

1 Elliptic Equation

The linearised elliptic equation is

$$G = Hu - \frac{H^3}{3}u_{xx}$$

Want to find out the FEM approximation factor \mathcal{G}_{FE_1} such that

$$G = \mathcal{G}_{FE_1}u$$

To do so we begin by first multiplying by an arbitrary test function v so that

$$Gv = Huv - \frac{H^3}{3}u_{xx}v$$

and then we integrate over the entire domain to get

$$\int_{\Omega} Gv dx = \int_{\Omega} Huv dx - \int_{\Omega} \frac{H^3}{3}u_{xx}v dx$$

for all v

We then make use of integration by parts, with Dirchlet boundaries to get

$$\int_{\Omega} Gv dx = \int_{\Omega} Huv dx + \int_{\Omega} \frac{H^3}{3}u_x v_x dx$$

For u we are going to use $x_{j+1/2}$ as the nodes, which generate the basis functions $\phi_{j+1/2}$, which for us will be the space of continuous linear elements. While for G we will choose basis functions w that are linear from $[x_{j-1/2}, x_{j+1/2}]$ but discontinuous at the edges.

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Gv dx = \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Huv dx + \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{H^3}{3}u_x v_x dx$$

for all v

2 P1 FEM

We are going to coordinate tranform from x space the interval $[x_{j-1/2}, x_{j+1/2}, x_{j+3/2}]$ to the ξ space interval $[-1, 0, 1]$. To accomplish this we have the following relation

$$x = \xi \Delta x + x_{j+1/2}$$

Taking the derivatives we see

$$dx = d\xi \Delta x, \quad \frac{dx}{d\xi} = \Delta x, \quad \frac{d\xi}{dx} = \frac{1}{\Delta x}.$$

We can describe the basis functions in the ξ space

$$\phi_{j+1/2} = \begin{cases} 1 + \xi & \xi < 0 \\ 1 - \xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

$$\phi_{j-1/2} = \begin{cases} -\xi & \xi < 0 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

$$\phi_{j+3/2} = \begin{cases} \xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

While the descriptions for w 's is

$$w_{j+1/2}^+ = \begin{cases} 1 - \xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

$$w_{j+1/2}^- = \begin{cases} 1 + \xi & \xi < 0 \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

$$w_{j-1/2}^+ = \begin{cases} -\xi & \xi < 0 \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

$$w_{j+3/2}^- = \begin{cases} \xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

We now replace our functions by our approximations to them

$$G \approx G' = \sum_j G_{j+1/2} w_{j+1/2}$$

$$u \approx u' = \sum_j u_{j+1/2} \phi_{j+1/2}$$

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} G' \phi_{j+1/2} dx - H \int_{x_{j-1/2}}^{x_{j+3/2}} u' \phi_{j+1/2} dx - \frac{H^3}{3} \int_{x_{j-1/2}}^{x_{j+3/2}} u'_x (\phi_x)_{j+1/2} dx = 0$$

For all $\phi_{j+1/2}$. For this analysis we choose a particular basis function $\phi_{j+1/2}$ and we look at all the integrals. Begining from the right

$$\begin{aligned} \int_{x_{j-1/2}}^{x_{j+3/2}} G'(x) \phi_{j+1/2} dx &= \int_{-1}^1 G'(\xi) \phi_{j+1/2}(\xi) \frac{dx}{d\xi} d\xi \\ &= \Delta x \int_{-1}^1 \left(G_{j-1/2}^+ w_{j-1/2}^+ + G_{j+1/2}^- w_{j+1/2}^- + G_{j+1/2}^+ w_{j+1/2}^+ + G_{j+3/2}^- w_{j+3/2}^- \right) \phi_{j+1/2} d\xi \\ &= \Delta x \left[G_{j-1/2}^+ \int_{-1}^1 w_{j-1/2}^+ \phi_{j+1/2} d\xi + G_{j+1/2}^- \int_{-1}^1 w_{j+1/2}^- \phi_{j+1/2} d\xi \right. \\ &\quad \left. + G_{j+1/2}^+ \int_{-1}^1 w_{j+1/2}^+ \phi_{j+1/2} d\xi + G_{j+3/2}^- \int_{-1}^1 w_{j+3/2}^- \phi_{j+1/2} d\xi \right] \quad (8) \end{aligned}$$

We have that

$$\int_{-1}^1 w_{j-1/2}^+ \phi_{j+1/2} d\xi = \int_{-1}^1 w_{j+3/2}^- \phi_{j+1/2} d\xi = \frac{1}{6}$$

and

$$\int_{-1}^1 w_{j+1/2}^- \phi_{j+1/2} d\xi = \int_{-1}^1 w_{j+1/2}^+ \phi_{j+1/2} d\xi = \frac{1}{3}$$

So

$$\begin{aligned} &= \Delta x \left[\frac{1}{6} G_{j-1/2}^+ + \frac{1}{3} G_{j+1/2}^- + \frac{1}{3} G_{j+1/2}^+ + \frac{1}{6} G_{j+3/2}^- \right] \\ &= \frac{\Delta x}{6} \left[G_{j-1/2}^+ + 2G_{j+1/2}^- + 2G_{j+1/2}^+ + G_{j+3/2}^- \right] \end{aligned}$$

Taking the next integral

$$\begin{aligned}
H \int_{x_{j-1/2}}^{x_{j+3/2}} u' \phi_{j+1/2} dx &= H \Delta x \int_{-1}^1 u'(\xi) \phi_{j+1/2}(\xi) d\xi \\
&= H \Delta x \int_{-1}^1 (u_{j-1/2} \phi_{j-1/2} + u_{j+1/2} \phi_{j+1/2} + u_{j+3/2} \phi_{j+3/2}) \phi_{j+1/2} d\xi \\
&= H \Delta x \left[u_{j-1/2} \int_{-1}^1 \phi_{j-1/2} \phi_{j+1/2} d\xi + u_{j+1/2} \int_{-1}^1 \phi_{j+1/2} \phi_{j+1/2} d\xi + u_{j+3/2} \int_{-1}^1 \phi_{j+3/2} \phi_{j+1/2} d\xi \right]
\end{aligned}$$

we have that

$$\begin{aligned}
\int_{-1}^1 \phi_{j-1/2} \phi_{j+1/2} d\xi &= \int_{-1}^1 \phi_{j+1/2} \phi_{j+3/2} d\xi = \frac{1}{6} \\
\int_{-1}^1 \phi_{j+1/2} \phi_{j+1/2} d\xi &= \frac{2}{3}
\end{aligned}$$

Then we have

$$= \frac{H \Delta x}{6} [u_{j-1/2} + 4u_{j+1/2} + u_{j+3/2}]$$

For the third integral we have

$$\begin{aligned}
\frac{H^3}{3} \int_{x_{j-1/2}}^{x_{j+3/2}} u'_x(\phi_{j+1/2})_x dx &= -\frac{H^3}{3} \int_{-1}^1 u'_\xi \frac{d\xi}{dx} (\phi_\xi)_{j+1/2} \frac{d\xi}{dx} \frac{dx}{d\xi} d\xi \\
&= \frac{H^3}{3\Delta x} \int_{-1}^1 u'_\xi (\phi_\xi)_{j+1/2} d\xi
\end{aligned}$$

We will now use ' to denote derivative

$$\begin{aligned}
&= \frac{H^3}{3\Delta x} \int_{-1}^1 (u'_{j-1/2} \phi'_{j-1/2} + u'_{j+1/2} \phi'_{j+1/2} + u'_{j+3/2} \phi'_{j+3/2}) \phi'_{j+1/2} d\xi \\
&= \frac{H^3}{3\Delta x} \left[u_{j-1/2} \int_{-1}^1 \phi'_{j-1/2} \phi'_{j+1/2} d\xi + u_{j+1/2} \int_{-1}^1 \phi'_{j+1/2} \phi'_{j+1/2} d\xi + u_{j+3/2} \int_{-1}^1 \phi'_{j+3/2} \phi'_{j+1/2} d\xi \right]
\end{aligned}$$

We have that

$$\int_{-1}^1 \phi'_{j-1/2} \phi'_{j+1/2} d\xi = -1 = \int_{-1}^1 \phi'_{j+3/2} \phi'_{j+1/2} d\xi$$

$$\int_{-1}^1 \phi'_{j+1/2} \phi'_{j+1/2} d\xi = 2$$

Therefore

$$= \frac{H^3}{3\Delta x} [-u_{j-1/2} + 2u_{j+1/2} - u_{j+3/2}]$$

Then our equation becomes

$$\frac{\Delta x}{6} [G_{j-1/2}^+ + 2G_{j+1/2}^- + 2G_{j+1/2}^+ + G_{j+3/2}^-] =$$

$$\frac{H\Delta x}{6} [u_{j-1/2} + 4u_{j+1/2} + u_{j+3/2}] + \frac{H^3}{3\Delta x} [-u_{j-1/2} + 2u_{j+1/2} - u_{j+3/2}] \quad (9)$$

$$[G_{j-1/2}^+ + 2G_{j+1/2}^- + 2G_{j+1/2}^+ + G_{j+3/2}^-] =$$

$$H [u_{j-1/2} + 4u_{j+1/2} + u_{j+3/2}] + \frac{2H^3}{\Delta x^2} [-u_{j-1/2} + 2u_{j+1/2} - u_{j+3/2}] \quad (10)$$

We now assume the following form for u and G .

Let quantity q is given by so that $q(x, t) = q(0, 0)e^{i(\omega t + kx)}$. The use of this comes when we use our uniform grid in space, so that $q(x_j, t) = q_j$ then $q_{j\pm l} = q_j e^{\pm ikl\Delta x}$. With reconstructions from the previous dispersion analysis

Then we have

$$[G_j e^{-ik\Delta x} \mathcal{R}^+ + 2G_j \mathcal{R}^- + 2G_j \mathcal{R}^+ + G_j e^{ik\Delta x} \mathcal{R}^-] =$$

$$H [u_j e^{-ik\Delta x/2} + 4u_j e^{ik\Delta x/2} + u_j e^{3ik\Delta x/2}] + \frac{2H^3}{\Delta x^2} [-u_j e^{-ik\Delta x/2} + 2u_j e^{ik\Delta x/2} - u_j e^{3ik\Delta x/2}] \quad (11)$$

$$[e^{-ik\Delta x} \mathcal{R}^+ + 2\mathcal{R}^- + 2\mathcal{R}^+ + e^{ik\Delta x} \mathcal{R}^-] G_j =$$

$$H [e^{-ik\Delta x} + 4 + e^{ik\Delta x}] u_j e^{ik\Delta x/2} + \frac{2H^3}{\Delta x^2} [-e^{ik\Delta x} + 2 - e^{-ik\Delta x}] u_j e^{ik\Delta x/2} \quad (12)$$

$$\begin{aligned} & \left[e^{-ik\Delta x} \mathcal{R}^+ + 2\mathcal{R}^- + 2\mathcal{R}^+ + e^{ik\Delta x} \mathcal{R}^- \right] G_j = \\ & H [2 \cos(k\Delta x) + 4] u_j e^{ik\Delta x/2} + \frac{2H^3}{\Delta x^2} [2 - 2 \cos(k\Delta x)] u_j e^{ik\Delta x/2} \quad (13) \end{aligned}$$

From the previous dispersion analysis we know that

$$\begin{aligned} \mathcal{R}^- &= 1 + \frac{i \sin(k\Delta x)}{2} \\ \mathcal{R}^+ &= e^{ik\Delta x} \left(1 - \frac{i \sin(k\Delta x)}{2} \right) \end{aligned}$$

So we have

$$\begin{aligned} & \left[\left(1 - \frac{i \sin(k\Delta x)}{2} \right) + 2 \left[1 + \frac{i \sin(k\Delta x)}{2} \right] \right. \\ & \quad \left. + 2 \left[e^{ik\Delta x} \left(1 - \frac{i \sin(k\Delta x)}{2} \right) \right] + e^{ik\Delta x} \left[1 + \frac{i \sin(k\Delta x)}{2} \right] \right] G_j = \\ & H [2 \cos(k\Delta x) + 4] u_j e^{ik\Delta x/2} + \frac{2H^3}{\Delta x^2} [2 - 2 \cos(k\Delta x)] u_j e^{ik\Delta x/2} \quad (14) \end{aligned}$$

$$\begin{aligned} & \left[3 + \frac{i \sin(k\Delta x)}{2} + e^{ik\Delta x} \left(3 - \frac{i \sin(k\Delta x)}{2} \right) \right] G_j = \\ & H [2 \cos(k\Delta x) + 4] u_j e^{ik\Delta x/2} + \frac{2H^3}{\Delta x^2} [2 - 2 \cos(k\Delta x)] u_j e^{ik\Delta x/2} \quad (15) \end{aligned}$$

$$G_j =$$

$$\begin{aligned} & 2H e^{ik\Delta x/2} \frac{\cos(k\Delta x) + 2}{3 + \frac{i \sin(k\Delta x)}{2} + e^{ik\Delta x} \left(3 - \frac{i \sin(k\Delta x)}{2} \right)} u_j \\ & + \frac{4H^3}{\Delta x^2} e^{ik\Delta x/2} \frac{1 - \cos(k\Delta x)}{3 + \frac{i \sin(k\Delta x)}{2} + e^{ik\Delta x} \left(3 - \frac{i \sin(k\Delta x)}{2} \right)} u_j \quad (16) \end{aligned}$$

$$G_j =$$

$$\begin{aligned} & H2e^{ik\Delta x/2} \frac{\cos(k\Delta x) + 2}{3 + \frac{i\sin(k\Delta x)}{2} + e^{ik\Delta x} \left(3 - \frac{i\sin(k\Delta x)}{2}\right)} u_j \\ & + \frac{H^3}{3} \frac{12}{\Delta x^2} e^{ik\Delta x/2} \frac{1 - \cos(k\Delta x)}{3 + \frac{i\sin(k\Delta x)}{2} + e^{ik\Delta x} \left(3 - \frac{i\sin(k\Delta x)}{2}\right)} u_j \end{aligned} \quad (17)$$

We want something like

$$1 \approx 2e^{ik\Delta x/2} \frac{\cos(k\Delta x) + 2}{3 + \frac{i\sin(k\Delta x)}{2} + e^{ik\Delta x} \left(3 - \frac{i\sin(k\Delta x)}{2}\right)}$$

and

$$k^2 \approx \frac{12}{\Delta x^2} e^{ik\Delta x/2} \frac{1 - \cos(k\Delta x)}{3 + \frac{i\sin(k\Delta x)}{2} + e^{ik\Delta x} \left(3 - \frac{i\sin(k\Delta x)}{2}\right)}$$

and we want to compare it to the FD approximation

$$k^2 \approx \frac{2}{\Delta x^2} (1 - \cos(k\Delta x))$$

and

$$1 \approx 1$$

For the FEM we have the Taylor series

$$\begin{aligned} & \frac{12}{\Delta x^2} e^{ik\Delta x/2} \frac{1 - \cos(k\Delta x)}{3 + \frac{i\sin(k\Delta x)}{2} + e^{ik\Delta x} \left(3 - \frac{i\sin(k\Delta x)}{2}\right)} = \\ & k^2 - \frac{k^4 \Delta x^2}{24} + \frac{91k^6 \Delta x^4}{5760} - \frac{1259k^8 \Delta x^6}{967680} + \frac{44327k^{10} \Delta x^8}{1454828800} + O(\Delta x^{10}) \end{aligned} \quad (18)$$

For the FD

$$\begin{aligned} & \frac{2}{\Delta x^2} (1 - \cos(k\Delta x)) = \\ & k^2 - \frac{k^4 \Delta x^2}{12} + \frac{k^6 \Delta x^4}{360} - \frac{k^8 \Delta x^6}{20160} - \frac{k^{10} \Delta x^8}{1814400} + O(\Delta x^{10}) \end{aligned} \quad (19)$$

We also have for the FEM

$$2e^{ik\Delta x/2} \frac{\cos(k\Delta x) + 2}{3 + \frac{i \sin(k\Delta x)}{2} + e^{ik\Delta x} \left(3 - \frac{i \sin(k\Delta x)}{2}\right)} =$$

$$1 - \frac{k^2 \Delta x^2}{8} + \frac{3k^4 \Delta x^4}{128} - \frac{121k^6 \Delta x^6}{46080} + \frac{14227k^8 \Delta x^8}{30965760} + O(\Delta x^{10}) \quad (20)$$

In our experiments we set $H = 1$ and so our expression for the \mathcal{G} factor is analytically

$$1 + \frac{k^3}{3}$$

For our FEM this is approximated by (taylor expansion)

$$1 - \frac{k^2 \Delta x^2}{8} + \frac{3k^4 \Delta x^4}{128} - \frac{121k^6 \Delta x^6}{46080} + \frac{14227k^8 \Delta x^8}{30965760} + O(\Delta x^{10}) +$$

$$\frac{1}{3} \left(k^2 - \frac{k^4 \Delta x^2}{24} + \frac{91k^6 \Delta x^4}{5760} - \frac{1259k^8 \Delta x^6}{967680} + \frac{44327k^{10} \Delta x^8}{1454828800} + O(\Delta x^{10}) \right) \quad (21)$$

$$1 - \frac{k^2 \Delta x^2}{8} + \frac{3k^4 \Delta x^4}{128} - \frac{121k^6 \Delta x^6}{46080} + \frac{14227k^8 \Delta x^8}{30965760} + O(\Delta x^{10}) +$$

$$\frac{k^2}{3} - \frac{k^4 \Delta x^2}{72} + \frac{91k^6 \Delta x^4}{17280} - \frac{1259k^8 \Delta x^6}{2903040} + \frac{44327k^{10} \Delta x^8}{4364486400} + O(\Delta x^{10}) \quad (22)$$

$$1 + \frac{k^2}{3} - \frac{(k^4 + 9k^2)\Delta x^2}{72} + \frac{(91k^6 + 405k^4)\Delta x^4}{17280} - \frac{(1259k^8 + 7623k^6)\Delta x^6}{2903040} + O(\Delta x^8) \quad (23)$$

For our FD this is approximated by (taylor expansion)

$$1 + \frac{k^2}{3} - \frac{k^4 \Delta x^2}{36} + \frac{k^6 \Delta x^4}{1080} - \frac{k^8 \Delta x^6}{60480} + O(\Delta x^8) \quad (24)$$

Taking $k = 0.5$
then we have

$$\frac{(k^4 + 9k^2)\Delta x^2}{72} = 0.03211805555 > 0.00173611111 = \frac{k^4\Delta x^2}{36}$$

And for $k = 2.5$ we have

$$\frac{(k^4 + 9k^2)\Delta x^2}{72} = 1.323784722 > 0.1736111111 = \frac{k^4\Delta x^2}{36}$$

So our FEM is worse