1 Linearised Equations

$$G = uh - \frac{h^3}{3}u_{xx}$$

$$\eta_t + hu_x = 0$$

$$hu_t - \frac{h^3}{3}u_{xxt} + gh\eta_x = 0$$

$$(G)_t + gh\eta_x = 0$$

2 Numerical Approximation

We investigate our numerical technique by adding in a fourier mode so $W_j = W_0 e^{i(vt+kx_j)}$, and rewriting the equations using our spatial discretisation

2.1 G

Analytic:

$$G_j = u_j h_j - (\frac{h_j^3}{3}u_{xx})_j$$

Numerical approximation, we used second order central differences so we replace the second derivative of u with this approximation to it So we get

$$G_j = u_j h_j - \frac{h_j^3}{3}\left(\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}\right)$$

$$G_j = u_0 e^{i(vt+kx_j)} h_0 - \frac{h_0}{3}u_0\left(\frac{e^{i(vt+kx_{j+1})} - 2e^{i(vt+kx_j)} + e^{i(vt+kx_{j-1})}}{\Delta x^2}\right)$$

$$G_j = u_0 e^{i(vt+kx_j)} h_0 - \frac{h_0}{3}u_0\left(\frac{e^{i(vt+kx_j)+ik\Delta x} - 2e^{i(vt+kx_j)} + e^{i(vt+kx_j)+ik\Delta x}}{\Delta x^2}\right)$$

$$G_j = u_j h_0 - \frac{h_0}{3}u_j\left(\frac{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}{\Delta x^2}\right)$$

$$G_j = u_j \left(h_0 - \frac{h_0^3}{3} \left(\frac{2\cos\left(k\Delta x\right) - 2}{\Delta x^2} \right) \right)$$

We are dealing with time continuous variables so, we first take the derivative in time exactly for the Fourier nodes so that:

So what we have is something that depends on the order used to approximate $u_x x$, lets call it $\mathcal{C}_2$ Thus:

$$\mathcal{C}_2 = \frac{2\cos\left(k\Delta x\right) - 2}{\Delta x^2}$$

$$G_j = u_j \left(h_0 - \frac{h_0^3}{3} \mathcal{C}_2 \right)$$

Furthermore we will call this whole thing $\mathcal{G}_2$ So we have

$$\mathcal{G}_2 = \left(h_0 - \frac{h_0^3}{3} \mathcal{C}_2 \right)$$

then

$$G_j = u_j \mathcal{G}_2$$

Now we move on to

$$\eta_t + h u_x = 0$$

our equations are time continuous so that:

$$\eta_t + h u_x = 0$$

$$iv\eta + h u_x = 0$$

next we approximate

our conservation equations of the form

$$q_t + \left[f(q)\right]_x = 0$$

by

$$q_t + \frac{1}{\Delta x}\left[F_{j+1/2} - F_{j-1/2}\right] = 0$$

where $F_{j\pm1/2}$ given by Kurganovs method. In this equation h is constant so $f(\eta, u) = hu$. We start Kurganovs method by doing a reconstruction, we start by doing a central differencing approximation to obtain that we note that the result is something like

$$q^-_{j+1/2} = q_j + \frac{q_{j+1} - q_{j-1}}{4}$$

$$q^+_{j+1/2} = q_{j+1} + \frac{q_{j+2} - q_j}{4}$$

Applying our fourier mode

$$q^-_{j+1/2} = q_j + \frac{q_j e^{ik\Delta x} - q_j e^{-ik\Delta x}}{4}$$

$$q^-_{j+1/2} = q_j \left(1 + \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{4}\right)$$

$$q^-_{j+1/2} = q_j \left(1 + \frac{2i\sin(k\Delta x)}{4}\right)$$

$$q^-_{j+1/2} = q_j \left(1 + \frac{i\sin(k\Delta x)}{2}\right)$$

for the plus we get the same result with a shift so that (because its around j+1) and a minus

$$q^+_{j+1/2} = q_j e^{ik\Delta x} \left(1 - \frac{i\sin(k\Delta x)}{2}\right)$$

So we have that

$$\mathcal{R}^-_2 = 1 + \frac{i\sin(k\Delta x)}{2}$$

$$\mathcal{R}^+_2 = e^{ik\Delta x} \left(1 - \frac{i\sin(k\Delta x)}{2}\right)$$

$$q^-_{j+1/2} = \mathcal{R}^-_2 q_j$$

$$q^+_{j+1/2} = \mathcal{R}^+_2 q_j$$

for u and η, this happens to G aswell just because G and u are related by a factor. Next we have to use the wavespeeds:

$$a^+_{j+1/2} = \max\left(0, u^+_{j+1/2} + \sqrt{gh_0}\sqrt{1 + \eta^+_{j+1/2}/h_0}, u^-_{j+1/2} + \sqrt{gh_0}\sqrt{1 + \eta^-_{j+1/2}/h_0}\right)$$

$$a^-_{j+1/2} = \min\left(0, u^+_{j+1/2} - \sqrt{gh_0}\sqrt{1+\eta^+_{j+1/2}/h_0}, u^-_{j+1/2} - \sqrt{gh_0}\sqrt{1+\eta^-_{j+1/2}/h_0}\right)$$

Up to order ϵ $\sqrt{1+\eta/h_0} = 1+\eta/h_0$, we will also simplify our method by only choosing the minus subscript wavespeeds so that

$$a^+_{j+1/2} = u^-_{j+1/2} + \sqrt{gh_0}\left(1+\eta^-_{j+1/2}/h_0\right)$$

$$a^-_{j+1/2} = u^-_{j+1/2} - \sqrt{gh_0}\left(1+\eta^-_{j+1/2}/h_0\right)$$

So we have that

$$a^+_{j+1/2} = u_j\mathcal{R}_2^- + \sqrt{gh_0}\left(1+\eta_j\mathcal{R}_2^-/h_0\right)$$

$$a^+_{j+1/2} = u_j\mathcal{R}_2^- + \sqrt{gh_0}\eta_j\mathcal{R}_2^-/h_0 + \sqrt{gh_0}$$

$$a^+_{j+1/2} = u_j\mathcal{R}_2^- + \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j + \sqrt{gh_0}$$

Similarly

$$a^-_{j+1/2} = u_j\mathcal{R}_2^- - \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j - \sqrt{gh_0}$$

$$F_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}} f\left(q^-_{j+\frac{1}{2}}\right) - a^-_{j+\frac{1}{2}} f\left(q^+_{j+\frac{1}{2}}\right)}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \frac{a^+_{j+\frac{1}{2}}\, a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\left[q^+_{j+\frac{1}{2}} - q^-_{j+\frac{1}{2}}\right] \quad (1)$$

$$\begin{aligned} F_{j+\frac{1}{2}} = {} & \frac{\left[u_j\mathcal{R}_2^- + \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j + \sqrt{gh_0}\right] h\mathcal{R}_2^- u_j - \left[u_j\mathcal{R}_2^- - \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j - \sqrt{gh_0}\right] h\mathcal{R}_2^+ u_j}{\left[u_j\mathcal{R}_2^- + \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j + \sqrt{gh_0}\right] - \left[u_j\mathcal{R}_2^- - \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j - \sqrt{gh_0}\right]} \\ & + \frac{\left[u_j\mathcal{R}_2^- + \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j + \sqrt{gh_0}\right]\left[u_j\mathcal{R}_2^- - \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j - \sqrt{gh_0}\right]}{\left[u_j\mathcal{R}_2^- + \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j + \sqrt{gh_0}\right] - \left[u_j\mathcal{R}_2^- - \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j - \sqrt{gh_0}\right]}\left[\mathcal{R}_2^+\eta_j - \mathcal{R}_2^-\eta_j\right] \end{aligned} \quad (2)$$

First lets work on the denominator

$$\left[u_j\mathcal{R}_2^- + \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j + \sqrt{gh_0}\right] - \left[u_j\mathcal{R}_2^- - \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j - \sqrt{gh_0}\right] = \\ \left[u_j\mathcal{R}_2^- + \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j\right] - \left[u_j\mathcal{R}_2^- - \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j\right] + 2\sqrt{gh_0} \quad (3)$$

$$\left[u_j\mathcal{R}_2^- + \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j\right] - \left[u_j\mathcal{R}_2^- - \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j\right] + 2\sqrt{gh_0} = \\ 2\sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j + 2\sqrt{gh_0} \quad (4)$$

So we have

$$F_{j+\frac{1}{2}} = \frac{\left[u_j\mathcal{R}_2^- + \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j + \sqrt{gh_0}\right] h\mathcal{R}_2^- u_j - \left[u_j\mathcal{R}_2^- - \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j - \sqrt{gh_0}\right] h\mathcal{R}_2^+ u_j}{2\sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j + 2\sqrt{gh_0}} \\ + \frac{\left[u_j\mathcal{R}_2^- + \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j + \sqrt{gh_0}\right]\left[u_j\mathcal{R}_2^- - \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j - \sqrt{gh_0}\right]}{2\sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j + 2\sqrt{gh_0}}\left[\mathcal{R}_2^+\eta_j - \mathcal{R}_2^-\eta_j\right] \quad (5)$$

The numerator on the first is

$$\left[u_j\mathcal{R}_2^- + \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j + \sqrt{gh_0}\right] h\mathcal{R}_2^- u_j - \left[u_j\mathcal{R}_2^- - \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j - \sqrt{gh_0}\right] h\mathcal{R}_2^+ u_j = \\ u_j\mathcal{R}_2^- h\mathcal{R}_2^- u_j + \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j h\mathcal{R}_2^- u_j + \sqrt{gh_0}h\mathcal{R}_2^- u_j - u_j\mathcal{R}_2^- h\mathcal{R}_2^+ u_j - \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j h\mathcal{R}_2^+ u_j - \sqrt{gh_0}h\mathcal{R}_2^+ u_j \quad (6)$$

$$hu_j^2\left[\mathcal{R}_2^-\right]^2 + \sqrt{gh_0}\left[\mathcal{R}_2^-\right]^2\eta_j u_j + h_0\sqrt{gh_0}\mathcal{R}_2^- u_j - hu_j^2\mathcal{R}_2^-\mathcal{R}_2^+ - \sqrt{gh_0}\mathcal{R}_2^-\mathcal{R}_2^+ u_j\eta_j - h_0\sqrt{gh_0}\mathcal{R}_2^+ u_j \\ = h_0\mathcal{R}_2^-\left(\mathcal{R}_2^- - \mathcal{R}_2^+\right)u_j^2 + \sqrt{gh_0}\mathcal{R}_2^-\left(\mathcal{R}_2^- - \mathcal{R}_2^+\right)\eta_j u_j + h_0\sqrt{gh_0}\left(\mathcal{R}_2^- - \mathcal{R}_2^+\right)u_j \quad (7)$$

So we have that

$$F_{j+\frac{1}{2}} = \left(\mathcal{R}_2^- - \mathcal{R}_2^+\right) \frac{h_0 \mathcal{R}_2^- u_j^2 + \sqrt{gh_0}\mathcal{R}_2^- \eta_j u_j + h_0\sqrt{gh_0} u_j}{2\sqrt{\frac{g}{h_0}}\mathcal{R}_2^- \eta_j + 2\sqrt{gh_0}}$$
$$+ \frac{\left[u_j \mathcal{R}_2^- + \sqrt{\frac{g}{h_0}}\mathcal{R}_2^- \eta_j + \sqrt{gh_0}\right]\left[u_j \mathcal{R}_2^- - \sqrt{\frac{g}{h_0}}\mathcal{R}_2^- \eta_j - \sqrt{gh_0}\right]}{2\sqrt{\frac{g}{h_0}}\mathcal{R}_2^- \eta_j + 2\sqrt{gh_0}}\left[\mathcal{R}_2^+ \eta_j - \mathcal{R}_2^- \eta_j\right] \tag{8}$$

$$F_{j+\frac{1}{2}} = \left(\mathcal{R}_2^- - \mathcal{R}_2^+\right) \frac{h_0 \mathcal{R}_2^- u_j^2 + \frac{h_0}{2} u_j \left(2\sqrt{g/h_0}\mathcal{R}_2^- \eta_j + 2\sqrt{gh_0}\right)}{2\sqrt{\frac{g}{h_0}}\mathcal{R}_2^- \eta_j + 2\sqrt{gh_0}}$$
$$+ \frac{\left[u_j \mathcal{R}_2^- + \sqrt{\frac{g}{h_0}}\mathcal{R}_2^- \eta_j + \sqrt{gh_0}\right]\left[u_j \mathcal{R}_2^- - \sqrt{\frac{g}{h_0}}\mathcal{R}_2^- \eta_j - \sqrt{gh_0}\right]}{2\sqrt{\frac{g}{h_0}}\mathcal{R}_2^- \eta_j + 2\sqrt{gh_0}}\left[\mathcal{R}_2^+ \eta_j - \mathcal{R}_2^- \eta_j\right] \tag{9}$$

$$F_{j+\frac{1}{2}} = \left(\mathcal{R}_2^- - \mathcal{R}_2^+\right) \frac{h_0}{2} u_j + \left(\mathcal{R}_2^- - \mathcal{R}_2^+\right) \frac{h_0 \mathcal{R}_2^- u_j^2}{2\sqrt{\frac{g}{h_0}}\mathcal{R}_2^- \eta_j + 2\sqrt{gh_0}}$$
$$+ \frac{\left[u_j \mathcal{R}_2^- + \sqrt{\frac{g}{h_0}}\mathcal{R}_2^- \eta_j + \sqrt{gh_0}\right]\left[u_j \mathcal{R}_2^- - \sqrt{\frac{g}{h_0}}\mathcal{R}_2^- \eta_j - \sqrt{gh_0}\right]}{2\sqrt{\frac{g}{h_0}}\mathcal{R}_2^- \eta_j + 2\sqrt{gh_0}}\left[\mathcal{R}_2^+ \eta_j - \mathcal{R}_2^- \eta_j\right] \tag{10}$$

$$F_{j+\frac{1}{2}} = \left(\mathcal{R}_2^- - \mathcal{R}_2^+\right) \frac{h_0}{2} u_j + \sqrt{h_0^3}\frac{\mathcal{R}_2^- - \mathcal{R}_2^+}{2\sqrt{g}} \frac{\mathcal{R}_2^- u_j^2}{\mathcal{R}_2^- \eta_j + h_0}$$
$$+ \frac{\left[u_j \mathcal{R}_2^- + \sqrt{\frac{g}{h_0}}\mathcal{R}_2^- \eta_j + \sqrt{gh_0}\right]\left[u_j \mathcal{R}_2^- - \sqrt{\frac{g}{h_0}}\mathcal{R}_2^- \eta_j - \sqrt{gh_0}\right]}{2\sqrt{\frac{g}{h_0}}\mathcal{R}_2^- \eta_j + 2\sqrt{gh_0}}\left[\mathcal{R}_2^+ \eta_j - \mathcal{R}_2^- \eta_j\right] \tag{11}$$

Next we look at the final numerator:

$$\left[u_j\mathcal{R}_2^- + \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j + \sqrt{gh_0}\right]\left[u_j\mathcal{R}_2^- - \sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j - \sqrt{gh_0}\right] = \left[u_j\mathcal{R}_2^-\right]^2 - \left[\sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j + \sqrt{gh_0}\right]^2 \tag{12}$$

$$\left[u_j\mathcal{R}_2^-\right]^2 - \left[\sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j + \sqrt{gh_0}\right]^2 = u_j^2\left[\mathcal{R}_2^-\right]^2 - \frac{g}{h_0}\mathcal{R}_2^-\mathcal{R}_2^-\eta_j^2 - 2\sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j\sqrt{gh_0} - gh_0 \tag{13}$$

$$u_j^2\left[\mathcal{R}_2^-\right]^2 - \frac{g}{h_0}\mathcal{R}_2^-\mathcal{R}_2^-\eta_j^2 - 2\sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j\sqrt{gh_0} - gh_0 = u_j^2\left[\mathcal{R}_2^-\right]^2 - \frac{g}{h_0}\left[\mathcal{R}_2^-\right]^2\eta_j^2 - 2g\mathcal{R}_2^-\eta_j - gh_0 \tag{14}$$

So we have that

$$F_{j+\frac{1}{2}} = \left(\mathcal{R}_2^- - \mathcal{R}_2^+\right)\frac{h_0}{2}u_j + \sqrt{h_0^3}\frac{\mathcal{R}_2^- - \mathcal{R}_2^+}{2\sqrt{g}}\frac{\mathcal{R}_2^-u_j^2}{\mathcal{R}_2^-\eta_j + h_0} + \frac{u_j^2\left[\mathcal{R}_2^-\right]^2 - \frac{g}{h_0}\left[\mathcal{R}_2^-\right]^2\eta_j^2 - 2g\mathcal{R}_2^-\eta_j - gh_0}{2\sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j + 2\sqrt{gh_0}}\left[\mathcal{R}_2^+\eta_j - \mathcal{R}_2^-\eta_j\right] \tag{15}$$

$$F_{j+\frac{1}{2}} = \left(\mathcal{R}_2^- - \mathcal{R}_2^+\right)\frac{h_0}{2}u_j + \sqrt{h_0^3}\frac{\mathcal{R}_2^- - \mathcal{R}_2^+}{2\sqrt{g}}\frac{\mathcal{R}_2^-u_j^2}{\mathcal{R}_2^-\eta_j + h_0} + \frac{u_j^2\left[\mathcal{R}_2^-\right]^2 - gh_0 - \frac{g}{h_0}\left[\mathcal{R}_2^-\right]^2\eta_j^2 - 2g\mathcal{R}_2^-\eta_j}{2\sqrt{\frac{g}{h_0}}\mathcal{R}_2^-\eta_j + 2\sqrt{gh_0}}\left[\mathcal{R}_2^+\eta_j - \mathcal{R}_2^-\eta_j\right] \tag{16}$$

$$F_{j+\frac{1}{2}} = \left(\mathcal{R}_2^- - \mathcal{R}_2^+\right)\frac{h_0}{2}u_j + \sqrt{h_0^3}\frac{\mathcal{R}_2^- - \mathcal{R}_2^+}{2\sqrt{g}}\frac{\mathcal{R}_2^-u_j^2}{\mathcal{R}_2^-\eta_j + h_0} + \frac{\sqrt{h_0}}{2\sqrt{g}}\frac{u_j^2\left[\mathcal{R}_2^-\right]^2 - gh_0 - \frac{g}{h_0}\left[\mathcal{R}_2^-\right]^2\eta_j^2 - 2g\mathcal{R}_2^-\eta_j}{\mathcal{R}_2^-\eta_j + h_0}\left[\mathcal{R}_2^+\eta_j - \mathcal{R}_2^-\eta_j\right] \tag{17}$$

$$F_{j+\frac{1}{2}} = \left(\mathcal{R}_2^- - \mathcal{R}_2^+\right)\frac{h_0}{2}u_j + \sqrt{h_0^3}\frac{\mathcal{R}_2^- - \mathcal{R}_2^+}{2\sqrt{g}}\frac{\mathcal{R}_2^- u_j^2}{\mathcal{R}_2^-\eta_j + h_0}$$
$$+ \left[\mathcal{R}_2^+ - \mathcal{R}_2^-\right]\frac{\sqrt{h_0}}{2\sqrt{g}}\frac{u_j^2\left[\mathcal{R}_2^-\right]^2 - gh_0 - \frac{g}{h_0}\left[\mathcal{R}_2^-\right]^2\eta_j^2 - 2g\mathcal{R}_2^-\eta_j}{\mathcal{R}_2^-\eta_j + h_0}\eta_j \quad (18)$$

Up to order ϵ we have

$$F_{j+\frac{1}{2}} = \left(\mathcal{R}_2^- - \mathcal{R}_2^+\right)\frac{h_0}{2}u_j + \sqrt{h_0^3}\frac{\mathcal{R}_2^- - \mathcal{R}_2^+}{2\sqrt{g}}\frac{\mathcal{R}_2^- u_j^2}{\mathcal{R}_2^-\eta_j + h_0}$$
$$+ \left[\mathcal{R}_2^+ - \mathcal{R}_2^-\right]\frac{\sqrt{h_0}}{2\sqrt{g}}\frac{-gh_0 - 2g\mathcal{R}_2^-\eta_j}{\mathcal{R}_2^-\eta_j + h_0}\eta_j \quad (19)$$

$$F_{j+\frac{1}{2}} = \left(\mathcal{R}_2^- - \mathcal{R}_2^+\right)\frac{h_0}{2}u_j + \sqrt{h_0^3}\frac{\mathcal{R}_2^- - \mathcal{R}_2^+}{2\sqrt{g}}\frac{\mathcal{R}_2^- u_j^2}{\mathcal{R}_2^-\eta_j + h_0}$$
$$- \left[\mathcal{R}_2^+ - \mathcal{R}_2^-\right]\frac{\sqrt{gh_0}}{2}\frac{2\mathcal{R}_2^-\eta_j + h_0}{\mathcal{R}_2^-\eta_j + h_0}\eta_j \quad (20)$$

So whats left as two probelms are

$$\frac{\mathcal{R}_2^- u_j^2}{\mathcal{R}_2^-\eta_j + h_0}$$

and

$$\frac{2\mathcal{R}_2^-\eta_j + h_0}{\mathcal{R}_2^-\eta_j + h_0}$$

We can again make use of Taylor expansions to get terms up to order ϵ

$$\frac{2\mathcal{R}_2^-\eta_j + h_0}{\mathcal{R}_2^-\eta_j + h_0} = 1 + \frac{\mathcal{R}_2^-\eta_j}{h_0}$$
$$\frac{\mathcal{R}_2^- u_j^2}{\mathcal{R}_2^-\eta_j + h_0} = \mathcal{R}_2^- u_j^2\frac{1}{\mathcal{R}_2^-\eta_j + h_0}$$

none of the terms in the taylor expansion of this are of order ϵ so we have that

$$F_{j+\frac{1}{2}} = \left(\mathcal{R}_2^- - \mathcal{R}_2^+\right) \frac{h_0}{2} u_j$$
$$- \left[\mathcal{R}_2^+ - \mathcal{R}_2^-\right] \left[1 + \frac{\mathcal{R}_2^- \eta_j}{h_0}\right] \eta_j \quad (21)$$

retaining only ϵ terms

$$F_{j+\frac{1}{2}} = \left(\mathcal{R}_2^- - \mathcal{R}_2^+\right) \frac{h_0}{2} u_j - \left[\mathcal{R}_2^+ - \mathcal{R}_2^-\right] \eta_j \quad (22)$$

We will use the following notation

$$\mathcal{F}_2^{(\eta,u)} = \left(\mathcal{R}_2^- - \mathcal{R}_2^+\right) \frac{h_0}{2} \quad (23)$$

$$\mathcal{F}_2^{(\eta,\eta)} = -\left[\mathcal{R}_2^+ - \mathcal{R}_2^-\right] \quad (24)$$

So

$$F_{j+\frac{1}{2}} = \mathcal{F}_2^{(\eta,u)} u_j - \mathcal{F}_2^{(\eta,\eta)} \eta_j \quad (25)$$

We can do a shift to get that

$$F_{j-\frac{1}{2}} = \left(\mathcal{R}_2^- - \mathcal{R}_2^+\right) \frac{h_0}{2} u_{j-1} - \left[\mathcal{R}_2^+ - \mathcal{R}_2^-\right] \eta_{j-1} \quad (26)$$

$$F_{j-\frac{1}{2}} = \left(\mathcal{R}_2^- - \mathcal{R}_2^+\right) \frac{h_0}{2} e^{-ik\Delta x} u_j - \left[\mathcal{R}_2^+ - \mathcal{R}_2^-\right] e^{-ik\Delta x} \eta_j \quad (27)$$

$$F_{j-\frac{1}{2}} = e^{-ik\Delta x} F_{j+\frac{1}{2}} \quad (28)$$

So we have that

$$iv\eta_j + \frac{1}{\Delta x}\left[\left(1 - e^{-ik\Delta x}\right) \mathcal{F}_2^{(\eta,u)} u_j + \left(1 - e^{-ik\Delta x}\right) \mathcal{F}_2^{(\eta,\eta)} \eta_j\right] = 0$$

Now we to deal with, since G is just a coefficient times u

$$(G)_t + gh\eta_x = 0$$

$$ivG_j + gh\eta_x = 0$$

Similar to the above process we get

$$\mathcal{F}_2^{(G,\eta)} = \left(\mathcal{R}_2^- - \mathcal{R}_2^+\right)\frac{gh_0}{2} \tag{29}$$

$$\mathcal{F}_2^{(G,u)} = -\left[\mathcal{R}_2^+ - \mathcal{R}_2^-\right]\mathcal{G}_2 \tag{30}$$

So we get that

$$iv\mathcal{G}_2 u_j + \frac{1}{\Delta x}\left[\left(1 - e^{-ik\Delta x}\right)\mathcal{F}_2^{(G,u)}u_j + \left(1 - e^{-ik\Delta x}\right)\mathcal{F}_2^{(G,\eta)}\eta_j\right] = 0$$

and

$$iv\eta_j + \frac{1}{\Delta x}\left[\left(1 - e^{-ik\Delta x}\right)\mathcal{F}_2^{(\eta,u)}u_j + \left(1 - e^{-ik\Delta x}\right)\mathcal{F}_2^{(\eta,\eta)}\eta_j\right] = 0$$

So we get

$$\begin{bmatrix} iv + \frac{1}{\Delta x}\left(1 - e^{-ik\Delta x}\right)\mathcal{F}_2^{(\eta,\eta)} & \frac{1}{\Delta x}\left(1 - e^{-ik\Delta x}\right)\mathcal{F}_2^{(\eta,u)} \\ \frac{1}{\Delta x}\left(1 - e^{-ik\Delta x}\right)\mathcal{F}_2^{(G,\eta)} & iv\mathcal{G} + \frac{1}{\Delta x}\left(1 - e^{-ik\Delta x}\right)\mathcal{F}_2^{(G,u)} \end{bmatrix}\begin{bmatrix} \eta_j \\ u_j \end{bmatrix} = 0$$

lets use $\mathcal{D} = \left(1 - e^{-ik\Delta x}\right)$ so we have

$$\begin{bmatrix} iv + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,\eta)} & \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,u)} \\ \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,\eta)} & iv\mathcal{G} + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,u)} \end{bmatrix}\begin{bmatrix} \eta_j \\ u_j \end{bmatrix} = 0$$

we get a nontrivial solution if

$$\left[iv + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,\eta)}\right]\left[iv\mathcal{G} + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,u)}\right] - \left[\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,u)}\right]\left[\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,\eta)}\right] = 0$$

$$\left[iv + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,\eta)}\right]\left[iv\mathcal{G} + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,u)}\right] - \frac{1}{\Delta x^2}\mathcal{D}^2\mathcal{F}_2^{(\eta,u)}\mathcal{F}_2^{(G,\eta)} = 0$$

$$-v^2\mathcal{G}+iv\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,u)}+iv\mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,\eta)}+\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,\eta)}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,u)}-\frac{1}{\Delta x^2}\mathcal{D}^2\mathcal{F}_2^{(\eta,u)}\mathcal{F}_2^{(G,\eta)} = 0$$

$$-\mathcal{G}v^2+\left(i\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,u)} + i\mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,\eta)}\right)v+\frac{1}{\Delta x^2}\mathcal{D}^2\left(\mathcal{F}_2^{(\eta,\eta)}\mathcal{F}_2^{(G,u)} - \mathcal{F}_2^{(\eta,u)}\mathcal{F}_2^{(G,\eta)}\right) = 0$$