1 Elliptic Equation

The linearised elliptic equation is

$$G = Hu - \frac{H^3}{3}u_{xx}$$

Want to find out the FEM approximation to this G' such that

$$G' = \mathcal{G}_{FE_1}u$$

for P^1 FEM

$$G' = \mathcal{G}_{FE_2}u$$

for P^2 FEM.

To do so we begin by first multiplying by an arbitrary test function v so that

$$Gv = Huv - \frac{H^3}{3}u_{xx}v$$

and then we integrate over the entire domain to get

$$\int_\Omega Gvdx = \int_\Omega Huvdx - \int_\Omega \frac{H^3}{3}u_{xx}vdx$$

for all v

We then make use of integration by parts, with Dirchlet boundaries to get

$$\int_\Omega Gvdx = \int_\Omega Huvdx + \int_\Omega \frac{H^3}{3}u_x v_x dx$$

Our FVM discretisation already has a natrual structure with linear functions intervals of $[x_{j-1/2}, x_{j+1/2}]$, to achieve this in P^1 we have our nodes at the boundaries, thus

So we can reformulate this as

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Gvdx = \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Huvdx + \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{H^3}{3}u_x v_x dx$$

or more aptly

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Gvdx - \int_{x_{j-1/2}}^{x_{j+3/2}} Huvdx - \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{H^3}{3} u_x v_x dx = 0$$

for all v

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Gvdx - H \int_{x_{j-1/2}}^{x_{j+3/2}} uvdx - \frac{H^3}{3} \int_{x_{j-1/2}}^{x_{j+3/2}} u_x v_x dx = 0$$

2 P1 FEM

We are going to corodainte tranform from x space the interval $[x_{i-1/2}, x_{i+1/2}, x_{i+3/2}]$ to the ξ space interval $[-1, 0, 1]$. To accomplish this we have the following relation

$$x = \xi \Delta x + x_{j+1/2}$$

Taking the derivative we see

$$dx = d\xi \Delta x$$

For this FEM we are intereseted in $G_{i+1/2}$ and then we can just get a shift operator to get the otherones. For FEM we replace the functions by their P1 approximations so

$$G \approx G' = \sum^{j} G_{j+1/2} v_{j+1/2}$$

$$u \approx u' = \sum^{j} u_{j+1/2} v_{j+1/2}$$

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} G'vdx - H \int_{x_{j-1/2}}^{x_{j+3/2}} u'vdx - \frac{H^3}{3} \int_{x_{j-1/2}}^{x_{j+3/2}} u'_x v_x dx = 0$$

We break this up into the integrals because of the domain of dependence of the basis functions is covered. We also just use a particular basis function as the test function, in particular we choose $v_{j+1/2}$

$$\int_{x_{j-1/2}}^{x_{j+3/2}} G'(x)v_{j+1/2}dx = \int_{-1}^{1} G'(\xi)v_{j+1/2}(\xi)\frac{d\xi}{dx}d\xi$$

$$= \Delta x \int_{-1}^{1} G'(\xi)v_{j+1/2}(\xi)d\xi$$

$$= \Delta x \int_{-1}^{1} \left(G_{j-1/2}v_{j-1/2} + G_{j+1/2}v_{j+1/2} + G_{j+3/2}v_{j+3/2}\right) v_{j+1/2}d\xi$$

$$= \Delta x \left[G_{j-1/2}\int_{-1}^{1} v_{j-1/2}v_{j+1/2}d\xi + G_{j+1/2}\int_{-1}^{1} v_{j+1/2}^2 d\xi + G_{j+3/2}\int_{-1}^{1} v_{j+3/2}v_{j+1/2}d\xi\right]$$

For linear elements it can be easily shown that

$$\int_{-1}^{1} v_{j+1/2}^2 d\xi = \frac{2}{3}$$

$$\int_{-1}^{1} v_{j-1/2}v_{j+1/2}d\xi = \frac{1}{6} = \int_{-1}^{1} v_{j+3/2}v_{j+1/2}d\xi$$

So

$$= \Delta x \left[G_{j-1/2}\frac{1}{6} + G_{j+1/2}\frac{2}{3} + G_{j+3/2}\frac{1}{6}\right]$$

$$= \frac{\Delta x}{6}\left[G_{j-1/2} + 4G_{j+1/2} + G_{j+3/2}\right]$$

$$-H\int_{x_{j-1/2}}^{x_{j+3/2}} u'vdx = -H\Delta x \int_{-1}^{1} u'vd\xi$$

$$= \Delta x \left[u_{j-1/2}\int_{-1}^{1} v_{j-1/2}v_{j+1/2}d\xi + u_{j+1/2}\int_{-1}^{1} v_{j+1/2}^2 d\xi + u_{j+3/2}\int_{-1}^{1} v_{j+3/2}v_{j+1/2}d\xi\right]$$

$$= -H\frac{\Delta x}{6}\left[u_{j-1/2} + 4u_{j+1/2} + u_{j+3/2}\right]$$

Also

$$-\frac{H^3}{3}\int_{x_{j-1/2}}^{x_{j+3/2}} u_x'v_x dx = \Delta x \frac{H^3}{3}\int_{-1}^{1} u_\xi' v_\xi (\frac{d\xi}{dx})^2 d\xi$$

$$= -\frac{1}{\Delta x}\frac{H^3}{3}\left[u_{j-1/2}\int_{-1}^{1}(v_{j-1/2})_\xi(v_{j+1/2})_\xi d\xi + u_{j+1/2}\int_{-1}^{1}(v_{j+1/2})^2_\xi d\xi + u_{j+3/2}\int_{-1}^{1}(v_{j+3/2}))_\xi(v_{j+1/2})\right.$$

It can be easily shown that

$$\int_{-1}^{1}(v_{j+1/2})^2_\xi d\xi = 2$$

$$\int_{-1}^{1}(v_{j-1/2})_\xi(v_{j+1/2})_\xi d\xi = -1 = \int_{-1}^{1}(v_{j+3/2}))_\xi(v_{j+1/2}))_\xi$$

$$= -\frac{H^3}{3\Delta x}\left[-u_{j-1/2} + 2u_{j+1/2} - u_{j+3/2}\right]$$

So we have

$$\Delta x\left[G_{j-1/2}\frac{1}{6} + G_{j+1/2}\frac{2}{3} + G_{j+3/2}\frac{1}{6}\right] =$$
$$H\frac{\Delta x}{6}\left[u_{j-1/2} + 2u_{j+1/2} + u_{j+3/2}\right] + \frac{H^3}{3\Delta x}\left[-u_{j-1/2} + 2u_{j+1/2} - u_{j+3/2}\right] \quad (1)$$

$$\left[G_{j-1/2} + 4G_{j+1/2} + G_{j+3/2}\right] =$$
$$H\left[u_{j-1/2} + 4u_{j+1/2} + u_{j+3/2}\right] + \frac{2H^3}{\Delta x^2}\left[-u_{j-1/2} + 2u_{j+1/2} - u_{j+3/2}\right] \quad (2)$$

let G and u be constant then