equations we are interested in some very basic equations namely

$$u_t + a(uv)_x = 0$$
$$v_t + b(uv)_x = 0$$

For stability we are interested in the difference between the exact solution and the numerical solution produced by a finite precision algorithm. We are going to investigate this using Von Nuemann stability which is best for linear equations so we linearise.

1 Stability of Methods for Linearised Equations

Linearising means we assume the form of u and v so that
$u = u_0 + u_1(x,t) + \ldots$, $v = v_0 + v_1(x,t) + \ldots$
where the $u_i >> u_{i+1}$ we then neglect all terms far smaller than u_1 so that

$$(u)_t + aU(v)_x + aV(u)_x = 0$$
$$(v)_t + bU(v)_x + bV(u)_x = 0$$

2 Silly FDM

Approximating all derivatives by centered FD methods gives

$$u_j^{n+1} = u_j^{n-1} - a\frac{\Delta t}{\Delta x}\left(U\left[v_{j+1}^n - v_{j-1}^n\right] + V\left[u_{j+1}^n - u_{j-1}^n\right]\right)$$

$$v_j^{n+1} = v_j^{n-1} - b\frac{\Delta t}{\Delta x}\left(U\left[v_{j+1}^n - v_{j-1}^n\right] + V\left[u_{j+1}^n - u_{j-1}^n\right]\right)$$

We assume $u_j^n = u(t)e^{ikx_j}$, $v_j^n = u(t)e^{ilx_j}$ Thus

$$u_j^{n+1} = u_j^{n-1} - a\frac{\Delta t}{\Delta x}\left(U\left[e^{il\Delta x} - e^{-il\Delta x}\right]v_j^n + V\left[e^{ik\Delta x} - e^{-ik\Delta x}\right]u_j^n\right)$$

using trig relations

$$u_j^{n+1} = u_j^{n-1} - a\frac{\Delta t}{\Delta x}\left(U 2i \sin\left(l\Delta x\right) v_j^n + V 2i \sin\left(k\Delta x\right) u_j^n\right)$$

$$u_j^{n+1} = u_j^{n-1} - \frac{2ia\Delta t}{\Delta x}\left(U \sin\left(l\Delta x\right) v_j^n + V \sin\left(k\Delta x\right) u_j^n\right)$$

$$v_j^{n+1} = v_j^{n-1} - \frac{2ib\Delta t}{\Delta x}\left(U \sin\left(l\Delta x\right) v_j^n + V \sin\left(k\Delta x\right) u_j^n\right)$$

We can write this in matrix form as

$$\begin{bmatrix} u_j^{n+1} \\ v_j^{n+1} \\ u_j^n \\ v_j^n \end{bmatrix} = \begin{bmatrix} -\frac{2ia\Delta t}{\Delta x}V\sin\left(l\Delta x\right) & -\frac{2ia\Delta t}{\Delta x}U\sin\left(k\Delta x\right) & 1 & 0 \\ -\frac{2ib\Delta t}{\Delta x}V\sin\left(l\Delta x\right) & -\frac{2ib\Delta t}{\Delta x}U\sin\left(k\Delta x\right) & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_j^n \\ v_j^n \\ u_j^{n-1} \\ v_j^{n-1} \end{bmatrix}$$

The easy situation is when u and v are the same amd $a = b$ in which case

$$\begin{bmatrix} u_j^{n+1} \\ v_j^{n+1} \\ u_j^n \\ v_j^n \end{bmatrix} = \begin{bmatrix} -\frac{2ia\Delta t}{\Delta x}U\sin\left(k\Delta x\right) & -\frac{2ia\Delta t}{\Delta x}U\sin\left(k\Delta x\right) & 1 & 0 \\ -\frac{2ia\Delta t}{\Delta x}U\sin\left(k\Delta x\right) & -\frac{2ia\Delta t}{\Delta x}U\sin\left(k\Delta x\right) & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_j^n \\ v_j^n \\ u_j^{n-1} \\ v_j^{n-1} \end{bmatrix}$$

Again we are interested in the 2 norm which in this case since the matrix is symmetric is the largest eigenvalue. We also have the eigenvalues for a matrix of the form

$$\begin{bmatrix} a & a & 1 & 0 \\ a & a & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

are $\lambda = \pm 1, a \pm \sqrt{a^2+1}$

But since the largest of these must be greater than or equal to 1 we are basically done, and this scheme is unconditionally unstable. We should however check that we don't have equality

if

$$a \pm \sqrt{a^2+1} = 1$$

then $a = 0$ which implies $\frac{2ia\Delta t}{\Delta x}U\sin\left(k\Delta x\right)$ which is not possible by choosing Δt, Δx for general U and a