1 All Definitions for Numerical Versions

$$\mathcal{C}_2 = \frac{2\cos(k\Delta x) - 2}{\Delta x^2}$$

$$\mathcal{C}_4 = \frac{-2\cos(2k\Delta x) + 32\cos(k\Delta x) - 30}{12\Delta x^2}$$

$$\mathcal{G} = \left[H - \frac{H^3}{3}\mathcal{C}\right]$$

$$\mathcal{M}_3 = \frac{24}{26 - 2\cos(k\Delta x)}$$

$$\mathcal{M}_1 = \mathcal{M}_2 = 1$$

$$\mathcal{R}_1^+ = e^{ik\Delta x} \quad , \quad \mathcal{R}_1^- = 1$$

$$\mathcal{R}_2^- = 1 + \frac{i\sin(k\Delta x)}{2}$$

$$\mathcal{R}_2^+ = e^{ik\Delta x}\left(1 - \frac{i\sin(k\Delta x)}{2}\right)$$

$$\mathcal{R}_3^- = \frac{\mathcal{M}_3}{6}\left[5 + -e^{-ik\Delta x} + 2e^{ik\Delta x}\right]$$

$$\mathcal{R}_3^+ = \frac{\mathcal{M}_3 e^{ik\Delta x}}{6}\left[5 + 2e^{-ik\Delta x} - e^{ik\Delta x}\right]$$

$$\mathcal{R}_2^u = \frac{e^{ik\Delta x} + 1}{2}$$

$$\mathcal{R}_3^u = \frac{-3e^{2ik\Delta x} + 27e^{ik\Delta x} + 27 - 3e^{-ik\Delta x}}{48}$$

$$\mathcal{F}^{h,u} = H\mathcal{R}^u$$

$$\mathcal{F}^{h,h} = -\frac{\sqrt{gH}}{2}\left[\mathcal{R}^+ - \mathcal{R}^-\right]$$

$$\mathcal{F}^{u,u} = -\frac{\sqrt{gH}}{2}\mathcal{G}\left[\mathcal{R}^{+} \ - \mathcal{R}^{-}\right]$$

$$\mathcal{F}^{u,h} = \frac{gH}{2}\left(\mathcal{R}^{+} + \mathcal{R}^{-}\right)$$

$$\mathcal{D} = 1 - e^{-ik\Delta x}$$

2 Taylor Expansions Of Analytic Values

We denote exact/analytic version with a subscript a

$$\mathcal{G}_a = H + \frac{H^3}{3}k^2$$

$$\mathcal{M}_a = \frac{2}{k\Delta x}\sin\left(\frac{k\Delta x}{2}\right)$$

$$\mathcal{M}_a = 1 - \frac{k^2}{24}(\Delta x)^2 + \frac{k^4}{1920}(\Delta x)^4 - \frac{k^6}{322560}(\Delta x)^6 + O(x^8)$$

$$\mathcal{R}_a = e^{i\frac{k\Delta x}{2}}$$

$$\mathcal{R}^{+}_a = \mathcal{R}^{-}_a = \mathcal{R}_a = 1 + \frac{ik}{2}\Delta x - \frac{k^2}{8}\Delta x^2 - \frac{ik^3}{48}\Delta x^3 + \frac{k^4}{384}\Delta x^4 + \frac{ik^5}{3840}\Delta x^5 + O(x^6)$$

For the fluxes I think its best to report the elements of our matrix $\boldsymbol{F}$, which now encapsulates all space approximations and is the update matrix for our system

$$\frac{\mathcal{D}_a}{\Delta x\mathcal{M}_a}\mathcal{F}^{h,u}_a = ikH$$

$$\frac{\mathcal{D}_a}{\Delta x\mathcal{M}_a}\mathcal{F}^{h,h}_a = 0$$

$$\frac{\mathcal{D}_a}{\mathcal{G}_a\Delta x\mathcal{M}_a}\mathcal{F}^{u,h}_a = \frac{ikgH}{H + \frac{H^3}{3}k^2} = igk\frac{3}{3 + H^2k^2} = i\frac{\omega^2}{kH}$$

$$\frac{\mathcal{D}_a}{\mathcal{G}_a\Delta x\mathcal{M}_a}\mathcal{F}^{u,u}_a = 0$$

3 First Order Values

$$\mathcal{G}_1 = H - \frac{H^3}{3}\left(\frac{2\cos(k\Delta x) - 2}{\Delta x^2}\right)$$

$$\mathcal{G}_1 = H + \frac{H^3k^2}{3} - \frac{H^3k^4}{36}(\Delta x)^2 + \frac{H^3k^6}{1080}(\Delta x)^4 + O(x^6)$$

$$\mathcal{M}_1 = 1$$

$$\mathcal{R}_1^- = 1$$

$$\mathcal{R}_1^+ = e^{ik\Delta x}$$

$$\mathcal{R}_1^+ = 1 + ik\Delta x - \frac{k^2}{2}(\Delta x)^2 - \frac{ik^3}{6}(\Delta x)^3 + \frac{k^4}{24}(\Delta x)^4 + O(\Delta x^5)$$

$$\mathcal{R}_1^u = \frac{e^{ik\Delta x} + 1}{2}$$

$$\mathcal{R}_1^u = 1 + \frac{ik}{2}\Delta x - \frac{k^2}{4}(\Delta x)^2 - \frac{ik^3}{12}(\Delta x)^3 + \frac{k^4}{48}(\Delta x)^4 + O(\Delta x^5)$$

$$\frac{\mathcal{D}}{\Delta x\mathcal{M}_1}\mathcal{F}_1^{h,u} = \frac{1 - e^{-ik\Delta x}}{\Delta x}H\frac{e^{ik\Delta x} + 1}{2}$$

$$\frac{\mathcal{D}}{\Delta x\mathcal{M}_1}\mathcal{F}_1^{h,u} = iHk - \frac{iHk^3}{6}(\Delta x)^2 + \frac{iHk^5}{120}(\Delta x)^4 + O(\Delta x^6)$$

$$\frac{\mathcal{D}}{\Delta x\mathcal{M}_1}\mathcal{F}_1^{h,h} = -\frac{\sqrt{gH}}{2}\frac{1 - e^{-ik\Delta x}}{\Delta x}\left[e^{ik\Delta x} - 1\right]$$

$$\frac{\mathcal{D}}{\Delta x\mathcal{M}_1}\mathcal{F}_1^{h,h} = \frac{k^2\sqrt{gH}}{2}\Delta x - \frac{k^4\sqrt{gH}}{24}\Delta x^3 + \frac{k^6\sqrt{gH}}{720}\Delta x^5 + O(\Delta x^7)$$

$$\frac{\mathcal{D}}{\mathcal{G}_1\Delta x\mathcal{M}_1}\mathcal{F}_1^{u,h} = \frac{gH}{2}\frac{1 - e^{-ik\Delta x}}{\Delta x}\left(1 + e^{ik\Delta x}\right)\left[H - \frac{H^3}{3}\left(\frac{2\cos(k\Delta x) - 2}{\Delta x^2}\right)\right]^{-1}$$

$$\frac{\mathcal{D}}{\mathcal{G}_1\Delta x\mathcal{M}_1}\mathcal{F}_1^{u,h} = igk\frac{3}{3 + H^3k^3} - \frac{igk^3\left(H^2k^2 + 6\right)}{4\left(3 + H^2k^2\right)^2}\Delta x^2 - \frac{igk^5\left(H^4k^4 - 54\right)}{240\left(3 + H^2k^2\right)^3}\Delta x^4 + O(\Delta x^6)$$

$$\frac{\mathcal{D}}{\mathcal{G}_1 \Delta x \mathcal{M}_1}\mathcal{F}_1^{u,u} = -\frac{\sqrt{gH}}{2}\frac{1-e^{-ik\Delta x}}{\Delta x}\left[e^{ik\Delta x}-1\right]$$

$$\frac{\mathcal{D}}{\mathcal{G}_1 \Delta x \mathcal{M}_1}\mathcal{F}_1^{u,u} = \frac{k^2\sqrt{gH}}{2}\Delta x - \frac{k^4\sqrt{gH}}{24}\Delta x^3 + \frac{k^6\sqrt{gH}}{720}\Delta x^5 + O(\Delta x^7)$$

4 Second Order Values

$$\mathcal{G}_2 = H - \frac{H^3}{3}\left(\frac{2\cos(k\Delta x)-2}{\Delta x^2}\right)$$

$$\mathcal{G}_2 = H + \frac{H^3k^2}{3} - \frac{H^3k^4}{36}(\Delta x)^2 + \frac{H^3k^6}{1080}(\Delta x)^4 + O(x^6)$$

$$\mathcal{M}_2 = 1$$

$$\mathcal{R}_2^- = 1 + \frac{i\sin(k\Delta x)}{2}$$

$$\mathcal{R}_2^- = 1 + \frac{ik}{2}(\Delta x) - \frac{ik^3}{12}(\Delta x)^3 + \frac{ik^5}{240}(\Delta x)^5 + O(x^7)$$

$$\mathcal{R}_2^+ = e^{ik\Delta x}\left(1 - \frac{i\sin(k\Delta x)}{2}\right)$$

$$\mathcal{R}_2^+ = 1 + \frac{ik}{2}\Delta x + \frac{ik^3}{6}\Delta x^3 - \frac{k^4}{8}\Delta x^4 - \frac{7ik^5}{120}\Delta x^5 + O\left(\Delta x^6\right)$$

$$\mathcal{R}_2^u = \frac{e^{ik\Delta x}+1}{2}$$

$$\mathcal{R}_2^u = 1 + \frac{ik}{2}\Delta x - \frac{k^2}{4}(\Delta x)^2 - \frac{ik^3}{12}(\Delta x)^3 + \frac{k^4}{48}(\Delta x)^4 + O(\Delta x^5)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_2}\mathcal{F}_2^{h,u} = \frac{1-e^{-ik\Delta x}}{\Delta x}H\frac{e^{ik\Delta x}+1}{2}$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_2}\mathcal{F}_2^{h,u} = iHk - \frac{iHk^3}{6}(\Delta x)^2 + \frac{iHk^5}{120}(\Delta x)^4 + O(\Delta x^6)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_2}\mathcal{F}_2^{h,h} = -\frac{\sqrt{gH}}{2}\frac{1-e^{-ik\Delta x}}{\Delta x}\left[e^{ik\Delta x}\left(1-\frac{i\sin\left(k\Delta x\right)}{2}\right)-\left(1+\frac{i\sin\left(k\Delta x\right)}{2}\right)\right]$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_2}\mathcal{F}_2^{h,h} = \frac{k^4\sqrt{gH}}{8}(\Delta x)^3 - \frac{k^6\sqrt{gH}}{48}(\Delta x)^5 + O(\Delta x^7)$$

$$\frac{\mathcal{D}}{\mathcal{G}_2\Delta x \mathcal{M}_2}\mathcal{F}_2^{u,u} = -\frac{\sqrt{gH}}{2}\frac{1-e^{-ik\Delta x}}{\Delta x}\left[e^{ik\Delta x}\left(1-\frac{i\sin\left(k\Delta x\right)}{2}\right)-\left(1+\frac{i\sin\left(k\Delta x\right)}{2}\right)\right]$$

$$\frac{\mathcal{D}}{\mathcal{G}_2\Delta x \mathcal{M}_2}\mathcal{F}_2^{u,u} = \frac{k^4\sqrt{gH}}{8}(\Delta x)^3 - \frac{k^6\sqrt{gH}}{48}(\Delta x)^5 + O(\Delta x^7)$$

$$\begin{aligned}\frac{\mathcal{D}}{\mathcal{G}_2\Delta x \mathcal{M}_2}\mathcal{F}_2^{u,h} = \frac{1-e^{-ik\Delta x}}{\Delta x}\frac{gH}{2}\left[e^{ik\Delta x}\left(1-\frac{i\sin\left(k\Delta x\right)}{2}\right)+\left(1+\frac{i\sin\left(k\Delta x\right)}{2}\right)\right]\\ \times\left[H-\frac{H^3}{3}\left(\frac{2\cos\left(k\Delta x\right)-2}{\Delta x^2}\right)\right]^{-1}\end{aligned}$$

$$\frac{\mathcal{D}}{\mathcal{G}_2\Delta x \mathcal{M}_2}\mathcal{F}_2^{u,h} =$$
$$igk\frac{3}{3+H^3k^3}+\frac{igk^3\left(2H^2k^2+3\right)}{4\left(3+H^2k^2\right)^2}\Delta x^2-\frac{igk^5\left(31H^4k^4+225H^2k^2+351\right)}{240\left(3+H^2k^2\right)^3}\Delta x^4+O(\Delta x^6)$$

5 Taylor Expansions Of Third Order Values

$$\mathcal{G}_3 = H - \frac{H^3}{3}\frac{-2\cos\left(2k\Delta x\right)+32\cos\left(k\Delta x\right)-30}{12\Delta x^2}$$

$$\mathcal{G}_3 = H + \frac{k^2H^3}{3} - \frac{k^6H^3}{270}(\Delta x)^4 + O(\Delta x^6)$$

$$\mathcal{M}_3 = \frac{24}{26-2\cos\left(k\Delta x\right)}$$

$$\mathcal{M}_3 = 1 - \frac{k^2}{24}(\Delta x)^2 + \frac{k^4}{192}(\Delta x)^4 + O(x^6)$$

$$\mathcal{R}_3^- = \frac{\mathcal{M}_3}{6}\left[5 - e^{-ik\Delta x} + 2e^{ik\Delta x}\right]$$

$$R_3^- = 1 + \frac{ik}{2}\Delta x - \frac{k^2}{8}(\Delta x)^2 - \frac{5ik^3}{48}(\Delta x)^3 + \frac{k^4}{64}(\Delta x)^4 + O(\Delta x^5)$$

$$\mathcal{R}_3^+ = \frac{\mathcal{M}_3 e^{ik\Delta x}}{6}\left[5 + 2e^{-ik\Delta x} - e^{ik\Delta x}\right]$$

$$R_3^+ = 1 + \frac{ik}{2}\Delta x - \frac{k^2}{8}(\Delta x)^2 + \frac{ik^3}{16}(\Delta x)^3 - \frac{13k^4}{192}(\Delta x)^4 + O(\Delta x^5)$$

$$\mathcal{R}_3^u = \frac{-3e^{2ik\Delta x} + 27e^{ik\Delta x} + 27 - 3e^{-ik\Delta x}}{48}$$

$$R_3^u = 1 + \frac{ik}{2}\Delta x - \frac{k^2}{8}(\Delta x)^2 - \frac{ik^3}{48}(\Delta x)^3 - \frac{k^4}{48}(\Delta x)^4 + O(\Delta x^5)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3}\mathcal{F}^{h,u} = \frac{1 - e^{-ik\Delta x}}{\Delta x}\frac{26 - 2\cos(k\Delta x)}{24}H\mathcal{R}^u$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3}\mathcal{F}_3^{h,u} = ikH - \frac{9ik^5 H}{320}\Delta x^4 - \frac{ik^7 H}{448}\Delta x^6 + O(\Delta x^9)$$

$$\begin{aligned}\frac{\mathcal{D}}{\Delta x \mathcal{M}_3}\mathcal{F}^{h,h} &= -\frac{1 - e^{-ik\Delta x}}{\Delta x \mathcal{M}_3}\frac{\sqrt{gH}}{2} \\ &\times \left[\left(\frac{\mathcal{M}_3 e^{ik\Delta x}}{6}\left[5 + 2e^{-ik\Delta x} - e^{ik\Delta x}\right]\right) - \left(\frac{\mathcal{M}_3}{6}\left[5 - e^{-ik\Delta x} + 2e^{ik\Delta x}\right]\right)\right]\end{aligned}$$

$$\begin{aligned}\frac{\mathcal{D}}{\Delta x \mathcal{M}_3}\mathcal{F}^{h,h} &= -\frac{1 - e^{-ik\Delta x}}{\Delta x}\frac{\sqrt{gH}}{2} \\ &\times \left[\left(\frac{e^{ik\Delta x}}{6}\left[5 + 2e^{-ik\Delta x} - e^{ik\Delta x}\right]\right) - \left(\frac{1}{6}\left[5 - e^{-ik\Delta x} + 2e^{ik\Delta x}\right]\right)\right]\end{aligned}$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_3}\mathcal{F}_3^{h,h} = \frac{k^4\sqrt{gH}}{12}\Delta x^3 - \frac{k^6\sqrt{gH}}{72}\Delta x^5 + \frac{k^8\sqrt{gH}}{960}\Delta x^7 + O(\Delta x^9)$$

$$\frac{\mathcal{D}}{\mathcal{G}_3\Delta x \mathcal{M}_3}\mathcal{F}^{u,u} = -\frac{1-e^{-ik\Delta x}}{\Delta x\mathcal{M}_3}\frac{\sqrt{gH}}{2}$$
$$\times\left[\left(\frac{\mathcal{M}_3 e^{ik\Delta x}}{6}\left[5+2e^{-ik\Delta x}-e^{ik\Delta x}\right]\right)-\left(\frac{\mathcal{M}_3}{6}\left[5-e^{-ik\Delta x}+2e^{ik\Delta x}\right]\right)\right]$$

$$\frac{\mathcal{D}}{\mathcal{G}_3\Delta x \mathcal{M}_3}\mathcal{F}^{u,u} = -\frac{1-e^{-ik\Delta x}}{\Delta x}\frac{\sqrt{gH}}{2}$$
$$\times\left[\left(\frac{e^{ik\Delta x}}{6}\left[5+2e^{-ik\Delta x}-e^{ik\Delta x}\right]\right)-\left(\frac{1}{6}\left[5-e^{-ik\Delta x}+2e^{ik\Delta x}\right]\right)\right]$$

$$\frac{\mathcal{D}}{\mathcal{G}_3\Delta x \mathcal{M}_3}\mathcal{F}^{u,u} = \frac{k^4\sqrt{gH}}{12}\Delta x^3 - \frac{k^6\sqrt{gH}}{72}\Delta x^5 + \frac{k^8\sqrt{gH}}{960}\Delta x^7 + O(\Delta x^9)$$

$$\frac{\mathcal{D}}{\mathcal{G}_3\Delta x \mathcal{M}_3}\mathcal{F}^{u,h} = \frac{gH}{2}\frac{1-e^{-ik\Delta x}}{\Delta x\mathcal{M}_3}\left[H-\frac{H^3}{3}\frac{-2\cos(2k\Delta x)+32\cos(k\Delta x)-30}{12\Delta x^2}\right]^{-1}$$
$$\times\left[\left(\frac{e^{ik\Delta x}\mathcal{M}_3}{6}\left[5+2e^{-ik\Delta x}-e^{ik\Delta x}\right]\right)+\left(\frac{\mathcal{M}_3}{6}\left[5-e^{-ik\Delta x}+2e^{ik\Delta x}\right]\right)\right]$$

$$\frac{\mathcal{D}}{\mathcal{G}_3\Delta x \mathcal{M}_3}\mathcal{F}^{u,h} = \frac{gH}{2}\frac{1-e^{-ik\Delta x}}{\Delta x}\left[H-\frac{H^3}{3}\frac{-2\cos(2k\Delta x)+32\cos(k\Delta x)-30}{12\Delta x^2}\right]^{-1}$$
$$\times\left[\left(\frac{e^{ik\Delta x}}{6}\left[5+2e^{-ik\Delta x}-e^{ik\Delta x}\right]\right)+\left(\frac{1}{6}\left[5-e^{-ik\Delta x}+2e^{ik\Delta x}\right]\right)\right]$$

$$\frac{\mathcal{D}}{\mathcal{G}_3\Delta x \mathcal{M}_3}\mathcal{F}^{u,h} = igk\frac{3}{3+H^2k^2} - \frac{igk^5\left(2H^2k^2+9\right)}{30\left(H^2k^2+3\right)^2}\Delta x^4 + \frac{igk^7\left(H^2k^2+4\right)}{112\left(H^2k^2+3\right)^2}\Delta x^6 + O(\Delta x^8)$$