1 Linearised Equations

$$G = uh - \frac{h^3}{3} u_{xx}$$

$$\eta_t + hu_x = 0$$

$$hu_t - \frac{h^3}{3} u_{xxt} + gh\eta_x = 0$$

$$(G)_t + gh\eta_x = 0$$

2 Numerical Approximation

We investigate our numerical technique by adding in a fourier mode so $W_j = W_0 e^{i(vt+kx_j)}$, and rewriting the equations using our spatial discretisation

2.1 G

Analytic:

$$G_j = u_j h_j - (\frac{h_j^3}{3} u_{xx})_j$$

Numerical approximation, we used second order central differences so we replace the second derivative of u with this approximation to it So we get

$$G_j = u_j h_j - \frac{h_j^3}{3} \left(\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} \right)$$

$$G_j = u_0 e^{i(vt+kx_j)} h_0 - \frac{h_0}{3} u_0 \left(\frac{e^{i(vt+kx_{j+1})} - 2e^{i(vt+kx_j)} + e^{i(vt+kx_{j-1})}}{\Delta x^2} \right)$$

$$G_j = u_0 e^{i(vt+kx_j)} h_0 - \frac{h_0}{3} u_0 \left(\frac{e^{i(vt+kx_j)+ik\Delta x} - 2e^{i(vt+kx_j)} + e^{i(vt+kx_j)+ik\Delta x}}{\Delta x^2} \right)$$

$$G_j = u_j h_0 - \frac{h_0}{3} u_j \left(\frac{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}{\Delta x^2} \right)$$

$$G_j = u_j \left(h_0 - \frac{h_0^3}{3} \left(\frac{2\cos\left(k\Delta x\right) - 2}{\Delta x^2} \right) \right)$$

We are dealing with time continuous variables so, we first take the derivative in time exactly for the Fourier nodes so that:

So what we have is something that depends on the order used to approximate $u_x x$, lets call it $\mathcal{C}_2$ Thus:

$$\mathcal{C}_2 = \frac{2\cos\left(k\Delta x\right) - 2}{\Delta x^2}$$

$$G_j = u_j \left(h_0 - \frac{h_0^3}{3} \mathcal{C}_2 \right)$$

Furthermore we will call this whole thing $\mathcal{G}_2$ So we have

$$\mathcal{G}_2 = \left(h_0 - \frac{h_0^3}{3} \mathcal{C}_2 \right)$$

then

$$G_j = u_j \mathcal{G}_2$$

Now we move on to

$$\eta_t + h u_x = 0$$

our equations are time continuous so that:

$$\eta_t + h u_x = 0$$

$$iv\eta + h u_x = 0$$

next we approximate

our conservation equations of the form

$$q_t + \left[f(q) \right]_x = 0$$

by

$$q_t + \frac{1}{\Delta x} \left[F_{j+1/2} - F_{j-1/2} \right] = 0$$

where $F_{j\pm 1/2}$ given by Kurganovs method. In this equation h is constant so $f(\eta, u) = hu$. We start Kurganovs method by doing a reconstruction, we start by doing a central differencing approximation to obtain that we note that the result is something like

$$q^{-}_{j+1/2} = q_j + \frac{q_{j+1} - q_{j-1}}{4}$$
$$q^{+}_{j+1/2} = q_{j+1} + \frac{q_{j+2} - q_j}{4}$$

Applying our fourier mode

$$q^{-}_{j+1/2} = q_j + \frac{q_j e^{ik\Delta x} - q_j e^{-ik\Delta x}}{4}$$
$$q^{-}_{j+1/2} = q_j \left(1 + \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{4}\right)$$
$$q^{-}_{j+1/2} = q_j \left(1 + \frac{2i \sin(k\Delta x)}{4}\right)$$
$$q^{-}_{j+1/2} = q_j \left(1 + \frac{i \sin(k\Delta x)}{2}\right)$$

for the plus we get the same result with a shift so that (because its around j+1) and a minus

$$q^{+}_{j+1/2} = q_j e^{ik\Delta x} \left(1 - \frac{i \sin(k\Delta x)}{2}\right)$$

So we have that

$$\mathcal{R}^{-}_{2} = 1 + \frac{i \sin(k\Delta x)}{2}$$
$$\mathcal{R}^{+}_{2} = e^{ik\Delta x} \left(1 - \frac{i \sin(k\Delta x)}{2}\right)$$

$$q^{-}_{j+1/2} = \mathcal{R}^{-}_{2} q_j$$
$$q^{+}_{j+1/2} = \mathcal{R}^{+}_{2} q_j$$

for u and η, this happens to G aswell just because G and u are related by a factor. Sanity checks: 1. uxx substitute behaves as it should (tick), so G seems correct c˙2 k**2 2. Reconstruction limits to 1 as delta x goes to zero.

Next we have to use the wavespeeds, up to our linearisation assuming still water the velocities are zero so

$$a^-_{j+1/2} = -\sqrt{gh^-_{j+1/2}}$$

$$a^+_{j+1/2} = +\sqrt{gh^-_{j+1/2}}$$

We have that

$$F_{i+\frac{1}{2}} = \frac{a^+_{i+\frac{1}{2}} f\left(q^-_{i+\frac{1}{2}}\right) - a^-_{i+\frac{1}{2}} f\left(q^+_{i+\frac{1}{2}}\right)}{a^+_{i+\frac{1}{2}} - a^-_{i+\frac{1}{2}}} + \frac{a^+_{i+\frac{1}{2}}\ a^-_{i+\frac{1}{2}}}{a^+_{i+\frac{1}{2}} - a^-_{i+\frac{1}{2}}} \left[q^+_{i+\frac{1}{2}} - q^-_{i+\frac{1}{2}}\right] \quad (1)$$

$$\begin{aligned} F_{i+\frac{1}{2}} = {} & \frac{\left(\sqrt{gh^-_{j+1/2}}\right) f\left(q^-_{i+\frac{1}{2}}\right) - \left(-\sqrt{gh^-_{j+1/2}}\right) f\left(q^+_{i+\frac{1}{2}}\right)}{\left(\sqrt{gh^-_{j+1/2}}\right) - \left(-\sqrt{gh^-_{j+1/2}}\right)} \\ & + \frac{\left(\sqrt{gh^-_{j+1/2}}\right)\ \left(-\sqrt{gh^-_{j+1/2}}\right)}{\left(+\sqrt{gh^-_{j+1/2}}\right) - \left(-\sqrt{gh^-_{j+1/2}}\right)} \left[q^+_{i+\frac{1}{2}} - q^-_{i+\frac{1}{2}}\right] \end{aligned} \quad (2)$$

$$\begin{aligned} F_{i+\frac{1}{2}} = {} & \frac{\left(+\sqrt{gh^-_{j+1/2}}\right) f\left(q^-_{i+\frac{1}{2}}\right) - \left(-\sqrt{gh^-_{j+1/2}}\right) f\left(q^+_{i+\frac{1}{2}}\right)}{2\sqrt{gh^-_{j+1/2}}} \\ & + \frac{\left(+\sqrt{gh^-_{j+1/2}}\right)\ \left(-\sqrt{gh^-_{j+1/2}}\right)}{2\sqrt{gh^-_{j+1/2}}} \left[q^+_{i+\frac{1}{2}} - q^-_{i+\frac{1}{2}}\right] \end{aligned} \quad (3)$$

$$\begin{aligned} F_{i+\frac{1}{2}} = {} & \frac{\left(+\sqrt{gh^-_{j+1/2}}\right) f\left(q^-_{i+\frac{1}{2}}\right) - \left(-\sqrt{gh^-_{j+1/2}}\right) f\left(q^+_{i+\frac{1}{2}}\right)}{2\sqrt{gh^-_{j+1/2}}} \\ & + \frac{-gh^-_{j+1/2}}{2\sqrt{gh^-_{j+1/2}}} \left[q^+_{i+\frac{1}{2}} - q^-_{i+\frac{1}{2}}\right] \end{aligned} \quad (4)$$

$$F_{i+\frac{1}{2}} = \frac{f\left(q^-_{i+\frac{1}{2}}\right) + f\left(q^+_{i+\frac{1}{2}}\right)}{2} - \frac{\sqrt{gh^-_{j+1/2}}}{2}\left[q^+_{i+\frac{1}{2}} - q^-_{i+\frac{1}{2}}\right] \quad (5)$$

for eta this becomes

$$F_{i+\frac{1}{2}}(\eta) = \frac{hu^-_{i+\frac{1}{2}} + hu^+_{i+\frac{1}{2}}}{2} - \frac{\sqrt{gh^-_{j+1/2}}}{2}\left[\eta^+_{i+\frac{1}{2}} - \eta^-_{i+\frac{1}{2}}\right]$$

up to order the last term becomes

$$F_{i+\frac{1}{2}}(\eta) = \frac{hu^-_{i+\frac{1}{2}} + hu^+_{i+\frac{1}{2}}}{2} - \frac{\sqrt{gh}}{2}\left[\eta^+_{i+\frac{1}{2}} - \eta^-_{i+\frac{1}{2}}\right]$$

$$F_{i+\frac{1}{2}}(\eta) = \frac{h\mathcal{R}^- u_j + h\mathcal{R}^+ u_j}{2} - \frac{\sqrt{gh}}{2}\left[\mathcal{R}^+\eta_j - \mathcal{R}^-\eta_j\right]$$

$$F_{i+\frac{1}{2}}(\eta) = \frac{h\mathcal{R}^- + h\mathcal{R}^+}{2}u_j - \frac{\sqrt{gh}}{2}\left[\mathcal{R}^+ - \mathcal{R}^-\right]\eta_j$$

$$F_{i+\frac{1}{2}}(\eta) = F_2^{\eta,u} u_j + F_2^{\eta,\eta}\eta_j$$

where

$$F_2^{\eta,u} = \frac{h\mathcal{R}^- + h\mathcal{R}^+}{2}$$

$$F_2^{\eta,\eta} = -\frac{\sqrt{gh}}{2}\left[\mathcal{R}^+ - \mathcal{R}^-\right]$$

For G this becomes

$$F_{i+\frac{1}{2}}(G) = \frac{gh\eta^-_{i+\frac{1}{2}} + gh\eta^+_{i+\frac{1}{2}}}{2} - \frac{\sqrt{gh^-_{j+1/2}}}{2}\left[\mathcal{G}u^+_{i+\frac{1}{2}} - \mathcal{G}u^-_{i+\frac{1}{2}}\right] \quad (6)$$

$$F_{i+\frac{1}{2}}(G) = \frac{gh\mathcal{R}^-\eta_j + gh\mathcal{R}^+\eta_j}{2} - \frac{\sqrt{gh^-_{j+1/2}}}{2}\left[\mathcal{G}\mathcal{R}^+ u_j - \mathcal{G}\mathcal{R}^- u_j\right] \quad (7)$$

up to order the last term becomes

$$F_{i+\frac{1}{2}}(G) = \frac{gh\mathcal{R}^{-}\eta_j + gh\mathcal{R}^{+}\eta_j}{2} - \frac{\sqrt{gh}}{2}\left[\mathcal{G}\mathcal{R}^{+}u_j - \mathcal{G}\mathcal{R}^{-}u_j\right] \quad (8)$$

$$F_{i+\frac{1}{2}}(G) = \frac{gh\mathcal{R}^{-} + gh\mathcal{R}^{+}}{2}\eta_j - \frac{\sqrt{gh}}{2}\left[\mathcal{G}\mathcal{R}^{+} - \mathcal{G}\mathcal{R}^{-}\right]u_j \quad (9)$$

$$F_{i+\frac{1}{2}}(G) = F_2^{G,u}u_j + F_2^{G,\eta}\eta_j$$

where

$$F_2^{G,u} = -\frac{\sqrt{gh}}{2}\left[\mathcal{G}\mathcal{R}^{+} - \mathcal{G}\mathcal{R}^{-}\right]$$

$$F_2^{G,\eta} = \frac{gh\mathcal{R}^{-} + gh\mathcal{R}^{+}}{2}$$

So we have

$$ivG_j + gh\eta_x = 0$$

$$iv\eta + hu_x = 0$$

become

$$iv\eta + \frac{1}{\Delta x}\left[(1 - e^{-ik\Delta x})F_2^{\eta,u}u_j + (1 - e^{-ik\Delta x})F_2^{\eta,\eta}\eta_j\right] = 0$$

We let $\mathcal{D} = (1 - e^{-ik\Delta x})$

$$iv\eta_j + \frac{1}{\Delta x}\left[\mathcal{D}F_2^{\eta,u}u_j + \mathcal{D}F_2^{\eta,\eta}\eta_j\right] = 0$$

$$\left[iv + \frac{1}{\Delta x}\mathcal{D}F_2^{\eta,\eta}\right]\eta_j + \frac{1}{\Delta x}\left[\mathcal{D}F_2^{\eta,u}\right]u_j = 0$$

Similarly for G

$$\mathcal{G}\left[iv + \frac{1}{\Delta x}\mathcal{D}F_2^{G,u}\right]u_j + \frac{1}{\Delta x}\left[\mathcal{D}F_2^{G,\eta}\right]\eta_j = 0$$

$$\begin{bmatrix} iv + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,\eta)} & \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,u)} \\ \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,\eta)} & iv\mathcal{G} + \mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,u)} \end{bmatrix} \begin{bmatrix} \eta_j \\ u_j \end{bmatrix} = 0$$

for a nontrivial solution

$$\left[iv + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,\eta)}\right]\left[iv\mathcal{G} + \mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,u)}\right] - \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,u)}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,\eta)}$$

$$\left[iv + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,\eta)}\right]\left[iv\mathcal{G} + \mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,u)}\right] - \frac{1}{\Delta x^2}\mathcal{D}^2\mathcal{F}_2^{(\eta,u)}\mathcal{F}_2^{(G,\eta)}$$

$$-v^2\mathcal{G} + iv\mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,u)} + iv\mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,\eta)} + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,\eta)}\mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,u)} - \frac{1}{\Delta x^2}\mathcal{D}^2\mathcal{F}_2^{(\eta,u)}\mathcal{F}_2^{(G,\eta)} = 0$$

$$-\mathcal{G}v^2 + i\left[\mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,u)} + \mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,\eta)}\right]v + \frac{1}{\Delta x^2}\mathcal{D}^2\mathcal{G}\mathcal{F}_2^{(\eta,\eta)}\mathcal{F}_2^{(G,u)} - \frac{1}{\Delta x^2}\mathcal{D}^2\mathcal{F}_2^{(\eta,u)}\mathcal{F}_2^{(G,\eta)} = 0$$

$$-\mathcal{G}v^2 + i\left[\mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,u)} + \mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,\eta)}\right]v + \frac{1}{\Delta x^2}\mathcal{D}^2\left[\mathcal{G}\mathcal{F}_2^{(\eta,\eta)}\mathcal{F}_2^{(G,u)} - \mathcal{F}_2^{(\eta,u)}\mathcal{F}_2^{(G,\eta)}\right] = 0$$