Dispersive Shock Waves of the Serre equations

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Introduction

Summary of our paper "Behaviour of the Serre equations in the presence of steep gradients revisited" (Wave Motion Volume 76, January 2018).

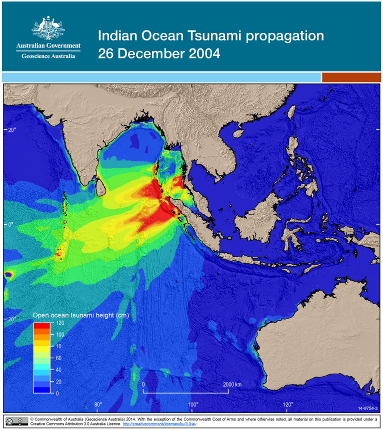
Outline of Presentation:

- Motivation.
- Serre Equations.
- Dispersive Shock Waves.
- Investigation.
- Results.

Our Background

- Interest : Numerical methods for water waves focusing on ocean hazards.

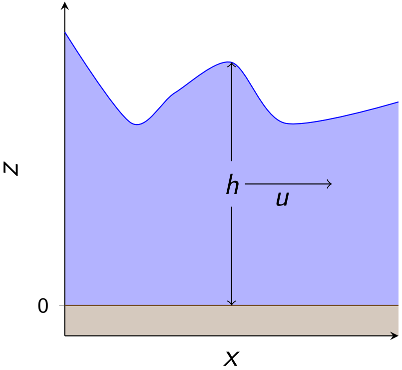
Indian ocean tsunami



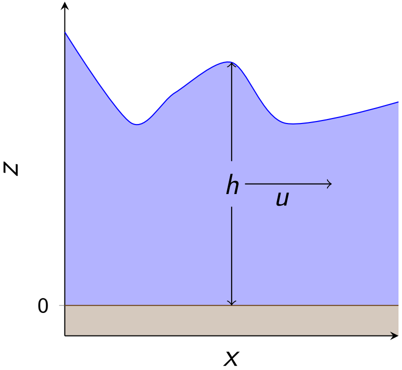
Our Background

- Interest : Numerical methods for water waves focusing on ocean hazards.
- Resulted In : Robust numerical method for the Shallow Water Wave Equations (ANUGA).

Shallow Water Wave Equations



Shallow Water Wave Equations



Conservation of mass

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0$$

Conservation of momentum

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2}\right) = 0$$

Serre equations

Conservation of mass

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0$$

Conservation of momentum

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^3}{3}\Phi\right) = 0$$

$$\Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}$$

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- Current Goal : Robust numerical method for the Serre equations.

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- Interest : Numerical methods for water waves focusing on ocean hazards.
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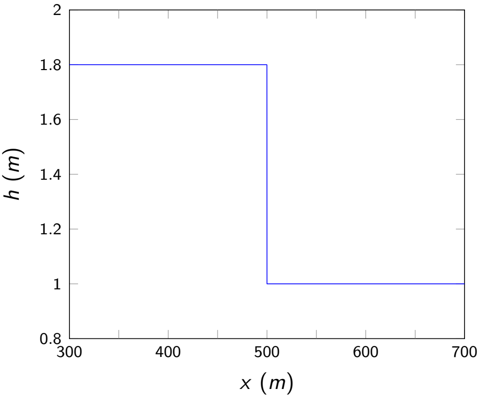
- Current Goal : Robust numerical method for the Serre equations.
- Problem : Handling discontinuous initial conditions (dam-break problem).

Model Problem : Dam Break Problem

$$h_0 = 1m \ , \ h_1 = 1.8m \text{ and } x_0 = 500m$$

$$h(x,0) = \begin{cases} h_1 & x \leq x_0 \\ h_0 & x > x_0 \end{cases}$$

$$u(x,0) = 0$$



Dambreak Problem Solutions

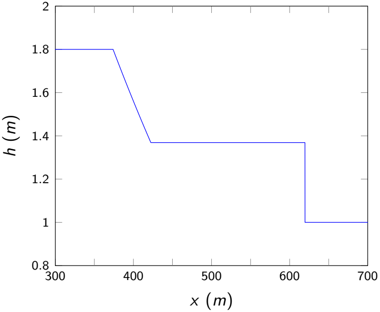


Figure: Shock wave (analytical solution of the shallow water wave equations).

Dambreak Problem Solutions

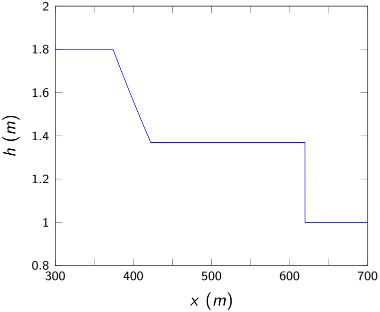


Figure: Shock wave (analytical solution of the shallow water wave equations).

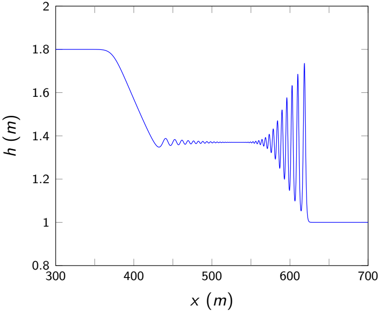


Figure: Dispersive shock wave (common numerical solution of the Serre equations in the literature).

New Observed Behaviour for Dispersive Shock Waves

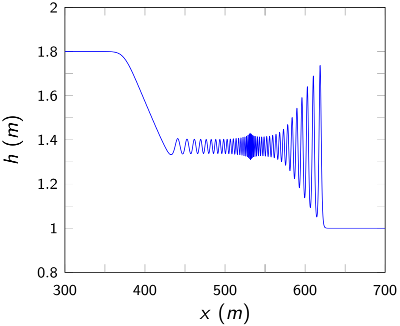


Figure: New observed structure.

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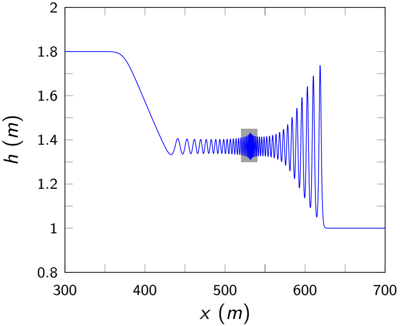

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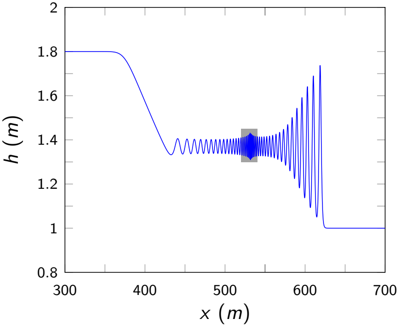

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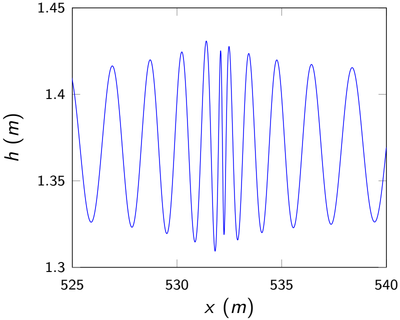


Figure: Zoomed in.

Properties of Dispersive Shock Waves of the Serre Equations

Asymptotic results for long times:

- Whitham modulation results for leading wave amplitude and location.

Whitham Modulation Results

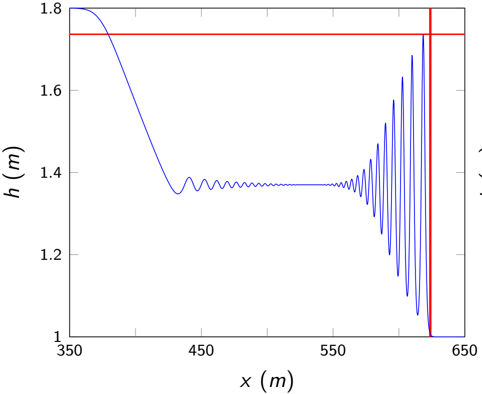
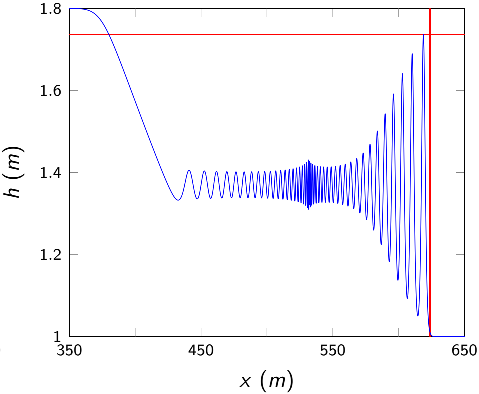


Figure: Common structure.



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Properties of Dispersive Shock Waves of the Serre Equations

Asymptotic results for long times:

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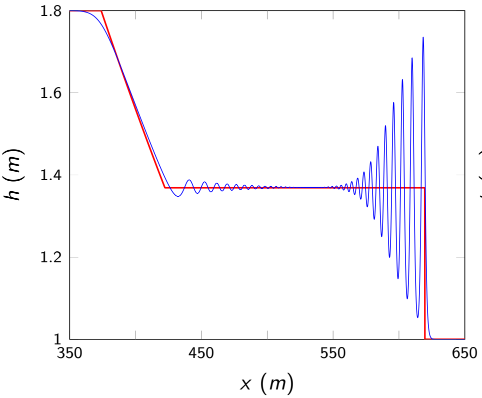
- Oscillations of the dispersive shock waves for the Serre equations oscillate around the shock waves of the shallow water wave equations.

Dispersive Shock Waves compared to Shock Waves



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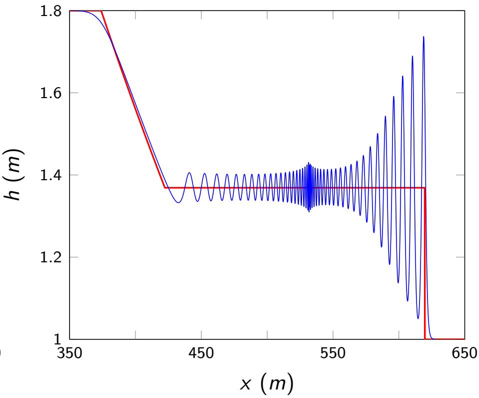

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Properties of Dispersive Shock Waves of the Serre Equations

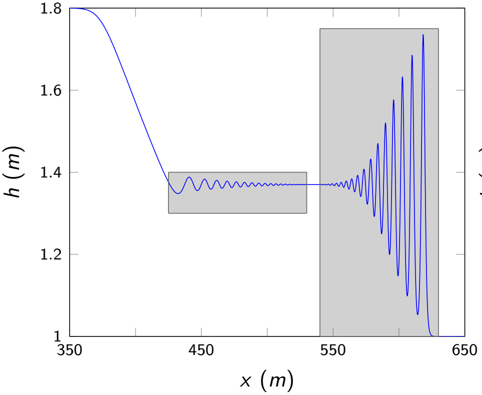
Asymptotic results for long times:

- Whitham modulation results for leading wave amplitude and location.
- Oscillations of the dispersive shock waves for the Serre equations oscillate around the shock waves of the shallow water wave equations.

Linear results:

- Separate dispersive wave trains.

Separation of Dispersive Wave Trains



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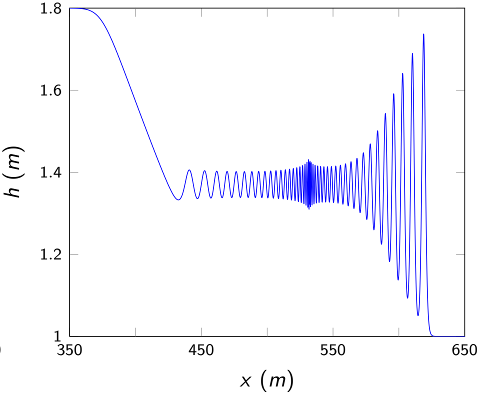


Figure: New structure.

Problem

- No analytic solution of the Serre equations for dispersive shock waves.
- Lack transient properties of dispersive shock waves.

Solution

Use numerical solvers for the Serre equations on a problem with a smooth approximation to the initial conditions of the dam-break problem.

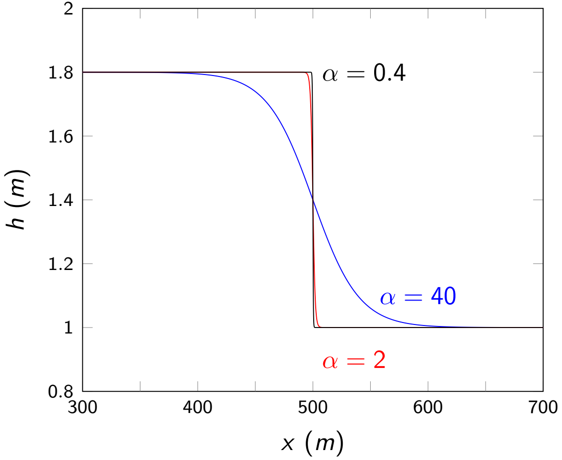
Smooth Dam Break Problem

$$h(x,0) = h_0 + \frac{h_1 - h_0}{2}\left(1 + \tanh\left(\frac{x_0 - x}{\alpha}\right)\right)$$

$$u(x,0) = 0$$

$$h_0 = 1m \; , \; h_1 = 1.8m \text{ and } x_0 = 500m$$

Smooth Dam Break Problem Examples



Observed Behaviours

We observed four different behaviours of the numerical solution as $\alpha \to 0$

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- Non Oscillatory Structure for large α values.

Non Oscillatory Structure $\alpha = 40$

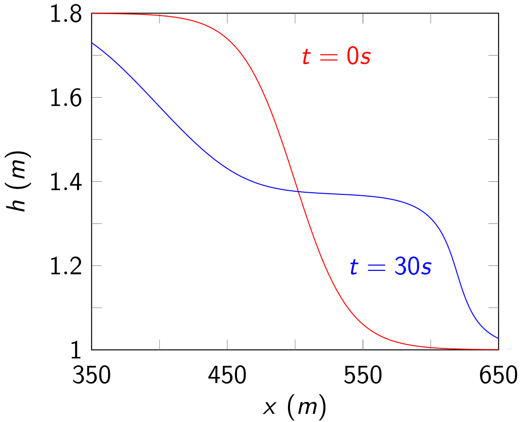


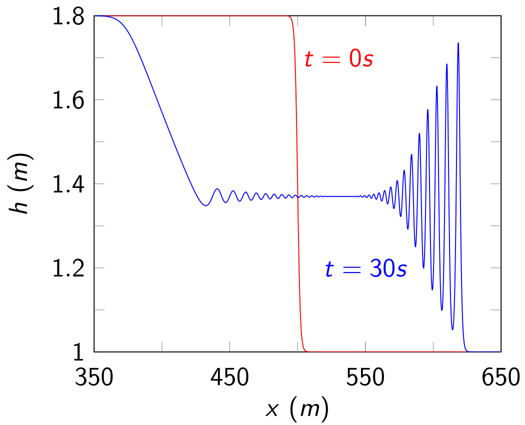
Figure: Highest resolution numerical solution at $t = 30s$

Observed Behaviours

We observed four different behaviours of the numerical solution as $\alpha \to 0$

- Non Oscillatory Structure for large α values.
- Flat Structure (common one in the literature).

Flat Structure $\alpha = 2$



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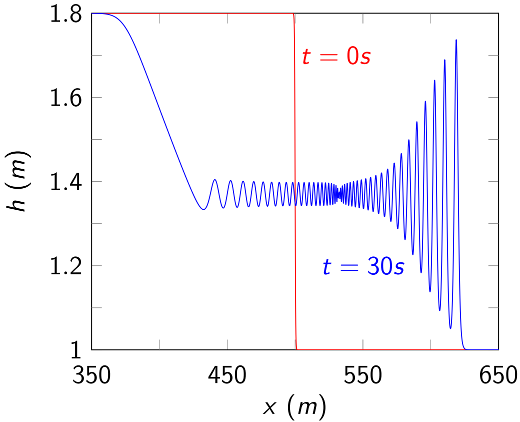
Observed Behaviours

We observed four different behaviours of the numerical solution as $\alpha \to 0$

- Non Oscillatory Structure for large α values.
- Flat Structure (common one in the literature).
- Node Structure (also present in literature).

Node Structure $\alpha = 0.4$



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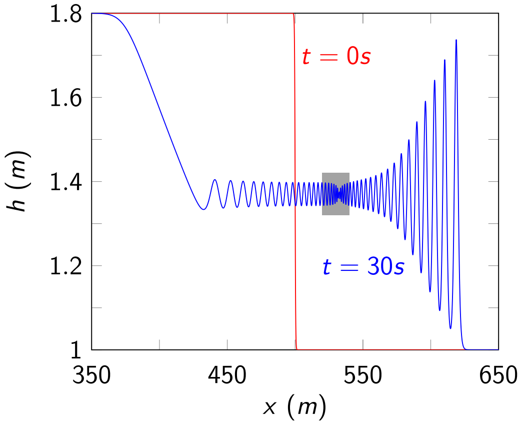
Node Structure $\alpha = 0.4$



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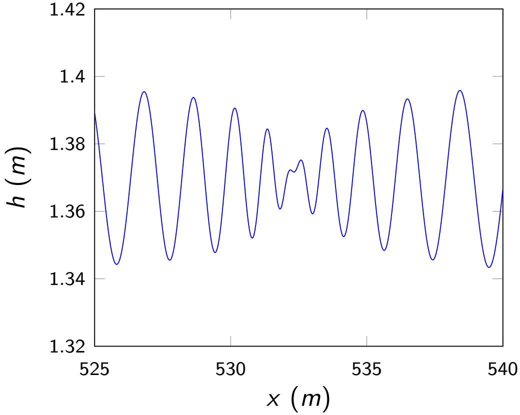
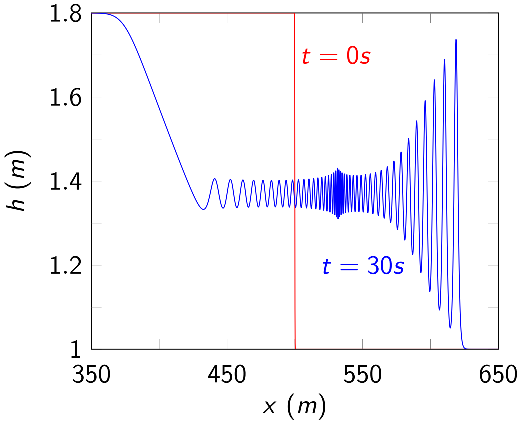


Observed Behaviours

We observed four different behaviours of the numerical solution as $\alpha \to 0$

- Non Oscillatory Structure for large α values.
- Flat Structure (common one in the literature).
- Node Structure (also present in literature).
- Growth Structure for small α values and the dam-break problem (New behaviour).

Growth Structure $\alpha = 0.1$



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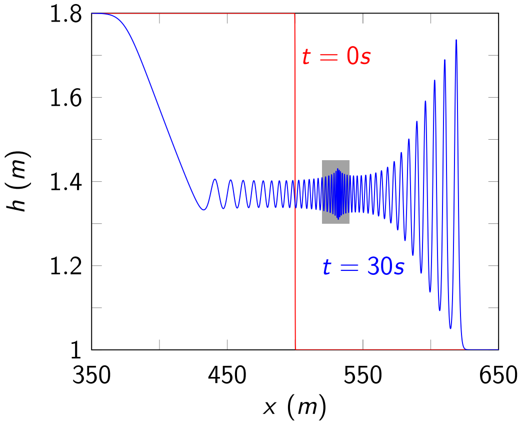
Growth Structure $\alpha = 0.1$



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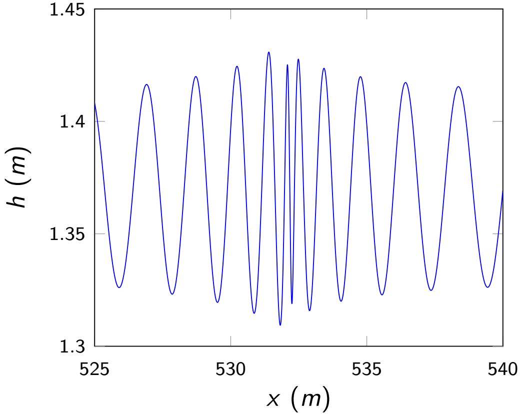
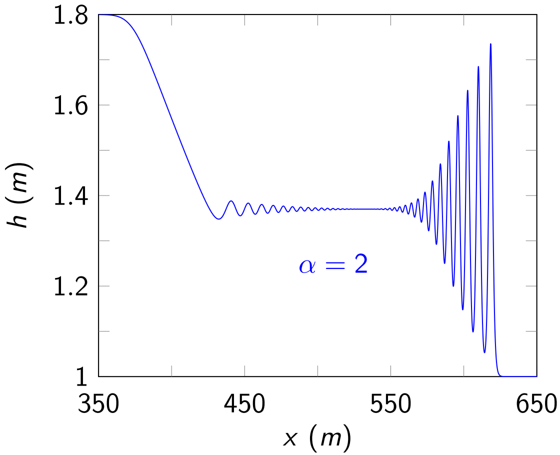
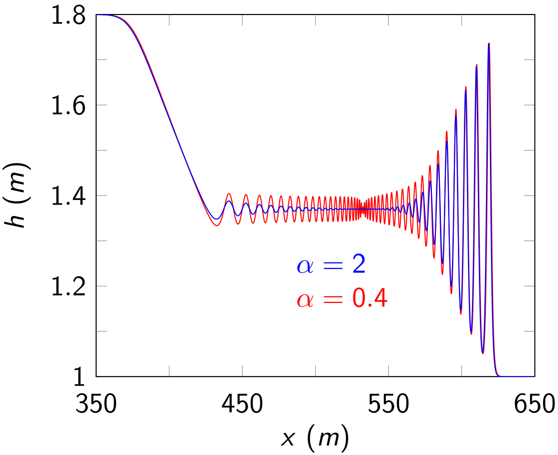
Growth Structure $\alpha = 0.1$



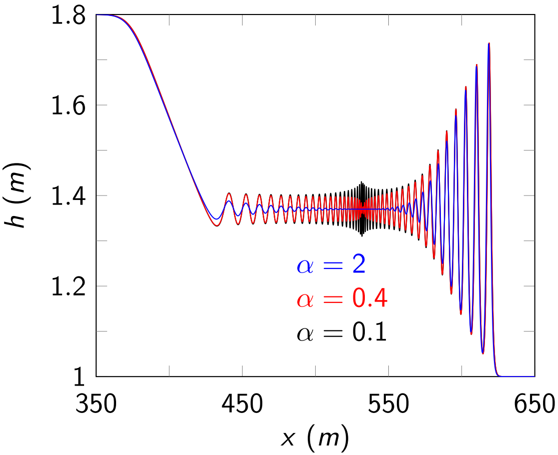
Structure Comparison: Flat



Structure Comparison: Flat & Node



Structure Comparison: Flat & Node & Growth



Structure Comparison Energy of Initial Conditions

Hamiltonian (Energy):

$$\mathcal{H}(x,t) = \frac{1}{2}\left(hu^2 + \frac{h^3}{3}\left(\frac{\partial u}{\partial x}\right)^2 + gh^2\right)$$

Comparison of Energy of Initial Conditions:

Structure	α	$\mathcal{H}$
Non-Oscillatory	40	10335.8
Flat	2	10395.5
Node	0.4	10398.0
Growth	0.1	10398.4

Justifying These Numerical Solutions

For a particular α value:

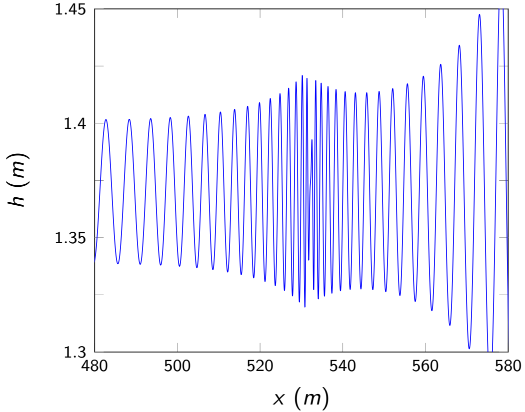
- Demonstrated convergence as the resolution of the method increases.
- Demonstrated numerical solutions conserve mass, momentum and energy.

These results demonstrated that the Growth Structure is correct structure of DSW for short time spans.

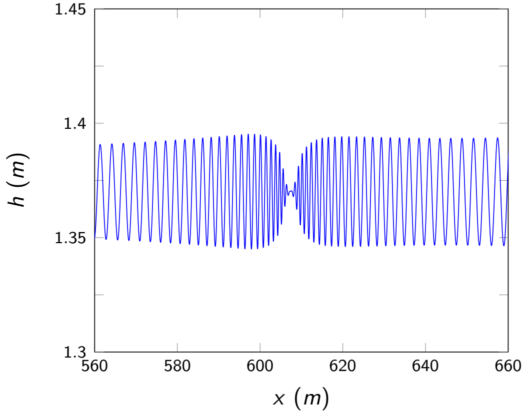
Long term solutions

- Growth structure agrees well with asymptotic results for short and long times.
- Growth structure decays to node structure which decays to the flat structure.
- Separation of dispersive wave trains over long times.

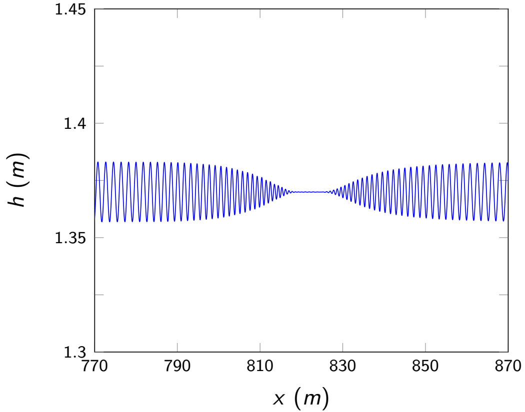
Dam break solution at $t = 30s$ ($\alpha = 0$)



Dam break solution at $t = 100s$ ($\alpha = 0$)



Dam break solution at $t = 300s$ ($\alpha = 0$)



Conclusion

Presentation

- Found new behaviour of dispersive shock waves for short time spans not previously published in the literature.
- Good agreement between numerical solutions and known properties of dispersive shock waves for long time periods.

Paper

- Explained why different behaviour published in the literature.
- Justified the robustness of our numerical methods.

Comparison of Node and Growth Structures