

1 Numerical Method for dispersion error Break Down

To do the first analysis we first construct the update matrix F , which plays a similar role to λ I showed you in the space continuous case. We then diagonalise this so that we get back to just factors (basically). This also means I would like to update what we present in the table, I think we should present all the elements of the matrices, for $\mathcal{F}^{h,u}$ and $\mathcal{F}^{h,h}$ there is no change but for $\mathcal{F}^{u,u}$ and $\mathcal{F}^{u,h}$ this means also dividing it by \mathcal{G} as well as \mathcal{M} and Δx . Anyway onto the method.

1.1 F

F comes from the FVM update scheme which for us is

$$\bar{q}_j^{n+1} = \bar{q}_j^n - \frac{\Delta t}{\Delta x} \left[F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}} \right]$$

This converts to (both analytical and numerical)

$$\mathcal{M}q_j^{n+1} = \mathcal{M}q_j^n - \frac{\Delta t}{\Delta x} [\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j - \mathcal{F}^{q,v}v_{j-1} - \mathcal{F}^{q,q}q_{j-1}]$$

$$\mathcal{M}q_j^{n+1} = \mathcal{M}q_j^n - \frac{\Delta t}{\Delta x} [\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j - \mathcal{F}^{q,v}e^{-ik\Delta x}v_j - \mathcal{F}^{q,q}e^{-ik\Delta x}q_j]$$

Defining $\mathcal{D}_x = 1 - e^{-ik\Delta x}$

$$\mathcal{M}q_j^{n+1} = \mathcal{M}q_j^n - \frac{\Delta t}{\Delta x} [\mathcal{D}_x \mathcal{F}^{q,v}v_j + \mathcal{D}_x \mathcal{F}^{q,q}q_j]$$

So we have

$$q_j^{n+1} = q_j^n - \frac{\mathcal{D}_x \Delta t}{\mathcal{M} \Delta x} [\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j]$$

Thus we have

$$\begin{bmatrix} h \\ \mathcal{G}u \end{bmatrix}_j^{n+1} = \begin{bmatrix} h \\ \mathcal{G}u \end{bmatrix}_j^n - \frac{\mathcal{D}_x \Delta t}{\mathcal{M} \Delta x} \begin{bmatrix} \mathcal{F}^{h,h} & \mathcal{F}^{h,u} \\ \mathcal{F}^{u,h} & \mathcal{F}^{u,u} \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \begin{bmatrix} h \\ u \end{bmatrix}_j^n - \frac{\mathcal{D}_x \Delta t}{\mathcal{M} \Delta x} \begin{bmatrix} \mathcal{F}^{h,h} & \mathcal{F}^{h,u} \\ \frac{1}{\mathcal{G}} \mathcal{F}^{u,h} & \frac{1}{\mathcal{G}} \mathcal{F}^{u,u} \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

Lets define

$$\begin{aligned} \mathbf{F} &= \frac{\mathcal{D}_x}{\mathcal{M} \Delta x} \begin{bmatrix} \mathcal{F}^{h,h} & \mathcal{F}^{h,u} \\ \frac{1}{\mathcal{G}} \mathcal{F}^{u,h} & \frac{1}{\mathcal{G}} \mathcal{F}^{u,u} \end{bmatrix} \\ \begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} &= \begin{bmatrix} h \\ u \end{bmatrix}_j^n - \Delta t \mathbf{F} \begin{bmatrix} h \\ u \end{bmatrix}_j^n \\ \begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} &= (\mathbf{I} - \Delta t \mathbf{F}) \begin{bmatrix} h \\ u \end{bmatrix}_j^n \end{aligned}$$

So we can see that ever time step produces a factor $(\mathbf{I} - \Delta t \mathbf{F})$, I will now demonstrate the method with the hardest example, third order.

1.2 Third Order

For third order the time stepping algorithm is

$$\begin{aligned} \begin{bmatrix} h \\ u \end{bmatrix}^1 &= (\mathbf{I} - \Delta t \mathbf{F}_3) \begin{bmatrix} h \\ u \end{bmatrix}^n \\ \begin{bmatrix} h \\ u \end{bmatrix}^2 &= (\mathbf{I} - \Delta t \mathbf{F}_3) \begin{bmatrix} h \\ u \end{bmatrix}^1 \\ \begin{bmatrix} h \\ u \end{bmatrix}^3 &= \frac{3}{4} \begin{bmatrix} h \\ u \end{bmatrix}^n + \frac{1}{4} \begin{bmatrix} h \\ u \end{bmatrix}^2 \\ \begin{bmatrix} h \\ u \end{bmatrix}^4 &= (\mathbf{I} - \Delta t \mathbf{F}_3) \begin{bmatrix} h \\ u \end{bmatrix}^3 \\ \begin{bmatrix} h \\ u \end{bmatrix}^{n+1} &= \frac{1}{3} \begin{bmatrix} h \\ u \end{bmatrix}^n + \frac{2}{3} \begin{bmatrix} h \\ u \end{bmatrix}^4 \end{aligned}$$

Thus

$$\begin{bmatrix} h \\ u \end{bmatrix}^3 = \frac{3}{4} \begin{bmatrix} h \\ u \end{bmatrix}^n + \frac{1}{4} (\mathbf{I} - \Delta t \mathbf{F}_3)^2 \begin{bmatrix} h \\ u \end{bmatrix}^n$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^3 = \left(\frac{3}{4}\mathbf{I} + \frac{1}{4}(\mathbf{I} - \Delta t \mathbf{F}_3)^2 \right) \begin{bmatrix} h \\ u \end{bmatrix}^n$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^{n+1} = \frac{1}{3} \begin{bmatrix} h \\ u \end{bmatrix}^n + \frac{2}{3}(\mathbf{I} - \Delta t \mathbf{F}_3) \left(\frac{3}{4}\mathbf{I} + \frac{1}{4}(\mathbf{I} - \Delta t \mathbf{F}_3)^2 \right) \begin{bmatrix} h \\ u \end{bmatrix}^n$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^{n+1} = \left(\frac{1}{3}\mathbf{I} + \frac{2}{3}(\mathbf{I} - \Delta t \mathbf{F}_3) \left(\frac{3}{4}\mathbf{I} + \frac{1}{4}(\mathbf{I} - \Delta t \mathbf{F}_3)^2 \right) \right) \begin{bmatrix} h \\ u \end{bmatrix}^n$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^{n+1} = \left(\frac{1}{3}\mathbf{I} + (\mathbf{I} - \Delta t \mathbf{F}_3) \left(\frac{1}{2}\mathbf{I} + \frac{1}{6}(\mathbf{I} - 2\Delta t \mathbf{F}_3 + \Delta t^2 \mathbf{F}_3^2) \right) \right) \begin{bmatrix} h \\ u \end{bmatrix}^n$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^{n+1} = \left(\frac{1}{3}\mathbf{I} + (\mathbf{I} - \Delta t \mathbf{F}_3) \left(\frac{2}{3}\mathbf{I} - \frac{1}{3}\Delta t \mathbf{F}_3 + \frac{1}{6}\Delta t^2 \mathbf{F}_3^2 \right) \right) \begin{bmatrix} h \\ u \end{bmatrix}^n$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^{n+1} = \left(\frac{1}{3}\mathbf{I} + \frac{2}{3}\mathbf{I} - \frac{1}{3}\Delta t \mathbf{F}_3 + \frac{1}{6}\Delta t^2 \mathbf{F}_3^2 + (-\Delta t \mathbf{F}_3) \left(\frac{2}{3}\mathbf{I} - \frac{1}{3}\Delta t \mathbf{F}_3 + \frac{1}{6}\Delta t^2 \mathbf{F}_3^2 \right) \right) \begin{bmatrix} h \\ u \end{bmatrix}^n$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^{n+1} = \left(\mathbf{I} - \frac{1}{3}\Delta t \mathbf{F}_3 + \frac{1}{6}\Delta t^2 \mathbf{F}_3^2 - \frac{2}{3}\Delta t \mathbf{F}_3 + \frac{1}{3}\Delta t \mathbf{F}_3 \Delta t \mathbf{F}_3 - \frac{1}{6}\Delta t^2 \mathbf{F}_3^2 \Delta t \mathbf{F}_3 \right) \begin{bmatrix} h \\ u \end{bmatrix}^n$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^{n+1} = \left(\mathbf{I} - \mathbf{F}_3 + \frac{1}{6}\Delta t^2 \mathbf{F}_3^2 + \frac{1}{3}\Delta t^2 \mathbf{F}_3^2 - \frac{1}{6}\Delta t^3 \mathbf{F}_3^3 \right) \begin{bmatrix} h \\ u \end{bmatrix}^n$$

Finally

$$\begin{bmatrix} h \\ u \end{bmatrix}^{n+1} = \left(\mathbf{I} - \Delta t \mathbf{F}_3 + \frac{1}{2}\Delta t^2 \mathbf{F}_3^2 - \frac{1}{6}\Delta t^3 \mathbf{F}_3^3 \right) \begin{bmatrix} h \\ u \end{bmatrix}^n$$

lets say we have an eigenvalue decomposition for \mathbf{F}_3

$$\mathbf{F}_3 = \mathbf{S}_3 \mathbf{D}_3 \mathbf{S}_3^{-1}$$

where

$$\mathbf{D}_3 = \begin{bmatrix} \lambda_{3,-} & 0 \\ 0 & \lambda_{3,+} \end{bmatrix}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^{n+1} = \left(\mathbf{I} - \Delta t \mathbf{S}_3 \mathbf{D}_3 \mathbf{S}_3^{-1} + \frac{1}{2} \Delta t^2 \mathbf{S}_3 \mathbf{D}_3^2 \mathbf{S}_3^{-1} - \frac{1}{6} \Delta t^3 \mathbf{S}_3 \mathbf{D}_3^3 \mathbf{S}_3^{-1} \right) \begin{bmatrix} h \\ u \end{bmatrix}^n$$

Multiply on the LHS by \mathbf{S}_3^{-1}

$$\mathbf{S}_3^{-1} \begin{bmatrix} h \\ u \end{bmatrix}^{n+1} = \left(\mathbf{S}_3^{-1} - \Delta t \mathbf{D}_3 \mathbf{S}_3^{-1} + \frac{1}{2} \Delta t^2 \mathbf{D}_3^2 \mathbf{S}_3^{-1} - \frac{1}{6} \Delta t^3 \mathbf{D}_3^3 \mathbf{S}_3^{-1} \right) \begin{bmatrix} h \\ u \end{bmatrix}^n$$

$$\mathbf{S}_3^{-1} e^{i\omega \Delta t} \begin{bmatrix} h \\ u \end{bmatrix}^n = \left(\mathbf{I} - \Delta t \mathbf{D}_3 + \frac{1}{2} \Delta t^2 \mathbf{D}_3^2 - \frac{1}{6} \Delta t^3 \mathbf{D}_3^3 \right) \mathbf{S}_3^{-1} \begin{bmatrix} h \\ u \end{bmatrix}^n$$

$$e^{i\omega \Delta t} \mathbf{S}_3^{-1} \begin{bmatrix} h \\ u \end{bmatrix}^n = \left(\mathbf{I} - \Delta t \mathbf{D}_3 + \frac{1}{2} \Delta t^2 \mathbf{D}_3^2 - \frac{1}{6} \Delta t^3 \mathbf{D}_3^3 \right) \mathbf{S}_3^{-1} \begin{bmatrix} h \\ u \end{bmatrix}^n$$

$$\begin{aligned} e^{i\omega \Delta t} \mathbf{S}_3^{-1} \begin{bmatrix} h \\ u \end{bmatrix}^n &= \\ \begin{bmatrix} 1 - \Delta t \lambda_{3,-} + \frac{\Delta t^2}{2} \lambda_{3,-}^2 - \frac{\Delta t^3}{6} \lambda_{3,-}^3 & 0 \\ 0 & 1 - \Delta t \lambda_{3,+} + \frac{\Delta t^2}{2} \lambda_{3,+}^2 - \frac{\Delta t^3}{6} \lambda_{3,+}^3 \end{bmatrix} \mathbf{S}_3^{-1} \begin{bmatrix} h \\ u \end{bmatrix}^n & \end{aligned} \quad (1)$$

Since $\mathbf{S}_3^{-1} \begin{bmatrix} h \\ u \end{bmatrix}^n$ is just a vector then $e^{i\omega \Delta t}$ an eigenvalue of this diagonal matrix, so it must be that

$$e^{i\omega \Delta t} = 1 - \Delta t \lambda_{3,\pm} + \frac{\Delta t^2}{2} \lambda_{3,\pm}^2 - \frac{\Delta t^3}{6} \lambda_{3,\pm}^3$$

$$i\omega\Delta t = \ln \left(1 - \Delta t\lambda_{3,\pm} + \frac{\Delta t^2}{2}\lambda_{3,\pm}^2 - \frac{\Delta t^3}{6}\lambda_{3,\pm}^3 \right)$$

$$\omega = \frac{1}{i\Delta t} \ln \left(1 - \Delta t\lambda_{3,\pm} + \frac{\Delta t^2}{2}\lambda_{3,\pm}^2 - \frac{\Delta t^3}{6}\lambda_{3,\pm}^3 \right)$$

We note that $\lambda_{3,\pm}$ means we get both the positive and negative ω . So then our method is to form the matrix \mathbf{F}_3 and calculate the eigenvalues numerically, then use the above operations to calculate ω for the numerical method.

1.3 First Order

We repeat this method for first order

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = (\mathbf{I} - \Delta t \mathbf{F}_1) \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

lets say we have

$$\mathbf{F}_1 = \mathbf{S}_1 \begin{bmatrix} \lambda_{1,-} & 0 \\ 0 & \lambda_{1,+} \end{bmatrix} \mathbf{S}_1^{-1}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \left(\mathbf{I} - \Delta t \mathbf{S}_1 \begin{bmatrix} \lambda_{1,-} & 0 \\ 0 & \lambda_{1,+} \end{bmatrix} \mathbf{S}_1^{-1} \right) \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$\mathbf{S}_1^{-1} \begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \left(\mathbf{S}_1^{-1} - \Delta t \begin{bmatrix} \lambda_{1,-} & 0 \\ 0 & \lambda_{1,+} \end{bmatrix} \mathbf{S}_1^{-1} \right) \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$e^{i\omega\Delta t} \mathbf{S}_1^{-1} \begin{bmatrix} h \\ u \end{bmatrix}_j^n = \left(\mathbf{I} - \Delta t \begin{bmatrix} \lambda_{1,-} & 0 \\ 0 & \lambda_{1,+} \end{bmatrix} \right) \mathbf{S}_1^{-1} \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$e^{i\omega\Delta t} \mathbf{S}_1^{-1} \begin{bmatrix} h \\ u \end{bmatrix}_j^n = \begin{bmatrix} 1 - \Delta t\lambda_{1,-} & 0 \\ 0 & 1 - \Delta t\lambda_{1,+} \end{bmatrix} \mathbf{S}_1^{-1} \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

So we have

$$e^{i\omega\Delta t} = 1 - \Delta t\lambda_{1,-}$$

$$\omega = \frac{1}{i\Delta t} \ln(1 - \Delta t\lambda_{1,-})$$

1.4 Second Order

$$\begin{aligned}
\begin{bmatrix} h \\ u \end{bmatrix}_j^1 &= (\mathbf{I} - \Delta t \mathbf{F}_2) \begin{bmatrix} h \\ u \end{bmatrix}_j^n \\
\begin{bmatrix} h \\ u \end{bmatrix}_j^2 &= (\mathbf{I} - \Delta t \mathbf{F}_2) \begin{bmatrix} h \\ u \end{bmatrix}_j^1 \\
\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} &= \frac{1}{2} \left(\begin{bmatrix} h \\ u \end{bmatrix}_j^n + \begin{bmatrix} h \\ u \end{bmatrix}_j^2 \right) \\
\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} &= \frac{1}{2} \left(\begin{bmatrix} h \\ u \end{bmatrix}_j^n + (\mathbf{I} - \Delta t \mathbf{F}_2)^2 \begin{bmatrix} h \\ u \end{bmatrix}_j^n \right) \\
\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} &= \frac{1}{2} (\mathbf{I} + (\mathbf{I} - \Delta t \mathbf{F}_2)^2) \begin{bmatrix} h \\ u \end{bmatrix}_j^n \\
\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} &= \frac{1}{2} (\mathbf{I} + \mathbf{I} - 2\Delta t \mathbf{F}_2 + \Delta t^2 \mathbf{F}_2^2) \begin{bmatrix} h \\ u \end{bmatrix}_j^n \\
\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} &= \frac{1}{2} (2\mathbf{I} - 2\Delta t \mathbf{F}_2 + \Delta t^2 \mathbf{F}_2^2) \begin{bmatrix} h \\ u \end{bmatrix}_j^n
\end{aligned}$$

lets say we have

$$\mathbf{F}_2 = \mathbf{S}_2 \begin{bmatrix} \lambda_{2,-} & 0 \\ 0 & \lambda_{2,+} \end{bmatrix} \mathbf{S}_2^{-1}$$

$$\begin{aligned}
\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} &= \frac{1}{2} \left(2\mathbf{I} - 2\Delta t \mathbf{S}_2 \begin{bmatrix} \lambda_{2,-} & 0 \\ 0 & \lambda_{2,+} \end{bmatrix} \mathbf{S}_2^{-1} + \Delta t^2 \mathbf{S}_2 \begin{bmatrix} \lambda_{2,-}^2 & 0 \\ 0 & \lambda_{2,+}^2 \end{bmatrix} \mathbf{S}_2^{-1} \right) \begin{bmatrix} h \\ u \end{bmatrix}_j^n \\
\mathbf{S}_2^{-1} \begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} &= \frac{1}{2} \left(2\mathbf{S}_2^{-1} - 2\Delta t \begin{bmatrix} \lambda_{2,-} & 0 \\ 0 & \lambda_{2,+} \end{bmatrix} \mathbf{S}_2^{-1} + \Delta t^2 \begin{bmatrix} \lambda_{2,-}^2 & 0 \\ 0 & \lambda_{2,+}^2 \end{bmatrix} \mathbf{S}_2^{-1} \right) \begin{bmatrix} h \\ u \end{bmatrix}_j^n \\
\mathbf{S}_2^{-1} \begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} &= \frac{1}{2} \left(2\mathbf{S}_2^{-1} + \begin{bmatrix} \Delta t^2 \lambda_{2,-}^2 & -2\Delta t \lambda_{2,-} \\ 0 & \Delta t^2 \lambda_{2,+}^2 - 2\Delta t \lambda_{2,-} \end{bmatrix} \mathbf{S}_2^{-1} \right) \begin{bmatrix} h \\ u \end{bmatrix}_j^n
\end{aligned}$$

$$e^{i\omega\Delta t} \left(\mathbf{S}_2^{-1} \begin{bmatrix} h \\ u \end{bmatrix}_j^n \right) = \frac{1}{2} \left(2 + \begin{bmatrix} \Delta t^2 \lambda_{2,-}^2 - 2\Delta t \lambda_{2,-} & 0 \\ 0 & \Delta t^2 \lambda_{2,+}^2 - 2\Delta t \lambda_{2,-} \end{bmatrix} \right) \left(\mathbf{S}_2^{-1} \begin{bmatrix} h \\ u \end{bmatrix}_j^n \right)$$

$$e^{i\omega\Delta t} \left(\mathbf{S}_2^{-1} \begin{bmatrix} h \\ u \end{bmatrix}_j^n \right) = \frac{1}{2} \begin{bmatrix} 2 + \Delta t^2 \lambda_{2,-}^2 - 2\Delta t \lambda_{2,-} & 0 \\ 0 & 2 + \Delta t^2 \lambda_{2,+}^2 - 2\Delta t \lambda_{2,-} \end{bmatrix} \left(\mathbf{S}_2^{-1} \begin{bmatrix} h \\ u \end{bmatrix}_j^n \right)$$

So we have

$$e^{i\omega\Delta t} = 1 + \frac{1}{2} \Delta t^2 \lambda_{2,\pm}^2 - \Delta t \lambda_{2,\pm}$$

$$\omega = \frac{1}{i\Delta t} \ln \left(1 + \frac{1}{2} \Delta t^2 \lambda_{2,\pm}^2 - \Delta t \lambda_{2,\pm} \right)$$