A Finite Element-Volume Method for the Serre Equations

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Outline

Outline

- Motivation

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- Motivation
- Method

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- Method
- Results

Ocean Wave Hazards

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- Tsunamis

Sulawesi 2018 Tsunami

Figure: Sulawesi Tsunami (Indonesia, 2018).

Ocean Wave Hazards

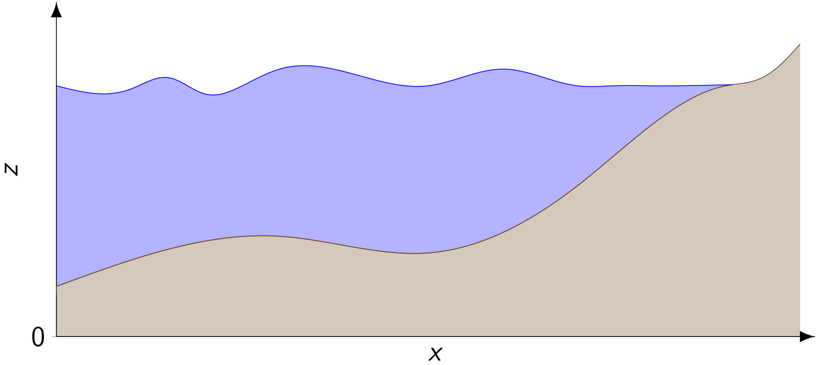
- Tsunamis
- Storm Surges

Storm Surge of Hurricane Florence and Michael

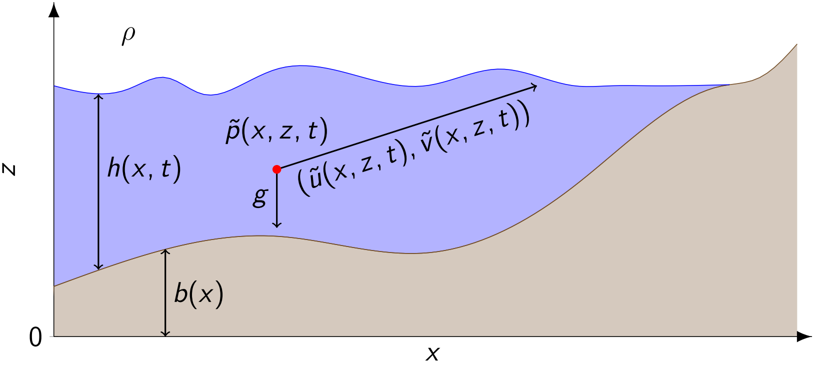
(a) Florence (U.S.A, 2018)

(b) Michael (U.S.A, 2018)

Two Dimensional Scenario



Navier-Stokes



Shallow Water Regime

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- Neglect viscosity

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- Shallow water - wavelengths far larger than water depth

Shallow Water Regime

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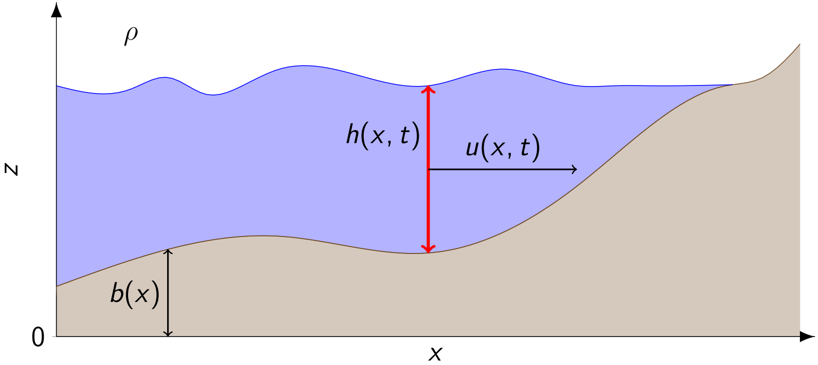
- Shallow water - wavelengths far larger than water depth
  - Tsunami wavelengths up to 100s of kilometres
  - Maximum ocean depth is 11 kilometres

Model Simplification: Serre Equations



Assumptions

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Quantity	Serre Equations
Particle: $\tilde{u}(x, z, t)$	$u(x, t)$

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Particle: $\tilde{u}(x, z, t)$	$u(x, t)$
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Particle: $\tilde{v}(x,z,t)$	$u\frac{\partial b}{\partial x} - (z-b)\frac{\partial b}{\partial x}$
Particle: $\tilde{p}(x,z,t)$	$g\rho\left[h+b-z\right] + \rho\left[h+b-z\right]\Psi$
	$+\frac{1}{2}\rho\left(h^2 - \left[z-b\right]^2\right)\Phi$

where

$$\Psi = \frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial^2 b}{\partial x^2}, \quad \Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}.$$

Equations

$$\text{Mass:} \qquad \frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

$$\text{Momentum:} \qquad \frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^2}{2}\Psi + \frac{h^3}{3}\Phi\right)$$

$$+ \frac{\partial b}{\partial x}\left(gh + h\Psi + \frac{h^2}{2}\Phi\right) = 0.$$

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Method

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Goal: Adapt Finite Volume Methods for the Serre Equations

Finite Volume Method

- Conservation law form

Finite Volume Method

- Conservation law form
- Finite volume update

Finite Volume Method

- Conservation law form

Conservation Law Form

$$\frac{\partial q}{\partial t}+\frac{\partial}{\partial x}\left[f\left(q,\frac{\partial q}{\partial x},\frac{\partial^2 q}{\partial x^2},\dots,\frac{\partial^n q}{\partial x^n}\right)\right]+s\left(q,\frac{\partial q}{\partial x},\frac{\partial^2 q}{\partial x^2},\dots,\frac{\partial^m q}{\partial x^m}\right)=0$$

Equations

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Conservation Law Form

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}h^3 \left[\frac{\partial u}{\partial x}\right]^2 + h^2 u \frac{\partial u}{\partial x}\frac{\partial b}{\partial x} \right)$$
$$+ \frac{1}{2}h^2 u \frac{\partial u}{\partial x}\frac{\partial^2 b}{\partial x^2} - hu^2 \frac{\partial b}{\partial x}\frac{\partial^2 b}{\partial x^2} + gh\frac{\partial b}{\partial x} = 0.$$

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$$\left. + \frac{1}{2}h^2 u \frac{\partial u}{\partial x}\frac{\partial^2 b}{\partial x^2} - hu^2 \frac{\partial b}{\partial x}\frac{\partial^2 b}{\partial x^2} + gh\frac{\partial b}{\partial x}\right) = 0.$$

with

$$G = hu\left(1 + \frac{\partial h}{\partial x}\frac{\partial b}{\partial x} + \frac{1}{2}h\frac{\partial^2 b}{\partial x^2} + \left[\frac{\partial b}{\partial x}\right]^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{3}h^3 \frac{\partial u}{\partial x}\right).$$

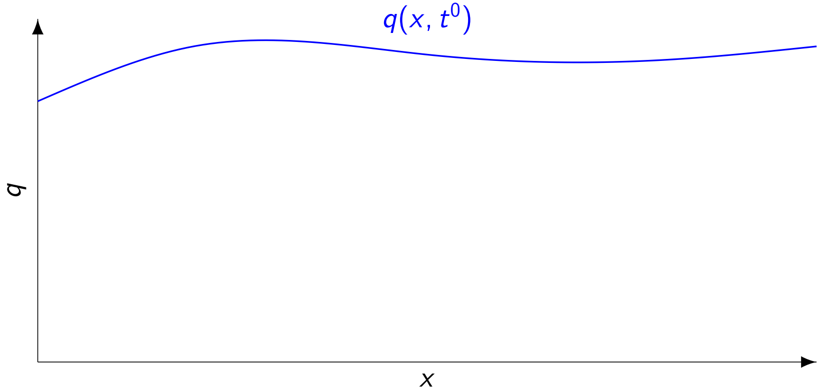
Finite Volume Method

- Conservation law form
- Finite volume update

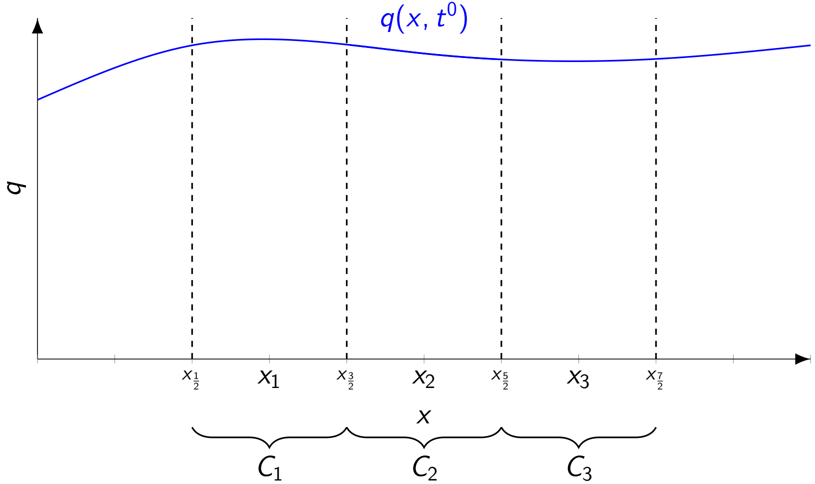
Conservation Law Form

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left[f\left(q, \frac{\partial q}{\partial x}, \frac{\partial^2 q}{\partial x^2}, \dots, \frac{\partial^n q}{\partial x^n}\right)\right] + s\left(q, \frac{\partial q}{\partial x}, \frac{\partial^2 q}{\partial x^2}, \dots, \frac{\partial^m q}{\partial x^m}\right) = 0$$

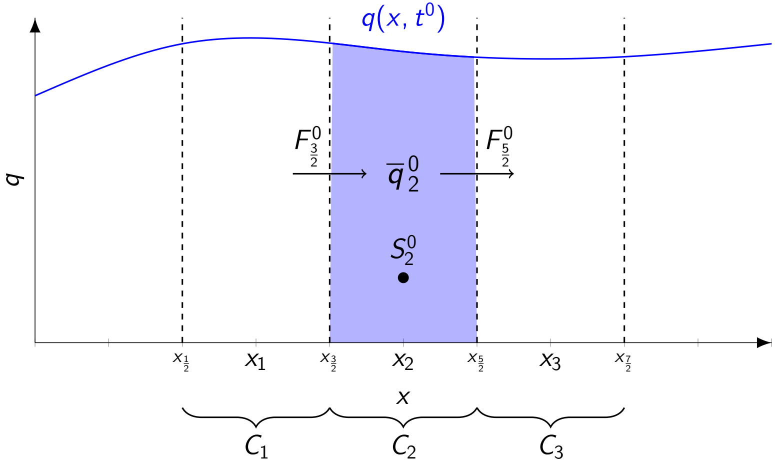
Finite Volume Method



Discretisation



Update



$q(x, t^0)$
q
$F^0_{\frac{3}{2}}$
$\overline{q}^{\,0}_{2}$
$F^0_{\frac{5}{2}}$
S^0_2
$x_{\frac{1}{2}}$
x_1
$x_{\frac{3}{2}}$
x_2
$x_{\frac{5}{2}}$
x_3
$x_{\frac{7}{2}}$
x
C_1
C_2
C_3

Finite Volume Method

$$
\begin{tikzcd}
\overline{h}_j^n, \overline{G}_j^n \arrow[r, "\text{Reconstruction}"] & \hat{h}_j^n, \hat{G}_j^n \arrow[r, "\text{Flux and Source Approximation}"] & F_{j\pm 1/2}^n, S_j^n \arrow[ll, bend left, "\text{Evolve}"]
\end{tikzcd}
$$

Require velocity to calculate flux and source

However to calculate the flux and source terms we require u

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$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

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Method

$$\begin{tikzcd} \overline{h}_j^n, \overline{G}_j^n \arrow[r, "\text{Reconstruction}"] & \hat{h}_j^n, \hat{G}_j^n \arrow[r, "\text{Calculate Velocity}"] & \hat{u}_j^n \arrow[r, "\text{Flux and Source Approximation}"] & F^n_{j\pm1/2}, S_j^n \arrow[lll, bend left, "\text{Evolve}"] \end{tikzcd}$$

Reconstruction

- Determines spatial order of accuracy

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Goal: Second-order accuracy

Reconstruction Spaces

Quantity	Number of spatial derivatives	Reconstructed functions

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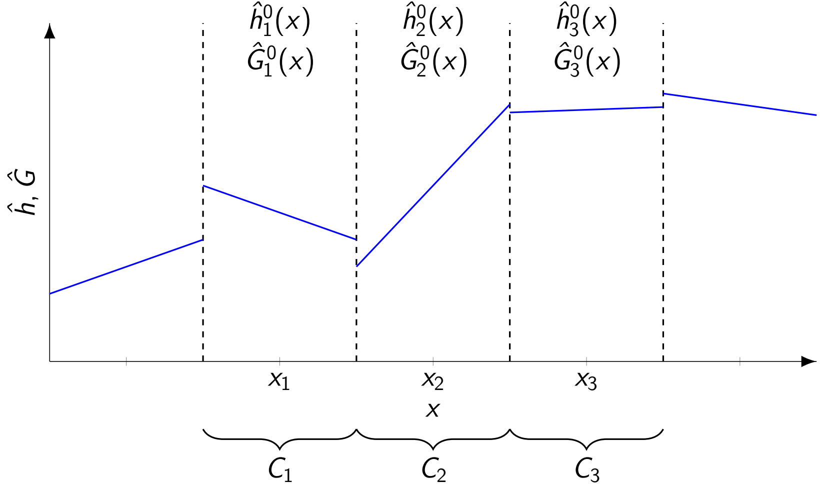
h	zero	linear over cell, discontinuous at edges
G	zero	linear over cell, discontinuous at edges

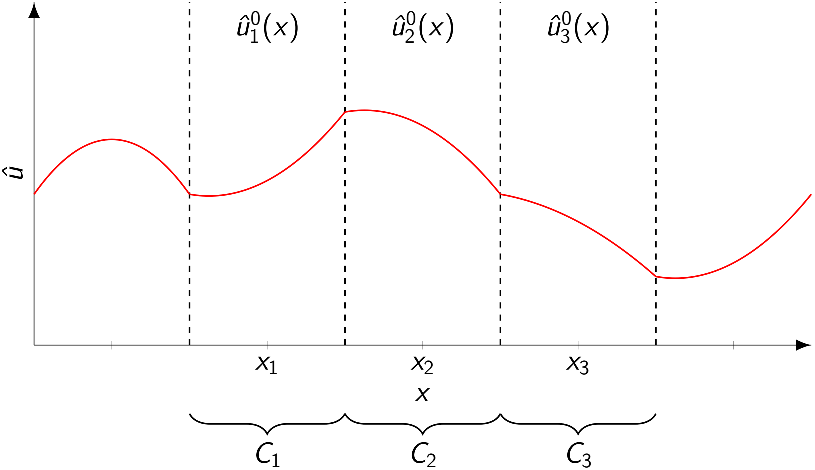
$\hat{h}$, $\hat{G}$


$\hat{h}_1^0(x)$
$\hat{G}_1^0(x)$
$\hat{h}_2^0(x)$
$\hat{G}_2^0(x)$
$\hat{h}_3^0(x)$
$\hat{G}_3^0(x)$
$\hat{h}$, $\hat{G}$
x_1
x_2
x_3
x
C_1
C_2
C_3

Reconstruction Spaces

Quantity	Number of spatial derivatives	Reconstructed functions
h	zero	linear over cell, discontinuous at edges
G	zero	linear over cell, discontinuous at edges
u	one	quadratic over cell, continuous at edges

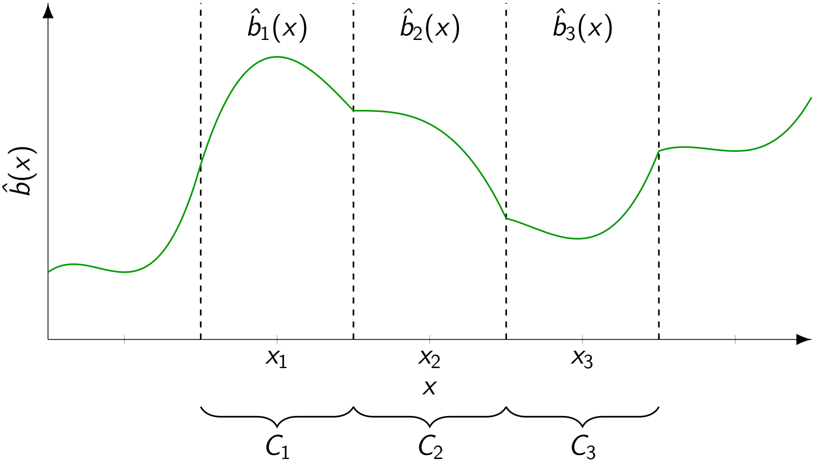
$\hat{u}$



Reconstruction Spaces

Quantity	Number of spatial derivatives	Reconstructed functions
h	zero	linear over cell, discontinuous at edges
G	zero	linear over cell, discontinuous at edges
u	one	quadratic over cell, continuous at edges
b	two	cubic over cell, continuous at edges

$\hat{b}$



Finite Element Calculation of Velocity

Finite Element Method to solve:

$$G = hu\left(1 + \frac{\partial h}{\partial x}\frac{\partial b}{\partial x} + \frac{1}{2}h\frac{\partial^2 b}{\partial x^2} + \left[\frac{\partial b}{\partial x}\right]^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{3}h^3\frac{\partial u}{\partial x}\right).$$

for u given h, G and b

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for u given h, G and b

Solves the weak form replacing all quantities with their reconstructions $\hat{h}$, $\hat{G}$ and $\hat{b}$ to get $\hat{u}$

Method

$$\begin{tikzcd}
\overline{h}_j^n, \overline{G}_j^n \arrow[r, "\text{Reconstruction}"] & \hat{h}_j^n, \hat{G}_j^n \arrow[r, "\text{Calculate Velocity}"] & \hat{u}_j^n \arrow[r, "\text{Flux and Source Approximation}"] & F^n_{j\pm1/2}, S_j^n \arrow[lll, bend left, "\text{Evolve}"]
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Validation

- Analytic Solution

Validation

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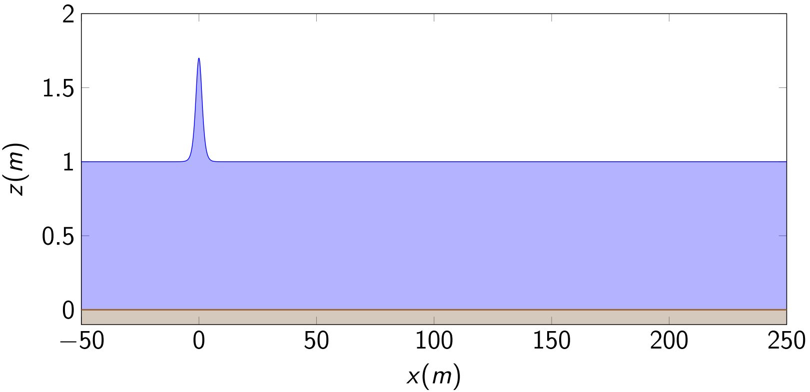
- Experimental Results

Validation

- Analytic Solution

Soliton Example



Soliton Equations

$$h(x,t) = a_0 + a_1 \text{sech}\left(\kappa\left(x - ct\right)\right),$$

$$u(x,t) = c\left(1 - \frac{a_0}{h(x,t)}\right),$$

$$b(x) = 0$$

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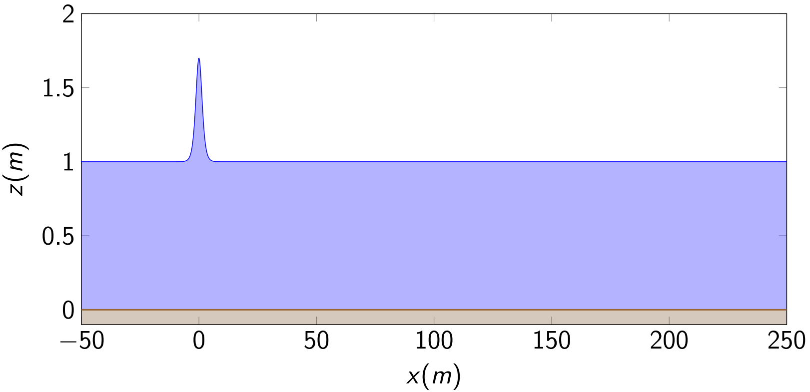
$$b(x) = 0$$

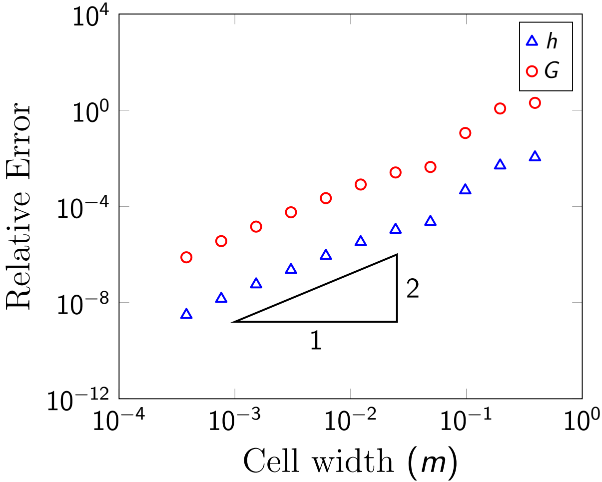
$$\kappa = \frac{\sqrt{3a_1}}{2a_0\sqrt{(a_0 + a_1)}},$$

$$c = \sqrt{g(a_0 + a_1)}.$$

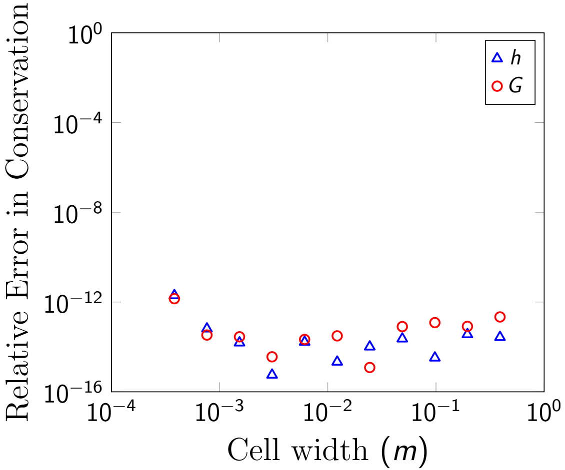
Numerical Solution $a_0 = 1m$, $a_1 = 0.7m$



Convergence



Conservation



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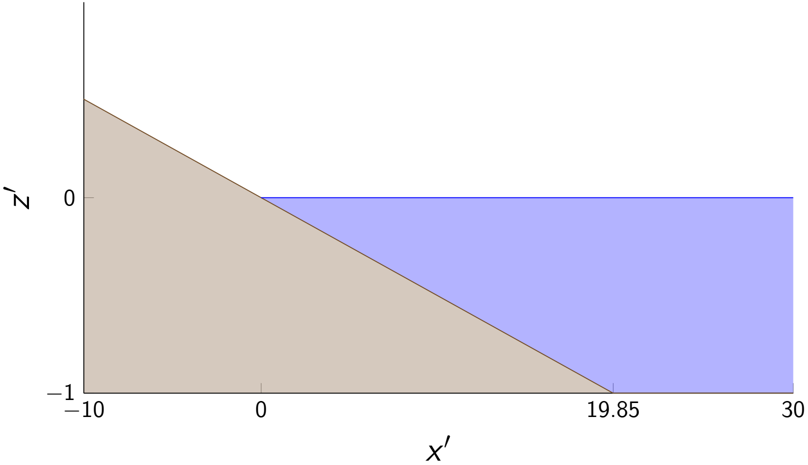
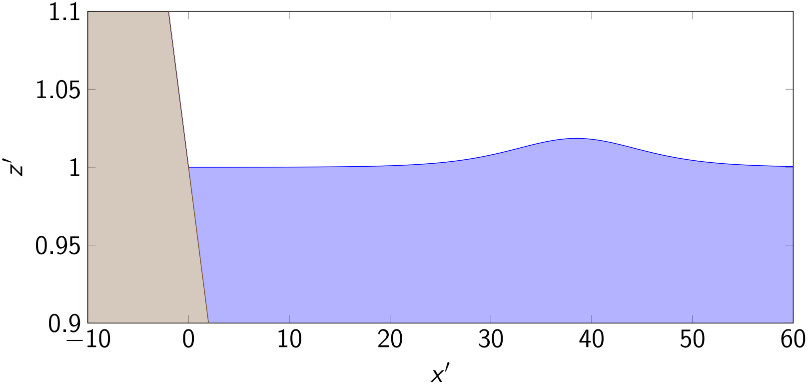
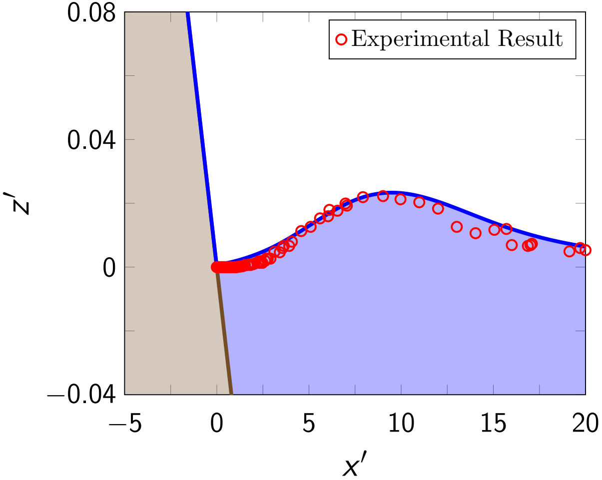
Synolakis Experiment



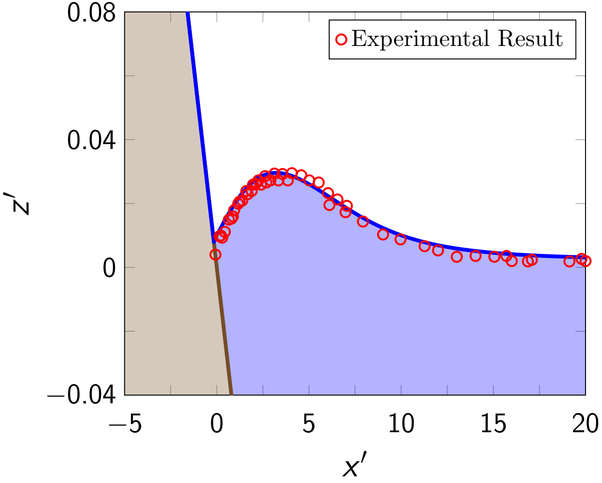
Numerical Solution



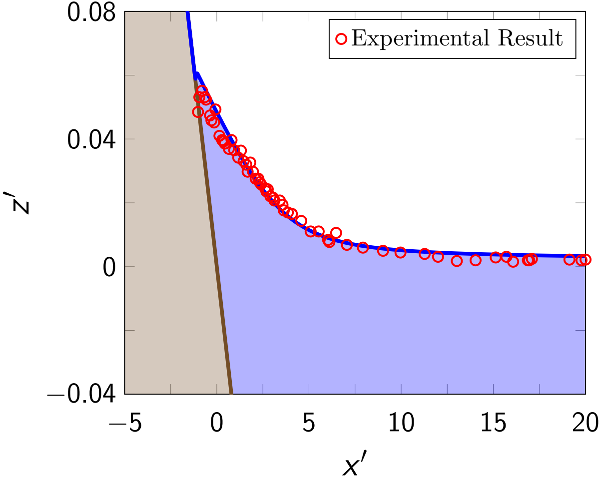
Comparison $t' = 30$



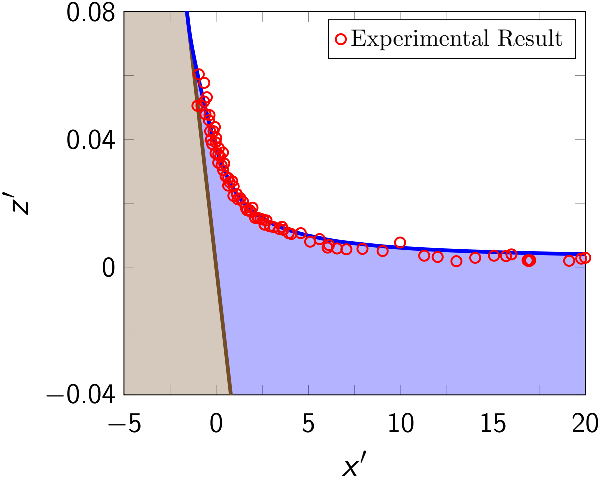
Comparison $t' = 40$



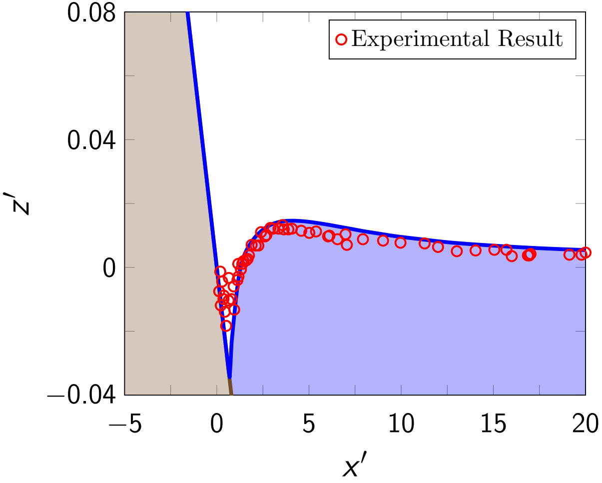
Comparison $t' = 50$



Comparison $t' = 60$



Comparison $t' = 70$



Conclusion

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Thanks!