1 Elliptic Equation

The linearised elliptic equation is

$$G = Hu - \frac{H^3}{3}u_{xx}$$

Want to find out the FEM approximation to this G' such that

$$G' = \mathcal{G}_{FE_1}u$$

for P^1 FEM

$$G' = \mathcal{G}_{FE_2}u$$

for P^2 FEM.

To do so we begin by first multiplying by an arbitrary test function v so that

$$Gv = Huv - \frac{H^3}{3}u_{xx}v$$

and then we integrate over the entire domain to get

$$\int_\Omega Gvdx = \int_\Omega Huvdx - \int_\Omega \frac{H^3}{3}u_{xx}vdx$$

for all v

We then make use of integration by parts, with Dirchlet boundaries to get

$$\int_\Omega Gvdx = \int_\Omega Huvdx + \int_\Omega \frac{H^3}{3}u_x v_x dx$$

Our FVM discretisation already has a natrual structure with linear functions intervals of $[x_{j-1/2}, x_{j+1/2}]$, to achieve this in P^1 we have our nodes at the boundaries, thus

So we can reformulate this as

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Gvdx = \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Huvdx + \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{H^3}{3}u_x v_x dx$$

or more aptly

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Gvdx - \int_{x_{j-1/2}}^{x_{j+3/2}} Huvdx - \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{H^3}{3} u_x v_x dx = 0$$

for all v

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Gvdx - H \int_{x_{j-1/2}}^{x_{j+3/2}} uvdx - \frac{H^3}{3} \int_{x_{j-1/2}}^{x_{j+3/2}} u_x v_x dx = 0$$

2 P1 FEM

We have the basis functions for linear elements:

$$v_{j+1/2} = \begin{cases} \frac{x - x_{j-1/2}}{\Delta x} & x_{j-1/2} \leq x < x_{j+1/2} \\ 1 - \frac{x - x_{j+1/2}}{\Delta x} & x_{j+1/2} \leq x < x_{j+3/2} \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$(v_{j+1/2})_x = \begin{cases} \frac{1}{\Delta x} & x_{j-1/2} \leq x < x_{j+1/2} \\ -\frac{1}{\Delta x} & x_{j+1/2} \leq x < x_{j+3/2} \\ 0 & \text{otherwise} \end{cases}$$

For this FEM we are intereseted in $G_{i+1/2}$ and then we can just get a shift operator to get the otherones. For FEM we replace the functions by their P1 approximations so

$$G \approx G' = \sum^{j} G_{j+1/2} v_{j+1/2}$$

$$u \approx u' = \sum^{j} u_{j+1/2} v_{j+1/2}$$

We hit our weak formulation with the test function $v_{j+1/2}$ with our P1 approximations

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} G' v_{j+1/2} dx - H \int_{x_{j-1/2}}^{x_{j+3/2}} u' v_{j+1/2} dx - \frac{H^3}{3} \int_{x_{j-1/2}}^{x_{j+3/2}} u'_x (v_{j+1/2})_x dx = 0$$

This test function is zero on all elements except j thus

$$\begin{aligned}\int_{x_{j-1/2}}^{x_{j+3/2}} &\left(G_{j-1/2}v_{j-1/2} + G_{j+1/2}v_{j+1/2} + G_{j+3/2}v_{j+3/2}\right) v_{j+1/2} dx \\ &- H \int_{x_{j-1/2}}^{x_{j+3/2}} \left(u_{j-1/2}v_{j-1/2} + u_{j+1/2}v_{j+1/2} + u_{j+3/2}v_{j+3/2}\right) v_{j+1/2} dx - \\ \frac{H^3}{3} \int_{x_{j-1/2}}^{x_{j+3/2}} &\left(u_{j-1/2}(v_{j-1/2})_x + u_{j+1/2}(v_{j+1/2})_x + u_{j+3/2}(v_{j+3/2})_x\right) (v_{j+1/2})_x dx = 0\end{aligned} \tag{1}$$

To calculate these we just need expressions for

$$\int_{x_{j-1/2}}^{x_{j+3/2}} v_{j-1/2}v_{j+1/2} dx$$
$$\int_{x_{j-1/2}}^{x_{j+3/2}} v_{j+1/2}v_{j+1/2} dx$$
$$\int_{x_{j-1/2}}^{x_{j+3/2}} v_{j+3/2}v_{j+1/2} dx$$

and

$$\int_{x_{j-1/2}}^{x_{j+3/2}} (v_{j-1/2})_x (v_{j+1/2})_x dx$$
$$\int_{x_{j-1/2}}^{x_{j+3/2}} (v_{j+1/2})_x (v_{j+1/2})_x dx$$
$$\int_{x_{j-1/2}}^{x_{j+3/2}} (v_{j+3/2})_x (v_{j+1/2})_x dx$$

$$v_{j+1/2} = \begin{cases} \frac{x - x_{j-1/2}}{\Delta x} & x_{j-1/2} \leq x < x_{j+1/2} \\ 1 - \frac{x - x_{j+1/2}}{\Delta x} & x_{j+1/2} \leq x < x_{j+3/2} \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$(v_{j+1/2})_x = \begin{cases} \frac{1}{\Delta x} & x_{j-1/2} \leq x < x_{j+1/2} \\ -\frac{1}{\Delta x} & x_{j+1/2} \leq x < x_{j+3/2} \\ 0 & \text{otherwise} \end{cases}$$

We begin:

$$\int_{x_{j-1/2}}^{x_{j+3/2}} v_{j-1/2}v_{j+1/2}dx = \int_{x_{j-1/2}}^{x_{j+1/2}} v_{j-1/2}v_{j+1/2}dx$$

$$= \int_{x_{j-1/2}}^{x_{j+1/2}} (1 - \frac{x - x_{j-1/2}}{\Delta x})\frac{x - x_{j-1/2}}{\Delta x}dx$$

$$= \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{x - x_{j-1/2}}{\Delta x} - \left(\frac{x - x_{j-1/2}}{\Delta x}\right)^2 dx$$

$$= \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{x - x_{j-1/2}}{\Delta x} - \left(\frac{x^2 - 2xx_{j-1/2} + x_{j-1/2}^2}{\Delta x^2}\right) dx$$

$$= \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\Delta x x - \Delta x x_{j-1/2}}{\Delta x^2} - \left(\frac{x^2 - 2xx_{j-1/2} + x_{j-1/2}^2}{\Delta x^2}\right) dx$$

$$= \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\Delta x x - \Delta x x_{j-1/2}}{\Delta x^2} - \frac{x^2 - 2xx_{j-1/2} + x_{j-1/2}^2}{\Delta x^2}dx$$

$$= \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\Delta x x - \Delta x x_{j-1/2} - x^2 + 2xx_{j-1/2} - x_{j-1/2}^2}{\Delta x^2}dx$$

$$= \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{-x^2 + (\Delta x + 2x_{j-1/2})x - \Delta x x_{j-1/2} - x_{j-1/2}^2}{\Delta x^2}dx$$

$$= \frac{1}{\Delta x^2} \int_{x_{j-1/2}}^{x_{j+1/2}} -x^2 + (\Delta x + 2x_{j-1/2})x - \Delta x x_{j-1/2} - x_{j-1/2}^2 dx$$

$$= \frac{1}{\Delta x^2} \left[-\frac{1}{3}x^3 + \frac{1}{2}(\Delta x + 2x_{j-1/2})x^2 + (-\Delta x x_{j-1/2} - x_{j-1/2}^2)x\right]_{x_{j-1/2}}^{x_{j+1/2}}$$

$$\begin{aligned}&\frac{1}{\Delta x^2}\left[-\frac{1}{3}x_{j+1/2}^3 + \frac{1}{2}(\Delta x + 2x_{j-1/2})x_{j+1/2}^2 + (-\Delta x x_{j-1/2} - x_{j-1/2}^2)x_{j+1/2}\right] \\ &-\frac{1}{\Delta x^2}\left[-\frac{1}{3}x_{j-1/2}^3 + \frac{1}{2}(\Delta x + 2x_{j-1/2})x_{j-1/2}^2 + (-\Delta x x_{j-1/2} - x_{j-1/2}^2)x_{j-1/2}\right]\end{aligned} \tag{2}$$