Importance of Dispersion for Shoaling Waves

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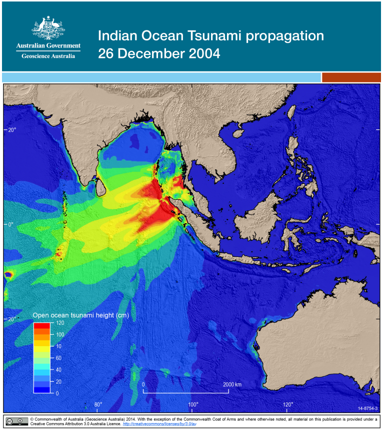
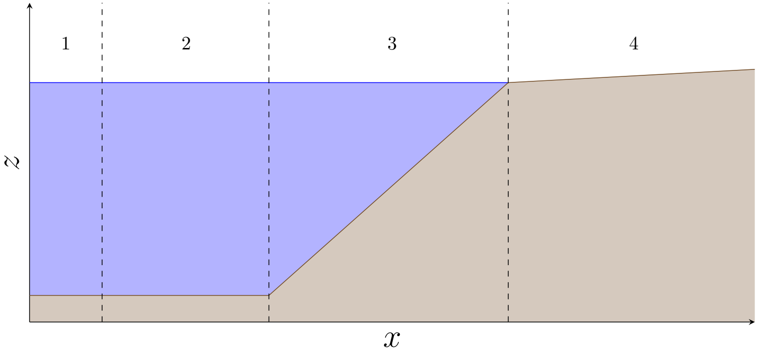
Introduction

- Motivation : Tsunamis
- Model : Shallow Water Wave and Serre equations
- Experiment : Comparison of numerical solutions

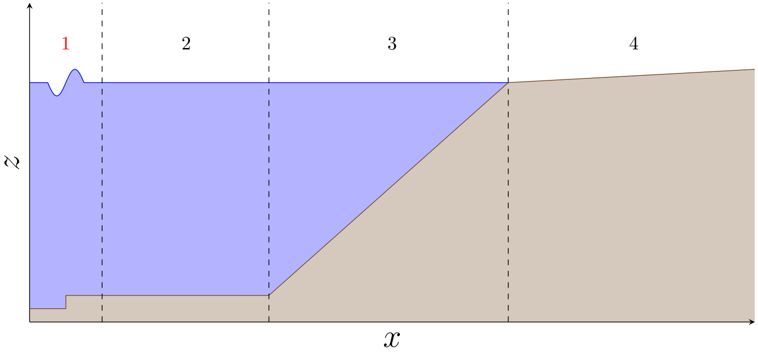
Indian ocean tsunami



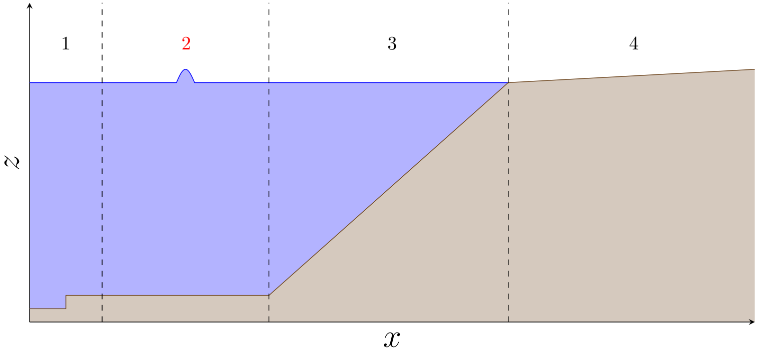
Tsunami diagram



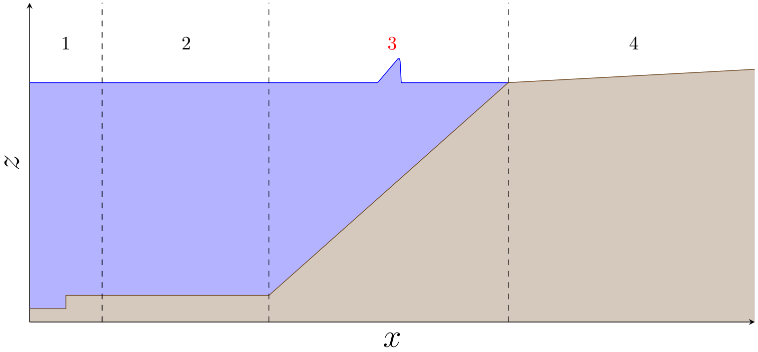
1 : Generation



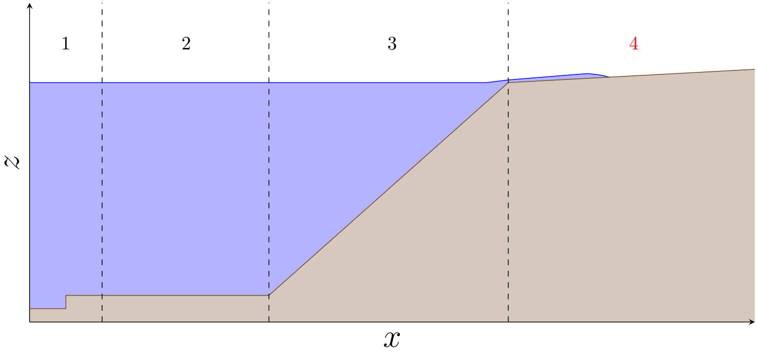
2 : Propagation far from coast



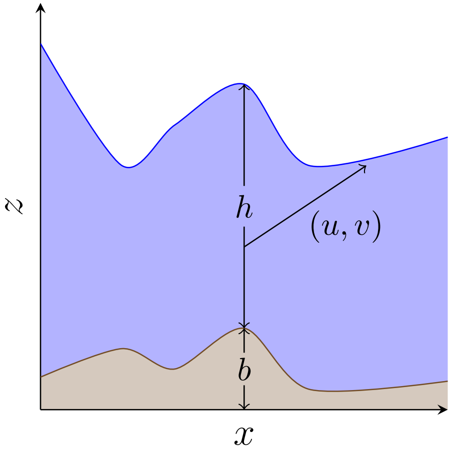
3 : Propagation near coast



4 : Inundation



Depth averaged equations



Shallow water wave equations

- Wavelengths (λ) $>>$ Water depth (H) ($\lambda \geq 20H$)
- Horizontal velocity constant over z
- Vertical velocity is 0
- Pressure is hydrostatic $p(z) = \rho g(h + b - z)$

Shallow water wave equations

Conservation of Mass

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0$$

Conservation of Momentum

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2}\right) + gh\frac{\partial b}{\partial x} = 0$$

Serre equations

- No restrictions
- Horizontal velocity is constant over z
- Vertical velocity is linear in z

$$v'(z) = u\frac{\partial b}{\partial x} - (z-b)\frac{\partial u}{\partial x}$$

- Pressure

$$p(z) = \rho g(h+b-z) + \rho(h+b-z)\Psi + \frac{\rho}{2}(h+b-z)(h-b+z)\Phi$$

$$\Psi = \frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial b}{\partial x}, \quad \Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}$$

Serre equations

Conservation of Mass

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0$$

Conservation of Momentum

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2}\right) + gh\frac{\partial b}{\partial x} + \frac{\partial}{\partial x}\left(\frac{h^2}{2}\Psi + \frac{h^3}{3}\Phi\right) + \frac{\partial b}{\partial x}\left(h\Psi + \frac{h^2}{2}\Phi\right) = 0$$

$$\Psi = \frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial b}{\partial x}, \quad \Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}$$

Differences

Differences:

- Dispersion
- Higher order terms

Are they important for tsunamis?

Aim

- Compare Shallow Water Wave and Serre equations
- Highlight different behaviours
- Highlight possible impacts these differences could make on current simulations

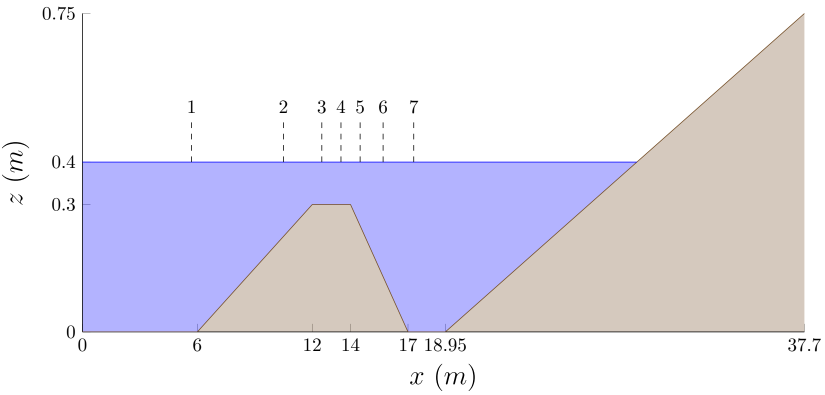
Numerical Solvers

- Shallow water wave equations: ANUGA, second-order finite volume method
- Serre equations: second-order finite volume method (same technique as ANUGA) and a second-order finite difference method

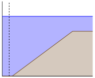
Experiments

- Experimental results of Beji and Battjes (1994)
- Artificial example replicating common phenomena

Periodic waves over a submerged bar: initial conditions



Wave gauge 1: boundary condition



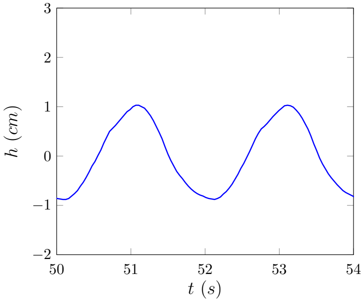


Figure: Low frequency $\lambda = 3.69m$ and $H = 0.4m$

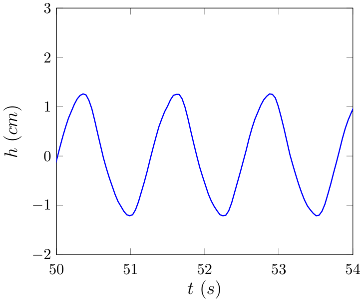
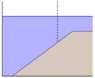


Figure: High frequency $\lambda = 2.05m$ and $H = 0.4m$

Wave gauge 2: experimental result



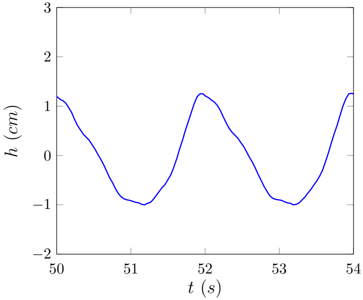
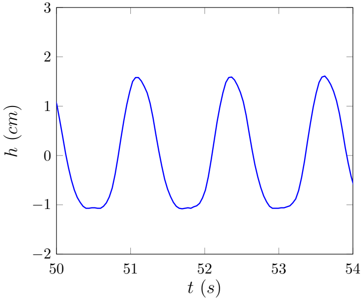


Figure: Low frequency



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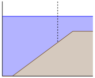
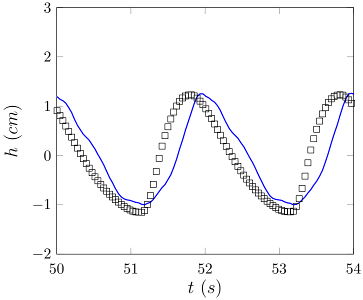


Wave gauge 2: shallow water wave equation



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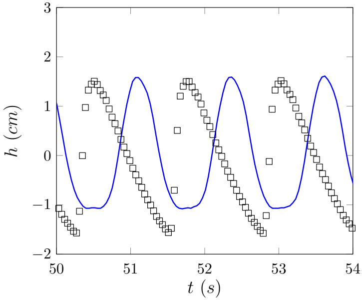
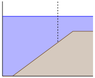
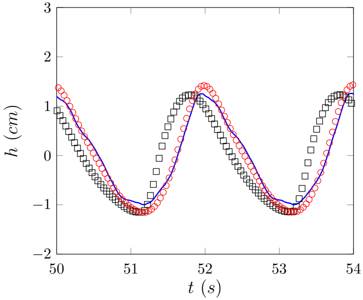


Figure: High frequency

Wave gauge 2: all results





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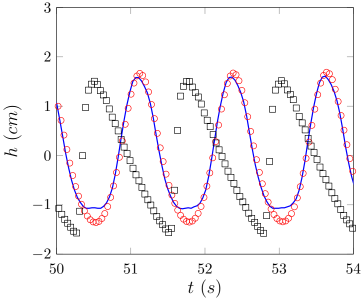

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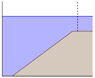
Wave gauge 3: experimental result



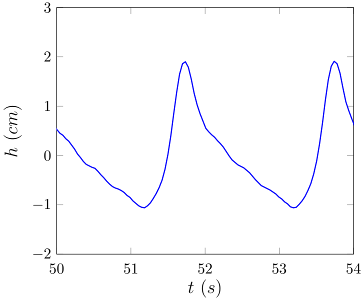
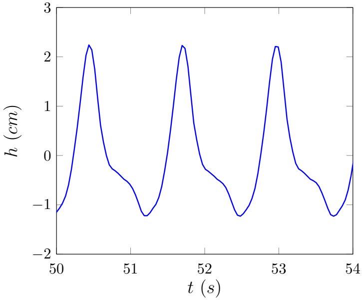


Figure: Low frequency



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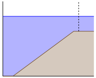
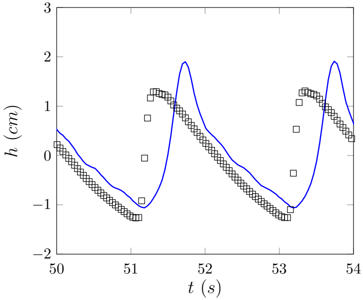
Wave gauge 3: shallow water wave equation





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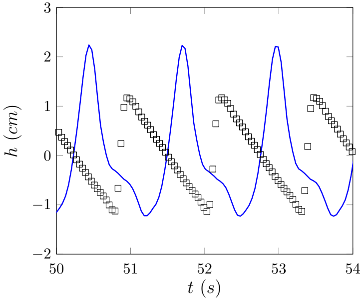

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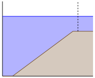
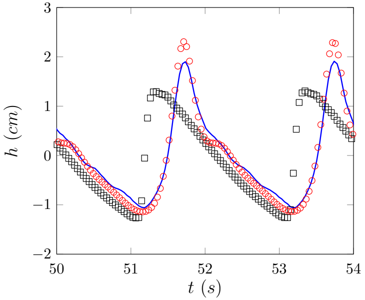
Wave gauge 3: all results





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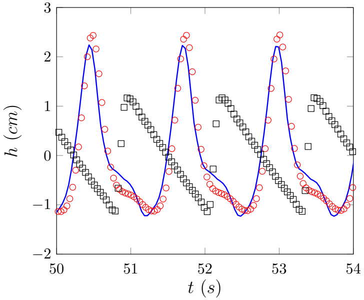
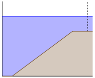
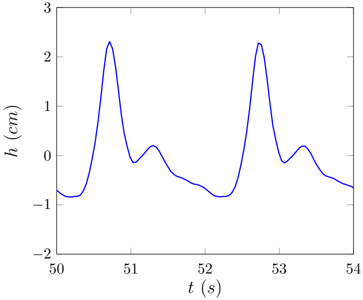


Figure: High frequency

Wave gauge 4: experimental result





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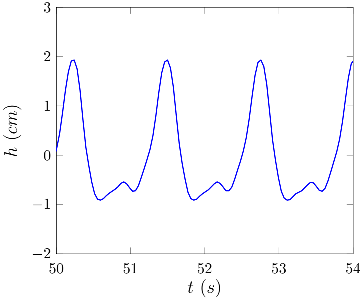
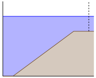


Figure: High frequency

Wave gauge 4: shallow water wave equation



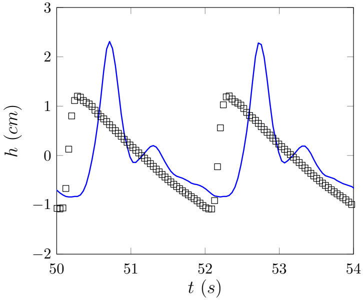


Figure: Low frequency

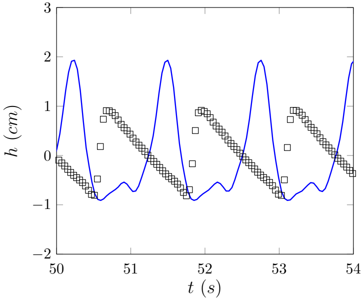
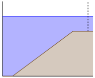


Figure: High frequency

Wave gauge 4: all results



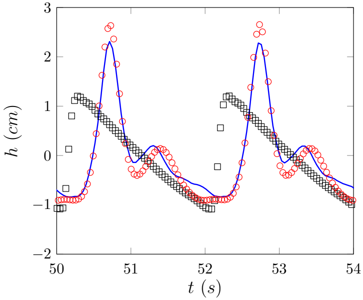


Figure: Low Frequency

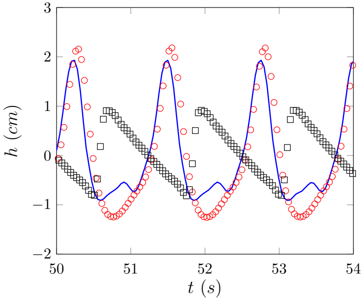
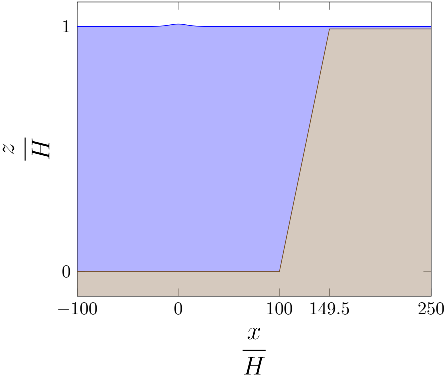
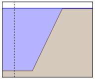


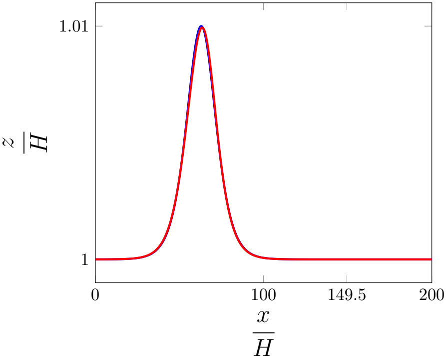
Figure: High Frequency

Solitary wave over a constant slope: initial conditions

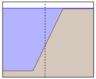


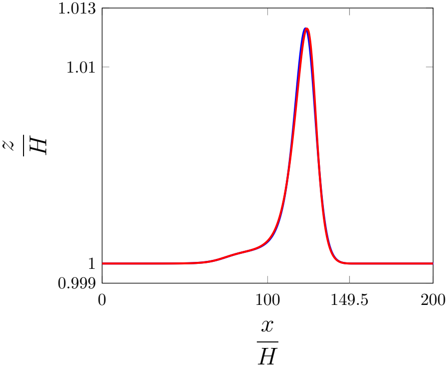
Before slope



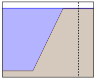


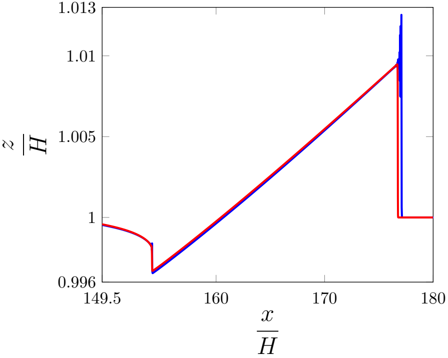
Shoaling



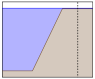


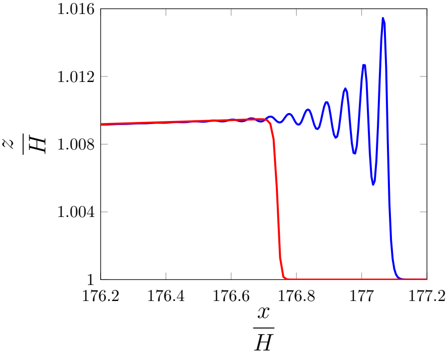
Bore formation





Front of bore





Conclusion

- Dispersion plays an important role when wavelengths are not long compared to water depths
- Dispersion is not important for shoaling of long wavelength waves
- Dispersion is an important effect for waves after shoaling has occurred.
- For shoaled waves our current models may underestimate wave amplitude and predict later arrival times.