

Euler's Formula

Maxwell Zen

A Proof of Euler's Formula

Euler's formula states that for any convex polyhedron, $V - E + F = 2$, where V is the number of vertices, E the number of edges, and F the number of faces. This formula seems to hold after a few test cases; a tetrahedron has 4 vertices, 6 edges, 4 faces, and $4 - 6 + 4 = 2$; a cube has 8 vertices, 12 edges, 6 faces, and $8 - 12 + 6 = 2$, and so on. We will prove this result and show some of its interesting applications.

Realize that it is possible to map any convex polyhedron to a sphere of radius 1. We can perform this mapping for two reasons.

First, two points on a sphere are connected by a single, unique line. This is the shortest distance between these two points and lies on a "great circle."

Second, convex polyhedra are topologically equivalent to spheres, so you can map one to the other without changing the numbers of vertices, edges, or faces. This also poses an explanation as to why Euler's Formula fails for concave polyhedra - mapping a polyhedron onto a sphere requires that every vertex can be connected to every other vertex by a line that is completely within the sphere, and a concave polyhedron by definition has a line between two vertices that doesn't lie within the solid.

Now that we know we can perform this mapping, let's define a spherical polytope as a set of points on the surface of a sphere connected to each other by arcs of great circles to form curved faces.

Lemma 1: We will prove that the area of a spherical triangle with angles α, β, γ (measured in radians) is equal to $\alpha + \beta + \gamma - \pi$.

Proof. First, we'll need to talk about the area of a lune. A lune is the intersection between two great circles, where the angle at the intersection of the great circles is α (Figure 1). Since angles traditionally exist on flat planes, we can define α as the angle between tangents of the two great circles at their intersection point. Note that the area of the lune increases proportionally to α . Also notice that when $\alpha = 2\pi$, the area of the lune is the surface area of the sphere, which is 4π for a sphere of radius 1. Thus, the area of a lune is $2 \cdot \alpha$.

Next, we'll need to talk about the inclusion-exclusion formula. If we have 3 areas A, B , and C , that (possibly) overlap, we can find the area of their intersection with the formula $A \cap B \cap C = A \cup B \cup C - A - B - C + A \cap B + A \cap C + B \cap C$.

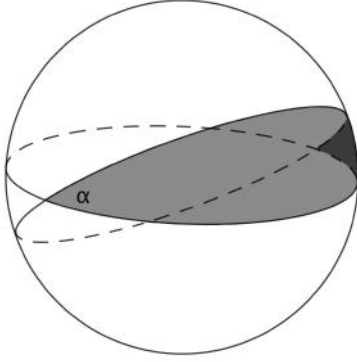


Figure 1: A lune with angle α on a sphere of radius 1

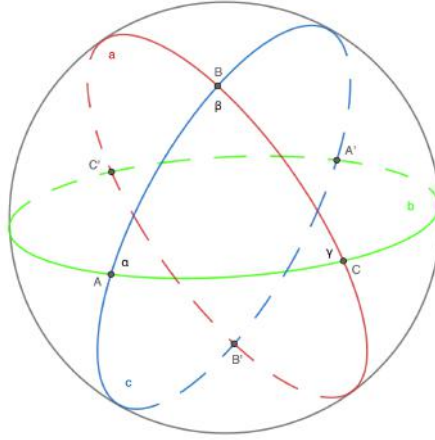


Figure 2: Triangle ABC on a sphere of radius 1

Now, we're finally prepared to tackle the area of a triangle on a sphere. We begin with triangle ABC on a sphere of radius 1, with angles α, β , and γ . Extend each side to form its corresponding great circle, and notice that these great circles also intersect at A', B' , and C' . We may define hemisphere a as the hemisphere whose base is the great circle defined by \overline{BC} and that includes A in its area. We define hemispheres b and c likewise (see Figure 2). Notice that the area of our triangle is the intersection of our 3 hemispheres, or $a \cap b \cap c$. Knowing this, we can use the inclusion-exclusion formula to get the following equation:

$$\Delta ABC = a \cup b \cup c - a - b - c + a \cap b + a \cap c + b \cap c$$

First, let's take care of the easy parts: the areas of a , b , and c , are each the area of a hemisphere, which is 2π . Thus, the equation becomes:

$$\Delta ABC = a \cup b \cup c - 6\pi + a \cap b + a \cap c + b \cap c$$

Next, $a \cap b$, $a \cap c$, and $b \cap c$ are all lunes whose area is twice the angle between their great circles. In this case, these angles are α , β , and γ .

$$\Delta ABC = a \cup b \cup c - 6\pi + 2\alpha + 2\beta + 2\gamma$$

Finally, notice that $a \cup b \cup c$ includes the entire sphere, minus triangle $A'B'C'$. Since we've defined a , b , and c so that triangle ABC is the only area covered by all 3, the only area excluded by all 3 should lie exactly opposite ΔABC . Thus, the area of $a \cup b \cup c$ is the area of the sphere minus the area of ΔABC , or $4\pi - \Delta ABC$. We now have the equation:

$$\Delta ABC = 4\pi - \Delta ABC - 6\pi + 2\alpha + 2\beta + 2\gamma$$

Solving for the area of ΔABC , we get

$$2 \cdot \Delta ABC = 4\pi - 6\pi + 2\alpha + 2\beta + 2\gamma$$

$$\Delta ABC = \alpha + \beta + \gamma - \pi$$

as desired. □

Now we've figured out how to find the area of a spherical triangle. What about any polygon on a sphere? Recall the proof you may have encountered for the sum of angles in a polygon - take the polygon, split it up into triangles, and add up the angles of each triangle. We'll take the same approach here - take any spherical n -gon, split it up into $n - 2$ triangles, and add up the areas. When we add up the angles of each triangle, we get the sum of angles of the n -gon, and when we subtract π for each of the triangles, we subtract $\pi n - 2$ times. Thus, the area of any n -gon is the sum of its angles minus $(n - 2)\pi$.

Finally, using that formula, we're ready to tackle the spherical polytope. Again, this polytope has V vertices, E edges, and F faces. We know that the surface area of the sphere is 4π , but we can also find this area by adding up the area of each polygon. First, we'll sum the angles of every polygon, then subtract $n\pi$ for each n -gon, and then add 2π for each polygon.

First, summing the angles of every polygon is equal to adding 2π for each of the V vertices, since the angles around each point form a full circle, so this contributes $2V\pi$.

Second, subtracting $n\pi$ for every n -gon is the same as subtracting π once for every side, but since each of the E edges contributes a side to 2 polygons, we subtract $2E\pi$.

Finally, adding 2π for each of the F faces contributes $2F\pi$. Now that we have all the pieces, let's put them together:

$$4\pi = 2V\pi - 2E\pi + 2F\pi$$

And finally, we divide both sides by 2π to get Euler's Formula:

$$2 = V - E + F$$

QED

Application to Platonic Solids

Let's use Euler's formula to prove that there are only 5 Platonic solids. A Platonic solid is a regular, convex polyhedron, meaning every face has the same number of sides and every vertex connects the same number of faces. Let's define k as the number of sides on each face and l as the number of faces that meet at each vertex. From Euler's formula, we have:

$$V - E + F = 2$$

Since each face coincides with k edges and each edge coincides with 2 faces, $k \cdot F = 2E$, and $F = \frac{2E}{k}$. This gives us:

$$V - E + \frac{2E}{k} = 2$$

Next, since l faces meet at each vertex, l edges meet at each vertex, but each edge connects to 2 vertices, so we get $l \cdot V = 2E$, and $V = \frac{2E}{l}$. This gives us:

$$\frac{2E}{l} - E + \frac{2E}{k} = 2$$

Bring E to the right-hand side:

$$\frac{2E}{k} + \frac{2E}{l} = 2 + E$$

Divide both sides by $2E$:

$$\begin{aligned} \frac{1}{k} + \frac{1}{l} &= \frac{1}{E} + \frac{1}{2} \\ \frac{1}{k} + \frac{1}{l} &> \frac{1}{2} \end{aligned}$$

Now we that we have these conditions, we can start limiting cases. We know that k and l are at least 3: each face must have at least 3 sides, and each vertex has to connect at least 3 faces. We also know that neither k nor l can be above 5: if one were 6, the sum of their reciprocals would be at most $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$, violating the inequality. Also note that at least one of k and l must be 3; if neither was equal to 3, the sum of their reciprocals would be at most $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, also violating the inequality. With this last condition, we can establish 2 cases: either $k = 3$ or $l = 3$.

If $k = 3$, then l can be 3, 4, or 5. If $l = 3$, we form a tetrahedron. Next, if $l = 4$, we form an octahedron. Finally, if $l = 5$, we form an icosahedron.

Our next case is $l = 3$, and k can once again be 3, 4, or 5. If $k = 3$, we get the tetrahedron again. If $k = 4$, we get the cube. If $k = 5$, we get the dodecahedron. Thus, we've identified the 5 Platonic solids and shown that there cannot be any others.

Application to Archimedean Solids

Now let's move on to the Archimedean solids. An Archimedean solid is a solid whose faces are all regular polygons and whose vertices are congruent. Notice that we have removed the restriction that all faces have the same number of sides. This means that each vertex has to connect the same sequence of n -gons in the same order under rotation and reflection. There are 13 Archimedean solids, as well as the two infinite classes of prisms and anti-prisms. The classes of prisms and anti-prisms are both infinite because you can take any two congruent polygons and connect them by squares (to form a prism) or triangles (to form an antiprism). Since it's more helpful to count Archimedean solids that aren't prisms or antiprisms, these two infinite classes will be excluded from our proof. We'll introduce some new variables: r is the number of edges that coincide at each vertex, F_n is the number of n -gons present in the solid, p_n will represent the number of sides on one of the faces present at a vertex, and q_n will be the number of n -gons that coincide at a single vertex.

Lemma 2: $1 - \frac{r}{2} + \frac{1}{p_1} + \dots + \frac{1}{p_r} = \frac{2}{V}$

Proof. First, we'll reuse the fact that $rV = 2E$ to get that $E = \frac{rV}{2}$. Next, notice that for all n , $n \cdot F_n$ counts each vertex once for each of the q_n n -gons present at each vertex. So we get $n \cdot F_n = V \cdot q_n$, and $F_n = \frac{V \cdot q_n}{n}$. Also notice that counting F_n for all n ends up counting all the faces. We can use that to form the following equations:

$$F_3 + F_4 + \dots + F_{V-1} = F$$

We start at F_3 because every polygon has at least 3 sides, and end at F_{V-1} because it's impossible for one face to connect to every single vertex in a solid.

$$\begin{aligned} \frac{V \cdot q_3}{3} + \frac{V \cdot q_4}{4} + \dots + \frac{V \cdot q_{V-1}}{V-1} &= F \\ V \cdot \left(\frac{q_3}{3} + \frac{q_4}{4} + \dots + \frac{q_{V-1}}{V-1} \right) &= F \end{aligned}$$

Notice that the sum of all $\frac{q_n}{n}$ for all n is the same as adding $\frac{1}{n}$ for each of the r polygons around a vertex. Thus we have:

$$V \cdot \left(\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_r} \right) = F$$

Now, we can substitute our values for E and F into Euler's formula:

$$\begin{aligned} V - E + F &= 2 \\ V - \frac{rV}{2} + V \cdot \left(\frac{1}{p_1} + \dots + \frac{1}{p_r} \right) &= 2 \\ V \cdot \left(1 - \frac{r}{2} + \frac{1}{p_1} + \dots + \frac{1}{p_r} \right) &= 2 \\ 1 - \frac{r}{2} + \frac{1}{p_1} + \dots + \frac{1}{p_r} &= \frac{2}{V} \end{aligned}$$

as desired. □

Lemma 3: $r < 6$

Proof. From Lemma 2 and the fact that $\frac{2}{V} > 0$, we can say that:

$$1 - \frac{r}{2} + \frac{1}{p_1} + \dots + \frac{1}{p_r} > 0$$

$$\frac{1}{p_1} + \dots + \frac{1}{p_r} > \frac{r}{2} - 1$$

Now we make the following observations:

$$p_n \geq 3$$

$$\frac{1}{p_n} \leq \frac{1}{3}$$

$$\frac{1}{p_1} + \dots + \frac{1}{p_r} \leq r \cdot \frac{1}{3}$$

$$\frac{r}{3} \geq \frac{1}{p_1} + \dots + \frac{1}{p_r} > \frac{r}{2} - 1$$

$$\frac{r}{3} > \frac{r}{2} - 1$$

$$2r > 3r - 6$$

$$r < 6$$

as desired. □

Since we also know that $r \geq 3$ (at least 3 faces must meet at every vertex), we now have 3 cases: $r = 5$, $r = 4$, and $r = 3$. From here, we just go through the cases. Each solution is going to be expressed in terms the faces present at each vertex: for example, a tetrahedron would be expressed as (3,3,3) because 3 triangles surround each vertex, a cube would be expressed as (4,4,4) because 3 squares surround each vertex, etc. If one vertex formula can be reached by a shift or reflection of another vertex formula, they produce the same solid and will only be listed once.

Case 1: $r = 5$

Start by plugging r into Lemma 2. For simplicity, we'll replace p_n with a, b, c, d, and e.

$$1 - \frac{5}{2} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} > 0$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} > \frac{3}{2}$$

From here, we know that all of a, b, c, d, e except one must be 3. If two of them are greater than 3, their sum is at most $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} = \frac{3}{2}$, violating the inequality. Let's say $b = c = d = e = 3$. Then we have:

$$\frac{1}{a} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} > \frac{3}{2}$$

$$\frac{1}{a} > \frac{1}{6}$$

$$a < 6$$

This gives us 3 solutions: (3,3,3,3,3), (4,3,3,3,3), and (5,3,3,3,3). Since (3,3,3,3,3) is a Platonic solid (the icosahedron), we'll exclude that solution.

Case 2: $r = 4$

Again, we'll plug r into Lemma 2 and replace p_n with a, b, c, d .

$$1 - \frac{4}{2} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} > 0$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} > 1$$

We know that at least one of a, b, c, d must be a triangle: if none of them were triangles, their sum would be at most $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$, violating the inequality.

Let's assume $d = 3$. We now have $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{3} > 1$, and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > \frac{2}{3}$.

We're going to introduce a new method where we focus on a single polygon and draw conclusions about its sides. In this case, we're going to focus on the sides of a triangle, since we're guaranteed at least one triangle at each vertex. Remember that at each vertex, there must be an a -gon, a b -gon, a c -gon, and a triangle, in that order.

Therefore, since every b -gon must lie opposite the triangle at each vertex, it cannot share a side with the triangle. Now let's look at the sides of the triangle. We already know that none of these sides can be adjacent to a b -gon, so they must be adjacent to either an a -gon or a c -gon. Notice that if one side of the triangle connects to an a -gon, the next side must connect to a c -gon: otherwise if they both connect to a -gons, the vertex between them would have two a -gons in its formula, violating the (a,b,c,d) formula.

Thus, if one side of the triangle connects to an a -gon, then the other two sides of the triangle must both connect to c -gons (else we'd have a vertex containing 2 a -gons). But now we have two adjacent sides that both connect to c -gons, so the vertex between them contains two c -gons and we've reached a contradiction. The only way to escape this contradiction is to let $a = c$ so that two adjacent sides can both connect to a -gons without a contradiction. Substituting in $a = c$ gives us $\frac{2}{a} + \frac{1}{b} > \frac{2}{3}$

Now we just run through the cases: $a = 3$ gives us $\frac{1}{b} > 0$ and b can be anything (this is the infinite class of antiprisms). $a = 4$ gives us $\frac{1}{b} > \frac{1}{6}$ and $b = 3, 4, 5$. $a = 5$ gives us $\frac{1}{b} > \frac{4}{15}$ and $b = 3$. If $a = 6$, we have $\frac{1}{b} > \frac{1}{3}$, which is impossible so any a greater than 5 fails.

This gives us an infinite class as well as 4 solutions: (3,3,3,m), (3,4,3,4), (3,4,4,4), (3,4,5,4), and (3,5,3,5). We'll exclude the antiprisms as an infinite class.

Case 3: $r = 3$

$$1 - \frac{3}{2} + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 0$$

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > \frac{1}{2}$$

Once again, we'll replace p_n with a , b , and c , giving us $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > \frac{1}{2}$. Note that one of a , b , or c must be 3, 4, or 5: if they were all at least 6, their sum would be at most $\frac{1}{2}$, violating the inequality. So now we can do cases on the minimum value:

3.1: $c = 3$

By traversing along the sides of the triangle and using the same logic we've previously encountered, we can conclude that $a = b$, giving us $\frac{2}{a} + \frac{1}{3} > \frac{1}{2}$, so $\frac{2}{a} > \frac{1}{6}$ and $3 \leq a \leq 11$. To further narrow these choices down, we can traverse the sides of an a -gon. Notice that if one side is adjacent to another a -gon, the next side must be adjacent to a triangle, giving us two possibilities: either a is even, or $a = 3$. The solutions we get here are: (3,3,3), (3,4,4), (3,6,6), (3,8,8), and (3,10,10). We'll exclude (3,3,3) since that's a Platonic solid, and (3,4,4) since that's a prism.

3.2: $c = 4$, $a, b \geq 4$

We may substitute c into our inequality to get $\frac{1}{a} + \frac{1}{b} + \frac{1}{4} > \frac{1}{2}$, so $\frac{1}{a} + \frac{1}{b} > \frac{1}{4}$. We can again traverse the sides of an a -gon to get two cases: either $b = c = 4$, in which case we get $\frac{1}{a} + \frac{1}{4} > \frac{1}{4}$ and a can be anything (this is the infinite class of prisms), or $b \neq 4$, in which case a has to be even. The same logic for the b -gon leads us to conclude that if $a \neq 4$, then b is also even. However, note that either a or b is less than 8; if both were 8, their sum is $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$, violating the inequality. Thus, assume $b = 6$. Our inequality gives $\frac{1}{a} + \frac{1}{6} > \frac{1}{4}$, so $\frac{1}{a} > \frac{1}{12}$ and $a = 6, 8, 10$. So the solutions here are: (4,4,m), (4,6,6), (4,6,8), and (4,6,10). We're excluding (4,4,m) as an infinite class.

3.3: $c = 5$, $a, b \geq 5$

Substituting c into our inequality gives us $\frac{1}{a} + \frac{1}{b} + \frac{1}{5} > \frac{1}{2}$, so $\frac{1}{a} + \frac{1}{b} > \frac{3}{10}$. We can traverse the sides of the pentagon to conclude that $a = b$. This gives us two solutions: $a = b = 5$ or $a = b = 6$. These solids are: (5,5,5) and (5,6,6). We'll exclude (5,5,5) because that's a Platonic solid.

Now that we've gone through all the cases, let's summarize the solutions(Figure 3):

Vertex Formula	Solid
(4,3,3,3,3)	Snubcube
(5,3,3,3,3)	Snub Dodecahedron
(3,4,3,4)	Cuboctahedron
(3,4,4,4)	Small Rhombicuboctahedron
(3,4,5,4)	Small Rhombicosidodecahedron
(3,5,3,5)	Icosidodecahedron
(3,6,6)	Truncated Tetrahedron
(3,8,8)	Truncated Cube
(3,10,10)	Truncated Dodecahedron
(4,6,6)	Truncated Octahedron
(4,6,8)	Great Rhombicuboctahedron
(4,6,10)	Great Rhombicosidodecahedron
(5,6,6)	Truncated Icosahedron

Archimedean solids

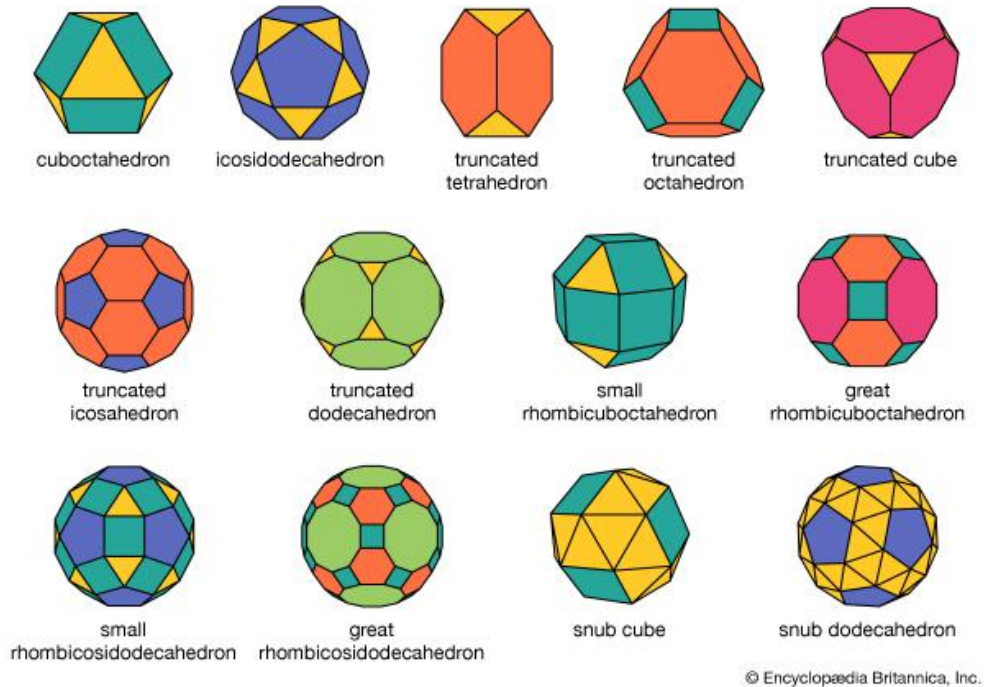


Figure 3: The 13 Archimedean Solids (taken from Brittanica Kids)

Application to Graph Theory

One of the more well-known applications of Euler's formula is through graph theory. A graph is a set of points connected to each other by edges. A planar graph is a graph that can be drawn on a plane so that its edges only intersect at vertices. As it turns out, Euler's formula also holds for planar graphs, but only if you include the infinite face (the area of the plane that's not contained by any edges) in your calculation. Knowing this, we can prove that some graphs are impossible to form without edges intersecting.

Problem 1: Can 5 points all be connected to each other by an edge without two edges intersecting?

Let's interpret this graph as a solid and plug in values into Euler's formula. There are 5 vertices given in the problem, and since there are 10 ways to pair two of these vertices, there are 10 edges.

$$V - E + F = 2$$

$$5 - 10 + F = 2$$

$$F = 7$$

Now we know there are 7 faces. Since every face coincides with at least 3 edges and each edge still connects two faces, we have $3F \leq 2E$. Plug in the numbers we have, and this gives us $3 \cdot 7 \leq 2 \cdot 10$, which is false and leads us to a contradiction. For that reason, it's

impossible to connect 5 points to each other without an intersection.

Problem 2: Can 3 points each be connected to all of 3 other points without any intersections?

You may have encountered this problem described as three houses that all need to get water, gas, and electricity from their respective facilities without any wires crossing each other. We're given 6 vertices, and connecting 3 utilities to 3 houses gives us 9 edges.

$$V - E + F = 2$$

$$6 - 9 + F = 2$$

$$F = 5$$

In this case, just knowing that each face has at least 3 edges isn't enough. However, the problem setup gives us a stronger condition: each face must have at least 4 sides. Since a house vertex can only be connected to facility vertices, each face must have an even number of vertices to accommodate pairs of houses and facilities. For that reason, every face has at least 4 sides, so $4F \leq 2E$. Plugging in values gives us $4 \cdot 5 \leq 2 \cdot 9$, which is false, showing that this is impossible.

Citations

1. Villarino, Mark. "On the Archimedean or Semiregular Polyhedra." *Elemente Der Mathematik*, 2008, pp. 76–87., doi:10.4171/em/90.
2. Hoang, Lê Nguyễn. "Euler's Formula and the Utilities Problem." *Science4All*, 29 Jan. 2016, www.science4all.org/article/eulers-formula-and-the-utilities-problem/.
3. Encyclopædia Britannica, Encyclopædia Britannica, Inc., kids.britannica.com/students/article/Archimedean-solid/316494.
4. IV, John Perry, et al. "Fig. 4 Platonic and Archimedean Solids. First Row (Left to Right):..." *ResearchGate*, 31 July 2018, www.researchgate.net/figure/Platonic-and-Archimedean-solids-First-row-left-to-right-tetrahedron-hexahedronfig124346584.