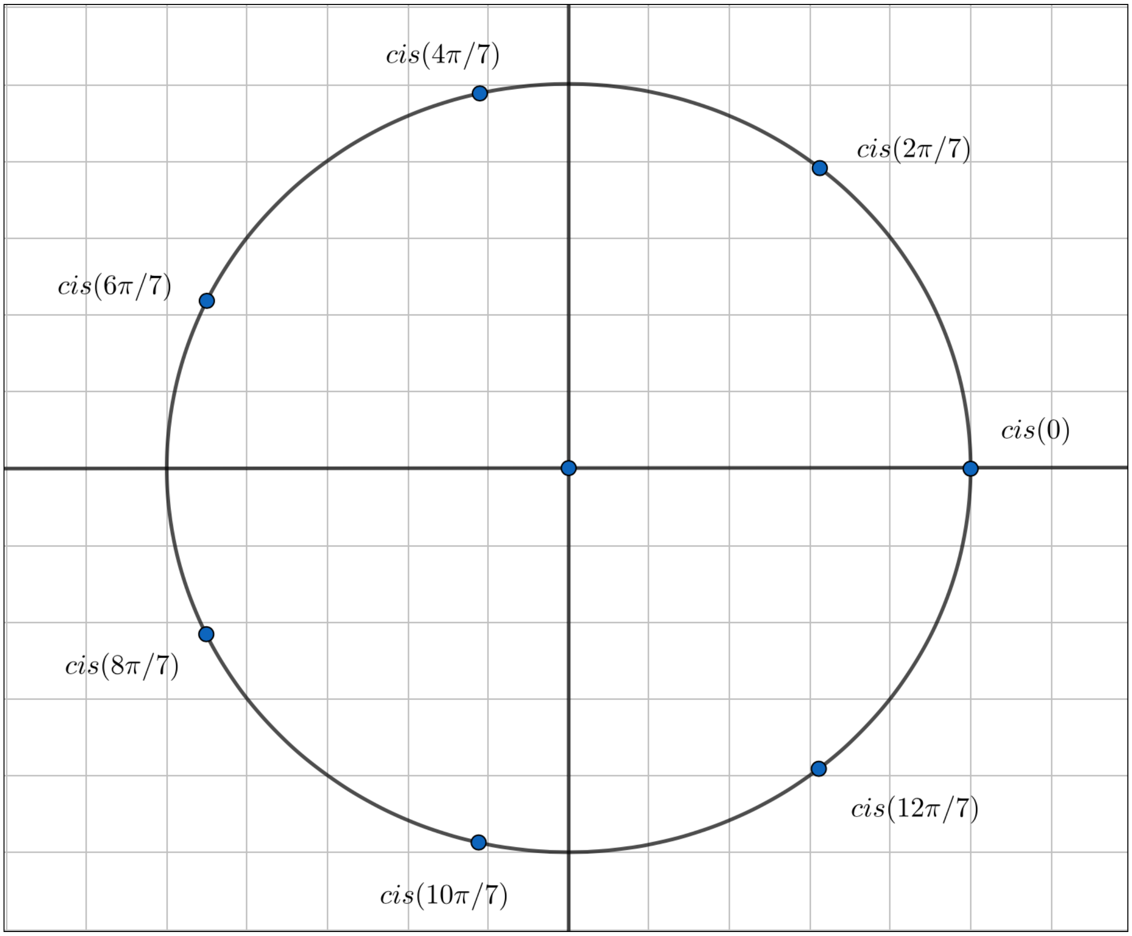
Taking roots is similar, except that instead of multiplying the argument by n, we divide the argument by n:

$$z^{\frac{1}{n}} = (re^{i\theta})^{\frac{1}{n}} = r^{\frac{1}{n}}e^{i(\frac{\theta}{n})} = r^{\frac{1}{n}}(cis(\frac{\theta}{n}))$$

This is not the final formula however. If the power of z is greater than 1, like in the previous example where the power was 3.54, there is only one principal solution to the problem. However, when dealing with an argument of $\frac{\theta}{n}$ where there are powers of $\frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$, there are several principal solutions. You can add $2\pi k$ to the argument several times and it would still be a unique principal complex number.

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}(cis(\frac{\theta + 2\pi k}{n})) \text{ where } k = 0, 1, 2..., n - 1$$

The 7th root of a complex number would have 7 unique solutions. The cube root of a complex number would have 3 unique solutions. $\sqrt[7]{1}$ has 7 unique solutions:



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2.2 Raising a complex number to a complex number

Raising a complex number to a complex power just combines what has been discussed so far. Suppose the following expression:

$$(2 + \tfrac{2\sqrt{3}}{3}i)^{\frac{1}{2}+3i} \text{ which is } (2 + \tfrac{2\sqrt{3}}{3}i)^{\frac{1}{2}}(2 + \tfrac{2\sqrt{3}}{3}i)^{3i}$$

Let's deal with the first factor first. It has two principal solutions:

$$(2 + \tfrac{2\sqrt{3}}{3}i)^{\frac{1}{2}} = [\sqrt{\tfrac{16}{3}}(cis(\tfrac{\pi}{6})]^{\frac{1}{2}} = \sqrt[4]{\tfrac{16}{3}}(cis(\tfrac{\pi}{12} + \pi k)) \text{ where } k = 0, 1$$
$$(2 + \tfrac{2\sqrt{3}}{3}i)^{\frac{1}{2}} = \sqrt[4]{\tfrac{16}{3}}(cis(\tfrac{\pi}{12})) \text{ or } \sqrt[4]{\tfrac{16}{3}}(cis(\tfrac{\pi}{12} + \pi))$$

Now let's deal with the second factor:

$$(2 + \tfrac{2\sqrt{3}}{3}i)^{3i} = [[\sqrt{\tfrac{16}{3}}(cis(\tfrac{\pi}{6})]^3]^i = [(\tfrac{16}{3})^{\frac{3}{2}}(cis(\tfrac{\pi}{2}))]^i = ((\tfrac{16}{3})^{\frac{3}{2}}i)^i = ((\tfrac{16}{3})^{\frac{3}{2}})^i(i)^i$$

For the i^i, writing the base, which is i, in the exponential polar form will assist in the evaluation; we surprisingly end up with a real number

$$i^i = (e^{i(\frac{\pi}{2})})^i = e^{\frac{-\pi}{2}}$$

For the $((\frac{16}{3})^{\frac{3}{2}})^i$, I used the $e^{lin(x)} = x$ identity to get the expression into the polar form.

$$((\frac{16}{3})^{\frac{3}{2}})^i = (e^{ln(\frac{16}{3}^{1.5})})^i = (e^{iln(\frac{16}{3})^{1.5}}) = cis(\frac{3}{2}ln\frac{16}{3})$$

The second factor is now:

$$(5^{\frac{3}{2}})^i(i)^i = e^{\frac{-\pi}{2}}cis(\frac{3}{2}ln\frac{16}{3})$$

Multiplying the factors, we find the final solution set:

$$\sqrt[4]{\frac{16}{3}}(cis(\frac{\pi}{12} + \pi k))e^{\frac{-\pi}{2}}cis(\frac{3}{2}ln\frac{16}{3}) = \sqrt[4]{\frac{16}{3}}e^{\frac{-\pi}{2}}cis(\frac{\pi}{12} + \frac{3}{2}ln\frac{16}{3} + \pi k) \text{ where } k \in \{1,0\}$$

To summarize this process, when raising a complex number to another complex number, it's best to separate the exponent into two factors - a factor with a real power and a factor with a complex power. To evaluate the factor with the real power, you just change the base into the polar form and employ the strategies from the last section to raise a complex number to a real power. For the second factor (the one with the complex power), you should first simplify until you have a complex number to the power of i. Then you should use the complex polar form and use the $e^{ln(x)} = x$ property to finish the evaluation of the second factor.

There's a general formula that can do this process much faster:

$$z = (c + di)^{(a+bi)}$$

$$ln(z) = (a + bi)ln(c + di)$$

$$\text{Let } (c + di) = re^{i\theta} \text{ where } r = \sqrt{c^2 + d^2} \text{ and } \theta = tan^{-1}(\frac{d}{c})$$

$$ln(z) = (a + bi)ln(re^{i\theta}) = (a + bi)(ln(r) + i\theta)$$

$$z = (c + di)^{(a+bi)} = e^{(a+bi)(ln(r)+i\theta)}$$

3 Miscellaneous

3.1 Complex form of trig functions

There are definitions for the cosine and sine functions that allow complex inputs. We'll derive the complex form of $\sin\theta$ using some algebraic manipulation. I use the evenness of cosine in line 2 to help reduce the expression

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i\theta} - e^{-i\theta} = \cos\theta + i\sin\theta - \cos(-\theta) - i\sin(-\theta) = 2i\sin(\theta)$$

$$sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Using a similar process, the complex definition of the cosine and tangent functions are found

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2i} \quad \tan\theta = \frac{i(e^{-i\theta} - e^{i\theta})}{(e^{i\theta} + e^{-i\theta})}$$

You can use these formulas to find solutions that would otherwise not exist in the real world. For example, $\sin z = 2$ has complex solutions.

3.2 Multiplication formulas for cosine and sine

While the polar form proves to be useful in many situations, there are some identity proofs that utilize the binomial expansion of the rectangular form of complex numbers. We'll be considering the formulas for $\sin(n\theta)$ and $\cos(n\theta)$ where $n \in N$

$$z^n = (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

From the top equation, we deduce a key equation in figuring out these identities.

$$Re((\cos\theta + i\sin\theta)^n) = \cos(n\theta)$$
$$Im((\cos\theta + i\sin\theta)^n) = \sin(n\theta)$$

This is when the binomial expansion of the rectangular form of complex numbers comes in. We'll let $\cos\theta = a$ and $\sin\theta = b$ so that $(\cos\theta + i\sin\theta)^n = (a + bi)^n$

$$(a + bi)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}(bi)^k$$

To deal with the i, the sum must be separated into 4 sections (cases for where i^k is 1, -1, i or -i). This is where the real and complex parts of the summation can be separated. For $k \in \{1, 5, 9...\}$,the term is multiplied by i. For For $k \in \{3, 7, 11...\}$,the term is multiplied by $-i$. For $k \in \{0, 4, 8...\}$,the term is multiplied by 1. For $k \in \{2, 6, 10...\}$,the term is multiplied by -1. We can separate this sum into its real and complex components.

$$(a+bi)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}(bi)^k = \sum_{k=0}^{[\frac{n}{2}]} (\binom{n}{2k})(-1)^k a^{n-2k} b^{2k} + i \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{2k+1} (-1)^k a^{n-2k-1} b^{2k+1}$$

Substituting back $\cos\theta = a$ and $\sin\theta = b$, we derive our formulas.

$$(a + bi)^n = (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

$$Re(\cos\theta + i\sin\theta)^n = \cos(n\theta) = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} (-1)^k \cos^{n-2k}\theta \sin^{2k}\theta$$

$$Im(\cos\theta + i\sin\theta)^n = \sin(n\theta) = \sum_{k=0}^{[\frac{n+1}{2}]} \binom{n}{2k+1} (-1)^k \cos^{n-2k-1}\theta \sin^{2k+1}\theta$$

Some of the common multiple angle formulas are listed below:

$$\sin(2x) = 2\cos x sinx$$
$$\sin(3x) = 3\cos^2 x sinx - sin^3 x$$
$$\sin(4x) = 4\cos^3 x \sin x - 4\cos x \sin^3 x$$
$$\sin(5x) = 5\cos^4 x \sin x - 10\cos^2 x \sin^3 x + \sin^5 x$$

$$\cos(2x) = \cos^2 x - sin^2 x$$
$$\cos(3x) = \cos^3 x - 3\cos x \sin^2 x$$
$$\cos(4x) = \cos^4 x - 6\cos^2 x \sin^2 x = \sin^4 x$$
$$\cos(5x) = \cos^5 x - 10\cos^3 x \sin^2 x + 5\cos x \sin^4 x$$

4 Challenge Problems

1. For $y = \arccos(4t^3 - 3t)$ with $t \in (-1, 1)$, find real numbers n, m, and c where $y = n\arccos(mt + c)$
2. Given complex number z and real number k, if $Re(zcis(\frac{-\pi}{4})) = Im(zcis(\frac{-\pi}{4}))$ and if $z^8 = 16 + ki$, what is the solution sets for k and z?(No calculator)
3. How many solutions of $\sqrt[2020]{i}$ have complex angle arguments in the open interval $(81°, 89°\ )$
4. If $(1+i)^{(1-i)} = (1+i)z$ where z is a complex number, what is z?

5 Answers

1. $n = 3, m = 1, c = 0$
2. $k \in \{0\}$ and $z \in \{i\sqrt{2}, -i\sqrt{2}\}]$
3. 45 solutions exist
4. Exact form: $z = e^{\frac{\pi}{4}} cis(\frac{-ln(2)}{2})$

   Approximation: $z = 2.063 - 0.745i$

References:

ICA: 1. Complex Numbers, pirate.shu.edu/ wachsmut/complex/numbers/index.html.
"Multiple-Angle Formulas." From Wolfram MathWorld, mathworld.wolfram.com/Multiple-AngleFormulas.html.