Importance of Dispersion for Shoaling Waves

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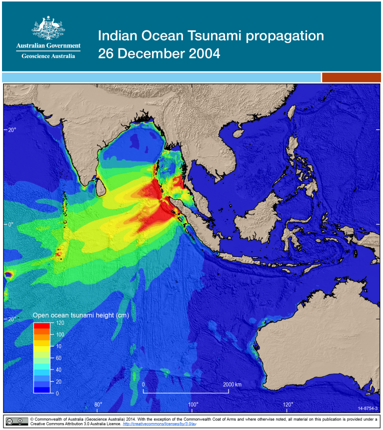
Definitions

- Dispersion:
  Waves of different frequencies travel at different speeds.
- Shoaling:
  Waves increase in height and steepness as they move into shallower water.

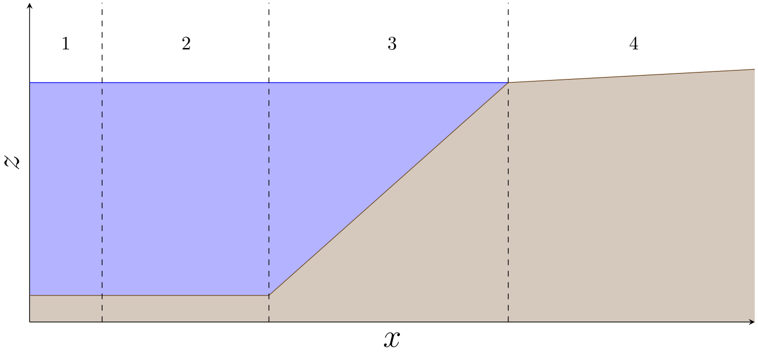
Introduction

- Motivation : Tsunamis
- Model : Shallow water wave and Serre equations
- Experiment : Comparison of numerical solutions

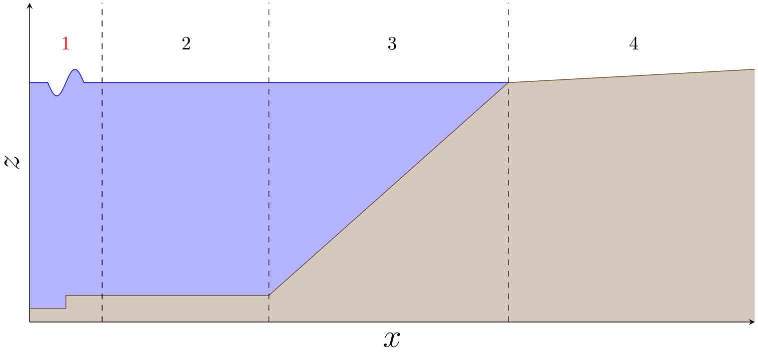
Indian ocean tsunami



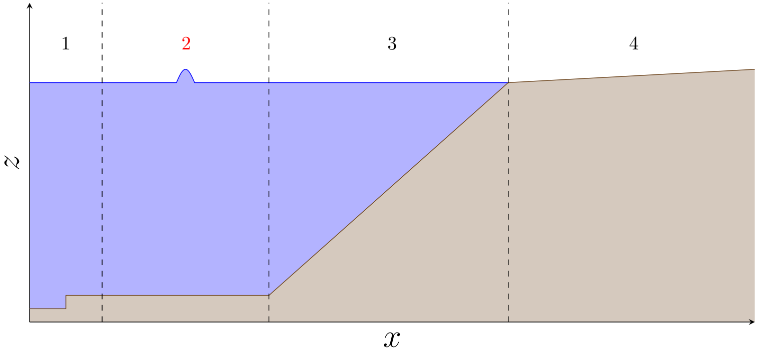
Tsunami diagram



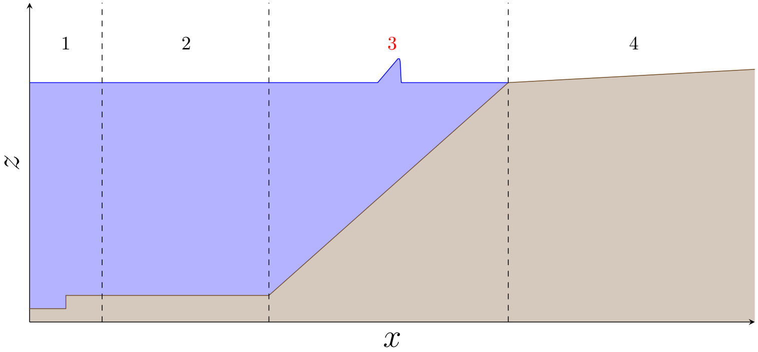
1 : Generation



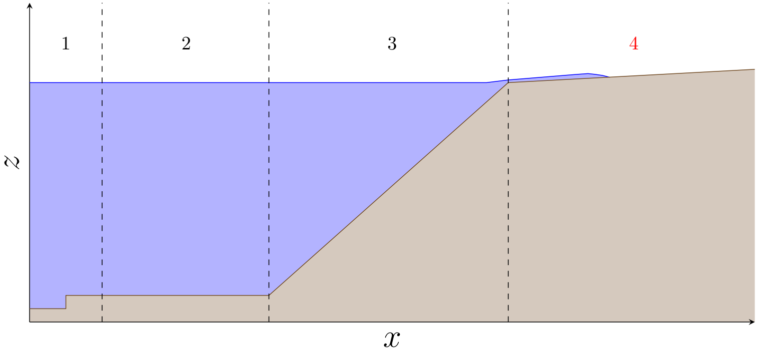
2 : Propagation far from coast



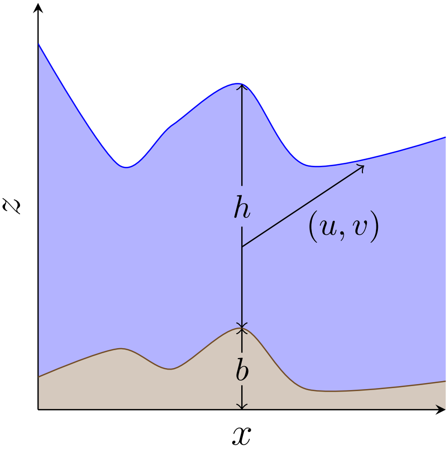
3 : Propagation near coast



4 : Inundation



Depth averaged equations



Shallow water wave equations

- Wavelengths (λ) $>>$ water depth (H) ($\lambda \geq 20H$).
- Horizontal velocity constant over z.
- Vertical velocity is 0.
- Pressure is hydrostatic $p(z) = \rho g(h + b - z)$.

Shallow water wave equations

Conservation of mass:

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0.$$

Conservation of momentum:

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2}\right) + gh\frac{\partial b}{\partial x} = 0.$$

Serre equations

- No restrictions.
- Horizontal velocity is constant over z.
- Vertical velocity is linear in z:

$$v'(z) = u\frac{\partial b}{\partial x} - (z - b)\frac{\partial u}{\partial x}.$$

- Pressure:

$$p(z) = \rho g(h + b - z) + \rho(h+b-z)\Psi + \frac{\rho}{2}(h+b-z)(h-b+z)\Phi,$$

$$\Psi = \frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial b}{\partial x}\,,\ \Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}.$$

Serre equations

Conservation of mass:

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0.$$

Conservation of momentum:

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2}\right) + gh\frac{\partial b}{\partial x} + \frac{\partial}{\partial x}\left(\frac{h^2}{2}\Psi + \frac{h^3}{3}\Phi\right) + \frac{\partial b}{\partial x}\left(h\Psi + \frac{h^2}{2}\Phi\right) = 0,$$

$$\Psi = \frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial b}{\partial x}, \quad \Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}.$$

Differences

Differences:

- Dispersion.
- Higher order terms.

Are they important for tsunamis?

Aim

- Compare shallow water wave and Serre equations.
- Highlight different behaviours.
- Highlight possible impacts these differences could make on current simulations.

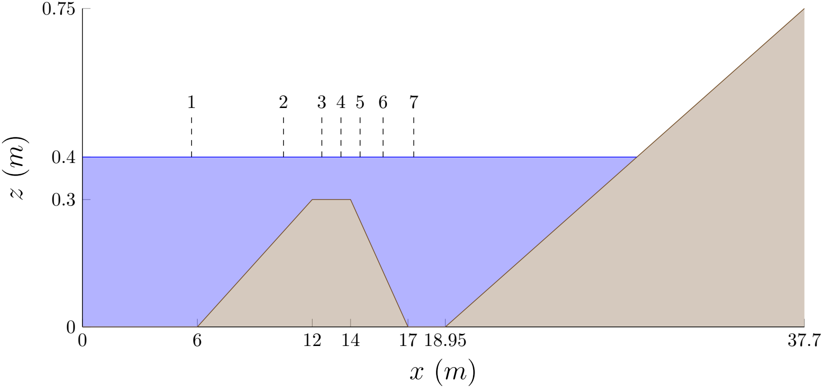
Numerical Solvers

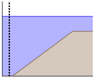
- Shallow water wave equations: ANUGA, second-order finite volume method.
- Serre equations: second-order finite volume method (same technique as ANUGA) and a second-order finite difference method.

Experiments

- Experimental results of Beji and Battjes (1994).
- Artificial example replicating common phenomena.

Periodic waves over a submerged bar: initial conditions





Wave gauge 1: boundary condition

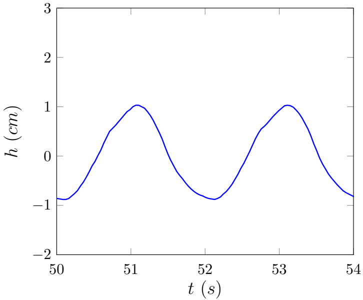


Figure: Low frequency $\lambda = 3.69m$ and $H = 0.4m$.

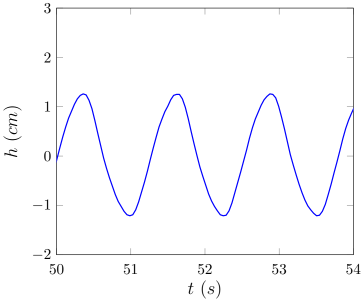
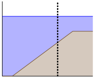


Figure: High frequency $\lambda = 2.05m$ and $H = 0.4m$.



Wave gauge 2: experimental result

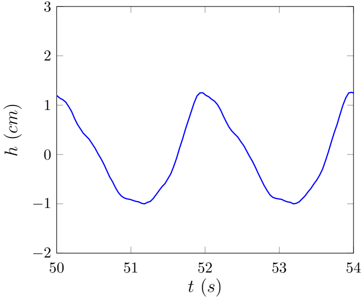


Figure: Low frequency.

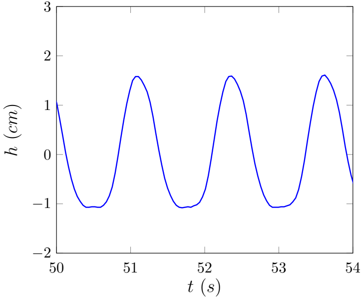
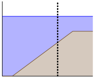


Figure: High frequency.



Wave gauge 2: shallow water wave equation

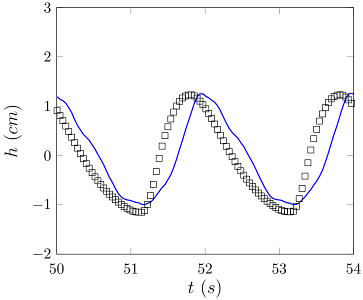
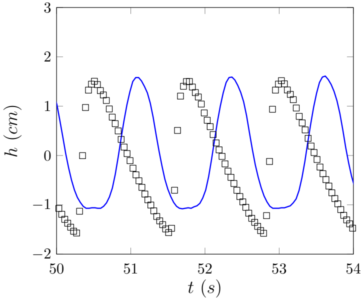


Figure: Low frequency.



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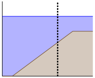


Wave gauge 2: all results

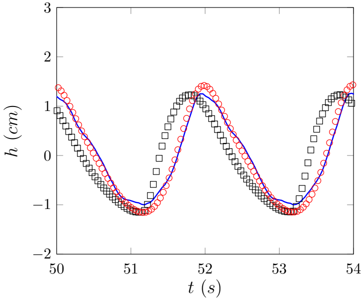


Figure: Low frequency.

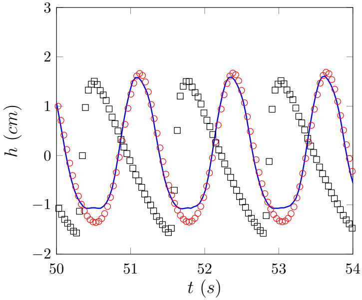
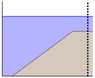


Figure: High frequency.



Wave gauge 4: experimental result

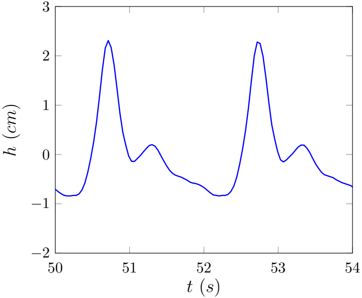


Figure: Low frequency.

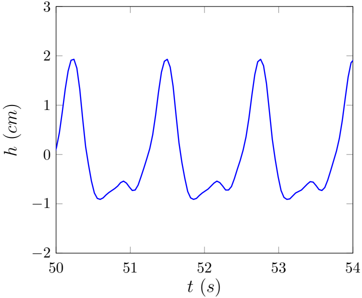
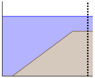


Figure: High frequency.



Wave gauge 4: shallow water wave equation

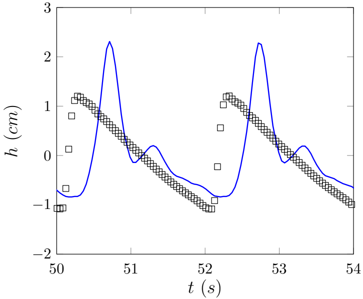


Figure: Low frequency.

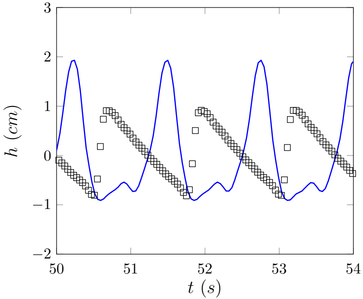
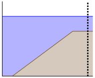


Figure: High frequency.



Wave gauge 4: all results

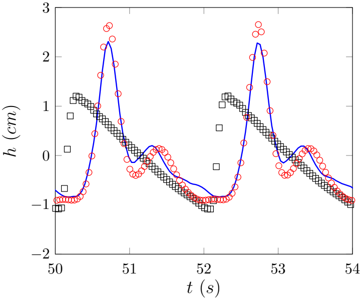


Figure: Low Frequency.

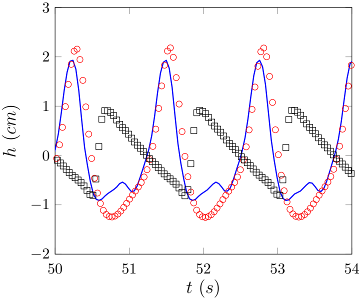
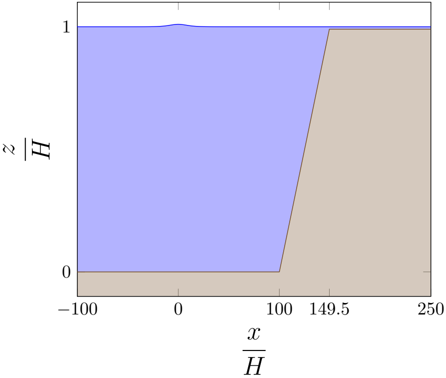
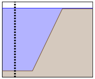


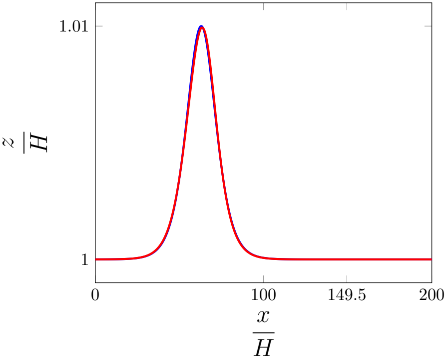
Figure: High Frequency.

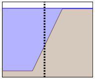
Solitary wave over a constant slope: initial conditions



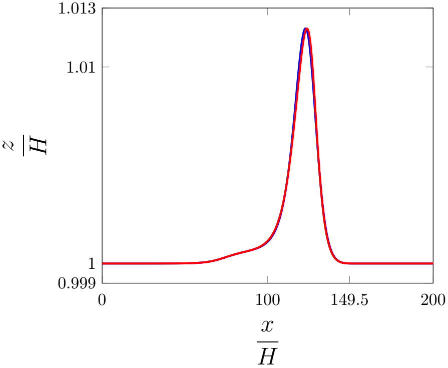


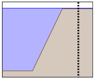
Before slope



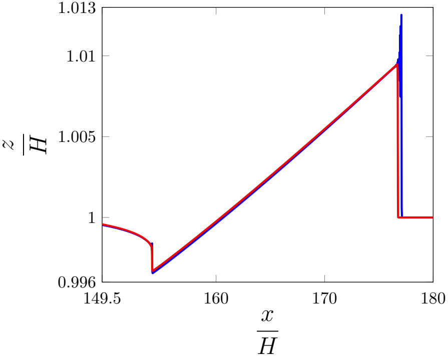


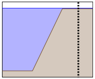
Shoaling



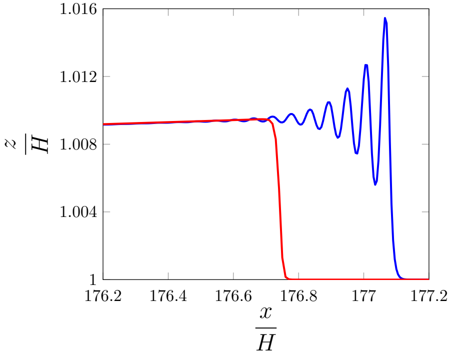


Bore formation





Front of bore



Tsunami wave trains

Figure: Indian ocean tsunami (2004).

Figure: Tohoku tsunami (2011).

Conclusion

- Dispersion plays an important role when wavelengths are not long compared to water depths.
- Dispersion is not important for shoaling of long wavelength waves.
- Dispersion is an important effect for waves after shoaling has occurred.
- For shoaled waves our current models may underestimate wave amplitude and predict later arrival times.