In[736]:=
```
MA = k*x / (2*Sin[k*x/2])
RA = Exp[I*k*x/2]*k*x / (2*Sin[k*x/2])
GA = k*x/ ((H + H^3/3*k^2) *Exp[-I*k*x/2] * (2*Sin[k*x/2]))
FnnA = 0
FnGA = I*k/ (1 + H^2*k^2/3)
FGnA = g*H*I*k
FGGA = 0
FmatA = {{FnnA, FnGA}, {FGnA, FGGA}}
wAp  =  Sqrt[g*H]*k*Sqrt[3 / (3 + H^2*k^2)]
wAm  =  -Sqrt[g*H]*k*Sqrt[3 / (3 + H^2*k^2)]
Eigenvalues[FmatA]
```

Out[736]= $\frac{1}{2}\,k\,x\,\mathrm{Csc}\left[\frac{k\,x}{2}\right]$

Out[737]= $\frac{1}{2}\,e^{\frac{i\,k\,x}{2}}\,k\,x\,\mathrm{Csc}\left[\frac{k\,x}{2}\right]$

Out[738]= $\frac{e^{\frac{i\,k\,x}{2}}\,k\,x\,\mathrm{Csc}\left[\frac{k\,x}{2}\right]}{2\left(H+\frac{H^3\,k^2}{3}\right)}$

Out[739]= 0

Out[740]= $\frac{i\,k}{1+\frac{H^2\,k^2}{3}}$

Out[741]= $i\,g\,H\,k$

Out[742]= 0

Out[743]= $\left\{\left\{0,\ \frac{i\,k}{1+\frac{H^2\,k^2}{3}}\right\},\ \{i\,g\,H\,k,\ 0\}\right\}$

Out[744]= $\sqrt{3}\ \sqrt{g\,H}\ k\sqrt{\frac{1}{3+H^2\,k^2}}$

Out[745]= $-\sqrt{3}\ \sqrt{g\,H}\ k\sqrt{\frac{1}{3+H^2\,k^2}}$

Out[746]= $\left\{-\frac{i\,\sqrt{3}\ k\,\sqrt{3\,g\,H+g\,H^3\,k^2}}{3+H^2\,k^2},\ \frac{i\,\sqrt{3}\ k\,\sqrt{3\,g\,H+g\,H^3\,k^2}}{3+H^2\,k^2}\right\}$

In[747]:=
```
M = 1
Series[M - MA, {x, 0, 10}]
```

Out[747]= 1

Out[748]= $-\frac{k^2\,x^2}{24}-\frac{7\,k^4\,x^4}{5760}-\frac{31\,k^6\,x^6}{967\,680}-\frac{127\,k^8\,x^8}{154\,828\,800}-\frac{73\,k^{10}\,x^{10}}{3\,503\,554\,560}+O[x]^{11}$

In[749]:=
```
Rm = 1 + I * Sin[k * x] / 2
Series[Rm - RA, {x, 0, 10}]
Rp = Exp[I * k * x] * (1 - I * Sin[k * x] / 2)
Series[Rp - RA, {x, 0, 10}]
```

Out[749]= $1 + \frac{1}{2}\, i \,\mathrm{Sin}[k\, x]$

Out[750]= $\frac{k^2 x^2}{12} - \frac{1}{12}\, i\, k^3 x^3 + \frac{k^4 x^4}{720} + \frac{1}{240}\, i\, k^5 x^5 + \frac{k^6 x^6}{30\,240} - \frac{i\, k^7 x^7}{10\,080} + \frac{k^8 x^8}{1\,209\,600} + \frac{i\, k^9 x^9}{725\,760} + \frac{k^{10} x^{10}}{47\,900\,160} + O[x]^{11}$

Out[751]= $e^{i\, k\, x} \left(1 - \frac{1}{2}\, i\, \mathrm{Sin}[k\, x]\right)$

Out[752]= $\frac{k^2 x^2}{12} + \frac{1}{6}\, i\, k^3 x^3 - \frac{89\, k^4 x^4}{720} - \frac{7}{120}\, i\, k^5 x^5 + \frac{631\, k^6 x^6}{30\,240} + \frac{31\, i\, k^7 x^7}{5040} - \frac{1889\, k^8 x^8}{1\,209\,600} - \frac{127\, i\, k^9 x^9}{362\,880} + \frac{481\, k^{10} x^{10}}{6\,842\,880} + O[x]^{11}$

In[753]:=
```
Ru = (1 + Exp[I*k*x])/2
Series[Ru - Exp[I*k*x/2] , {x, 0, 10}]
Gold = H - H^3/3*(2*Cos[k*x] - 2)/x^2
G = Ru / Gold
Series[G, {x, 0, 3} ]
Series[GA, {x, 0, 3} ]
Series[G - GA, {x, 0, 5}]
```

Out[753]= $\frac{1}{2}\left(1+e^{i k x}\right)$

Out[754]= $-\frac{k^2 x^2}{8}-\frac{1}{16} i k^3 x^3+\frac{7 k^4 x^4}{384}+\frac{1}{256} i k^5 x^5-\frac{31 k^6 x^6}{46080}-\frac{i k^7 x^7}{10240}+\frac{127 k^8 x^8}{10321920}+\frac{17 i k^9 x^9}{12386304}-\frac{73 k^{10} x^{10}}{530841600}+O[x]^{11}$

Out[755]= $H-\frac{H^3 (-2+2 \mathrm{Cos}[k x])}{3 x^2}$

Out[756]= $\frac{1+e^{i k x}}{2\left(H-\frac{H^3 (-2+2 \mathrm{Cos}[k x])}{3 x^2}\right)}$

Out[757]= $\frac{1}{H+\frac{H^3 k^2}{3}}+\frac{i k x}{2\left(H+\frac{H^3 k^2}{3}\right)}+\frac{\left(-9 k^2-2 H^2 k^4\right) x^2}{4 H\left(3+H^2 k^2\right)^2}-\frac{i\left(6 k^3+H^2 k^5\right) x^3}{8 H\left(3+H^2 k^2\right)^2}+O[x]^4$

Out[758]= $\frac{1}{H+\frac{H^3 k^2}{3}}+\frac{i k x}{2\left(H+\frac{H^3 k^2}{3}\right)}-\frac{k^2 x^2}{12\left(H+\frac{H^3 k^2}{3}\right)}+O[x]^4$

Out[759]= $\frac{\left(-6 k^2-H^2 k^4\right) x^2}{4 H\left(3+H^2 k^2\right)^2}-\frac{i\left(6 k^3+H^2 k^5\right) x^3}{8 H\left(3+H^2 k^2\right)^2}+\frac{\left(144 k^4+45 H^2 k^6+4 H^4 k^8\right) x^4}{240 H\left(3+H^2 k^2\right)^3}-\frac{i\left(-54 k^5+H^4 k^9\right) x^5}{480 H\left(3+H^2 k^2\right)^3}+O[x]^6$

In[791]:=
```
fnn = - Sqrt[g * H] / 2 * (Rp - Rm);
fng = H * G;
fgg = - Sqrt[g * H] / 2 * (Rp - Rm);
fgn = g * H * (Rp + Rm) / 2;

Fnn = (1 - Exp[-I * k * x]) / x * fnn
Series[Fnn - FnnA, {x, 0, 5}]
Fng = (1 - Exp[-I * k * x]) / x * fng
Series[Fng - FnGA, {x, 0, 5}]
Fgg = (1 - Exp[-I * k * x]) / x * fgg
Series[Fgg - FGGA, {x, 0, 5}]
Fgn = (1 - Exp[-I * k * x]) / x * fgn
Series[Fgn - FGnA, {x, 0, 5}]

Fmat = {{Fnn , Fng}, {Fgn , Fgg}}
EigvFmat = Eigenvalues[Fmat];
 Simplify[Series[EigvFmat , {x, 0, 5}]]
t =  x / (2 * Sqrt[g * H])
RKStep = Log[ 1 - t * EigvFmat + (t * EigvFmat) ^2 / 2] / (I * t);
RKstepTay = Series[RKStep, {x, 0, 5}]
Simplify[RKstepTay, k * H > 0 ]
Simplify[RKstepTay - {wAp, wAm}, k * H > 0]
```

Out[795]= $-\frac{1}{2\,x}\left(1-e^{-i\,k\,x}\right)\sqrt{g\,H}\,\left(-1+e^{i\,k\,x}\left(1-\frac{1}{2}\,i\,\mathrm{Sin}[k\,x]\right)-\frac{1}{2}\,i\,\mathrm{Sin}[k\,x]\right)$

Out[796]= $\frac{1}{8}\sqrt{g\,H}\;k^4\,x^3-\frac{1}{48}\left(\sqrt{g\,H}\;k^6\right)x^5+O[x]^6$

Out[797]= $\frac{\left(1-e^{-i\,k\,x}\right)\left(1+e^{i\,k\,x}\right)H}{2\,x\left(H-\frac{H^3\,(-2+2\,\mathrm{Cos}[k\,x])}{3\,x^2}\right)}$

Out[798]= $-\frac{i\left(6\,k^3+H^2\,k^5\right)x^2}{4\left(3+H^2\,k^2\right)^2}-\frac{i\left(-54\,k^5+H^4\,k^9\right)x^4}{240\left(3+H^2\,k^2\right)^3}+O[x]^6$

Out[799]= $-\frac{\left(1-e^{-i\,k\,x}\right)\sqrt{g\,H}\,\left(-1+e^{i\,k\,x}\left(1-\frac{1}{2}\,i\,\mathrm{Sin}[k\,x]\right)-\frac{1}{2}\,i\,\mathrm{Sin}[k\,x]\right)}{2\,x}$

Out[800]= $\frac{1}{8}\sqrt{g\,H}\;k^4\,x^3-\frac{1}{48}\left(\sqrt{g\,H}\;k^6\right)x^5+O[x]^6$

Out[801]= $\frac{\left(1-e^{-i\,k\,x}\right)g\,H\left(1+e^{i\,k\,x}\left(1-\frac{1}{2}\,i\,\mathrm{Sin}[k\,x]\right)+\frac{1}{2}\,i\,\mathrm{Sin}[k\,x]\right)}{2\,x}$

Out[802]= $\frac{1}{12}\,i\,g\,H\,k^3\,x^2-\frac{13}{240}\,i\,g\,H\,k^5\,x^4+O[x]^6$

Out[803]= $$\left\{\left\{-\frac{\left(1-e^{-i k x}\right)\sqrt{g H}\left(-1+e^{i k x}\left(1-\frac{1}{2} i \operatorname{Sin}[k x]\right)-\frac{1}{2} i \operatorname{Sin}[k x]\right)}{2 x}, \frac{\left(1-e^{-i k x}\right)\left(1+e^{i k x}\right) H}{2 x\left(H-\frac{H^3(-2+2 \operatorname{Cos}[k x])}{3 x^2}\right)}\right\}, \left\{\frac{\left(1-e^{-i k x}\right) g H\left(1+e^{i k x}\left(1-\frac{1}{2} i \operatorname{Sin}[k x]\right)+\frac{1}{2} i \operatorname{Sin}[k x]\right)}{2 x}, -\frac{\left(1-e^{-i k x}\right)\sqrt{g H}\left(-1+e^{i k x}\left(1-\frac{1}{2} i \operatorname{Sin}[k x]\right)-\frac{1}{2} i \operatorname{Sin}[k x]\right)}{2 x}\right\}\right\}$$

Out[805]= $$\left\{-\frac{i\sqrt{3}\, g H k}{\sqrt{g H\left(3+H^2 k^2\right)}}+\frac{i\sqrt{3}\, g^2 H^2 k^3 x^2}{8\left(g H\left(3+H^2 k^2\right)\right)^{3/2}}+\frac{1}{8}\sqrt{g H}\, k^4 x^3+\frac{i\sqrt{3}\, k^5\sqrt{g H\left(3+H^2 k^2\right)}\left(177+124 H^2 k^2+20 H^4 k^4\right) x^4}{640\left(3+H^2 k^2\right)^3}-\frac{1}{48}\left(\sqrt{g H}\, k^6\right) x^5+O[x]^6, \frac{i\sqrt{3}\, g H k}{\sqrt{g H\left(3+H^2 k^2\right)}}-\frac{i\sqrt{3}\, g^2 H^2 k^3 x^2}{8\left(g H\left(3+H^2 k^2\right)\right)^{3/2}}+\frac{1}{8}\sqrt{g H}\, k^4 x^3-\frac{i\sqrt{3}\, k^5\sqrt{g H\left(3+H^2 k^2\right)}\left(177+124 H^2 k^2+20 H^4 k^4\right) x^4}{640\left(3+H^2 k^2\right)^3}-\frac{1}{48}\left(\sqrt{g H}\, k^6\right) x^5+O[x]^6\right\}$$

Out[806]= $$\frac{x}{2\sqrt{g H}}$$

Out[808]= $$\left\{\frac{k\sqrt{144 g H+48 g H^3 k^2}}{4\left(3+H^2 k^2\right)}-\frac{i\sqrt{g H}\left(-63 k^4-48 H^2 k^6-8 H^4 k^8\right) x^3}{64\left(3+H^2 k^2\right)^2}-\frac{\left(g H\left(225\sqrt{3}\, k^5+124\sqrt{3}\, H^2 k^7+20\sqrt{3}\, H^4 k^9\right)\right) x^4}{640\left(\left(3+H^2 k^2\right)^2\sqrt{g H\left(3+H^2 k^2\right)}\right)}-\frac{i\sqrt{g H}\left(27 k^6+108 H^2 k^8+54 H^4 k^{10}+8 H^6 k^{12}\right) x^5}{384\left(3+H^2 k^2\right)^3}+O[x]^6, -\frac{k\sqrt{144 g H+48 g H^3 k^2}}{4\left(3+H^2 k^2\right)}-\frac{i\sqrt{g H}\left(-63 k^4-48 H^2 k^6-8 H^4 k^8\right) x^3}{64\left(3+H^2 k^2\right)^2}+\frac{g H\left(225\sqrt{3}\, k^5+124\sqrt{3}\, H^2 k^7+20\sqrt{3}\, H^4 k^9\right) x^4}{640\left(3+H^2 k^2\right)^2\sqrt{g H\left(3+H^2 k^2\right)}}-\frac{i\sqrt{g H}\left(27 k^6+108 H^2 k^8+54 H^4 k^{10}+8 H^6 k^{12}\right) x^5}{384\left(3+H^2 k^2\right)^3}+O[x]^6\right\}$$

Out[809]=
$$\left\{\frac{\sqrt{3}\, g H k}{\sqrt{g H\left(3+H^2 k^2\right)}}+\frac{i \sqrt{g H}\, k^4\left(63+48 H^2 k^2+8 H^4 k^4\right) x^3}{64\left(3+H^2 k^2\right)^2}-\frac{\left(\sqrt{3}\, \sqrt{g H}\, k^5\left(225+124 H^2 k^2+20 H^4 k^4\right)\right) x^4}{640\left(3+H^2 k^2\right)^{5/2}}-\frac{i \sqrt{g H}\, k^6\left(27+108 H^2 k^2+54 H^4 k^4+8 H^6 k^6\right) x^5}{384\left(3+H^2 k^2\right)^3}+O[x]^6,\ -\frac{\sqrt{3}\, g H k}{\sqrt{g H\left(3+H^2 k^2\right)}}+\frac{i \sqrt{g H}\, k^4\left(63+48 H^2 k^2+8 H^4 k^4\right) x^3}{64\left(3+H^2 k^2\right)^2}+\frac{\sqrt{3}\, \sqrt{g H}\, k^5\left(225+124 H^2 k^2+20 H^4 k^4\right) x^4}{640\left(3+H^2 k^2\right)^{5/2}}-\frac{i \sqrt{g H}\, k^6\left(27+108 H^2 k^2+54 H^4 k^4+8 H^6 k^6\right) x^5}{384\left(3+H^2 k^2\right)^3}+O[x]^6\right\}$$

Out[810]=
$$\left\{\frac{i \sqrt{g H}\, k^4\left(63+48 H^2 k^2+8 H^4 k^4\right) x^3}{64\left(3+H^2 k^2\right)^2}-\frac{\left(\sqrt{3}\, \sqrt{g H}\, k^5\left(225+124 H^2 k^2+20 H^4 k^4\right)\right) x^4}{640\left(3+H^2 k^2\right)^{5/2}}-\frac{i \sqrt{g H}\, k^6\left(27+108 H^2 k^2+54 H^4 k^4+8 H^6 k^6\right) x^5}{384\left(3+H^2 k^2\right)^3}+O[x]^6,\ \frac{i \sqrt{g H}\, k^4\left(63+48 H^2 k^2+8 H^4 k^4\right) x^3}{64\left(3+H^2 k^2\right)^2}+\frac{\sqrt{3}\, \sqrt{g H}\, k^5\left(225+124 H^2 k^2+20 H^4 k^4\right) x^4}{640\left(3+H^2 k^2\right)^{5/2}}-\frac{i \sqrt{g H}\, k^6\left(27+108 H^2 k^2+54 H^4 k^4+8 H^6 k^6\right) x^5}{384\left(3+H^2 k^2\right)^3}+O[x]^6\right\}$$

In[1017]:=
```
t =  x / (4 * Sqrt[g * H])
RKStep = Log[1 - t * EigvFmat + (t * EigvFmat) ^2 / 2] / (I * t);
RKstepTay = Series[RKStep, {x, 0, 5}]
Simplify[RKstepTay, k * H > 0 ]
Simplify[RKstepTay - {wAp, wAm}, k * H > 0]
```

Out[1017]=
$$\frac{x}{4\sqrt{g H}}$$

Out[1019]=
$$\left\{\frac{\sqrt{3}\,k\sqrt{g\,H\left(3+H^2k^2\right)}}{3+H^2k^2}+\frac{\sqrt{3}\,k^3\sqrt{g\,H\left(3+H^2k^2\right)}\,x^2}{32\left(3+H^2k^2\right)^2}-i\sqrt{g\,H}\left(-\frac{k^4}{12}+\frac{9\,k^4}{512\left(3+H^2k^2\right)^2}\right)x^3-\frac{\left(g\,H\left(12825\sqrt{3}\,k^5+7728\sqrt{3}\,H^2k^7+1160\sqrt{3}\,H^4k^9\right)\right)x^4}{46080\left(\left(3+H^2k^2\right)^2\sqrt{g\,H\left(3+H^2k^2\right)}\right)}-\frac{i\sqrt{g\,H}\left(39\,k^6+16\,H^2k^8\right)x^5}{1152\left(3+H^2k^2\right)}+O[x]^6,\right.$$
$$\left.-\frac{\sqrt{3}\,k\sqrt{g\,H\left(3+H^2k^2\right)}}{3+H^2k^2}-\frac{\left(\sqrt{3}\,k^3\sqrt{g\,H\left(3+H^2k^2\right)}\right)x^2}{32\left(3+H^2k^2\right)^2}-i\sqrt{g\,H}\left(-\frac{k^4}{12}+\frac{9\,k^4}{512\left(3+H^2k^2\right)^2}\right)x^3+\frac{g\,H\left(12825\sqrt{3}\,k^5+7728\sqrt{3}\,H^2k^7+1160\sqrt{3}\,H^4k^9\right)x^4}{46080\left(3+H^2k^2\right)^2\sqrt{g\,H\left(3+H^2k^2\right)}}-\frac{i\sqrt{g\,H}\left(39\,k^6+16\,H^2k^8\right)x^5}{1152\left(3+H^2k^2\right)}+O[x]^6\right\}$$

Out[1020]=
$$\left\{\frac{\sqrt{3}\,g\,H\,k}{\sqrt{g\,H\left(3+H^2k^2\right)}}+\frac{\sqrt{3}\sqrt{g\,H}\,k^3x^2}{32\left(3+H^2k^2\right)^{3/2}}-\frac{i\sqrt{g\,H}\,k^4\left(-128+\frac{27}{\left(3+H^2k^2\right)^2}\right)x^3}{1536}-\frac{\left(\sqrt{g\,H}\,k^5\left(12825+7728\,H^2k^2+1160\,H^4k^4\right)\right)x^4}{15360\left(\sqrt{3}\left(3+H^2k^2\right)^{5/2}\right)}-\frac{i\sqrt{g\,H}\,k^6\left(39+16\,H^2k^2\right)x^5}{1152\left(3+H^2k^2\right)}+O[x]^6,\right.$$
$$\left.-\frac{\sqrt{3}\,g\,H\,k}{\sqrt{g\,H\left(3+H^2k^2\right)}}-\frac{\left(\sqrt{3}\sqrt{g\,H}\,k^3\right)x^2}{32\left(3+H^2k^2\right)^{3/2}}-\frac{i\sqrt{g\,H}\,k^4\left(-128+\frac{27}{\left(3+H^2k^2\right)^2}\right)x^3}{1536}+\frac{\sqrt{g\,H}\,k^5\left(12825+7728\,H^2k^2+1160\,H^4k^4\right)x^4}{15360\sqrt{3}\left(3+H^2k^2\right)^{5/2}}-\frac{i\sqrt{g\,H}\,k^6\left(39+16\,H^2k^2\right)x^5}{1152\left(3+H^2k^2\right)}+O[x]^6\right\}$$

Out[1021]=
$$\left\{\frac{\sqrt{3}\sqrt{g\,H}\,k^3x^2}{32\left(3+H^2k^2\right)^{3/2}}-\frac{i\sqrt{g\,H}\,k^4\left(-128+\frac{27}{\left(3+H^2k^2\right)^2}\right)x^3}{1536}-\frac{\left(\sqrt{g\,H}\,k^5\left(12825+7728\,H^2k^2+1160\,H^4k^4\right)\right)x^4}{15360\left(\sqrt{3}\left(3+H^2k^2\right)^{5/2}\right)}-\frac{i\sqrt{g\,H}\,k^6\left(39+16\,H^2k^2\right)x^5}{1152\left(3+H^2k^2\right)}+O[x]^6,\right.$$
$$\left.-\frac{\left(\sqrt{3}\sqrt{g\,H}\,k^3\right)x^2}{32\left(3+H^2k^2\right)^{3/2}}-\frac{i\sqrt{g\,H}\,k^4\left(-128+\frac{27}{\left(3+H^2k^2\right)^2}\right)x^3}{1536}+\frac{\sqrt{g\,H}\,k^5\left(12825+7728\,H^2k^2+1160\,H^4k^4\right)x^4}{15360\sqrt{3}\left(3+H^2k^2\right)^{5/2}}-\frac{i\sqrt{g\,H}\,k^6\left(39+16\,H^2k^2\right)x^5}{1152\left(3+H^2k^2\right)}+O[x]^6\right\}$$