1 Elliptic Equation

The linearised elliptic equation is

$$G = H\upsilon - \frac{H^3}{3}\left(\frac{\partial^2 \upsilon}{\partial x^2}\right)$$

Taking the weak version of this we get that

$$\int_\Omega Gv \, dx = H \int_\Omega \upsilon v \, dx - \frac{H^3}{3} \int_\Omega \frac{\partial^2 \upsilon}{\partial x^2} v \, dx$$

$$\int_\Omega Gv \, dx = H \int_\Omega \upsilon v \, dx + \frac{H^3}{3} \int_\Omega \frac{\partial \upsilon}{\partial x}\frac{\partial v}{\partial x} \, dx$$

We use the FEM discretisation from []

$$G = \sum_j G^+_{j-1/2}\psi^+_{j-1/2} + G^-_{j+1/2}\psi^-_{j+1/2}$$

and

$$\upsilon = \sum_j \upsilon_{j-1/2}\phi_{j-1/2} + \upsilon_j\phi_j + \upsilon_{j+1/2}\phi_{j+1/2} \tag{1}$$

Now for our evolution equations we only need to to get the errors introduced from our calculation of $\upsilon_{j+1/2}$ and υ_j, as we can get $\upsilon_{j-1/2}$ from just a shift. We previously demonstrated how the coefficient matrices are calculated for the FEM so we now just skip ahead to give the equations.

The FEM gives

$$\sum_j \frac{\Delta x}{6}\begin{bmatrix} G^+_{j-1/2} \\ 2G^+_{j-1/2} + 2G^-_{j+1/2} \\ G^-_{j+1/2} \end{bmatrix} =$$

$$\sum_j \left(H\frac{\Delta x}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} + \frac{H^3}{9\Delta x}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}\right)\begin{bmatrix} u_{j-1/2} \\ u_j \\ u_{j+1/2} \end{bmatrix} \tag{2}$$

$$\sum_j \frac{\Delta x}{6} \begin{bmatrix} e^{-ik\Delta x}\mathcal{R}_2^+ \\ 2e^{-ik\Delta x}\mathcal{R}_2^+ + 2\mathcal{R}_2^- \\ \mathcal{R}_2^- \end{bmatrix} G_j =$$
$$\sum_j \left(H\frac{\Delta x}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} + \frac{H^3}{9\Delta x} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \right) \begin{bmatrix} e^{-ik\frac{\Delta x}{2}} \\ 1 \\ e^{ik\frac{\Delta x}{2}} \end{bmatrix} u_j \quad (3)$$

$$\sum_j \frac{\Delta x}{6} \begin{bmatrix} e^{-ik\Delta x}\mathcal{R}_2^+ \\ 2e^{-ik\Delta x}\mathcal{R}_2^+ + 2\mathcal{R}_2^- \\ \mathcal{R}_2^- \end{bmatrix} G_j =$$
$$\sum_j \left(H\frac{\Delta x}{30} \begin{bmatrix} 5i\sin\left(k\frac{\Delta x}{2}\right) + 3\cos\left(k\frac{\Delta x}{2}\right) + 2 \\ 16 + 4\cos\left(k\frac{\Delta x}{2}\right) \\ -5i\sin\left(k\frac{\Delta x}{2}\right) + 3\cos\left(k\frac{\Delta x}{2}\right) + 2 \end{bmatrix} \right.$$
$$\left. + \frac{H^3}{9\Delta x} \begin{bmatrix} 6i\sin\left(k\frac{\Delta x}{2}\right) + 8\cos\left(k\frac{\Delta x}{2}\right) - 8 \\ -16\cos\left(k\frac{\Delta x}{2}\right) + 16 \\ -6i\sin\left(k\frac{\Delta x}{2}\right) + 8\cos\left(k\frac{\Delta x}{2}\right) - 8 \end{bmatrix} \right) u_j \quad (4)$$

For the calculation of $u_{j+1/2}$ we have

$$\frac{\Delta x}{6}\left(\mathcal{R}_2^- + \mathcal{R}_2^+\right) G_j =$$
$$\left[H\frac{\Delta x}{30}\left(-5i\sin\left(k\frac{\Delta x}{2}\right) + 3\cos\left(k\frac{\Delta x}{2}\right) + 2 + e^{ik\frac{\Delta x}{2}}\left(5i\sin\left(k\frac{\Delta x}{2}\right) + 3\cos\left(k\frac{\Delta x}{2}\right)\right)\right) \right.$$
$$+ \frac{H^3}{9\Delta x}\left(-6i\sin\left(k\frac{\Delta x}{2}\right) + 8\cos\left(k\frac{\Delta x}{2}\right) - 8 + e^{ik\frac{\Delta x}{2}}\left(6i\sin\left(k\frac{\Delta x}{2}\right) + 8\cos\left(k\frac{\Delta x}{2}\right) - 8\right)\right)$$
$$\left. \vphantom{\frac{\Delta x}{2}} \right] u_j \quad (5)$$

$$
\begin{aligned}
\frac{\Delta x}{6}\left(\mathcal{R}_2^- + \mathcal{R}_2^+\right) G_j = \\
&\left[H\frac{\Delta x}{30}\left(4e^{-ik\frac{\Delta x}{2}} + e^{ik\frac{\Delta x}{2}} + 4e^{ik\Delta x} + 1\right)\right. \\
&\left. + \frac{H^3}{9\Delta x}\left(7e^{-ik\frac{\Delta x}{2}} - 7e^{ik\frac{\Delta x}{2}} + 7e^{ik\Delta x} - 7\right)\right] u_j
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
\frac{\Delta x}{6}\left(\mathcal{R}_2^- + \mathcal{R}_2^+\right) G_j = \\
&\left[H\frac{\Delta x}{30}\left(4e^{-ik\Delta x} + 1 + 4e^{ik\frac{\Delta x}{2}} + e^{-ik\frac{\Delta x}{2}}\right)\right. \\
&\left. + \frac{H^3}{9\Delta x}\left(7e^{-ik\Delta x} - 7 + 7e^{ik\frac{\Delta x}{2}} - 7e^{-ik\frac{\Delta x}{2}}\right)\right] u_{j+1/2}
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\frac{\Delta x}{6}\left(\mathcal{R}_2^- + \mathcal{R}_2^+\right) G_j = \\
&\left[H\frac{\Delta x}{30}\left(4e^{-ik\Delta x} + 1 + 4e^{ik\frac{\Delta x}{2}} + e^{-ik\frac{\Delta x}{2}}\right)\right. \\
&\left. + \frac{H^3}{9\Delta x}\left(7e^{-ik\Delta x} - 7 + 14i\sin\left(k\frac{\Delta x}{2}\right)\right)\right] u_{j+1/2}
\end{aligned} \tag{8}
$$