1 Linearised Equations

We begin with the linearised equations from Chris's thesis/papers.

continuity:

$$\frac{\partial h_1}{\partial t} + h_0 \frac{\partial u_1}{\partial x} + u_0 \frac{\partial h_1}{\partial x} = 0$$

velocity:

$$\frac{\partial u_1}{\partial t} + g \frac{\partial h_1}{\partial x} + u_0 \frac{\partial u_1}{\partial x} - \frac{h_0^2}{3} \left(u_0 \frac{\partial^3 u_1}{\partial x^3} + \frac{\partial^3 u_1}{\partial x^3 \partial t} \right) = 0$$

Also G

$$G = u_0 h_0 + u_0 h_1 + h_0 u_1 - \frac{h_0^3}{3} \frac{\partial^2 u_1}{\partial x^2}$$

Now in the Fillipine paper, we assume the water is still (except for the pertubations) so that $u_0 = 0$ thus we get:

$$\frac{\partial h_1}{\partial t} + h_0 \frac{\partial u_1}{\partial x} = 0$$

$$\frac{\partial u_1}{\partial t} + g \frac{\partial h_1}{\partial x} - \frac{h_0^2}{3} \frac{\partial^3 u_1}{\partial x^3 \partial t} = 0$$

$$G = h_0 u_1 - \frac{h_0^3}{3} \frac{\partial^2 u_1}{\partial x^2}$$

Importantly by multiplying the velocity by h_0 to get the momentum equation we have

$$\frac{\partial u_1}{\partial t} h_0 + g \frac{\partial h_1}{\partial x} h_0 - \frac{h_0^3}{3} \frac{\partial^3 u_1}{\partial x^3 \partial t} = 0$$

and thus

$$\frac{\partial G}{\partial t} + g \frac{\partial h_1}{\partial x} h_0 = 0$$

So we finally have

$$\frac{\partial h_1}{\partial t} + h_0 \frac{\partial u_1}{\partial x} = 0$$

$$\frac{\partial G}{\partial t} + g\frac{\partial h_1}{\partial x}h_0 = 0$$

$$G = h_0 u_1 - \frac{h_0^3}{3}\frac{\partial^2 u_1}{\partial x^2}$$

For convenience I will make the following notational changes $H = h_0$, $h = h_1$ and $u = u_1$. So that

$$\frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial G}{\partial t} + gH\frac{\partial h}{\partial x} = 0$$

$$G = Hu - \frac{H^3}{3}\frac{\partial^2 u}{\partial x^2}$$

2 Dispersion Error

To perform the dispersion error we replace both h and u by fourier modes, which for some quantity q is given by so that $q(x,t) = q(0,0)e^{i(\omega t + kx)}$. The use of this comes when we use our uniform grid in space, so that $q(x_j,t) = q_j$ then $q_{j\pm l} = q_j e^{\pm ikl\Delta x}$ and we use this to work with just a point and then factors.

2.1 Elliptic Equation

I will start by analysing the dispersion error on the elliptic equation

$$G_j = Hu_j - \frac{H^3}{3}\left(\frac{\partial^2 u}{\partial x^2}\right)_j$$

For the different order schemes we approximate the derivative of u differently. For the first and second order method we use second order central differencing so that

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}$$

for the third order method we use 4th order central differencing so that

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_j = \frac{-u_{j+2} + 16u_{j+1} - 30u_j + 16u_{j-1} - u_{j-2}}{12\Delta x^2}$$

Using our relations from above for the second order central difference

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_j = \frac{u_j e^{ik\Delta x} - 2u_j + u_j e^{-ik\Delta x}}{\Delta x^2}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_j = \frac{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}{\Delta x^2} u_j$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_j = \frac{2\cos(k\Delta x) - 2}{\Delta x^2} u_j$$

We introduce the following notation:

$$\mathcal{C}_2 = \frac{2\cos(k\Delta x) - 2}{\Delta x^2}$$

thus

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_j = \mathcal{C}_2 u_j$$

This agrees with the a result in the Fillipine paper.
For the fourth order central difference

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_j = \frac{-u_j e^{2ik\Delta x} + 16u_j e^{ik\Delta x} - 30u_j + 16u_j e^{-ik\Delta x} - u_j e^{-2ik\Delta x}}{12\Delta x^2}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_j = \frac{-e^{2ik\Delta x} + 16e^{ik\Delta x} - 30 + 16e^{-ik\Delta x} - e^{-2ik\Delta x}}{12\Delta x^2} u_j$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_j = \frac{-2\cos(2k\Delta x) + 32\cos(k\Delta x) - 30}{12\Delta x^2} u_j$$

We introduce the following notation:

$$\mathcal{C}_4 = \frac{-2\cos(2k\Delta x) + 32\cos(k\Delta x) - 30}{12\Delta x^2}$$

thus

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_j = \mathcal{C}_4 u_j$$

An easy way to check this is to see how well this relaiton holds $k^2 \approx \mathcal{C}_4$ as well as what happens as $\Delta x \to 0$. Both of these check out for these approximations so we continue.

Thus

$$G_j = Hu_j - \frac{H^3}{3}\mathcal{C}u_j$$

(we leave the order ambigious here)

Thus

$$G_j = \left[H - \frac{H^3}{3}\mathcal{C}\right] u_j$$

We define:

$$\mathcal{G} = \left[H - \frac{H^3}{3}\mathcal{C}\right]$$

In particular for the first($\mathcal{G}_1$) and second($\mathcal{G}_2$) order we use $\mathcal{C}_2$, while for the third ($\mathcal{G}_3$) order we use $\mathcal{C}_4$ as $\mathcal{C}$.

Thus

$$G_j = \mathcal{G}u_j$$

2.2 Conservation Equations

$$\frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial G}{\partial t} + gH\frac{\partial h}{\partial x} = 0$$

Our analysis has continuous time, so we can do this time derivatives properly

$$i\omega h + H\frac{\partial u}{\partial x} = 0$$

Since G is some factor times u we can do this to G as well [pretty lax on this, but yes it would, we could derive a space continuous factor similar to above]

$$i\omega G_j + gH\frac{\partial h}{\partial x} = 0$$

For our method we approximate the flux, and not the individual derivative terms so we get

$$i\omega h_j + \frac{1}{\Delta x}\left[F^h_{j+\frac{1}{2}} - F^h_{j-\frac{1}{2}}\right] = 0$$

$$i\omega G_j + \frac{1}{\Delta x}\left[F^u_{j+\frac{1}{2}} - F^u_{j-\frac{1}{2}}\right] = 0$$

where F^h and F^u are given by Kurganovs method. WE also use the cell averages for that time derivative term, so that

$$i\omega \mathcal{G}\bar{u} + gH\frac{\partial h}{\partial x} = 0$$

For our method we approximate the flux, and not the individual derivative terms so we get

$$i\omega \bar{h}_j + \frac{1}{\Delta x}\left[F^h_{j+\frac{1}{2}} - F^h_{j-\frac{1}{2}}\right] = 0$$

$$i\omega \bar{G}_j + \frac{1}{\Delta x}\left[F^u_{j+\frac{1}{2}} - F^u_{j-\frac{1}{2}}\right] = 0$$

[should have made this clear earlier, since the average is an integral it comes out as a factor anyway, but what we are interested in is the numerical factor from conversion].

For the first and second order methods this distinction is trivial, for the third order method we have the following relation

$$q_j = \frac{-\bar{q}_{j+1} + 26\bar{q}_j - \bar{q}_{j-1}}{24}$$

$$q_j = \bar{q}_j \frac{-e^{ik\Delta x} + 26 - e^{-ik\Delta x}}{24}$$

$$q_j = \bar{q}_j \frac{26 - 2\cos(k\Delta x)}{24}$$

Defining

$$\mathcal{M}_3 = \frac{26 - 2\cos(k\Delta x)}{24}$$

We again will suppress order subscripts further on, but we also have $\mathcal{M}_1 = \mathcal{M}_2 = 1$. So we have

$$i\omega\mathcal{M}h_j + \frac{1}{\Delta x}\left[F^h_{j+\frac{1}{2}} - F^h_{j-\frac{1}{2}}\right] = 0$$

$$i\omega\mathcal{M}\mathcal{G}u_j + \frac{1}{\Delta x}\left[F^u_{j+\frac{1}{2}} - F^u_{j-\frac{1}{2}}\right] = 0$$

Now the only thing that changes between the different orders for the calculation of the flux is the reconstruction process.

2.2.1 Reconstruction

For the first order method

$$q^+_{j+1/2} = q_{j+1} = e^{ik\Delta x}q_j$$

$$q^-_{j+1/2} = q_j$$

So we define $\mathcal{R}^+_1 = e^{ik\Delta x}$ and $\mathcal{R}^-_1 = 1$.

For the second order method we have

$$q^-_{j+1/2} = q_j + \frac{q_{j+1} - q_{j-1}}{4}$$

$$q^+_{j+1/2} = q_{j+1} + \frac{q_{j+2} - q_j}{4}$$

Applying our fourier mode

$$q^-_{j+1/2} = q_j + \frac{q_j e^{ik\Delta x} - q_j e^{-ik\Delta x}}{4}$$

$$q^-_{j+1/2} = q_j\left(1 + \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{4}\right)$$

$$q^-_{j+1/2} = q_j\left(1 + \frac{2i\sin(k\Delta x)}{4}\right)$$

$$q^-_{j+1/2} = q_j\left(1 + \frac{i\sin(k\Delta x)}{2}\right)$$

for the plus we get the same result with a shift so that (because its around j+1) and a minus

$$q_{j+1/2}^{+} = q_j e^{ik\Delta x}\left(1 - \frac{i\sin(k\Delta x)}{2}\right)$$

So we have that

$$\mathcal{R}_2^- = 1 + \frac{i\sin(k\Delta x)}{2}$$

$$\mathcal{R}_2^+ = e^{ik\Delta x}\left(1 - \frac{i\sin(k\Delta x)}{2}\right)$$

For the third order method we have

$$q_{j+1/2}^{-} = \bar{q}_j + \frac{1}{6}\left(\bar{q}_j - \bar{q}_{j-1}\right) + \frac{1}{3}\left(\bar{q}_{j+1} - \bar{q}_j\right)$$

$$q_{j+1/2}^{+} = \bar{q}_{j+1} - \frac{1}{3}\left(\bar{q}_{j+1} - \bar{q}_j\right) - \frac{1}{6}\left(\bar{q}_{j+2} - \bar{q}_{j+1}\right)$$

So we have

$$q_{j+1/2}^{-} = \mathcal{M}_3\left[q_j + \frac{1}{6}\left(q_j - q_{j-1}\right) + \frac{1}{3}\left(q_{j+1} - q_j\right)\right]$$

$$q_{j+1/2}^{-} = \mathcal{M}_3 q_j\left[1 + \frac{1}{6}\left(1 - e^{-ik\Delta x}\right) + \frac{1}{3}\left(e^{ik\Delta x} - 1\right)\right]$$

$$q_{j+1/2}^{-} = \mathcal{M}_3 q_j\left[\frac{5}{6} + \frac{1}{6}\left(-e^{-ik\Delta x}\right) + \frac{1}{3}\left(e^{ik\Delta x}\right)\right]$$

$$q_{j+1/2}^{-} = \frac{\mathcal{M}_3}{6}\left[5 + -e^{-ik\Delta x} + 2e^{ik\Delta x}\right] q_j$$

So defining

$$R_3^- = \frac{\mathcal{M}_3}{6}\left[5 + -e^{-ik\Delta x} + 2e^{ik\Delta x}\right]$$

for plus

$$q_{j+1/2}^{+} = \mathcal{M}_3\left[q_{j+1} - \frac{1}{3}\left(q_{j+1} - q_j\right) - \frac{1}{6}\left(q_{j+2} - q_{j+1}\right)\right]$$

$$q_{j+1/2}^{+} = \mathcal{M}_3\left[1 - \frac{1}{3}\left(1 - e^{-k\Delta x}\right) - \frac{1}{6}\left(e^{k\Delta x} - 1\right)\right] q_{j+1}$$

$$q_{j+1/2}^{+} = \mathcal{M}_3\left[\frac{5}{6} + \frac{1}{3}e^{-k\Delta x} - \frac{1}{6}e^{k\Delta x}\right] q_{j+1}$$

$$q^+_{j+1/2} = \frac{\mathcal{M}_3}{6}\left[5 + 2e^{-k\Delta x} - e^{k\Delta x}\right] q_{j+1}$$

$$q^+_{j+1/2} = \frac{\mathcal{M}_3 e^{ik\Delta x}}{6}\left[5 + 2e^{-ik\Delta x} - e^{ik\Delta x}\right] q_j$$

Defining

$$R^+_3 = \frac{\mathcal{M}_3 e^{ik\Delta x}}{6}\left[5 + 2e^{-ik\Delta x} - e^{ik\Delta x}\right]$$

So for the reconstruction we have

$$q^-_{j+1/2} = \mathcal{R}^- q_j$$
$$q^+_{j+1/2} = \mathcal{R}^+ q_j$$

Actually, we do reconstruction of u differently.
For the first and second order method

$$u^-_{j+1/2} = u^+_{j+1/2} = \frac{u_{j+1} + u_j}{2}$$

$$u^-_{j+1/2} = \frac{u_j e^{ik\Delta x} + u_j}{2} = \frac{e^{ik\Delta x} + 1}{2} u_j$$

so we define

$$R^u_2 = \frac{e^{ik\Delta x} + 1}{2}$$

For the third order method we use

$$u^-_{j+1/2} = u^+_{j+1/2} = \frac{-3u_{j+2} + 27u_{j+1} + 27u_j - 3u_{j-1}}{48}$$

$$u^-_{j+1/2} = \frac{-3e^{2ik\Delta x} + 27e^{ik\Delta x} + 27 - 3e^{-ik\Delta x}}{48} u_j$$

[We could probably do something smarter here] so we define

$$R^u_3 = \frac{-3e^{2ik\Delta x} + 27e^{ik\Delta x} + 27 - 3e^{-ik\Delta x}}{48}$$

2.2.2 Kurganovs Method

Up to the order of our linearisation we can just use the background wavespeed, instead of the wavespeed at a point so that

$$a^{-}_{j+1/2} = -\sqrt{gH}$$

$$a^{+}_{j+1/2} = \sqrt{gH}$$

We have that

$$F_{i+\frac{1}{2}} = \frac{a^{+}_{i+\frac{1}{2}} f\left(q^{-}_{i+\frac{1}{2}}\right) - a^{-}_{i+\frac{1}{2}} f\left(q^{+}_{i+\frac{1}{2}}\right)}{a^{+}_{i+\frac{1}{2}} - a^{-}_{i+\frac{1}{2}}} + \frac{a^{+}_{i+\frac{1}{2}}\, a^{-}_{i+\frac{1}{2}}}{a^{+}_{i+\frac{1}{2}} - a^{-}_{i+\frac{1}{2}}} \left[q^{+}_{i+\frac{1}{2}} - q^{-}_{i+\frac{1}{2}}\right] \tag{1}$$

$$F_{i+\frac{1}{2}} = \frac{\left(\sqrt{gH}\right) f\left(q^{-}_{i+\frac{1}{2}}\right) - \left(-\sqrt{gH}\right) f\left(q^{+}_{i+\frac{1}{2}}\right)}{\left(\sqrt{gH}\right) - \left(-\sqrt{gH}\right)} + \frac{\left(\sqrt{gH}\right)\,\left(-\sqrt{gH}\right)}{\left(+\sqrt{gH}\right) - \left(-\sqrt{gH}\right)} \left[q^{+}_{i+\frac{1}{2}} - q^{-}_{i+\frac{1}{2}}\right] \tag{2}$$

$$F_{i+\frac{1}{2}} = \frac{\left(+\sqrt{gH}\right) f\left(q^{-}_{i+\frac{1}{2}}\right) - \left(-\sqrt{gH}\right) f\left(q^{+}_{i+\frac{1}{2}}\right)}{2\sqrt{gH}} + \frac{\left(+\sqrt{gH}\right)\,\left(-\sqrt{gH}\right)}{2\sqrt{gH}} \left[q^{+}_{i+\frac{1}{2}} - q^{-}_{i+\frac{1}{2}}\right] \tag{3}$$

$$F_{i+\frac{1}{2}} = \frac{\left(+\sqrt{gH}\right) f\left(q^{-}_{i+\frac{1}{2}}\right) - \left(-\sqrt{gH}\right) f\left(q^{+}_{i+\frac{1}{2}}\right)}{2\sqrt{gH}} + \frac{-gH}{2\sqrt{gH}} \left[q^{+}_{i+\frac{1}{2}} - q^{-}_{i+\frac{1}{2}}\right] \tag{4}$$

$$F_{i+\frac{1}{2}} = \frac{f\left(q^{-}_{i+\frac{1}{2}}\right) + f\left(q^{+}_{i+\frac{1}{2}}\right)}{2} - \frac{\sqrt{gH}}{2} \left[q^{+}_{i+\frac{1}{2}} - q^{-}_{i+\frac{1}{2}}\right] \tag{5}$$

$$F_{j+\frac{1}{2}} = \frac{f\left(q^-_{j+\frac{1}{2}}\right) + f\left(q^+_{j+\frac{1}{2}}\right)}{2} - \frac{\sqrt{gH}}{2}\left[q^+_{j+\frac{1}{2}} - q^-_{j+\frac{1}{2}}\right] \quad (6)$$

For the mass equation $f = Hu$

$$F_{j+\frac{1}{2}} = \frac{Hu^-_{j+\frac{1}{2}} + Hu^+_{j+\frac{1}{2}}}{2} - \frac{\sqrt{gH}}{2}\left[h^+_{j+\frac{1}{2}} - h^-_{j+\frac{1}{2}}\right] \quad (7)$$

$$F_{j+\frac{1}{2}} = \frac{H\mathcal{R}^u u_j + H\mathcal{R}^u u_j}{2} - \frac{\sqrt{gH}}{2}\left[\mathcal{R}^+ h_j - \mathcal{R}^- h_j\right] \quad (8)$$

$$F_{j+\frac{1}{2}} = H\mathcal{R}^u u_j - \frac{\sqrt{gH}}{2}\left[\mathcal{R}^+ - \mathcal{R}^-\right] h_j \quad (9)$$

From this we define

$$\mathcal{F}^{h,u} = H\mathcal{R}^u$$

$$\mathcal{F}^{h,h} = -\frac{\sqrt{gH}}{2}\left[\mathcal{R}^+ - \mathcal{R}^-\right]$$

So that

$$F_{j+\frac{1}{2}} = \mathcal{F}^{h,u} u_j + \mathcal{F}^{h,h} h_j \quad (10)$$

For the momentum equation $f = gHh$

$$F_{j+\frac{1}{2}} = \frac{gHh^-_{j+\frac{1}{2}} + gHh^+_{j+\frac{1}{2}}}{2} - \frac{\sqrt{gH}}{2}\left[G^+_{j+\frac{1}{2}} - G^-_{j+\frac{1}{2}}\right] \quad (11)$$

Because we reconstruct G, we dont need to use u's reconstruction, just the old one.

$$F_{j+\frac{1}{2}} = \frac{gH\mathcal{R}^{-}h_j + gH\mathcal{R}^{+}h_j}{2} - \frac{\sqrt{gH}}{2}\left[\mathcal{R}^{+}G_j - \mathcal{R}^{-}G_j\right] \quad (12)$$

$$F_{j+\frac{1}{2}} = \frac{gH\mathcal{R}^{-}h_j + gH\mathcal{R}^{+}h_j}{2} - \frac{\sqrt{gH}}{2}\left[\mathcal{R}^{+}\mathcal{G}u_j - \mathcal{R}^{-}\mathcal{G}u_j\right] \quad (13)$$

$$F_{j+\frac{1}{2}} = \frac{gH\mathcal{R}^{-} + gH\mathcal{R}^{+}}{2}h_j - \frac{\sqrt{gH}}{2}\mathcal{G}\left[\mathcal{R}^{+} \ - \mathcal{R}^{-}\right]u_j \quad (14)$$

From this we define

$$\mathcal{F}^{u,u} = -\frac{\sqrt{gH}}{2}\mathcal{G}\left[\mathcal{R}^{+} \ - \mathcal{R}^{-}\right]$$

$$\mathcal{F}^{u,h} = \frac{gH\mathcal{R}^{-} + gH\mathcal{R}^{+}}{2}$$

So we have

$$F_{j+\frac{1}{2}} = \mathcal{F}^{u,u}u_j + \mathcal{F}^{u,h}h_j \quad (15)$$

To get the flux at $j-1/2$ we just shift everything back so we pick up one factor

2.2.3 Solving

So our equations become

for mass

$$i\omega\mathcal{M}h_j + \frac{1}{\Delta x}\left[\mathcal{F}^{h,u}u_j + \mathcal{F}^{h,h}h_j - e^{-ik\Delta x}\mathcal{F}^{h,u}u_j - e^{-ik\Delta x}\mathcal{F}^{h,h}h_j\right] = 0$$

$$\left[i\omega\mathcal{M} + \frac{1}{\Delta x}\mathcal{F}^{h,h} - e^{-ik\Delta x}\mathcal{F}^{h,h}\frac{1}{\Delta x}\right]h_j + \frac{1}{\Delta x}\left[\mathcal{F}^{h,u} - e^{-ik\Delta x}\mathcal{F}^{h,u}\right]u_j = 0$$

Defining $\mathcal{D} = 1 - e^{-ik\Delta x}$

$$\left[i\omega\mathcal{M} + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}^{h,h}\right] h_j + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}^{h,u} u_j = 0$$

for momentum

$$i\omega\mathcal{M}\mathcal{G}u_j + \frac{1}{\Delta x}\left[\mathcal{D}\mathcal{F}^{u,u}u_j + \mathcal{D}\mathcal{F}^{u,h}h_j\right] = 0$$

$$\left(i\omega\mathcal{M}\mathcal{G} + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}^{u,u}\right) u_j + \frac{1}{\Delta x}\left[\mathcal{D}\mathcal{F}^{u,h}\right] h_j = 0$$

So we have

$$\left[\begin{array}{cc} i\omega\mathcal{M} + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}^{h,h} & \frac{1}{\Delta x}\mathcal{D}\mathcal{F}^{h,u} \\ \frac{1}{\Delta x}\mathcal{D}\mathcal{F}^{u,h} & i\omega\mathcal{M}\mathcal{G} + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}^{u,u} \end{array}\right]\left[\begin{array}{c} h_j \\ u_j \end{array}\right] = 0$$

The nontrivial solutions are given when

$$\left[i\omega\mathcal{M} + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}^{h,h}\right]\left[i\omega\mathcal{M}\mathcal{G} + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}^{u,u}\right] - \frac{1}{\Delta x}\mathcal{D}\mathcal{F}^{h,u}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}^{u,h} = 0$$

$$-\omega^2\mathcal{M}^2\mathcal{G} + i\omega\mathcal{M}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}^{u,u} + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}^{h,h} i\omega\mathcal{M}\mathcal{G} + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}^{h,h}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}^{u,u} - \frac{1}{\Delta x^2}\mathcal{D}^2\mathcal{F}^{h,u}\mathcal{F}^{u,h} = 0$$

$$-\mathcal{M}^2\mathcal{G}\omega^2 + i\mathcal{M}\frac{1}{\Delta x}\mathcal{D}\left(\mathcal{F}^{u,u} + \mathcal{F}^{h,h}\mathcal{G}\right)\omega + \frac{1}{\Delta x^2}\mathcal{D}^2\left(\mathcal{F}^{h,h}\mathcal{F}^{u,u} - \mathcal{F}^{h,u}\mathcal{F}^{u,h}\right) = 0$$

We solve this quadratic with a program.