We are going to perform a Von Neumann stability analysis for the FD methods presented in the DB paper.

We have the following from the paper

To get a second-order approximation to momentuim is built by first expanding all the derivatives out and making use of the continuity equation [], this results in:

$$h\frac{\partial u}{\partial t} + X - h^2\frac{\partial^2 u}{\partial x \partial t} - \frac{h^3}{3}\frac{\partial^3 u}{\partial x^2 \partial t} = 0 \tag{1a}$$

where X contains only spatial derivatives and is

$$X = uh\frac{\partial u}{\partial x} + gh\frac{\partial h}{\partial x} + h^2\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{h^3}{3}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - h^2 u\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{3}u\frac{\partial^3 u}{\partial x^3}. \tag{1b}$$

Then taking second-order approximations to the time derivatives for [] gives

$$h^n\frac{u^{n+1} - u^{n-1}}{2\Delta t} + X^n - (h^n)^2\frac{\left(\frac{\partial u}{\partial x}\right)^{n+1} - \left(\frac{\partial u}{\partial x}\right)^{n-1}}{2\Delta t} - \frac{(h^n)^3}{3}\frac{\left(\frac{\partial^2 u}{\partial x^2}\right)^{n+1} - \left(\frac{\partial^2 u}{\partial x^2}\right)^{n-1}}{2\Delta t} = 0 \tag{2}$$

$$h^n\left(u^{n+1} - u^{n-1}\right) + 2\Delta t X^n - (h^n)^2\left(\left(\frac{\partial u}{\partial x}\right)^{n+1} - \left(\frac{\partial u}{\partial x}\right)^{n-1}\right) - \frac{(h^n)^3}{3}\left(\left(\frac{\partial^2 u}{\partial x^2}\right)^{n+1} - \left(\frac{\partial^2 u}{\partial x^2}\right)^{n-} \tag{3}$$

$$h^n u^{n+1} - (h^n)^2\left(\frac{\partial u}{\partial x}\right)^{n+1} - \frac{(h^n)^3}{3}\left(\frac{\partial^2 u}{\partial x^2}\right)^{n+1} + 2\Delta t X^n - h^n u^{n-1} + (h^n)^2\left(\frac{\partial u}{\partial x}\right)^{n-1} + \frac{(h^n)^3}{3}\left(\frac{\partial}{\partial x}\right) \tag{4}$$

Let

$$Y^n = 2\Delta t X^n - h^n u^{n-1} + (h^n)^2\left(\frac{\partial u}{\partial x}\right)^{n-1} + \frac{(h^n)^3}{3}\left(\frac{\partial^2 u}{\partial x^2}\right)^{n-1} \tag{5}$$

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Taking second-order approximations to the spatial derivatives gives

$$h_i^n u_i^{n+1} - (h_i^n)^2 \left(\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} \right) - \frac{(h_i^n)^3}{3} \left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} \right) = -Y_i^n$$

(6)

This can be rearranged into a tri-diagonal matrix that updates u given its current and previous values. So that

$$\begin{bmatrix} u_0^{n+1} \\ \vdots \\ u_m^{n+1} \end{bmatrix} = A^{-1} \begin{bmatrix} -Y_0^n \\ \vdots \\ -Y_m^n \end{bmatrix} =: \mathcal{G}_u \left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{u}^{n-1}, \Delta t \right).$$

Where

$$A = \begin{bmatrix} b_0 & c_0 & & & & & \\ a_0 & b_1 & c_1 & & & & \\ & a_1 & b_2 & c_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & a_{m-3} & b_{m-2} & c_{m-2} & \\ & & & & a_{m-2} & b_{m-1} & c_{m-1} \\ & & & & & a_{m-1} & b_m \end{bmatrix}$$

with

$$a_{i-1} = \frac{(h_i^n)^2}{2\Delta x} \frac{h_{i+1}^n - h_{i-1}^n}{2\Delta x} - \frac{(h_i^n)^3}{3\Delta x^2},$$

(7a)

$$b_i = h_i^n + \frac{2h_i^n}{3\Delta x^2}$$

(7b)

and

$$c_i = -\frac{(h_i^n)^2}{2\Delta x} \frac{h_{i+1}^n - h_{i-1}^n}{2\Delta x} - \frac{(h_i^n)^3}{3\Delta x^2}.$$

(7c)

Lastly the final expression for Y_i^n is given by:

$$Y_i^n = 2\Delta t X_i^n - h_i^n u_i^{n-1} + (h_i^n)^2 \frac{u_{i+1}^{n-1} - u_{i-1}^{n-1}}{2\Delta x} + \frac{(h_i^n)^3}{3} \frac{u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}}{\Delta x^2}$$

$$(8)$$

$$
\begin{aligned}
Y_i^n = 2\Delta t &\left[u_i^n h_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + g h_i^n \frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{2\Delta x} + (h_i^n)^2 \left(\frac{u_{i+1}^{n-1} - u_{i-1}^{n-1}}{2\Delta x} \right)^2 \right. \\
&+ \frac{(h_i^n)^3}{3} \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} - (h_i^n)^2 u_i^n \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \\
&\left. - \frac{(h_i^n)^3}{3} u_i^n \frac{u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n}{2\Delta x^3} \right] \\
&- h_i^n u_i^{n-1} + (h_i^n)^2 \frac{u_{i+1}^{n-1} - u_{i-1}^{n-1}}{2\Delta x} + \frac{(h_i^n)^3}{3} \frac{u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}}{\Delta x^2}
\end{aligned}
$$

$$(9)$$

1   Second Order Finite Difference Method

Above a second order finite difference method for updating u was given, thus replacing the numerical method for h by replacing derivatives with second order finite differences will give another full finite difference method. From (??) we expand derivatives and then approximate them by second order finite differences to give

$$\frac{h_i^{n+1} - h_i^{n-1}}{2\Delta t} + u_i^n \frac{h_{i+1}^n - h_{i-1}^n}{2\Delta x} + h_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0 \qquad (10)$$

After rearranging this to give an update formula one obtains

$$h_i^{n+1} = h_i^{n-1} - \Delta t \left(u_i^n \frac{h_{i+1}^n - h_{i-1}^n}{\Delta x} + h_i^n \frac{u_{i+1}^n - u_{i-1}^n}{\Delta x} \right) \qquad (11)$$

Combining this with the update formula for u [] gives a full finite difference method for the Serre equations.

1.1 The Lax Wendroff Method for Conservation of Mass Equation

The conservation of mass equation (??) has no mixed derivative term allowing standard numerical techniques for conservation laws to be used. In particular the Lax-Wendroff method as was done by [?], here we present the method in replicable detail.

Note that (??) is in conservative law form for h where the flux is uh. Thus the two step Lax-Wendroff update for h is

$$h_{i+1/2}^{n+1/2} = \frac{1}{2}\left(h_{i+1}^n + h_i^n\right) - \frac{\Delta t}{2\Delta x}\left(u_{i+1}^n h_{i+1}^n - h_i^n u_i^n\right), \tag{12}$$

$$h_{i-1/2}^{n+1/2} = \frac{1}{2}\left(h_i^n + h_{i-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(u_i^n h_i^n - h_{i-1}^n u_{i-1}^n\right) \tag{13}$$

and

$$h_i^{n+1} = h_i^n - \frac{\Delta t}{\Delta x}\left(u_{i+1/2}^{n+1/2} h_{i+1/2}^{n+1/2} - u_{i-1/2}^{n+1/2} h_{i-1/2}^{n+1/2}\right). \tag{14}$$

The quantities $u_{i\pm1/2}^{n+1/2}$ are calculated using u^{n+1} obtained by appling $\mathcal{G}_u$ to u^n then linearly interpolating in space and time to give

$$u_{i+1/2}^{n+1/2} = \frac{u_{i+1}^{n+1} + u_{i+1}^n + u_i^{n+1} + u_i^n}{4} \tag{15}$$

and

$$u_{i-1/2}^{n+1/2} = \frac{u_i^n + u_i^n + u_{i-1}^{n+1} + u_{i-1}^n}{4}. \tag{16}$$

Thus we have the following update scheme for (??)

$$\boldsymbol{h}^{n+1} = \mathcal{E}_h\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{u}^{n+1}, \Delta x, \Delta t\right). \tag{17}$$

The update scheme for all of (??) is

$$\left.\begin{array}{ll}\boldsymbol{u}^{n+1} & = \mathcal{G}_u\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{u}^{n-1}, \Delta x, \Delta t\right) \\ \boldsymbol{h}^{n+1} & = \mathcal{E}_h\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{u}^{n+1}, \Delta x, \Delta t\right)\end{array}\right\}\mathcal{E}\left(\boldsymbol{u}^n, \boldsymbol{h}^n, \boldsymbol{u}^{n-1}, \boldsymbol{h}^{n-1}, \Delta x, \Delta t\right). \tag{18}$$