

# 1 Elliptic Equation

The linearised elliptic equation is

$$G = Hu - \frac{H^3}{3}u_{xx}$$

Want to find out the FEM approximation to this  $G'$  such that

$$G' = \mathcal{G}_{FE_1} u$$

for  $P^1$  FEM

$$G' = \mathcal{G}_{FE_2} u$$

for  $P^2$  FEM.

To do so we begin by first multiplying by an arbitrary test function  $v$  so that

$$Gv = Huv - \frac{H^3}{3}u_{xx}v$$

and then we integrate over the entire domain to get

$$\int_{\Omega} Gv dx = \int_{\Omega} Huv dx - \int_{\Omega} \frac{H^3}{3}u_{xx}v dx$$

for all  $v$

We then make use of integration by parts, with Dirichlet boundaries to get

$$\int_{\Omega} Gv dx = \int_{\Omega} Huv dx + \int_{\Omega} \frac{H^3}{3}u_x v_x dx$$

Our FVM discretisation already has a natural structure with linear functions intervals of  $[x_{j-1/2}, x_{j+1/2}]$ , to achieve this in  $P^1$  we have our nodes at the boundaries, thus

So we can reformulate this as

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Gv dx = \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Huv dx + \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{H^3}{3}u_x v_x dx$$

or more aptly

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} G v dx - \int_{x_{j-1/2}}^{x_{j+3/2}} H u v dx - \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{H^3}{3} u_x v_x dx = 0$$

for all v

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} G v dx - H \int_{x_{j-1/2}}^{x_{j+3/2}} u v dx - \frac{H^3}{3} \int_{x_{j-1/2}}^{x_{j+3/2}} u_x v_x dx = 0$$

## 2 P1 FEM

We are going to corodainte tranform from x space the interval  $[x_{i-1/2}, x_{i+1/2}, x_{i+3/2}]$  to the  $\xi$  space interval  $[-1, 0, 1]$ . To accomplish this we have the following relation

$$x = \xi \Delta x + x_{j+1/2}$$

Taking the derivative we see

$$dx = d\xi \Delta x, \quad \frac{dx}{d\xi} = \Delta x, \quad \frac{d\xi}{dx} = \frac{1}{\Delta x}$$

For this FEM we are intereseted in  $G_{i+1/2}$  and then we can just get a shift operator to get the otherones. For FEM we replace the functions by their P1 approximations so

$$G \approx G' = \sum_j G_{j+1/2} v_{j+1/2}$$

$$u \approx u' = \sum_j u_{j+1/2} v_{j+1/2}$$

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} G' v dx - H \int_{x_{j-1/2}}^{x_{j+3/2}} u' v dx - \frac{H^3}{3} \int_{x_{j-1/2}}^{x_{j+3/2}} u'_x v_x dx = 0$$

We break this up into the integrals because of the domain of dependence of the basis functions is covered. We also just use a particular basis function as the test function, in particular we choose  $v_{j+1/2}$

$$\begin{aligned}
& \int_{x_{j-1/2}}^{x_{j+3/2}} G'(x) v_{j+1/2} dx = \int_{-1}^1 G'(\xi) v_{j+1/2}(\xi) \frac{dx}{d\xi} d\xi \\
& = \Delta x \int_{-1}^1 (G_{j-1/2} v_{j-1/2} + G_{j+1/2} v_{j+1/2} + G_{j+3/2} v_{j+3/2}) v_{j+1/2} d\xi \\
& = \Delta x \left[ G_{j-1/2} \int_{-1}^1 v_{j-1/2} v_{j+1/2} d\xi + G_{j+1/2} \int_{-1}^1 v_{j+1/2} v_{j+1/2} d\xi + G_{j+3/2} \int_{-1}^1 v_{j+3/2} v_{j+1/2} d\xi \right]
\end{aligned}$$

We have the

$$\begin{aligned}
& \int_{-1}^1 v_{j-1/2} v_{j+1/2} d\xi = \int_{-1}^1 v_{j+3/2} v_{j+1/2} d\xi = \frac{1}{6} \\
& \int_{-1}^1 v_{j+1/2} v_{j+1/2} d\xi = \frac{2}{3} \\
& = \Delta x \left[ G_{j-1/2} \frac{1}{6} + G_{j+1/2} \frac{2}{3} + G_{j+3/2} \frac{1}{6} \right] \\
& = \frac{\Delta x}{6} [G_{j-1/2} + 4G_{j+1/2} + G_{j+3/2}]
\end{aligned}$$

Similarly we have

$$\begin{aligned}
& -H \int_{x_{j-1/2}}^{x_{j+3/2}} u' v_{j+1/2} dx = -\frac{H\Delta x}{6} [u_{j-1/2} + 4u_{j+1/2} + u_{j+3/2}] \\
& -\frac{H^3}{3} \int_{x_{j-1/2}}^{x_{j+3/2}} u'_x (v_{j+1/2})_x dx = -\frac{H^3}{3} \int_{-1}^1 u'_\xi \frac{d\xi}{dx} (v_{j+1/2})_\xi \frac{d\xi}{dx} \frac{dx}{d\xi} d\xi \\
& = -\frac{H^3}{3\Delta x} \int_{-1}^1 u'_\xi (v_{j+1/2})_\xi d\xi
\end{aligned}$$

where ' denotes derivative

$$= -\frac{H^3}{3\Delta x} \int_{-1}^1 (u'_{j-1/2} v'_{j-1/2} + u'_{j+1/2} v'_{j+1/2} + u'_{j+3/2} v'_{j+3/2}) v'_{j+1/2} d\xi$$

$$= -\frac{H^3}{3\Delta x} \left[ u_{j-1/2} \int_{-1}^1 v'_{j-1/2} v'_{j+1/2} d\xi + u_{j+1/2} \int_{-1}^1 v'_{j+1/2} v'_{j+1/2} d\xi + u_{j+3/2} \int_{-1}^1 v'_{j+3/2} v'_{j+1/2} d\xi \right]$$

We have that

$$\begin{aligned} \int_{-1}^1 v'_{j-1/2} v'_{j+1/2} d\xi &= -1 = \int_{-1}^1 v'_{j+3/2} v'_{j+1/2} d\xi \\ \int_{-1}^1 v'_{j+1/2} v'_{j+1/2} d\xi &= 2 \\ &= -\frac{H^3}{3\Delta x} [-u_{j-1/2} + 2u_{j+1/2} - u_{j+3/2}] \end{aligned}$$

Then our equation becomes

$$\begin{aligned} \frac{\Delta x}{6} [G_{j-1/2} + 4G_{j+1/2} + G_{j+3/2}] &= \\ \frac{H\Delta x}{6} [u_{j-1/2} + 4u_{j+1/2} + u_{j+3/2}] &+ \frac{H^3}{3\Delta x} [-u_{j-1/2} + 2u_{j+1/2} - u_{j+3/2}] \quad (1) \end{aligned}$$

$$\begin{aligned} [G_{j-1/2} + 4G_{j+1/2} + G_{j+3/2}] &= \\ H [u_{j-1/2} + 4u_{j+1/2} + u_{j+3/2}] &+ \frac{2H^3}{\Delta x^2} [-u_{j-1/2} + 2u_{j+1/2} - u_{j+3/2}] \quad (2) \end{aligned}$$

This formula is correct for  $u = 1, x, x^2$

By shifting we also get

$$\begin{aligned} [G_{j-3/2} + 4G_{j-1/2} + G_{j+1/2}] &= \\ H [u_{j-3/2} + 4u_{j-1/2} + u_{j+1/2}] &+ \frac{2H^3}{\Delta x^2} [-u_{j-3/2} + 2u_{j-1/2} - u_{j+1/2}] \quad (3) \end{aligned}$$

$$\begin{aligned} [G_{j+1/2} + 4G_{j+3/2} + G_{j+5/2}] &= \\ H [u_{j+1/2} + 4u_{j+3/2} + u_{j+5/2}] &+ \frac{2H^3}{\Delta x^2} [-u_{j+1/2} + 2u_{j+3/2} - u_{j+5/2}] \quad (4) \end{aligned}$$

We begin by assuming the analytic structure for  $G$  and  $u$  (to get easy shift operators). Let quantity  $q$  is given by so that  $q(x, t) = q(0, 0)e^{i(\omega t + kx)}$ . The use of this comes when we use our uniform grid in space, so that  $q(x_j, t) = q_j$  then  $q_{j \pm l} = q_j e^{\pm ikl\Delta x}$

Then we have

$$\begin{aligned} & \left[ G_j e^{-ik\frac{1}{2}\Delta x} + 4G_j e^{ik\frac{1}{2}\Delta x} + G_j e^{ik\frac{3}{2}\Delta x} \right] = \\ H & \left[ u_j e^{-ik\frac{1}{2}\Delta x} + 4u_j e^{ik\frac{1}{2}\Delta x} + u_j e^{ik\frac{3}{2}\Delta x} \right] + \frac{2H^3}{\Delta x^2} \left[ -u_j e^{-ik\frac{1}{2}\Delta x} + 2u_j e^{ik\frac{1}{2}\Delta x} - u_j e^{ik\frac{3}{2}\Delta x} \right] \end{aligned} \quad (5)$$

$$\begin{aligned} & G_j \left[ e^{-ik\frac{1}{2}\Delta x} + 4e^{ik\frac{1}{2}\Delta x} + e^{ik\frac{3}{2}\Delta x} \right] = \\ & \left( H \left[ e^{-ik\frac{1}{2}\Delta x} + 4e^{ik\frac{1}{2}\Delta x} + e^{ik\frac{3}{2}\Delta x} \right] + \frac{2H^3}{\Delta x^2} \left[ -e^{-ik\frac{1}{2}\Delta x} + 2e^{ik\frac{1}{2}\Delta x} - e^{ik\frac{3}{2}\Delta x} \right] \right) u_j \end{aligned} \quad (6)$$

$$G_j = \left( H + \frac{2H^3}{\Delta x^2} \frac{\left[ -e^{-ik\frac{1}{2}\Delta x} + 2e^{ik\frac{1}{2}\Delta x} - e^{ik\frac{3}{2}\Delta x} \right]}{\left[ e^{-ik\frac{1}{2}\Delta x} + 4e^{ik\frac{1}{2}\Delta x} + e^{ik\frac{3}{2}\Delta x} \right]} \right) u_j \quad (7)$$

$$G_j = \left( H + \frac{2H^3}{\Delta x^2} \frac{\left[ 2i \sin(k\frac{1}{2}\Delta x) + e^{ik\frac{1}{2}\Delta x} - e^{ik\frac{3}{2}\Delta x} \right]}{\left[ e^{-ik\frac{1}{2}\Delta x} + 4e^{ik\frac{1}{2}\Delta x} + e^{ik\frac{3}{2}\Delta x} \right]} \right) u_j \quad (8)$$

$$G_j = \left( H + \frac{2H^3}{\Delta x^2} \frac{2i \sin(k\frac{1}{2}\Delta x) + e^{ik\Delta x} \left( e^{-ik\frac{1}{2}\Delta x} - e^{ik\frac{1}{2}\Delta x} \right)}{\left[ e^{-ik\frac{1}{2}\Delta x} + 4e^{ik\frac{1}{2}\Delta x} + e^{ik\frac{3}{2}\Delta x} \right]} \right) u_j \quad (9)$$

$$G_j = \left( H + \frac{2H^3}{\Delta x^2} \frac{2i \sin(k_{\frac{1}{2}}\Delta x) - 2ie^{ik\Delta x} \sin(k_{\frac{1}{2}}\Delta x)}{\left[ e^{-ik_{\frac{1}{2}}\Delta x} + 4e^{ik_{\frac{1}{2}}\Delta x} + e^{ik_{\frac{3}{2}}\Delta x} \right]} \right) u_j \quad (10)$$

$$G_j = \left( H + \frac{2H^3}{\Delta x^2} \frac{2i \sin(k_{\frac{1}{2}}\Delta x) - 2ie^{ik\Delta x} \sin(k_{\frac{1}{2}}\Delta x)}{\left[ 2 \cos(k_{\frac{1}{2}}\Delta x) + 2e^{ik_{\frac{1}{2}}\Delta x} + e^{ik_{\frac{1}{2}}\Delta x} + e^{ik_{\frac{3}{2}}\Delta x} \right]} \right) u_j \quad (11)$$

$$G_j = \left( H + \frac{2H^3}{\Delta x^2} \frac{2i \sin(k_{\frac{1}{2}}\Delta x) - 2ie^{ik\Delta x} \sin(k_{\frac{1}{2}}\Delta x)}{\left[ 2 \cos(k_{\frac{1}{2}}\Delta x) + 2e^{ik_{\frac{1}{2}}\Delta x} + 2e^{ik\Delta x} \cos(k_{\frac{1}{2}}\Delta x) \right]} \right) u_j \quad (12)$$

$$G_j = \left( H + \frac{2H^3 i}{\Delta x^2} \frac{\sin(k_{\frac{1}{2}}\Delta x) - e^{ik\Delta x} \sin(k_{\frac{1}{2}}\Delta x)}{\left[ \cos(k_{\frac{1}{2}}\Delta x) + e^{ik_{\frac{1}{2}}\Delta x} + e^{ik\Delta x} \cos(k_{\frac{1}{2}}\Delta x) \right]} \right) u_j \quad (13)$$

$$G_j = \left( H + \frac{2H^3 i}{\Delta x^2} \frac{(1 - e^{ik\Delta x}) \sin(k_{\frac{1}{2}}\Delta x)}{(1 + e^{ik\Delta x}) \cos(k_{\frac{1}{2}}\Delta x) + e^{ik_{\frac{1}{2}}\Delta x}} \right) u_j \quad (14)$$

So we have

$$\mathcal{G}_{FEM_2} = \left( H + \frac{2H^3 i}{\Delta x^2} \frac{(1 - e^{ik\Delta x}) \sin(k_{\frac{1}{2}}\Delta x)}{(1 + e^{ik\Delta x}) \cos(k_{\frac{1}{2}}\Delta x) + e^{ik_{\frac{1}{2}}\Delta x}} \right)$$

With taylor expansion

$$\mathcal{G}_{FEM_2} = H + \frac{H^3 k^2}{3} + O(\Delta x^2)$$

as desired.