1 $\mathcal{M}$

We will demonstrate how the tables for $\mathcal{M}$, $\mathcal{R}^-$, $\mathcal{R}^+$ and $\mathcal{G}$ are constructed by using $\mathcal{M}$ as an example and then just present the tables for the other factors. First we calculate the analytic value for $\mathcal{M}$. For a general quantity q we have by definition [] that

$$\overline{q}_j = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} q \; dx.$$

Assuming q is a Fourier mode by (**??**) we have that

$$\begin{aligned}
\overline{q}_j &= \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} q(0,0) e^{i(\omega t + kx)} \; dx = \frac{q(0,0)e^{i\omega t}}{\Delta x} \left[\frac{1}{ik} e^{ikx} \right]_{x_{j-1/2}}^{x_{j+1/2}} \\
&= \frac{q(0,0)e^{i\omega t}}{\Delta x} \frac{1}{ik} e^{ikx_j} \left[e^{ik\frac{\Delta x}{2}} - e^{-ik\frac{\Delta x}{2}} \right] = \frac{q(0,0)e^{i(\omega t + kx_j)}}{\Delta x} \frac{1}{ik} \left[2i \sin\left(k \frac{\Delta x}{2} \right) \right] \\
&= \frac{2}{k\Delta x} \sin\left(k \frac{\Delta x}{2} \right) q_j.
\end{aligned}$$

Therefore,

$$q_j = \frac{k\Delta x}{2\sin\left(k\frac{\Delta x}{2}\right)} \overline{q}_j = \mathcal{M}\overline{q}_j. \tag{1}$$

This is the analytic value of $\mathcal{M}$ with which we want to compare the derived $\mathcal{M}$ from our numerical methods. To compare them we take their Taylor series expansion and compare those to get the lowest order term of the error. For the analytic value we have

$$\mathcal{M} = \frac{k\Delta x}{2\sin\left(k\frac{\Delta x}{2}\right)} = 1 + \frac{1}{24} k^2 \Delta x^2 + \frac{7}{5760} k^4 \Delta x^4 + \cdots. \tag{2}$$

Since our value for $\mathcal{M}$ for the second-order FEVM is 1 (**??**) we can see that the lowest order term of the error between the second-order FEVM and the analytical value is $-\frac{1}{24}k^2\Delta x^2$. These results have been summarised in Table **??** for all FDVM and the FEVM.

2 $\mathcal{R}^-$ and $\mathcal{R}^+$

For both $\mathcal{R}^-$ and $\mathcal{R}^+$ we are approximating the value of the quantity at $x_{j+1/2}$ in terms of the cell average. Now from [] we have that

$$q_{j+1/2} = e^{ik\Delta x/2} q_j$$

and from above we have [], so we have

$$q_{j+1/2} = e^{ik\Delta x/2}\mathcal{M}\overline{q}_j = \mathcal{R}^{-}\overline{q}_j = \mathcal{R}^{+}\overline{q}_j$$

where for the analytic solution we have that $\mathcal{R}^{-} = \mathcal{R}^{+}$.

3 $\mathcal{G}$

For $\mathcal{G}$ we desire a relation between the cell average values $\overline{\eta}_j$ and $\overline{G}_j$ to the cell edge value $\upsilon_{j+1/2}$. Equation [] holds for all x values and so in particular we have

$$G_{j+1/2} = UH + U\eta_{j+1/2} + \left(H + \frac{H^3}{3}k^2\right)\upsilon_{j+1/2} \tag{3}$$

From [] we have

$$e^{ik\Delta x/2}\mathcal{M}\overline{G}_j + UH + Ue^{ik\Delta x/2}\mathcal{M}\overline{\eta}_j = \left(H + \frac{H^3}{3}k^2\right)\upsilon_{j+1/2} \tag{4}$$

So

$$\begin{aligned}\upsilon_{j+1/2} &= \frac{e^{ik\Delta x/2}\mathcal{M}}{H + \frac{H^3}{3}k^2}\overline{G}_j + \frac{UH}{H + \frac{H^3}{3}k^2} + \frac{Ue^{ik\Delta x/2}\mathcal{M}}{H + \frac{H^3}{3}k^2}\overline{\eta}_j & (5)\\ &= \mathcal{G}^{G}\overline{G}_j + \mathcal{G}^{\eta}\overline{\eta}_j + \mathcal{G}^{c} & (6)\end{aligned}$$

4 F

For a Finite volume method we have that

$$F^{n}_{j+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(x_{j+1/2}, t)dt \tag{7}$$

4.1 η

For η we have

$$f(x_{j+1/2}, t) = H\upsilon_{j+1/2} + U\eta_{j+1/2}$$

Since we desire our fluxes to be written in terms of the cell averages of the conserved variables η and G we use [] and [] to rewrite this as

$$
\begin{aligned}
f(x_{j+1/2},t) &= H\left(\mathcal{G}^G\overline{G}_j + \mathcal{G}^\eta\overline{\eta}_j + \mathcal{G}^c\right) + Ue^{ik\Delta x/2}\mathcal{M}\overline{\eta}_j && (8)\\
&= H\mathcal{G}^G\overline{G}_j + U\left(H\mathcal{G}^\eta + e^{ik\Delta x/2}\mathcal{M}\right)\overline{\eta}_j + H\mathcal{G}^c && (9)
\end{aligned}
$$

$$
F^n_{j+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \left[H\mathcal{G}^G\overline{G}_j + U\left(H\mathcal{G}^\eta + e^{ik\Delta x/2}\mathcal{M}\right)\overline{\eta}_j + H\mathcal{G}^c\right] dt \qquad (10)
$$

for a general Fourier mode q we have that

$$
\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} q(x_j,t)\, dt = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} q(0,0)e^{i(\omega t + kx_j)}\, dt = \frac{q(0,0)e^{ikx_j}}{\Delta t}\left[\frac{1}{i\omega}e^{i\omega t}\right]_{t^n}^{t^{n+1}} \qquad (11)
$$

$$
= \frac{q(0,0)e^{ikx_j}}{\Delta t}\frac{1}{i\omega}e^{i\omega t^n}\left[e^{i\omega\Delta t} - 1\right] = \frac{e^{i\omega\Delta t} - 1}{i\omega\Delta t}q^n_j \qquad (12)
$$

Substituting this into [] we have that

$$
\begin{aligned}
F^n_{j+1/2} &= \frac{e^{i\omega\Delta t} - 1}{i\omega\Delta t}\left[H\mathcal{G}^G\overline{G}_j + U\left(H\mathcal{G}^\eta + e^{ik\Delta x/2}\mathcal{M}\right)\overline{\eta}_j\right] + H\mathcal{G}^c && (13)\\
&= \frac{e^{i\omega\Delta t} - 1}{i\omega\Delta t}H\mathcal{G}^G\overline{G}_j + U\left(H\mathcal{G}^\eta + e^{ik\Delta x/2}\mathcal{M}\right)\frac{e^{i\omega\Delta t} - 1}{i\omega\Delta t}\overline{\eta}_j + H\mathcal{G}^c && (14)\\
&= \mathcal{F}^{\eta,\eta}\overline{\eta}_j + \mathcal{F}^{\eta,G}\overline{G}_j + \mathcal{F}^{\eta,c} && (15)
\end{aligned}
$$

Since $F^n_{j-1/2}$ is just $F^n_{j+1/2}$ translated by Δx to the left we get that

$$
F^n_{j-1/2} = e^{-ik\Delta x}F^n_{j+1/2} \qquad (16)
$$

4.2 G

For G we have

$$
f(x_{j+1/2},t) = UG_{j+1/2} + UH\upsilon_{j+1/2} + gH\eta_{j+1/2}
$$

Since we desire our fluxes to be written in terms of the cell averages of the conserved variables η and G we use [] and [] to rewrite this as

$$f(x_{j+1/2},t) = Ue^{ik\Delta x/2}\mathcal{M}\overline{G}_j + UH\left(\mathcal{G}^G\overline{G}_j + \mathcal{G}^\eta\overline{\eta}_j + \mathcal{G}^c\right) + gHe^{ik\Delta x/2}\mathcal{M}\overline{\eta}_j \tag{17}$$

$$= \left(Ue^{ik\Delta x/2}\mathcal{M} + UH\mathcal{G}^G\right)\overline{G}_j + \left(UH\mathcal{G}^\eta + gHe^{ik\Delta x/2}\mathcal{M}\right)\overline{\eta}_j + UH\mathcal{G}^c \tag{18}$$

$$F^n_{j+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \left[\left(Ue^{ik\Delta x/2}\mathcal{M} + UH\mathcal{G}^G\right)\overline{G}_j + \left(UH\mathcal{G}^\eta + gHe^{ik\Delta x/2}\mathcal{M}\right)\overline{\eta}_j + UH\mathcal{G}^c\right] dt \tag{19}$$

Using [] this reduces to

$$F^n_{j+1/2} = \frac{e^{i\omega\Delta t} - 1}{i\omega\Delta t}\left[\left(Ue^{ik\Delta x/2}\mathcal{M} + UH\mathcal{G}^G\right)\overline{G}_j + \left(UH\mathcal{G}^\eta + gHe^{ik\Delta x/2}\mathcal{M}\right)\overline{\eta}_j\right] + UH\mathcal{G}^c \tag{20}$$

Since $F^n_{j-1/2}$ is just $F^n_{j+1/2}$ translated by Δx to the left we get that

$$F^n_{j-1/2} = e^{-ik\Delta x}F^n_{j+1/2} \tag{21}$$

5 Update

From [] our update scheme is

$$\overline{\eta}^{n+1}_j = \overline{\eta}^n_j - \frac{\Delta t}{\Delta x}\left[\left(\mathcal{F}^{\eta,\eta}\overline{\eta}_j + \mathcal{F}^{\eta,G}\overline{G}_j + \mathcal{F}^{\eta,c}\right) - \left(\mathcal{F}^{\eta,\eta}\overline{\eta}_{j-1} + \mathcal{F}^{\eta,G}\overline{G}_{j-1} + \mathcal{F}^{\eta,c}\right)\right],$$
$$\overline{G}^{n+1}_j = \overline{G}^n_j - \frac{\Delta t}{\Delta x}\left[\left(\mathcal{F}^{G,\eta}\overline{\eta}_j + \mathcal{F}^{G,G}\overline{G}_j + \mathcal{F}^{G,c}\right) - \left(\mathcal{F}^{G,\eta}\overline{\eta}_{j-1} + \mathcal{F}^{G,G}\overline{G}_{j-1} + \mathcal{F}^{G,c}\right)\right].$$

$$\overline{\eta}^{n+1}_j = \overline{\eta}^n_j - \frac{\Delta t}{\Delta x}\left[\left(1 - e^{-ik\Delta x}\right)\left(\mathcal{F}^{\eta,\eta}\overline{\eta}_j + \mathcal{F}^{\eta,G}\overline{G}_j\right)\right],$$
$$\overline{G}^{n+1}_j = \overline{G}^n_j - \frac{\Delta t}{\Delta x}\left[\left(1 - e^{-ik\Delta x}\right)\left(\mathcal{F}^{G,\eta}\overline{\eta}_j + \mathcal{F}^{G,G}\overline{G}_j\right)\right].$$

This can be written in matrix form as

$$\begin{bmatrix}\overline{\eta}\\ \overline{G}\end{bmatrix}_j^{n+1} = \begin{bmatrix}\overline{\eta}\\ \overline{G}\end{bmatrix}_j^{n} - \frac{\left(1-e^{-ik\Delta x}\right)\Delta t}{\Delta x}\begin{bmatrix}\mathcal{F}^{\eta,\eta} & \mathcal{F}^{\eta,G}\\ \mathcal{F}^{G,\eta} & \mathcal{F}^{G,G}\end{bmatrix}\begin{bmatrix}\overline{\eta}\\ \overline{G}\end{bmatrix}_j^{n}$$

$$= \left(\mathbf{I} - \Delta t\mathbf{F}\right)\begin{bmatrix}\overline{\eta}\\ \overline{G}\end{bmatrix}_j^{n} \tag{22}$$

$$e^{i\omega\Delta t}\begin{bmatrix}\overline{\eta}\\ \overline{G}\end{bmatrix}_j^{n} = \left(\mathbf{I} - \Delta t\mathbf{F}\right)\begin{bmatrix}\overline{\eta}\\ \overline{G}\end{bmatrix}_j^{n} \tag{23}$$

if $\lambda_\pm$ are the eigenvalues of $\mathbf{F}$ then

$$e^{i\omega_\pm\Delta t} = 1 - \Delta t\lambda_\pm$$

$\mathbf{F}$ is

$$\frac{\left(1-e^{-ik\Delta x}\right)\Delta t}{\Delta x}\begin{bmatrix} U\left(H\mathcal{G}^\eta + e^{ik\Delta x/2}\mathcal{M}\right)\frac{e^{i\omega\Delta t}-1}{i\omega\Delta t} & \frac{e^{i\omega\Delta t}-1}{i\omega\Delta t}H\mathcal{G}^G\\ \frac{e^{i\omega\Delta t}-1}{i\omega\Delta t}\left[\left(UH\mathcal{G}^\eta + gHe^{ik\Delta x/2}\mathcal{M}\right)\right] & \frac{e^{i\omega\Delta t}-1}{i\omega\Delta t}\left[\left(Ue^{ik\Delta x/2}\mathcal{M} + UH\mathcal{G}^G\right)\right]\end{bmatrix} \tag{24}$$

$$\frac{\left(1-e^{-ik\Delta x}\right)\Delta t}{\Delta x}\frac{e^{i\omega\Delta t}-1}{i\omega\Delta t}\begin{bmatrix} U\left(H\mathcal{G}^\eta + e^{ik\Delta x/2}\mathcal{M}\right) & H\mathcal{G}^G\\ \left(UH\mathcal{G}^\eta + gHe^{ik\Delta x/2}\mathcal{M}\right) & \left(Ue^{ik\Delta x/2}\mathcal{M} + UH\mathcal{G}^G\right)\end{bmatrix} \tag{25}$$

$$\frac{\left(1-e^{-ik\Delta x}\right)\left(e^{i\omega\Delta t}-1\right)}{i\omega\Delta x}\begin{bmatrix} U^2H\frac{e^{ik\Delta x/2}\mathcal{M}}{H+\frac{H^3}{3}k^2} + Ue^{ik\Delta x/2}\mathcal{M} & H\frac{e^{ik\Delta x/2}\mathcal{M}}{H+\frac{H^3}{3}k^2}\\ U^2H\frac{e^{ik\Delta x/2}\mathcal{M}}{H+\frac{H^3}{3}k^2} + gHe^{ik\Delta x/2}\mathcal{M} & Ue^{ik\Delta x/2}\mathcal{M} + UH\frac{e^{ik\Delta x/2}\mathcal{M}}{H+\frac{H^3}{3}k^2}\end{bmatrix} \tag{26}$$

$$\frac{\left(1-e^{-ik\Delta x}\right)\left(e^{i\omega\Delta t}-1\right)}{i\omega\Delta x}\mathcal{M}e^{ik\Delta x/2}\begin{bmatrix} U^2H\frac{1}{H+\frac{H^3}{3}k^2} + U & H\frac{1}{H+\frac{H^3}{3}k^2}\\ U^2H\frac{1}{H+\frac{H^3}{3}k^2} + gH & U + UH\frac{1}{H+\frac{H^3}{3}k^2}\end{bmatrix} \tag{27}$$

$$\frac{\left(e^{ik\Delta x/2}-e^{-ik\Delta x/2}\right)\left(e^{i\omega\Delta t}-1\right)}{i\omega\Delta x}\mathcal{M}\begin{bmatrix}\frac{U^2}{1+\frac{H^2}{3}k^2}+U & \frac{1}{1+\frac{H^2}{3}k^2}\\ \frac{U^2}{1+\frac{H^2}{3}k^2}+gH & U+\frac{U}{1+\frac{H^2}{3}k^2}\end{bmatrix} \tag{28}$$

$$\frac{2i\sin\left(k\Delta x/2\right)\left(e^{i\omega\Delta t}-1\right)}{i\omega\Delta x}\mathcal{M}\begin{bmatrix}\frac{U^2}{1+\frac{H^2}{3}k^2}+U & \frac{1}{1+\frac{H^2}{3}k^2}\\ \frac{U^2}{1+\frac{H^2}{3}k^2}+gH & U+\frac{U}{1+\frac{H^2}{3}k^2}\end{bmatrix} \tag{29}$$

$$\frac{\left(e^{i\omega\Delta t}-1\right)k}{\omega}\begin{bmatrix}\frac{U^2}{1+\frac{H^2}{3}k^2}+U & \frac{1}{1+\frac{H^2}{3}k^2}\\ \frac{U^2}{1+\frac{H^2}{3}k^2}+gH & U+\frac{U}{1+\frac{H^2}{3}k^2}\end{bmatrix} \tag{30}$$

λ eigenvalues

$$\begin{bmatrix}\frac{U^2}{1+\frac{H^2}{3}k^2}+U-\lambda & \frac{1}{1+\frac{H^2}{3}k^2}\\ \frac{U^2}{1+\frac{H^2}{3}k^2}+gH & U+\frac{U}{1+\frac{H^2}{3}k^2}-\lambda\end{bmatrix} \tag{31}$$

Determinant is

$$\left[\frac{U^2}{1+\frac{H^2}{3}k^2}+U-\lambda\right]\left[U+\frac{U}{1+\frac{H^2}{3}k^2}-\lambda\right]-\frac{U^2}{\left(1+\frac{H^2}{3}k^2\right)^2}-\frac{gH}{1+\frac{H^2}{3}k^2}=0 \tag{32}$$

$$\lambda^2-\lambda\left(\frac{U^2}{1+\frac{H^2}{3}k^2}+U+U+\frac{U}{1+\frac{H^2}{3}k^2}\right)+\left(\frac{U^2}{1+\frac{H^2}{3}k^2}+U\right)\left(U+\frac{U}{1+\frac{H^2}{3}k^2}\right) \tag{33}$$

$$-\frac{U^2}{\left(1+\frac{H^2}{3}k^2\right)^2}-\frac{gH}{1+\frac{H^2}{3}k^2}=0 \tag{34}$$

$$\lambda^2 - \lambda\left(\frac{U^2+U}{1+\frac{H^2}{3}k^2}+2U\right)+\left(\frac{U^2}{1+\frac{H^2}{3}k^2}+U\right)\left(U+\frac{U}{1+\frac{H^2}{3}k^2}\right) \tag{35}$$

$$-\frac{U^2}{\left(1+\frac{H^2}{3}k^2\right)^2}-\frac{gH}{1+\frac{H^2}{3}k^2}=0 \tag{36}$$

$$\lambda^2 - \lambda\left(\frac{U^2+U}{1+\frac{H^2}{3}k^2}+2U\right) \tag{37}$$

$$+\frac{U^3}{1+\frac{H^2}{3}k^2}+\frac{U^3}{\left(1+\frac{H^2}{3}k^2\right)^2}+U^2+\frac{U^2}{1+\frac{H^2}{3}k^2} \tag{38}$$

$$-\frac{U^2}{\left(1+\frac{H^2}{3}k^2\right)^2}-\frac{gH}{1+\frac{H^2}{3}k^2}=0 \tag{39}$$

$$\lambda^2 - \lambda\left(\frac{U^2+U}{1+\frac{H^2}{3}k^2}+2U\right) \tag{40}$$

$$+U^2+\frac{U^3+U^2-gH}{1+\frac{H^2}{3}k^2}+\frac{U^3-U^2}{\left(1+\frac{H^2}{3}k^2\right)^2}=0 \tag{41}$$

b^2-4ac

$$\left(\frac{U^2+U}{1+\frac{H^2}{3}k^2}+2U\right)^2-4U^2-4\frac{U^3+U^2-gH}{1+\frac{H^2}{3}k^2}-4\frac{U^3-U^2}{\left(1+\frac{H^2}{3}k^2\right)^2} \tag{42}$$

$$\left[\frac{U^2+U}{1+\frac{H^2}{3}k^2}\right]^2+4U\frac{U^2+U}{1+\frac{H^2}{3}k^2}+4U^2 \tag{43}$$

$$-4U^2-4\frac{U^3+U^2-gH}{1+\frac{H^2}{3}k^2}-4\frac{U^3-U^2}{\left(1+\frac{H^2}{3}k^2\right)^2} \tag{44}$$

$$\frac{U^4+2U^3+U^2}{\left(1+\frac{H^2}{3}k^2\right)^2}+4\frac{gH}{1+\frac{H^2}{3}k^2}-4\frac{U^3-U^2}{\left(1+\frac{H^2}{3}k^2\right)^2} \tag{45}$$

$$\frac{U^4 - 2U^3 + 5U^2}{\left(1 + \frac{H^2}{3}k^2\right)^2} + \frac{4gH}{1 + \frac{H^2}{3}k^2} \tag{46}$$

$$U^2 \frac{U^2 - 2U + 5}{\left(1 + \frac{H^2}{3}k^2\right)^2} + \frac{4gH}{1 + \frac{H^2}{3}k^2} \tag{47}$$