# 1 Elliptic Equation

The linearised elliptic equation is

$$G = H\upsilon - \frac{H^3}{3}\left(\frac{\partial^2 \upsilon}{\partial x^2}\right)$$

Taking the weak version of this we get that

$$\int_\Omega G v \; dx = H \int_\Omega \upsilon v \; dx - \frac{H^3}{3}\int_\Omega \frac{\partial^2 \upsilon}{\partial x^2} v \; dx$$

$$\int_\Omega G v \; dx = H \int_\Omega \upsilon v \; dx + \frac{H^3}{3}\int_\Omega \frac{\partial \upsilon}{\partial x}\frac{\partial v}{\partial x} \; dx$$

In particular for the basis function $\phi_j$ we must have

$$\int_\Omega G \phi_j \; dx = H \int_\Omega \upsilon \phi_j \; dx + \frac{H^3}{3}\int_\Omega \frac{\partial \upsilon}{\partial x}\frac{\partial \left(\phi_j\right)}{\partial x} \; dx$$

We use the FEM discretisation from []

$$G = \sum_j G^+_{j-1/2}\psi^+_{j-1/2} + G^-_{j+1/2}\psi^-_{j+1/2}$$

and

$$\upsilon = \sum_j \upsilon_{j-1/2}\phi_{j-1/2} + \upsilon_{j+1/2}\phi_{j+1/2} \tag{1}$$

From our equation making the substitutions and integrating the basis functions we get that

$$\sum_j \frac{\Delta x}{2}\begin{bmatrix}\frac{1}{3}\\ \frac{1}{3}\end{bmatrix}\begin{bmatrix}G^+_{j-1/2}\\ G^-_{j+1/2}\end{bmatrix} = \sum_j H\frac{\Delta x}{2}\begin{bmatrix}\frac{4}{15} & \frac{-1}{15}\\ \frac{-1}{15} & \frac{14}{15}\end{bmatrix}\begin{bmatrix}u_{j-1/2}\\ u_{j+1/2}\end{bmatrix} + \frac{2H^3}{3\Delta x}\begin{bmatrix}\frac{7}{6} & \frac{1}{6}\\ \frac{1}{6} & \frac{7}{6}\end{bmatrix}\begin{bmatrix}u_{j-1/2}\\ u_{j+1/2}\end{bmatrix} \tag{2}$$

$$\sum_j \frac{1}{3}\begin{bmatrix}e^{-ik\Delta x}\mathcal{R}^+_2\\ \mathcal{R}^-_2\end{bmatrix} G_j = \sum_j H\begin{bmatrix}\frac{4}{15} & \frac{-1}{15}\\ \frac{-1}{15} & \frac{14}{15}\end{bmatrix}\begin{bmatrix}e^{-ik\frac{\Delta x}{2}}\\ e^{ik\frac{\Delta x}{2}}\end{bmatrix} u_j + \frac{4H^3}{3\Delta x^2}\begin{bmatrix}\frac{7}{6} & \frac{1}{6}\\ \frac{1}{6} & \frac{7}{6}\end{bmatrix}\begin{bmatrix}e^{-ik\frac{\Delta x}{2}}\\ e^{ik\frac{\Delta x}{2}}\end{bmatrix} u_j \tag{3}$$

$$\sum_j \frac{1}{3}\begin{bmatrix} e^{-ik\Delta x}\mathcal{R}_2^+ \\ \mathcal{R}_2^- \end{bmatrix} G_j = \sum_j H\frac{1}{15}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} e^{-ik\frac{\Delta x}{2}} \\ e^{ik\frac{\Delta x}{2}} \end{bmatrix} u_j + \frac{4H^3}{3\Delta x^2}\frac{1}{6}\begin{bmatrix} 7 & 1 \\ 1 & 7 \end{bmatrix}\begin{bmatrix} e^{-ik\frac{\Delta x}{2}} \\ e^{ik\frac{\Delta x}{2}} \end{bmatrix} u_j \tag{4}$$

$$\begin{aligned}\sum_j \frac{1}{3}\begin{bmatrix} e^{-ik\Delta x}\mathcal{R}_2^+ \\ \mathcal{R}_2^- \end{bmatrix} G_j = \sum_j H\frac{1}{15}&\begin{bmatrix} 3\cos\left(k\frac{\Delta x}{2}\right) - 5i\sin\left(k\frac{\Delta x}{2}\right) \\ 3\cos\left(k\frac{\Delta x}{2}\right) + 5i\sin\left(k\frac{\Delta x}{2}\right) \end{bmatrix} u_j \\ &+ \frac{4H^3}{3\Delta x^2}\frac{1}{6}\begin{bmatrix} 8\cos\left(k\frac{\Delta x}{2}\right) - 6i\sin\left(k\frac{\Delta x}{2}\right) \\ 8\cos\left(k\frac{\Delta x}{2}\right) + 6i\sin\left(k\frac{\Delta x}{2}\right) \end{bmatrix} u_j \end{aligned} \tag{5}$$

If we add the contributions from the overlapping elements we get

$$\begin{aligned}\sum_j \frac{1}{3}\begin{bmatrix} e^{-ik\Delta x}\left(\mathcal{R}_2^- + \mathcal{R}_2^+\right) \\ \mathcal{R}_2^- + \mathcal{R}_2^+ \end{bmatrix} G_j = \sum_j H\frac{1}{15}&\begin{bmatrix} -2e^{-ik\frac{\Delta x}{2}}\left(\cos\left(2k\frac{\Delta x}{2}\right) - 4\right) \\ -2e^{ik\frac{\Delta x}{2}}\left(\cos\left(2k\frac{\Delta x}{2}\right) - 4\right) \end{bmatrix} u_j \\ &+ \frac{4H^3}{3\Delta x^2}\frac{1}{6}\begin{bmatrix} 2e^{-ik\frac{\Delta x}{2}}\left(\cos\left(2k\frac{\Delta x}{2}\right) + 7\right) \\ 2e^{ik\frac{\Delta x}{2}}\left(\cos\left(2k\frac{\Delta x}{2}\right) + 7\right) \end{bmatrix} u_j \end{aligned} \tag{6}$$