1 Elliptic Equation

The linearised elliptic equation is

$$G = Hu - \frac{H^3}{3} u_{xx}$$

Want to find out the FEM approximation to this G' such that

$$G' = \mathcal{G}_{FE_1} u$$

for P^1 FEM

$$G' = \mathcal{G}_{FE_2} u$$

for P^2 FEM.

To do so we begin by first multiplying by an arbitrary test function v so that

$$Gv = Huv - \frac{H^3}{3} u_{xx} v$$

and then we integrate over the entire domain to get

$$\int_\Omega Gv dx = \int_\Omega Huv dx - \int_\Omega \frac{H^3}{3} u_{xx} v dx$$

for all v

We then make use of integration by parts, with Dirchlet boundaries to get

$$\int_\Omega Gv dx = \int_\Omega Huv dx + \int_\Omega \frac{H^3}{3} u_x v_x dx$$

Our FVM discretisation already has a natrual structure with linear functions intervals of like $[x_{j-1/2}, x_{j+1/2}]$

So we can reformulate this as

$$\sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} Gv dx = \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} Huv dx + \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{H^3}{3} u_{xx} v dx$$

or more aptly

$$\sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} Gvdx - \int_{x_{j-1/2}}^{x_{j+1/2}} Huvdx - \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{H^3}{3} u_{xx} vdx = 0$$

for all v

$$\sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} Gvdx - H \int_{x_{j-1/2}}^{x_{j+1/2}} uvdx - \frac{H^3}{3} \int_{x_{j-1/2}}^{x_{j+1/2}} u_{xx} vdx = 0$$

Each of these integrals has been computed by Chris previously, for the basis functions that are non-zero on the element. We include just the basis function which is 1 at x_j

$$\int_{x_{j-1/2}}^{x_{j+1/2}} Gvdx = \frac{dx}{d\xi} \int_{-1}^{1} Gvd\xi = \frac{\Delta x}{15} \left[G^+_{j-1/2} + 8G_j + G^-_{j+1/2} \right]$$

$$H \int_{x_{j-1/2}}^{x_{j+1/2}} uvdx = \frac{dx}{d\xi} H \int_{-1}^{1} uvd\xi = \frac{H\Delta x}{15} \left[u_{j-1/2} + 8u_j + u_{j+1/2} \right]$$

$$\frac{H^3}{3} \int_{x_{j-1/2}}^{x_{j+1/2}} u_x v_x dx = \frac{d\xi}{dx} \frac{H^3}{3} \int_{-1}^{1} u_\xi v_\xi d\xi = \frac{H^3}{3} \frac{2}{3\Delta x} \left[-4u_{j-1/2} + 8u_j - 4u_{j+1/2} \right]$$

So we ahve

$$\begin{aligned} &\frac{\Delta x}{15} \left[G^+_{j-1/2} + 8G_j + G^-_{j+1/2} \right] \\ &= \frac{H\Delta x}{15} \left[u_{j-1/2} + 8u_j + u_{j+1/2} \right] + \frac{H^3}{3} \frac{2}{3\Delta x} \left[-4u_{j-1/2} + 8u_j - 4u_{j+1/2} \right] \end{aligned} \quad (1)$$

$$\begin{aligned} &\frac{\Delta x}{15} \left[G^+_{j-1/2} + 8G_j + G^-_{j+1/2} \right] \\ &= \frac{H\Delta x}{15} \left[u_{j-1/2} + 8u_j + u_{j+1/2} \right] + \frac{H^3}{3} \frac{8}{3\Delta x} \left[-u_{j-1/2} + 2u_j - u_{j+1/2} \right] \end{aligned} \quad (2)$$

$$\frac{1}{5}\left[G^{+}_{j-1/2} + 8G_j + G^{-}_{j+1/2}\right]$$
$$= \frac{H\Delta x}{5}\left[u_{j-1/2} + 8u_j + u_{j+1/2}\right] + \frac{H^3}{3}\frac{8}{\Delta x^2}\left[-u_{j-1/2} + 2u_j - u_{j+1/2}\right] \quad (3)$$

$$\left[G^{+}_{j-1/2} + 8G_j + G^{-}_{j+1/2}\right]$$
$$= H\left[u_{j-1/2} + 8u_j + u_{j+1/2}\right] + \frac{H^3}{3}\frac{40}{\Delta x^2}\left[-u_{j-1/2} + 2u_j - u_{j+1/2}\right] \quad (4)$$

This formula works, then using our shift operators we can do

$$\left[e^{-ik\Delta x}\mathcal{R}^{+} + 8 + \mathcal{R}^{-}\right] G_j$$
$$= \left(H\left[e^{-ik\Delta x/2} + 8 + e^{ik\Delta x/2}\right] + \frac{H^3}{3}\frac{40}{\Delta x^2}\left[-e^{-ik\Delta x/2} + 2 - e^{ik\Delta x/2}\right]\right) u_j \quad (5)$$

$$G_j$$
$$= \frac{1}{e^{-ik\Delta x}\mathcal{R}^{+} + 8 + \mathcal{R}^{-}}\left(H\left[8 + 2\cos\left(k\Delta x/2\right)\right] + \frac{H^3}{3}\frac{40}{\Delta x^2}\left[2 - 2\cos\left(k\Delta x/2\right)\right]\right) u_j \quad (6)$$

$$G_j$$
$$= \frac{2}{e^{-ik\Delta x}\mathcal{R}^{+} + 8 + \mathcal{R}^{-}}\left(H\left[4 + \cos\left(k\Delta x/2\right)\right] + \frac{H^3}{3}\frac{40}{\Delta x^2}\left[1 - \cos\left(k\Delta x/2\right)\right]\right) u_j \quad (7)$$