1 Elliptic Equation

The linearised elliptic equation is

$$G = Hu - \frac{H^3}{3}u_{xx}$$

Want to find out the FEM approximation to this G' such that

$$G' = \mathcal{G}_{FE_1}u$$

for P^1 FEM

$$G' = \mathcal{G}_{FE_2}u$$

for P^2 FEM.

To do so we begin by first multiplying by an arbitrary test function v so that

$$Gv = Huv - \frac{H^3}{3}u_{xx}v$$

and then we integrate over the entire domain to get

$$\int_\Omega Gvdx = \int_\Omega Huvdx - \int_\Omega \frac{H^3}{3}u_{xx}vdx$$

for all v

We then make use of integration by parts, with Dirchlet boundaries to get

$$\int_\Omega Gvdx = \int_\Omega Huvdx + \int_\Omega \frac{H^3}{3}u_x v_x dx$$

Our FVM discretisation already has a natrual structure with linear functions intervals of $[x_{j-1/2}, x_{j+1/2}]$, to achieve this in P^1 we have our nodes at the boundaries, thus

So we can reformulate this as

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Gvdx = \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Huvdx + \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{H^3}{3}u_x v_x dx$$

or more aptly

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Gvdx - \int_{x_{j-1/2}}^{x_{j+3/2}} Huvdx - \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{H^3}{3} u_x v_x dx = 0$$

for all v

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Gvdx - H\int_{x_{j-1/2}}^{x_{j+3/2}} uvdx - \frac{H^3}{3} \int_{x_{j-1/2}}^{x_{j+3/2}} u_x v_x dx = 0$$

2 P1 FEM

We are going to corodainte tranform from x space the interval $[x_{i-1/2}, x_{i+1/2}, x_{i+3/2}]$ to the ξ space interval $[-1, 0, 1]$. To accomplish this we have the following relation

$$x = \xi \Delta x + x_{j+1/2}$$

Taking the derivative we see
$dx = d\xi \Delta x$, $\frac{dx}{d\xi} = \Delta x$, $\frac{d\xi}{dx} = \frac{1}{\Delta x}$
For this FEM we are intereseted in $G_{i+1/2}$ and then we can just get a shift operator to get the otherones. For FEM we replace the functions by their P1 approximations so

$$G \approx G' = \sum_j G_{j+1/2} v_{j+1/2}$$

$$u \approx u' = \sum_j u_{j+1/2} v_{j+1/2}$$

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} G'vdx - H\int_{x_{j-1/2}}^{x_{j+3/2}} u'vdx - \frac{H^3}{3} \int_{x_{j-1/2}}^{x_{j+3/2}} u'_x v_x dx = 0$$

We break this up into the integrals because of the domain of dependence of the basis functions is covered. We also just use a particular basis function as the test function, in particular we choose $v_{j+1/2}$

$$\int_{x_{j-1/2}}^{x_{j+3/2}} G'(x)v_{j+1/2}dx = \int_{-1}^{1} G'(\xi)v_{j+1/2}(\xi)\frac{dx}{d\xi}d\xi$$

$$= \Delta x \int_{-1}^{1} \left(G_{j-1/2}v_{j-1/2} + G_{j+1/2}v_{j+1/2} + G_{j+3/2}v_{j+3/2}\right) v_{j+1/2}d\xi$$

$$= \Delta x \left[G_{j-1/2}\int_{-1}^{1} v_{j-1/2}v_{j+1/2}d\xi + G_{j+1/2}\int_{-1}^{1} v_{j+1/2}v_{j+1/2}d\xi + G_{j+3/2}\int_{-1}^{1} v_{j+3/2}v_{j+1/2}d\xi\right]$$

We have the

$$\int_{-1}^{1} v_{j-1/2}v_{j+1/2}d\xi = \int_{-1}^{1} v_{j+3/2}v_{j+1/2}d\xi = \frac{1}{6}$$

$$\int_{-1}^{1} v_{j+1/2}v_{j+1/2}d\xi = \frac{2}{3}$$

$$= \Delta x \left[G_{j-1/2}\frac{1}{6} + G_{j+1/2}\frac{2}{3} + G_{j+3/2}\frac{1}{6}\right]$$

$$= \frac{\Delta x}{6}\left[G_{j-1/2} + 4G_{j+1/2} + G_{j+3/2}\right]$$

Similarly we have

$$-H\int_{x_{j-1/2}}^{x_{j+3/2}} u'v_{j+1/2}dx = -\frac{H\Delta x}{6}\left[u_{j-1/2} + 4u_{j+1/2} + u_{j+3/2}\right]$$

$$-\frac{H^3}{3}\int_{x_{j-1/2}}^{x_{j+3/2}} u'_x(v_{j+1/2})_x dx = -\frac{H^3}{3}\int_{-1}^{1} u'_\xi\frac{d\xi}{dx}(v_{j+1/2})_\xi\frac{d\xi}{dx}\frac{dx}{d\xi}d\xi$$

$$= -\frac{H^3}{3\Delta x}\int_{-1}^{1} u'_\xi(v_{j+1/2})_\xi d\xi$$

where $'$ denotes derivative

$$= -\frac{H^3}{3\Delta x}\int_{-1}^{1} \left(u'_{j-1/2}v'_{j-1/2} + u'_{j+1/2}v'_{j+1/2} + u'_{j+3/2}v'_{j+3/2}\right) v'_{j+1/2}d\xi$$

$$= -\frac{H^3}{3\Delta x}\left[u_{j-1/2}\int_{-1}^{1} v'_{j-1/2}v'_{j+1/2}d\xi + u_{j+1/2}\int_{-1}^{1} v'_{j+1/2}v'_{j+1/2}d\xi + u_{j+3/2}\int_{-1}^{1} v'_{j+3/2}v'_{j+1/2}d\xi\right]$$

We have that

$$\int_{-1}^{1} v'_{j-1/2}v'_{j+1/2}d\xi = -1 = \int_{-1}^{1} v'_{j+3/2}v'_{j+1/2}d\xi$$

$$\int_{-1}^{1} v'_{j+1/2}v'_{j+1/2}d\xi = 2$$

$$= -\frac{H^3}{3\Delta x}\left[-u_{j-1/2} + 2u_{j+1/2} - u_{j+3/2}\right]$$

Then our equation becomes

$$\begin{aligned}&\frac{\Delta x}{6}\left[G_{j-1/2} + 4G_{j+1/2} + G_{j+3/2}\right] = \\ &\frac{H\Delta x}{6}\left[u_{j-1/2} + 4u_{j+1/2} + u_{j+3/2}\right] + \frac{H^3}{3\Delta x}\left[-u_{j-1/2} + 2u_{j+1/2} - u_{j+3/2}\right]\end{aligned} \tag{1}$$

$$\begin{aligned}&\left[G_{j-1/2} + 4G_{j+1/2} + G_{j+3/2}\right] = \\ &\quad H\left[u_{j-1/2} + 4u_{j+1/2} + u_{j+3/2}\right] + \frac{2H^3}{\Delta x^2}\left[-u_{j-1/2} + 2u_{j+1/2} - u_{j+3/2}\right]\end{aligned} \tag{2}$$

This formula is correct for $u = 1, x, x^2$
By shifting we also get

$$\begin{aligned}&\left[G_{j-3/2} + 4G_{j-1/2} + G_{j+1/2}\right] = \\ &\quad H\left[u_{j-3/2} + 4u_{j-1/2} + u_{j+1/2}\right] + \frac{2H^3}{\Delta x^2}\left[-u_{j-3/2} + 2u_{j-1/2} - u_{j+1/2}\right]\end{aligned} \tag{3}$$

$$\begin{aligned}&\left[G_{j+1/2} + 4G_{j+3/2} + G_{j+5/2}\right] = \\ &\quad H\left[u_{j+1/2} + 4u_{j+3/2} + u_{j+5/2}\right] + \frac{2H^3}{\Delta x^2}\left[-u_{j+1/2} + 2u_{j+3/2} - u_{j+5/2}\right]\end{aligned} \tag{4}$$

We begin by assuming the analytic structure for G and u (to get easy shift operators). Let quantity q is given by so that $q(x,t) = q(0,0)e^{i(\omega t + kx)}$. The use of this comes when we use our uniform grid in space, so that $q(x_j, t) = q_j$ then $q_{j\pm l} = q_j e^{\pm ikl\Delta x}$

Then we have

$$\left[G_j e^{-ik\frac{1}{2}\Delta x} + 4G_j e^{ik\frac{1}{2}\Delta x} + G_j e^{ik\frac{3}{2}\Delta x}\right] =$$
$$H\left[u_j e^{-ik\frac{1}{2}\Delta x} + 4u_j e^{ik\frac{1}{2}\Delta x} + u_j e^{ik\frac{3}{2}\Delta x}\right] + \frac{2H^3}{\Delta x^2}\left[-u_j e^{-ik\frac{1}{2}\Delta x} + 2u_j e^{ik\frac{1}{2}\Delta x} - u_j e^{ik\frac{3}{2}\Delta x}\right] \tag{5}$$

$$G_j\left[e^{-ik\frac{1}{2}\Delta x} + 4e^{ik\frac{1}{2}\Delta x} + e^{ik\frac{3}{2}\Delta x}\right] =$$
$$\left(H\left[e^{-ik\frac{1}{2}\Delta x} + 4e^{ik\frac{1}{2}\Delta x} + e^{ik\frac{3}{2}\Delta x}\right] + \frac{2H^3}{\Delta x^2}\left[-e^{-ik\frac{1}{2}\Delta x} + 2e^{ik\frac{1}{2}\Delta x} - e^{ik\frac{3}{2}\Delta x}\right]\right) u_j \tag{6}$$

$$G_j =$$
$$\left(H + \frac{2H^3}{\Delta x^2}\frac{\left[-e^{-ik\frac{1}{2}\Delta x} + 2e^{ik\frac{1}{2}\Delta x} - e^{ik\frac{3}{2}\Delta x}\right]}{\left[e^{-ik\frac{1}{2}\Delta x} + 4e^{ik\frac{1}{2}\Delta x} + e^{ik\frac{3}{2}\Delta x}\right]}\right) u_j \tag{7}$$

$$G_j =$$
$$\left(H + \frac{2H^3}{\Delta x^2}\frac{\left[2i\sin\left(k\frac{1}{2}\Delta x\right) + e^{ik\frac{1}{2}\Delta x} - e^{ik\frac{3}{2}\Delta x}\right]}{\left[e^{-ik\frac{1}{2}\Delta x} + 4e^{ik\frac{1}{2}\Delta x} + e^{ik\frac{3}{2}\Delta x}\right]}\right) u_j \tag{8}$$

$$G_j =$$
$$\left(H + \frac{2H^3}{\Delta x^2}\frac{2i\sin\left(k\frac{1}{2}\Delta x\right) + e^{ik\Delta x}\left(e^{-ik\frac{1}{2}\Delta x} - e^{ik\frac{1}{2}\Delta x}\right)}{\left[e^{-ik\frac{1}{2}\Delta x} + 4e^{ik\frac{1}{2}\Delta x} + e^{ik\frac{3}{2}\Delta x}\right]}\right) u_j \tag{9}$$

$G_j =$

$$\left(H + \frac{2H^3}{\Delta x^2} \frac{2i\sin\left(k\frac{1}{2}\Delta x\right) - 2ie^{ik\Delta x}\sin\left(k\frac{1}{2}\Delta x\right)}{\left[e^{-ik\frac{1}{2}\Delta x} + 4e^{ik\frac{1}{2}\Delta x} + e^{ik\frac{3}{2}\Delta x}\right]}\right) u_j \quad (10)$$

$G_j =$

$$\left(H + \frac{2H^3}{\Delta x^2} \frac{2i\sin\left(k\frac{1}{2}\Delta x\right) - 2ie^{ik\Delta x}\sin\left(k\frac{1}{2}\Delta x\right)}{\left[2\cos\left(k\frac{1}{2}\Delta x\right) + 2e^{ik\frac{1}{2}\Delta x} + e^{ik\frac{1}{2}\Delta x} + e^{ik\frac{3}{2}\Delta x}\right]}\right) u_j \quad (11)$$

$G_j =$

$$\left(H + \frac{2H^3}{\Delta x^2} \frac{2i\sin\left(k\frac{1}{2}\Delta x\right) - 2ie^{ik\Delta x}\sin\left(k\frac{1}{2}\Delta x\right)}{\left[2\cos\left(k\frac{1}{2}\Delta x\right) + 2e^{ik\frac{1}{2}\Delta x} + 2e^{ik\Delta x}\cos\left(k\frac{1}{2}\Delta x\right)\right]}\right) u_j \quad (12)$$

$G_j =$

$$\left(H + \frac{2H^3 i}{\Delta x^2} \frac{\sin\left(k\frac{1}{2}\Delta x\right) - e^{ik\Delta x}\sin\left(k\frac{1}{2}\Delta x\right)}{\left[\cos\left(k\frac{1}{2}\Delta x\right) + e^{ik\frac{1}{2}\Delta x} + e^{ik\Delta x}\cos\left(k\frac{1}{2}\Delta x\right)\right]}\right) u_j \quad (13)$$

$G_j =$

$$\left(H + \frac{2H^3 i}{\Delta x^2} \frac{\left(1 - e^{ik\Delta x}\right)\sin\left(k\frac{1}{2}\Delta x\right)}{\left(1 + e^{ik\Delta x}\right)\cos\left(k\frac{1}{2}\Delta x\right) + e^{ik\frac{1}{2}\Delta x}}\right) u_j \quad (14)$$

So we have

$$\mathcal{G}_{FEM_2} = \left(H + \frac{2H^3 i}{\Delta x^2} \frac{\left(1 - e^{ik\Delta x}\right)\sin\left(k\frac{1}{2}\Delta x\right)}{\left(1 + e^{ik\Delta x}\right)\cos\left(k\frac{1}{2}\Delta x\right) + e^{ik\frac{1}{2}\Delta x}}\right)$$

With taylor expansion

$$\mathcal{G}_{FEM_2} = H + \frac{H^3 k^2}{3} + O(\Delta x^2)$$

as desired.