

# 1 All Definitions for Numerical Versions

$$\mathcal{C}_2 = \frac{2 \cos(k\Delta x) - 2}{\Delta x^2}$$

$$\mathcal{C}_4 = \frac{-2 \cos(2k\Delta x) + 32 \cos(k\Delta x) - 30}{12\Delta x^2}$$

$$\mathcal{G} = \left[ H - \frac{H^3}{3} \mathcal{C} \right]$$

$$\mathcal{M}_3 = \frac{24}{26 - 2 \cos(k\Delta x)}$$

$$\mathcal{M}_1 = \mathcal{M}_2 = 1$$

$$\mathcal{R}_1^+ = e^{ik\Delta x} \quad , \quad \mathcal{R}_1^- = 1$$

$$\mathcal{R}_2^- = 1 + \frac{i \sin(k\Delta x)}{2}$$

$$\mathcal{R}_2^+ = e^{ik\Delta x} \left( 1 - \frac{i \sin(k\Delta x)}{2} \right)$$

$$\mathcal{R}_3^- = \frac{\mathcal{M}_3}{6} [5 + -e^{-ik\Delta x} + 2e^{ik\Delta x}]$$

$$\mathcal{R}_3^+ = \frac{\mathcal{M}_3 e^{ik\Delta x}}{6} [5 + 2e^{-ik\Delta x} - e^{ik\Delta x}]$$

$$\mathcal{R}_2^u = \frac{e^{ik\Delta x} + 1}{2}$$

$$\mathcal{R}_3^u = \frac{-3e^{2ik\Delta x} + 27e^{ik\Delta x} + 27 - 3e^{-ik\Delta x}}{48}$$

$$\mathcal{F}^{h,u} = H\mathcal{R}^u$$

$$\mathcal{F}^{h,h} = -\frac{\sqrt{gH}}{2} [\mathcal{R}^+ - \mathcal{R}^-]$$

$$\mathcal{F}^{u,u} = -\frac{\sqrt{gH}}{2} \mathcal{G} [\mathcal{R}^+ - \mathcal{R}^-]$$

$$\mathcal{F}^{u,h} = \frac{gH}{2} (\mathcal{R}^+ + \mathcal{R}^-)$$

$$\mathcal{D} = 1 - e^{-ik\Delta x}$$

## 2 Taylor Expansions Of Analytic Values

We denote exact/analytic version with a subscript  $a$

$$\mathcal{G}_a = H + \frac{H^3}{3} k^2$$

$$\mathcal{M}_a = \frac{2}{k\Delta x} \sin\left(\frac{k\Delta x}{2}\right)$$

$$\mathcal{M}_a = 1 - \frac{k^2}{24}(\Delta x)^2 + \frac{k^4}{1920}(\Delta x)^4 - \frac{k^6}{322560}(\Delta x)^6 + O(x^8)$$

$$\mathcal{R}_a = e^{i\frac{k\Delta x}{2}}$$

$$\mathcal{R}_a^+ = \mathcal{R}_a^- = \mathcal{R}_a = 1 + \frac{ik}{2}\Delta x - \frac{k^2}{8}\Delta x^2 - \frac{ik^3}{48}\Delta x^3 + \frac{k^4}{384}\Delta x^4 + \frac{ik^5}{3840}\Delta x^5 + O(x^6)$$

For the fluxes I think its best to group the  $\frac{\mathcal{D}}{\Delta x \mathcal{M}} \mathcal{F}$  because its collects all the terms using spatial approximations and has a nice form. In fact these terms approximate the derivative of the flux.

$$\frac{\mathcal{D}_a}{\Delta x \mathcal{M}_a} \mathcal{F}_a^{h,u} = ikH$$

$$\frac{\mathcal{D}_a}{\Delta x \mathcal{M}_a} \mathcal{F}_a^{h,h} = 0$$

$$\frac{\mathcal{D}_a}{\Delta x \mathcal{M}_a} \mathcal{F}_a^{u,h} = ikgH$$

$$\frac{\mathcal{D}_a}{\Delta x \mathcal{M}_a} \mathcal{F}_a^{u,u} = 0$$

So in particular for the mass equation

$$h_t + Hu_x = 0$$

then

$$i\omega h + \frac{\mathcal{D}_a}{\Delta x \mathcal{M}_a} \mathcal{F}_a^{h,u} u_j + \frac{\mathcal{D}_a}{\Delta x \mathcal{M}_a} \mathcal{F}_a^{h,h} h_j = 0$$

Indeed it is these the values of our approximations to these terms that confirm that our methods have the correct spatial accuracy.

### 3 First Order Values

$$\mathcal{G}_1 = H - \frac{H^3}{3} \left( \frac{2 \cos(k\Delta x) - 2}{\Delta x^2} \right)$$

$$\mathcal{G}_1 = H + \frac{H^3 k^2}{3} - \frac{H^3 k^4}{36} (\Delta x)^2 + \frac{H^3 k^6}{1080} (\Delta x)^4 + O(x^6)$$

$$\mathcal{M}_1 = 1$$

$$\mathcal{R}_1^- = 1$$

$$\mathcal{R}_1^+ = e^{ik\Delta x}$$

$$\mathcal{R}_1^+ = 1 + ik\Delta x - \frac{k^2}{2} (\Delta x)^2 - \frac{ik^3}{6} (\Delta x)^3 + \frac{k^4}{24} (\Delta x)^4 + O(\Delta x^5)$$

$$\mathcal{R}_1^u = \frac{e^{ik\Delta x} + 1}{2}$$

$$\mathcal{R}_1^u = 1 + \frac{ik}{2} \Delta x - \frac{k^2}{4} (\Delta x)^2 - \frac{ik^3}{12} (\Delta x)^3 + \frac{k^4}{48} (\Delta x)^4 + O(\Delta x^5)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1} \mathcal{F}_1^{h,u} = \frac{1 - e^{-ik\Delta x}}{\Delta x} H \frac{e^{ik\Delta x} + 1}{2}$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1} \mathcal{F}_1^{h,u} = iHk - \frac{iHk^3}{6} (\Delta x)^2 + \frac{iHk^5}{120} (\Delta x)^4 + O(\Delta x^6)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1} \mathcal{F}_1^{h,h} = -\frac{\sqrt{gH}}{2} \frac{1 - e^{-ik\Delta x}}{\Delta x} [e^{ik\Delta x} - 1]$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1} \mathcal{F}_1^{h,h} = \frac{k^2 \sqrt{gH}}{2} \Delta x - \frac{k^4 \sqrt{gH}}{24} \Delta x^3 + \frac{k^6 \sqrt{gH}}{720} \Delta x^5 + O(\Delta x^7)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1} \mathcal{F}_1^{u,h} = \frac{gH}{2} \frac{1 - e^{-ik\Delta x}}{\Delta x} (1 + e^{ik\Delta x})$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1} \mathcal{F}_1^{u,h} = igHk - \frac{igHk^3}{6} (\Delta x)^2 + \frac{igHk^5}{120} (\Delta x)^4 + O(\Delta x^6)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1} \mathcal{F}_1^{u,u} = -\frac{\sqrt{gH}}{2} \left[ H - \frac{H^3}{3} \left( \frac{2 \cos(k\Delta x) - 2}{\Delta x^2} \right) \right] \frac{1 - e^{-ik\Delta x}}{\Delta x} [e^{ik\Delta x} - 1]$$

$$\mathcal{D} \mathcal{F}_1^{u,u} = \frac{k^2 H \sqrt{gH} (H^2 k^2 + 3)}{6} \Delta x - \frac{k^4 H \sqrt{gH} (2H^2 k^2 + 3)}{72} \Delta x^3 + O(\Delta x^5)$$

## 4 Second Order Values

$$\mathcal{G}_2 = H - \frac{H^3}{3} \left( \frac{2 \cos(k\Delta x) - 2}{\Delta x^2} \right)$$

$$\mathcal{G}_2 = H + \frac{H^3 k^2}{3} - \frac{H^3 k^4}{36} (\Delta x)^2 + \frac{H^3 k^6}{1080} (\Delta x)^4 + O(x^6)$$

$$\mathcal{M}_2 = 1$$

$$\mathcal{R}_2^- = 1 + \frac{i \sin(k\Delta x)}{2}$$

$$\mathcal{R}_2^- = 1 + \frac{ik}{2} (\Delta x) - \frac{ik^3}{12} (\Delta x)^3 + \frac{ik^5}{240} (\Delta x)^5 + O(x^7)$$

$$\mathcal{R}_2^+ = e^{ik\Delta x} \left( 1 - \frac{i \sin(k\Delta x)}{2} \right)$$

$$\mathcal{R}_2^+ = 1 + \frac{ik}{2} \Delta x + \frac{ik^3}{6} \Delta x^3 - \frac{k^4}{8} \Delta x^4 - \frac{7ik^5}{120} \Delta x^5 + O(\Delta x^6)$$

$$\begin{aligned}
\mathcal{R}_2^u &= \frac{e^{ik\Delta x} + 1}{2} \\
\mathcal{R}_2^u &= 1 + \frac{ik}{2}\Delta x - \frac{k^2}{4}(\Delta x)^2 - \frac{ik^3}{12}(\Delta x)^3 + \frac{k^4}{48}(\Delta x)^4 + O(\Delta x^5) \\
\frac{\mathcal{D}}{\Delta x \mathcal{M}_2} \mathcal{F}_2^{h,u} &= \frac{1 - e^{-ik\Delta x}}{\Delta x} H \frac{e^{ik\Delta x} + 1}{2} \\
\frac{\mathcal{D}}{\Delta x \mathcal{M}_2} \mathcal{F}_2^{h,u} &= iHk - \frac{iHk^3}{6}(\Delta x)^2 + \frac{iHk^5}{120}(\Delta x)^4 + O(\Delta x^6) \\
\frac{\mathcal{D}}{\Delta x \mathcal{M}_2} \mathcal{F}_2^{h,h} &= -\frac{\sqrt{gH}}{2} \frac{1 - e^{-ik\Delta x}}{\Delta x} \left[ e^{ik\Delta x} \left( 1 - \frac{i \sin(k\Delta x)}{2} \right) - \left( 1 + \frac{i \sin(k\Delta x)}{2} \right) \right] \\
\frac{\mathcal{D}}{\Delta x \mathcal{M}_2} \mathcal{F}_2^{h,h} &= \frac{k^4 \sqrt{gH}}{8}(\Delta x)^3 - \frac{k^6 \sqrt{gH}}{48}(\Delta x)^5 + O(\Delta x^7) \\
\frac{\mathcal{D}}{\Delta x \mathcal{M}_2} \mathcal{F}_2^{u,u} &= \\
&\quad - \frac{1 - e^{-ik\Delta x}}{\Delta x} \frac{\sqrt{gH}}{2} \\
&\times \left[ H - \frac{H^3}{3} \left( \frac{2 \cos(k\Delta x) - 2}{\Delta x^2} \right) \right] \left[ e^{ik\Delta x} \left( 1 - \frac{i \sin(k\Delta x)}{2} \right) - \left( 1 + \frac{i \sin(k\Delta x)}{2} \right) \right] \\
\frac{\mathcal{D}}{\Delta x \mathcal{M}_2} \mathcal{F}_2^{u,u} &= \frac{k^4 H \sqrt{gH} (H^2 k^2 + 3)}{24} \Delta x^3 - \frac{k^6 H \sqrt{gH} (H^2 k^2 + 2)}{96} \Delta x^5 + O(\Delta x^7) \\
\frac{\mathcal{D}}{\Delta x \mathcal{M}_2} \mathcal{F}_2^{u,h} &= \frac{1 - e^{-ik\Delta x}}{\Delta x} \frac{gH}{2} \left[ e^{ik\Delta x} \left( 1 - \frac{i \sin(k\Delta x)}{2} \right) + \left( 1 + \frac{i \sin(k\Delta x)}{2} \right) \right] \\
\mathcal{D} \mathcal{F}_2^{u,h} &= igkH + \frac{igk^3 H}{12} \Delta x^2 - \frac{13ik^5 gH}{240} \Delta x^4 + O(\Delta x^6)
\end{aligned}$$

## 5 Taylor Expansions Of Third Order Values

$$\mathcal{G}_3 = H - \frac{H^3}{3} \frac{-2 \cos(2k\Delta x) + 32 \cos(k\Delta x) - 30}{12\Delta x^2}$$

$$\mathcal{G}_3 = H + \frac{k^2 H^3}{3} - \frac{k^6 H^3}{270} (\Delta x)^4 + O(\Delta x^6)$$

$$\mathcal{M}_3 = \frac{24}{26 - 2 \cos(k\Delta x)}$$

$$\mathcal{M}_3 = 1 - \frac{k^2}{24} (\Delta x)^2 + \frac{k^4}{192} (\Delta x)^4 + O(x^6)$$

$$\mathcal{R}_3^- = \frac{\mathcal{M}_3}{6} [5 - e^{-ik\Delta x} + 2e^{ik\Delta x}]$$

$$R_3^- = 1 + \frac{ik}{2} \Delta x - \frac{k^2}{8} (\Delta x)^2 - \frac{5ik^3}{48} (\Delta x)^3 + \frac{k^4}{64} (\Delta x)^4 + O(\Delta x^5)$$

$$\mathcal{R}_3^+ = \frac{\mathcal{M}_3 e^{ik\Delta x}}{6} [5 + 2e^{-ik\Delta x} - e^{ik\Delta x}]$$

$$R_3^+ = 1 + \frac{ik}{2} \Delta x - \frac{k^2}{8} (\Delta x)^2 + \frac{ik^3}{16} (\Delta x)^3 - \frac{13k^4}{192} (\Delta x)^4 + O(\Delta x^5)$$

$$\mathcal{R}_3^u = \frac{-3e^{2ik\Delta x} + 27e^{ik\Delta x} + 27 - 3e^{-ik\Delta x}}{48}$$

$$R_3^u = 1 + \frac{ik}{2} \Delta x - \frac{k^2}{8} (\Delta x)^2 - \frac{ik^3}{48} (\Delta x)^3 - \frac{k^4}{48} (\Delta x)^4 + O(\Delta x^5)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3} \mathcal{F}^{h,u} = \frac{1 - e^{-ik\Delta x}}{\Delta x} \frac{26 - 2 \cos(k\Delta x)}{24} H \mathcal{R}^u$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3} \mathcal{F}_3^{h,u} = ikH - \frac{9ik^5 H}{320} \Delta x^4 - \frac{ik^7 H}{448} \Delta x^6 + O(\Delta x^9)$$

$$\begin{aligned} \frac{\mathcal{D}}{\Delta x \mathcal{M}_3} \mathcal{F}^{h,h} &= -\frac{1 - e^{-ik\Delta x}}{\Delta x \mathcal{M}_3} \frac{\sqrt{gH}}{2} \\ &\times \left[ \left( \frac{\mathcal{M}_3 e^{ik\Delta x}}{6} [5 + 2e^{-ik\Delta x} - e^{ik\Delta x}] \right) - \left( \frac{\mathcal{M}_3}{6} [5 - e^{-ik\Delta x} + 2e^{ik\Delta x}] \right) \right] \end{aligned}$$

$$\begin{aligned} \frac{\mathcal{D}}{\Delta x \mathcal{M}_3} \mathcal{F}^{h,h} &= -\frac{1 - e^{-ik\Delta x}}{\Delta x} \frac{\sqrt{gH}}{2} \\ &\times \left[ \left( \frac{e^{ik\Delta x}}{6} [5 + 2e^{-ik\Delta x} - e^{ik\Delta x}] \right) - \left( \frac{1}{6} [5 - e^{-ik\Delta x} + 2e^{ik\Delta x}] \right) \right] \end{aligned}$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3} \mathcal{F}_3^{h,h} = \frac{k^4 \sqrt{gH}}{12} \Delta x^3 - \frac{k^6 \sqrt{gH}}{72} \Delta x^5 + \frac{k^8 \sqrt{gH}}{960} \Delta x^7 + O(\Delta x^9)$$

$$\begin{aligned} \frac{\mathcal{D}}{\Delta x \mathcal{M}_3} \mathcal{F}^{u,u} &= -\frac{1 - e^{-ik\Delta x}}{\Delta x \mathcal{M}_3} \frac{\sqrt{gH}}{2} \left[ H - \frac{H^3 - 2 \cos(2k\Delta x) + 32 \cos(k\Delta x) - 30}{12\Delta x^2} \right] \\ &\times \left[ \left( \frac{\mathcal{M}_3 e^{ik\Delta x}}{6} [5 + 2e^{-ik\Delta x} - e^{ik\Delta x}] \right) - \left( \frac{\mathcal{M}_3}{6} [5 - e^{-ik\Delta x} + 2e^{ik\Delta x}] \right) \right] \end{aligned}$$

$$\begin{aligned} \frac{\mathcal{D}}{\Delta x \mathcal{M}_3} \mathcal{F}^{u,u} &= -\frac{1 - e^{-ik\Delta x}}{\Delta x} \frac{\sqrt{gH}}{2} \left[ H - \frac{H^3 - 2 \cos(2k\Delta x) + 32 \cos(k\Delta x) - 30}{12\Delta x^2} \right] \\ &\times \left[ \left( \frac{e^{ik\Delta x}}{6} [5 + 2e^{-ik\Delta x} - e^{ik\Delta x}] \right) - \left( \frac{1}{6} [5 - e^{-ik\Delta x} + 2e^{ik\Delta x}] \right) \right] \end{aligned}$$

$$\begin{aligned} &\frac{\mathcal{D}}{\Delta x \mathcal{M}_3} \mathcal{F}^{u,u} \\ &= \frac{\sqrt{gH}}{2} \left[ -\frac{ikH(k^2 H^2 + 3)}{3} - \frac{k^2 H(k^2 H^2 + 3)}{3} \Delta x + \frac{ik^3 H(k^2 H^2 + 3)}{18} \Delta x^2 + O(\Delta x^3) \right] \\ &\quad \times \left[ \frac{ik^3}{6} \Delta x^3 - \frac{k^4}{12} \Delta x^4 + O(\Delta x^5) \right] \end{aligned}$$

$$\begin{aligned} & \frac{\mathcal{D}}{\Delta x \mathcal{M}_3} \mathcal{F}^{u,u} \\ &= \frac{\sqrt{gH}}{2} \left[ -\frac{ikH(k^2H^2+3)}{3} - \frac{k^2H(k^2H^2+3)}{6} \Delta x + \frac{ik^3H(k^2H^2+3)}{18} \Delta x^2 + O(\Delta x^3) \right] \\ & \quad \times \left[ \frac{ik^3}{6} \Delta x^3 - \frac{k^4}{12} \Delta x^4 + O(\Delta x^5) \right] \end{aligned}$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3} \mathcal{F}^{u,u} = \frac{\sqrt{gH}}{2} \left[ \frac{k^4H(k^2H^2+3)}{18} \Delta x^3 + \frac{k^6H(k^2H^2+3)}{216} \Delta x^5 + O(\Delta x^6) \right]$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3} \mathcal{F}^{u,u} = \frac{k^4H\sqrt{gH}(k^2H^2+3)}{36} \Delta x^3 + \frac{k^6H\sqrt{gH}(k^2H^2+3)}{532} \Delta x^5 + O(\Delta x^6)$$

$$\begin{aligned} & \frac{\mathcal{D}}{\Delta x \mathcal{M}_3} \mathcal{F}^{u,h} = \frac{gH}{2} \frac{1 - e^{-ik\Delta x}}{\Delta x \mathcal{M}_3} \\ & \quad \times \left[ \left( \frac{e^{ik\Delta x} \mathcal{M}_3}{6} [5 + 2e^{-ik\Delta x} - e^{ik\Delta x}] \right) + \left( \frac{\mathcal{M}_3}{6} [5 - e^{-ik\Delta x} + 2e^{ik\Delta x}] \right) \right] \end{aligned}$$

$$\begin{aligned} & \frac{\mathcal{D}}{\Delta x \mathcal{M}_3} \mathcal{F}^{u,h} = \frac{gH}{2} \frac{1 - e^{-ik\Delta x}}{\Delta x} \\ & \quad \times \left[ \left( \frac{e^{ik\Delta x}}{6} [5 + 2e^{-ik\Delta x} - e^{ik\Delta x}] \right) + \left( \frac{1}{6} [5 - e^{-ik\Delta x} + 2e^{ik\Delta x}] \right) \right] \end{aligned}$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3} \mathcal{F}_3^{u,h} = igkH - \frac{ik^5gH}{30} \Delta x^4 + O(\Delta x^6)$$

## 6 Numerical Method Break Down

Our conservative update is, for our equations is

$$\bar{q}_j^{n+1} = \bar{q}_j^n - \frac{\Delta t}{\Delta x} \left[ F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}} \right]$$



This converts to (both analytical and numerical)

$$\mathcal{M}q_j^{n+1} = \mathcal{M}q_j^n - \frac{\Delta t}{\Delta x} [\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j - \mathcal{F}^{q,v}v_{j-1} - \mathcal{F}^{q,q}q_{j-1}]$$

$$\mathcal{M}q_j^{n+1} = \mathcal{M}q_j^n - \frac{\Delta t}{\Delta x} [\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j - \mathcal{F}^{q,v}e^{-ik\Delta x}v_j - \mathcal{F}^{q,q}e^{-ik\Delta x}q_j]$$

Defining  $\mathcal{D}_x = 1 - e^{-ik\Delta x}$

$$\mathcal{M}q_j^{n+1} - \mathcal{M}q_j^n = -\frac{\Delta t}{\Delta x} [\mathcal{D}_x\mathcal{F}^{q,v}v_j + \mathcal{D}_x\mathcal{F}^{q,q}q_j]$$

$$\mathcal{M}(q_j^{n+1} - q_j^n) = -\frac{\Delta t}{\Delta x} [\mathcal{D}_x\mathcal{F}^{q,v}v_j + \mathcal{D}_x\mathcal{F}^{q,q}q_j]$$

$$\mathcal{M}(e^{i\omega\Delta t} - 1)q_j^n = -\frac{\Delta t}{\Delta x} [\mathcal{D}_x\mathcal{F}^{q,v}v_j + \mathcal{D}_x\mathcal{F}^{q,q}q_j]$$

Defining  $\mathcal{D}_t = e^{i\omega\Delta t} - 1$

$$\mathcal{M}\mathcal{D}_tq_j^n = -\frac{\Delta t}{\Delta x} [\mathcal{D}_x\mathcal{F}^{q,v}v_j + \mathcal{D}_x\mathcal{F}^{q,q}q_j]$$

$$\mathcal{M}\mathcal{D}_tq_j^n = -\frac{\Delta t}{\Delta x} [\mathcal{D}_x\mathcal{F}^{q,v}v_j + \mathcal{D}_x\mathcal{F}^{q,q}q_j]$$

$$\frac{\mathcal{D}_t}{\Delta t}q_j^n = -\frac{\mathcal{D}_x}{\mathcal{M}\Delta x} [\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j]$$

$$\frac{\mathcal{D}_t}{\Delta t}q_j^n = -\left[ \frac{\mathcal{D}_x}{\mathcal{M}\Delta x}\mathcal{F}^{q,v}v_j + \frac{\mathcal{D}_x}{\mathcal{M}\Delta x}\mathcal{F}^{q,q}q_j \right]$$

So as we've said we have an approximation to the time derivative on the left and an approximation to the space derivative on the right. However, this is a bit muddled by the fact that in the numerical methods  $\mathcal{F}^{q,v}v_j$  does not have an explicit time relation, while for the analytic one it does, for example the continuity equation under this scheme, substituting the analytic expression will be

$$\frac{\mathcal{D}_t}{\Delta t}h_j^n = -\left[ \frac{\mathcal{D}_x}{\mathcal{M}\Delta x}\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}Hu_{j+1/2} \right]$$

$$\frac{\mathcal{D}_t}{\Delta t} h_j^n = - \left[ \frac{\mathcal{D}_x}{\mathcal{M}\Delta x} \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} H e^{ik\frac{\Delta x}{2}} u_j \right]$$

here I just use  $u = u_0 e^{i(\omega t + kx)}$

$$\frac{\mathcal{D}_t}{\Delta t} h_j^n = - \left[ \frac{\mathcal{D}_x}{\mathcal{M}\Delta x} e^{ik\frac{\Delta x}{2}} H u_0 e^{ikx_j} \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} e^{i\omega t} \right]$$

$$\frac{\mathcal{D}_t}{\Delta t} h_j^n = - \left[ \frac{\mathcal{D}_x}{\mathcal{M}\Delta x} e^{ik\frac{\Delta x}{2}} H u_0 e^{ikx_j} \frac{1}{\Delta t} \left[ \frac{1}{i\omega} e^{i\omega t} \right]_{t_n}^{t_{n+1}} \right]$$

$$\frac{\mathcal{D}_t}{\Delta t} h_j^n = - \left[ \frac{\mathcal{D}_x}{\mathcal{M}\Delta x} e^{ik\frac{\Delta x}{2}} H u_0 e^{ikx_j} \frac{1}{i\omega} \frac{1}{\Delta t} [e^{i\omega t_{n+1}} - e^{i\omega t_n}] \right]$$

$$\frac{\mathcal{D}_t}{\Delta t} h_j^n = - \left[ \frac{\mathcal{D}_x}{\mathcal{M}\Delta x} e^{ik\frac{\Delta x}{2}} H u_0 e^{ikx_j + i\omega t_n} \frac{1}{i\omega} \frac{1}{\Delta t} [e^{i\omega \Delta t} - 1] \right]$$

$$\frac{\mathcal{D}_t}{\Delta t} h_j^n = - \left[ \frac{\mathcal{D}_x}{\mathcal{M}\Delta x} e^{ik\frac{\Delta x}{2}} H u_j^n \frac{1}{i\omega} \frac{\mathcal{D}_t}{\Delta t} \right]$$

$$i\omega h_j^n = - \left[ \frac{\mathcal{D}_x}{\mathcal{M}\Delta x} e^{ik\frac{\Delta x}{2}} H u_j^n \right]$$

$$i\omega h_j^n = - \left[ \frac{1 - e^{-ik\Delta x}}{\frac{2}{k\Delta x} \sin(k\frac{\Delta x}{2}) \Delta x} e^{ik\frac{\Delta x}{2}} H u_j^n \right]$$

$$i\omega h_j^n = - \left[ k \frac{e^{ik\frac{\Delta x}{2}} - e^{-ik\frac{\Delta x}{2}}}{2 \sin(k\frac{\Delta x}{2})} H u_j^n \right]$$

$$i\omega h_j^n = - \left[ k \frac{2i \sin(k\frac{\Delta x}{2})}{2 \sin(k\frac{\Delta x}{2})} H u_j^n \right]$$

$$i\omega h_j^n = -ik H u_j^n$$

As desired. The problem is that when we use a numerical method we end up losing time accuracy as a result of the approximation of the flux. Does this clear it up Chris?