Robust Computational Models for Water Waves

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Outline of the Presentation

- Motivation
- History
- Contribution
  - Method
  - Validation

Physical Phenomena: Water Waves

Water wave hazards:

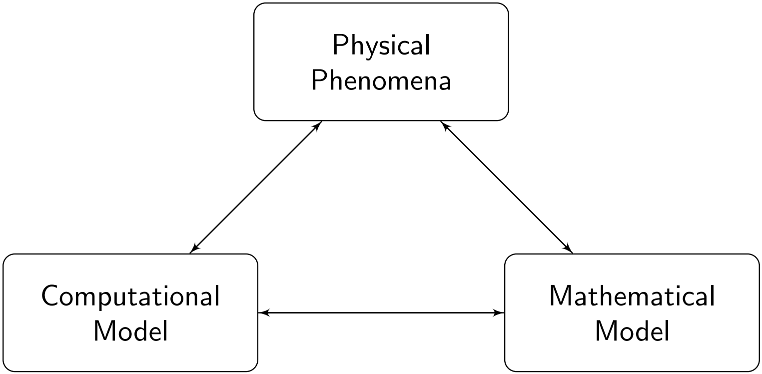
- Tsunamis
- Storm Surges
- Rogue Waves

Phenomena caused by water waves:

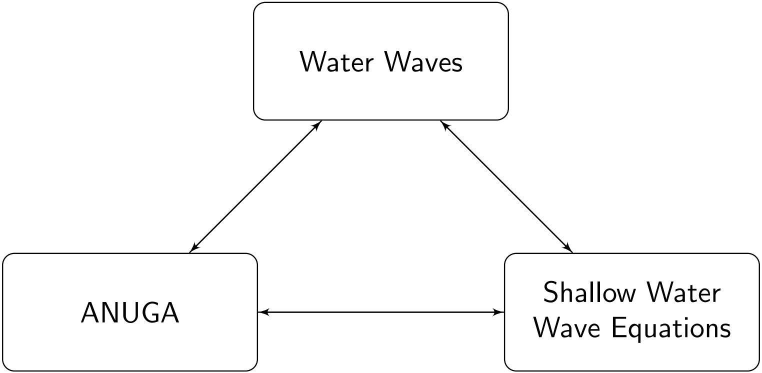
- Nutrient Transport
- Beach Erosion
- Breakup of Sea Ice

Computational Modelling

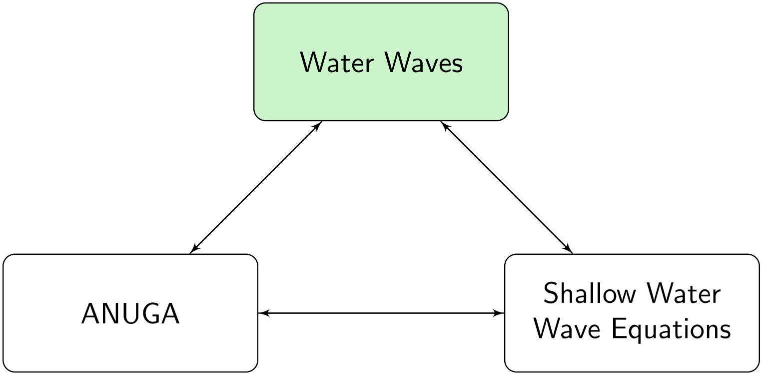
Goal: Model Physics On Computers



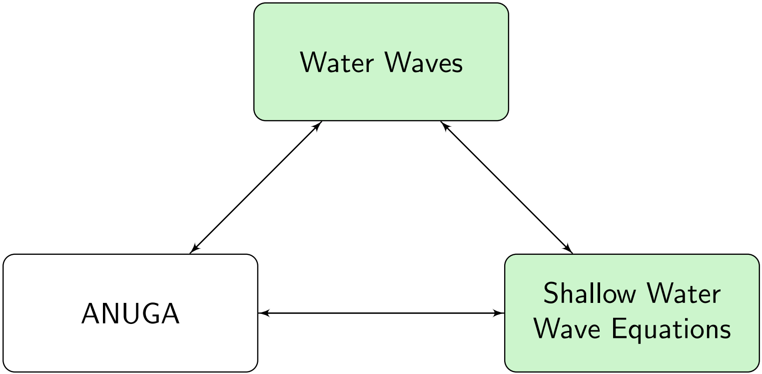
ANUGA



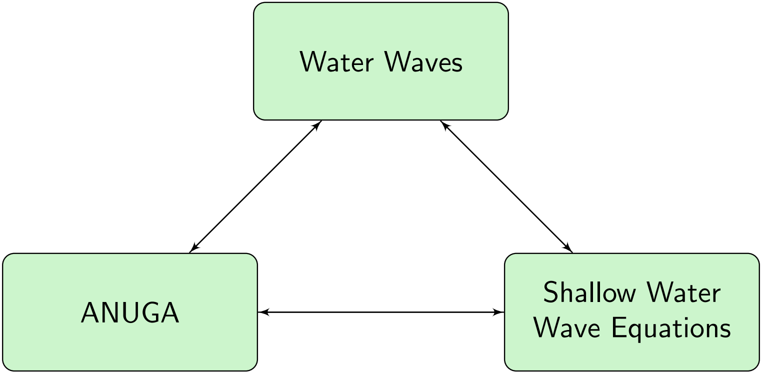
ANUGA: Water Waves



ANUGA: Shallow Water Wave Equations



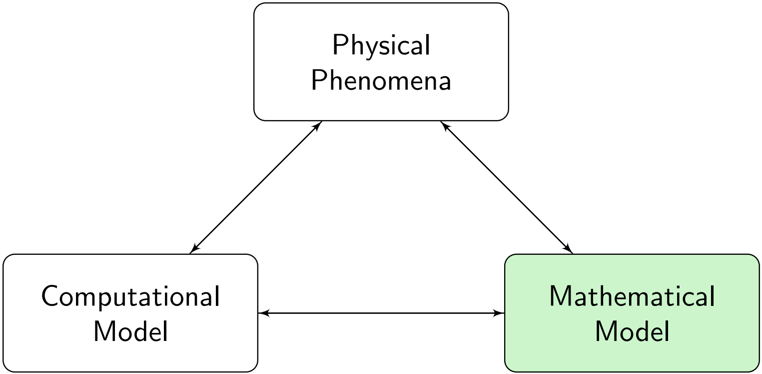
ANUGA



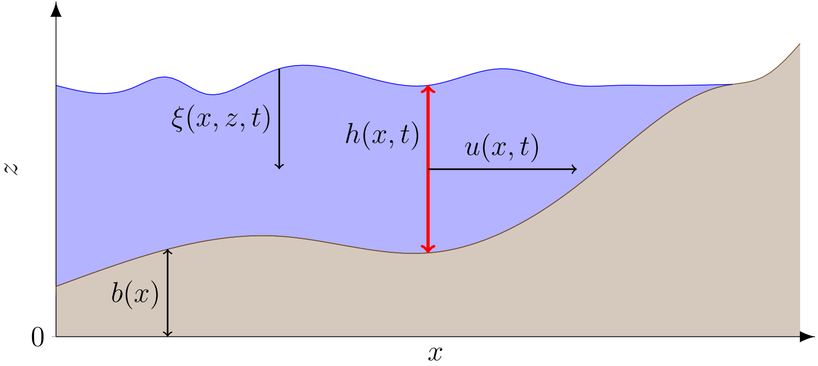
Outcome

New project at the ANU to build robust computational model from dispersive mathematical models

Mathematical Model



Serre Model



Assumptions

- Particle: $u'(x,z,t)$ constant in z
- Particle: $v'(x,z,t) = u\frac{\partial b}{\partial x} - (z-b)\frac{\partial b}{\partial x}$
- Particle: $p'(x,z,t) = g\xi + \xi\Psi + \frac{1}{2}\xi\left(2h-\xi\right)\Phi$

with

$$\Psi = \frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial^2 b}{\partial x^2},$$

$$\Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}.$$

Equations

Mass:
$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

Momentum:
$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^2}{2}\Psi + \frac{h^3}{3}\Phi\right)$$
$$+ \frac{\partial b}{\partial x}\left(gh + h\Psi + \frac{h^2}{2}\Phi\right) = 0.$$

$$\Psi = \frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial^2 b}{\partial x^2}, \quad \Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}.$$

Pros and Cons

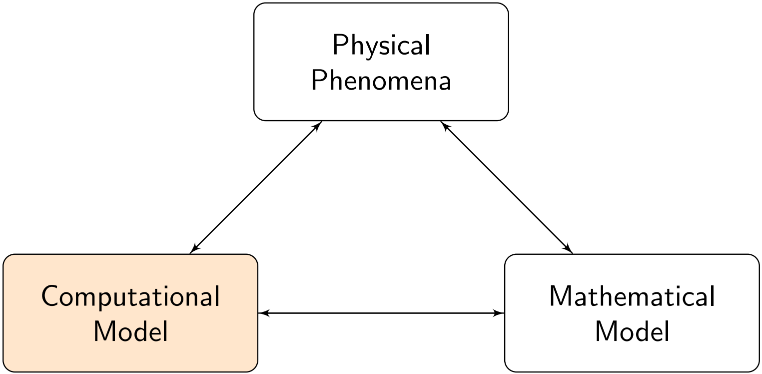
Pro:

- Far simpler than the Euler equations
- Includes dispersive effects
- Still a good model for long wavelength waves and also a good model for shorter wavelengths
- Considered one of the best models for water waves up to wave breaking

Cons:

- More complicated than the Shallow Water Wave Equations
- Cannot model breaking waves

Computational Model



Previous Work at the ANU

- 2014: Chris Zoppou's PhD thesis
  Demonstrated computational model based on Finite Volume Method for the Serre equations with varying bathymetry in 2D.
- 2014: My Honours thesis
  Independent reproduction of Chris Zoppou's computational model

Open problems:

3D: Extension of the method to 3D flows

Robust: Validation of model with steep gradients in free surface

Robust: Validation of model in the presence of dry beds

Thesis Goals

Solve these open problems:

3D: Extension of the method to 3D flows

Robust: Validation of model with steep gradients in free surface

Robust: Validation of model in the presence of dry beds

Technique: Develop a robust computational model from the 2D Serre equations that can be easily extended to 3D.

Method

Brief description of the method which is a combination of a Finite Volume Method and a Finite Element method.

Finite Volume Method

3D: Extends well to 3D

Robust: Stable in the presence of steep gradients

Robust: Stable in the presence of dry beds

- Maintains conservation properties of the equations

Chris Zoppou's thesis demonstrated how an adaptation of the Finite Volume Method to the Serre Equations.

Equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^2}{2}\Psi + \frac{h^3}{3}\Phi\right) + \frac{\partial b}{\partial x}\left(gh + h\Psi + \frac{h^2}{2}\Phi\right) = 0$$

$$\Psi = \frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial^2 b}{\partial x^2}, \quad \Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}.$$

For a Finite Volume Method we require equations in the form

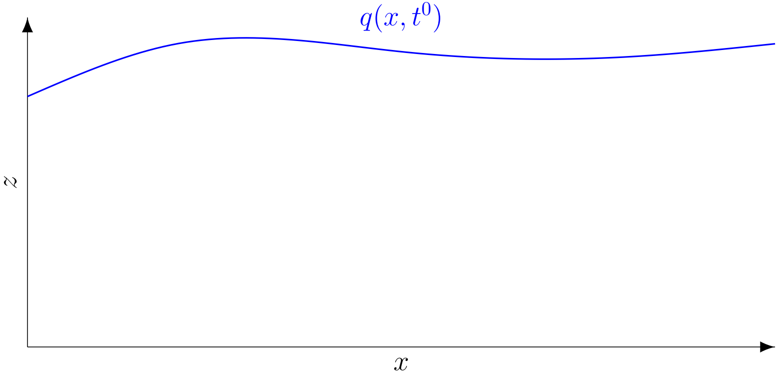
$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} + s(q) = 0$$

where $f(q)$ and $s(q)$ do not contain temporal derivatives

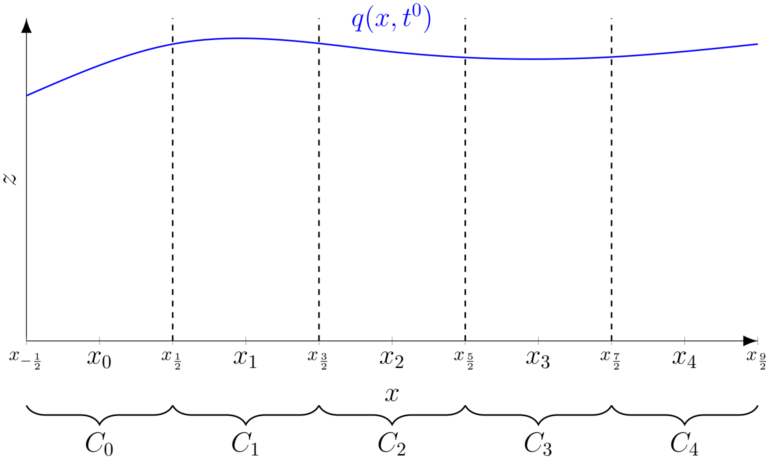
Finite Volume Method Example

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} + s(q) = 0$$

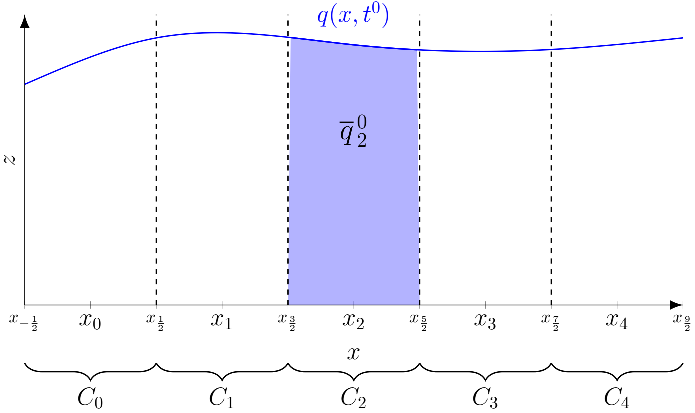
Function at $t = t_0$



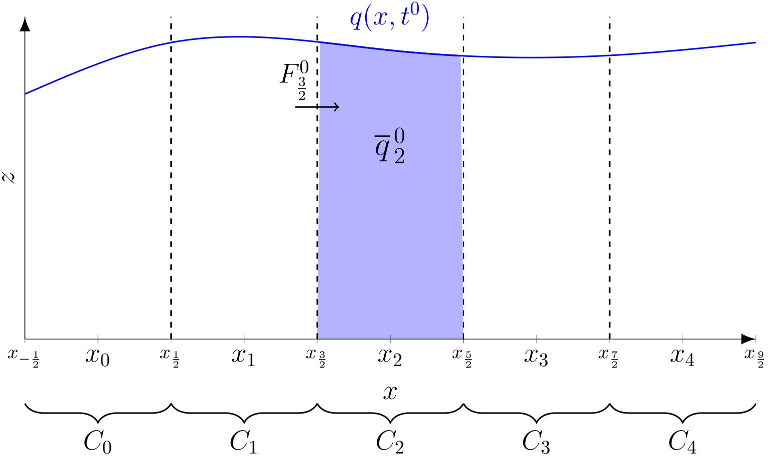
Cell Discretisation



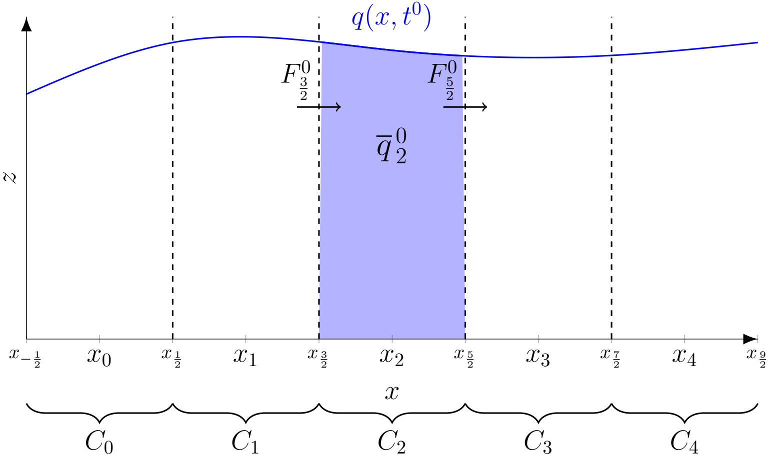
Total Amount



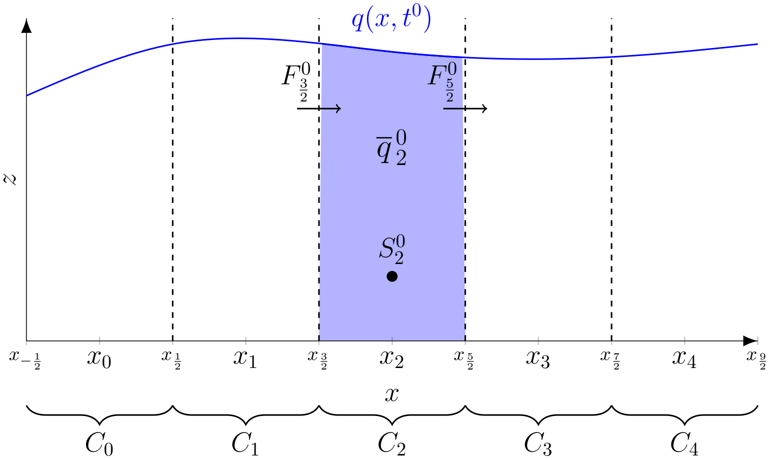
Flux Left


$q(x, t^0)$
$F^0_{\frac{3}{2}}$
$\overline{q}^0_2$
z
$x_{-\frac{1}{2}}$
x_0
$x_{\frac{1}{2}}$
x_1
$x_{\frac{3}{2}}$
x_2
$x_{\frac{5}{2}}$
x_3
$x_{\frac{7}{2}}$
x_4
$x_{\frac{9}{2}}$
x
C_0
C_1
C_2
C_3
C_4

Flux Right


$q(x,t^0)$
$F^0_{\frac{3}{2}}$
$F^0_{\frac{5}{2}}$
$\bar{q}^{\,0}_2$
z
$x_{-\frac{1}{2}}$ x_0 $x_{\frac{1}{2}}$ x_1 $x_{\frac{3}{2}}$ x_2 $x_{\frac{5}{2}}$ x_3 $x_{\frac{7}{2}}$ x_4 $x_{\frac{9}{2}}$
x
C_0 C_1 C_2 C_3 C_4

Source


$q(x, t^0)$
$F^0_{\frac{3}{2}}$
$F^0_{\frac{5}{2}}$
$\overline{q}^{\,0}_{2}$
S^0_2
z
$x_{-\frac{1}{2}}$
x_0
$x_{\frac{1}{2}}$
x_1
$x_{\frac{3}{2}}$
x_2
$x_{\frac{5}{2}}$
x_3
$x_{\frac{7}{2}}$
x_4
$x_{\frac{9}{2}}$
x
C_0
C_1
C_2
C_3
C_4

Finite Volume Update

$$\overline{q}_2^1 = \overline{q}_2^0 - \left(F_{\frac{5}{2}}^0 - F_{\frac{3}{2}}^0\right) - \left(S_2^0\right)$$

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} + s(q) = 0$$

Finite Volume Update

$$\overline{q}_2^1 = \overline{q}_2^0 - \left(F^0_{\frac{5}{2}} - F^0_{\frac{3}{2}}\right) - \left(S_2^0\right)$$

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} + s(q) = 0$$

$$\overbrace{\int_{C_2} q(x, t^1)\, dx}^{\overline{q}_2^1} = \overbrace{\int_{C_2} q(x, t^0)\, dx}^{\overline{q}_2^0} - \Bigg(\overbrace{\int_{t^0}^{t^1} f(q(x_{5/2}, t))\, dt}^{F^0_{\frac{5}{2}}}$$
$$- \overbrace{\int_{t^0}^{t^1} f(q(x_{3/2}, t))\, dt}^{F^0_{\frac{3}{2}}} \Bigg) - \overbrace{\int_{t^0}^{t^1} \int_{C_2} s(q(x, t))\, dt}^{S_2^0}$$

Update Formula for Serre Equations

$$\overline{h}_j^{n+1} = \overline{h}_j^{n} - \left[F^n_{j+1/2} - F^n_{j-1/2} \right]$$

$$\overline{G}_j^{n+1} = \overline{G}_j^{n} - \left[F^n_{j+1/2} - F^n_{j-1/2} \right] - S^n_j$$

- All the fluxes $F^n_{j+1/2}$ and $F^n_{j-1/2}$ and the source term S^n_j require u
- require a method to obtain u from $\overline{h}_j^n$, $\overline{G}_j^n$ and b

Calculate Velocity

$$G = hu\left(1 + \frac{\partial h}{\partial x}\frac{\partial b}{\partial x} + \frac{1}{2}h\frac{\partial^2 b}{\partial x^2} + \left[\frac{\partial b}{\partial x}\right]^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{3}h^3\frac{\partial u}{\partial x}\right).$$

- Chris Zoppou's Thesis used a Finite Difference Method
- Goal: Solve using a Finite Element Method

Finite Element Method

3D: Extends well to 3D

Robust: Stable in the presence of steep gradients

- Maintains conservation properties for conservation equations

Finite Element Method Example

Example:

$$-\frac{\partial^2 u}{\partial x^2} = f.$$

Weak Form

$$-\int_\Omega \frac{\partial^2 u}{\partial x^2} v = \int_\Omega fv \; dx$$

Integrate by parts (Dirichlet boundary conditions):

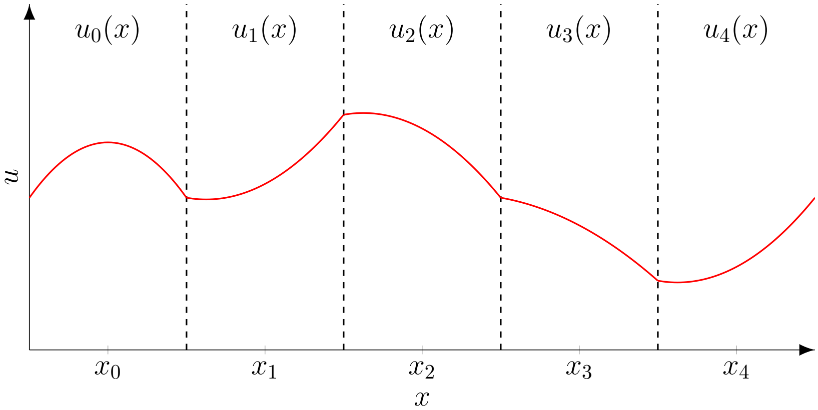
$$\int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} \; dx = \int_\Omega fv \; dx$$

Finite Element Method

$$\int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\;dx = \int_\Omega fv\;dx$$

$$\sum_j \left[\int_{C_j} \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\;dx\right] = \sum_j \left[\int_{C_j} fv\;dx\right]$$

Piecewise Polynomial Representation



Finite Element Method

$$\sum_j \left[\int_{C_j} \frac{\partial u_j}{\partial x} \frac{\partial v_j}{\partial x} \, dx \right] = \sum_j \left[\int_{C_j} f_j v_j \, dx \right]$$

$$\mathbf{A}\vec{u} = \vec{c}$$

where

- $\mathbf{A}$ depends on the polynomial representation of v
- $\vec{u}$ determines the polynomial representation of u
- $\vec{c}$ depends on polynomial representation of f and v

Finite Element Method for Serre Equations

$$G = hu\left(1 + \frac{\partial h}{\partial x}\frac{\partial b}{\partial x} + \frac{1}{2}h\frac{\partial^2 b}{\partial x^2} + \left[\frac{\partial b}{\partial x}\right]^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{3}h^3\frac{\partial u}{\partial x}\right).$$

$$\mathbf{A}\vec{u} = \vec{c}$$

where

- $\mathbf{A}$ depends on the polynomial representation of h, b and test function
- $\vec{u}$ determines the polynomial representation of u
- $\vec{c}$ depends on polynomial representation of G and test function

Method

- Reconstruction: Calculate the representations of h and G over the cells from the averages $\bar{h}$ and $\bar{G}$
- Finite Element: use the representations of h and G over the cells to calculate the representation of u over the cell
- Finite Volume Method: Update h and G to the next time using the Finite Volume Method update

Progress

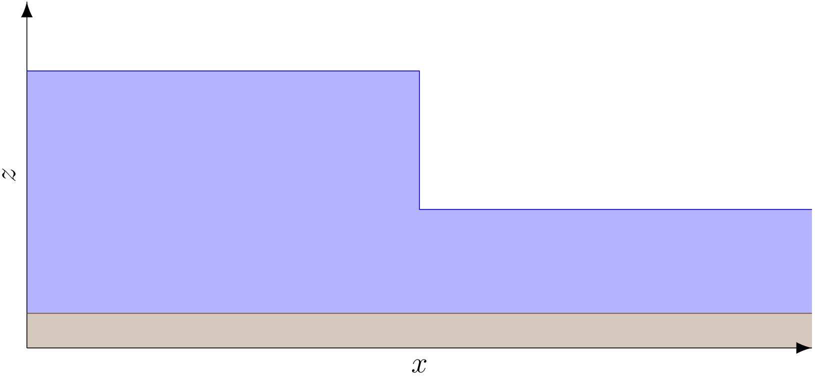
3D: Extension of the method to 3D flows ✓

Robust: Validation of model with steep gradients in free surface

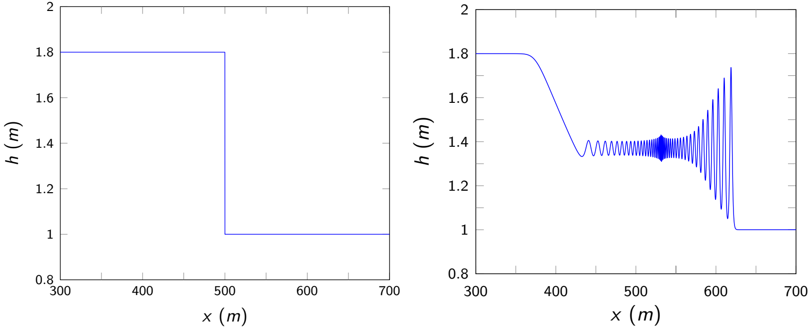
Robust: Validation of model in the presence of dry beds

Statement of Problem

How does this initially still body of water evolve?



Our New Numerical Solution



Pitt, J., Zoppou, C., and Roberts, S. (2018). Behaviour of the Serre Equations in the Presence of Steep Gradients Revisited. Wave Motion, 76(1):6177.

What was known

- No analytic solutions
- Asymptotic results for step gradient problems as $t \to \infty$
- Some experimental comparisons [1]
- Other numerical solutions from the literature; some solving the actual steep gradient problem and some solving a smoothed initial water profile.

[1] Zoppou, C. (2014). Numerical Solution of the One-dimensional and Cylindrical Serre Equations for Rapidly Varying Free Surface Flows. PhD thesis, Australian National University.

Solution

- Demonstrate convergence to one solution for many numerical methods
- Demonstrate good agreement with asymptotic results
- Comprehensively review many numerical methods and smoothing techniques from the literature

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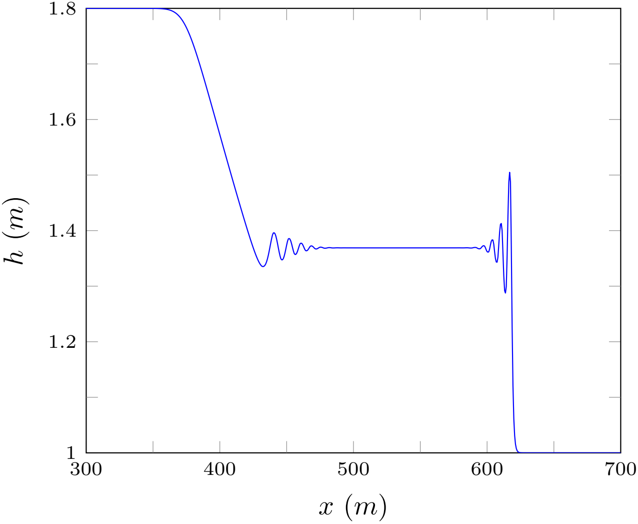
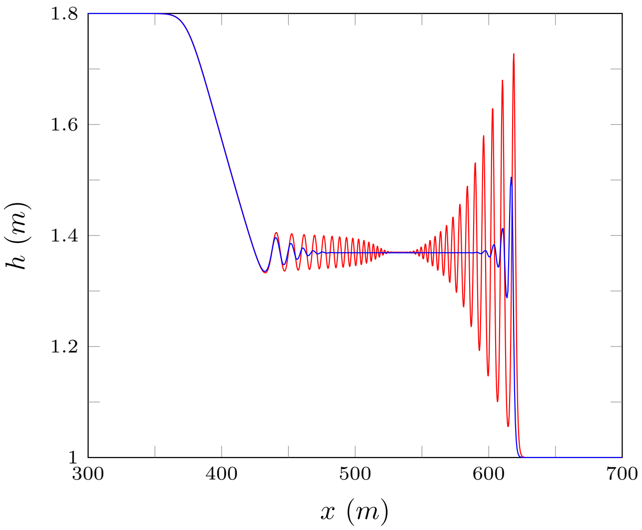
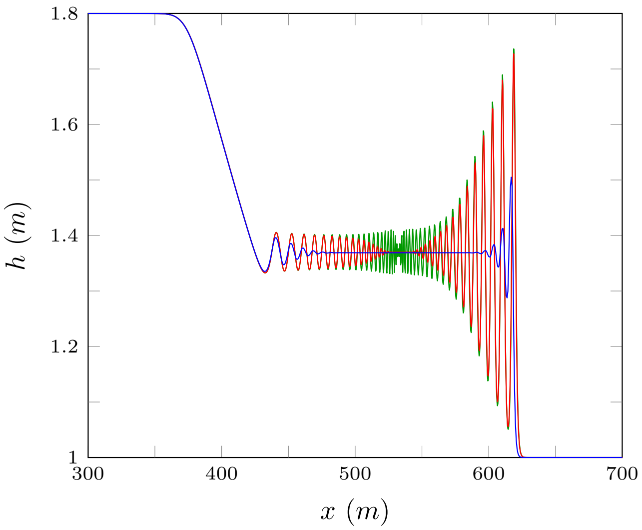
Pitt, J., Zoppou, C., and Roberts, S. (2018). Behaviour of the Serre Equations in the Presence of Steep Gradients Revisited. Wave Motion, 76(1):6177.

Convergence




1.8
1.6
1.4
1.2
1
300
400
500
600
700
$h\ (m)$
$x\ (m)$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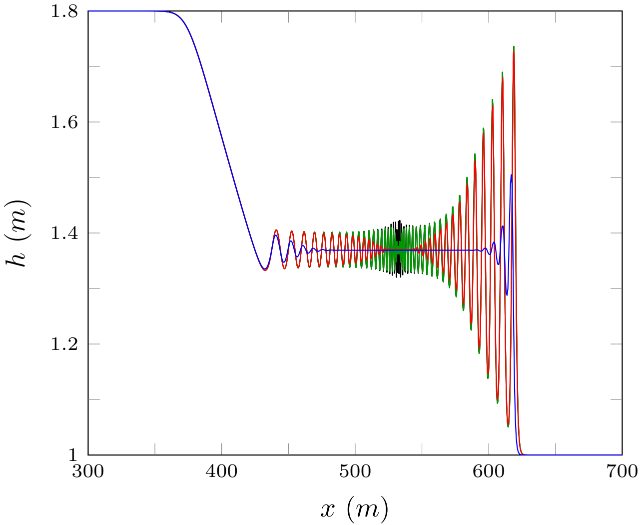
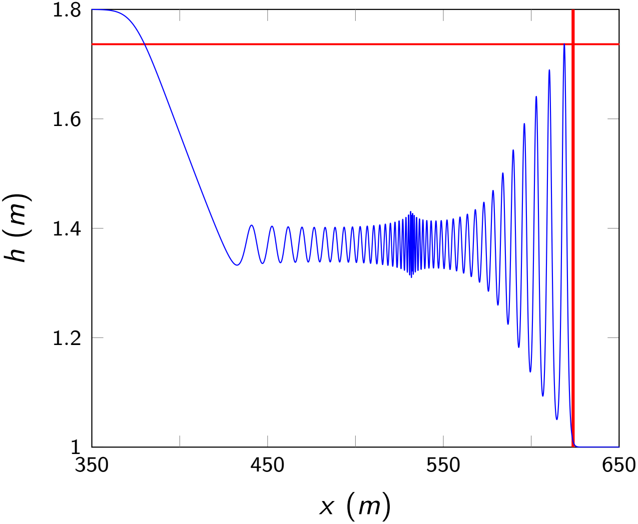

1.8
1.6
1.4
1.2
1
$h\ (m)$
300
400
500
600
700
$x\ (m)$

Asymptotic Results



Review of Smoothing and Methods

- Demonstrated that behaviour is consistent across many numerical methods
- Were able to explain why the behaviour had not previously been observed

Result

Our numerical solutions for the steep gradient problems are well validated [2]

[2] Pitt, J., Zoppou, C., and Roberts, S. (2018). Behaviour of the Serre Equations in the Presence of Steep Gradients Revisited. Wave Motion, 76(1):6177.

Progress

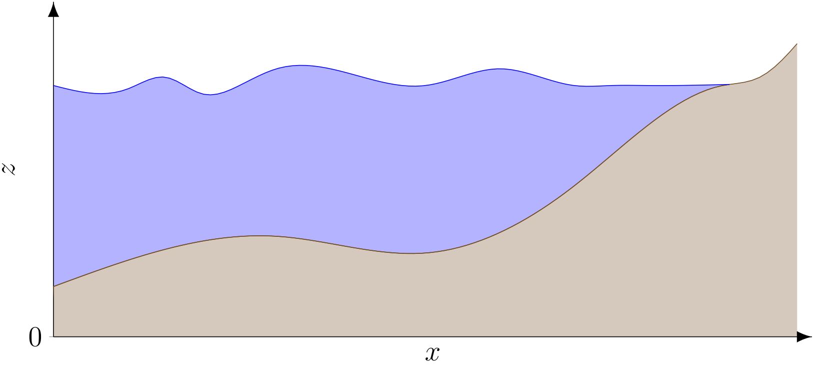
3D: Extension of the method to 3D flows ✓

Robust: Validation of model with steep gradients in free surface ✓

Robust: Validation of model in the presence of dry beds

Statement of Problem

Properly handle interaction of waves and the dry bed

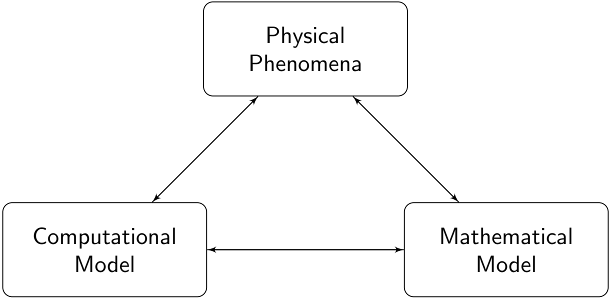


What was known

- No analytic solutions
- A variety of numerical techniques only validated against experimental data

Solution

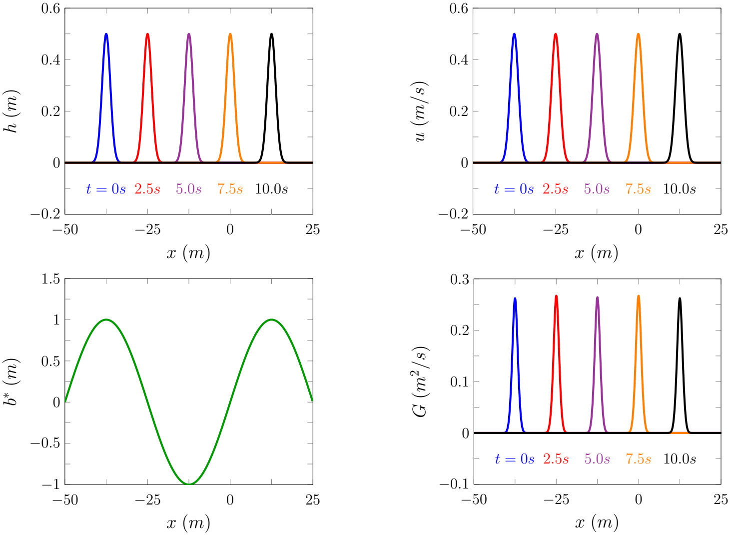
- Solved modified equations that did possess analytic solutions
- Compared with experimental data



Constructing Modified Equations

- Pick functions for height, velocity and bed: h^*, u^* and b^*
- Add Source terms to Serre equations that force a solution for h^*, u^* and b^*
- Validation tests

Pick Functions



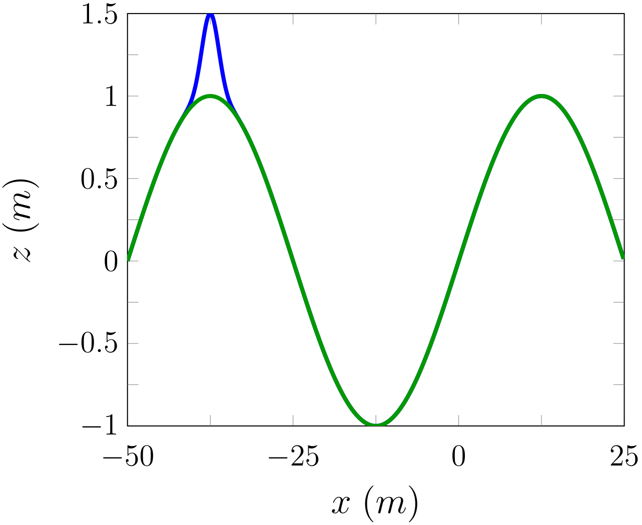
Modify Equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = S_h^*,$$

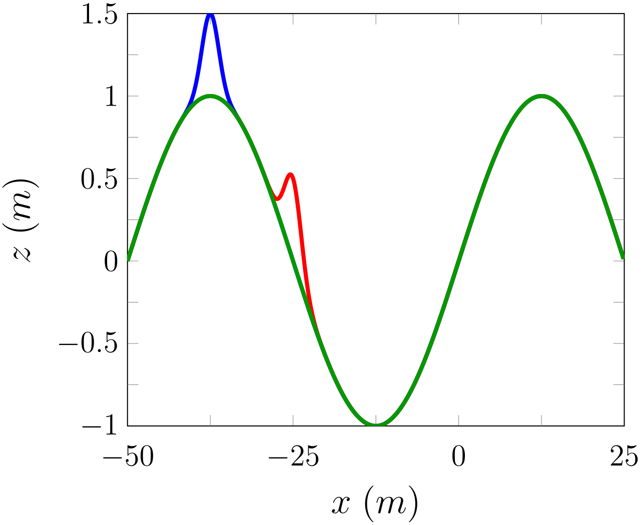
$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}h^3\left[\frac{\partial u}{\partial x}\right]^2 + h^2 u \frac{\partial u}{\partial x}\frac{\partial b}{\partial x}\right)$$
$$+\frac{1}{2}h^2 u \frac{\partial u}{\partial x}\frac{\partial^2 b}{\partial x^2} - hu^2\frac{\partial b}{\partial x}\frac{\partial^2 b}{\partial x^2} + gh\frac{\partial b}{\partial x} = S_G^*.$$

S_h^* and S_G^* are just the LHS with the quantities replaced by their associated chosen function. We solve the LHS using our method and add in the source terms on the RHS analytically.

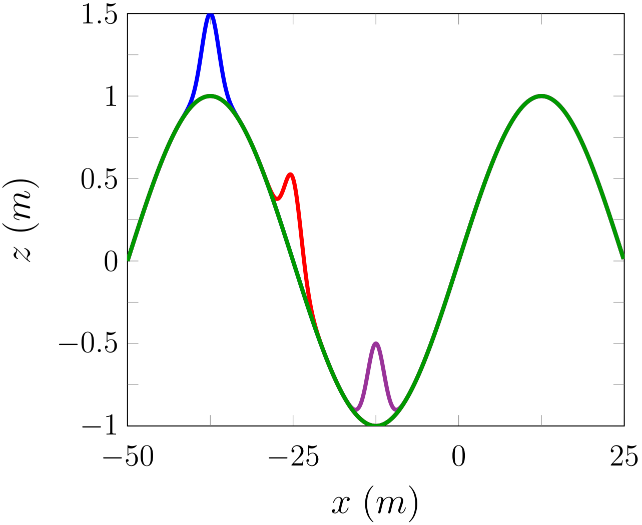
Results $t = 0s$



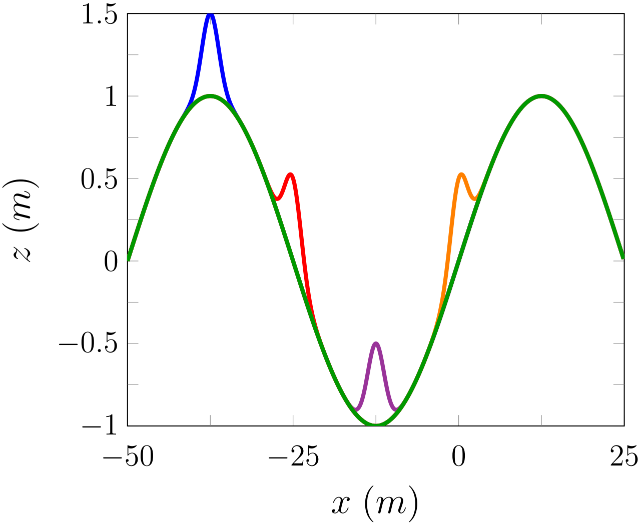
Results $t = 0s, 2.5s$



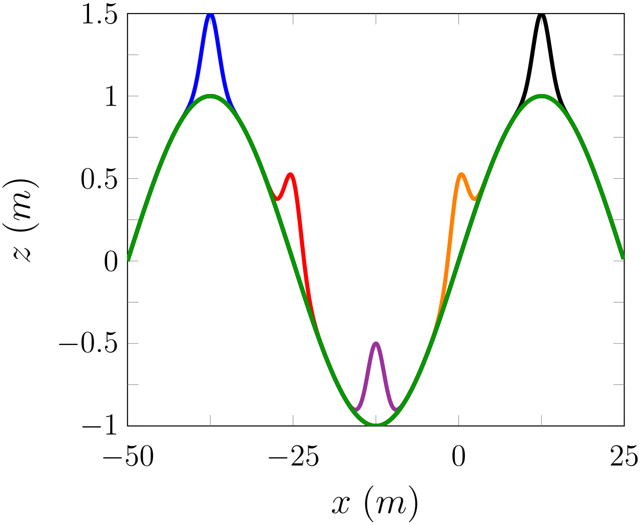
Results $t = 0s, 2.5s, 5.0s$



Results $t = 0s, 2.5s, 5.0s, 7.5s$



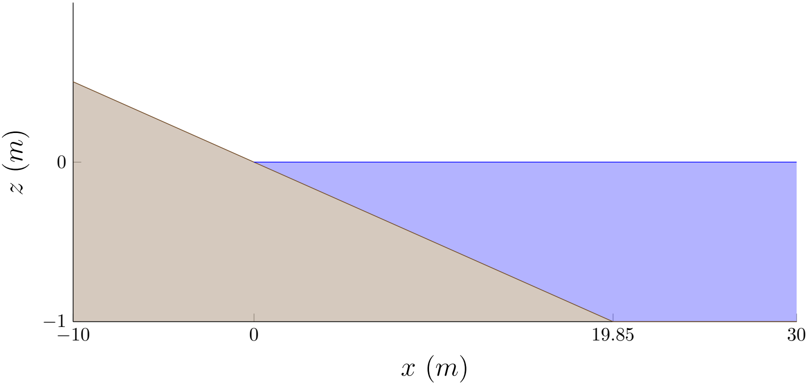
Results $t = 0s, 2.5s, 5.0s, 7.5s, 10.0s$



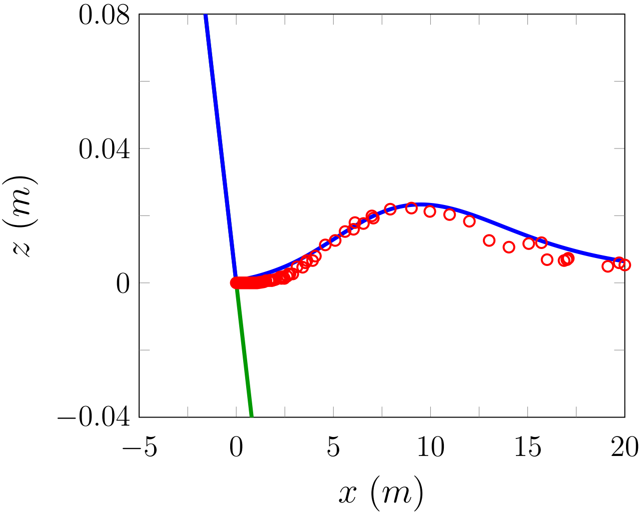
Modified Equations Validation Conclusions

- Very strong test that we are actually solving the Serre equations accurately as all terms must be accurately approximated
- Can measure the convergence of numerical solutions to the force solutions

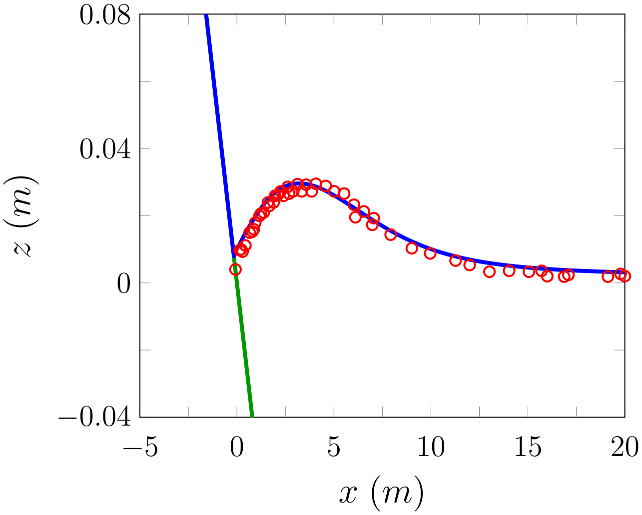
Experimental Data



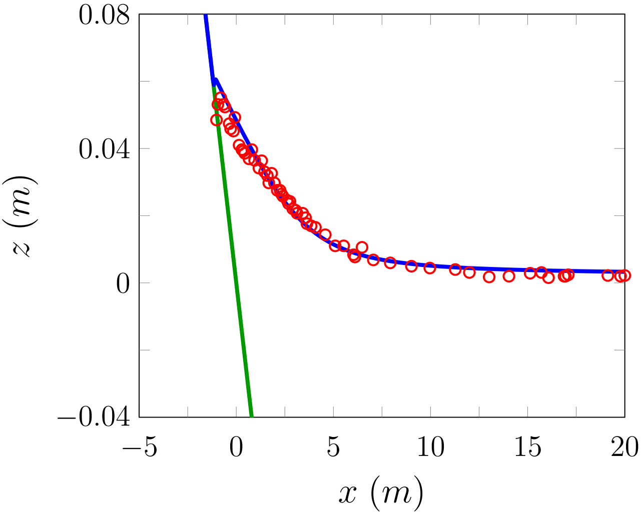
$t = 30s$



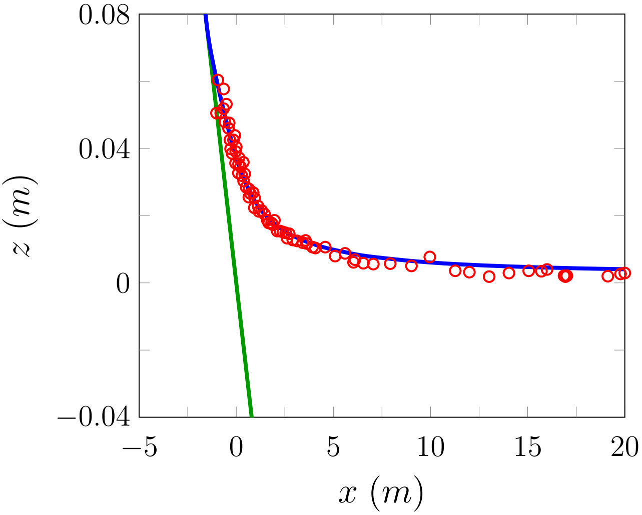
$t = 40s$



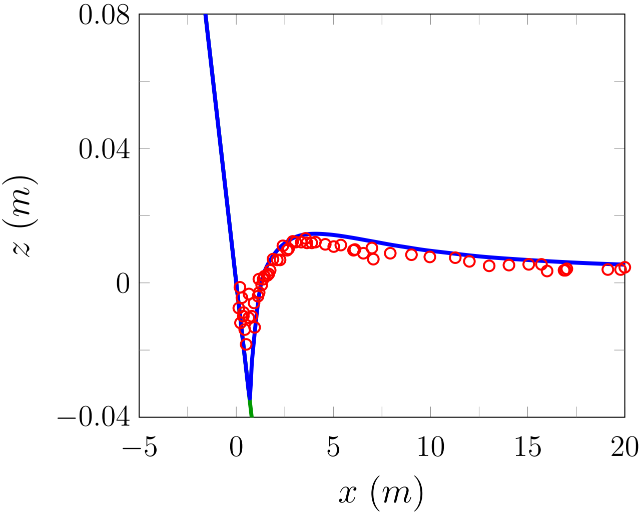
$t = 50s$



$t = 60s$



$t = 70s$



Experimental Validation Conclusions

- Demonstrates that our computational model agrees with the physical process
- not a very stringent test as there are many source of errors
- few experimental results for non breaking waves

Progress

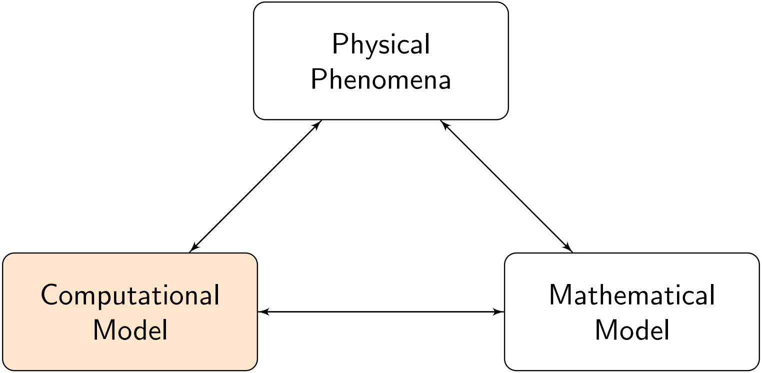
3D: Extension of the method to 3D flows ✓

Robust: Validation of model with steep gradients in free surface ✓

Robust: Validation of model in the presence of dry beds ✓

Conclusions

- Developed a Robust Computational Model from the Serre equations for the 2D water wave problem



References I

Pitt, J., Zoppou, C., and Roberts, S. (2018).
Behaviour of the serre equations in the presence of steep gradients revisited.
Wave Motion, 76(1):61–77.

Zoppou, C. (2014).
Numerical Solution of the One-dimensional and Cylindrical Serre Equations for Rapidly Varying Free Surface Flows.
PhD thesis, Australian National University, Mathematical Sciences Institute, College of Physical and Mathematical Sciences, Australian National University, Canberra, ACT 2600, Australia.