Dispersive Shock Waves of the Serre equations

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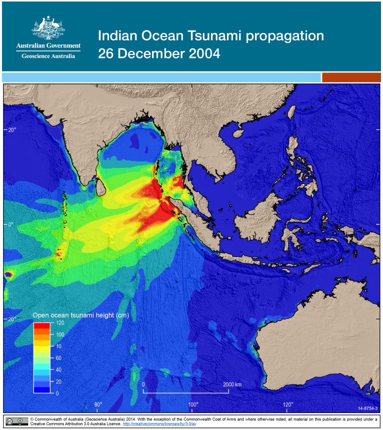
Outline of the Presentation

- Motivation
- Serre Equations
- Dispersive Shock Waves
- Numerical Experiment
- Results

Our Background

- Interest : Numerical methods for water waves. Focusing on ocean hazards.

Indian ocean tsunami



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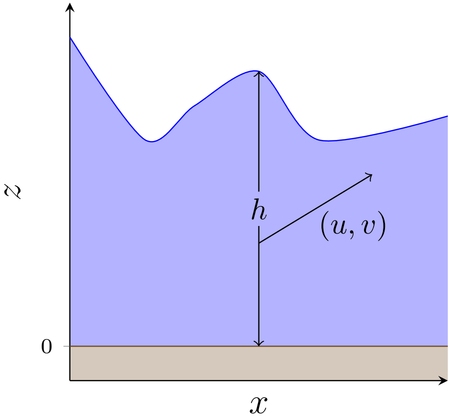
- Resulted In : Robust numerical method for the Shallow Water Wave Equations (ANUGA)

Depth Averaged Equations



Shallow Water Wave Equations

Conservation of Mass

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0$$

Conservation of Momentum

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2}\right) = 0$$

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- Current Goal : Robust numerical method for the Serre equations

Serre equations

Conservation of Mass

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0$$

Conservation of Momentum

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^3}{3}\Phi\right) = 0$$

$$\Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}$$

Our Background

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- Current Goal : Robust numerical method for the Serre equations
- Problem : Evolution of shocks for the Serre equations

Dispersive Shock Waves

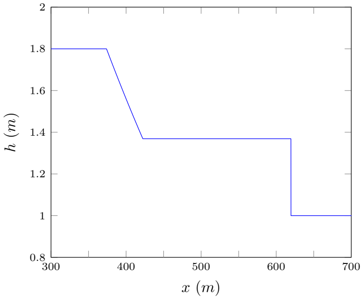


Figure: Shock Wave (analytical solution of the shallow water wave equations).

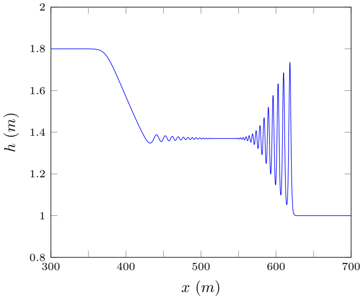


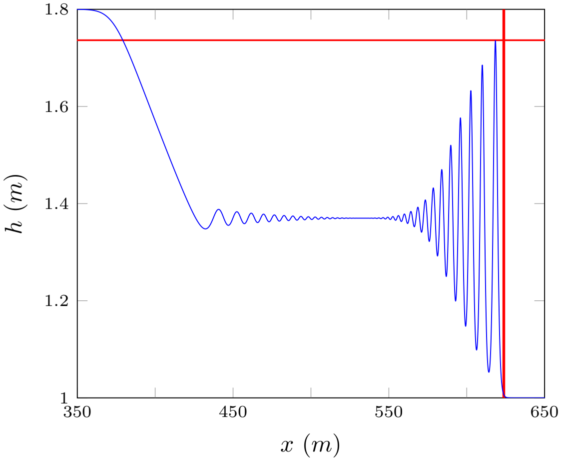
Figure: Dispersive Shock Wave (numerical solution of the Serre equations).

Properties of DSW for the Serre Equations

Asymptotic results for long times

- Whitham modulation results for leading wave amplitude and speed

Whitham Modulation Results



Properties of DSW for the Serre Equations

Asymptotic results for long times

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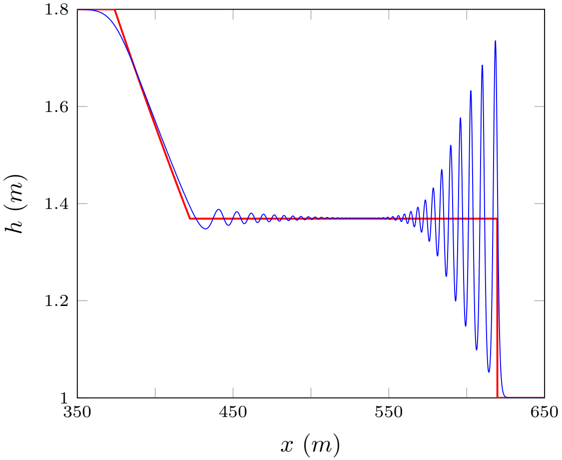
- Oscillations of the DSW for the Serre equations oscillate around the SW of the SWWE

DSW comparison to SW



Properties of DSW for the Serre Equations

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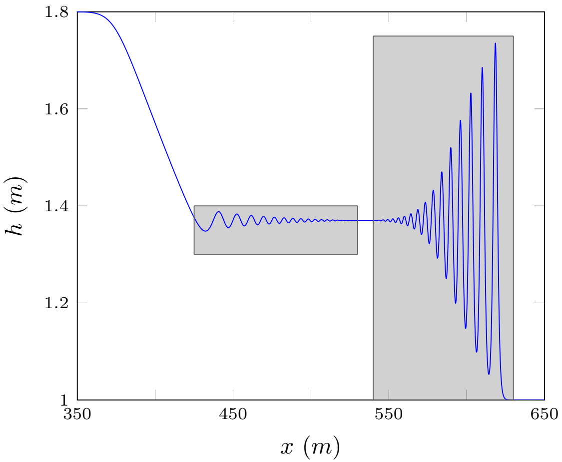
Linear results

- Separate dispersive tails

Separation of Dispersive Tails



Problem

No analytic solution of the Serre equations for DSW

Numerical Solutions

- Few numerical solutions in the literature for DSW
- Most common numerical solutions are for the dam-break problem or a smooth approximation to it

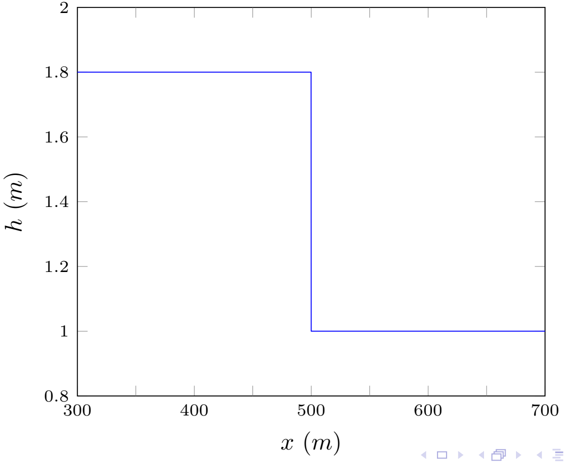
Model Problem : Dam Break Problem

$$h(x,0) = \begin{cases} h_1 & x \leq x_0 \\ h_0 & x > x_0 \end{cases}$$

$$u(x,0) = 0.0.$$

Dam Break Problem Example

$h_0 = 1m$, $h_1 = 1.8m$ and $x_0 = 500m$



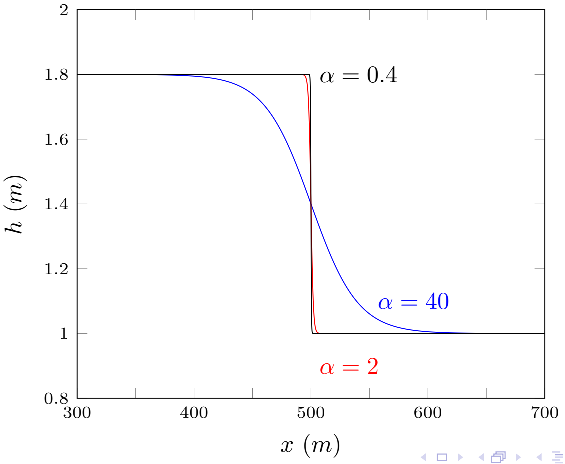
Smoothed Approximation : Smoothed Dam Break Problem

$$h(x,0) = h_0 + \frac{h_1 - h_0}{2}\left(1 + \tanh\left(\frac{x_0 - x}{\alpha}\right)\right)$$

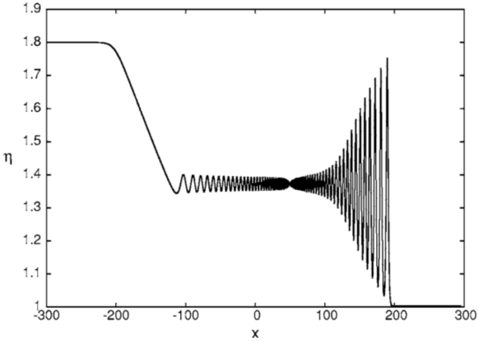
$$u(x,0) = 0.0.$$

Smoothed Dam Break Problem Example

$h_0 = 1m$, $h_1 = 1.8m$ and $x_0 = 500m$

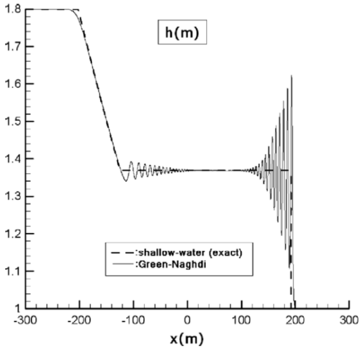


Grimshaw's Results (2006)



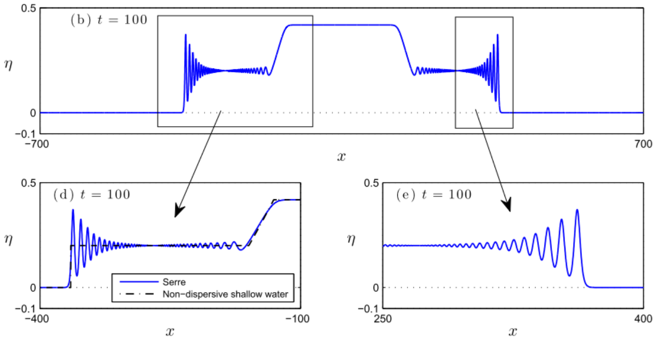
Numerical solution of second-order finite difference method for a smooth approximation to the dam break problem

Hanks Results (2010)



Numerical solution of first-order hybrid finite difference volume method for dam break problem

Mitsotakis Results (2014)



Numerical solution of fourth-order finite element method for smoothed dam break problem ($\alpha = 1$)

Problem

Numerical results in the literature have different behaviours, although most publications report the completely separated dispersive tails of Hank and Mitsotakis[]
Questions:

- Which behaviour is correct?
- What is the effect of the numerical method?
- What is the effect of the smoothing of the dam-break problem?

Aim

Investigate numerical solutions of the Serre equations to the smoothed dam-break problem with various

- values of α
- numerical methods

Methods

Finite Difference Methods

- Naive second-order centered finte difference
- Finite difference of Grimshaw[]

Hybrid Finite Difference Volume Methods

- First-order (same method as Hank[])
- Second-order
- Third-order

Observed Behaviours

We observed and justified four different behaviours of the numerical solution for the DSW

Main cause of the different behaviours was the α value

We demonstrate the different observed behaviours here using the third-order hybrid method's highest resolution numerical solution for a particular α value

Non Oscillatory Structure $\alpha = 40$

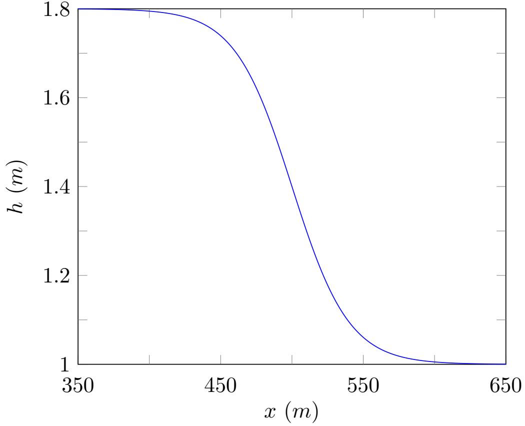


Figure: Initial conditions

Non Oscillatory Structure $\alpha = 40$

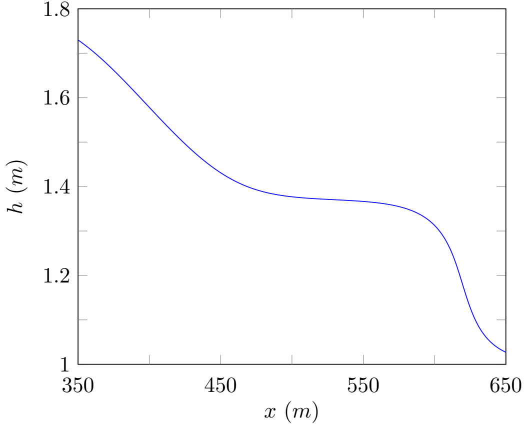
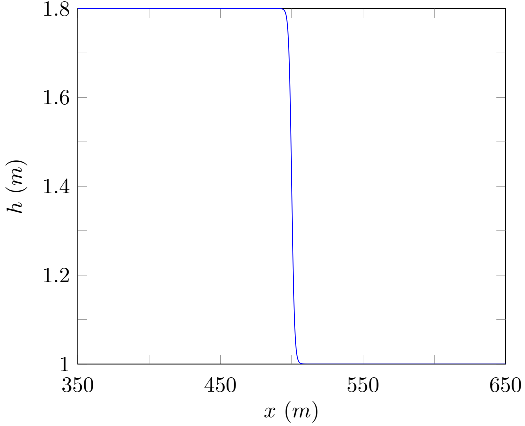


Figure: Highest resolution third-order numerical solution at $t = 30s$

Flat Structure $\alpha = 2$



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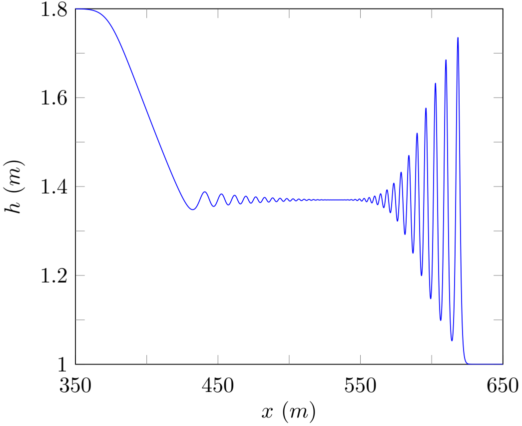
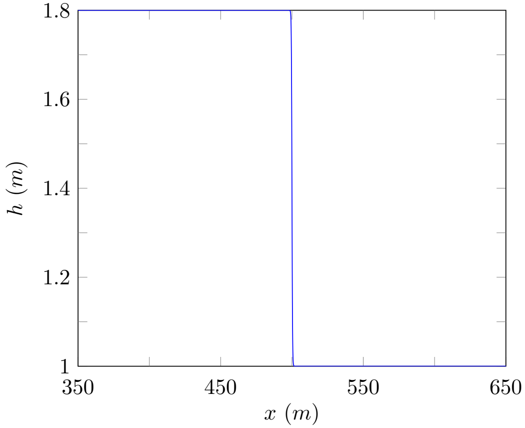


Figure: Highest resolution third-order numerical solution at $t = 30s$

Node Structure $\alpha = 0.4$



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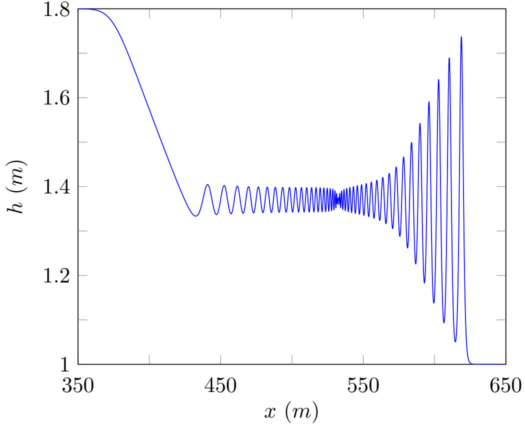

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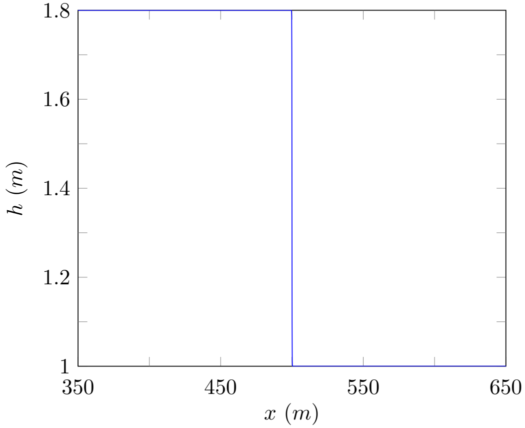
Growth Structure $\alpha = 0.1$



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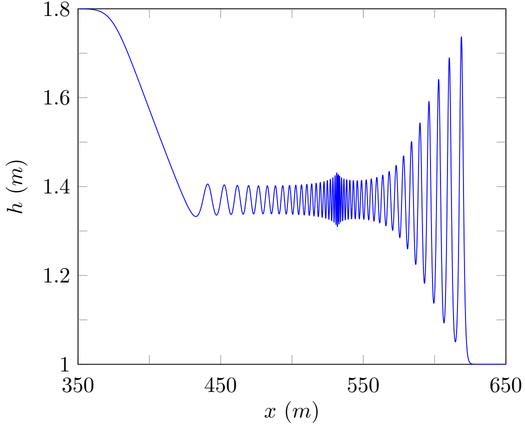

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Justifying These Numerical Solutions

For a particular numerical method and α value:

- Demonstrated convergence as the resolution of the method increases
- Demonstrated numerical solutions conserve mass, momentum and the Hamiltonian

Answers

Questions:

- Which behaviour is correct?
- What is the effect of the numerical method?
- What is the effect of the smoothing of the dam-break problem?

Which behaviour is correct?

- Depends on the α value.
- These results demonstrate that for solving the dam-break problem we expect the growth structure for short time periods.
- For longer time periods as time increases the growth structure decays to the node structure which then decays to the flat structure.

What is the effect of the numerical method?

- Diffusive first-order methods limit the observable behaviours with reasonable resolutions. In particular we will not get the growth structure for the DSW of the dam-break problem unless we use very fine resolutions.
- All higher-order methods reproduce all the observed behaviours
- Hybrid finite difference volume methods more robust than the finite difference methods for small α values

What is the effect of the smoothing of the dam-break problem?

- Most important factor determining the observed behaviour
- Observed behaviour sensitive to the smoothing of the problem

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Asymptotic results [] (long time solutions)

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Linear results []

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Conclusion

- Explained the differences in behaviour for numerical solutions published in the literature
- Found new behaviour of DSW for short time spans not previously published in the literature
- Good agreement between numerical solutions and known properties of DSW for long time periods
- Justified the robustness of the hybrid finite difference volume methods