1 Elliptic Equation

The linearised elliptic equation is

$$G = Hu - \frac{H^3}{3}u_{xx}$$

Want to find out the FEM approximation factor $\mathcal{G}_{FE_1}$ such that

$$G = \mathcal{G}_{FE_1}u$$

To do so we begin by first multiplying by an arbitrary test function v so that

$$Gv = Huv - \frac{H^3}{3}u_{xx}v$$

and then we integrate over the entire domain to get

$$\int_\Omega Gvdx = \int_\Omega Huvdx - \int_\Omega \frac{H^3}{3}u_{xx}vdx$$

for all v

We then make use of integration by parts, with Dirchlet boundaries to get

$$\int_\Omega Gvdx = \int_\Omega Huvdx + \int_\Omega \frac{H^3}{3}u_x v_x dx$$

We are going to use x_j as the nodes, which generate the basis functions ϕ_j, which for us will be the space of continuous linear elements. These are such that $\phi_j(x) \neq 0$ when $x_{j-1} < x < x_{j+1}$ and are the usual hat functions centered at x_j. So we can reformulate this as

$$\sum_j \int_{x_{j-1}}^{x_{j+1}} Gvdx = \sum_j \int_{x_{j-1}}^{x_{j+1}} Huvdx + \sum_j \int_{x_{j-1}}^{x_{j+1}} \frac{H^3}{3}u_x v_x dx$$

for all v

2 P1 FEM

We are going to coordinate tranform from x space the interval $[x_{j-1}, x_j, x_{j+1}]$ to the ξ space interval $[-1, 0, 1]$. To accomplish this we have the following relation

$$x = \xi \Delta x + x_j$$

Taking the derivatives we see
$dx = d\xi \Delta x$, $\frac{dx}{d\xi} = \Delta x$, $\frac{d\xi}{dx} = \frac{1}{\Delta x}$.

Our ϕ_j can be described in ξ space as

$$\phi_j = \begin{cases} 1+\xi & \xi < 0 \\ 1-\xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$\phi_{j-1} = \begin{cases} -\xi & \xi < 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$\phi_{j+1} = \begin{cases} \xi & \xi > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

For FEM we replace the functions by their P1 approximations so

$$G \approx G' = \sum_j G_j \phi_j$$

$$u \approx u' = \sum_j u_j \phi_j$$

$$\sum_j \int_{x_{j-1}}^{x_{j+1}} G' \phi_j dx - H \int_{x_{j-1}}^{x_{j+1}} u' \phi_j dx - \frac{H^3}{3} \int_{x_{j-1}}^{x_{j+1}} u'_x (\phi_x)_j dx = 0$$

For all ϕ_j. For this analysis we choose a particular basis function ϕ_j and we look at all the integrals. Beginning with the first one:

$$\int_{x_{j-1}}^{x_{j+1}} G'(x) v_j dx = \int_{-1}^{1} G'(\xi) v_j(\xi) \frac{dx}{d\xi} d\xi$$

$$= \Delta x \int_{-1}^{1} \left(G_{j-1}v_{j-1} + G_j v_j + G_{j+1}v_{j+1}\right) v_j d\xi$$

$$= \Delta x \left[G_{j-1} \int_{-1}^{1} v_{j-1}v_j d\xi + G_j \int_{-1}^{1} v_j v_j d\xi + G_{j+1} \int_{-1}^{1} v_{j+1}v_j d\xi \right]$$

We have that

$$\int_{-1}^{1} v_{j-1}v_j d\xi = \int_{-1}^{1} v_{j+1}v_j d\xi = \frac{1}{6}$$

$$\int_{-1}^{1} v_j v_j d\xi = \frac{2}{3}$$

So

$$= \Delta x \left[G_{j-1}\frac{1}{6} + G_j \frac{2}{3} + G_{j+1}\frac{1}{6} \right]$$

$$= \frac{\Delta x}{6} \left[G_{j-1} + 4G_j + G_{j+1} \right]$$

Similarly we have

$$-H \int_{x_{j-1}}^{x_{j+1}} u' v_j dx = -\frac{H\Delta x}{6} \left[u_{j-1} + 4u_j + u_{j+1} \right]$$

$$-\frac{H^3}{3} \int_{x_{j-1}}^{x_{j+1}} u'_x (v_j)_x dx = -\frac{H^3}{3} \int_{-1}^{1} u'_\xi \frac{d\xi}{dx} (v_\xi)_j \frac{d\xi}{dx} \frac{dx}{d\xi} d\xi$$

$$= -\frac{H^3}{3\Delta x} \int_{-1}^{1} u'_\xi (v_\xi)_j d\xi$$

We will now use $'$ to denote derivative

$$= -\frac{H^3}{3\Delta x} \int_{-1}^{1} \left(u'_{j-1}v'_{j-1} + u'_j v'_j + u'_{j+1}v'_{j+1} \right) v'_j d\xi$$

$$= -\frac{H^3}{3\Delta x} \left[u_{j-2} \int_{-1}^{1} v'_{j-1}v'_j d\xi + u_j \int_{-1}^{1} v'_j v'_j d\xi + u_{j+1} \int_{-1}^{1} v'_{j+1}v'_j d\xi \right]$$

We have that

$$\int_{-1}^{1} v'_{j-1} v'_j d\xi = -1 = \int_{-1}^{1} v'_{j+1} v'_j d\xi$$

$$\int_{-1}^{1} v'_j v'_j d\xi = 2$$

Therefore

$$= -\frac{H^3}{3\Delta x}\left[-u_{j-1} + 2u_j - u_{j+1}\right]$$

Then our equation becomes

$$\frac{\Delta x}{6}\left[G_{j-1} + 4G_j + G_{j+1}\right] = \frac{H\Delta x}{6}\left[u_j + 4u_j + u_{j+1}\right] + \frac{H^3}{3\Delta x}\left[-u_{j-1} + 2u_j - u_{j+1}\right] \quad (4)$$

$$\left[G_{j-1} + 4G_j + G_{j+1}\right] = H\left[u_{j-1} + 4u_j + u_{j+1}\right] + \frac{2H^3}{\Delta x^2}\left[-u_{j-1} + 2u_j - u_{j+1}\right] \quad (5)$$

This formula is correct for $u = 1, x, x^2$

We now assume the following form for u and G.

Let quantity q is given by so that $q(x,t) = q(0,0)e^{i(\omega t + kx)}$. The use of this comes when we use our uniform grid in space, so that $q(x_j, t) = q_j$ then $q_{j\pm l} = q_j e^{\pm ikl\Delta x}$

Then we have

$$\left[G_j e^{-ik\Delta x} + 4G_j + G_j e^{ik\Delta x}\right] = H\left[u_j e^{-ik\Delta x} + 4u_j + u_j e^{ik\Delta x}\right] + \frac{2H^3}{\Delta x^2}\left[-u_j e^{-ik\Delta x} + 2u_j - u_j e^{ik\Delta x}\right] \quad (6)$$

$$G_j \left[e^{-ik\Delta x} + 4 + e^{ik\Delta x}\right] = \left(H \left[e^{-ik\Delta x} + 4 + e^{ik\Delta x}\right] + \frac{2H^3}{\Delta x^2} \left[-e^{-ik\Delta x} + 2 - e^{ik\Delta x}\right] \right) u_j \quad (7)$$

$$G_j = \left(H + \frac{2H^3}{\Delta x^2} \frac{\left[-e^{-ik\Delta x} + 2 - e^{ik\Delta x}\right]}{\left[e^{-ik\Delta x} + 4 + e^{ik\Delta x}\right]} \right) u_j \quad (8)$$

$$G_j = \left(H + \frac{2H^3}{\Delta x^2} \frac{2 - 2\cos(k\Delta x)}{4 + 2\cos(k\Delta x)} \right) u_j \quad (9)$$

$$G_j = \left(H + \frac{2H^3}{\Delta x^2} \frac{1 - \cos(k\Delta x)}{2 + \cos(k\Delta x)} \right) u_j$$

We want something like

$$\frac{k^2}{3} \approx \frac{2}{\Delta x^2} \frac{1 - \cos(k\Delta x)}{2 + \cos(k\Delta x)}$$

and we want to compare it to the FD approximation

$$\frac{k^2}{3} \approx \frac{2}{3\Delta x^2} (1 - \cos(k\Delta x))$$

For the FEM we have the taylor series

$$\frac{2}{\Delta x^2} \frac{1 - \cos(k\Delta x)}{2 + \cos(k\Delta x)} = \frac{k^2}{3} + \frac{k^4 \Delta x^2}{36} + \frac{k^6 \Delta x^4}{1080} - \frac{17k^8 \Delta x^6}{181440} - \frac{11k^{10} \Delta x^8}{604800} + O(\Delta x^{10}) \quad (10)$$

$$\frac{2}{3\Delta x^2} (1 - \cos(k\Delta x)) = \frac{k^2}{3} - \frac{k^4 \Delta x^2}{36} + \frac{k^6 \Delta x^4}{1080} - \frac{k^8 \Delta x^6}{60480} - \frac{k^{10} \Delta x^8}{5443200} + O(\Delta x^{10}) \quad (11)$$

We can see that because the FD error alternates earlier its error is actualy slightly smaler than the FEM error, hence why it is worse. Although this is a better approximationg than the discontinuous edges one.