Robust Computational Models for Water Waves

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Australian National University

September 3, 2018

Outline of the Presentation

- Motivation
- History
- Contribution
  - Method
  - Validation

Water Waves

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Water wave hazards:

- Tsunamis

Tsunamis

Figure: 2004 Indian Ocean Tsunami (Banda Aceh)

Tsunamis

Figure: 2011 Tohoku Tsunami

Water Waves

Water wave hazards:

- Tsunamis
- Storm Surges

Storm Surges

Figure: 2012 Hurricane Sandy Storm Surge

Water Waves

Water wave hazards:

- Tsunamis
- Storm Surges

Phenomena caused by water waves:

- Nutrient Transport
- Beach Erosion
- Breakup of Sea Ice

Computational Modelling

Goal: Model Physics On Computers

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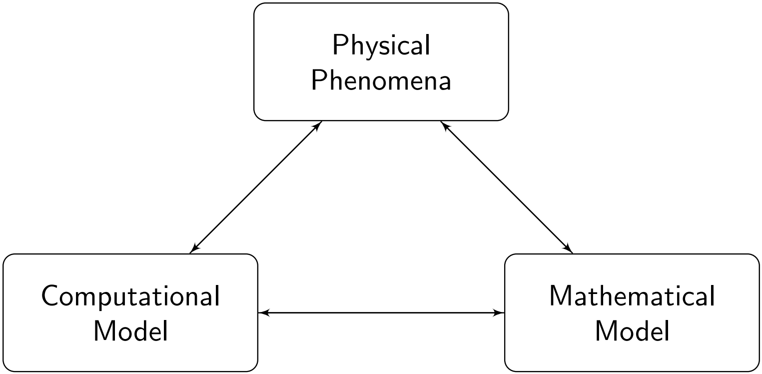
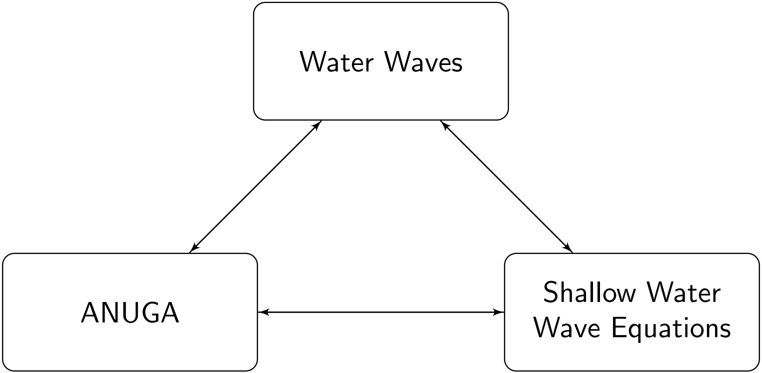
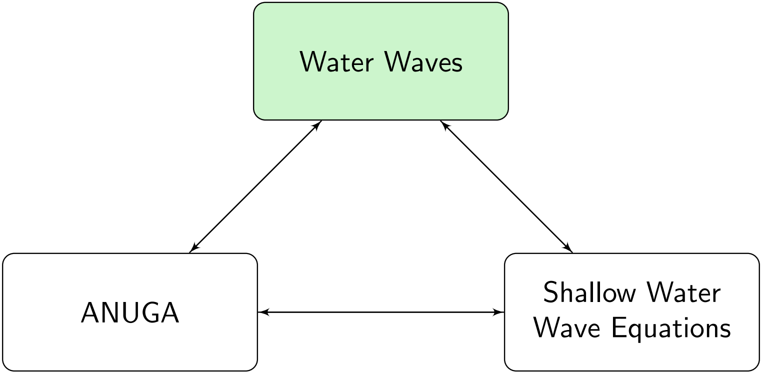


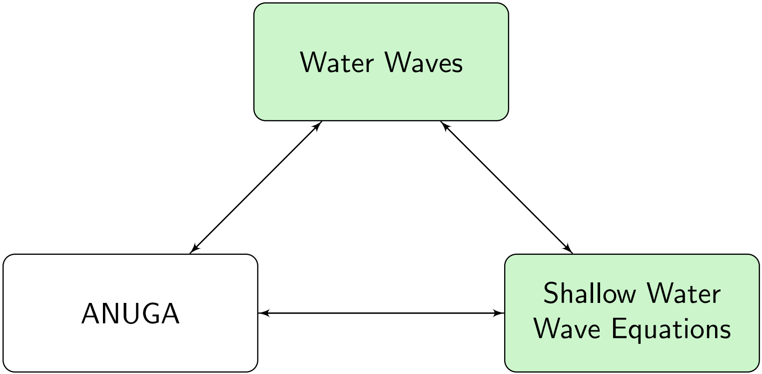
ANUGA



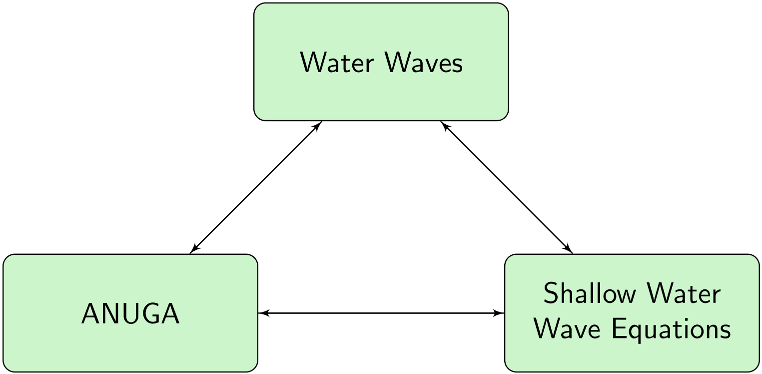
ANUGA: Water Waves



ANUGA: Shallow Water Wave Equations



ANUGA



Outcome

New project at the ANU to develop a robust computational model for the Serre equations.

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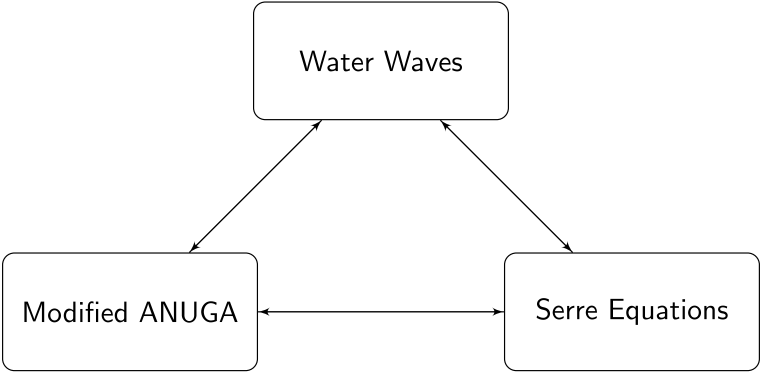
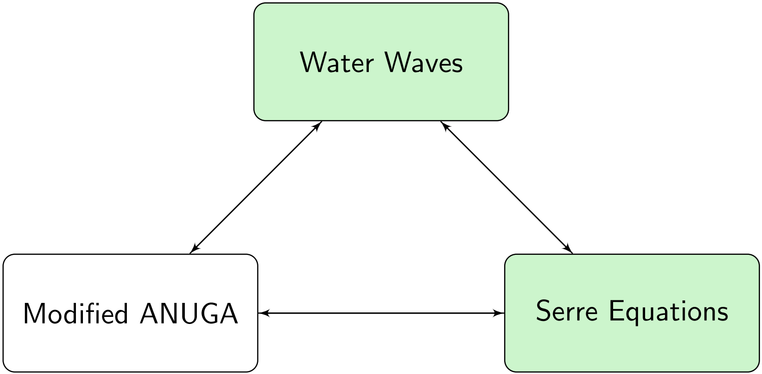
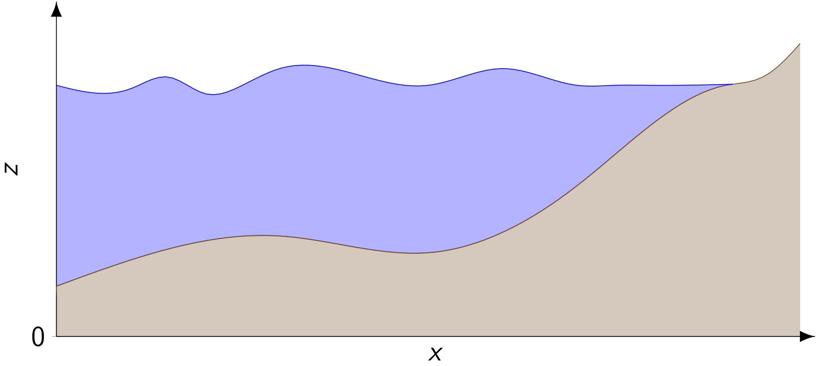


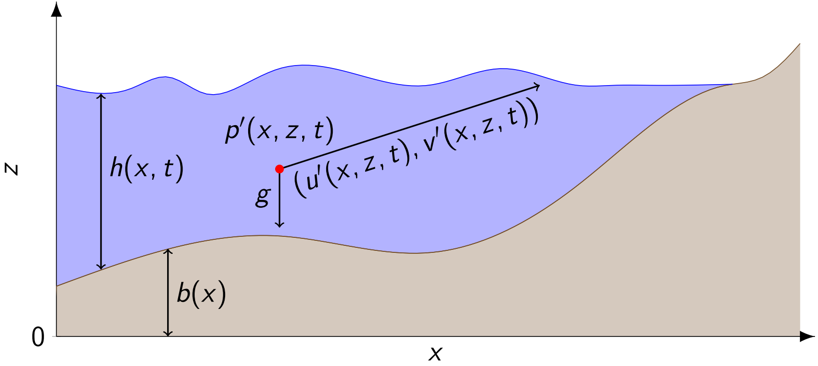
Mathematical Model



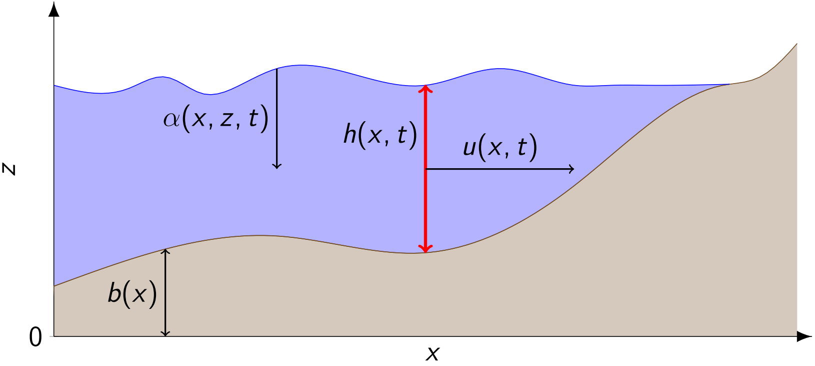
Typical Scenario



Navier Stokes Model



Serre Model



Assumptions

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Quantity	Shallow Water Wave Equations	Serre Equations
Particle: $v'(x,z,t)$	0	$u\frac{\partial b}{\partial x} - (z-b)\frac{\partial b}{\partial x}$

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where

$$\alpha(x,z,t) = (h(x,t) + b(x)) - z$$

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$$\Psi = \frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial^2 b}{\partial x^2}, \quad \Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}.$$

Equations

Mass: $$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

Momentum: $$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^2}{2}\Psi + \frac{h^3}{3}\Phi\right)$$

$$+ \frac{\partial b}{\partial x}\left(gh + h\Psi + \frac{h^2}{2}\Phi\right) = 0.$$

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Pros and Cons

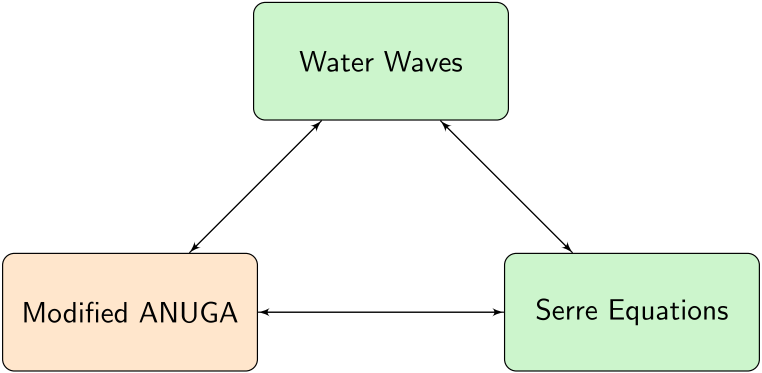
Pros:

- Includes dispersive effects
- Considered one of the best models for water waves
- Can apply techniques of ANUGA

Cons:

- More complex than the Shallow Water Wave Equations

Computational Model



Previous Work at the ANU

- 2014: Chris Zoppou's PhD thesis
  Demonstrated computational model for the 1D Serre equations.
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  Independent reproduction of Chris Zoppou's computational model

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Open problems:

2D: Extension of the method to 2D equations

Robust: Validation for steep gradients in free surface

Robust: Inclusion and validation of dry beds

Thesis Goals

Solve these open problems:

2D: 1D method that extends well to 2D

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Solve these open problems:

2D: 1D method that extends well to 2D

Robust: Validation for steep gradients in free surface

Robust: Inclusion and validation of dry beds

Technique: Develop a robust computational model from the 1D Serre equations that can be easily extended to 2D.

Method

- Finite Volume Method (ANUGA)
- Finite Element Method

Finite Volume Method

2D: Extends well to 2D

Robust: Stable in the presence of steep gradients

Robust: Stable in the presence of dry beds

- Maintains conservation properties of the equations

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Robust: Stable in the presence of dry beds

- Maintains conservation properties of the equations

Chris Zoppou's thesis demonstrated an adaptation of the Finite Volume Method to solve the Serre Equations.

Equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^2}{2}\Psi + \frac{h^3}{3}\Phi\right) + \frac{\partial b}{\partial x}\left(gh + h\Psi + \frac{h^2}{2}\Phi\right) = 0$$

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For a Finite Volume Method we require equations in the form

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} + s(q) = 0$$

where $f(q)$ and $s(q)$ do not contain temporal derivatives

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Reformulation

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}h^3 \left[\frac{\partial u}{\partial x}\right]^2 + h^2 u \frac{\partial u}{\partial x}\frac{\partial b}{\partial x}\right) + \frac{1}{2}h^2 u \frac{\partial u}{\partial x}\frac{\partial^2 b}{\partial x^2} - hu^2 \frac{\partial b}{\partial x}\frac{\partial^2 b}{\partial x^2} + gh\frac{\partial b}{\partial x} = 0.$$

with

$$G = hu\left(1 + \frac{\partial h}{\partial x}\frac{\partial b}{\partial x} + \frac{1}{2}h\frac{\partial^2 b}{\partial x^2} + \left[\frac{\partial b}{\partial x}\right]^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{3}h^3\frac{\partial u}{\partial x}\right).$$

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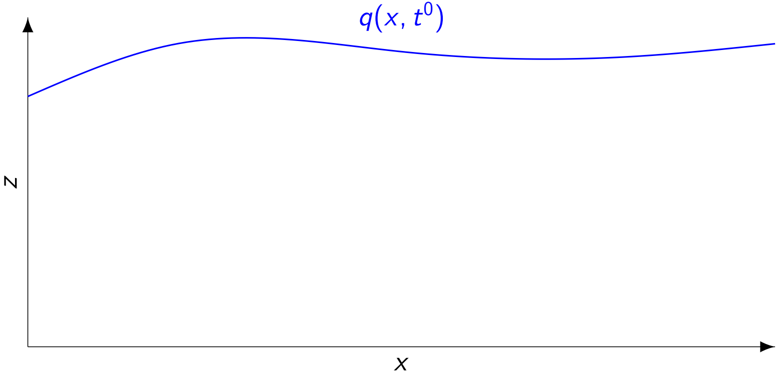
Finite Volume Method Example

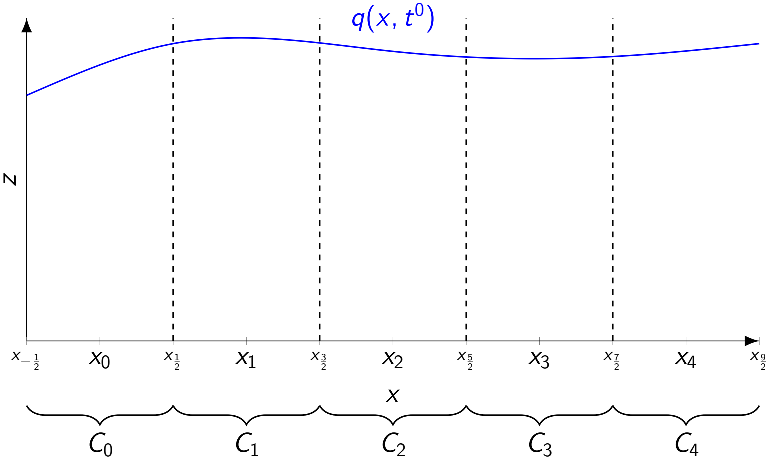
Conservation equations with a source term:

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} + s(q) = 0$$

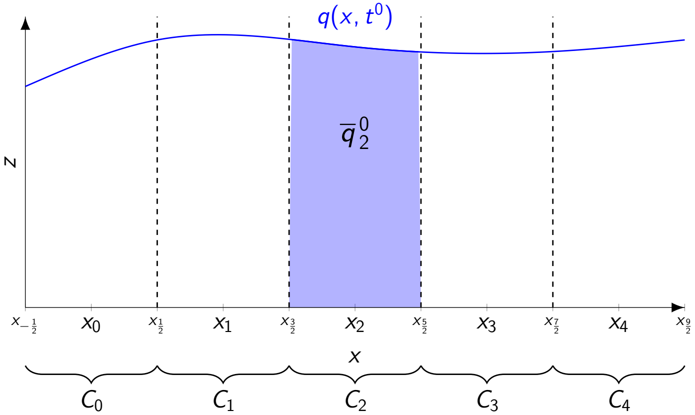
Function at $t = t^0$



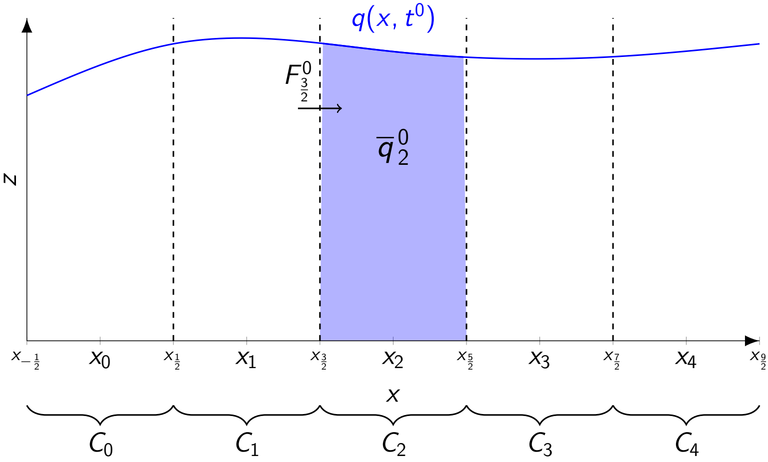
Cell Discretisation



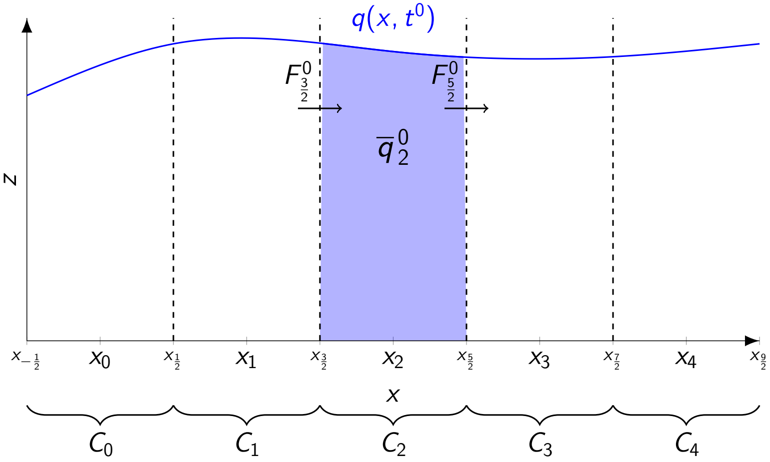
Total Amount



Flux Left


$q(x, t^0)$
$F^0_{\frac{3}{2}}$
$\bar{q}^0_2$
z
$x_{-\frac{1}{2}}$
x_0
$x_{\frac{1}{2}}$
x_1
$x_{\frac{3}{2}}$
x_2
$x_{\frac{5}{2}}$
x_3
$x_{\frac{7}{2}}$
x_4
$x_{\frac{9}{2}}$
x
C_0
C_1
C_2
C_3
C_4

Flux Right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$F^0_{\frac{3}{2}}$
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z
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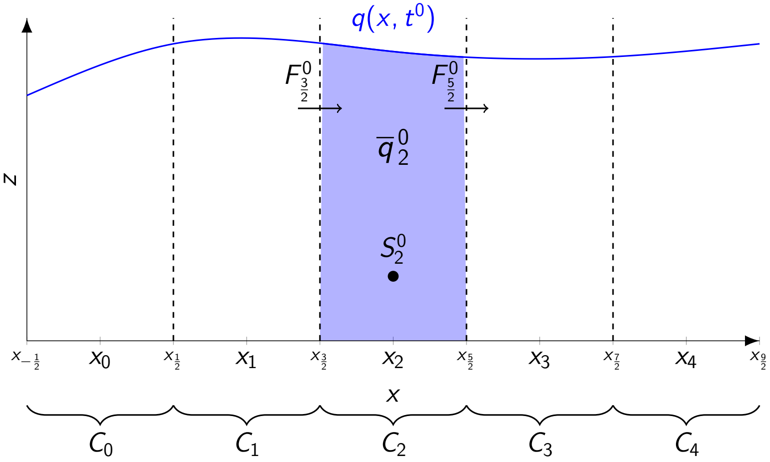
x_4
$x_{\frac{9}{2}}$
x
C_0
C_1
C_2
C_3
C_4

Source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x
C_0
C_1
C_2
C_3
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Finite Volume Update

$$\overline{q}_2^1 = \overline{q}_2^0 - \left(F^0_{\frac{5}{2}} - F^0_{\frac{3}{2}}\right) - \left(S^0_2\right)$$

Finite Volume Update

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$$\overbrace{\int_{C_2} q(x, t^1)\, dx}^{\overline{q}_2^1} - \overbrace{\int_{C_2} q(x, t^0)\, dx}^{\overline{q}_2^0} + \Bigg(\overbrace{\int_{t^0}^{t^1} f(q(x_{5/2}, t))\, dt}^{F^0_{\frac{5}{2}}}$$

$$- \overbrace{\int_{t^0}^{t^1} f(q(x_{3/2}, t))\, dt}^{F^0_{\frac{3}{2}}} \Bigg) + \overbrace{\int_{t^0}^{t^1} \int_{C_2} s(q(x, t))\, dt}^{S_2^0}$$

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Update Formula for Serre Equations

$$\overline{h}_j^{n+1} = \overline{h}_j^{n} - \left[F^n_{j+1/2} - F^n_{j-1/2}\right]$$

$$\overline{G}_j^{n+1} = \overline{G}_j^{n} - \left[F^n_{j+1/2} - F^n_{j-1/2}\right] - S^n_j$$

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- All the fluxes $F^n_{j+1/2}$ and $F^n_{j-1/2}$ and the source term S_j^n require u at t^n

Calculate Velocity

$$G = hu\left(1 + \frac{\partial h}{\partial x}\frac{\partial b}{\partial x} + \frac{1}{2}h\frac{\partial^2 b}{\partial x^2} + \left[\frac{\partial b}{\partial x}\right]^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{3}h^3\frac{\partial u}{\partial x}\right).$$

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- Previously used a Finite Difference Method [1]
- Contribution: use a Finite Element Method

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Finite Element Method

2D: Extends well to 2D

Robust: Stable in the presence of steep gradients

- Maintains conservation properties

Finite Element Method Example

Example:

$$-\frac{\partial^2 u}{\partial x^2} = f$$

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Weak Form:

$$-\int_\Omega \frac{\partial^2 u}{\partial x^2} v \ dx = \int_\Omega fv \ dx$$

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$$-\int_\Omega \frac{\partial^2 u}{\partial x^2} v \, dx = \int_\Omega f v \, dx$$

Integrate by parts

$$\int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} \, dx = \int_\Omega f v \, dx$$

Finite Element Method

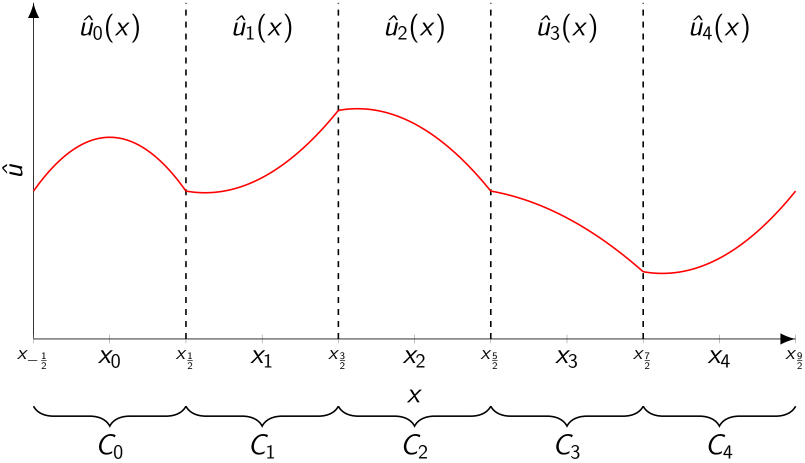
$$\int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\, dx = \int_\Omega fv\, dx$$

Finite Element Method

$$\int_{\Omega} \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\ dx = \int_{\Omega} fv\ dx$$

$$\sum_{j}\left[\int_{C_j} \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\ dx\right] = \sum_{j}\left[\int_{C_j} fv\ dx\right]$$

Piecewise Polynomial Representation



Finite Element Method

$$\int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\; dx = \int_\Omega fv\; dx$$

$$\sum_j \left[\int_{C_j} \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\; dx \right] = \sum_j \left[\int_{C_j} fv\; dx \right]$$

$$\sum_j \left[\int_{C_j} \frac{\partial \hat{u}_j}{\partial x}\frac{\partial \hat{v}_j}{\partial x}\; dx \right] = \sum_j \left[\int_{C_j} \hat{f}_j \hat{v}_j\; dx \right]$$

Finite Element Method

$$\int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\ dx = \int_\Omega fv\ dx$$

$$\sum_j \left[\int_{C_j} \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\ dx \right] = \sum_j \left[\int_{C_j} fv\ dx \right]$$

$$\sum_j \left[\int_{C_j} \frac{\partial \hat{u}_j}{\partial x}\frac{\partial \hat{v}_j}{\partial x}\ dx \right] = \sum_j \left[\int_{C_j} \hat{f}_j \hat{v}_j\ dx \right]$$

$$\mathbf{A}\vec{u} = \vec{c}$$

Finite Element Method

$$\int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dx = \int_\Omega fv\,dx$$

$$\sum_j \left[\int_{C_j} \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dx\right] = \sum_j \left[\int_{C_j} fv\,dx\right]$$

$$\sum_j \left[\int_{C_j} \frac{\partial \hat{u}_j}{\partial x}\frac{\partial \hat{v}_j}{\partial x}\,dx\right] = \sum_j \left[\int_{C_j} \hat{f}_j\hat{v}_j\,dx\right]$$

$$\mathbf{A}\vec{u} = \vec{c}$$

where

- $\mathbf{A}$ depends on $\hat{v}_j$
- $\vec{u}$ determines $\hat{u}_j$
- $\vec{c}$ depends on $\hat{f}_j$ and $\hat{v}_j$

Finite Element Method for Serre Equations

$$G = hu\left(1 + \frac{\partial h}{\partial x}\frac{\partial b}{\partial x} + \frac{1}{2}h\frac{\partial^2 b}{\partial x^2} + \left[\frac{\partial b}{\partial x}\right]^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{3}h^3\frac{\partial u}{\partial x}\right)$$

Finite Element Method for Serre Equations

$$G = hu\left(1 + \frac{\partial h}{\partial x}\frac{\partial b}{\partial x} + \frac{1}{2}h\frac{\partial^2 b}{\partial x^2} + \left[\frac{\partial b}{\partial x}\right]^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{3}h^3\frac{\partial u}{\partial x}\right)$$

$$\mathbf{A}\vec{u} = \vec{c}$$

where

- $\mathbf{A}$ depends on the polynomial representation of h, b and test function
- $\vec{u}$ determines the polynomial representation of u
- $\vec{c}$ depends on polynomial representation of G and test function

Method

$$
\begin{tikzcd}
 & \overline{h}_j^n, \overline{G}_j^n \arrow[dr, "\text{Reconstruction}"] & \\
\hat{u}_j^n \arrow[ur, "\text{Finite Volume}"] & & \hat{h}_j^n, \hat{G}_j^n \arrow[ll, "\text{Finite Element}"]
\end{tikzcd}
$$

Progress

2D: 1D method that extends well to 2D ✓

Robust: Validation for steep gradients in free surface

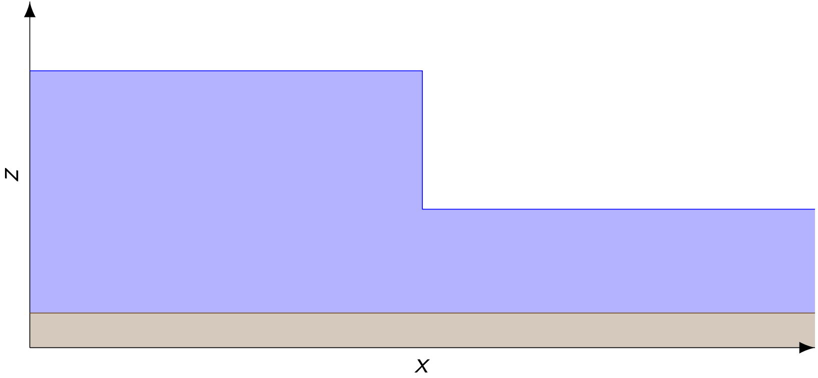
Robust: Inclusion and validation of dry beds

Validation

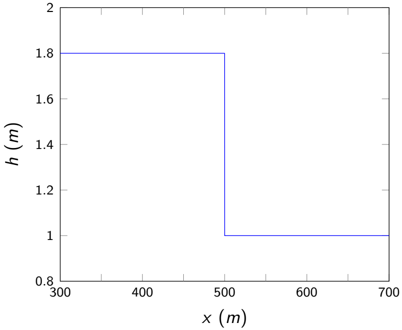
- Steep gradients in the free surface
- Dry beds

Statement of Problem

How does this initially still body of water evolve?

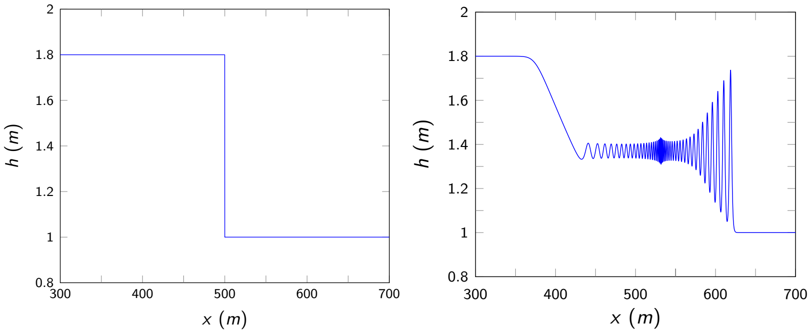


Our New Numerical Solution



Pitt, J., Zoppou, C., and Roberts, S. (2018). Behaviour of the Serre Equations in the Presence of Steep Gradients Revisited. Wave Motion, 76(1):6177.

Our New Numerical Solution



Pitt, J., Zoppou, C., and Roberts, S. (2018). Behaviour of the Serre Equations in the Presence of Steep Gradients Revisited. Wave Motion, 76(1):6177.

What was known

- No analytic solutions
- Some experimental comparisons [2]
- Other numerical solutions from the literature

[2] Zoppou, C. (2014). Numerical Solution of the One-dimensional and Cylindrical Serre Equations for Rapidly Varying Free Surface Flows. PhD thesis, Australian National University.

Contribution

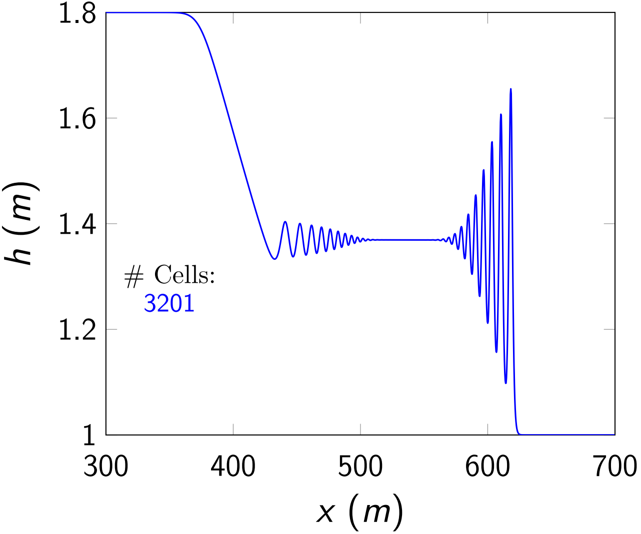
- Observed this new behaviour
- Demonstrated convergence
- Comprehensive review of numerical solutions from the literature

Pitt, J., Zoppou, C., and Roberts, S. (2018). Behaviour of the Serre Equations in the Presence of Steep Gradients Revisited. Wave Motion, 76(1):6177.

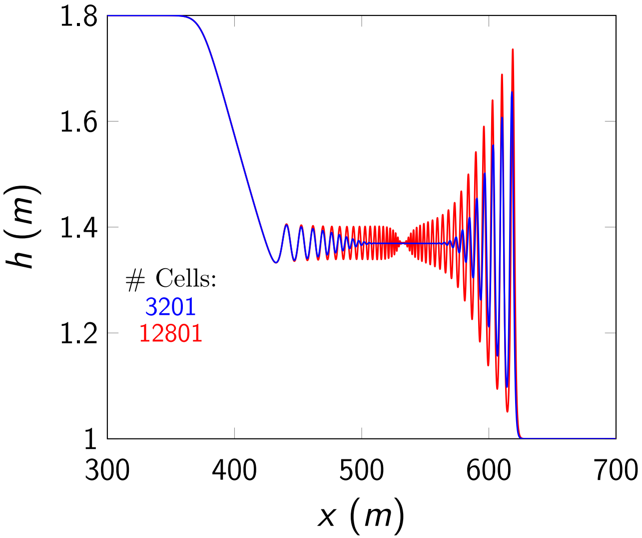
Convergence

As we increase number of cells the numerical solutions should converge to the true solution

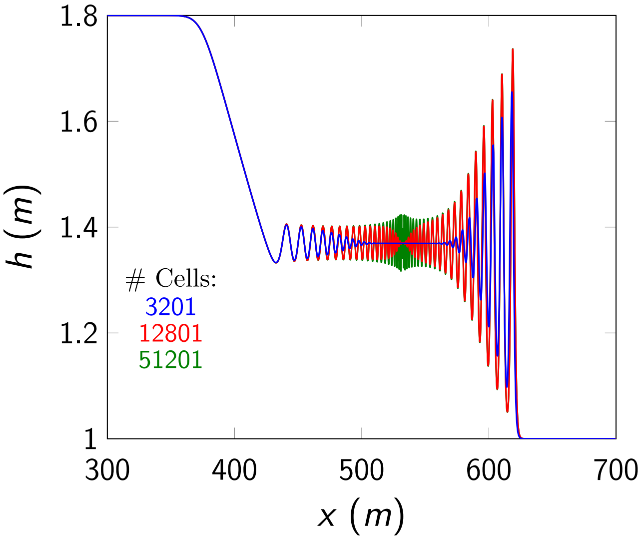
Convergence



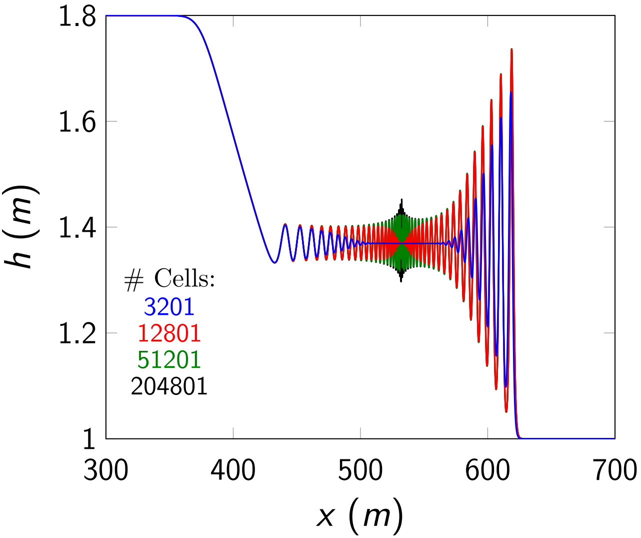
Convergence



Convergence



Convergence



Comprehensive Review

- Demonstrated consistent behaviour across many numerical methods
- Were able to explain why the behaviour had not previously been observed

Result

Validated our computational model when steep gradients are present in the free surface. [3]

[3] Pitt, J., Zoppou, C., and Roberts, S. (2018). Behaviour of the Serre Equations in the Presence of Steep Gradients Revisited. Wave Motion, 76(1):6177.

Progress

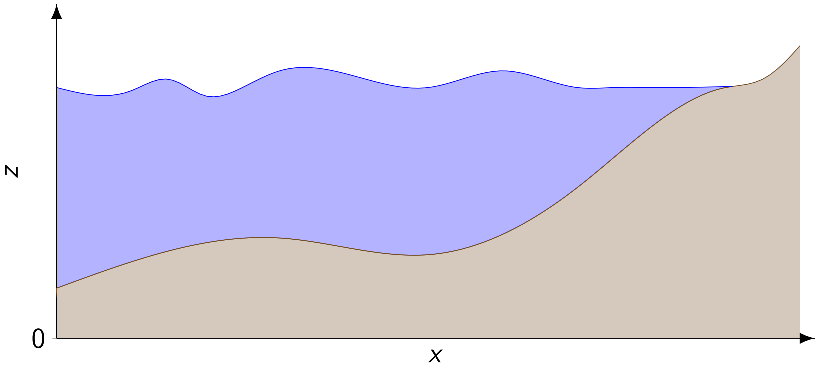
2D: 1D method that extends well to 2D ✓

Robust: Validation for steep gradients in free surface ✓

Robust: Inclusion and validation of dry beds

Statement of Problem

Properly handle interaction of waves and the dry bed



What was known

- No analytic solutions
- A variety of numerical techniques only compared to experimental data

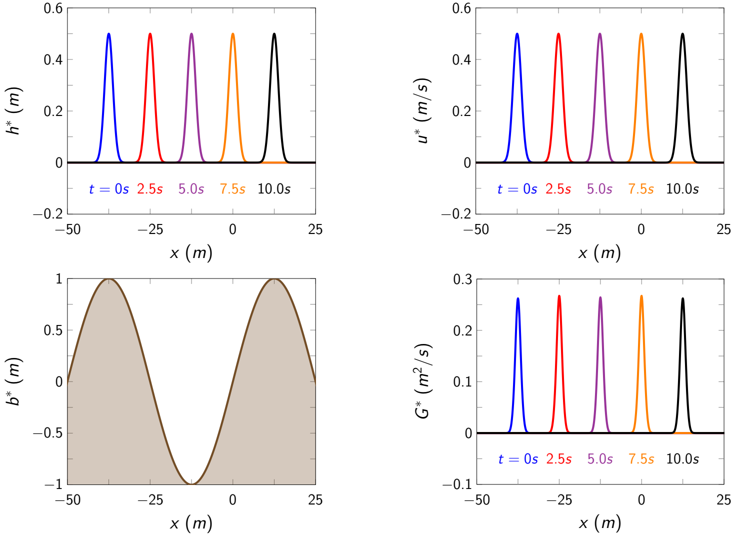
Contribution

- Solved modified equations that did possess analytic solutions
- Compared with experimental data

Constructing Modified Equations

- Pick functions for height, velocity and bed: h^*, u^* and b^*
- Add source terms to Serre equations that force h^*, u^* and b^* to be solutions
- Validation tests

Pick Functions



Modify Equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = S_h^*,$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}h^3\left[\frac{\partial u}{\partial x}\right]^2 + h^2 u \frac{\partial u}{\partial x}\frac{\partial b}{\partial x}\right) + \frac{1}{2}h^2 u \frac{\partial u}{\partial x}\frac{\partial^2 b}{\partial x^2} - hu^2 \frac{\partial b}{\partial x}\frac{\partial^2 b}{\partial x^2} + gh\frac{\partial b}{\partial x} = S_G^*.$$

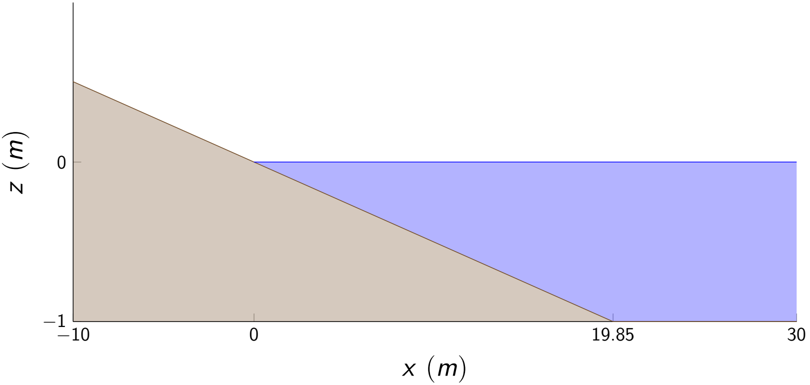
S_h^* and S_G^* are just the LHS with the quantities replaced by their associated chosen function.

Results

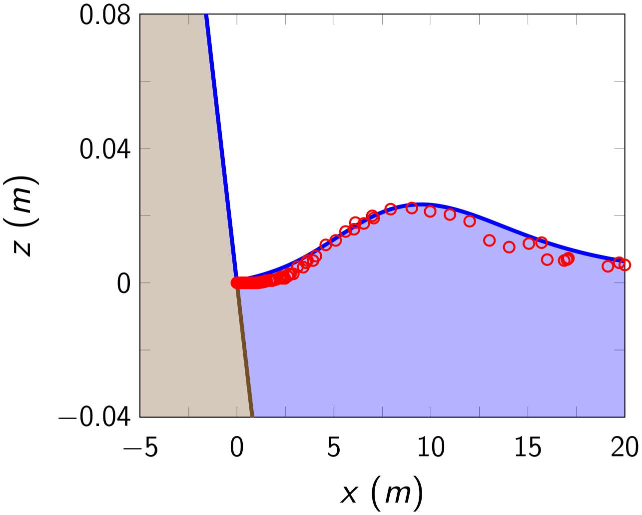
Modified Equations Validation Conclusions

- Very strong test as all terms must be accurately approximated
- Only source of error is the method

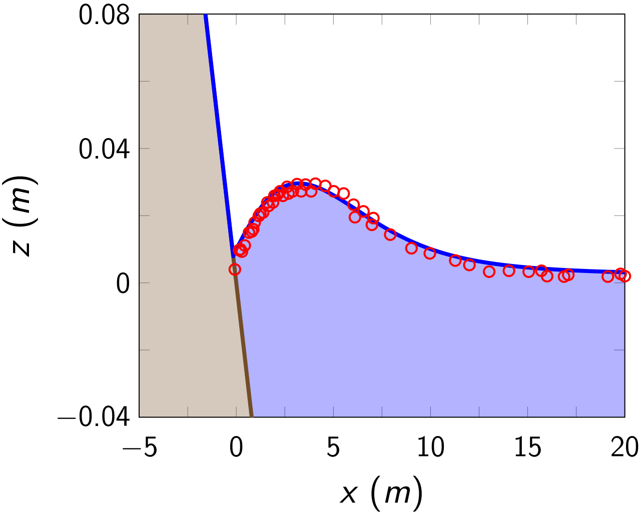
Experimental Data



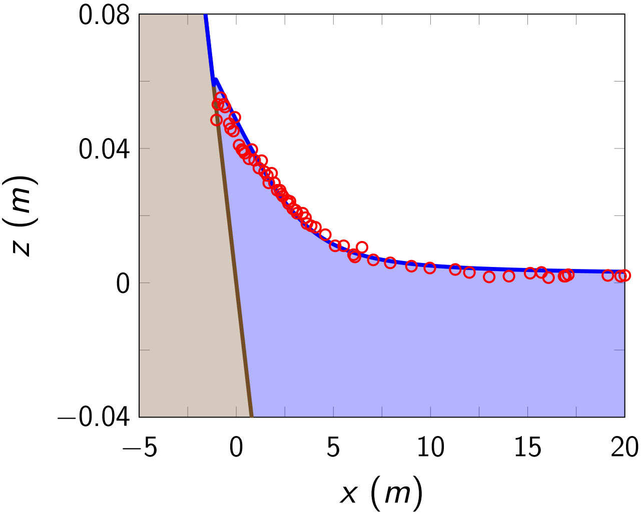
$t = 30s$



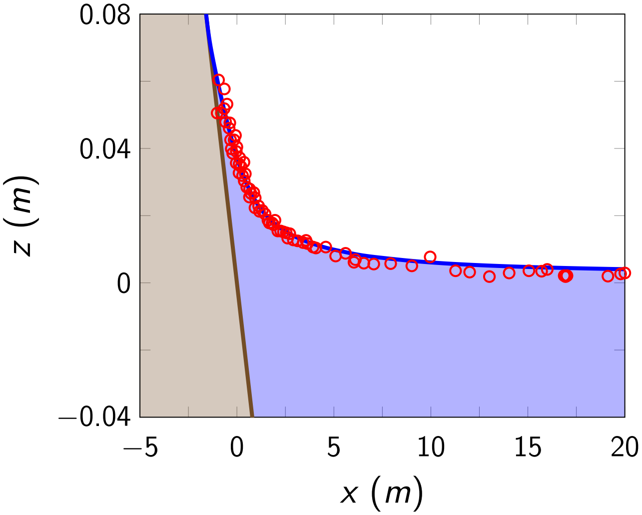
$t = 40s$



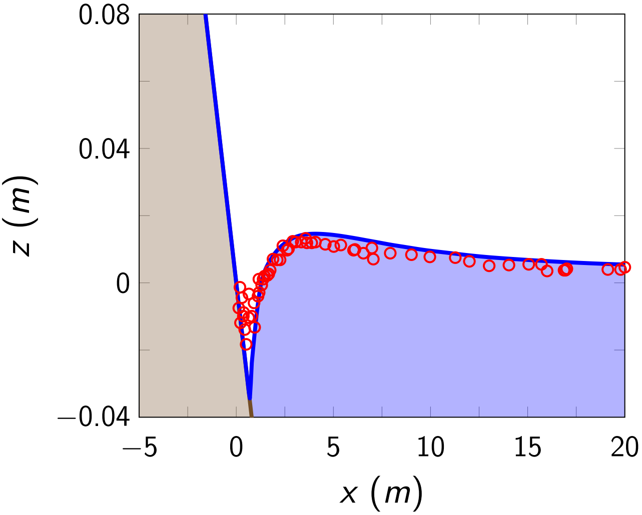
$t = 50s$



$t = 60s$



$t = 70s$



Experimental Validation Conclusions

- Demonstrates agreement of computational model and physical process
- Many sources of errors

Progress

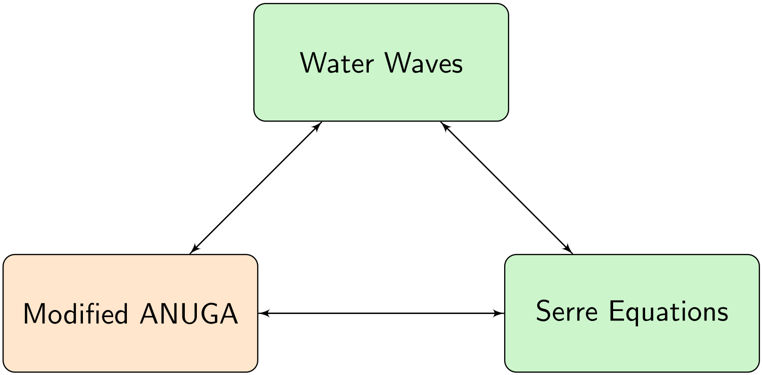
2D: 1D method that extends well to 2D ✓

Robust: Validation for steep gradients in free surface ✓

Robust: Inclusion and validation of dry beds ✓

Conclusions

- Developed a Robust Computational Model from the 1D Serre equations for the 2D water wave problem



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Wave Motion, 76(1):61–77.

Zoppou, C. (2014).
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PhD thesis, Australian National University, Mathematical Sciences Institute, College of Physical and Mathematical Sciences, Australian National University, Canberra, ACT 2600, Australia.