Undular Bores of the Serre Equations

Jordan Pitt, Stephen Roberts and Christopher Zoppou
Australian National University

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Undular Bores

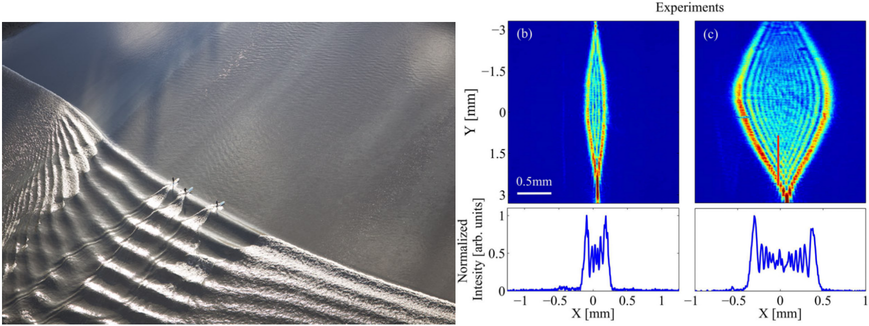

Figure: examples of undular bores from tidal flows to even optics.

Dam Break

Fluid depth (h) :

$$h(x,0) = \begin{cases} h_1 & x \leq x_0 \\ h_0 & x > x_0 \end{cases}$$

Fluid velocity (u) :

$$u(x,0) = 0.0.$$

Smoothing

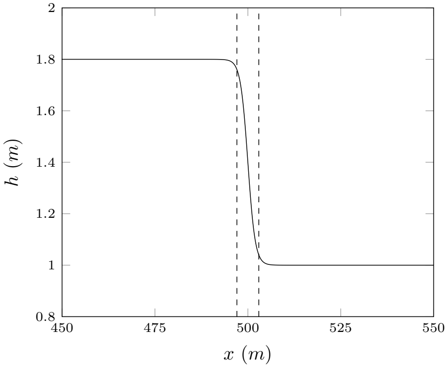

Figure: Example of water profile of a smoothed dam break with a transition width β of 5.8888.

Serre Equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

$$\underbrace{\underbrace{\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2}\right)}_{\text{Shallow Water Wave Equations}} + \underbrace{\frac{\partial}{\partial x}\left(\frac{h^3}{3}\left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}\right]\right)}_{\text{Dispersion Terms}}}_{\text{Serre Equations}} = 0$$

Literature Results

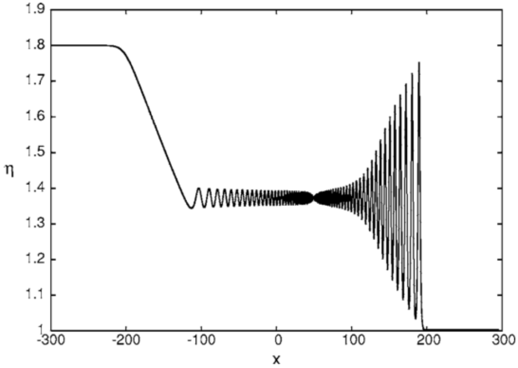

Figure: Fluid depth at $150s$ obtained from numerical method by El and Grimshaw (El et al., 2006).

Numerical

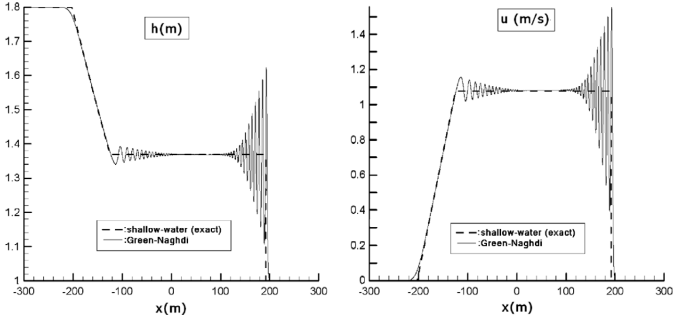

Figure: Fluid depth at 48s obtained from numerical method by Le Métayer (Le Métayer et al., 2010) (The Serre equations are also known as the Green Naghdi equations).

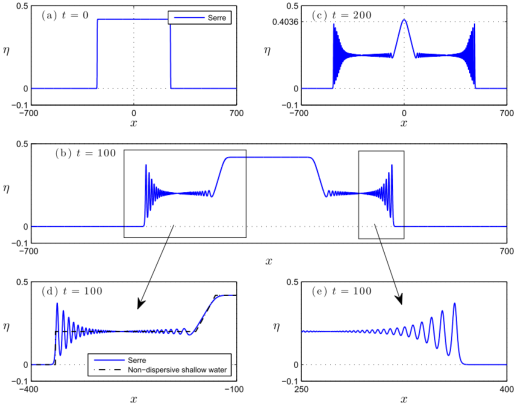

Figure: Wave height at various times for the smoothed dam break problem obtained from numerical method by Mitsotakis (Mitsotakis et al., 2014).

SWW equations

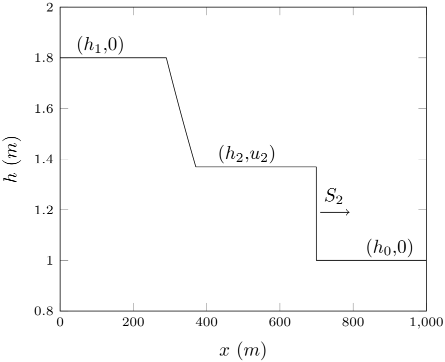

Figure: SWW analytic solution to dam break problem.

$$h_2 = \frac{h_0}{2}\left[\sqrt{1 + 8\left(\frac{2h_2}{h_2 - h_0}\frac{\sqrt{gh_1} - \sqrt{gh_2}}{\sqrt{gh_0}}\right)^2} - 1\right],$$

$$u_2 = 2\left(\sqrt{gh_1} - \sqrt{gh_2}\right),$$

$$S_2 = \frac{h_2 u_2}{h_2 - h_0}.$$

El and Grimshaws Whitham Modulation

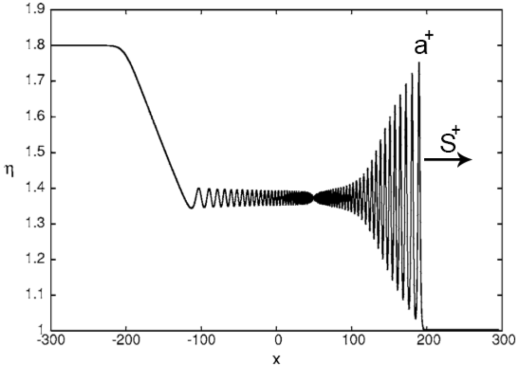

Figure: Whitham modulation values demonstrated on El and Grimshaws numerical results

$$\frac{\Delta}{\left(a^{+}+1\right)^{1/4}} - \left(\frac{3}{4-\sqrt{a^{+}+1}}\right)^{21/10}\left(\frac{2}{1+\sqrt{a^{+}+1}}\right)^{2/5} = 0$$

$$S^{+} = \sqrt{g\left(a^{+}+1\right)}$$

where $\Delta = \frac{h_1 - h_0}{h_0}$. Appropriate when $\Delta \leq 1.43$.

Findings

Literature

- El and Grimshaws numerical and analytic results supported, but do not give the full picture.
- Le Métayers first order scheme is too diffusive.
- Mistotakis initial conditions were not sufficiently steep.
- SWW analytic solution is a useful guide for the mean behaviour of the fluid.

Methods

Le Métayer methods.

- First order
- Second order
- Third order

Finite Difference Method

- El and Grimshaws

Initial Conditions

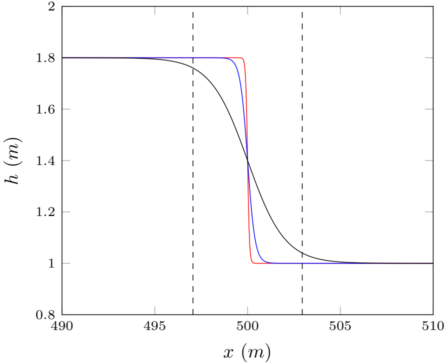

Figure: Initial Conditions where $\beta = 0.294$ (–), $\beta = 1.17778$ (–), $\beta = 5.8888$ (–) with reference β interval(– –).

$\beta = 5.8888$

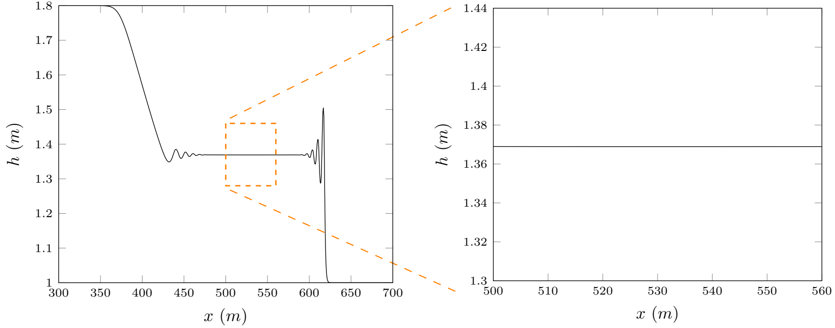

Figure: Numerical results of third order Le Métayer method at 30s with $\Delta x = \frac{10}{2^4}$ (—).

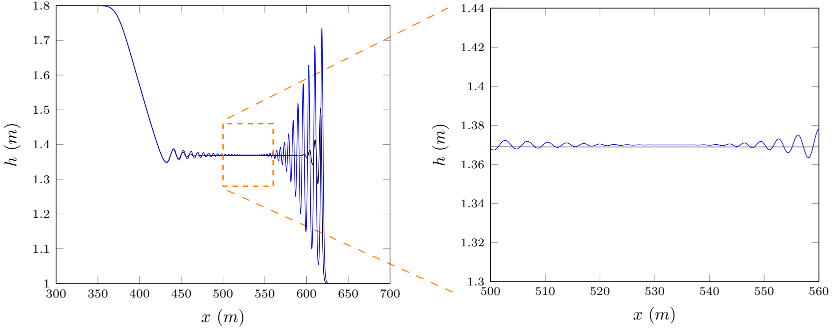

Figure: Numerical results of third order Le Métayer method at 30s with a Δx of $\frac{10}{2^4}$ (—) and $\frac{10}{2^7}$ (—).

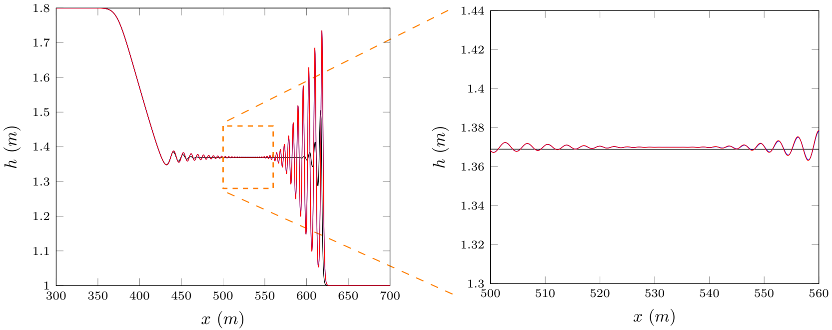

Figure: Numerical results of third order Le Métayer method at 30s with a Δx of $\frac{10}{2^4}$ (—) , $\frac{10}{2^7}$ (—) and $\frac{10}{2^{10}}$ (—).

$\beta = 1.17778$

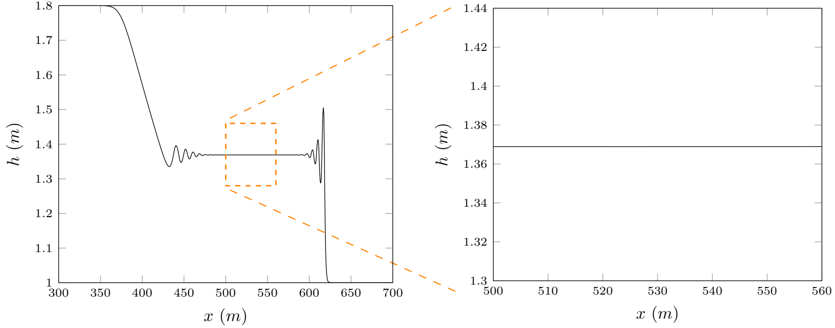

Figure: Numerical results of third order Le Métayer method at 30s with a Δx of $\frac{10}{2^4}$ (—).

Water Profile

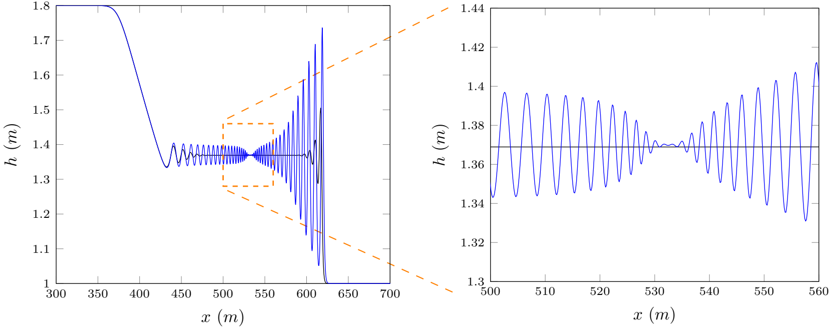

Figure: Numerical results of third order Le Métayer method at $30s$ with a Δx of $\frac{10}{2^4}$ (—) and $\frac{10}{2^7}$ (—).

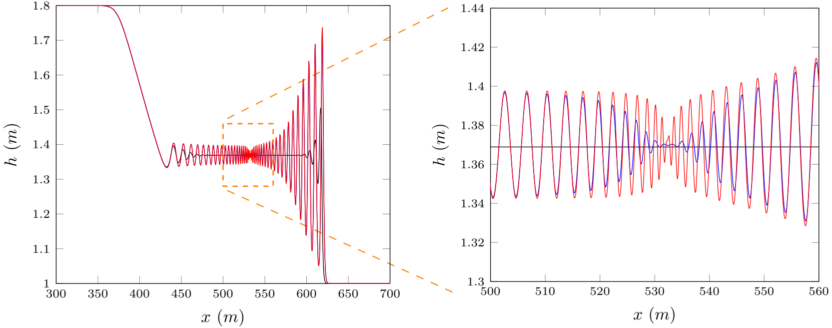

Figure: Numerical results of third order Le Métayer method at 30s with a Δx of $\frac{10}{2^4}$ (—), $\frac{10}{2^7}$ (—) and $\frac{10}{2^{10}}$ (—).

Dispersion Relation

The dispersion relation for the linearised Serre equations is

$$\omega = u_0 k \pm k\sqrt{gh_0}\sqrt{\frac{3}{h_0^2 k^2 + 3}}$$

Thus the phase speed is

$$v_p = u_0 \pm \sqrt{gh_0}\sqrt{\frac{3}{h_0^2 k^2 + 3}}$$

Taking $k \to 0$ we see $v_p \to u_0 \pm \sqrt{gh_0}$
Taking $k \to \infty$ we see $v_p \to u_0$

Contact Discontinuity

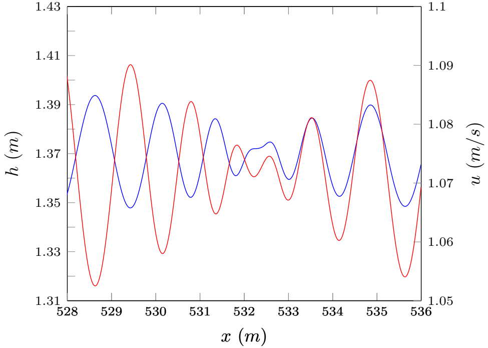

Figure: plot of h (—) and u (—) around contact discontinuity for third order Le Métayer method with $\Delta x = \frac{10}{2^{10}}$ at $30s$.

$\beta = 0.294$

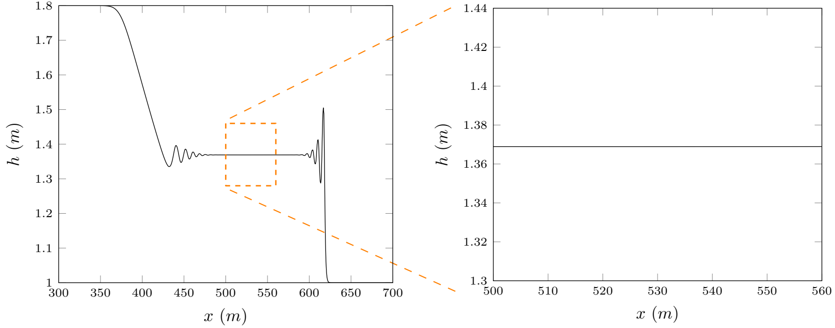

Figure: Numerical results of third order Le Métayer method at 30s with a Δx of $\frac{10}{2^4}$ (—).

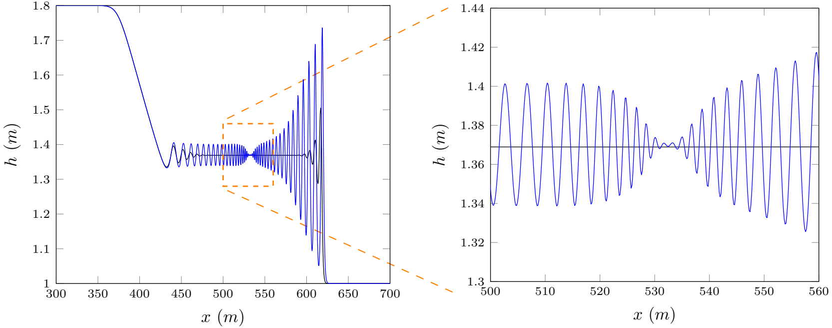

Figure: Numerical results of third order Le Métayer method at 30s with a Δx of $\frac{10}{2^4}$ (—) and $\frac{10}{2^7}$ (—).

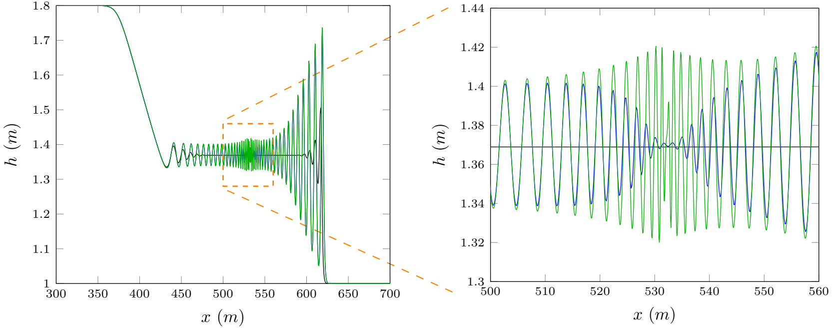

Figure: Numerical results of third order Le Métayer method at 30s with a Δx of $\frac{10}{2^4}$ (—) , $\frac{10}{2^7}$ (—) and $\frac{10}{2^9}$ (—).

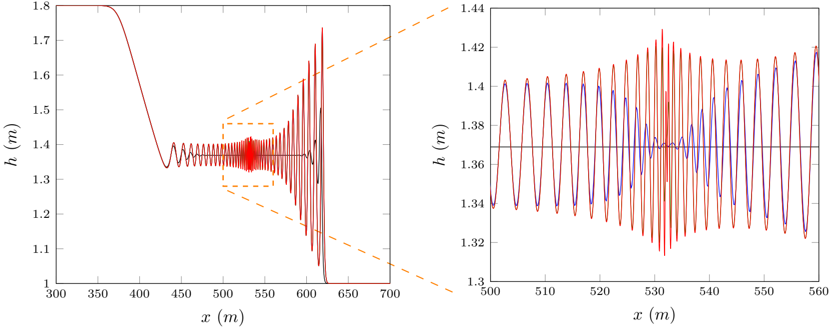

Figure: Numerical results of third order Le Métayer method at 30s with a Δx of $\frac{10}{2^4}$ (—) , $\frac{10}{2^7}$ (—) , $\frac{10}{2^9}$ (—) and $\frac{10}{2^{10}}$ (—).

$\beta = 0.294$ Various Models

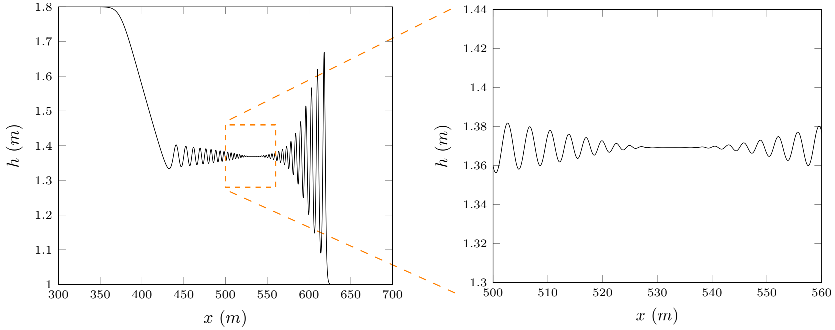

Figure: Numerical results at 30s with a $\Delta x = \frac{10}{2^{10}}$ for the first order(—) Le Métayer method.

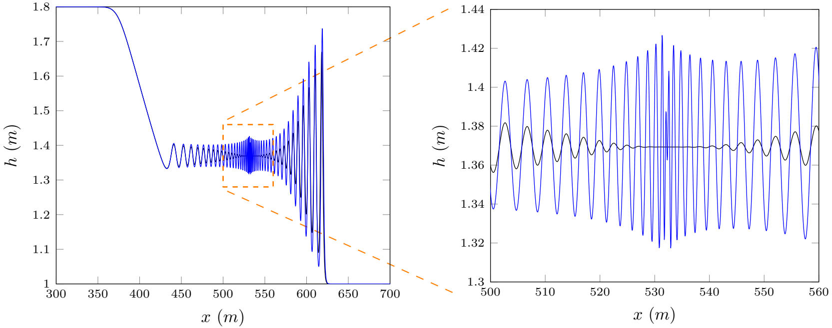

Figure: Numerical results at 30s with a $\Delta x = \frac{10}{2^{10}}$ for the first order (—) and second order (—) Le Métayer method.

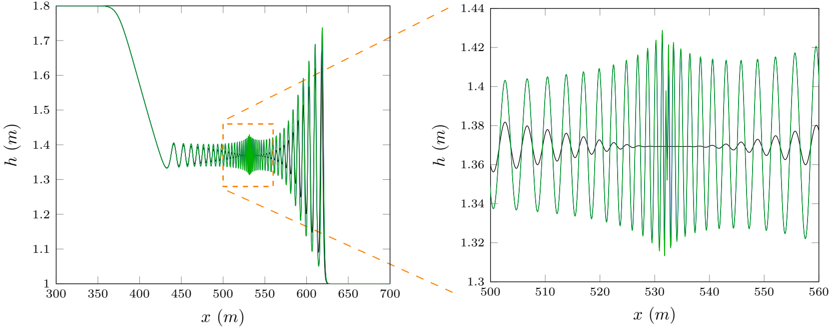

Figure: Numerical results at $30s$ with a $\Delta x = \frac{10}{2^{10}}$ for the first order (—), second order (—) and third order (—) Le Métayer method.

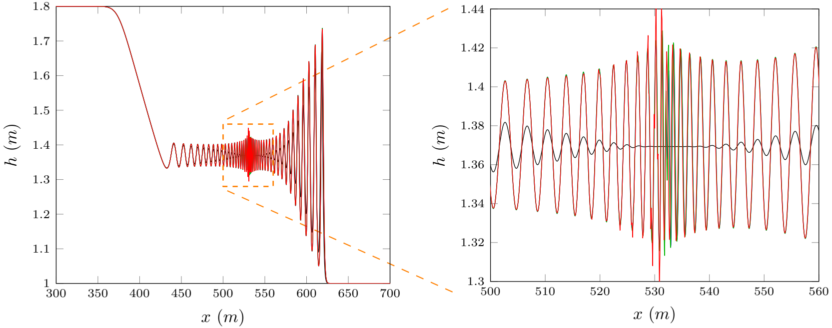

Figure: Numerical results at 30s with a $\Delta x = \frac{10}{2^{10}}$ for the first order (—), second order (—) and third order (—) Le Métayer method and El and Grimshaws method (—).

$\beta = 0.294$ Long Time

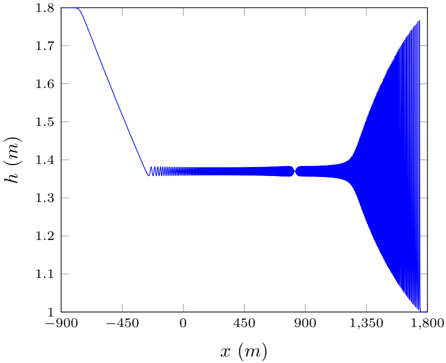

Figure: Numerical results at 300s with $\Delta x = 10/2^9$ for third-order Le Métayer Method.

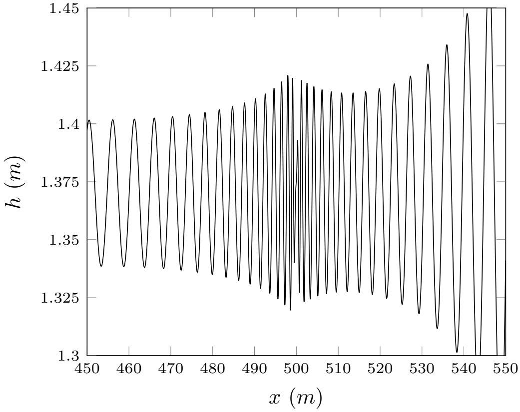

Figure: Translated numerical results with $\Delta x = 10/2^9$ at $30s$ (—) using third order Le Métayer method.

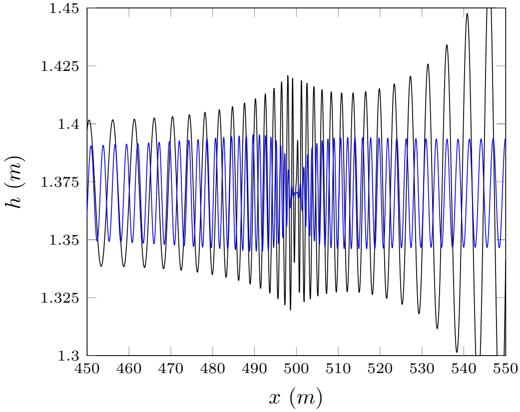

Figure: Translated numerical results with $\Delta x = 10/2^9$ at $30s$ (—) , $100s$ (—) using third order Le Métayer method.

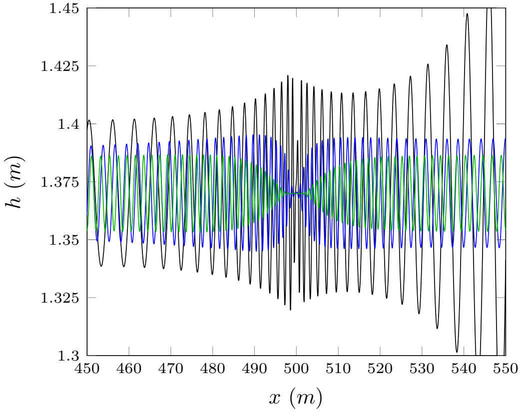

Figure: Translated numerical results with $\Delta x = 10/2^9$ at $30s$ (—) , $100s$ (—) , $200s$ (—) using third order Le Métayer method.

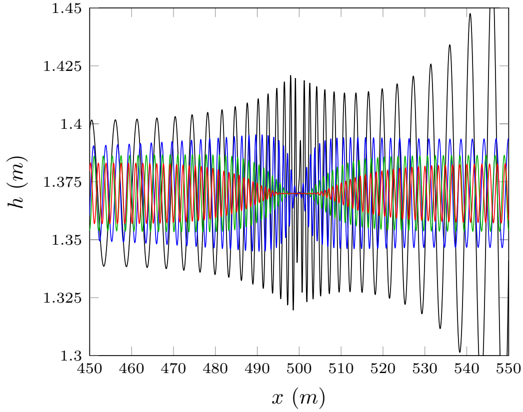

Figure: Translated numerical results with $\Delta x = 10/2^9$ at $30s$ (—) , $100s$ (—) , $200s$ (—) and $300s$ (—) using third order Le Métayer method.

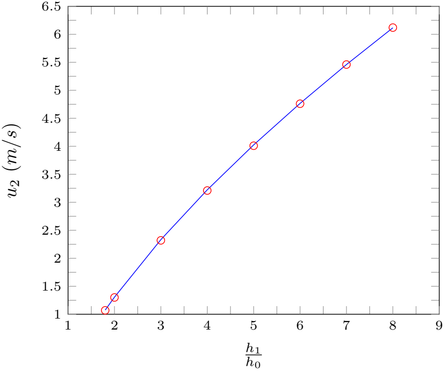

Figure: compares u_2 (—) to the average speed of the contact discontinuity (○) for third order Le Métayer method with $\Delta x = \frac{10}{2^9}$ at 300s.

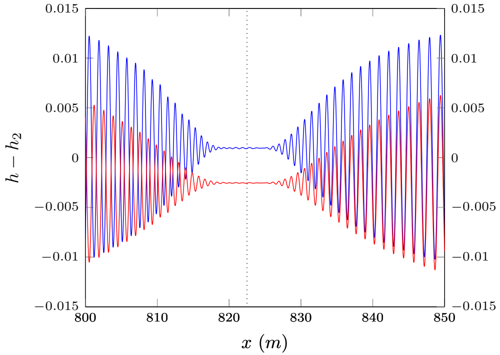

Figure: plot of $h - h_2$ (—) and $u - u_2$ (—) with x_2 ($\cdots$) for third order Le Métayer method with $\Delta x = \frac{10}{2^9}$ at $300s$.

SWWE Solution Comparison

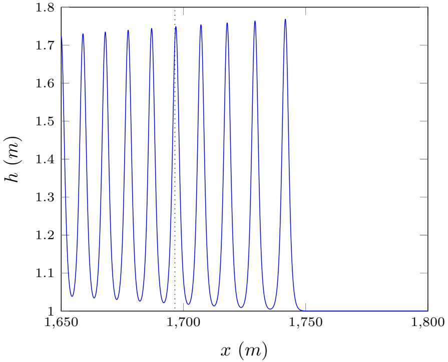

Figure: Plot comparing numerical results for shock front of the Serre equations to S_2 for third order Le Métayer method with $\Delta x = \frac{10}{2^9}$ at $300s$.

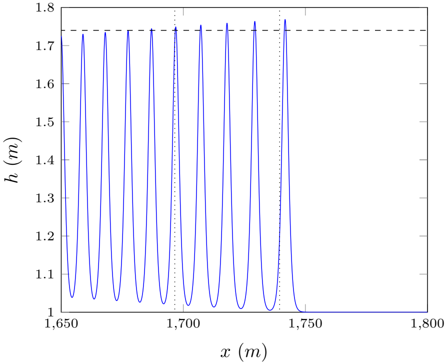

Figure: Plot comparing numerical results for shock front of the Serre equations to S^+ ($\cdots$), S_2($\cdots$)and a^+ (– –) for third order Le Métayer method with $\Delta x = \frac{10}{2^9}$ at 300s.

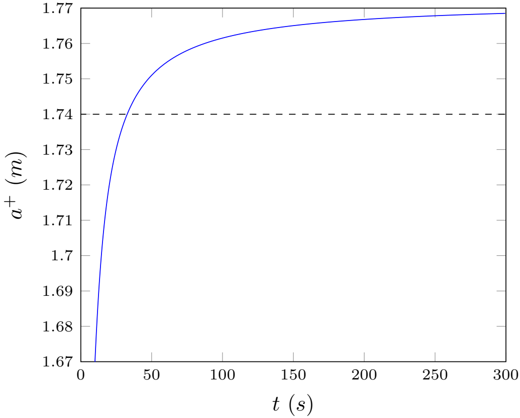

Figure: Plot of lead oscillation amplitude over time for third order Le Métayer method with $\Delta x = \frac{10}{2^9}$ at 300s with analytic comparison (− −)

Conclusions

Literature

- Supports Els results
  - Best numerical results for the dam break problem
  - a^+ seems to underestimate lead oscillation amplitude
  - S^+ underestimates speed
- Le Métayers first order scheme is too diffusive
- Mistotakis initial conditions were not sufficiently steep
- SWW analytic solution is a useful guide for the mean bore height h_2 (underestimate), mean bore velocity u_2 (overestimate) and speed of the bore S_2 (underestimate)

References I

El, G., Grimshaw, R. H. J., and Smyth, N. F. (2006).
Unsteady undular bores in fully nonlinear shallow-water theory.
Physics of Fluids, 18(027104).

Le Métayer, O., Gavrilyuk, S., and Hank, S. (2010).
A numerical scheme for the GreenNaghdi model.
Journal of Computational Physics, 229(6):2034–2045.

Mitsotakis, D., Dutykh, D., and Carter, J. (2014).
On the nonlinear dynamics of the traveling-wave solutions of the serre equations.
arXiv preprint arXiv:1404.6725.

Zoom in on u and h

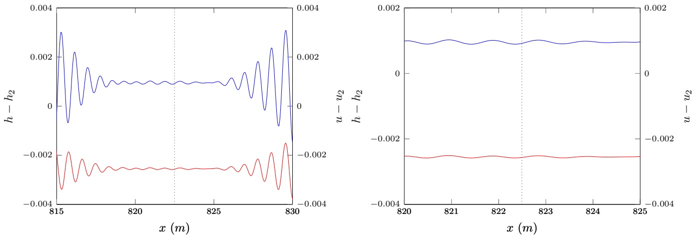

Figure: plot of $h - h_2$ (—) and $u - u_2$ (—) with x_2 ($\cdots$)

Zoom in on all Models

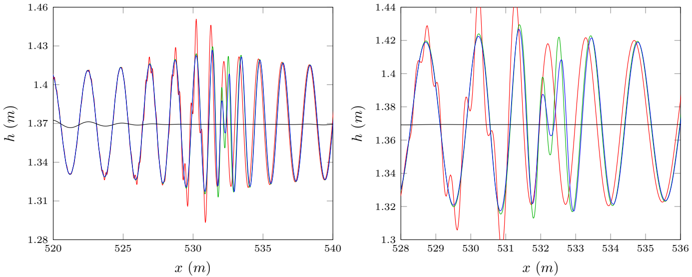

Figure: Numerical results at 30s with a $\Delta x = \frac{10}{2^{10}}$ for the first order (—), second order (—) and third order (—) Le Métayer method and El and Grimshaws method (—).

Serre Equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

$$\underbrace{\underbrace{\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2}\right)}_{\text{Shallow Water Wave Equations}} + \underbrace{\frac{\partial}{\partial x}\left(\frac{h^3}{3}\left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}\right]\right)}_{\text{Dispersion Terms}}}_{\text{Serre Equations}} = 0$$

Conservation Law Form

New conserved quantity

$$G = uh - h^2 \frac{\partial h}{\partial x}\frac{\partial u}{\partial x} - \frac{h^3}{3}\frac{\partial^2 u}{\partial x^2}. \tag{6}$$

Reformulated equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{7a}$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(Gu + \frac{gh^2}{2} - \frac{2h^3}{3}\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right) = 0 \tag{7b}$$

Basic Overview

Vector of conserved quantities:

$$\mathbf{U} = \begin{bmatrix} h \\ G \end{bmatrix}$$

Algorithm:

$$\mathcal{H}\left(\bar{\mathbf{U}}^n, \Delta x, \Delta t\right) = \begin{cases} \mathbf{U}^n & = & \mathcal{M}\left(\bar{\mathbf{U}}^n\right) \\ \mathbf{u}^n & = & \mathcal{A}\left(\mathbf{U}^n, \Delta x\right) \\ \bar{\mathbf{U}}^{n+1} & = & \mathcal{L}\left(\bar{\mathbf{U}}^n, \mathbf{u}^n, \Delta x, \Delta t\right) \end{cases} .$$