Robust Computational Models for Water Waves

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Outline of the Presentation

- Motivation for interest in water waves
- History of the computational water modelling project at the ANU
- Contribution of the Thesis
  - Method
  - Validation

Computational Modelling

Goal: Model Physics On Computers

Computational Modelling

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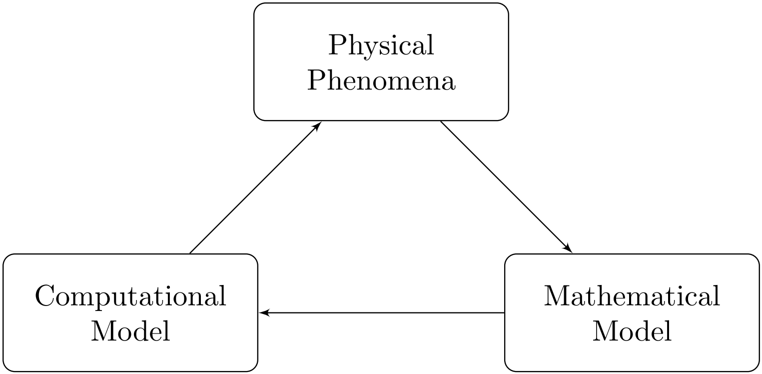
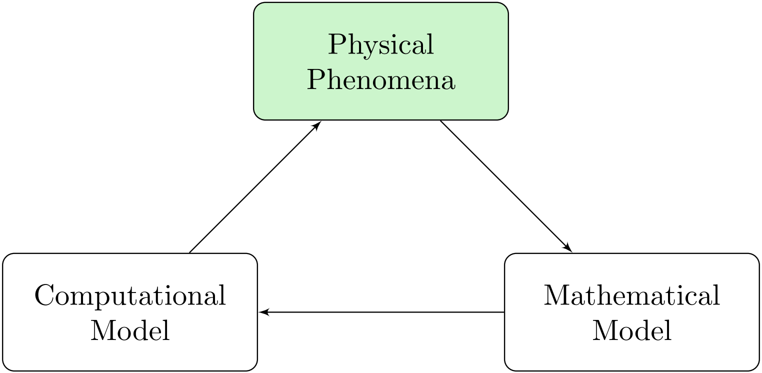


Physical Phenomena



Physical Phenomena: Water Waves

Water wave hazards:

- Tsunamis
- Storm Surges
- Rogue Waves

Physical Phenomena: Water Waves

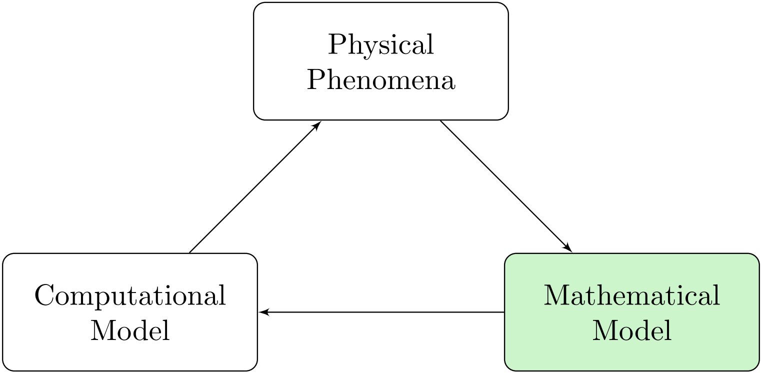
Water wave hazards:

- Tsunamis
- Storm Surges
- Rogue Waves

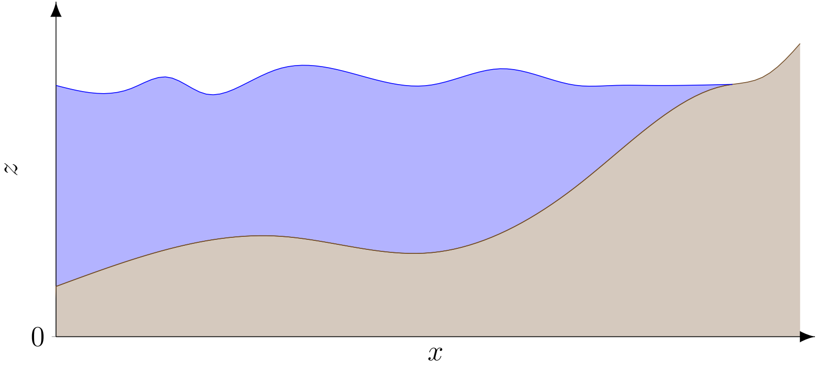
Phenomena caused by water waves:

- Nutrient Transport
- Beach Erosion
- Breakup of Sea Ice

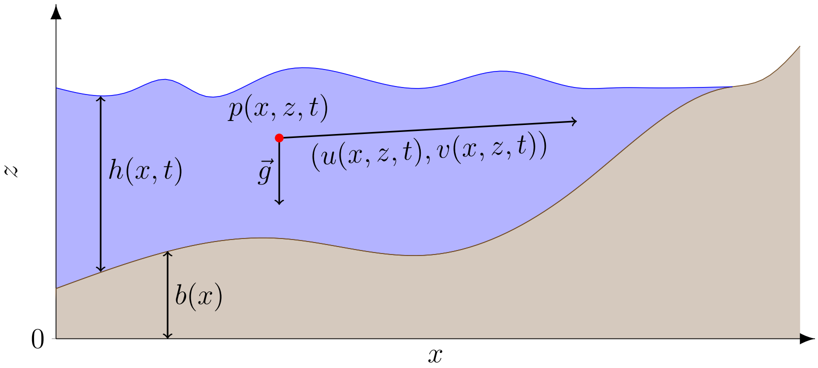
Mathematical Model



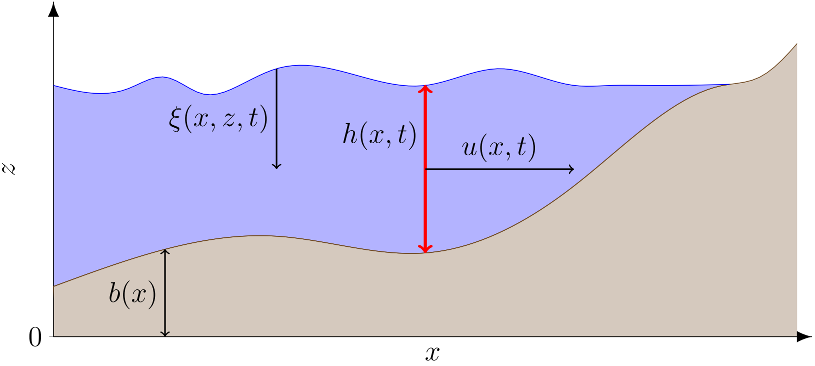
Typical Scenario



Full Water Model



Shallow Water Wave Model



Assumptions

- $u(x, z, t)$ constant in z
- $v(x, z, t) = 0$
- $p(x, z, t) = g\xi$

with

$$\xi(x, z, t) = (h(x, t) + b(x)) - z$$

Equations

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0$$

$$\frac{\partial uh}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{1}{2} g h^2\right) + gh\frac{\partial b}{\partial x} = 0$$

Pros and Cons

Pros:

- Far simpler than the full water wave model
- Models waves with long wavelengths very well
- Shows good agreement with experimental results

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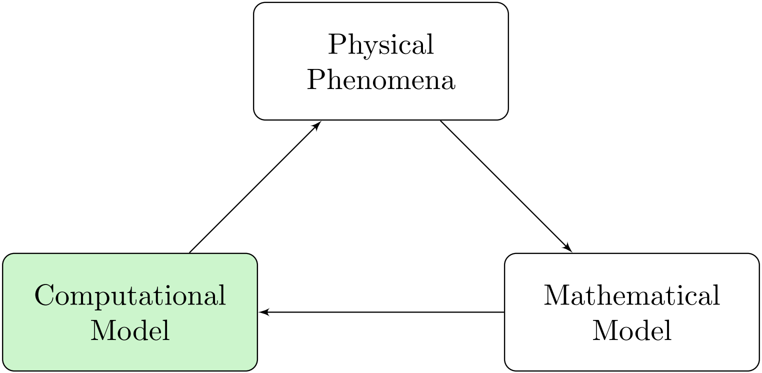
- Models waves with long wavelengths very well
- Shows good agreement with experimental results

Cons:

- No dispersion
- Poor model for short waves
- Cannot model breaking waves

Computational Model



ANUGA

- 1999 : Stephen Roberts and Chris Zoppou Paper solving SWWE with a Finite Volume Method
- 2004 : ANUGA development begins originally focusing on storm surges
- 2005 : ANUGA refocused to tsunamis
- 2006 : ANUGA has first public release

Pros and Cons

Pros

- Robust computational model for water waves based on the Shallow Water Wave Equations

Pros and Cons

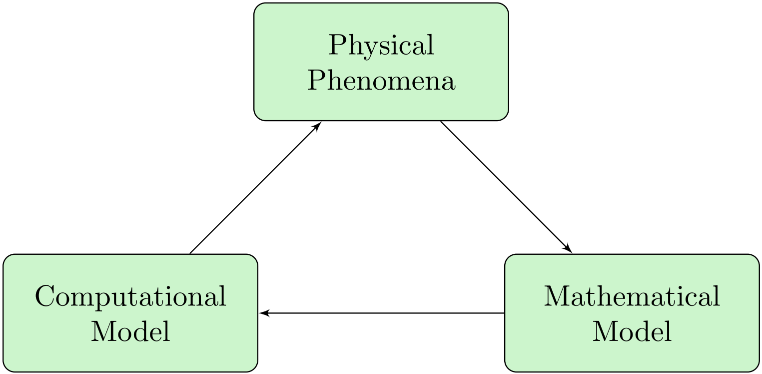
Pros

- Robust computational model for water waves based on the Shallow Water Wave Equations

Cons

- Limited by the Shallow Water Wave Equations:
  - no dispersion of waves (recent papers suggest dispersion important for tsunami modelling)
  - not appropriate for shorter waves

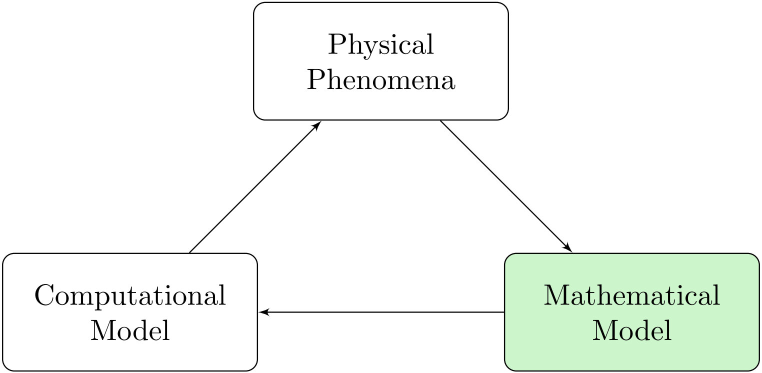
ANUGA


Physical Phenomena
Computational Model
Mathematical Model

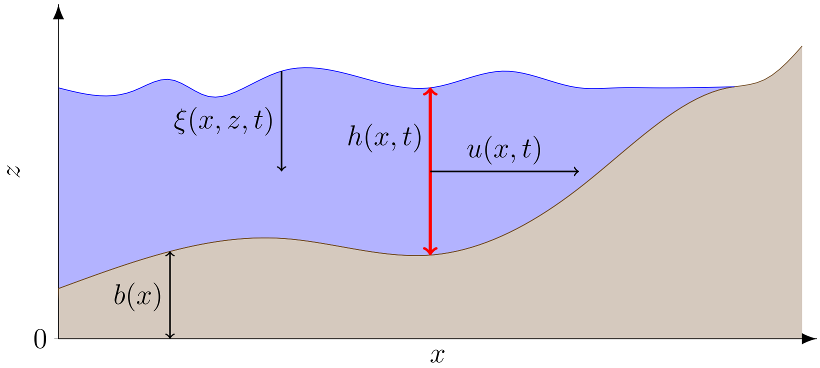
Outcome

New Project at the ANU to build robust computational model from dispersive mathematical models

Mathematical Model



Serre Model



Assumptions

- $u(x, z, t)$ constant in z
- $v(x, z, t) = u\frac{\partial b}{\partial x} - (z - b)\frac{\partial b}{\partial x}$
- $p(x, z, t) = g\xi + \xi\Psi + \frac{1}{2}\xi\left(2h - \xi\right)\Phi$

with

$$\Psi = \frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial^2 b}{\partial x^2},$$

$$\Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}.$$

Equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^2}{2}\Psi + \frac{h^3}{3}\Phi\right)$$

$$+ \frac{\partial b}{\partial x}\left(gh + h\Psi + \frac{h^2}{2}\Phi\right) = 0.$$

$$\Psi = \frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial^2 b}{\partial x^2}, \quad \Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}.$$

Pros and Cons

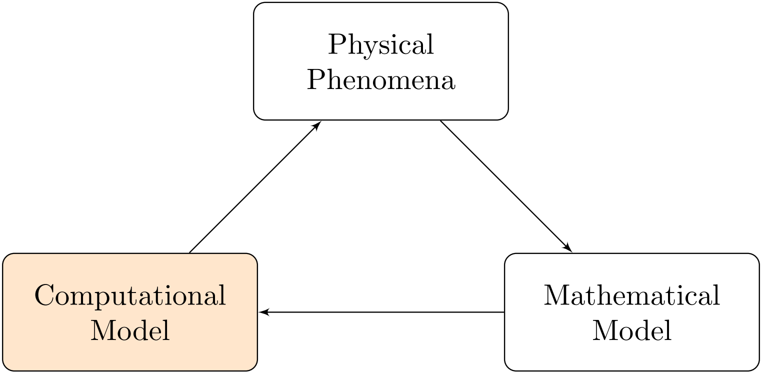
Pro:

- Far simpler than the Euler equations
- Includes dispersive effects
- Still a good model for long wavelength waves and also a good model for shorter wavelengths
- Considered one of the best models for water waves up to wave breaking

Cons:

- More complicated than the Shallow Water Wave Equations
- Cannot model breaking waves

Computational Model



Previous Work at the ANU

- 2014: Chris Zoppou's PhD thesis
  Demonstrated computational model based on Finite Volume Method for the Serre equations with varying bathymetry in 1D.
- 2014: My Honours thesis
  Independent reproduction of Chris's work

Open problems:

3D: Extension of the method to 3D flows

Robust: Validation of model with steep gradients in free surface

Robust: Validation of model in the presence of dry beds

Thesis Goals

Solve these open problems:

3D: Extension of the method to 3D flows

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Thesis Goals

Solve these open problems:

3D: Extension of the method to 3D flows

Robust: Validation of model with steep gradients in free surface

Robust: Validation of model in the presence of dry beds

Technique: Develop a robust computational model from the 2D Serre equations that can be easily extended to 3D.

Finite Volume Method

3D: Extends well to 3D

Robust: Stable in the presence of steep gradients

Robust: Stable in the presence of dry beds

- Maintains conservation properties of the equations

Finite Volume Method

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- Maintains conservation properties of the equations

Adaptation of the Finite Volume Method to the Serre Equations was a major achievement of Chris Zoppou's thesis.

Equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^2}{2}\Psi + \frac{h^3}{3}\Phi\right) + \frac{\partial b}{\partial x}\left(gh + h\Psi + \frac{h^2}{2}\Phi\right) = 0$$

$$\Psi = \frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial^2 b}{\partial x^2}, \quad \Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}.$$

For a Finite Volume Method we require equations in the form

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} + s(q) = 0$$

where $f(q)$ and $s(q)$ do not contain temporal derivatives

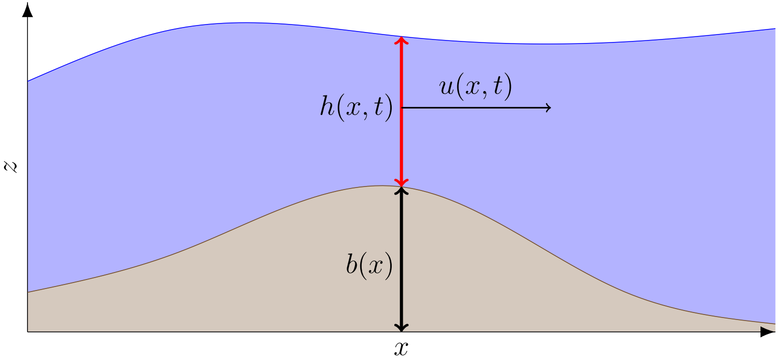
Reformulation (Chris Zoppous Thesis)

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}h^3\left[\frac{\partial u}{\partial x}\right]^2 + h^2 u \frac{\partial u}{\partial x}\frac{\partial b}{\partial x}\right) + \frac{1}{2}h^2 u \frac{\partial u}{\partial x}\frac{\partial^2 b}{\partial x^2} - hu^2 \frac{\partial b}{\partial x}\frac{\partial^2 b}{\partial x^2} + gh\frac{\partial b}{\partial x} = 0.$$

$$G = hu\left(1 + \frac{\partial h}{\partial x}\frac{\partial b}{\partial x} + \frac{1}{2}h\frac{\partial^2 b}{\partial x^2} + \left[\frac{\partial b}{\partial x}\right]^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{3}h^3\frac{\partial u}{\partial x}\right).$$

Finite Volume Method



Equations

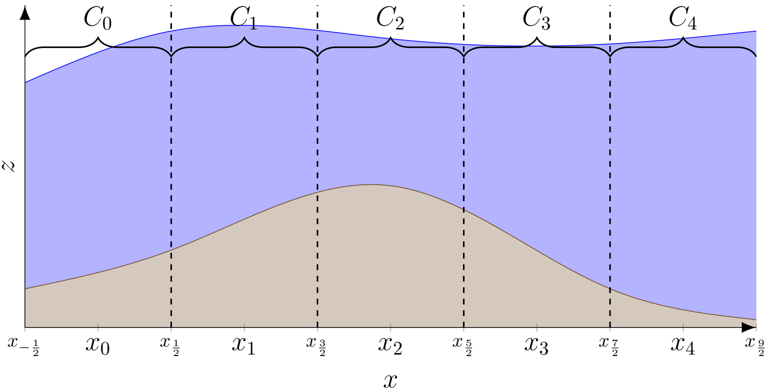
$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}h^3\left[\frac{\partial u}{\partial x}\right]^2 + h^2 u \frac{\partial u}{\partial x}\frac{\partial b}{\partial x}\right) + \frac{1}{2}h^2 u \frac{\partial u}{\partial x}\frac{\partial^2 b}{\partial x^2} - hu^2\frac{\partial b}{\partial x}\frac{\partial^2 b}{\partial x^2} + gh\frac{\partial b}{\partial x} = 0.$$

General Conservation Form with Source Term

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} + s(q) = 0$$

Cell Discretisation



Cell Integration

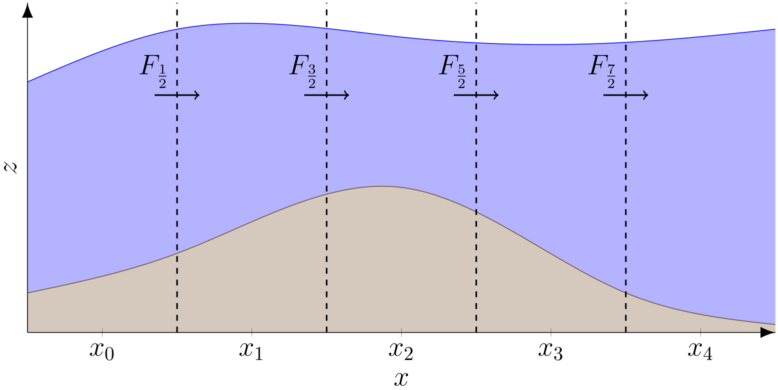
$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} + s(q) = 0$$

$$\frac{\partial}{\partial t}\int_{C_j} q\ dx + \left[f(q(x_{j+1/2},t)) - f(q(x_{j-1/2},t))\right] + \int_{C_j} s(q)\ dx = 0$$

$$\bar{q}(x_j,t) = \int_{C_j} q(x,t)\ dx$$

$$\frac{\partial}{\partial t}\bar{q}(x_j,t) + \left[f(q(x_{j+1/2},t)) - f(q(x_{j-1/2},t))\right] + \int_{C_j} s(q)\ dx = 0$$

Fluxes



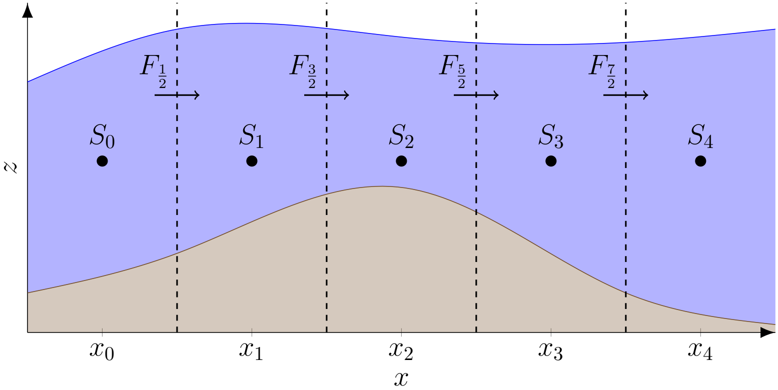
Time Integration

$$\frac{\partial}{\partial t}\bar{q}(x_j, t) + \left[f(q(x_{j+1/2}, t)) - f(q(x_{j-1/2}, t))\right] + \int_{C_j} s(q)\ dx = 0$$

$$F_{j\pm1/2} = \int_{t^n}^{t^{n+1}} f(q(x_{j\pm,1/2}, t))\ dt \tag{3}$$

$$\left[\bar{q}(x_j, t^{n+1}) - \bar{q}(x_j, t^n)\right] + \left[F_{j+1/2} - F_{j-1/2}\right] + \int_{t^n}^{t^{n+1}} \int_{C_j} s(q)\, dx\, dt = 0 \tag{4}$$

Source Terms



Update Formula

$$\left[\bar{q}(x_j, t^{n+1}) - \bar{q}(x_j, t^n)\right] + \left[F_{j+1/2} - F_{j-1/2}\right] + \int_{t^n}^{t^{n+1}} \int_{C_j} s(q)\, dx\, dt = 0$$

$$S_j = \int_{t^n}^{t^{n+1}} \int_{C_j} s(q)\ dx$$

$$\left[\bar{q}(x_j, t^{n+1}) - \bar{q}(x_j, t^n)\right] + \left[F_{j+1/2} - F_{j-1/2}\right] + S_j = 0$$

$$\bar{q}(x_j, t^{n+1}) = \bar{q}(x_j, t^n) - \left[F_{j+1/2} - F_{j-1/2}\right] - S_j$$

Update Formula for Serre Equations

$$\bar{h}(x_j, t^{n+1}) = \bar{h}(x_j, t^n) - \left[F_{j+1/2} - F_{j-1/2}\right]$$

$$\bar{G}(x_j, t^{n+1}) = \bar{G}(x_j, t^n) - \left[F_{j+1/2} - F_{j-1/2}\right] - S_j$$

- All the fluxes $F_{j+1/2}$ and $F_{j-1/2}$ and the source term for G S_j require $u(x_j, t^n)$
- require a method to obtain $u(x_j, t^n)$ from $\bar{h}(x_j, t^n)$, $\bar{G}(x_j, t^n)$ and $b(x_j)$

Calculate Velocity

$$G = hu\left(1 + \frac{\partial h}{\partial x}\frac{\partial b}{\partial x} + \frac{1}{2}h\frac{\partial^2 b}{\partial x^2} + \left[\frac{\partial b}{\partial x}\right]^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{3}h^3\frac{\partial u}{\partial x}\right).$$

- Chris's thesis used a finite difference method, but this doesn't extend well to 3D flows
- Solution: Solve this using a Finite Element Method

Finite Element Method

3D: Extends well to 3D

Robust: Stable in the presence of steep gradients

- Maintains conservation properties for conservation equations

Finite Element Method Example

Example:

$$-\frac{\partial^2 u}{\partial x^2} = f.$$

Weak Form

$$-\int_\Omega \frac{\partial^2 u}{\partial x^2} v = \int_\Omega fv \ dx$$

Integrate by parts (Dirichlet boundary conditions):

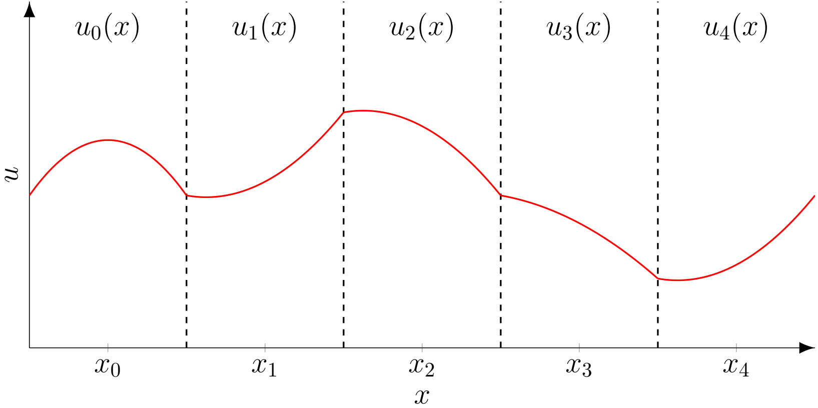
$$\int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} \ dx = \int_\Omega fv \ dx$$

Finite Element Method

$$\int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\;dx = \int_\Omega fv\;dx$$

$$\sum_j \left[\int_{C_j} \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\;dx\right] = \sum_j \left[\int_{C_j} fv\;dx\right]$$

Piecewise Polynomial Representation



Finite Element Method

$$\sum_j \left[\int_{C_j} \frac{\partial u_j}{\partial x} \frac{\partial v_j}{\partial x} \, dx \right] = \sum_j \left[\int_{C_j} f_j v_j \, dx \right]$$

$$\mathbf{A}\vec{u} = \vec{c}$$

where

- **A** depends on the polynomial representation of v
- $\vec{u}$ determines the polynomial representation of u
- $\vec{c}$ depends on polynomial representation of f and v

Method

- Reconstruction: Calculate the representations of h and G over the cells from the averages $\bar{h}$ and $\bar{G}$
- Finite Element: use the representations of h and G over the cells to calculate the representation of u over the cell
- Finite Volume Method: Update h and G to the next time using the Finite Volume Method update

Progress

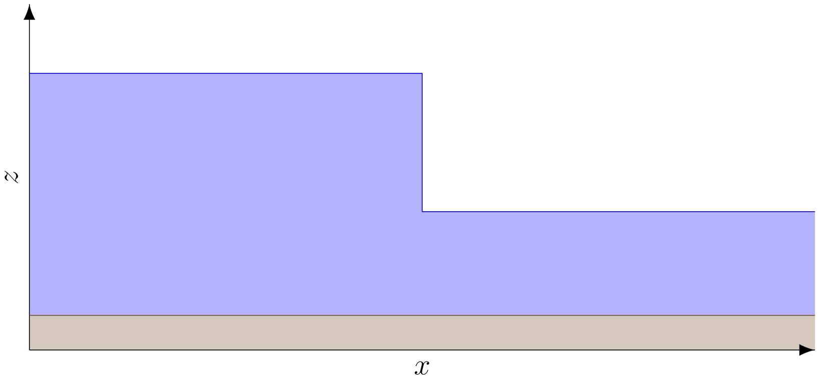
3D: Extension of the method to 3D flows ✓

Robust: Validation of model with steep gradients in free surface

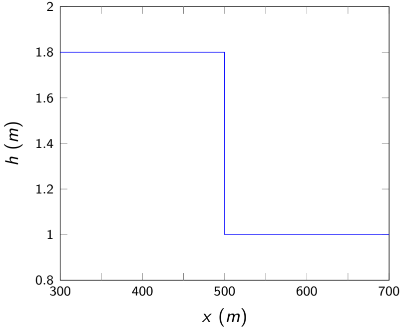
Robust: Validation of model in the presence of dry beds

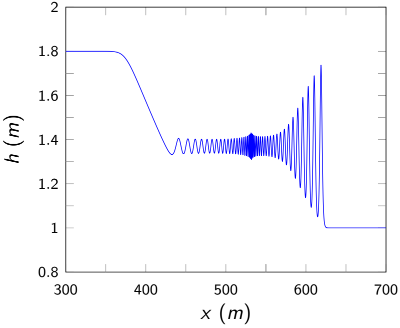
Statement of Problem

How does this initially still body of water evolve?



Our Numerical Solution





Pitt, J., Zoppou, C., and Roberts, S. (2018). Behaviour of the Serre Equations in the Presence of Steep Gradients Revisited. Wave Motion, 76(1):6177.

What was known

- No analytic solutions
- Asymptotic results for step gradient problems as $t \to \infty$
- Some experimental comparisons [1]
- Other numerical solutions from the literature; some solving the actual steep gradient problem and some solving a smoothed initial water profile.

[1] Zoppou, C. (2014). Numerical Solution of the One-dimensional and Cylindrical Serre Equations for Rapidly Varying Free Surface Flows. PhD thesis, Australian National University.

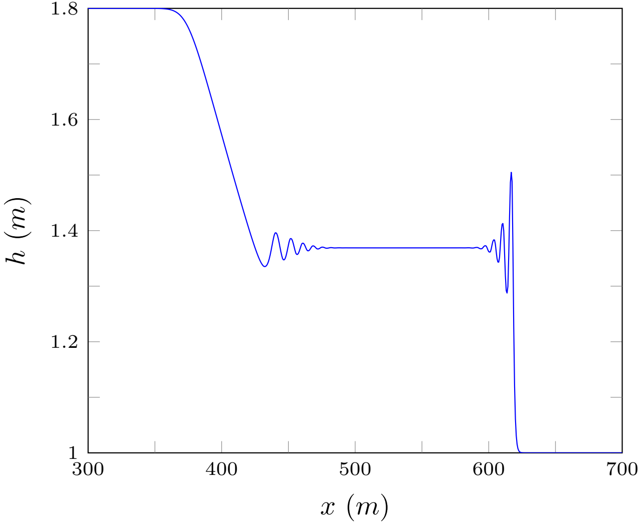
Solution

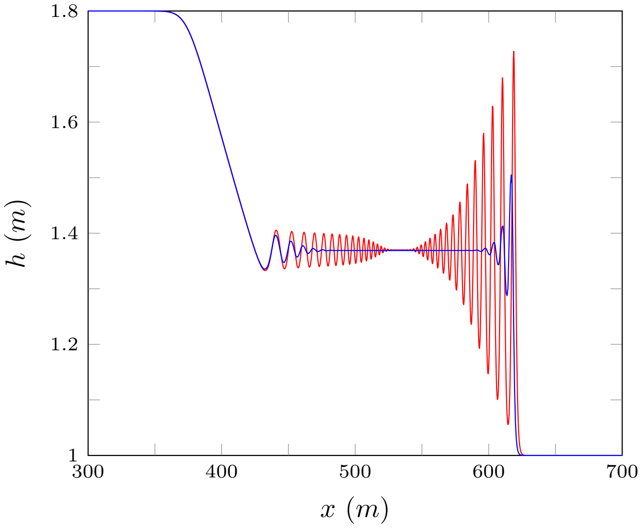
Paper [2]

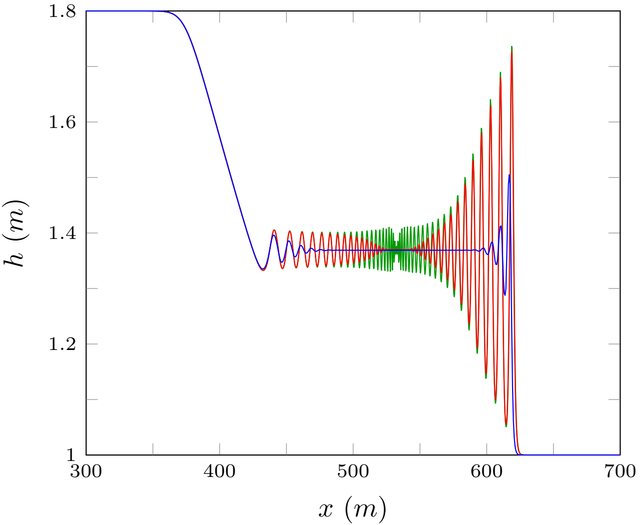
- Demonstrate convergence to one solution for many numerical methods
- Demonstrate good agreement with asymptotic results
- Comprehensively review many numerical methods and smoothing techniques from the literature

[2] Pitt, J., Zoppou, C., and Roberts, S. (2018). Behaviour of the Serre Equations in the Presence of Steep Gradients Revisited. Wave Motion, 76(1):6177.

Convergence




1.8
1.6
1.4
1.2
1
300
400
500
600
700
h (m)
x (m)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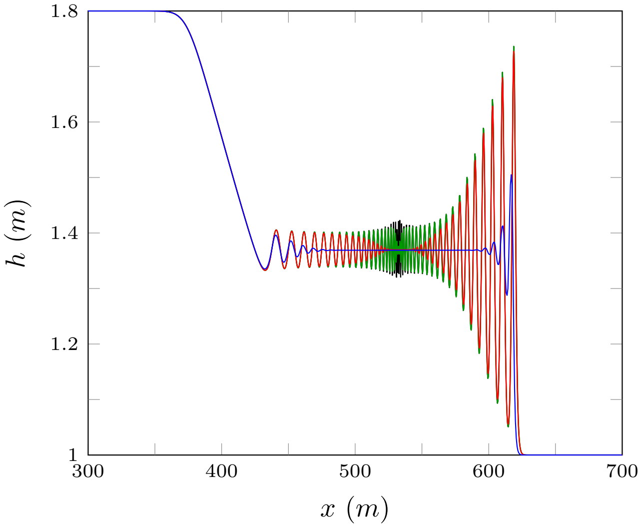
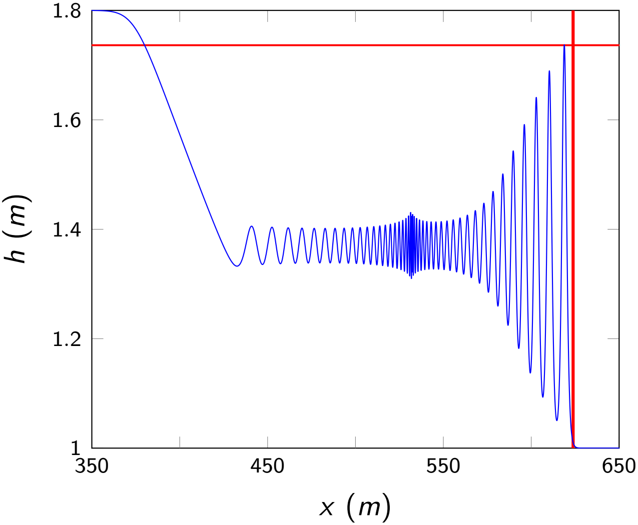

1.8
1.6
1.4
1.2
1
$h\ (m)$
300
400
500
600
700
$x\ (m)$

Asymptotic Results



Review of Smoothing and Methods

- Demonstrated that behaviour is consistent across many numerical methods
- Were able to explain why the behaviour had not previously been observed

Result

Our numerical solutions for the steep gradient problems are well validated [3]

[3] Pitt, J., Zoppou, C., and Roberts, S. (2018). Behaviour of the Serre Equations in the Presence of Steep Gradients Revisited. Wave Motion, 76(1):6177.

Progress

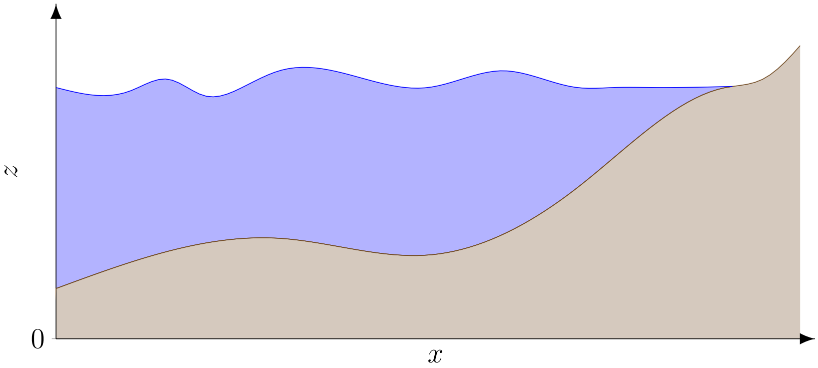
3D: Extension of the method to 3D flows ✓

Robust: Validation of model with steep gradients in free surface ✓

Robust: Validation of model in the presence of dry beds

Statement of Problem

Properly handle interaction of waves and the dry bed

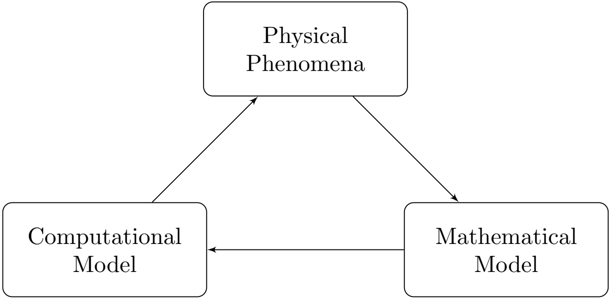


What was known

- No analytic solutions
- A variety of numerical techniques only validated against experimental data

Solution

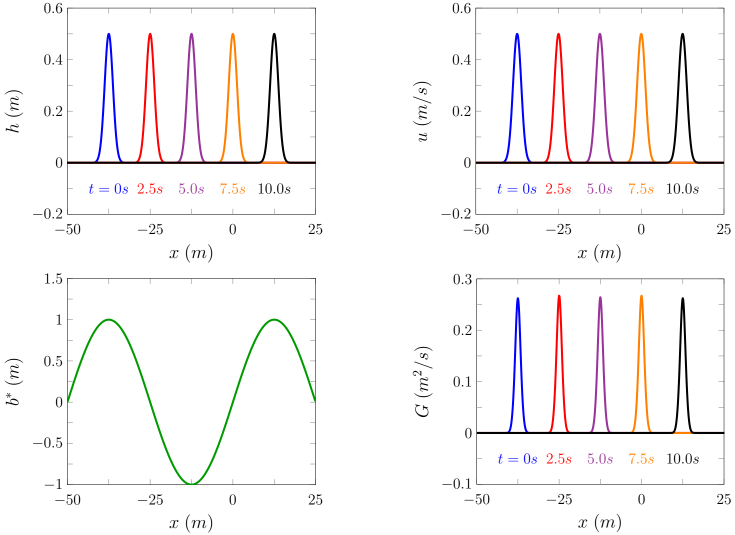
- Solved modified equations that did possess analytic solutions
- Compared with experimental data



Constructing Modified Equations

- Pick functions for height, velocity and bed: h^*, u^* and b^*
- Add Source terms to Serre equations that force a solution for h^*, u^* and b^*
- Validation tests

Pick Functions



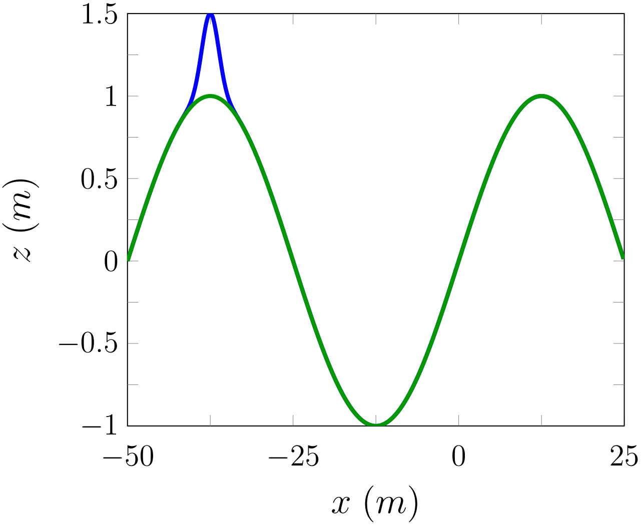
Modify Equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = S_h^*,$$

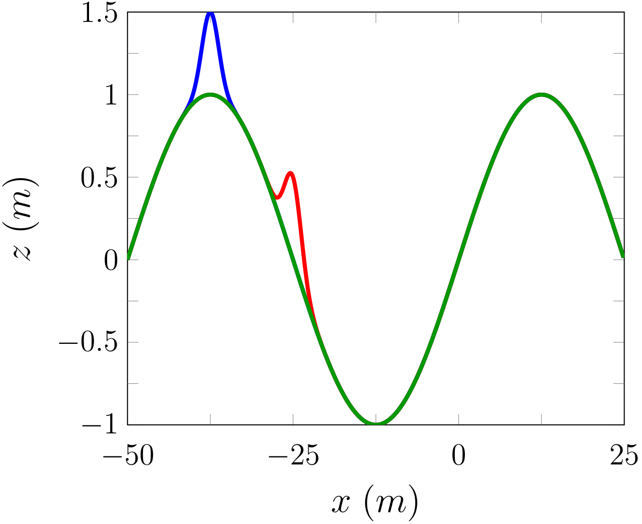
$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}h^3 \left[\frac{\partial u}{\partial x}\right]^2 + h^2 u \frac{\partial u}{\partial x}\frac{\partial b}{\partial x}\right)$$
$$+ \frac{1}{2}h^2 u \frac{\partial u}{\partial x}\frac{\partial^2 b}{\partial x^2} - hu^2 \frac{\partial b}{\partial x}\frac{\partial^2 b}{\partial x^2} + gh\frac{\partial b}{\partial x} = S_G^*.$$

S_h^* and S_G^* are just the LHS with the quantities replaced by their associated chosen function. We solve the LHS using our method and add in the source terms on the RHS analytically.

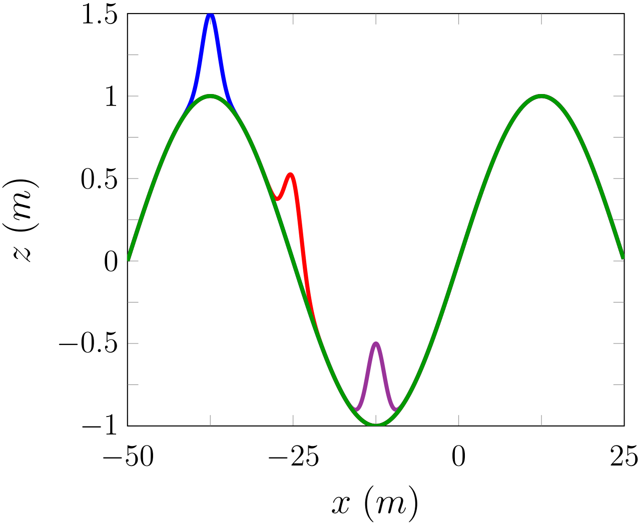
Results $t = 0s$



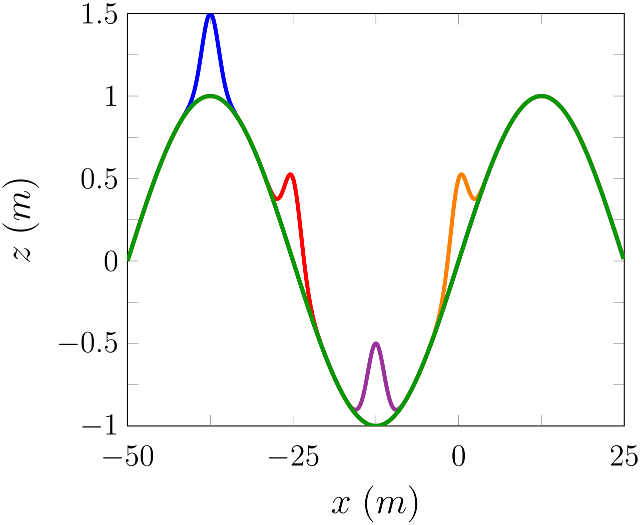
Results $t = 0s, 2.5s$



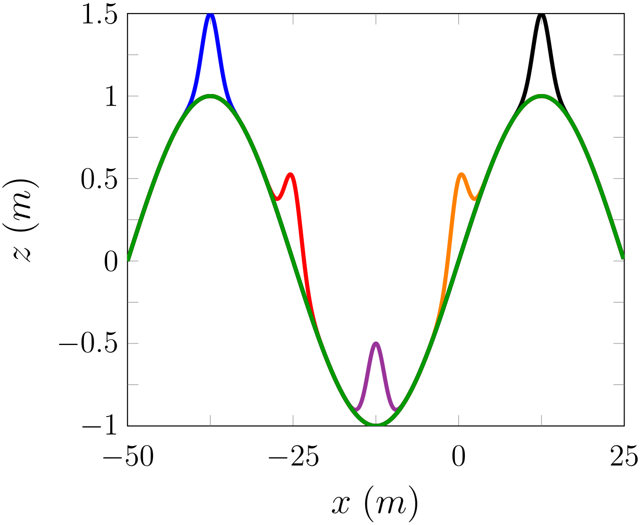
Results $t = 0s, 2.5s, 5.0s$



Results $t = 0s, 2.5s, 5.0s, 7.5s$



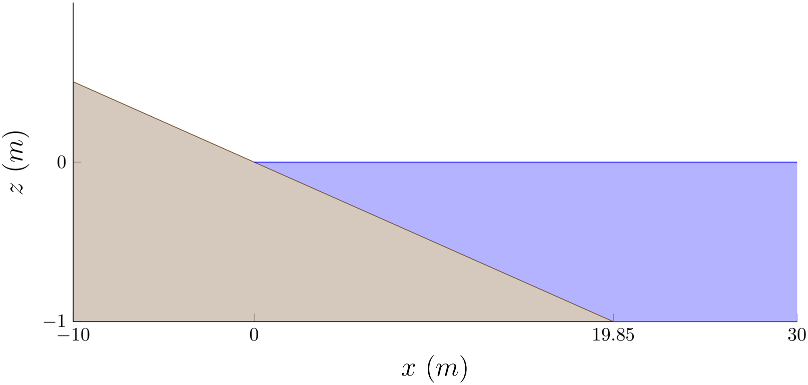
Results $t = 0s, 2.5s, 5.0s, 7.5s, 10.0s$



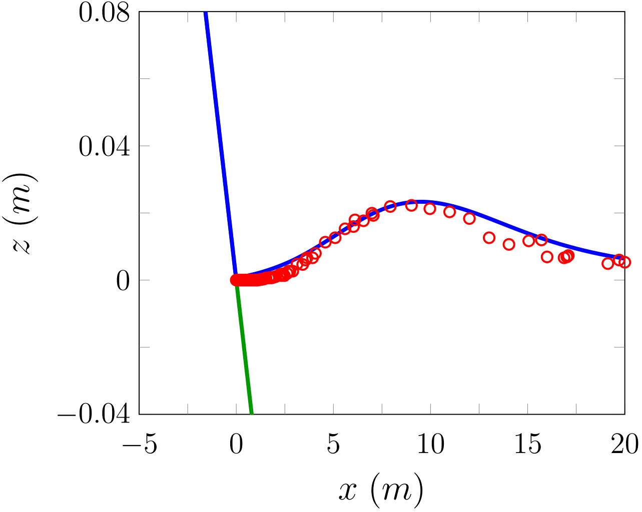
Modified Equations Validation Conclusions

- Very strong test that we are actually solving the Serre equations accurately as all terms must be accurately approximated
- Can measure the convergence of numerical solutions to the force solutions

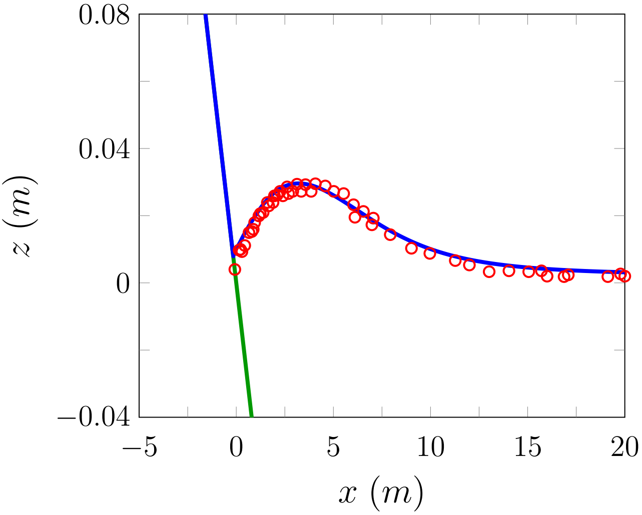
Experimental Data



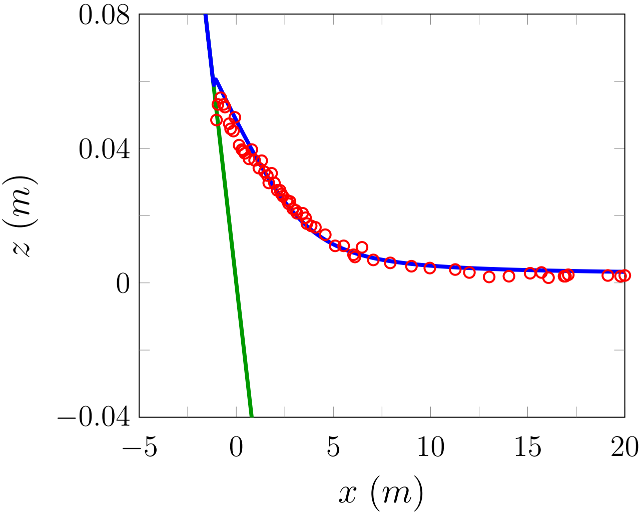
$t = 30s$



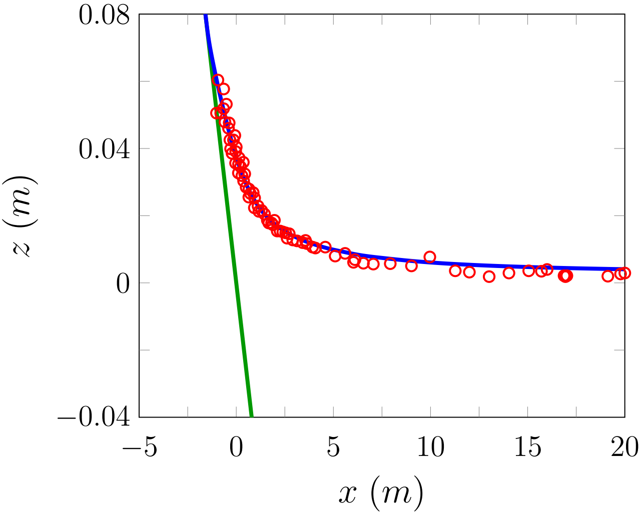
$t = 40s$



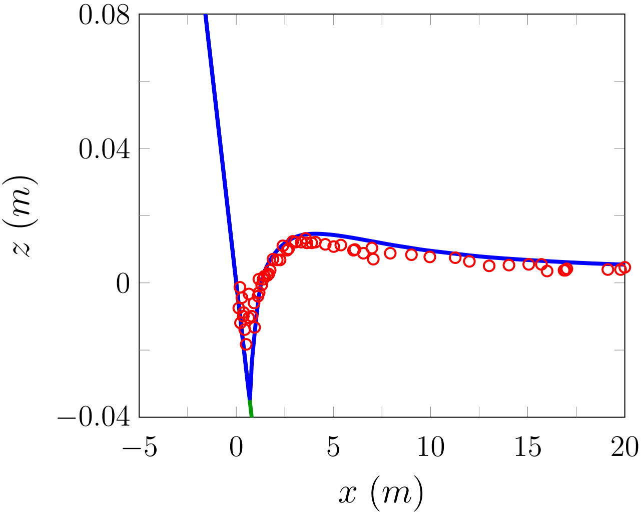
$t = 50s$



$t = 60s$



$t = 70s$



Experimental Validation Conclusions

- Demonstrates that our computational model agrees with the physical process
- not a very stringent test as there are many source of errors
- few experimental results for non breaking waves

Progress

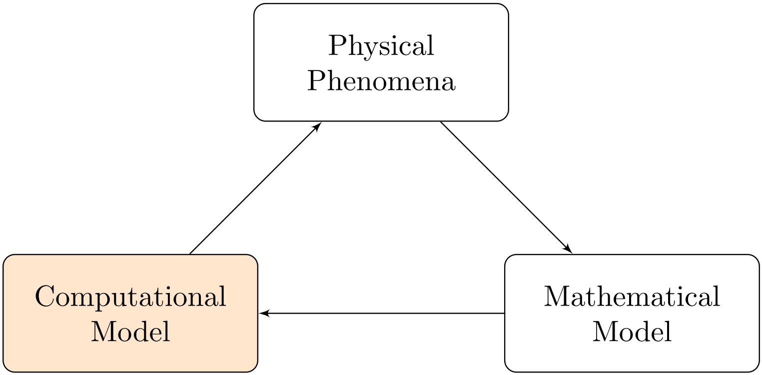
3D: Extension of the method to 3D flows ✓

Robust: Validation of model with steep gradients in free surface ✓

Robust: Validation of model in the presence of dry beds ✓

Conclusions

- Developed a Robust Computational Model from the Serre equations for the 2D water wave problem



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