Poles and Polars

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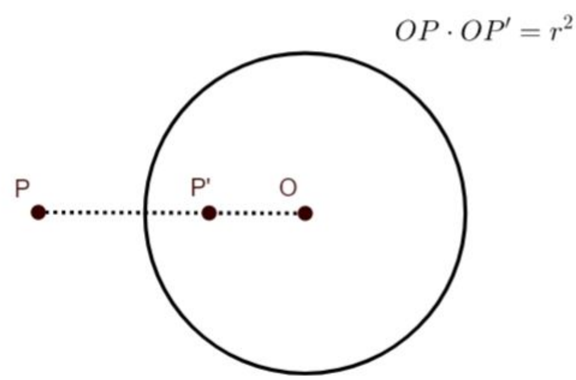
1 Introduction

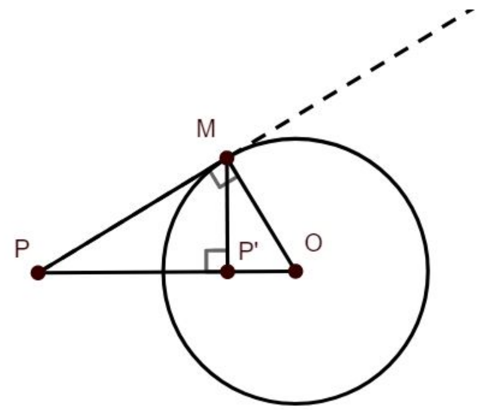
Poles and polars are a powerful geometrical concept that can be used to complete interesting proofs in often elegant ways. But what exactly are they?

Before we can define what poles and polars are, we must first familiarize ourselves with the concept of an inverse. If we start with a circle centered at O with radius r and a point P which isn't on the center of the circle, then the inverse of point P is the point P' such that O, P, and P' are collinear and $OP \cdot OP' = r^2$.

It's important to note that the way we are currently thinking about lengths, the above definition would have two solutions in which P' is reflected about O. But we can introduce directed lengths, which claims that the product of two collinear segments is positive if they point in the same direction, and negative if they point in opposite directions. Under that concept of lengths, the condition that O, P, and P' are collinear and $OP \cdot OP' = r^2$ now has only one solution.



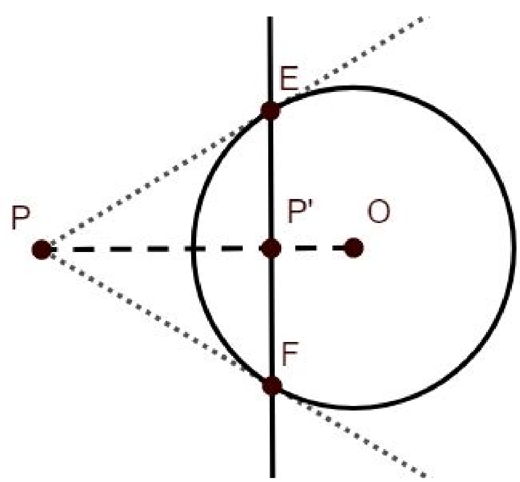
It's also important to know how to construct the inverse. If we are given a point P outside a circle centered at O, then we can construct a tangent from P to the circle and drop an altitude from the point of tangency to the line $\overline{OP}$. The point where the altitude intersects $\overline{OP}$ will be the inverse of point P.



We can prove this by using similar triangles. If we call the point of tangency M, then we know that ΔPMO is a right triangle. The altitude of a right triangle splits that triangle into two similar triangles. Since $\Delta MP'O$ and ΔPMO are similar, we know that $\frac{OP'}{OM} = \frac{OM}{OP}$. M is on the circle centered at O, so $OM = r$, and $\frac{OP'}{r} = \frac{r}{OP}$. Multiplying out the denominators gives $OP \cdot OP' = r^2$, which satisfies the definition of the inverse.

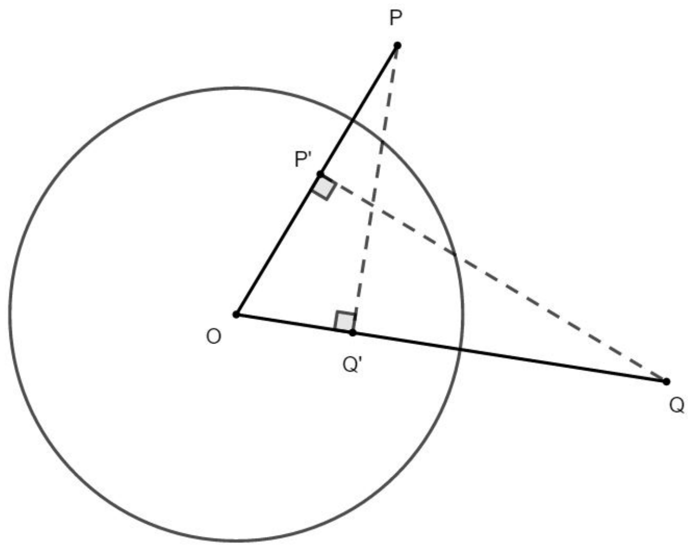
Now, we'll define poles and polars: if we start with a circle centered at O and a point P which we will call a pole, then the polar of pole P is the line perpendicular to OP which passes through the inverse of P.

Again, we'll start with a construction. If we start with a circle centered at O and a pole P that lies outside the circle, then we can draw the tangents from P that intersect the circle at points E and F. As it turns out, line EF goes through the inverse of P, and is perpendicular to OP, so therefore it is the polar of pole P. Likewise, if we start with a chord EF, we can draw the tangents to the circle at points E and F, and they will intersect at the pole P.



2 La Hire's Theorem

Now we can go on to prove La Hire's Theorem, an important statement about poles and polars: if we have poles P and Q such that Q lies on the polar of P, then P must also lie on the polar of Q.

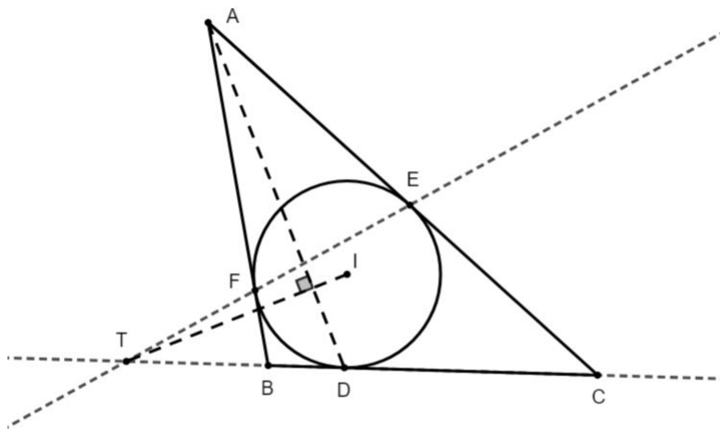


Proving this statement becomes easier when we break it down into simpler geometric statements. Saying Q is on the polar of P is equivalent to saying that $\angle OP'Q$ is a right angle. Likewise, to prove that P is on the polar of Q, we just have to show that $\angle OQ'P$ is a right angle. We know that $OP \cdot OP' = r^2$ and $OQ \cdot OQ' = r^2$, so we also know that $OP \cdot OP' = OQ \cdot OQ'$. Therefore, $\frac{OP}{OQ'} = \frac{OQ}{OP'}$. Next, we can use SAS similarity to prove that $\triangle OP'Q \sim \triangle OQ'P$. We know $\angle P'OQ \cong Q'OP$ since they're the same angle, and we know that the surrounding sides are proportional based on what we just showed. Therefore, the two triangles are similar, and so their angles are equal. We're given that $\angle OP'Q$ is a right angle, so because of the similar triangles we've established, $\angle OQ'P$ must also be a right angle. Therefore, if Q is on the polar of P, then P is on the polar of Q.

2.1 Example Problem

Here's an example of a problem that's made significantly easier by using the concept of poles and polars.

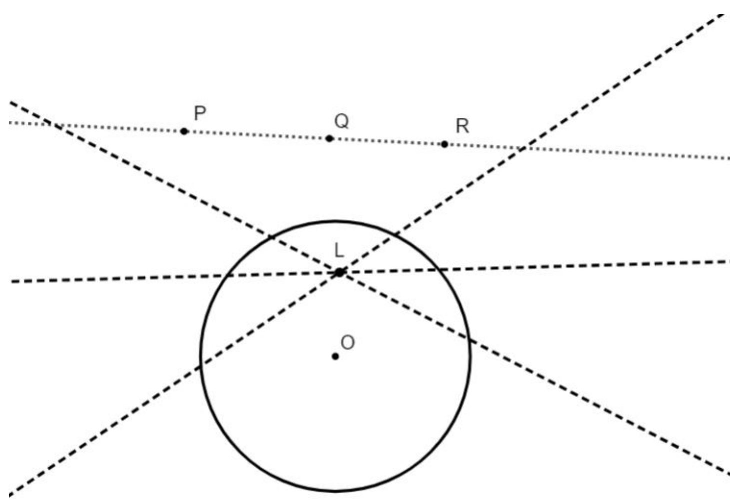
Problem: Given a triangle $\triangle ABC$, with an incircle centered at incenter I which intersects BC, AC, and AB at points D, E, and F respectively, prove that if line EF intersects BC at point T, then $\overline{IT} \perp \overline{AD}$.



Proof: First, notice that $\overline{EF}$ is the polar of A, since we've already shown that polars are defined by the points of tangency from the pole. Therefore, if T lies on $\overline{EF}$, then T is on the polar of A. By La Hire's, A must also lie on the polar of T. We also know that D must also be on the polar of T, since it is a point of tangency. Therefore, $\overline{AD}$ must be the polar of T, and so by the definition of a polar, it must be perpendicular to $\overline{IT}$.

2.2 Corollary

Now we can go on to prove an extension of La Hire's Theorem: three points P, Q, and R are collinear if and only if their polars are concurrent.



There are two things we have to prove here. Since the statement is bidirectional, we have to prove both directions separately.

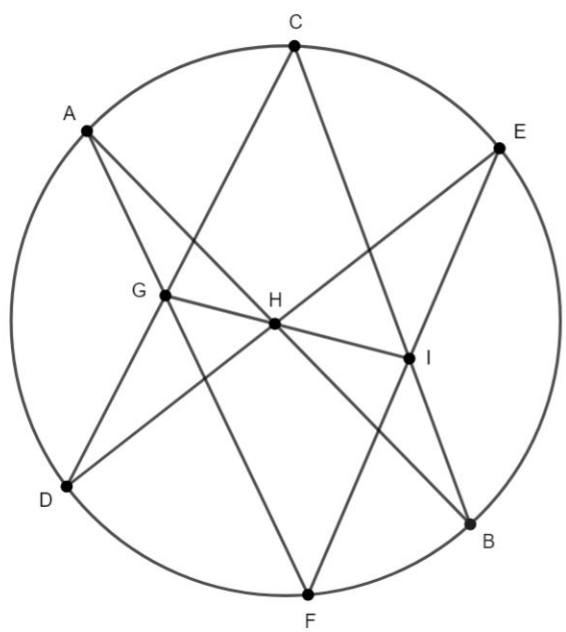
First, we will prove that if P, Q, and R are collinear, then their polars are concurrent. Let's say the the line that goes through all three is labelled ℓ. Then we can construct the pole of polar ℓ and label it L.

Next, we can apply La Hire's Theorem three times; if P lies on the polar of L, then L lies on the polar of P; if Q lies on the polar of L, then L lies on the polar of Q; if R lies on the polar of L, then L lies on the polar of R. Therefore, we've proved that L lies on the polars of P, Q, and R, so therefore the three polars are concurrent.

Next, we'll use a similar strategy to prove that if the polars of P, Q, and R are concurrent, then they are collinear. Since the three lines are concurrent, call their intersection L, construct the polar of L and call it ℓ, and apply La Hire's three times to show that P, Q, and R all lie on the polar of L, ℓ, and so they must be collinear.

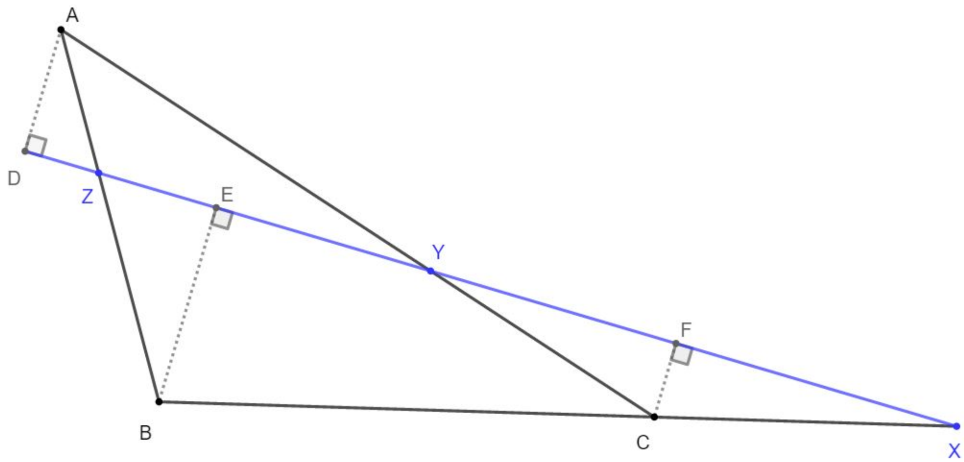
3 Pascal's Theorem

Before we prove more theorems about poles and polars, we have to get familiar with Pascal's Theorem. It states the following: given a hexagon $ABCDEF$ inscribed in a circle, then the intersections of opposite sides are collinear. To clarify, if we label $AF \cap CD$ as G, $AB \cap DE$ as H, and $BC \cap EF$ as I, then G, H, and I are collinear. It is important to note that a hexagon does not have to be a convex geometric shape, but rather any collection of 6 points that are connected by 6 lines in a cycle.



3.1 Menelaus's Theorem

Before we prove Pascal's Theorem, we must familiarize ourselves with Menelaus's Theorem. This is a generalization of Ceva's Theorem, but can be super useful in interesting situations. It states that given triangle $\triangle ABC$ and points X, Y, and Z on sides BC, AC, and AB respectively (with X on the extension of side BC rather than on the line segment itself), then X, Y, and Z are collinear if and only if $\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = -1$. Note that the product is negative because since BX and XC run in opposite directions, dividing one by the other yields a negative result.



This is also a bidirectional statement so we must prove both directions separately. The first direction we'll prove is that if the points are collinear, then the equation is true. We can drop altitudes to line XYZ from A, B, and C, which meet the line at points D, E, and F respectively. From here, we can find multiple similar triangles to get ratios that equal each other. $\triangle ADZ \sim \triangle BEZ$ because they share a right angle and vertical angles, so $\frac{AZ}{ZB} = \frac{AD}{BE}$. Likewise, $\triangle XEB \sim \triangle XFC$ because of two shared angles, so $\frac{BX}{XC} = -\frac{BE}{CF}$ (Note the negative sign to indicate that $\frac{BX}{XC}$ is negative). Finally, $\triangle ADY \sim \triangle CFY$ by angle-angle similarity, so $\frac{CY}{YA} = \frac{CF}{AD}$. Finally we can start considering the equation we are trying to prove:

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = -1$$

We've found that each of these factors is equal to another ratio, so we can substitute those in:

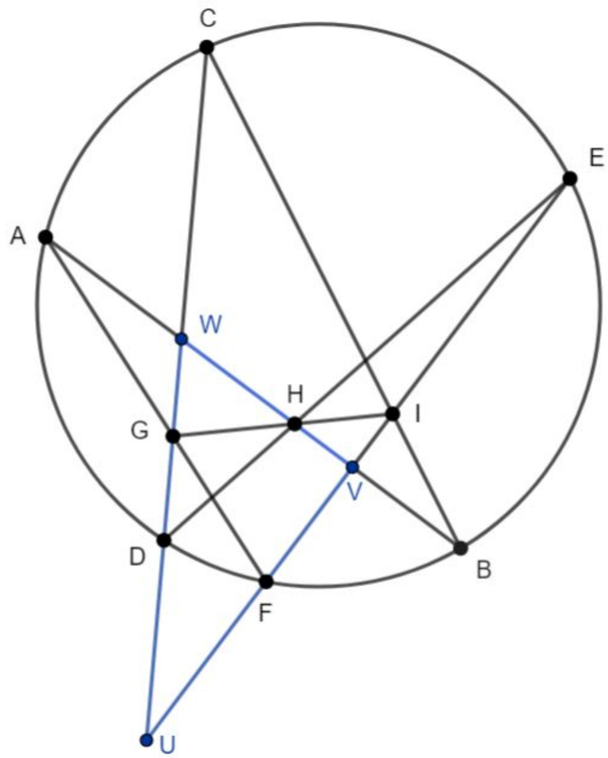
$$\frac{AD}{BE} \cdot \frac{-BE}{CF} \cdot \frac{CF}{AD} = -1$$

And now that the factors in the numerators and denominators are the same except for a negative sign, the left side is equal to -1 and the theorem has been proved true.

Now we just have to prove the other direction, but this is a relatively easy proof by contradiction: let's assume that three points X, Y, and Z satisfy the equation without being collinear. Then there must exist a unique point X' that also lies on BC while being collinear with Y and Z. Since we've already proved the other direction of Menelaus, we know that X', Y, and Z satisfy the equation. If both sets of points satisfy the equation, then it must be true that $\frac{BX}{XC} = \frac{BX'}{X'C}$ in order for both equations to simultaneously hold. And that is a contradiction, because under the assumption that X and X' are distinct points on BC, the two ratios cannot be equal.

3.2 Proof of Pascal's Theorem

First, we'll define some more points: let $CD \cap EF$ be labelled U, and $AB \cap EF$ be labelled V, and $AB \cap CD$ be labelled W.



Now we can apply Menelaus with $\triangle UVW$ and various lines. First, if we use Menelaus with $\triangle UVW$ and line EHD, we get

$$\frac{VH}{WH} \cdot \frac{WD}{UD} \cdot \frac{UE}{VE} = -1$$

Next, we can apply Menelaus with $\triangle UVW$ and line AGF to get

$$\frac{VA}{WA} \cdot \frac{WG}{UG} \cdot \frac{UF}{VF} = -1$$

Next, we'll apply Menelaus with $\triangle UVW$ and line BIC to get

$$\frac{VB}{WB} \cdot \frac{WC}{UC} \cdot \frac{UI}{VI} = -1$$

Now we can multiply all three equations together:

$$\frac{VH}{WH} \cdot \frac{WD}{UD} \cdot \frac{UE}{VE} \cdot \frac{VA}{WA} \cdot \frac{WG}{UG} \cdot \frac{UF}{VF} \cdot \frac{VB}{WB} \cdot \frac{WC}{UC} \cdot \frac{UI}{VI} = -1$$

That's a lot of factors being multiplied, but a little bit of organizing helps us cancel out most of them:

$$\frac{WD \cdot WC}{WA \cdot WB} \cdot \frac{VA \cdot VB}{VE \cdot VF} \cdot \frac{UE \cdot UF}{UC \cdot UD} \cdot \frac{VH}{WH} \cdot \frac{WG}{UG} \cdot \frac{UI}{VI} = -1$$

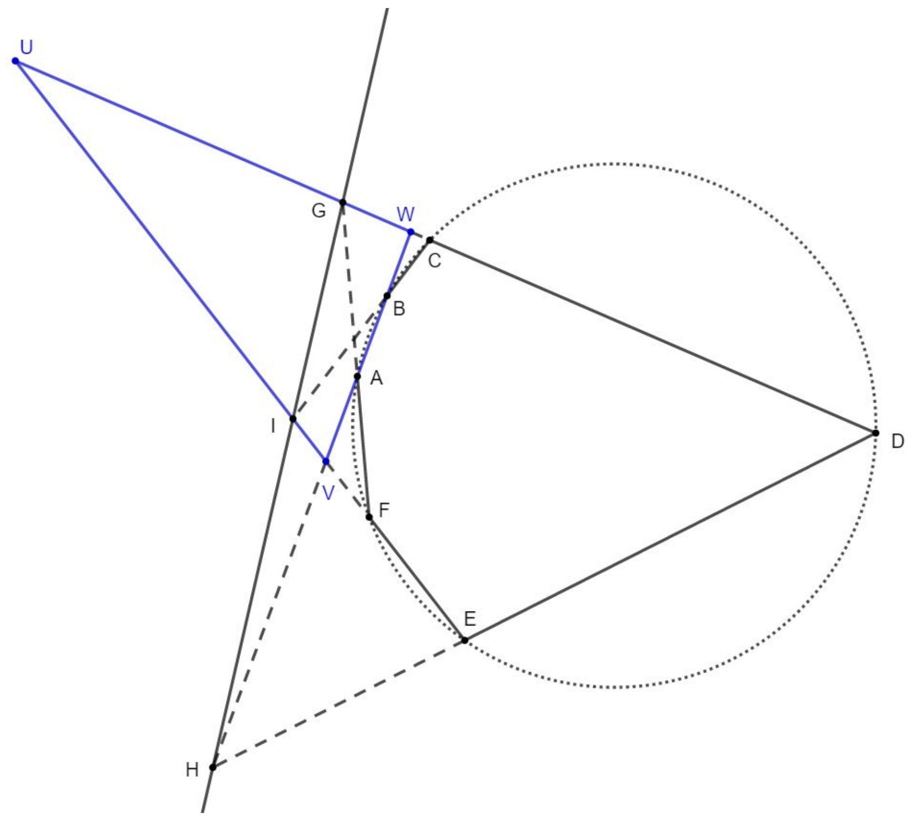
We can use power of a point here to cancel out the first three fractions. As a reminder, power of a point claims that for a fixed point P and circle ω, every chord that passes through

P and intersects ω at points E and F will have a constant product $PE \cdot PF$. In this case, it helps us show that $WD \cdot WC = WA \cdot WB$, and $VA \cdot VB = VE \cdot VF$, and $UE \cdot UF = UC \cdot UD$. Cancelling out these equal factors gives us

$$\frac{VH}{WH} \cdot \frac{WG}{UG} \cdot \frac{UI}{VI} = -1$$

Notice that this is exactly what Menelaus says when applied to $\triangle UVW$ and line GHI. Since we've shown that Menelaus is bidirectional, this proves that G, H, and I are collinear, and our proof of Pascal's Theorem is complete.

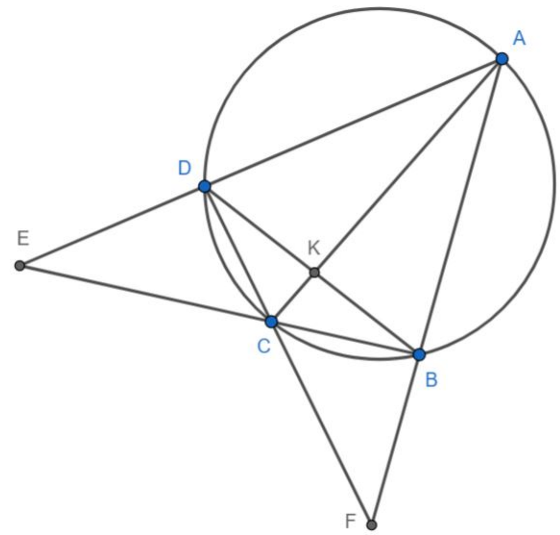
Just to show that Pascal's Theorem is true for all cyclic hexagons, including convex hexagons, here's a diagram of the same proof drawn out for a convex hexagon.



Note that all the steps in the proof are the same and still hold for this configuration, except for one complication: line EHD now has 3 points that lie outside the borders of the triangle. But that's okay, because we can actually show that we can apply Menelaus to a set of three collinear points where every point lies outside the borders of the triangle. The proof is exactly the same: if we draw perpendiculars from points U, V, and W to line EHD, we can use the resulting similar triangles as outlined in the proof of Menelaus above, which confirms that $\frac{VH}{WH} \cdot \frac{WD}{UD} \cdot \frac{UE}{VE} = -1$ and completes the proof.

4 Fundamental Theorem of Poles and Polars

Now we can go on to prove another theorem about poles and polars: given a cyclic quadrilateral $ABCD$ inscribed in a circle, let $AB \cap CD$ be labelled F, and $AD \cap BC$ be labelled E, and $AC \cap BD$ be labelled K. Then EK is the polar of F, KF is the polar of E, and EF is the polar of K.



We can apply Pascal's Theorem here, but we must first establish how we interpret "lines" defined by a single point on a circle. For example, we know what line AB looks like, but how would we interpret line AA? For the purposes of this article, this will represent the line tangent to the circle at point A. You can conceptualize it this way: if Pascal's Theorem is true for any hexagon, then it is true for hexagons where A and B are extremely close together. When A and B are extremely close together, the line between them very closely resembles the tangent to the circle at point A. So if Pascal's Theorem is true when line AB is very similar to the tangent at A, then it should also be true when line AB is exactly the tangent at A, which we'll denote as line AA.

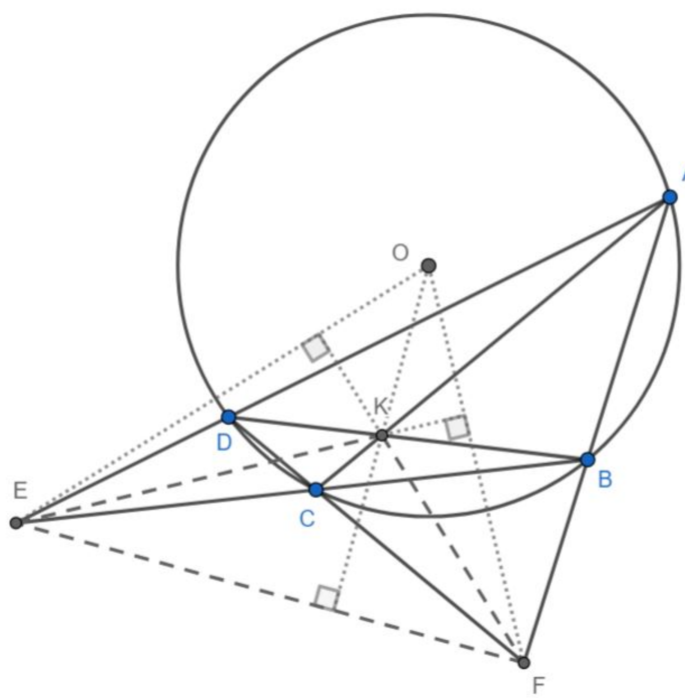
Now we can prove that EK is the polar of F by applying Pascal's Theorem. If we look at hexagon $ACCBDD$, Pascal's Theorem states that $AC \cap BD$, $CC \cap DD$, and $CB \cap DA$ are collinear, so K, E, and $CC \cap DD$ are collinear. We can also apply Pascal's Theorem to hexagon $CAADBB$ to find that $CA \cap DB$, $AA \cap BB$, and $AD \cap BC$ are collinear, so K, E, and $AA \cap BB$ are collinear. Combining the two applications of Pascal's Theorem shows that K, E, $CC \cap DD$, and $AA \cap BB$ are all collinear. Notice that $CC \cap DD$ is the pole of polar CD, since we've shown that poles can be constructed by the intersection of two tangents to a circle. And now we can apply La Hire's: since F lies on CD, which is the polar of $CC \cap DD$, then $CC \cap DD$ must lie on the polar of F. For the same reason, if F lies on AB, which is the polar of $AA \cap BB$, then $AA \cap BB$ must lie on the polar of F. Therefore, the line containing E, K, $CC \cap DD$, and $AA \cap BB$ must be the polar of F, and therefore EK is the polar of F.

Very similar proofs can show that KF is the polar of E and EF is the polar of K, but we can skip the tedious work by making a clever observation: we've already proven that EK is the polar of F, which means that in general, for any cyclic quadrilateral $ABCD$, the line between $AD \cap BC$ and $AC \cap BD$ will be the polar of $AB \cap CD$. This means that we can take the same diagram and change the cyclic quadrilateral we choose to consider.

For example, in the given diagram, we can choose to consider quadrilateral $ADCB$ instead, and what we've already proved can be used to demonstrate that the line between $AB \cap DC$ and $AD \cap DB$ is the polar of $AD \cap BC$, so KF is the polar of E. We can even choose to consider quadrilateral $ACBD$ — poles, polars, La Hire's Theorem, and Pascal's Theorem do not depend on polygons in question being convex, so we can let our quadrilateral intersect itself. Considering what we've proved in the context of this new quadrilateral shows that the line between $AD \cap CB$ and $AB \cap CD$ will be the polar of $AC \cap BD$, so EF is the polar of K.

4.1 Brocard's Theorem

Brocard's Theorem is a corollary of the Fundamental Theorem of Poles and Polars. It starts off with the same setup: a cyclic quadrilateral $ABCD$ with $AD \cap BC$ labelled as E, $AB \cap CD$ labelled as F, and $AC \cap BD$ labelled as K. Brocard's Theorem then states that the orthocenter of triangle $\triangle EFK$ is the circumcenter of $ABCD$, which we'll call O.

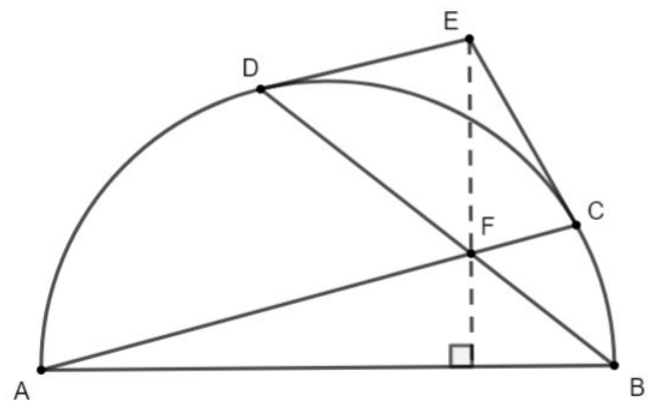


We can first apply the Fundamental Theorem of Poles and Polars that we have just proved, which shows that each side of the triangle is the polar of the opposite vertex. Next, we can look at the altitudes to each side. For example, if EF is the polar of K, then line OK must be perpendicular to EF. That means the altitude from K must pass through O. For the same reason, the altitudes through E and F must also pass through O. Therefore, the orthocenter of triangle $\triangle EFK$ must be the circumcenter of $ABCD$, O.

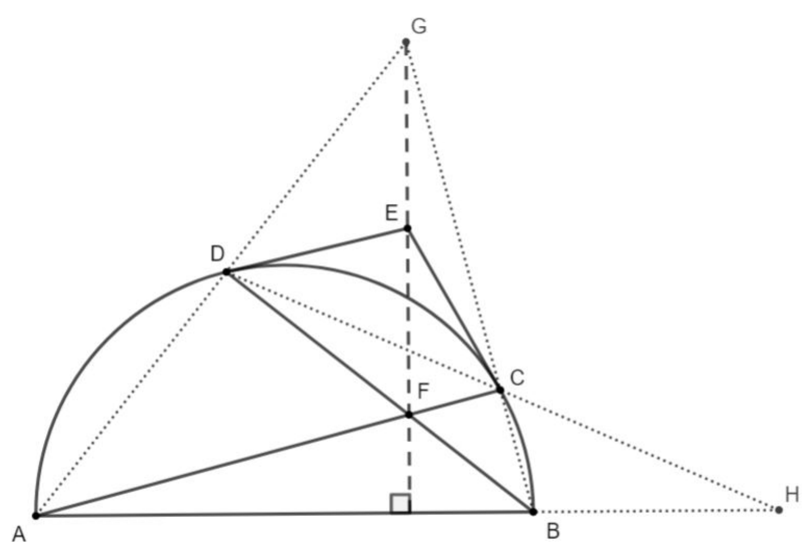
4.2 Example Problem

Here's a problem that uses the theorems we've just discussed.

Problem: Given a semicircle with diameter AB and points C and D on the semicircle, define E as the intersection between the tangents to the semicircle at C and D (in other words, $E = CC \cap DD$, and define F as $AC \cap BD$. Prove that $EF \perp AB$.



The first step is to use Pascal's Theorem on hexagon $ACCBDD$ to show that $AC \cap BD$, $CC \cap DD$, and $CB \cap DA$ are collinear. If we call $CB \cap DA$ point G, then that means E, F, and G are collinear. Let's also label $AB \cap CD$ as H. By the Fundamental Theorem of Poles and Polars, line FG is the polar of H. Since E, F, and G are all collinear, we can also say that line EF is the polar of H. Finally, by the definition of polars, line EF must be perpendicular to the line connecting the center of the circle to H. Since line AB goes through both the center of the circle and point H, then we have proven that EF is perpendicular to AB.



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