Behaviour of The Serre Equations In The Presence of Steep Gradients

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- Numerically Model: want a numerical method for partial differential equations that model fluids
- Tsunami
- Efficiently: our method must not be too computationally demanding
- Robustly: our method should handle all types of initial conditions and situations that arise from them, the main difficulty usually is the presence of steep gradients.

Problem

Goal: numerically model a tsunami throughout its evolution efficiently and robustly.

Step 1: Pick partial differential equations that model tsunamis well and have robust and efficient methods. We focus on two dimensional flow first.

PDEs

Non-exhaustive list of PDES that describe fluid dynamics well for tsunamis from most difficult to solve to easiest

- Navier-Stokes equations
- inviscid incompressible Euler equations
- Serre equations
- shallow water wave equations

Navier-Stokes / Euler Equations

Coordinates: (x,z) over time t
Function of ρ (density), $\mathbf{u} = (u, w)$ (velocity), p (pressure)
, $\boldsymbol{\tau}$ (stress tensor) and $\mathbf{g}$ (acceleration due to gravity)

Conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Conservation of momentum:

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot \left(\rho \mathbf{u} \times \mathbf{u}^T\right) = -\nabla \cdot p\mathbf{I} + \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{g}$$

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4. No analytic solutions for problems containing steep gradients

Shallow Water Wave Equations

Coordinates are (x,z) over time t
$h(x,t)$: water depth
$\mathbf{u} = (u, w)$: velocities over water depth
g: acceleration due to gravity
assumption that $w = 0$. Conservation of mass:

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

Conservation of momentum:

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2}\right) = 0$$

Shallow Water Wave Equations

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Compromise: Serre Equations

Coordinates are (x,z) over time t

$h(x,t)$: water depth

$\mathbf{u} = (u, w)$: velocities over water depth

g: acceleration due to gravity

assumption that $w = -z\frac{\partial u}{\partial x}$.

Conservation of mass:

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

Conservation of momentum:

$$\underbrace{\underbrace{\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2}\right)}_{\text{Shallow Water Wave Equations}} + \underbrace{\frac{\partial}{\partial x}\left(\frac{h^3}{3}\left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}\right]\right)}_{\text{Dispersion Terms}}}_{\text{Serre Equations}} = 0$$

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1. Free surface approximation
2. Vertical velocity of fluid is linear across depth
3. Captures more behaviour than the shallow water wave equations such as dispersion
4. Efficient and robust methods
5. Cannot model wave breaking

Problem

Goal: Numerically model a tsunami throughout its evolution efficiently and robustly.

Step 1: We choose the Serre equations as a compromise between the shallow water wave equations and the Navier-Stokes/ Euler equations

New Problem: Efficient numerical method

3 Main Types of Numerical Methods

- Finite Difference
- Finite Element
- Finite Volume

Finite Difference

We approximate derivatives at a point x_0 using function evaluations at nearby points, for example

$$f'(x_0) \approx \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

The accuracy of this approximation depends on both f and Δx, although we focus on Δx.

Main Tool Taylor Series

Provided f infinitely differentiable around x_0

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots$$

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$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0) + \frac{f''(x_0)}{2!}(\Delta x) + \dots$$

Main Tool Taylor Series

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$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0) + \frac{f''(x_0)}{2!}(\Delta x) + \dots$$

$$f'(x_0) = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} - \frac{f''(x_0)}{2!}(\Delta x) - \dots$$

Application

Serre equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2}\right) + \frac{\partial}{\partial x}\left(\frac{h^3}{3}\left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}\right]\right) = 0$$

Expand all terms then approximate them as finite differences.

Finite Volume

Equation in conservative form

$$\frac{\partial}{\partial t}u = -\frac{\partial}{\partial x}f(u)$$

Spatial grid: x_i uniform so that $\Delta x = x_i - x_{i-1}$ for all i

Temporal grid: t^n uniform so that $\Delta t = t^n - t^{n-1}$ for all n

Cells: ith cell centred around x_i is $\mathcal{C}_i = \left[x_i - \frac{\Delta x}{2}, x_i + \frac{\Delta x}{2}\right]$

Integrating our PDE over $\mathcal{C}_i$ and time step gives

$$\int_{\mathcal{C}_i} u(x, t^{n+1})\, dx - \int_{\mathcal{C}_i} u(x, t^{n})\, dx$$
$$= \int_{t^n}^{t^{n+1}} f\left(u\left(x_i - \frac{\Delta x}{2}, t\right)\right)\, dt - \int_{t^n}^{t^{n+1}} f\left(u\left(x_i + \frac{\Delta x}{2}, t\right)\right)\, dt$$

$$U_i^n = \frac{1}{\Delta x}\int_{\mathcal{C}_i} u(x, t^n)\, dx$$

$$F_{i-1/2}^n = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f\left(u\left(x_i - \frac{\Delta x}{2}, t\right)\right)$$

$$F_{i+1/2}^n = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f\left(u\left(x_i + \frac{\Delta x}{2}, t\right)\right)$$

$$U_i^n = \frac{1}{\Delta x} \int_{\mathcal{C}_i} u(x, t^n)\, dx$$

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$$\Delta x U_i^{n+1} = \Delta x U_i^n - \Delta t \left[F_{i+1/2}^n - F_{i-1/2}^n\right]$$

$$U_i^n = \frac{1}{\Delta x} \int_{\mathcal{C}_i} u(x, t^n)\, dx$$

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$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x} \left[F_{i+1/2}^n - F_{i-1/2}^n\right]$$

This equation is exact, usually approximate $F_{i+1/2}^n$ and $F_{i-1/2}^n$.
These methods conserve quantities very well.

Application

Serre equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2}\right) + \frac{\partial}{\partial x}\left(\frac{h^3}{3}\left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}\right]\right) = 0$$

Not in conservative form.

Conservative form for Serre equations

Introducing $G = uh - \frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial u}{\partial x}\right)$
Serre equations can be rearranged into conservative form

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(Gu + \frac{gh^2}{2} - \frac{2h^3}{3}\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right) = 0$$

Method: Use finite volume to update h and G, with u from a finite difference approximation of equation for G

Problem

Goal: numerically model a tsunami throughout its evolution efficiently and robustly.

Step 1: We choose the Serre equations

Step 2: Presented some efficient methods

New Problem: Check robustness without analytic solutions for Serre equations involving steep gradients

Attempt: Numerically solve a toy problem. Literature has already done this.

Toy Problem: Dam-Break Problem

Solve the Serre equations for $h(x,t)$ and $u(x,t)$ given these initial conditions. Fluid depth (h) :

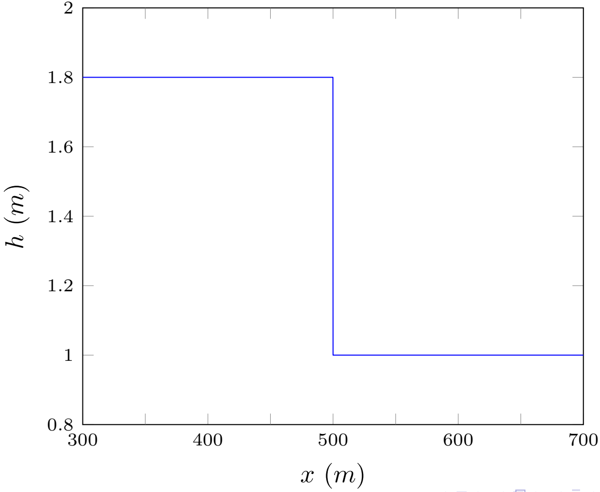
$$h(x,0) = \begin{cases} h_1 & x \leq x_0 \\ h_0 & x > x_0 \end{cases}$$

Fluid velocity (u) :

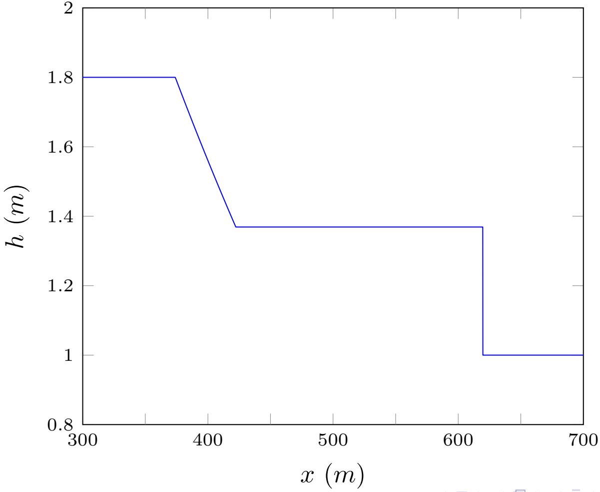
$$u(x,0) = 0.0.$$

We are going to be looking at the results of the dam-break with $h_1 = 1.8m$, $h_0 = 1m$ at $t = 30s$ initially centered around $x_0 = 500m$.

Initial conditions



Shallow water wave equations analytic solution



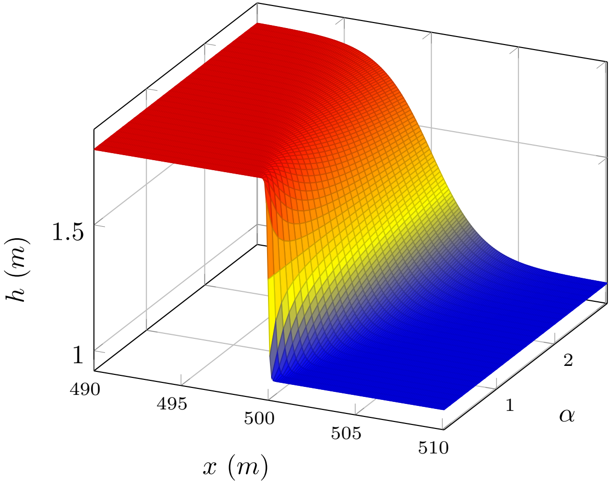
Smoothed Dam-Break Problem

Solve the Serre equations for $h(x,t)$ and $u(x,t)$ given these initial conditions.

$$h(x,0) = h_0 + \frac{h_1 - h_0}{2}\left(1 + \tanh\left(\frac{x_0 - x}{\alpha}\right)\right),$$

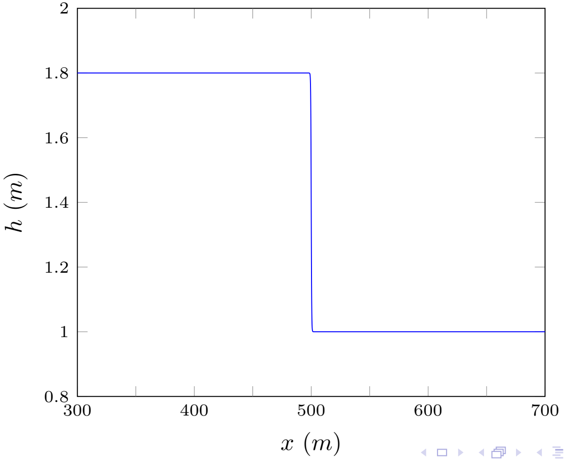
$$u(x,0) = 0.0m/s$$

Smoothed dam-break for $h_1 = 1.8m$, $h_0 = 1m$ and $x_0 = 500m$



Literature Solutions initial conditions when $\alpha = 0.4m$

G. A. El , Roger HJ Grimshaw and Noel F. Smyth (2006)



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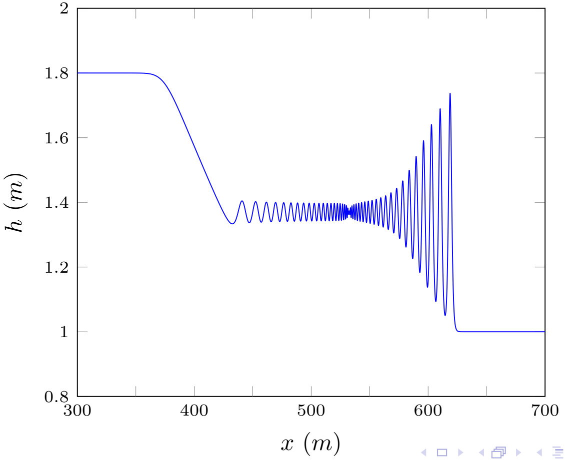
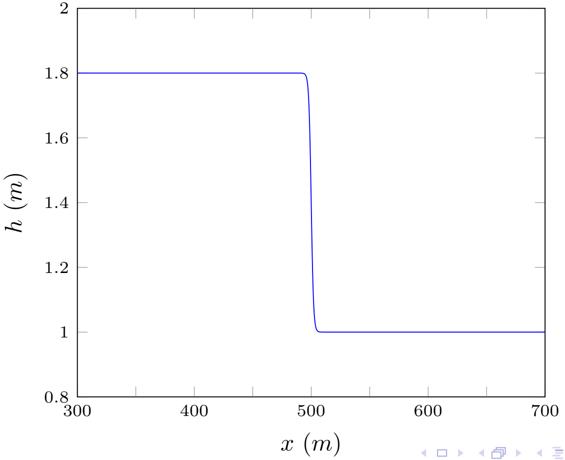


Literature Solutions initial conditions when $\alpha = 2m$

D. Mitsotakis, B. Ilan and D. Dutykh (2014)



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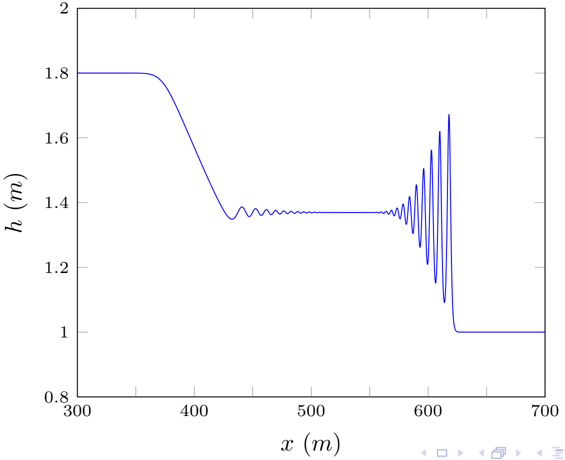


Problem

Goal: numerically model a tsunami throughout its evolution efficiently and robustly.

Step 1: We choose the Serre equations

Step 2: Presented some efficient methods

Problem: Check robustness without analytic solutions for Serre equations involving steep gradients

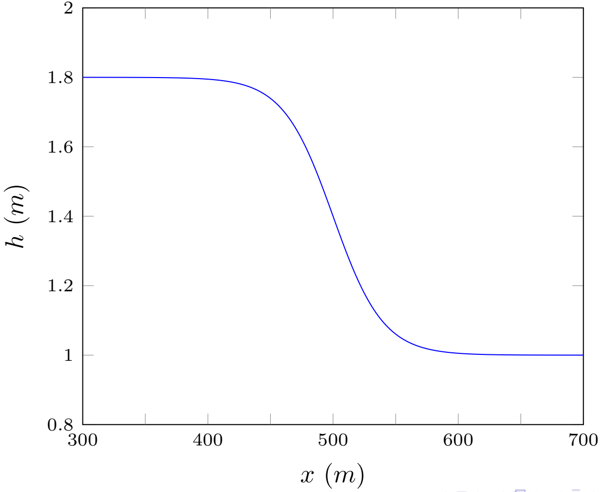
Attempt: Numerically solve simplest problem with a steep gradient.

Structures

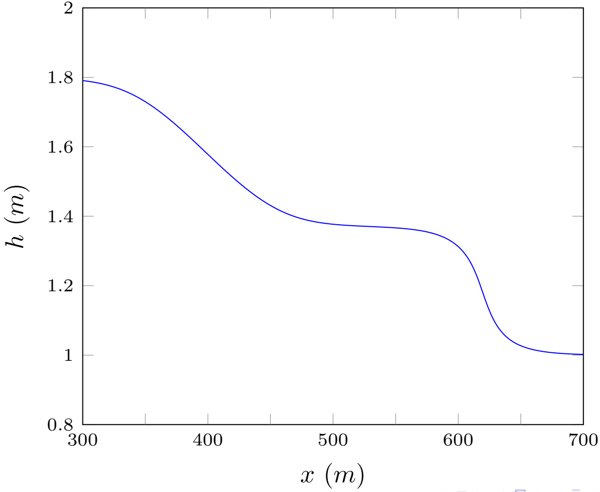
We found 4 different structures for numerical solutions of the Serre equations to various smoothed dam-break problems using our highest order method finite volume method with highest resolution.

- Non-oscillatory structure
- Flat structure
- Node structure
- Growth structure

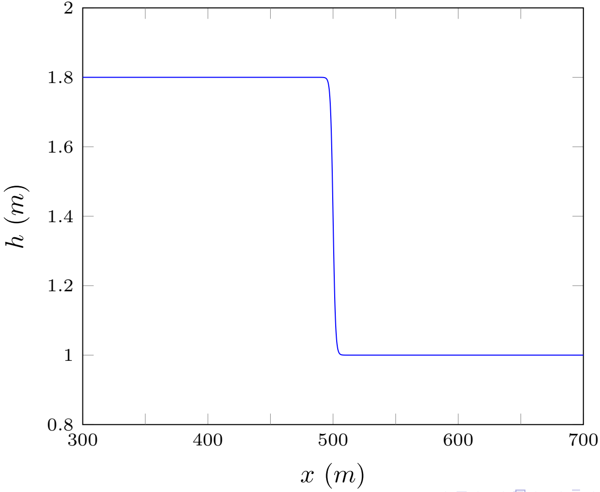
Initial conditions when $\alpha = 40m$



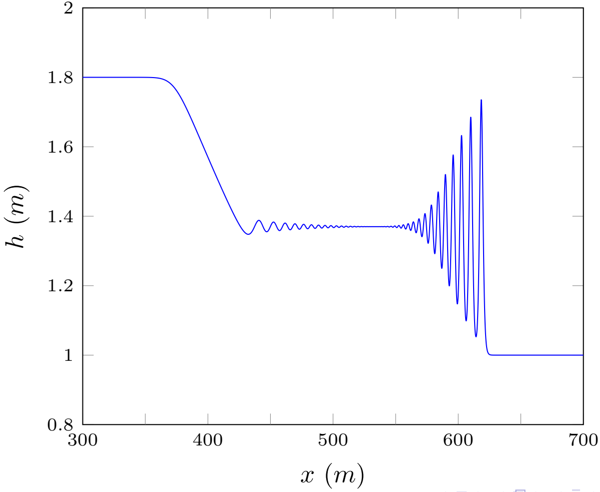
Non-Oscillatory structure $\alpha = 40m$



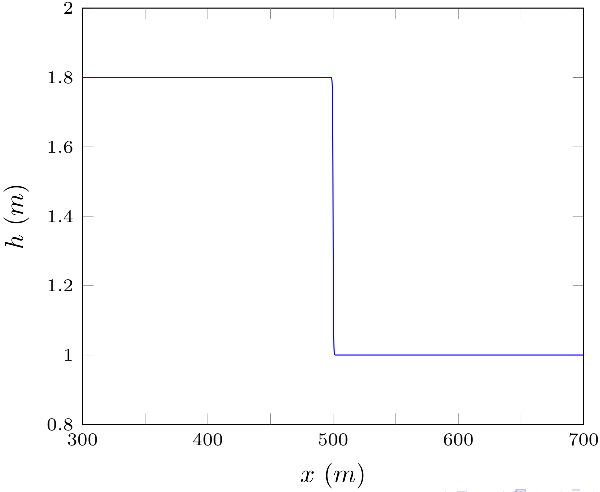
Initial conditions when $\alpha = 2m$



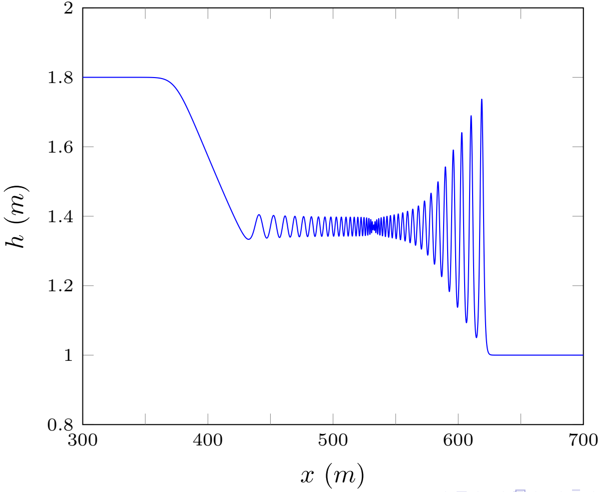
Flat $\alpha = 2m$



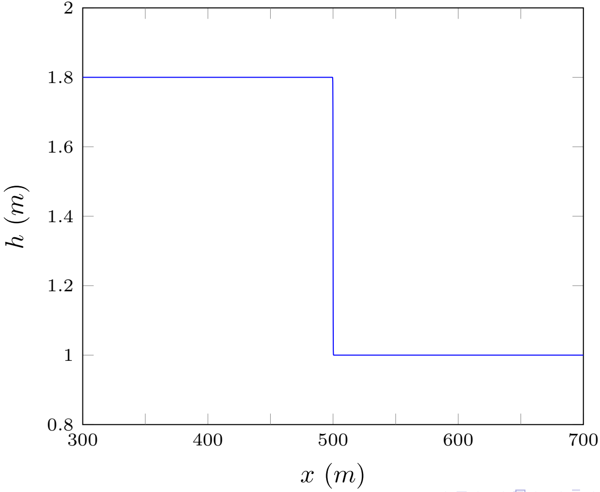
Initial conditions when $\alpha = 0.4m$



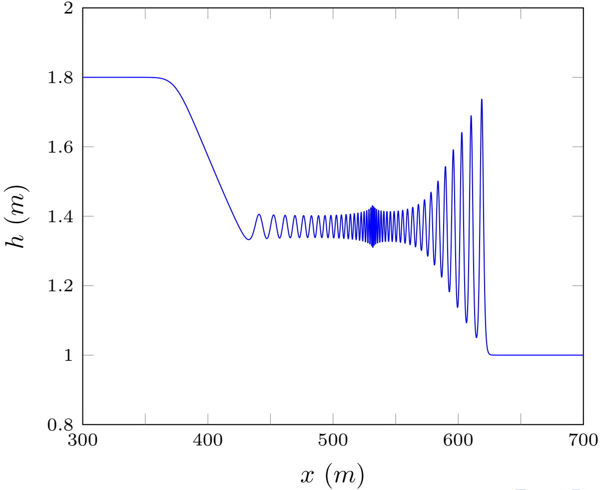
Node $\alpha = 0.4m$



Initial conditions when $\alpha = 0.1m$



Growth ($\alpha = 0.1m$)



Problem

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Step 1: We choose the Serre equations

Step 2: Presented some efficient methods

Problem: Check robustness without analytic solutions for Serre equations involving steep gradients

Attempt: Numerically solve simplest problem with a steep gradient.

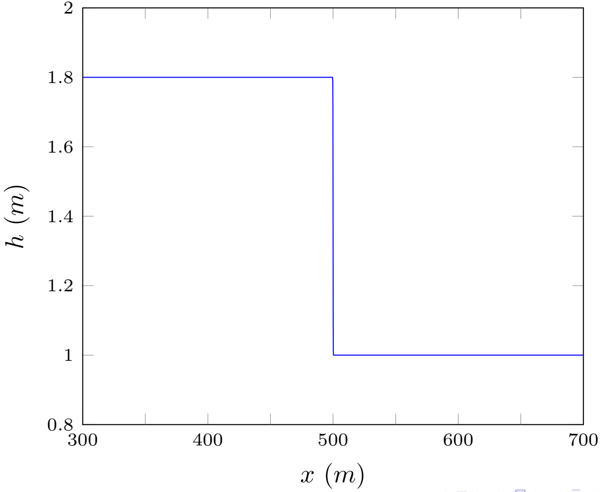
New Problem: Are these numerical results correct?

Growth structure: changing resolutions

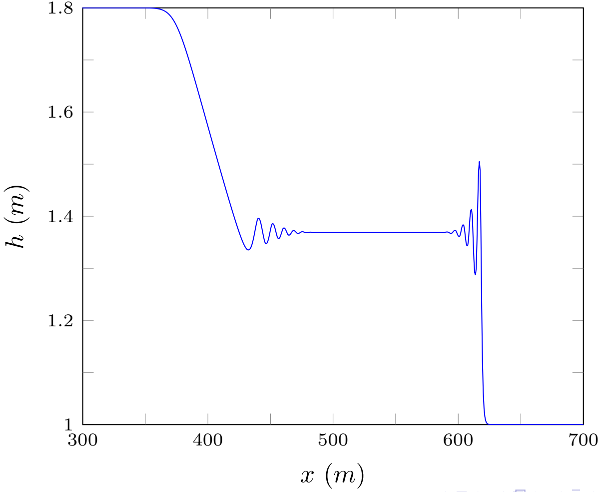
Although we cannot check if our numerical solutions are converging to a true solution as one is not known for the Serre equations we can check if the numerical solutions converge to one another as $\Delta x \to 0$.

We will now perform this for our highest order finite volume method, with an initial Δx of $0.5m$, resolution is increased by dividing Δx by 4.

Initial conditions when $\alpha = 0.1m$



Growth structure: changing resolutions



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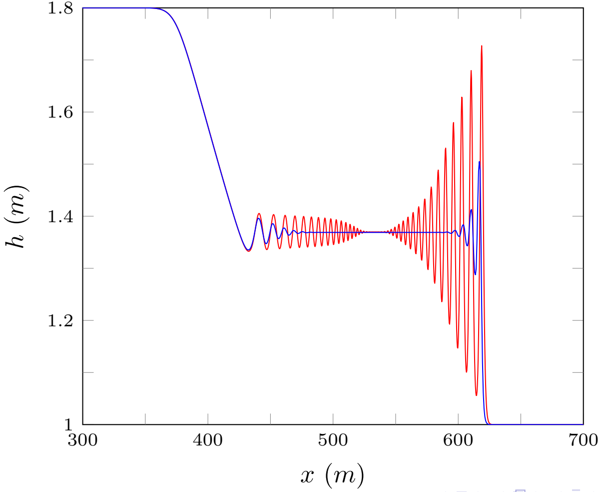

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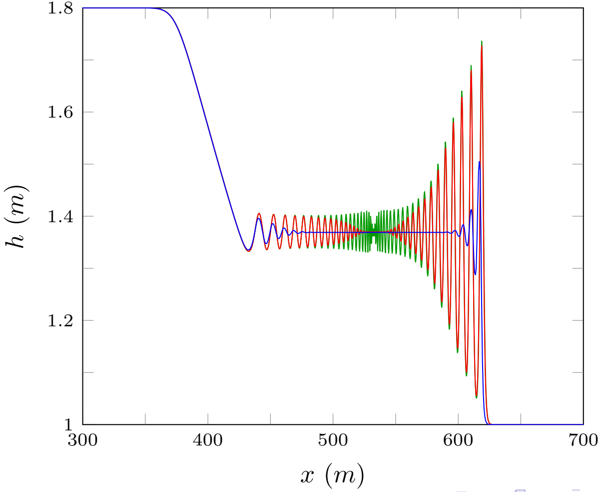

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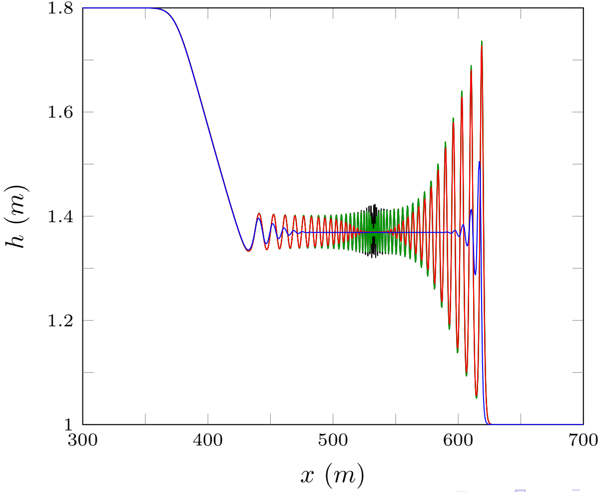

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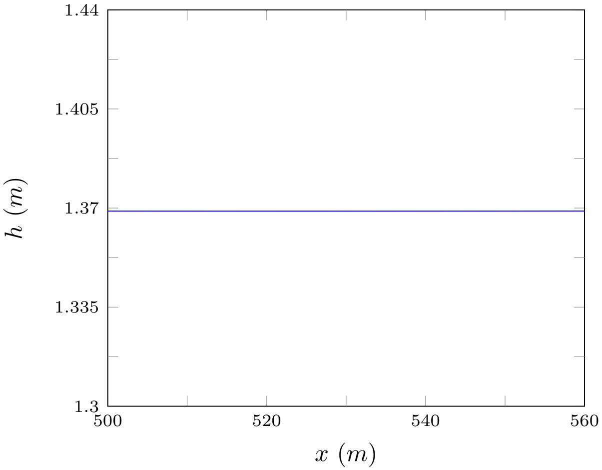

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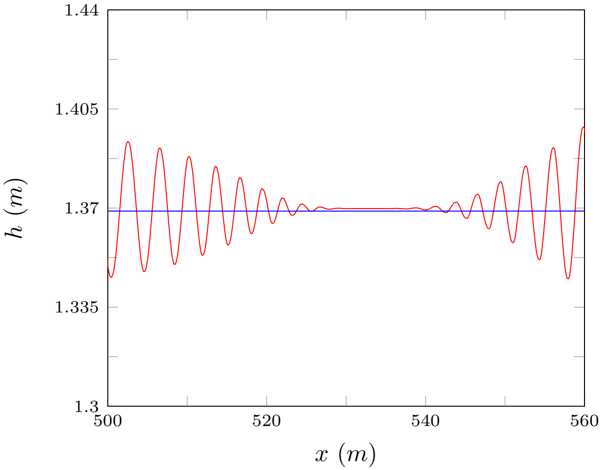

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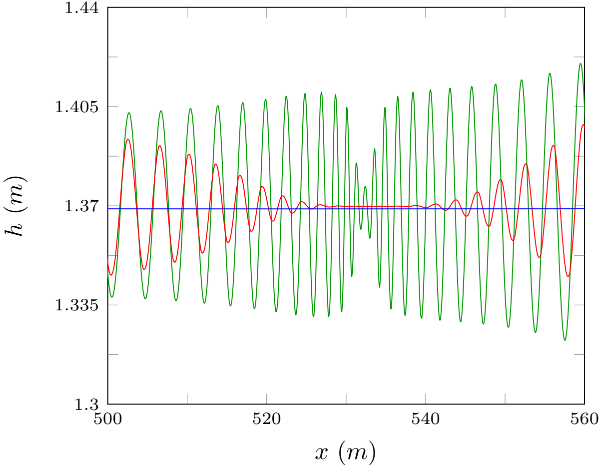

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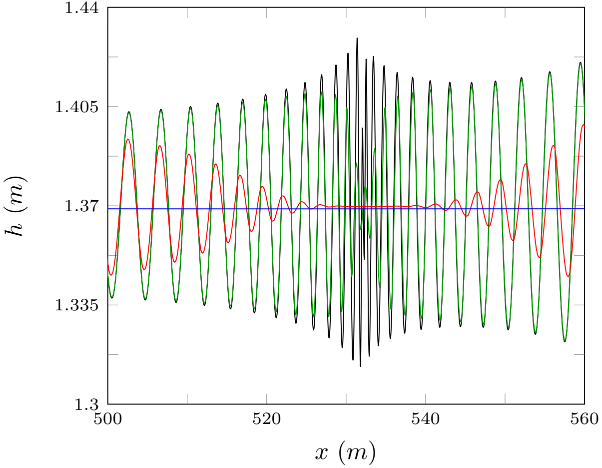

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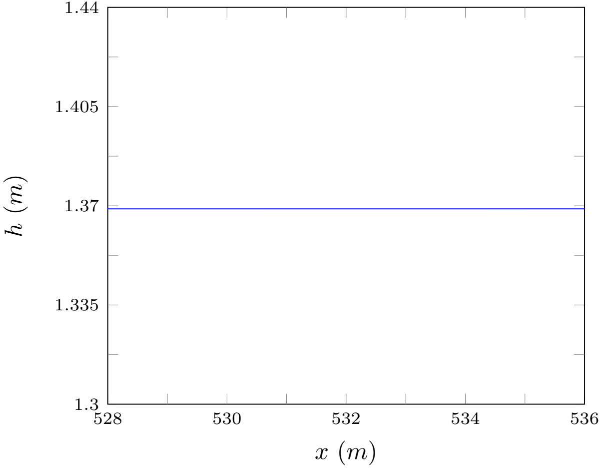

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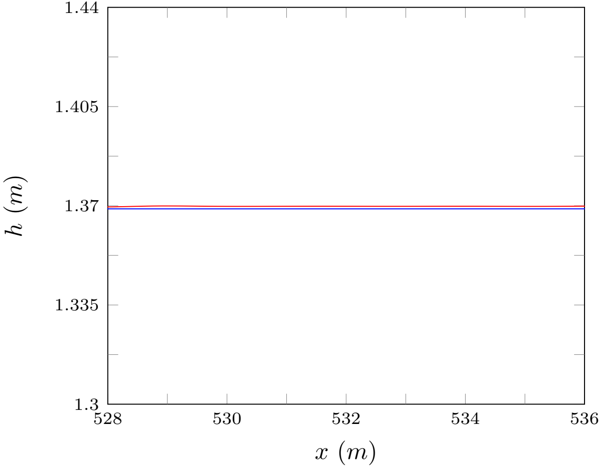

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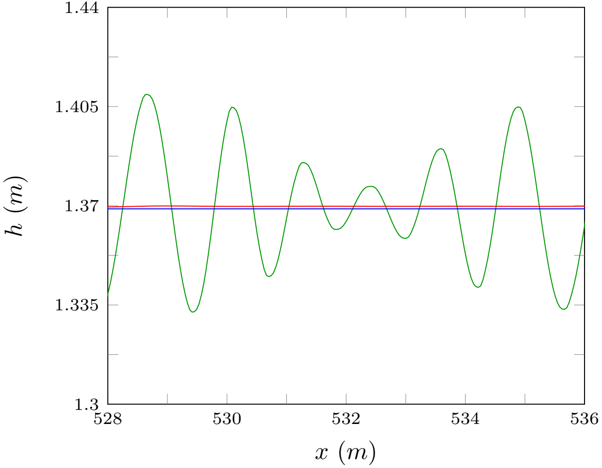

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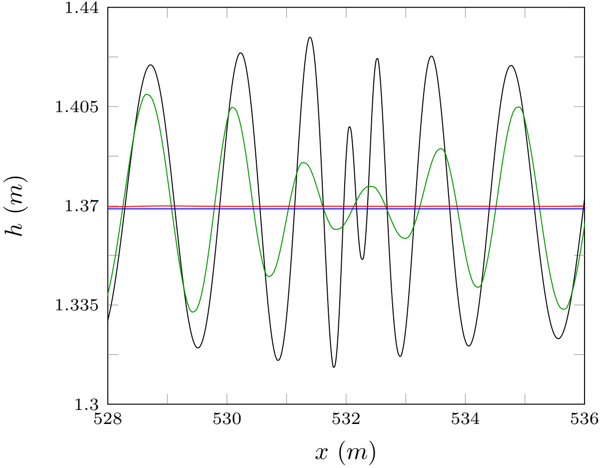


Growth structure: conservation while changing resolutions

α	Δx	Conservation error h	Conservation error uh
0.1	0.2	$7.6 \cdot 10^{-14}$	$4.82 \cdot 10^{-3}$
0.1	0.05	$2.4 \cdot 10^{-13}$	$2.39 \cdot 10^{-4}$
0.1	0.0125	$9.79 \cdot 10^{-13}$	$2.21 \cdot 10^{-7}$
0.1	0.003125	$3.92 \cdot 10^{-12}$	$4.46 \cdot 10^{-8}$

Growth structure: changing resolutions

This shows that:

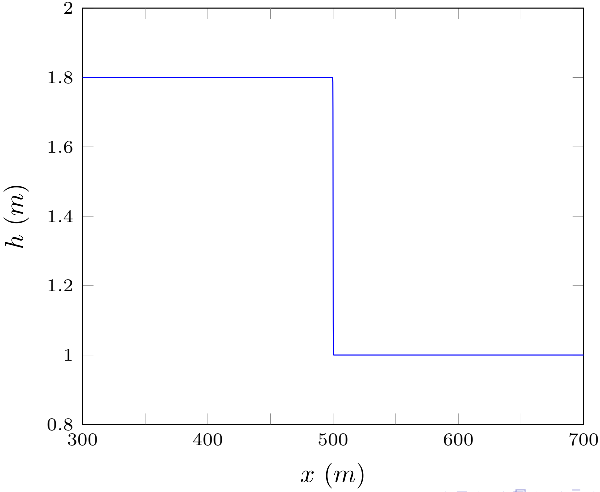
1. Away from the growth in oscillations the numerical solutions converge quite well
2. Main difference in solutions is amplitude of oscillations (indicating that these are not of numerical origin)
3. Our numerical methods are approaching a solution which is conservative

Growth structure: changing methods

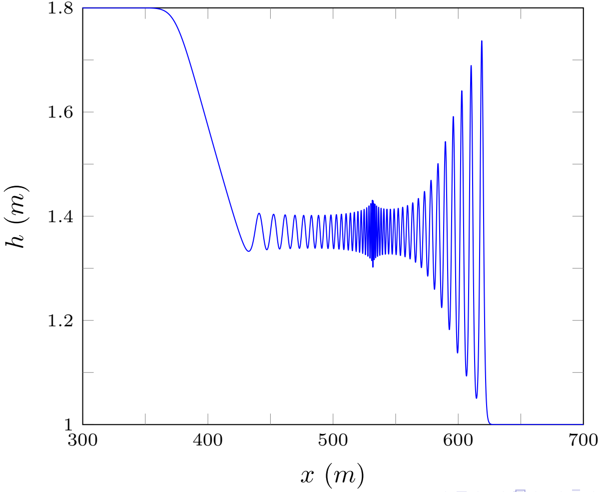
We want our numerical solutions to not be model dependent for high resolution grids and fixed α.

We compare two models, the highest order finite volume from before, and a finite difference method.

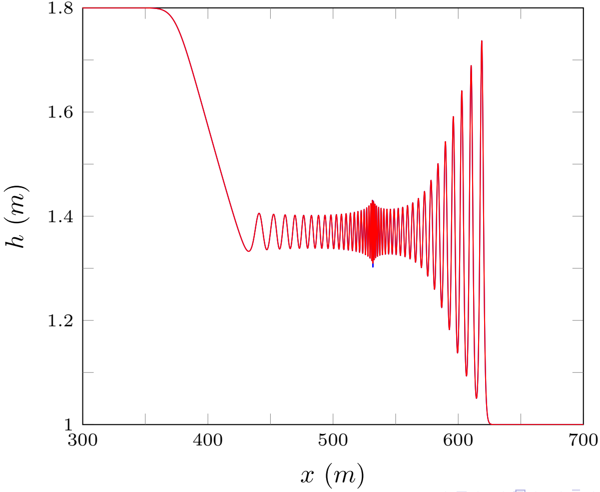
Initial conditions when $\alpha = 0.1m$



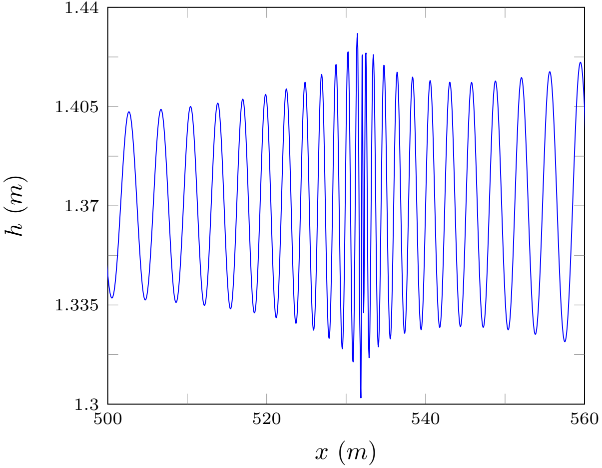
Growth structure: changing methods



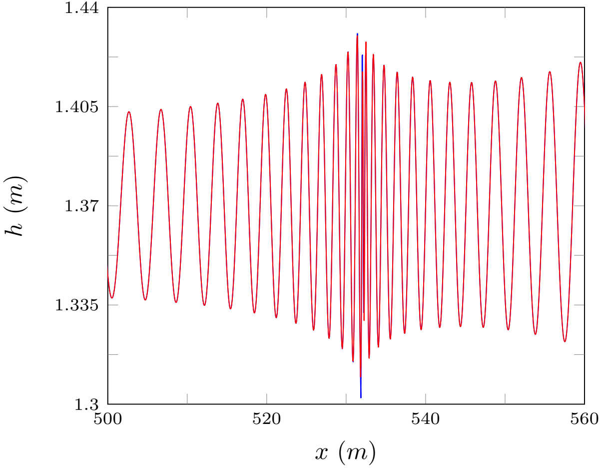
Growth structure: changing methods



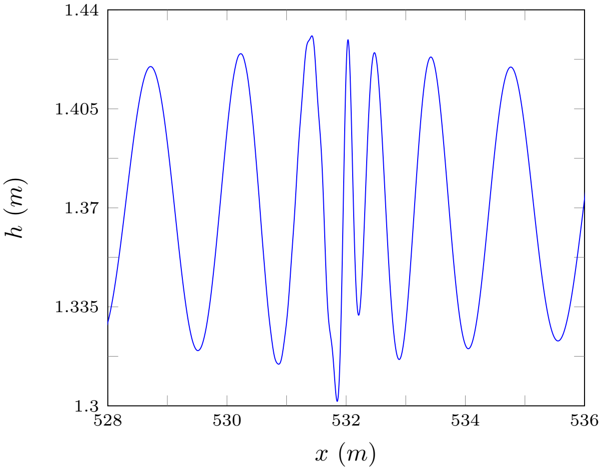
Growth structure: changing methods



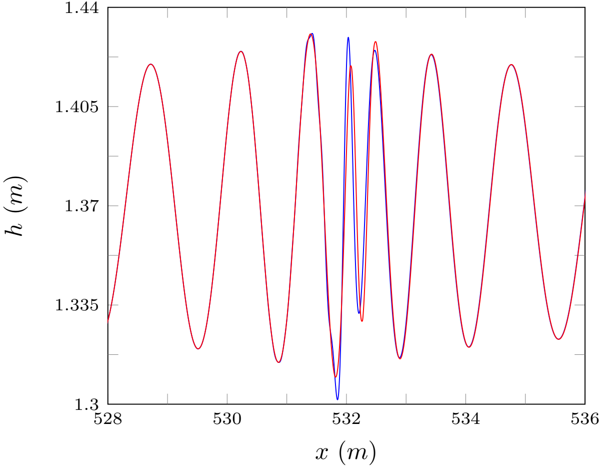
Growth structure: changing methods



Growth structure: changing methods



Growth structure: changing methods



Growth structure: changing methods

This shows that:

1. Just a very few oscillations are different across the models
2. Again only difference is amplitude of oscillations
3. Structure independent of method

Growth structure: conclusion

- Numerical solutions for a method converge to one another as $\Delta x \to 0$
- Numerical solutions for a method converge to a conservative solution as $\Delta x \to 0$
- Growth structure is found for different methods

Conclusion: solutions of the Serre equations should exhibit the growth structure for the smoothed dam-break problem with small α and even the dam-break problem.

Problem

Goal: numerically model a tsunami throughout its evolution efficiently and robustly.

Step 1: We choose the Serre equations

Step 2: Presented some efficient methods

Step 3: Numerically solved Serre equations in the presence of steep gradients and verified our solutions.

Further Work

Goal: numerically model a tsunami throughout its evolution efficiently and robustly.

Sub Goal: Inundation of land, have to solve the dry bed problem for our numerical solutions

Sub Goal: Full three dimensional flow