1 Elliptic Equation

The linearised elliptic equation is

$$G = H\upsilon - \frac{H^3}{3}\left(\frac{\partial^2 \upsilon}{\partial x^2}\right)$$

Taking the weak version of this we get that

$$\int_\Omega G v \; dx = H \int_\Omega \upsilon v \; dx - \frac{H^3}{3} \int_\Omega \frac{\partial^2 \upsilon}{\partial x^2} v \; dx$$

$$\int_\Omega G v \; dx = H \int_\Omega \upsilon v \; dx + \frac{H^3}{3} \int_\Omega \frac{\partial \upsilon}{\partial x}\frac{\partial v}{\partial x} \; dx$$

In particular for the basis function ϕ_j we must have

$$\int_\Omega G \phi_j \; dx = H \int_\Omega \upsilon \phi_j \; dx + \frac{H^3}{3} \int_\Omega \frac{\partial \upsilon}{\partial x}\frac{\partial \left(\phi_j\right)}{\partial x} \; dx$$

We use the FEM discretisation from []

$$G = \sum_j G^+_{j-1/2}\psi^+_{j-1/2} + G^-_{j+1/2}\psi^-_{j+1/2}$$

and

$$\upsilon = \sum_j \upsilon_{j-1/2}\phi_{j-1/2} + \upsilon_j\phi_j + \upsilon_{j+1/2}\phi_{j+1/2} \tag{1}$$

Now for our evolution equations we only need to to get the errors introduced from our calculation of $\upsilon_{j+1/2}$ and υ_j, as we can get $\upsilon_{j-1/2}$ from just a shift. We previously demonstrated how the coefficient matrices are calculated for the FEM so we now just skip ahead to give the equations.

The FEM gives

$$\sum_j \frac{\Delta x}{2}\begin{bmatrix} \frac{1}{3}G^+_{j-1/2} \\ \frac{2}{3}G^+_{j-1/2} + \frac{2}{3}G^-_{j+1/2} \\ \frac{1}{3}G^-_{j+1/2} \end{bmatrix} =$$

$$\sum_j \left(H\frac{\Delta x}{2}\begin{bmatrix} \frac{4}{15} & \frac{2}{15} & -\frac{1}{15} \\ \frac{2}{15} & \frac{16}{15} & \frac{2}{15} \\ -\frac{1}{15} & \frac{2}{15} & \frac{4}{15} \end{bmatrix} + \frac{2H^3}{3\Delta x}\begin{bmatrix} \frac{7}{6} & -\frac{4}{3} & \frac{1}{6} \\ -\frac{4}{3} & \frac{8}{3} & -\frac{4}{3} \\ \frac{1}{6} & -\frac{4}{3} & 7 \end{bmatrix}\right)\begin{bmatrix} u_{j-1/2} \\ u_j \\ u_{j+1/2} \end{bmatrix} \tag{2}$$

$$\sum_j \frac{\Delta x}{6}\begin{bmatrix} G^+_{j-1/2} \\ 2G^+_{j-1/2}+2G^-_{j+1/2} \\ G^-_{j+1/2}\end{bmatrix} =$$

$$\sum_j \left(H\frac{\Delta x}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} + \frac{H^3}{9\Delta x}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7\end{bmatrix}\right)\begin{bmatrix} u_{j-1/2} \\ u_j \\ u_{j+1/2}\end{bmatrix} \tag{3}$$

$$\sum_j \frac{\Delta x}{6}\begin{bmatrix} G^+_{j-1/2} \\ 2G^+_{j-1/2}+2G^-_{j+1/2} \\ G^-_{j+1/2}\end{bmatrix} =$$

$$\sum_j \left(H\frac{\Delta x}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} + \frac{H^3}{9\Delta x}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7\end{bmatrix}\right)\begin{bmatrix} e^{-ik\frac{\Delta x}{2}} \\ 1 \\ e^{ik\frac{\Delta x}{2}}\end{bmatrix} u_j \tag{4}$$

$$\sum_j \frac{\Delta x}{6}\begin{bmatrix} e^{-ik\Delta x}\mathcal{R}^+_2 \\ 2e^{-ik\Delta x}\mathcal{R}^+_2+2\mathcal{R}^-_2 \\ \mathcal{R}^-_2\end{bmatrix} G_j =$$

$$\sum_j \left(H\frac{\Delta x}{30}\begin{bmatrix} -5i\sin\left(k\frac{\Delta x}{2}\right)+3\cos\left(k\frac{\Delta x}{2}\right)+2 \\ 16+4\cos\left(k\frac{\Delta x}{2}\right) \\ 5i\sin\left(k\frac{\Delta x}{2}\right)+3\cos\left(k\frac{\Delta x}{2}\right)+2 \end{bmatrix} \right.$$
$$\left. + \frac{H^3}{9\Delta x}\begin{bmatrix} -6i\sin\left(k\frac{\Delta x}{2}\right)+8\cos\left(k\frac{\Delta x}{2}\right)-8 \\ 32\sin^2\left(\frac{k\Delta x}{4}\right) \\ 6i\sin\left(k\frac{\Delta x}{2}\right)+8\cos\left(k\frac{\Delta x}{2}\right)-8 \end{bmatrix}\right) u_j \tag{5}$$

assemble into complete matrices so we get, change notation to denote we sum the corresponding matrices together

$$
\begin{aligned}
&\sum_j \frac{\Delta x}{6} \begin{bmatrix} e^{-ik\Delta x}\mathcal{R}_2^+ + e^{-ik\Delta x}\mathcal{R}_2^- \\ 2e^{-ik\Delta x}\mathcal{R}_2^+ + 2\mathcal{R}_2^- \\ \mathcal{R}_2^- + \mathcal{R}_2^+ \end{bmatrix} G_j = \\
&\qquad \sum_j \left(H\frac{\Delta x}{30} \begin{bmatrix} e^{-ik\frac{\Delta x}{2}} \left(4\cos\left(\frac{k\Delta x}{2}\right) - 2\cos\left(k\Delta x\right) + 8\right) \\ 16 + 4\cos\left(k\frac{\Delta x}{2}\right) \\ e^{ik\frac{\Delta x}{2}} \left(4\cos\left(\frac{k\Delta x}{2}\right) - 2\cos\left(k\Delta x\right) + 8\right) \end{bmatrix} \right. \\
&\qquad\quad \left. + \frac{H^3}{9\Delta x} \begin{bmatrix} e^{-ik\frac{\Delta x}{2}} \left(-16\cos\left(\frac{k\Delta x}{2}\right) + 2\cos\left(k\Delta x\right) + 14\right) \\ 32\sin^2\left(\frac{k\Delta x}{4}\right) \\ e^{ik\frac{\Delta x}{2}} \left(-16\cos\left(\frac{k\Delta x}{2}\right) + 2\cos\left(k\Delta x\right) + 14\right) \end{bmatrix} \right) u_j
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
&\sum_j \frac{\Delta x}{6} \begin{bmatrix} \mathcal{R}_2^+ + \mathcal{R}_2^- \\ 2 \\ \mathcal{R}_2^- + \mathcal{R}_2^+ \end{bmatrix}^T \begin{bmatrix} e^{-ik\Delta x} \\ 1 \\ 1 \end{bmatrix} G_j = \\
&\qquad \sum_j \left(H\frac{\Delta x}{30} \begin{bmatrix} 4\cos\left(\frac{k\Delta x}{2}\right) - 2\cos\left(k\Delta x\right) + 8 \\ 16 + 4\cos\left(k\frac{\Delta x}{2}\right) \\ 4\cos\left(\frac{k\Delta x}{2}\right) - 2\cos\left(k\Delta x\right) + 8 \end{bmatrix} \right. \\
&\qquad\quad \left. + \frac{H^3}{9\Delta x} \begin{bmatrix} -16\cos\left(\frac{k\Delta x}{2}\right) + 2\cos\left(k\Delta x\right) + 14 \\ 32\sin^2\left(\frac{k\Delta x}{4}\right) \\ -16\cos\left(\frac{k\Delta x}{2}\right) + 2\cos\left(k\Delta x\right) + 14 \end{bmatrix} \right) \begin{bmatrix} e^{-ik\frac{\Delta x}{2}} \\ 1 \\ e^{ik\frac{\Delta x}{2}} \end{bmatrix} u_j
\end{aligned}
\tag{7}
$$

The equation for $u_{j+1/2}$ is then

$$
\begin{aligned}
&\frac{\Delta x}{6}\left(\mathcal{R}_2^+ + \mathcal{R}_2^-\right) G_j = \\
&\qquad \left(H\frac{\Delta x}{30}\left(4\cos\left(\frac{k\Delta x}{2}\right) - 2\cos\left(k\Delta x\right) + 8\right) \right. \\
&\qquad\quad \left. + \frac{H^3}{9\Delta x}\left(-16\cos\left(\frac{k\Delta x}{2}\right) 2\cos\left(k\Delta x\right) + 14\right) \right) e^{ik\frac{\Delta x}{2}} u_j.
\end{aligned}
\tag{8}
$$