

# 1 Linearised Equations

From Chris's paper we have, where  $h_0$  is constant and we let  $h_1 = h$  (same with velocity)

For mass:

$$\frac{\partial h}{\partial t} + h_0 \frac{\partial u}{\partial x} + u_0 \frac{\partial h}{\partial x} = 0$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (h_0 u + u_0 h) = 0$$

For momentum:

$$\frac{\partial u}{\partial t} + g \frac{\partial h}{\partial x} + u_0 \frac{\partial u}{\partial x} - \frac{h_0^2}{3} \left( u_0 \frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x^2 \partial t} \right) = 0$$

# 2 Actual Work

We do a Von Neumann stability analysis, we assume two different errors for  $h$  and  $u$  otherwise everything else is the same. We jsut run the errors of known structure through the method, for convenience we know use  $h$  and  $u$  to refer to their respective errors, and we use  $q$  top refer to a general quantity ( $k$ , a different for  $u$  and  $l$  and  $b$  for  $h$ )

$$q_{j+1}^n = e^{ik\Delta x} q_j^n$$

$$q_{j+2}^n = e^{2ik\Delta x} q_j^n$$

$$q_{j-1}^n = e^{-ik\Delta x} q_j^n$$

$$q_{j-2}^n = e^{-2ik\Delta x} q_j^n$$

$$\frac{\partial q}{\partial x} = \frac{q_{j+1}^n - q_{j-1}^n}{2\Delta x} = \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x} q_j^n = \frac{i \sin(k\Delta x)}{\Delta x} q_j^n$$

S

$$\frac{\partial^2 q}{\partial x^2} = \frac{q_{j+1}^n - 2q_j^n + q_{j-1}^n}{\Delta x^2} = \frac{e^{ik\Delta x} + e^{-ik\Delta x} - 2}{\Delta x^2} q_j^n = \frac{2 \cos(k\Delta x) - 2}{\Delta x^2} q_j^n$$

$$= -\frac{4}{\Delta x^2} \sin^2 \left( \frac{k\Delta x}{2} \right) q_j^n$$

S

$$\begin{aligned} \frac{\partial^3 q}{\partial x^3} &= \frac{-q_{j-2}^n + 2q_{j-1}^n - 2q_{j+1}^n + q_{j+2}^n}{2\Delta x^3} = \frac{-e^{-2ik\Delta x} + 2e^{-ik\Delta x} - 2e^{ik\Delta x} + e^{2ik\Delta x}}{2\Delta x^3} q_j^n \\ &= \frac{e^{2ik\Delta x} - e^{-2ik\Delta x} - 2e^{ik\Delta x} + 2e^{-ik\Delta x}}{2\Delta x^3} q_j^n \\ &= \frac{i \sin(2k\Delta x) - 2i \sin(k\Delta x)}{\Delta x^3} q_j^n \\ &= i \frac{2 \sin(k\Delta x) \cos(k\Delta x) - 2 \sin(k\Delta x)}{\Delta x^3} q_j^n \\ &= 2i \sin(k\Delta x) \frac{\cos(k\Delta x) - 1}{\Delta x^3} q_j^n \\ &= 2i \sin(k\Delta x) \frac{-2 \sin^2 \left( \frac{k\Delta x}{2} \right)}{\Delta x^3} q_j^n \\ &= -4i \sin(k\Delta x) \frac{\sin^2 \left( \frac{k\Delta x}{2} \right)}{\Delta x^3} q_j^n \end{aligned}$$

## 2.1 FD for u

$$\begin{aligned} \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + g \frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x} + u_0 \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \\ - \frac{h_0^2}{3} \left( u_0 \frac{-u_{j-2}^n + 2u_{j-1}^n - 2u_{j+1}^n + u_{j+2}^n}{2\Delta x^3} \right) \\ - \frac{h_0^2}{3} \frac{\partial \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}}{\partial t} \\ = 0 \quad (1) \end{aligned}$$

$$\begin{aligned}
& \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + g \frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x} + u_0 \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \\
& - \frac{h_0^2}{3} \left( u_0 \frac{-u_{j-2}^n + 2u_{j-1}^n - 2u_{j+1}^n + u_{j+2}^n}{2\Delta x^3} \right) \\
& - \frac{h_0^2}{3} \frac{\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} - \frac{u_{j+1}^{n-1} - 2u_j^{n-1} + u_{j-1}^{n-1}}{\Delta x^2}}{2\Delta t} \\
& = 0 \quad (2)
\end{aligned}$$

$$\begin{aligned}
& \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + g \frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x} + u_0 \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \\
& - \frac{h_0^2}{3} \left( u_0 \frac{-u_{j-2}^n + 2u_{j-1}^n - 2u_{j+1}^n + u_{j+2}^n}{2\Delta x^3} \right) \\
& - \frac{h_0^2}{3} \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1} - u_{j+1}^{n-1} + 2u_j^{n-1} - u_{j-1}^{n-1}}{2\Delta x^2 \Delta t} \\
& = 0 \quad (3)
\end{aligned}$$

$$\begin{aligned}
& \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + g \frac{i \sin(l\Delta x)}{\Delta x} h_j^n + u_0 \frac{i \sin(k\Delta x)}{\Delta x} u_j^n \\
& + \frac{h_0^2}{3} \left( 4iu_0 \sin(k\Delta x) \frac{\sin^2\left(\frac{k\Delta x}{2}\right)}{\Delta x^3} \right) u_j^n \\
& - \frac{h_0^2}{6\Delta t} \left( \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} - \frac{u_{j+1}^{n-1} - 2u_j^{n-1} + u_{j-1}^{n-1}}{\Delta x^2} \right) \\
& = 0 \quad (4)
\end{aligned}$$

$$\begin{aligned}
& \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + g \frac{i \sin(l\Delta x)}{\Delta x} h_j^n + u_0 \frac{i \sin(k\Delta x)}{\Delta x} u_j^n \\
& \quad + \frac{h_0^2}{3} \left( 4iu_0 \sin(k\Delta x) \frac{\sin^2\left(\frac{k\Delta x}{2}\right)}{\Delta x^3} \right) u_j^n \\
& \quad - \frac{h_0^2}{6\Delta t} \left( -\frac{4}{\Delta x^2} \sin^2\left(\frac{k\Delta x}{2}\right) u_j^{n+1} - \frac{4}{\Delta x^2} \sin^2\left(\frac{k\Delta x}{2}\right) u_j^{n-1} \right) \\
& \hspace{15em} = 0 \quad (5)
\end{aligned}$$

$$\begin{aligned}
& \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} - \frac{h_0^2}{6\Delta t} \left( -\frac{4}{\Delta x^2} \sin^2\left(\frac{k\Delta x}{2}\right) u_j^{n+1} - \frac{4}{\Delta x^2} \sin^2\left(\frac{k\Delta x}{2}\right) u_j^{n-1} \right) \\
& = + \frac{h_0^2}{3} \left( 4iu_0 \sin(k\Delta x) \frac{\sin^2\left(\frac{k\Delta x}{2}\right)}{\Delta x^3} \right) u_j^n - g \frac{i \sin(l\Delta x)}{\Delta x} h_j^n + u_0 \frac{i \sin(k\Delta x)}{\Delta x} u_j^n \\
& \hspace{15em} (6)
\end{aligned}$$

$$\begin{aligned}
& \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} - \frac{h_0^2}{6\Delta t} \left( -\frac{4}{\Delta x^2} \sin^2\left(\frac{k\Delta x}{2}\right) u_j^{n+1} - \frac{4}{\Delta x^2} \sin^2\left(\frac{k\Delta x}{2}\right) u_j^{n-1} \right) \\
& = \left[ \frac{h_0^2}{3} \left( 4iu_0 \sin(k\Delta x) \frac{\sin^2\left(\frac{k\Delta x}{2}\right)}{\Delta x^3} \right) + u_0 \frac{i \sin(k\Delta x)}{\Delta x} \right] u_j^n - g \frac{i \sin(l\Delta x)}{\Delta x} h_j^n \\
& \hspace{15em} (7)
\end{aligned}$$

$$\begin{aligned}
& u_j^{n+1} - u_j^{n-1} + \frac{h_0^2}{3} \left( \frac{4}{\Delta x^2} \sin^2\left(\frac{k\Delta x}{2}\right) u_j^{n+1} - \frac{4}{\Delta x^2} \sin^2\left(\frac{k\Delta x}{2}\right) u_j^{n-1} \right) \\
& = 2\Delta t \left( u_0 \frac{i \sin(k\Delta x)}{\Delta x} \left[ \frac{4h_0^2}{3\Delta x^2} \left( \sin^2\left(\frac{k\Delta x}{2}\right) \right) + 1 \right] u_j^n - g \frac{i \sin(l\Delta x)}{\Delta x} h_j^n \right) \\
& \hspace{15em} (8)
\end{aligned}$$

$$\begin{aligned}
u_j^{n+1} & \left[ 1 + \frac{h_0^2}{3} \frac{4}{\Delta x^2} \sin^2 \left( \frac{k\Delta x}{2} \right) \right] \\
& = u_j^{n-1} \left[ 1 + \frac{h_0^2}{3} \frac{4}{\Delta x^2} \sin^2 \left( \frac{k\Delta x}{2} \right) \right] + \\
2\Delta t & \left( u_0 \frac{i \sin(k\Delta x)}{\Delta x} \left[ \frac{4h_0^2}{3\Delta x^2} \left( \sin^2 \left( \frac{k\Delta x}{2} \right) \right) + 1 \right] u_j^n - g \frac{i \sin(l\Delta x)}{\Delta x} h_j^n \right) \quad (9)
\end{aligned}$$

$$\begin{aligned}
u_j^{n+1} & \\
& = u_j^{n-1} \\
& + 2\Delta t \left( u_0 \frac{i \sin(k\Delta x)}{\Delta x} u_j^n - g \frac{1}{\left[ 1 + \frac{h_0^2}{3} \frac{4}{\Delta x^2} \sin^2 \left( \frac{k\Delta x}{2} \right) \right]} \frac{i \sin(l\Delta x)}{\Delta x} h_j^n \right) \quad (10)
\end{aligned}$$

## 2.2 FD for h

$$\begin{aligned}
\frac{h_j^{n+1} - h_j^{n-1}}{2\Delta t} + h_0 \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + u_0 \frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x} & = 0 \\
\frac{h_j^{n+1} - h_j^{n-1}}{2\Delta t} + h_0 \frac{i \sin(k\Delta x)}{\Delta x} u_j^n + u_0 \frac{i \sin(l\Delta x)}{\Delta x} h_j^n & = 0 \\
h_j^{n+1} - h_j^{n-1} = -2\Delta t \left[ h_0 \frac{i \sin(k\Delta x)}{\Delta x} u_j^n + u_0 \frac{i \sin(l\Delta x)}{\Delta x} h_j^n \right] & = 0 \\
h_j^{n+1} = h_j^{n-1} - \frac{2i\Delta t}{\Delta x} [h_0 \sin(k\Delta x) u_j^n + u_0 \sin(l\Delta x) h_j^n] & = 0
\end{aligned}$$

### 2.2.1 Together

We can formulate these schemes together to get

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \begin{bmatrix} h \\ u \end{bmatrix}_j^{n-1} + \begin{bmatrix} -\frac{2i\Delta t}{\Delta x} u_0 \sin(l\Delta x) & -\frac{2i\Delta t}{\Delta x} h_0 \sin(k\Delta x) \\ -\frac{\frac{h_0^2}{3} \frac{4}{\Delta x^2} \sin^2(\frac{k\Delta x}{2})}{1 + \frac{h_0^2}{3} \frac{4}{\Delta x^2} \sin^2(\frac{k\Delta x}{2})} g \frac{i \sin(l\Delta x)}{\Delta x} & \frac{2i\Delta t}{\Delta x} u_0 \sin(k\Delta x) \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^n \quad (11)$$

Also we now have  $k = l$

$$\text{Let } A = \begin{bmatrix} -\frac{2i\Delta t}{\Delta x} u_0 \sin(k\Delta x) & -\frac{2i\Delta t}{\Delta x} h_0 \sin(k\Delta x) \\ -\frac{h_0^2}{1 + \frac{4}{3} \frac{\Delta t^2}{\Delta x^2} \sin^2(\frac{k\Delta x}{2})} g \frac{i \sin(k\Delta x)}{\Delta x} & \frac{2i\Delta t}{\Delta x} u_0 \sin(k\Delta x) \end{bmatrix}$$

Then

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \begin{bmatrix} h \\ u \end{bmatrix}_j^{n-1} + A \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$\text{Assume } \begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = G \begin{bmatrix} h \\ u \end{bmatrix}_j^n \text{ So } \begin{bmatrix} h \\ u \end{bmatrix}_j^{n-1} = \frac{1}{G} \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

Then

$$\begin{aligned} G \begin{bmatrix} h \\ u \end{bmatrix}_j^n &= \frac{1}{G} \begin{bmatrix} h \\ u \end{bmatrix}_j^n + A \begin{bmatrix} h \\ u \end{bmatrix}_j^n \\ 0 &= \left[ \frac{1}{G} I - GI + A \right] \begin{bmatrix} h \\ u \end{bmatrix}_j^n \\ 0 &= \left[ A - \frac{G^2 - 1}{G} I \right] \begin{bmatrix} h \\ u \end{bmatrix}_j^n \end{aligned}$$

So we have that  $\frac{G^2 - 1}{G} = \lambda_{A,1}, \lambda_{A,2}$

$$G^2 - 1 = \lambda_{A,1} G$$

$$G^2 - \lambda_{A,1} G - 1 = 0$$

Thus

$$G = \frac{1}{2} \left( \lambda_{A,1} \pm \sqrt{\lambda_{A,1}^2 + 4} \right)$$

Also

$$G = \frac{1}{2} \left( \lambda_{A,2} \pm \sqrt{\lambda_{A,2}^2 + 4} \right)$$

Want  $\max \{|G|\} \leq 1$

This is the equivalent to writing it like so:

$$\begin{bmatrix} h_j^{n+1} \\ u_j^{n+1} \\ h_j^n \\ u_j^n \end{bmatrix} = \begin{bmatrix} -\frac{2i\Delta t}{\Delta x} u_0 \sin(k\Delta x) & -\frac{2i\Delta t}{\Delta x} h_0 \sin(k\Delta x) & 1 & 0 \\ -\frac{2\Delta t}{1 + \frac{h_0^2}{3} \frac{4}{\Delta x^2} \sin^2(\frac{k\Delta x}{2})} g \frac{i \sin(k\Delta x)}{\Delta x} & \frac{2i\Delta t}{\Delta x} u_0 \sin(k\Delta x) & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_j^n \\ u_j^n \\ h_j^{n-1} \\ u_j^{n-1} \end{bmatrix} \quad (12)$$

and having the eigenvalues of this less than 1 because if we let

$$A = \begin{bmatrix} -\frac{2i\Delta t}{\Delta x} u_0 \sin(k\Delta x) & -\frac{2i\Delta t}{\Delta x} h_0 \sin(k\Delta x) & 1 & 0 \\ -\frac{2\Delta t}{1 + \frac{h_0^2}{3} \frac{4}{\Delta x^2} \sin^2(\frac{k\Delta x}{2})} g \frac{i \sin(k\Delta x)}{\Delta x} & \frac{2i\Delta t}{\Delta x} u_0 \sin(k\Delta x) & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and we have a growth factor  $G$  then

$$G \begin{bmatrix} h_j^n \\ u_j^n \\ h_j^{n-1} \\ u_j^{n-1} \end{bmatrix} = A \begin{bmatrix} h_j^n \\ u_j^n \\ h_j^{n-1} \\ u_j^{n-1} \end{bmatrix}$$

Then

$$(A - GI) \begin{bmatrix} h_j^n \\ u_j^n \\ h_j^{n-1} \\ u_j^{n-1} \end{bmatrix} = 0$$

So  $G$  are the eigenvalues of  $A$  and so as long as  $\rho(A) \leq 1$  then  $G \leq 1$  as well.

### 3 Lax Wendroff Nonlinear

We have

$$\begin{aligned}
u_j^{n+1} &= u_j^{n-1} \\
&+ 2\Delta t \left( u_0 \frac{i \sin(k\Delta x)}{\Delta x} u_j^n - g \frac{1}{\left[1 + \frac{h_0^2}{3} \frac{4}{\Delta x^2} \sin^2\left(\frac{k\Delta x}{2}\right)\right]} \frac{i \sin(k\Delta x)}{\Delta x} h_j^n \right) \quad (13)
\end{aligned}$$

lets define

$$\begin{aligned}
c &= -g \frac{2\Delta t}{\left[1 + \frac{h_0^2}{3} \frac{4}{\Delta x^2} \sin^2\left(\frac{k\Delta x}{2}\right)\right]} \frac{i \sin(k\Delta x)}{\Delta x} \\
d &= u_0 \frac{2i\Delta t \sin(k\Delta x)}{\Delta x}
\end{aligned}$$

Then

$$u_j^{n+1} = u_j^{n-1} + du_j^n + ch_j^n$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (h_0 u + u_0 h) = 0$$

Using the two step richtemyer LW method

$$h_{j+1/2}^{n+1/2} = \frac{1}{2} (h_{j+1}^n + h_j^n) - \frac{\Delta t}{2\Delta x} [H(u_{j+1}^n - u_j^n) + U(h_{j+1}^n - h_j^n)]$$

$$h_{j-1/2}^{n+1/2} = \frac{1}{2} (h_j^n + h_{j-1}^n) - \frac{\Delta t}{2\Delta x} [H(u_j^n - u_{j-1}^n) + U(h_j^n - h_{j-1}^n)]$$

$$h_j^{n+1} = h_j^n - \frac{\Delta t}{\Delta x} [H(u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2}) + U(h_{j+1/2}^{n+1/2} - h_{j-1/2}^{n+1/2})]$$

We calculate  $u_{j+1/2}^{n+1/2}$  by taking the average over the updated u and the current u. So we have

$$u_{j+1/2}^{n+1/2} = \frac{u_{j+1}^{n+1} + u_j^{n+1} + u_{j+1}^n + u_j^n}{4}$$



$$u_{j-1/2}^{n+1/2} = \frac{u_j^{n+1} + u_{j-1}^{n+1} + u_j^n + u_{j-1}^n}{4}$$

Using Fourier Nodes

$$u_{j+1/2}^{n+1/2} = \frac{e^{ik\Delta x} u_j^{n+1} + u_j^{n+1} + e^{ik\Delta x} u_j^n + u_j^n}{4}$$

$$u_{j+1/2}^{n+1/2} = \frac{(e^{ik\Delta x} + 1)(u_j^{n+1} + u_j^n)}{4}$$

$$u_{j-1/2}^{n+1/2} = \frac{(e^{-ik\Delta x} + 1)(u_j^{n+1} + u_j^n)}{4}$$

$$u_{j-1/2}^{n+1/2} = e^{-ik\Delta x} \frac{(1 + e^{ik\Delta x})(u_j^{n+1} + u_j^n)}{4}$$

$$u_{j-1/2}^{n+1/2} = e^{-ik\Delta x} u_{j+1/2}^{n+1/2}$$

$$h_{j+1/2}^{n+1/2} = \frac{1}{2} (e^{ik\Delta x} + 1) h_j^n - \frac{\Delta t}{2\Delta x} [H (e^{ik\Delta x} - 1) u_j^n + U (e^{ik\Delta x} - 1) h_j^n]$$

$$h_{j-1/2}^{n+1/2} = \frac{1}{2} (1 + e^{-ik\Delta x}) h_j^n - \frac{\Delta t}{2\Delta x} [H (1 - e^{-ik\Delta x}) u_j^n + U (1 - e^{-ik\Delta x}) h_j^n]$$

$$h_{j-1/2}^{n+1/2} = e^{-ik\Delta x} \left[ \frac{1}{2} (e^{ik\Delta x} + 1) h_j^n - \frac{\Delta t}{2\Delta x} [H (e^{ik\Delta x} - 1) u_j^n + U (e^{ik\Delta x} - 1) h_j^n] \right]$$

$$h_{j-1/2}^{n+1/2} = e^{-ik\Delta x} h_{j+1/2}^{n+1/2}$$

$$h_{j+1/2}^{n+1/2} = \left( \frac{1}{2} (e^{ik\Delta x} + 1) - \frac{\Delta t}{2\Delta x} U (e^{ik\Delta x} - 1) \right) h_j^n - \left( \frac{\Delta t}{2\Delta x} H (e^{ik\Delta x} - 1) \right) u_j^n$$

$$u_{j+1/2}^{n+1/2} = \frac{(e^{ik\Delta x} + 1)(u_j^{n+1} + u_j^n)}{4}$$

$$u_{j+1/2}^{n+1/2} = \frac{(e^{ik\Delta x} + 1)(u_j^{n-1} + du_j^n + ch_j^n + u_j^n)}{4}$$

$$u_{j+1/2}^{n+1/2} = \frac{e^{ik\Delta x} + 1}{4} (u_j^{n-1} + (d+1)u_j^n + ch_j^n)$$

So we have

$$h_j^{n+1} = h_j^n - \frac{\Delta t}{\Delta x} \left[ H(1 - e^{-ik\Delta x})u_{j+1/2}^{n+1/2} + U(1 - e^{-ik\Delta x})h_{j+1/2}^{n+1/2} \right]$$

$$\begin{aligned} h_j^{n+1} = h_j^n - \frac{\Delta t}{\Delta x} H(1 - e^{-ik\Delta x}) \frac{e^{ik\Delta x} + 1}{4} (u_j^{n-1} + (d+1)u_j^n + ch_j^n) \\ - \frac{\Delta t}{\Delta x} U(1 - e^{-ik\Delta x}) \left[ \left( \frac{1}{2} (e^{ik\Delta x} + 1) - \frac{\Delta t}{2\Delta x} U(e^{ik\Delta x} - 1) \right) h_j^n - \left( \frac{\Delta t}{2\Delta x} H(e^{ik\Delta x} - 1) \right) u_j^n \right] \end{aligned} \quad (14)$$

$$\begin{aligned} h_j^{n+1} = \\ \left[ 1 - \frac{\Delta t}{\Delta x} cH \frac{2i \sin(k\Delta x)}{4} - \frac{\Delta t}{\Delta x} U(1 - e^{-ik\Delta x}) \left( \frac{1}{2} (e^{ik\Delta x} + 1) - \frac{\Delta t}{2\Delta x} U(e^{ik\Delta x} - 1) \right) \right] h_j^n \\ - \frac{\Delta t}{\Delta x} H(1 - e^{-ik\Delta x}) \frac{e^{ik\Delta x} + 1}{4} (u_j^{n-1} + (d+1)u_j^n) \\ - \frac{\Delta t}{\Delta x} U(1 - e^{-ik\Delta x}) \left[ - \left( \frac{\Delta t}{2\Delta x} H(e^{ik\Delta x} - 1) \right) u_j^n \right] \end{aligned} \quad (15)$$

$$\begin{aligned} h_j^{n+1} = \\ \left[ 1 - \frac{\Delta t}{\Delta x} cH \frac{i \sin(k\Delta x)}{2} - \frac{\Delta t}{\Delta x} U \left( \frac{1}{2} (2i \sin(k\Delta x)) - \frac{\Delta t}{2\Delta x} U(2 \cos(ik\Delta x) - 2) \right) \right] h_j^n \\ - \frac{\Delta t}{\Delta x} H \frac{i \sin(k\Delta x)}{2} (u_j^{n-1} + (d+1)u_j^n) \\ - \frac{\Delta t}{\Delta x} U \left[ - \left( \frac{\Delta t}{2\Delta x} H(2 \cos(ik\Delta x) - 2) \right) u_j^n \right] \end{aligned} \quad (16)$$

$$\begin{aligned}
h_j^{n+1} = & \left[ 1 - \frac{\Delta t}{\Delta x} c H \frac{i \sin(k \Delta x)}{2} - \frac{\Delta t}{\Delta x} U \left( (i \sin(k \Delta x)) - \frac{\Delta t}{\Delta x} U (\cos(ik \Delta x) - 1) \right) \right] h_j^n \\
& - \frac{\Delta t}{\Delta x} H \frac{i \sin(k \Delta x)}{2} (u_j^{n-1} + (d+1)u_j^n) \\
& - \frac{\Delta t}{\Delta x} U \left[ - \left( \frac{\Delta t}{\Delta x} H (\cos(ik \Delta x) - 1) \right) u_j^n \right] \quad (17)
\end{aligned}$$

$$\begin{aligned}
h_j^{n+1} = & \left[ 1 - \frac{\Delta t}{\Delta x} c H \frac{i \sin(k \Delta x)}{2} - \frac{\Delta t}{\Delta x} U \left( (i \sin(k \Delta x)) - \frac{\Delta t}{\Delta x} U (\cos(ik \Delta x) - 1) \right) \right] h_j^n \\
& - \frac{\Delta t}{\Delta x} H \frac{i \sin(k \Delta x)}{2} u_j^{n-1} \\
& - \frac{\Delta t}{\Delta x} H \frac{i \sin(k \Delta x)}{2} (d+1)u_j^n - \frac{\Delta t}{\Delta x} U \left[ - \left( \frac{\Delta t}{\Delta x} H (\cos(ik \Delta x) - 1) \right) u_j^n \right] \quad (18)
\end{aligned}$$

$$\begin{aligned}
h_j^{n+1} = & \left[ 1 - \frac{\Delta t}{\Delta x} c H \frac{i \sin(k \Delta x)}{2} - \frac{\Delta t}{\Delta x} U \left( (i \sin(k \Delta x)) - \frac{\Delta t}{\Delta x} U (\cos(ik \Delta x) - 1) \right) \right] h_j^n \\
& - \frac{\Delta t}{\Delta x} H \frac{i \sin(k \Delta x)}{2} u_j^{n-1} \\
& - \frac{\Delta t}{\Delta x} \left[ H \frac{i \sin(k \Delta x)}{2} (d+1) + U \left[ - \left( \frac{\Delta t}{\Delta x} H (\cos(ik \Delta x) - 1) \right) \right] \right] u_j^n \quad (19)
\end{aligned}$$

$$\begin{aligned}
h_j^{n+1} = & \left[ 1 - \frac{\Delta t}{\Delta x} c H \frac{i \sin(k \Delta x)}{2} - \frac{\Delta t}{\Delta x} U \left( (i \sin(k \Delta x)) - \frac{\Delta t}{\Delta x} U (\cos(ik \Delta x) - 1) \right) \right] h_j^n \\
& - \frac{\Delta t}{\Delta x} H \frac{i \sin(k \Delta x)}{2} u_j^{n-1} \\
& - \frac{\Delta t}{\Delta x} \left[ H \frac{i \sin(k \Delta x)}{2} (d+1) - U \left( \frac{\Delta t}{\Delta x} H (\cos(ik \Delta x) - 1) \right) \right] u_j^n \quad (20)
\end{aligned}$$

defining  $a = \left[ 1 - \frac{\Delta t}{\Delta x} c H \frac{i \sin(k \Delta x)}{2} - \frac{\Delta t}{\Delta x} U ((i \sin(k \Delta x)) - \frac{\Delta t}{\Delta x} U (\cos(ik \Delta x) - 1)) \right]$   
 $b = -\frac{\Delta t}{\Delta x} \left[ H \frac{i \sin(k \Delta x)}{2} (d + 1) - U \left( \frac{\Delta t}{\Delta x} H (\cos(ik \Delta x) - 1) \right) \right]$   
 $e = -\frac{\Delta t}{\Delta x} H \frac{i \sin(k \Delta x)}{2}$   
Then we have

$$\begin{bmatrix} h_j^{n+1} \\ u_j^{n+1} \\ h_j^n \\ u_j^n \end{bmatrix} = \begin{bmatrix} a & b & 0 & e \\ c & d & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_j^n \\ u_j^n \\ h_j^{n-1} \\ u_j^{n-1} \end{bmatrix}$$

equivalently we can write this as

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \begin{bmatrix} 0 & e \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^{n-1} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

Taking out the growth factors we have

$$G \begin{bmatrix} h \\ u \end{bmatrix}_j^n = \frac{1}{G} \begin{bmatrix} 0 & e \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^n + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$\left( \begin{bmatrix} aG & bG + e \\ cG & dG + 1 \end{bmatrix} - G^2 I \right) \begin{bmatrix} h \\ u \end{bmatrix}_j^n = 0$$

So  $G^2$  is the eigenvalues of  $\begin{bmatrix} aG & bG + e \\ cG & dG + 1 \end{bmatrix}$  we have that the eigenvalues of this matrix are

$$\lambda = \frac{1}{2} \left( \pm \sqrt{(d-a)^2 G^2 + 4bcG^2 + 2(d-a)G + 4ceG + 1 + (d+a)G + 1} \right)$$

Therefore

$$G^2 = \frac{1}{2} \left( \pm \sqrt{(d-a)^2 G^2 + 4bcG^2 + 2(d-a)G + 4ceG + 1 + (d+a)G + 1} \right)$$

$$2G^2 - (d+a)G - 1 = \pm \sqrt{(d-a)^2 G^2 + 4bcG^2 + 2(d-a)G + 4ceG + 1}$$

$$(2G^2 - (d + a)G - 1)^2 = (d - a)^2 G^2 + 4bcG^2 + 2(d - a)G + 4ceG + 1$$

$$4G^4 - 4(a + d)G^3 + (a + d)^2 G^2 - 4G^2 + 2(a + d)G + 1 = (d - a)^2 G^2 + 4bcG^2 + 2(d - a)G + 4ceG + 1$$

$$4G^4 - 4(a + d)G^3 + 4adG^2 - 4G^2 + 2(a)G = +4bcG^2 + 2(-a)G + 4ceG$$

$$4G^4 - 4(a + d)G^3 + 4(ad - bc)G^2 - 4G^2 + 4(a - ce)G = 0$$

$$G^4 - (a + d)G^3 + (ad - bc)G^2 - G^2 + (a - ce)G = 0$$

$$G [G^3 - (a + d)G^2 + (ad - bc - 1)G + (a - ce)] = 0$$

## 4 LWL

we have

$$u_j^{n+1} = u_j^{n-1} + 2\Delta t \left( u_0 \frac{i \sin(k\Delta x)}{\Delta x} u_j^n - g \frac{1}{\left[1 + \frac{h_0^2}{3} \frac{4}{\Delta x^2} \sin^2\left(\frac{k\Delta x}{2}\right)\right]} \frac{i \sin(k\Delta x)}{\Delta x} h_j^n \right) \quad (21)$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (h_0 u + u_0 h) = 0$$

Since  $h$  equation is linear we can use the linear LW method

$$h_j^{n+1} = h_j^n - \frac{\Delta t}{2\Delta x} [H (u_{j+1}^n - u_{j-1}^n) + U (h_{j+1}^n - h_{j-1}^n)] + \frac{\Delta t^2}{2\Delta x^2} [H^2 (u_{j+1}^n - 2u_j^n + u_{j-1}^n) + U^2 (h_{j+1}^n - 2h_j^n + h_{j-1}^n)]$$

$$h_j^{n+1} = h_j^n - \frac{\Delta t}{2\Delta x} [H (e^{ik\Delta x} - e^{-ik\Delta x}) u_j^n + U (e^{ik\Delta x} - e^{-ik\Delta x}) h_j^n] + \frac{\Delta t^2}{2\Delta x^2} [H^2 (e^{ik\Delta x} - 2 + e^{-ik\Delta x}) u_j^n + U^2 (e^{ik\Delta x} - 2 + e^{-ik\Delta x}) h_j^n]$$

$$h_j^{n+1} = h_j^n - \frac{\Delta t}{\Delta x} [H i \sin(k\Delta x) u_j^n + U i \sin(k\Delta x) h_j^n] + \frac{\Delta t^2}{\Delta x^2} [H^2 (\cos(k\Delta x) - 1) u_j^n + U^2 (\cos(k\Delta x) - 1) h_j^n]$$

$$h_j^{n+1} = h_j^n - \frac{\Delta t}{\Delta x} [H i \sin(k\Delta x) u_j^n + U i \sin(k\Delta x) h_j^n] - \frac{\Delta t^2}{\Delta x^2} \left[ 2H^2 \sin^2\left(\frac{k\Delta x}{2}\right) u_j^n + 2U^2 \sin^2\left(\frac{k\Delta x}{2}\right) h_j^n \right]$$

$$h_j^{n+1} = \left[ 1 - \frac{\Delta t}{\Delta x} U i \sin(k\Delta x) + \frac{\Delta t^2}{\Delta x^2} 2U^2 \sin^2\left(\frac{k\Delta x}{2}\right) \right] h_j^n - \left[ -\frac{\Delta t}{\Delta x} H i \sin(k\Delta x) + \frac{\Delta t^2}{\Delta x^2} 2H^2 \sin^2\left(\frac{k\Delta x}{2}\right) \right] u_j^n$$

So we have

$$\begin{aligned} \begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^{n-1} \\ + \begin{bmatrix} 1 - \frac{\Delta t}{\Delta x} U i \sin(k\Delta x) - \frac{\Delta t^2}{\Delta x^2} 2U^2 \sin^2\left(\frac{k\Delta x}{2}\right) & -\frac{\Delta t}{\Delta x} H i \sin(k\Delta x) - \frac{\Delta t^2}{\Delta x^2} 2H^2 \sin^2\left(\frac{k\Delta x}{2}\right) \\ -\frac{2\Delta t}{1 + \frac{h_0^2}{3} \frac{4}{\Delta x^2} \sin^2\left(\frac{k\Delta x}{2}\right)} g \frac{i \sin(l\Delta x)}{\Delta x} & \frac{2i\Delta t}{\Delta x} u_0 \sin(k\Delta x) \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^n \end{aligned} \quad (22)$$

Lets define

$$\begin{aligned} a &= 1 - \frac{\Delta t}{\Delta x} U i \sin(k\Delta x) - \frac{\Delta t^2}{\Delta x^2} 2U^2 \sin^2\left(\frac{k\Delta x}{2}\right) \\ b &= -\frac{\Delta t}{\Delta x} H i \sin(k\Delta x) - \frac{\Delta t^2}{\Delta x^2} 2H^2 \sin^2\left(\frac{k\Delta x}{2}\right) \\ c &= -\frac{2\Delta t}{1 + \frac{h_0^2}{3} \frac{4}{\Delta x^2} \sin^2\left(\frac{k\Delta x}{2}\right)} g \frac{i \sin(l\Delta x)}{\Delta x} \\ d &= \frac{2i\Delta t}{\Delta x} u_0 \sin(k\Delta x) \end{aligned}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^{n-1} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^n \quad (23)$$

As before we have a growth factor  $G$

$$G \begin{bmatrix} h \\ u \end{bmatrix}_j^n = \frac{1}{G} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^n + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^n \quad (24)$$

$$\left(-GI + \frac{1}{G} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \begin{bmatrix} h \\ u \end{bmatrix}_j^n = 0 \quad (25)$$

$$\left(-G^2I + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} aG & bG \\ cG & dG \end{bmatrix}\right) \begin{bmatrix} h \\ u \end{bmatrix}_j^n = 0 \quad (26)$$

$$\left(\begin{bmatrix} aG & bG \\ cG & dG + 1 \end{bmatrix} - G^2I\right) \begin{bmatrix} h \\ u \end{bmatrix}_j^n = 0 \quad (27)$$

So  $G^2$  are the eigenvalues of  $\begin{bmatrix} aG & bG \\ cG & dG + 1 \end{bmatrix}$

We have

$$\lambda_1 = \frac{1}{2} \left( -\sqrt{a^2G^2 - 2adG^2 - 2aG + 4bcG^2 + d^2G^2 + 2dG + 1} + aG + dG + 1 \right)$$

$$\lambda_2 = \frac{1}{2} \left( \sqrt{a^2G^2 - 2adG^2 - 2aG + 4bcG^2 + d^2G^2 + 2dG + 1} + aG + dG + 1 \right)$$

$$G^2 = \frac{1}{2} \left( \pm \sqrt{(a^2 - 2ad + d^2 + 4bc)G^2 + (2d - 2a)G + 1} + aG + dG + 1 \right)$$

$$2G^2 = \pm \sqrt{(a^2 - 2ad + d^2 + 4bc)G^2 + (2d - 2a)G + 1} + aG + dG + 1$$

$$2G^2 - (a + d)G - 1 = \pm \sqrt{(a^2 - 2ad + d^2 + 4bc)G^2 + (2d - 2a)G + 1}$$

$$(2G^2 - (a + d)G - 1)^2 = (a^2 - 2ad + d^2 + 4bc)G^2 + (2d - 2a)G + 1$$



$$\begin{aligned}
& 4G^4 - 4aG^3 - 4dG^3 + a^2G^2 + 2adG^2 + d^2G^2 - 4G^2 + 2aG + 2dG + 1 \\
&= (a^2 - 2ad + d^2 + 4bc) G^2 + (2d - 2a) G + 1
\end{aligned}$$

$$4G^4 - 4(a + d) G^3 + 4adG^2 - 4G^2 + 2aG = 4bcG^2 - 2aG$$

$$4G^4 - 4(a + d) G^3 + 4(ad - bc - 1)G^2 + 4aG = 0$$

$$G^4 - (a + d) G^3 + (ad - bc - 1)G^2 + aG = 0$$

$$G (G^3 - (a + d) G^2 + (ad - bc - 1)G + a) = 0$$