1 Linearised Equations

From Chris's paper we have, where h_0 is constant and we let $h_1 = h$ (same with velocity)

For mass:

$$\frac{\partial h}{\partial t} + h_0 \frac{\partial u}{\partial x} + u_0 \frac{\partial h}{\partial x} = 0$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(h_0 u + u_0 h\right) = 0$$

For momentum:

$$\frac{\partial u}{\partial t} + g\frac{\partial h}{\partial x} + u_0 \frac{\partial u}{\partial x} - \frac{h_0^2}{3}\left(u_0 \frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x^2 \partial t}\right) = 0$$

2 Actual Work

We do a Von Neumann stability analysis, we assume two different errors for h and u otherwise everything else is the same. We jsut run the errrors of known structure through the method, for convenience we know use h and u to refer to their respective errors, and we use q top refer to a general quantity (k, a different for u and l and b for h)

$$q^n_{j+1} = e^{ik\Delta x} q^n_j$$
$$q^n_{j+2} = e^{2ik\Delta x} q^n_j$$
$$q^n_{j-1} = e^{-ik\Delta x} q^n_j$$
$$q^n_{j-2} = e^{-ik\Delta x} q^n_j$$

$$\frac{\partial q}{\partial x} = \frac{q^n_{j+1} - q^n_{j-1}}{2\Delta x} = \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x} q^n_{j+1} = \frac{i \sin\left(k\Delta x\right)}{\Delta x} q^n_j$$

S

$$\frac{\partial^2 q}{\partial x^2} = \frac{q^n_{j+1} - 2q^n_j + q^n_{j-1}}{\Delta x^2} = \frac{e^{ik\Delta x} + e^{-ik\Delta x} - 2}{\Delta x^2} q^n_j = \frac{2\cos\left(k\Delta x\right) - 2}{\Delta x^2} q^n_j$$

$$= -\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)q_j^n$$

S

$$\frac{\partial^3 q}{\partial x^2} = \frac{-q_{j-2}^n + 2q_{j-1}^n - 2q_{j+1}^n + q_{j+2}^n}{2\Delta x^3} = \frac{2e^{ik\Delta x} - 2e^{-ik\Delta x} + e^{2ik\Delta x} - e^{-2ik\Delta x}}{2\Delta x^3}q_j^n$$

$$= \frac{4i\sin(k\Delta x) + 2i\sin(2k\Delta x)}{2\Delta x^3}q_j^n$$

$$= i\frac{2\sin(k\Delta x) + \sin(2k\Delta x)}{\Delta x^3}q_j^n$$

$$= i\frac{2\sin(k\Delta x) + 2\sin(k\Delta x)\cos(k\Delta x)}{\Delta x^3}q_j^n$$

$$= 2i\sin(k\Delta x)\frac{1+\cos(k\Delta x)}{\Delta x^3}q_j^n$$

$$= 2i\sin(k\Delta x)\,2\cos^2\left(\frac{k\Delta x}{2}\right)\frac{1}{\Delta x^3}q_j^n$$

$$= \frac{4i}{\Delta x^3}\sin(k\Delta x)\cos^2\left(\frac{k\Delta x}{2}\right)q_j^n$$

2.1 FD for u

$$\begin{aligned}\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + g\frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x} + u_0\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \\ - \frac{h_0^2}{3}\left(u_0\frac{-u_{j-2}^n + 2u_{j-1}^n - 2u_{j+1}^n + u_{j+2}^n}{2\Delta x^3}\right) \\ - \frac{h_0^2}{3}\frac{\partial\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}}{\partial t} \\ = 0 \quad (1)\end{aligned}$$

$$
\begin{aligned}
\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t} + g\frac{h_{j+1}^n-h_{j-1}^n}{2\Delta x} + u_0\frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x} \\
-\frac{h_0^2}{3}\left(u_0\frac{-u_{j-2}^n+2u_{j-1}^n-2u_{j+1}^n+u_{j+2}^n}{2\Delta x^3}\right) \\
-\frac{h_0^2}{3}\frac{\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{\Delta x^2}-\frac{u_{j+1}^{n-1}-2u_j^{n-1}+u_{j-1}^{n-1}}{\Delta x^2}}{2\Delta t} \\
=0
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t} + g\frac{h_{j+1}^n-h_{j-1}^n}{2\Delta x} + u_0\frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x} \\
-\frac{h_0^2}{3}\left(u_0\frac{-u_{j-2}^n+2u_{j-1}^n-2u_{j+1}^n+u_{j+2}^n}{2\Delta x^3}\right) \\
-\frac{h_0^2}{3}\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}-u_{j+1}^{n-1}+2u_j^{n-1}-u_{j-1}^{n-1}}{2\Delta x^2\Delta t} \\
=0
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t} + g\frac{i\sin(l\Delta x)}{\Delta x}h_j^n + u_0\frac{i\sin(k\Delta x)}{\Delta x}u_j^n \\
-\frac{h_0^2}{3}\left(u_0\frac{4i}{\Delta x^3}\sin(k\Delta x)\cos^2\left(\frac{k\Delta x}{2}\right)u_j^n\right) \\
-\frac{h_0^2}{6\Delta t}\left(\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{\Delta x^2}-\frac{u_{j+1}^{n-1}-2u_j^{n-1}+u_{j-1}^{n-1}}{\Delta x^2}\right) \\
=0
\end{aligned} \tag{4}
$$

$$\begin{aligned}\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t}+g\frac{i\sin\left(l\Delta x\right)}{\Delta x}h_j^n+u_0\frac{i\sin\left(k\Delta x\right)}{\Delta x}u_j^n \\ -\frac{h_0^2}{3}\left(u_0\frac{4i}{\Delta x^3}\sin\left(k\Delta x\right)\cos^2\left(\frac{k\Delta x}{2}\right)u_j^n\right) \\ -\frac{h_0^2}{6\Delta t}\left(-\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)u_j^{n+1}--\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)u_j^{n-1}\right) \\ =0\end{aligned}\tag{5}$$

$$\begin{aligned}\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t}-\frac{h_0^2}{6\Delta t}\left(-\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)u_j^{n+1}--\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)u_j^{n-1}\right) \\ =\frac{h_0^2}{3}\left(u_0\frac{4i}{\Delta x^3}\sin\left(k\Delta x\right)\cos^2\left(\frac{k\Delta x}{2}\right)u_j^n\right)-g\frac{i\sin\left(l\Delta x\right)}{\Delta x}h_j^n+u_0\frac{i\sin\left(k\Delta x\right)}{\Delta x}u_j^n\end{aligned}\tag{6}$$

$$\begin{aligned}\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t}-\frac{h_0^2}{6\Delta t}\left(-\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)u_j^{n+1}--\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)u_j^{n-1}\right) \\ =\left[\frac{h_0^2}{3}\left(u_0\frac{4i}{\Delta x^3}\sin\left(k\Delta x\right)\cos^2\left(\frac{k\Delta x}{2}\right)\right)+u_0\frac{i\sin\left(k\Delta x\right)}{\Delta x}\right]u_j^n-g\frac{i\sin\left(l\Delta x\right)}{\Delta x}h_j^n\end{aligned}\tag{7}$$

$$\begin{aligned}u_j^{n+1}-u_j^{n-1}+\frac{h_0^2}{3}\left(\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)u_j^{n+1}-\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)u_j^{n-1}\right) \\ =2\Delta t\left(\left[\frac{h_0^2}{3}\left(u_0\frac{4i}{\Delta x^3}\sin\left(k\Delta x\right)\cos^2\left(\frac{k\Delta x}{2}\right)\right)+u_0\frac{i\sin\left(k\Delta x\right)}{\Delta x}\right]u_j^n-g\frac{i\sin\left(l\Delta x\right)}{\Delta x}h_j^n\right)\end{aligned}\tag{8}$$

$$u_j^{n+1}\left[1+\frac{h_0^2}{3}\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)\right]$$
$$=u_j^{n-1}\left[1+\frac{h_0^2}{3}\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)\right]$$
$$2\Delta t\left(\left[\frac{h_0^2}{3}\left(u_0\frac{4i}{\Delta x^3}\sin\left(k\Delta x\right)\cos^2\left(\frac{k\Delta x}{2}\right)\right)+u_0\frac{i\sin\left(k\Delta x\right)}{\Delta x}\right]u_j^n-g\frac{i\sin\left(l\Delta x\right)}{\Delta x}h_j^n\right) \tag{9}$$

$$u_j^{n+1}=$$
$$u_j^{n-1}+\frac{2\Delta t}{1+\frac{4h_0^2}{3\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)}$$
$$\times\left(\frac{iu_0\sin\left(k\Delta x\right)}{\Delta x}\left[\frac{4h_0^2}{3\Delta x^2}\cos^2\left(\frac{k\Delta x}{2}\right)+1\right]u_j^n-g\frac{i\sin\left(l\Delta x\right)}{\Delta x}h_j^n\right) \tag{10}$$

2.2 FD for h

$$\frac{h_j^{n+1}-h_j^{n-1}}{2\Delta t}+h_0\frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x}+u_0\frac{h_{j+1}^n-h_{j-1}^n}{2\Delta x}=0$$

$$\frac{h_j^{n+1}-h_j^{n-1}}{2\Delta t}+h_0\frac{i\sin\left(k\Delta x\right)}{\Delta x}u_j^n+u_0\frac{i\sin\left(l\Delta x\right)}{\Delta x}h_j^n=0$$

$$h_j^{n+1}-h_j^{n-1}=-2\Delta t\left[h_0\frac{i\sin\left(k\Delta x\right)}{\Delta x}u_j^n+u_0\frac{i\sin\left(l\Delta x\right)}{\Delta x}h_j^n\right]=0$$

$$h_j^{n+1}=h_j^{n-1}-\frac{2i\Delta t}{\Delta x}\left[h_0\sin\left(k\Delta x\right)u_j^n+u_0\sin\left(l\Delta x\right)h_j^n\right]=0$$

2.2.1 Together

We can formulate these schemes together to get

$$
\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = \left[\begin{array}{c} h \\ u \end{array}\right]_j^{n-1}
$$

$$
+\left[\begin{array}{cc} -\frac{2i\Delta t}{\Delta x} u_0 \sin\left(l\Delta x\right) & -\frac{2i\Delta t}{\Delta x} h_0 \sin\left(k\Delta x\right) \\ -\frac{2\Delta t}{1+\frac{h_0^2}{3}\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)} g \frac{i\sin(l\Delta x)}{\Delta x} & \frac{2\Delta t}{1+\frac{h_0^2}{3}\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)} \left[\frac{h_0^2}{3}\left(u_0 \frac{4i}{\Delta x^3}\sin\left(k\Delta x\right)\cos^2\left(\frac{k\Delta x}{2}\right)\right) + u_0 \frac{i\sin(k\Delta x)}{\Delta x}\right] \end{array}\right]
$$

$$
\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n} \quad (11)
$$

$$
\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = \left[\begin{array}{c} h \\ u \end{array}\right]_j^{n-1} + 2\Delta t
$$

$$
\times\left[\begin{array}{cc} -\frac{i}{\Delta x} u_0 \sin\left(l\Delta x\right) & -\frac{i}{\Delta x} h_0 \sin\left(k\Delta x\right) \\ -\frac{1}{1+\frac{h_0^2}{3}\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)} g \frac{i\sin(l\Delta x)}{\Delta x} & \frac{1}{1+\frac{4h_0^2}{3\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)} \frac{iu_0\sin(k\Delta x)}{\Delta x}\left[\frac{4h_0^2}{3\Delta x^2}\cos^2\left(\frac{k\Delta x}{2}\right)+1\right] \end{array}\right]
$$

$$
\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n} \quad (12)
$$

$$
\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = \left[\begin{array}{c} h \\ u \end{array}\right]_j^{n-1} +
$$

$$
2\Delta t \left[\begin{array}{cc} -\frac{i}{\Delta x} u_0 \sin\left(l\Delta x\right) & -\frac{i}{\Delta x} h_0 \sin\left(k\Delta x\right) \\ -\frac{1}{1+\frac{h_0^2}{3}\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)} g \frac{i\sin(l\Delta x)}{\Delta x} & \frac{1+\frac{4h_0^2}{3\Delta x^2}\cos^2\left(\frac{k\Delta x}{2}\right)}{1+\frac{4h_0^2}{3\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)} \frac{iu_0\sin(k\Delta x)}{\Delta x} \end{array}\right]
$$

$$
\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n} \quad (13)
$$

Defining the growth matrix as $\mathcal{G}(k, l, \Delta x, \Delta t)$, so that

$$
\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = \mathcal{G} \left[\begin{array}{c} h \\ u \end{array}\right]_j^{n}
$$

we have that

$$
\mathcal{G} = \mathcal{G} - 2\Delta t \left[\begin{array}{cc} -\frac{i}{\Delta x} u_0 \sin\left(l\Delta x\right) & -\frac{i}{\Delta x} h_0 \sin\left(k\Delta x\right) \\ -\frac{1}{1+\frac{h_0^2}{3}\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)} g \frac{i\sin(l\Delta x)}{\Delta x} & \frac{1+\frac{4h_0^2}{3\Delta x^2}\cos^2\left(\frac{k\Delta x}{2}\right)}{1+\frac{4h_0^2}{3\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)} \frac{iu_0\sin(k\Delta x)}{\Delta x} \end{array}\right]
$$