A Combinatorial Identity

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Figure 1: Image by Reya Miller

Question

Let S be the set of n-tuples $A = (a_1, a_2, \ldots, a_n)$ such that $a_i \geq 0$, $\sum_{i=1}^{n} a_i = k$, and $\sum_{i=1}^{n} i \cdot a_i = n$. Prove $\sum_{A \in S} \frac{k!}{a_1! a_2! \cdots a_n!} = \binom{n-1}{k-1}$.

Solution

One solution is to look at the number of ways to partition n identical objects into k ordered groups, where each group has at least one object, which is $\binom{n-1}{k-1}$. (If you are not familiar with this type of "stars and bars" argument, think of a row of n objects. There are $n-1$ empty spaces between them, and you can choose $k-1$ of those spaces to place a separator. These separators split the n objects into k groups of objects, each with at least 1 object.)

For each partition P, let a_i be the number of groups with i objects in them for $i = 1, 2, \ldots, n$. Since there are k groups, and each group must have between 1 and n objects, $\sum_{i=1}^{n} a_i = k$. Since the sum over all groups of the number of objects in each group is n, $\sum_{i=1}^{n} i \cdot a_i = n$. Thus, $A_P := (a_1, a_2, \ldots, a_n) \in S$. Notice that two different partitions can have the same A, since our definition of A_P does not reference the ordering of the groups. Any partition with $A_P = A$ must have a_1 groups with 1 object, a_2 groups with 2 objects, etc. Thus, the number of partitions such that $A_P = A$ is the number of k-tuples of a_1 1's, a_2 2's, ..., and a_n n's, which is $\frac{(\sum_{i=1}^{n} a_i)!}{a_1! a_2! \cdots a_n!} = \frac{k!}{a_1! a_2! \cdots a_n!}$ (since there are $k!$ ways to arrange the k numbers, but there are $a_1!$ indistinguishable ways to permute the a_1 1's, $a_2!$ indistinguishable ways to permute the a_2 2's, etc.). Thus, the number of ways to partition n identical objects into k ordered groups, where each group has at least one object, can also be counted by the sum over all A of the number of partitions P with $A_P = A$, which is $\sum_{A \in S} \frac{k!}{a_1! a_2! \cdots a_n!}$, which concludes the proof.