Robust Computational Models for Water Waves

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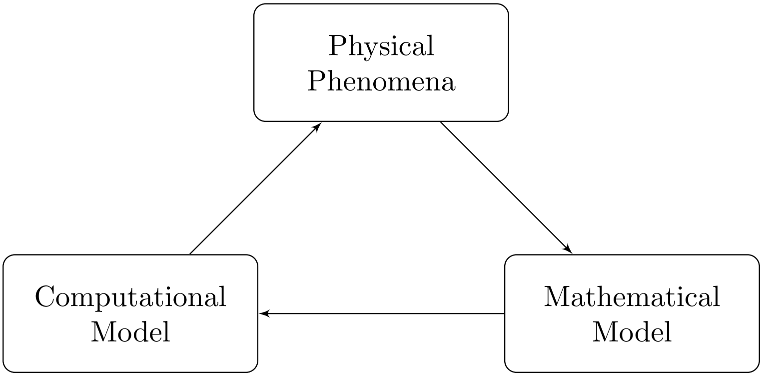
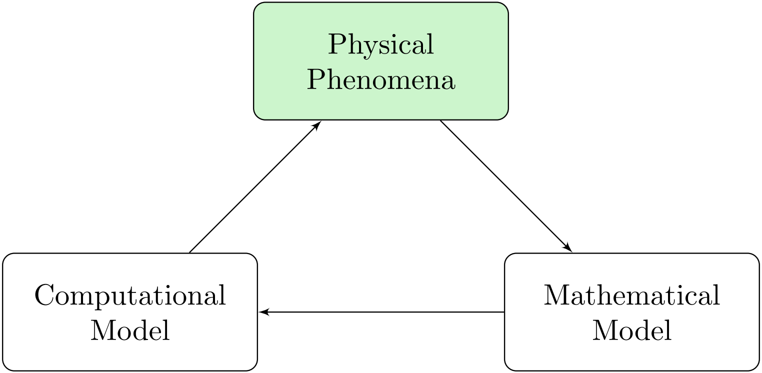
Outline of the Presentation

Modelling


Physical Phenomena
Mathematical Model
Computational Model

Physical Phenomena



Physical Phenomena: Water Waves

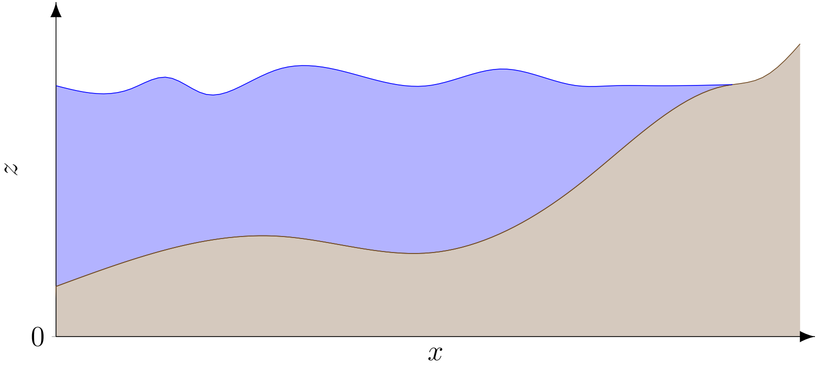
Water wave hazards:

- Tsunamis
- Storm Surges
- Rogue Waves

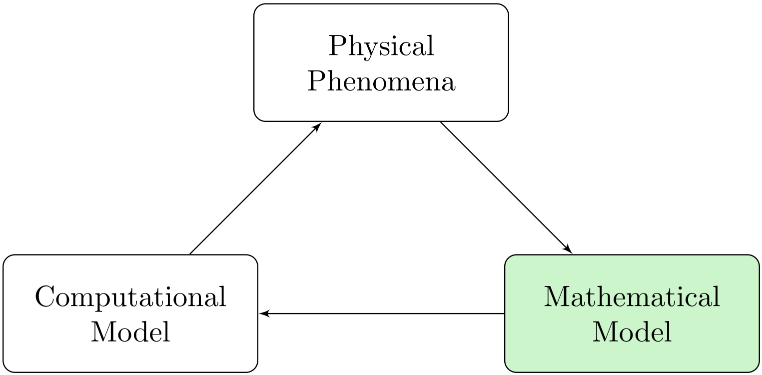
Phenomena caused by water waves:

- Nutrient Transport
- Beach Erosion
- Breakup of Sea Ice

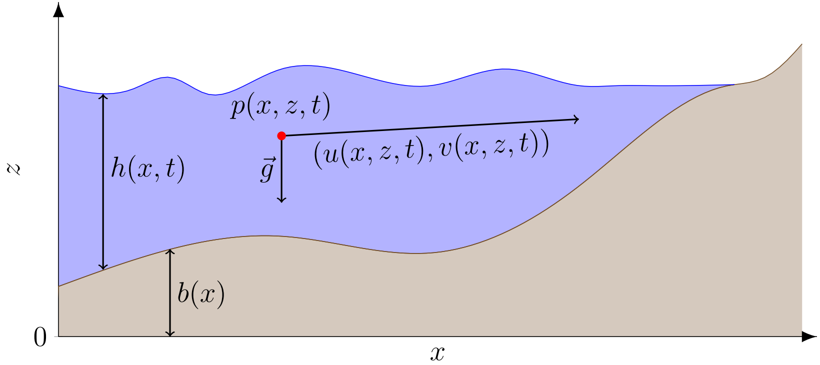
Typical Scenario



Mathematical Model



Euler Model (Navier Stokes)



Equations

Vectors: $\vec{u} = (u, v), \quad \vec{g} = (0, -9.81 m/s^2)$

$$\text{Mass}: \quad \frac{\partial \rho}{\partial t} + \rho \nabla \cdot \vec{u} = 0 \tag{1}$$

$$\text{Momentum}: \quad \frac{\partial \vec{u}}{\partial t} + \nabla \cdot \left(\vec{u} \vec{u}^T \right) + \nabla p - \vec{g} = 0 \tag{2}$$

Pros and Cons

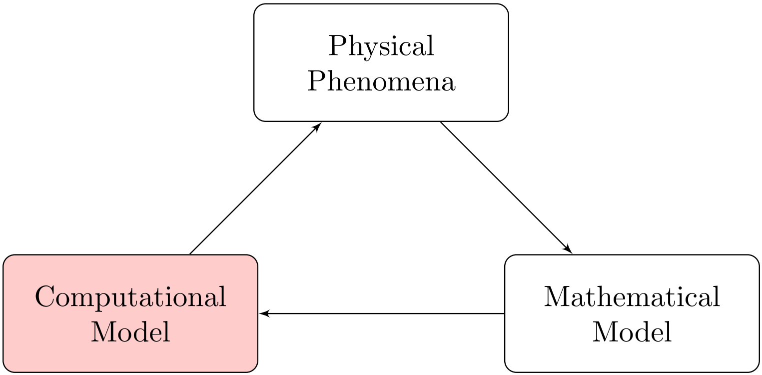
Pro:

- Models all water behaviour very well (density differences, viscosity, temperature negligible)

Con:

- Complex mainly due to number of terms u, v, p, h and b.

Computational Model

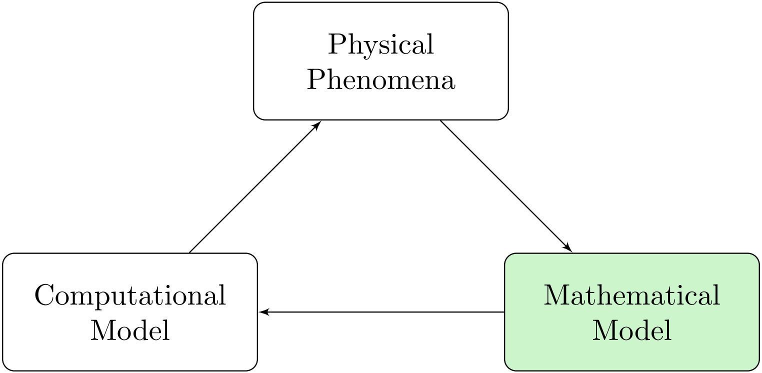


Computational Model

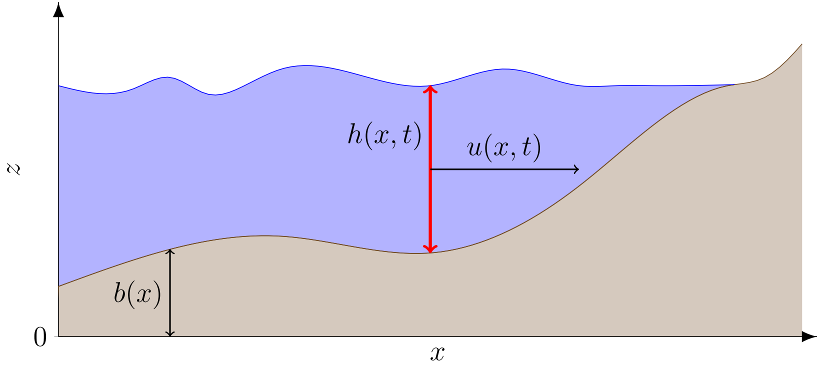
- Lots of good Computational Models for the Euler equations
- Computational Models only efficient and accurate over scale of metres
- We are interested in phenomena over the scale of kilometres to hundreds of kilometres

Require a different Mathematical Model to efficiently model water over the large extent of physical processes we are interested in. To do this we must reduce the number of quantities.

Mathematical Model



Shallow Water Wave Model



Assumptions

- $u(x, z, t)$ constant in z
- $v(x, z, t) = 0$
- $p(x, z, t) = g\left[(h(x, t) - b(x)) - z\right]$

Equations

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0 \tag{3}$$

$$\frac{\partial uh}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{1}{2}gh^2\right) + gh\frac{\partial b}{\partial x} = 0 \tag{4}$$

Pros and Cons

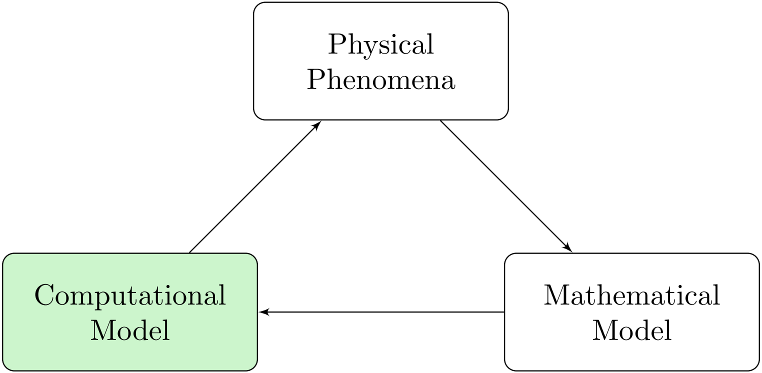
Pro:

- Far simpler than the Euler equations
- Models waves with long wavelengths very well
- Show good agreement with experimental results

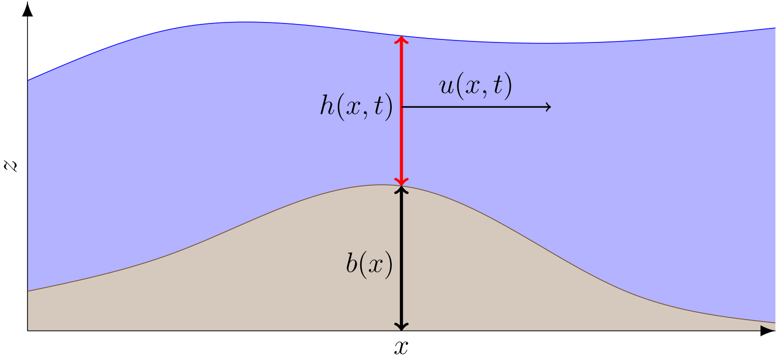
Cons:

- No dispersion
- Poor model for short waves

Computational Model



Finite Volume Method



Equations

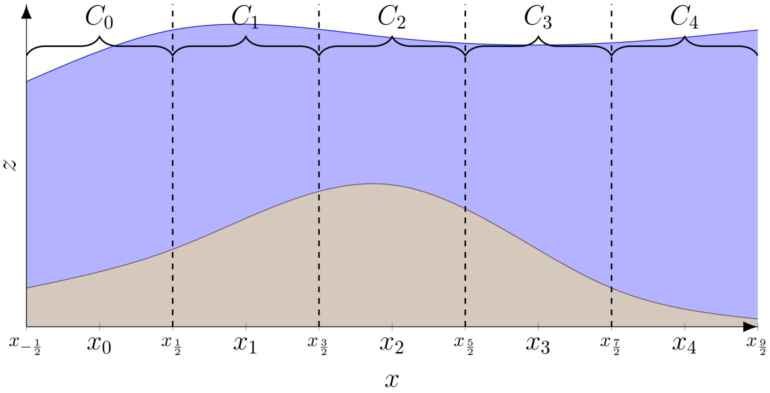
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0 \tag{5}$$

$$\frac{\partial uh}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{1}{2}gh^2\right) + gh\frac{\partial b}{\partial x} = 0 \tag{6}$$

General Conservation Form with Source Term

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} + s(q) = 0 \tag{7}$$

Cell Discretisation



Cell Integration

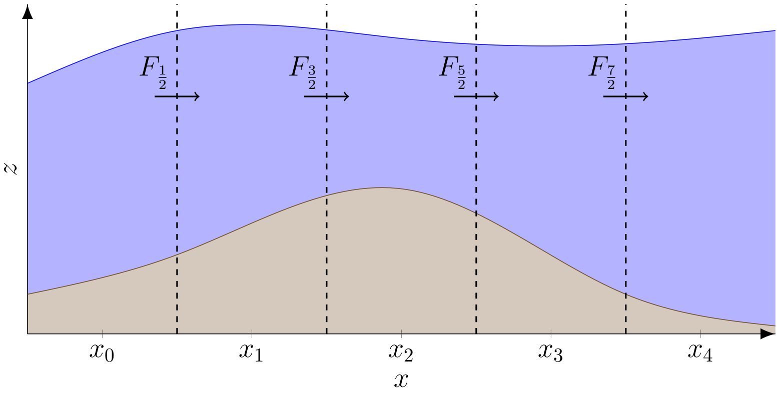
$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} + s(q) = 0$$

$$\frac{\partial}{\partial t}\int_{C_j} q \; dx + \left[f(q(x_{j+1/2}, t)) - f(q(x_{j-1/2}, t))\right] + \int_{C_j} s(q) \; dx = 0$$

$$\bar{q}(x_j, t) = \int_{C_j} q(x, t) \; dx$$

$$\frac{\partial}{\partial t}\bar{q}(x_j, t) + \left[f(q(x_{j+1/2}, t)) - f(q(x_{j-1/2}, t))\right] + \int_{C_j} s(q) \; dx = 0$$

Fluxes



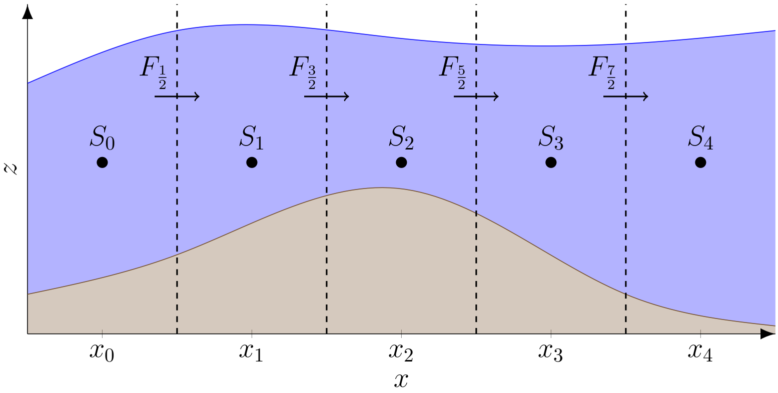
Time Integration

$$\frac{\partial}{\partial t}\bar{q}(x_j, t) + \left[f(q(x_{j+1/2}, t)) - f(q(x_{j-1/2}, t))\right] + \int_{C_j} s(q)\, dx = 0$$

$$F_{j\pm1/2} = \int_{t^n}^{t^{n+1}} f(q(x_{j\pm,1/2}, t))\, dt \tag{8}$$

$$\left[\bar{q}(x_j, t^{n+1}) - \bar{q}(x_j, t^n)\right] + \left[F_{j+1/2} - F_{j-1/2}\right] + \int_{t^n}^{t^{n+1}} \int_{C_j} s(q)\, dx\, dt = 0 \tag{9}$$

Source Terms



Update Formula

$$\left[\bar{q}(x_j, t^{n+1}) - \bar{q}(x_j, t^n)\right] + \left[F_{j+1/2} - F_{j-1/2}\right] + \int_{t^n}^{t^{n+1}} \int_{C_j} s(q)\, dx\, dt = 0 \tag{10}$$

$$S_j = \int_{t^n}^{t^{n+1}} \int_{C_j} s(q)\ dx \tag{11}$$

$$\left[\bar{q}(x_j, t^{n+1}) - \bar{q}(x_j, t^n)\right] + \left[F_{j+1/2} - F_{j-1/2}\right] + S_j = 0 \tag{12}$$

$$\bar{q}(x_j, t^{n+1}) = \bar{q}(x_j, t^n) - \left[F_{j+1/2} - F_{j-1/2}\right] - S_j \tag{13}$$

ANUGA

- 1999 : Stephen Roberts and Chris Zoppou Paper solving SWWE
- 2004 : ANUGA development begins originally focusing on storm surges
- 2005 : ANUGA refocused to tsunamis
- 2006 : ANUGA has first public release

Pros and Cons

Pros

- Efficient, robust and accurate computational model based on the SWWE

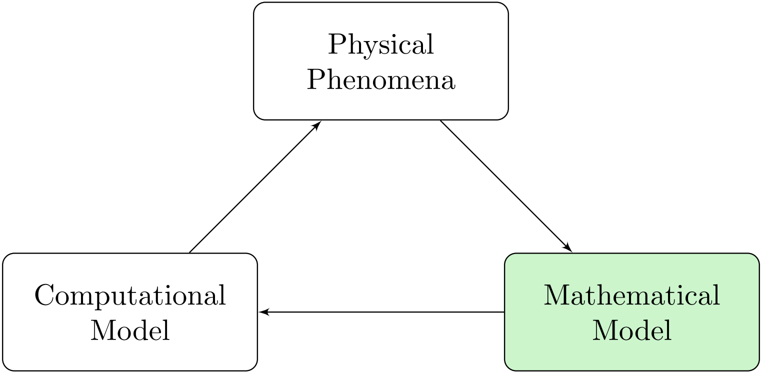
Cons

- No dispersion because of the SWWE
- SWWE not valid for shorter wavelengths

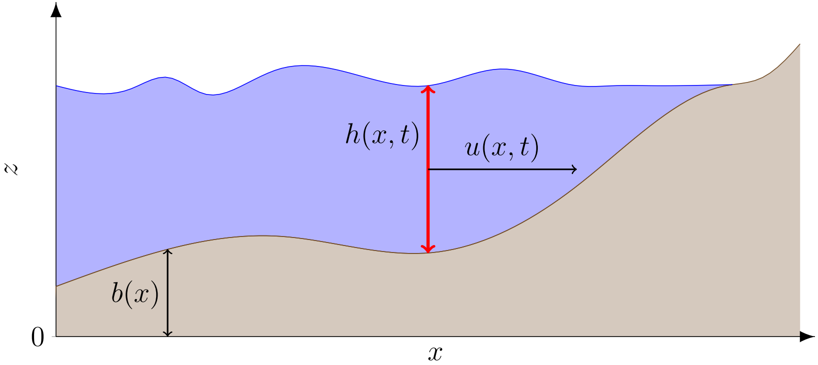
Outcome

New Project at the ANU to build computational models from dispersive mathematical models

Mathematical Model



Serre Model



Assumptions

- $u(x,z,t)$ constant in z
- $v(x,z,t) = u\frac{\partial b}{\partial x} - (z-b)\frac{\partial b}{\partial x}$
- $p(x,z,t) = g\xi + \xi\Psi + \frac{1}{2}\xi\left(2h-\xi\right)\Phi$

Equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^2}{2}\Psi + \frac{h^3}{3}\Phi\right) + \frac{\partial b}{\partial x}\left(gh + h\Psi + \frac{h^2}{2}\Phi\right) = 0$$

with

$$\Psi = \frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial^2 b}{\partial x^2},$$

$$\Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}.$$

Pros and Cons

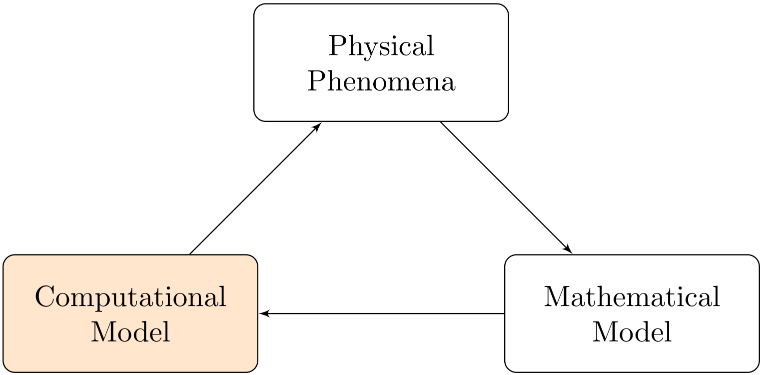
Pro:

- Far simpler than the Euler equations
- Maintains good model for long wavelength waves and extends our range of applicability to shorter wavelengths
- Has dispersion
- Considered one of the most appropriate models of water waves up to breaking

Cons:

- More complicated than the SWWE due to extra terms
- No well developed, efficient Robust Computational Models for three dimensional flows over complex geometries.

Computational Model



Problem 1: Not in Conservative Form with Source Term

Recall: General Conservation Form with Source Term

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} + s(q) = 0 \tag{15}$$

Serre Equations:

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^2}{2}\Psi + \frac{h^3}{3}\Phi\right) + \frac{\partial b}{\partial x}\left(gh + h\Psi + \frac{h^2}{2}\Phi\right) = 0$$

with Ψ and Φ containing temporal derivatives of u

Previous Work

- 2014: Chris Zoppou's PhD thesis
  Demonstrated Computational Model for the Serre equations with varying bathymetry in 1D.
- 2014: My Honours thesis
  Independent reproduction of Chris's work

Open problems:

- Are solutions in the presence of steep gradient correct?
- Solution in the presence of dry beds
- Extension to 3D flows

Thesis Goals

Answer these open problems:

- Are solutions in the presence of steep gradient correct?
- Solution in the presence of dry beds
- Extension to 3D flows

Technique: Develop a numerical method for the 2D Serre equations with a Finite Volume method at its core that can readily be extended to 3D flows and is well validated for dry beds and steep gradients.

Steep Gradients in the Flow

Dry Beds