# 1 All Definitions for Numerical Versions

$$\mathcal{C}_2 = \frac{2\cos(k\Delta x) - 2}{\Delta x^2}$$

$$\mathcal{C}_4 = \frac{-2\cos(2k\Delta x) + 32\cos(k\Delta x) - 30}{12\Delta x^2}$$

$$\mathcal{G} = \left[H - \frac{H^3}{3}\mathcal{C}\right]$$

$$\mathcal{M}_3 = \frac{24}{26 - 2\cos(k\Delta x)}$$

$$\mathcal{M}_1 = \mathcal{M}_2 = 1$$

$$\mathcal{R}_1^+ = e^{ik\Delta x} \quad , \quad \mathcal{R}_1^- = 1$$

$$\mathcal{R}_2^- = 1 + \frac{i\sin(k\Delta x)}{2}$$

$$\mathcal{R}_2^+ = e^{ik\Delta x}\left(1 - \frac{i\sin(k\Delta x)}{2}\right)$$

$$\mathcal{R}_3^- = \frac{\mathcal{M}_3}{6}\left[5 + -e^{-ik\Delta x} + 2e^{ik\Delta x}\right]$$

$$\mathcal{R}_3^+ = \frac{\mathcal{M}_3 e^{ik\Delta x}}{6}\left[5 + 2e^{-ik\Delta x} - e^{ik\Delta x}\right]$$

$$\mathcal{R}_2^u = \frac{e^{ik\Delta x} + 1}{2}$$

$$\mathcal{R}_3^u = \frac{-3e^{2ik\Delta x} + 27e^{ik\Delta x} + 27 - 3e^{-ik\Delta x}}{48}$$

$$\mathcal{F}^{h,u} = H\mathcal{R}^u$$

$$\mathcal{F}^{h,h} = -\frac{\sqrt{gH}}{2}\left[\mathcal{R}^+ - \mathcal{R}^-\right]$$

$$\mathcal{F}^{u,u} = -\frac{\sqrt{gH}}{2}\mathcal{G}\left[\mathcal{R}^+ \ - \mathcal{R}^-\right]$$

$$\mathcal{F}^{u,h} = \frac{gH}{2}\left(\mathcal{R}^+ + \mathcal{R}^-\right)$$

$\mathcal{D} = 1 - e^{-ik\Delta x}$

# 2 Taylor Expansions Of Analytic Values

We denote exact/analytic version with a subscript $a$

$$\mathcal{G}_a = H + \frac{H^3}{3}k^2$$

$$\mathcal{M}_a = \frac{2}{k\Delta x}\sin\left(\frac{k\Delta x}{2}\right)$$

$$\mathcal{M}_a = 1 - \frac{k^2}{24}(\Delta x)^2 + \frac{k^4}{1920}(\Delta x)^4 - \frac{k^6}{322560}(\Delta x)^6 + O(x^8)$$

$$\mathcal{R}_a = e^{i\frac{k\Delta x}{2}}$$

$$\mathcal{R}_a^+ = \mathcal{R}_a^- = \mathcal{R}_a = 1+\frac{ik}{2}\Delta x-\frac{k^2}{8}\Delta x^2-\frac{ik^3}{48}\Delta x^3+\frac{k^4}{384}\Delta x^4+\frac{ik^5}{3840}\Delta x^5+O(x^6)$$

For the fluxes I think its best to group the $\frac{\mathcal{D}}{\Delta x\mathcal{M}}\mathcal{F}$ because its collects all the terms using spatial approximations and has a nice form. In fact these terms approximate the derivative of the flux.

$$\frac{\mathcal{D}_a}{\Delta x\mathcal{M}_a}\mathcal{F}_a^{h,u} = ikH$$

$$\frac{\mathcal{D}_a}{\Delta x\mathcal{M}_a}\mathcal{F}_a^{h,h} = 0$$

$$\frac{\mathcal{D}_a}{\Delta x\mathcal{M}_a}\mathcal{F}_a^{u,h} = ikgH$$

$$\frac{\mathcal{D}_a}{\Delta x\mathcal{M}_a}\mathcal{F}_a^{u,u} = 0$$

So in particular for the mass equation

$$h_t + Hu_x = 0$$

then

$$i\omega h + \frac{\mathcal{D}_a}{\Delta x \mathcal{M}_a}\mathcal{F}_a^{h,u} u_j + \frac{\mathcal{D}_a}{\Delta x \mathcal{M}_a}\mathcal{F}_a^{h,h} h_j = 0$$

Indeed it is these the values of our approximations to these terms that confirm that our methods have the correct spatial accuracy.

# 3 First Order Values

$$\mathcal{G}_1 = H - \frac{H^3}{3}\left(\frac{2\cos\left(k\Delta x\right) - 2}{\Delta x^2}\right)$$

$$\mathcal{G}_1 = H + \frac{H^3k^2}{3} - \frac{H^3k^4}{36}(\Delta x)^2 + \frac{H^3k^6}{1080}(\Delta x)^4 + O(x^6)$$

$$\mathcal{M}_1 = 1$$

$$\mathcal{R}_1^- = 1$$

$$\mathcal{R}_1^+ = e^{ik\Delta x}$$

$$\mathcal{R}_1^+ = 1 + ik\Delta x - \frac{k^2}{2}(\Delta x)^2 - \frac{ik^3}{6}(\Delta x)^3 + \frac{k^4}{24}(\Delta x)^4 + O(\Delta x^5)$$

$$\mathcal{R}_1^u = \frac{e^{ik\Delta x} + 1}{2}$$

$$\mathcal{R}_1^u = 1 + \frac{ik}{2}\Delta x - \frac{k^2}{4}(\Delta x)^2 - \frac{ik^3}{12}(\Delta x)^3 + \frac{k^4}{48}(\Delta x)^4 + O(\Delta x^5)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1}\mathcal{F}_1^{h,u} = \frac{1 - e^{-ik\Delta x}}{\Delta x} H \frac{e^{ik\Delta x} + 1}{2}$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1}\mathcal{F}_1^{h,u} = iHk - \frac{iHk^3}{6}(\Delta x)^2 + \frac{iHk^5}{120}(\Delta x)^4 + O(\Delta x^6)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1}\mathcal{F}_1^{h,h} = -\frac{\sqrt{gH}}{2}\frac{1 - e^{-ik\Delta x}}{\Delta x}\left[e^{ik\Delta x} - 1\right]$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1}\mathcal{F}_1^{h,h} = \frac{k^2\sqrt{gH}}{2}\Delta x - \frac{k^4\sqrt{gH}}{24}\Delta x^3 + \frac{k^6\sqrt{gH}}{720}\Delta x^5 + O(\Delta x^7)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1}\mathcal{F}_1^{u,h} = \frac{gH}{2}\frac{1-e^{-ik\Delta x}}{\Delta x}\left(1+e^{ik\Delta x}\right)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1}\mathcal{F}_1^{u,h} = igHk - \frac{igHk^3}{6}(\Delta x)^2 + \frac{igHk^5}{120}(\Delta x)^4 + O(\Delta x^6)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1}\mathcal{F}_1^{u,u} = -\frac{\sqrt{gH}}{2}\left[H - \frac{H^3}{3}\left(\frac{2\cos(k\Delta x)-2}{\Delta x^2}\right)\right]\frac{1-e^{-ik\Delta x}}{\Delta x}\left[e^{ik\Delta x}-1\right]$$

$$\mathcal{D}\mathcal{F}_1^{u,u} = \frac{k^2H\sqrt{gH}(H^2k^2+3)}{6}\Delta x - \frac{k^4H\sqrt{gH}(2H^2k^2+3)}{72}\Delta x^3 + O(\Delta x^5)$$

# 4 Second Order Values

$$\mathcal{G}_2 = H - \frac{H^3}{3}\left(\frac{2\cos(k\Delta x)-2}{\Delta x^2}\right)$$

$$\mathcal{G}_2 = H + \frac{H^3k^2}{3} - \frac{H^3k^4}{36}(\Delta x)^2 + \frac{H^3k^6}{1080}(\Delta x)^4 + O(x^6)$$

$$\mathcal{M}_2 = 1$$

$$\mathcal{R}_2^- = 1 + \frac{i\sin(k\Delta x)}{2}$$

$$\mathcal{R}_2^- = 1 + \frac{ik}{2}(\Delta x) - \frac{ik^3}{12}(\Delta x)^3 + \frac{ik^5}{240}(\Delta x)^5 + O(x^7)$$

$$\mathcal{R}_2^+ = e^{ik\Delta x}\left(1 - \frac{i\sin(k\Delta x)}{2}\right)$$

$$\mathcal{R}_2^+ = 1 + \frac{ik}{2}\Delta x + \frac{ik^3}{6}\Delta x^3 - \frac{k^4}{8}\Delta x^4 - \frac{7ik^5}{120}\Delta x^5 + O\left(\Delta x^6\right)$$

$$\mathcal{R}_2^u = \frac{e^{ik\Delta x}+1}{2}$$

$$\mathcal{R}_2^u = 1 + \frac{ik}{2}\Delta x - \frac{k^2}{4}(\Delta x)^2 - \frac{ik^3}{12}(\Delta x)^3 + \frac{k^4}{48}(\Delta x)^4 + O(\Delta x^5)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_2}\mathcal{F}_2^{h,u} = \frac{1-e^{-ik\Delta x}}{\Delta x}H\frac{e^{ik\Delta x}+1}{2}$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_2}\mathcal{F}_2^{h,u} = iHk - \frac{iHk^3}{6}(\Delta x)^2 + \frac{iHk^5}{120}(\Delta x)^4 + O(\Delta x^6)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_2}\mathcal{F}_2^{h,h} = -\frac{\sqrt{gH}}{2}\frac{1-e^{-ik\Delta x}}{\Delta x}\left[e^{ik\Delta x}\left(1-\frac{i\sin(k\Delta x)}{2}\right)-\left(1+\frac{i\sin(k\Delta x)}{2}\right)\right]$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_2}\mathcal{F}_2^{h,h} = \frac{k^4\sqrt{gH}}{8}(\Delta x)^3 - \frac{k^6\sqrt{gH}}{48}(\Delta x)^5 + O(\Delta x^7)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_2}\mathcal{F}_2^{u,u} =$$

$$-\frac{1-e^{-ik\Delta x}}{\Delta x}\frac{\sqrt{gH}}{2}$$

$$\times\left[H-\frac{H^3}{3}\left(\frac{2\cos(k\Delta x)-2}{\Delta x^2}\right)\right]\left[e^{ik\Delta x}\left(1-\frac{i\sin(k\Delta x)}{2}\right)-\left(1+\frac{i\sin(k\Delta x)}{2}\right)\right]$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_2}\mathcal{F}_2^{u,u} = \frac{k^4H\sqrt{gH}(H^2k^2+3)}{24}\Delta x^3 - \frac{k^6H\sqrt{gH}(H^2k^2+2)}{96}\Delta x^5 + O(\Delta x^7)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_2}\mathcal{F}_2^{u,h} = \frac{1-e^{-ik\Delta x}}{\Delta x}\frac{gH}{2}\left[e^{ik\Delta x}\left(1-\frac{i\sin(k\Delta x)}{2}\right)+\left(1+\frac{i\sin(k\Delta x)}{2}\right)\right]$$

$$\mathcal{D}\mathcal{F}_2^{u,h} = igkH + \frac{igk^3H}{12}\Delta x^2 - \frac{13ik^5gH}{240}\Delta x^4 + O(\Delta x^6)$$

# 5 Taylor Expansions Of Third Order Values

$$\mathcal{G}_3 = H - \frac{H^3}{3}\frac{-2\cos(2k\Delta x) + 32\cos(k\Delta x) - 30}{12\Delta x^2}$$

$$\mathcal{G}_3 = H + \frac{k^2H^3}{3} - \frac{k^6H^3}{270}(\Delta x)^4 + O(\Delta x^6)$$

$$\mathcal{M}_3 = \frac{24}{26 - 2\cos(k\Delta x)}$$

$$\mathcal{M}_3 = 1 - \frac{k^2}{24}(\Delta x)^2 + \frac{k^4}{192}(\Delta x)^4 + O(x^6)$$

$$\mathcal{R}_3^- = \frac{\mathcal{M}_3}{6}\left[5 - e^{-ik\Delta x} + 2e^{ik\Delta x}\right]$$

$$R_3^- = 1 + \frac{ik}{2}\Delta x - \frac{k^2}{8}(\Delta x)^2 - \frac{5ik^3}{48}(\Delta x)^3 + \frac{k^4}{64}(\Delta x)^4 + O(\Delta x^5)$$

$$\mathcal{R}_3^+ = \frac{\mathcal{M}_3 e^{ik\Delta x}}{6}\left[5 + 2e^{-ik\Delta x} - e^{ik\Delta x}\right]$$

$$R_3^+ = 1 + \frac{ik}{2}\Delta x - \frac{k^2}{8}(\Delta x)^2 + \frac{ik^3}{16}(\Delta x)^3 - \frac{13k^4}{192}(\Delta x)^4 + O(\Delta x^5)$$

$$\mathcal{R}_3^u = \frac{-3e^{2ik\Delta x} + 27e^{ik\Delta x} + 27 - 3e^{-ik\Delta x}}{48}$$

$$R_3^u = 1 + \frac{ik}{2}\Delta x - \frac{k^2}{8}(\Delta x)^2 - \frac{ik^3}{48}(\Delta x)^3 - \frac{k^4}{48}(\Delta x)^4 + O(\Delta x^5)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3}\mathcal{F}^{h,u} = \frac{1 - e^{-ik\Delta x}}{\Delta x}\frac{26 - 2\cos(k\Delta x)}{24}H\mathcal{R}^u$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3}\mathcal{F}_3^{h,u} = ikH - \frac{9ik^5H}{320}\Delta x^4 - \frac{ik^7H}{448}\Delta x^6 + O(\Delta x^9)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3}\mathcal{F}^{h,h} = -\frac{1-e^{-ik\Delta x}}{\Delta x \mathcal{M}_3}\frac{\sqrt{gH}}{2}$$
$$\times\left[\left(\frac{\mathcal{M}_3 e^{ik\Delta x}}{6}\left[5+2e^{-ik\Delta x}-e^{ik\Delta x}\right]\right)-\left(\frac{\mathcal{M}_3}{6}\left[5-e^{-ik\Delta x}+2e^{ik\Delta x}\right]\right)\right]$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3}\mathcal{F}^{h,h} = -\frac{1-e^{-ik\Delta x}}{\Delta x}\frac{\sqrt{gH}}{2}$$
$$\times\left[\left(\frac{e^{ik\Delta x}}{6}\left[5+2e^{-ik\Delta x}-e^{ik\Delta x}\right]\right)-\left(\frac{1}{6}\left[5-e^{-ik\Delta x}+2e^{ik\Delta x}\right]\right)\right]$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3}\mathcal{F}_3^{h,h} = \frac{k^4\sqrt{gH}}{12}\Delta x^3 - \frac{k^6\sqrt{gH}}{72}\Delta x^5 + \frac{k^8\sqrt{gH}}{960}\Delta x^7 + O(\Delta x^9)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3}\mathcal{F}^{u,u} = -\frac{1-e^{-ik\Delta x}}{\Delta x \mathcal{M}_3}\frac{\sqrt{gH}}{2}\left[H - \frac{H^3}{3}\frac{-2\cos(2k\Delta x)+32\cos(k\Delta x)-30}{12\Delta x^2}\right]$$
$$\times\left[\left(\frac{\mathcal{M}_3 e^{ik\Delta x}}{6}\left[5+2e^{-ik\Delta x}-e^{ik\Delta x}\right]\right)-\left(\frac{\mathcal{M}_3}{6}\left[5-e^{-ik\Delta x}+2e^{ik\Delta x}\right]\right)\right]$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3}\mathcal{F}^{u,u} = -\frac{1-e^{-ik\Delta x}}{\Delta x}\frac{\sqrt{gH}}{2}\left[H - \frac{H^3}{3}\frac{-2\cos(2k\Delta x)+32\cos(k\Delta x)-30}{12\Delta x^2}\right]$$
$$\times\left[\left(\frac{e^{ik\Delta x}}{6}\left[5+2e^{-ik\Delta x}-e^{ik\Delta x}\right]\right)-\left(\frac{1}{6}\left[5-e^{-ik\Delta x}+2e^{ik\Delta x}\right]\right)\right]$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3}\mathcal{F}^{u,u}$$
$$= \frac{\sqrt{gH}}{2}\left[-\frac{ikH\left(k^2H^2+3\right)}{3} - \frac{k^2H\left(k^2H^2+3\right)}{3}\Delta x + \frac{ik^3H\left(k^2H^2+3\right)}{18}\Delta x^2 + O(\Delta x^3)\right]$$
$$\times\left[\frac{ik^3}{6}\Delta x^3 - \frac{k^4}{12}\Delta x^4 + O(\Delta x^5)\right]$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3}\mathcal{F}^{u,u}$$
$$= \frac{\sqrt{gH}}{2}\left[-\frac{ikH\left(k^2H^2+3\right)}{3} - \frac{k^2H\left(k^2H^2+3\right)}{6}\Delta x + \frac{ik^3H\left(k^2H^2+3\right)}{18}\Delta x^2 + O(\Delta x^3)\right]$$
$$\times\left[\frac{ik^3}{6}\Delta x^3 - \frac{k^4}{12}\Delta x^4 + O(\Delta x^5)\right]$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3}\mathcal{F}^{u,u} = \frac{\sqrt{gH}}{2}\left[\frac{k^4H\left(k^2H^2+3\right)}{18}\Delta x^3 + \frac{k^6H\left(k^2H^2+3\right)}{216}\Delta x^5 + O(\Delta x^6)\right]$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3}\mathcal{F}^{u,u} = \frac{k^4H\sqrt{gH}\left(k^2H^2+3\right)}{36}\Delta x^3 + \frac{k^6H\sqrt{gH}\left(k^2H^2+3\right)}{532}\Delta x^5 + O(\Delta x^6)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3}\mathcal{F}^{u,h} = \frac{gH}{2}\frac{1-e^{-ik\Delta x}}{\Delta x \mathcal{M}_3}$$
$$\times\left[\left(\frac{e^{ik\Delta x}\mathcal{M}_3}{6}\left[5+2e^{-ik\Delta x} - e^{ik\Delta x}\right]\right) + \left(\frac{\mathcal{M}_3}{6}\left[5 - e^{-ik\Delta x} + 2e^{ik\Delta x}\right]\right)\right]$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3}\mathcal{F}^{u,h} = \frac{gH}{2}\frac{1-e^{-ik\Delta x}}{\Delta x}$$
$$\times\left[\left(\frac{e^{ik\Delta x}}{6}\left[5+2e^{-ik\Delta x} - e^{ik\Delta x}\right]\right) + \left(\frac{1}{6}\left[5 - e^{-ik\Delta x} + 2e^{ik\Delta x}\right]\right)\right]$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3}\mathcal{F}_3^{u,h} = igkH - \frac{ik^5 gH}{30}\Delta x^4 + O(\Delta x^6)$$

# 6 Numerical Method Break Down

Our conservative update is, for our equations is

$$\bar{q}_j^{n+1} = \bar{q}_j^n - \frac{\Delta t}{\Delta x}\left[F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right]$$

This converts to (both analytical and numerical)

$$\mathcal{M}q_j^{n+1} = \mathcal{M}q_j^n - \frac{\Delta t}{\Delta x}\left[\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j - \mathcal{F}^{q,v}v_{j-1} - \mathcal{F}^{q,q}q_{j-1}\right]$$

$$\mathcal{M}q_j^{n+1} = \mathcal{M}q_j^n - \frac{\Delta t}{\Delta x}\left[\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j - \mathcal{F}^{q,v}e^{-ik\Delta x}v_j - \mathcal{F}^{q,q}e^{-ik\Delta x}q_j\right]$$

Defining $\mathcal{D}_x = 1 - e^{-ik\Delta x}$

$$\mathcal{M}q_j^{n+1} - \mathcal{M}q_j^n = -\frac{\Delta t}{\Delta x}\left[\mathcal{D}_x\mathcal{F}^{q,v}v_j + \mathcal{D}_x\mathcal{F}^{q,q}q_j\right]$$

$$\mathcal{M}\left(q_j^{n+1} - q_j^n\right) = -\frac{\Delta t}{\Delta x}\left[\mathcal{D}_x\mathcal{F}^{q,v}v_j + \mathcal{D}_x\mathcal{F}^{q,q}q_j\right]$$

$$\mathcal{M}\left(e^{i\omega\Delta t} - 1\right)q_j^n = -\frac{\Delta t}{\Delta x}\left[\mathcal{D}_x\mathcal{F}^{q,v}v_j + \mathcal{D}_x\mathcal{F}^{q,q}q_j\right]$$

Defining $\mathcal{D}_t = e^{i\omega\Delta t} - 1$

$$\mathcal{M}\mathcal{D}_t q_j^n = -\frac{\Delta t}{\Delta x}\left[\mathcal{D}_x\mathcal{F}^{q,v}v_j + \mathcal{D}_x\mathcal{F}^{q,q}q_j\right]$$

$$\mathcal{M}\mathcal{D}_t q_j^n = -\frac{\Delta t}{\Delta x}\left[\mathcal{D}_x\mathcal{F}^{q,v}v_j + \mathcal{D}_x\mathcal{F}^{q,q}q_j\right]$$

$$\frac{\mathcal{D}_t}{\Delta t}q_j^n = -\frac{\mathcal{D}_x}{\mathcal{M}\Delta x}\left[\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j\right]$$

$$\frac{\mathcal{D}_t}{\Delta t}q_j^n = -\left[\frac{\mathcal{D}_x}{\mathcal{M}\Delta x}\mathcal{F}^{q,v}v_j + \frac{\mathcal{D}_x}{\mathcal{M}\Delta x}\mathcal{F}^{q,q}q_j\right]$$

So as we've said we have an approximation to the time derivative on the left and an approximation to the space derivative on the right. However, this is a bit muddied by the fact that in the numerical methods $\mathcal{F}^{q,v}v_j$ does not have an explicit time relation, while for the analytic one it does, for example the continuity equation under this scheme, substituting the analytic experssion will be

$$\frac{\mathcal{D}_t}{\Delta t}h_j^n = -\left[\frac{\mathcal{D}_x}{\mathcal{M}\Delta x}\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} Hu_{j+1/2}\right]$$

$$\frac{\mathcal{D}_t}{\Delta t} h_j^n = -\left[\frac{\mathcal{D}_x}{\mathcal{M}\Delta x}\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} He^{ik\frac{\Delta x}{2}}u_j\right]$$

here I just use $u = u_0 e^{i(\omega t + kx)}$

$$\frac{\mathcal{D}_t}{\Delta t} h_j^n = -\left[\frac{\mathcal{D}_x}{\mathcal{M}\Delta x} e^{ik\frac{\Delta x}{2}} H u_0 e^{ikx_j} \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} e^{i\omega t}\right]$$

$$\frac{\mathcal{D}_t}{\Delta t} h_j^n = -\left[\frac{\mathcal{D}_x}{\mathcal{M}\Delta x} e^{ik\frac{\Delta x}{2}} H u_0 e^{ikx_j} \frac{1}{\Delta t}\left[\frac{1}{i\omega} e^{i\omega t}\right]_{t_n}^{t_{n+1}}\right]$$

$$\frac{\mathcal{D}_t}{\Delta t} h_j^n = -\left[\frac{\mathcal{D}_x}{\mathcal{M}\Delta x} e^{ik\frac{\Delta x}{2}} H u_0 e^{ikx_j} \frac{1}{i\omega}\frac{1}{\Delta t}\left[e^{i\omega t_{n+1}} - e^{i\omega t_n}\right]\right]$$

$$\frac{\mathcal{D}_t}{\Delta t} h_j^n = -\left[\frac{\mathcal{D}_x}{\mathcal{M}\Delta x} e^{ik\frac{\Delta x}{2}} H u_0 e^{ikx_j + i\omega t_n} \frac{1}{i\omega}\frac{1}{\Delta t}\left[e^{i\omega \Delta t} - 1\right]\right]$$

$$\frac{\mathcal{D}_t}{\Delta t} h_j^n = -\left[\frac{\mathcal{D}_x}{\mathcal{M}\Delta x} e^{ik\frac{\Delta x}{2}} H u_j^n \frac{1}{i\omega}\frac{\mathcal{D}_t}{\Delta t}\right]$$

$$i\omega h_j^n = -\left[\frac{\mathcal{D}_x}{\mathcal{M}\Delta x} e^{ik\frac{\Delta x}{2}} H u_j^n\right]$$

$$i\omega h_j^n = -\left[\frac{1 - e^{-ik\Delta x}}{\frac{2}{k\Delta x}\sin\left(k\frac{\Delta x}{2}\right)\Delta x} e^{ik\frac{\Delta x}{2}} H u_j^n\right]$$

$$i\omega h_j^n = -\left[k\frac{e^{ik\frac{\Delta x}{2}} - e^{-ik\frac{\Delta x}{2}}}{2\sin\left(k\frac{\Delta x}{2}\right)} H u_j^n\right]$$

$$i\omega h_j^n = -\left[k\frac{2i\sin\left(k\frac{\Delta x}{2}\right)}{2\sin\left(k\frac{\Delta x}{2}\right)} H u_j^n\right]$$

$$i\omega h_j^n = -ikHu_j^n$$

As desired. The problem is that when we use a numerical method we end up losing time accuracy as a result of the approximation of the flux. Does this clear it up Chris?