1 Linearised Equations

From Chris's paper we have, where h_0 is constant and we let $h_1 = h$ (same with velocity)

For mass:

$$\frac{\partial h}{\partial t} + h_0 \frac{\partial u}{\partial x} + u_0 \frac{\partial h}{\partial x} = 0$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(h_0 u + u_0 h\right) = 0$$

For momentum:

$$\frac{\partial u}{\partial t} + g\frac{\partial h}{\partial x} + u_0 \frac{\partial u}{\partial x} - \frac{h_0^2}{3}\left(u_0 \frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x^2 \partial t}\right) = 0$$

2 Actual Work

We do a Von Neumann stability analysis, we assume two different errors for h and u otherwise everything else is the same. We jsut run the errrors of known structure through the method, for convenience we know use h and u to refer to their respective errors, and we use q top refer to a general quantity (k, a different for u and l and b for h)

$$q_{j+1}^n = e^{ik\Delta x} q_j^n$$
$$q_{j+2}^n = e^{2ik\Delta x} q_j^n$$
$$q_{j-1}^n = e^{-ik\Delta x} q_j^n$$
$$q_{j-2}^n = e^{-ik\Delta x} q_j^n$$

$$\frac{\partial q}{\partial x} = \frac{q_{j+1}^n - q_{j-1}^n}{2\Delta x} = \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x} q_{j+1}^n = \frac{i \sin\left(k\Delta x\right)}{\Delta x} q_j^n$$

S

$$\frac{\partial^2 q}{\partial x^2} = \frac{q_{j+1}^n - 2q_j^n + q_{j-1}^n}{\Delta x^2} = \frac{e^{ik\Delta x} + e^{-ik\Delta x} - 2}{\Delta x^2} q_j^n = \frac{2\cos\left(k\Delta x\right) - 2}{\Delta x^2} q_j^n$$

$$= -\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)q_j^n$$

S

$$\frac{\partial^3 q}{\partial x^3} = \frac{-q_{j-2}^n + 2q_{j-1}^n - 2q_{j+1}^n + q_{j+2}^n}{2\Delta x^3} = \frac{-e^{-2ik\Delta x} + 2e^{-ik\Delta x} - 2e^{ik\Delta x} + e^{2ik\Delta x}}{2\Delta x^3}q_j^n$$

$$= \frac{e^{2ik\Delta x} - e^{-2ik\Delta x} - 2e^{ik\Delta x} + 2e^{-ik\Delta x}}{2\Delta x^3}q_j^n$$

$$= \frac{i\sin(2k\Delta x) - 2i\sin(k\Delta x)}{\Delta x^3}q_j^n$$

$$= i\frac{2\sin(k\Delta x)\cos(k\Delta x) - 2\sin(k\Delta x)}{\Delta x^3}q_j^n$$

$$= 2i\sin(k\Delta x)\frac{\cos(k\Delta x) - 1}{\Delta x^3}q_j^n$$

$$= 2i\sin(k\Delta x)\frac{-2\sin^2\left(\frac{k\Delta x}{2}\right)}{\Delta x^3}q_j^n$$

$$= -4i\sin(k\Delta x)\frac{\sin^2\left(\frac{k\Delta x}{2}\right)}{\Delta x^3}q_j^n$$

2.1 FD for u

$$\begin{aligned}\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + g\frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x} + u_0\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \\ - \frac{h_0^2}{3}\left(u_0\frac{-u_{j-2}^n + 2u_{j-1}^n - 2u_{j+1}^n + u_{j+2}^n}{2\Delta x^3}\right) \\ - \frac{h_0^2}{3}\frac{\partial\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}}{\partial t} \\ = 0 \quad (1)\end{aligned}$$

$$
\begin{aligned}
\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t} + g\frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x} + u_0\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \\
- \frac{h_0^2}{3}\left(u_0\frac{-u_{j-2}^n + 2u_{j-1}^n - 2u_{j+1}^n + u_{j+2}^n}{2\Delta x^3}\right) \\
- \frac{h_0^2}{3}\frac{\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{\Delta x^2} - \frac{u_{j+1}^{n-1}-2u_j^{n-1}+u_{j-1}^{n-1}}{\Delta x^2}}{2\Delta t} \\
= 0 \quad (2)
\end{aligned}
$$

$$
\begin{aligned}
\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t} + g\frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x} + u_0\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \\
- \frac{h_0^2}{3}\left(u_0\frac{-u_{j-2}^n + 2u_{j-1}^n - 2u_{j+1}^n + u_{j+2}^n}{2\Delta x^3}\right) \\
- \frac{h_0^2}{3}\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1} - u_{j+1}^{n-1} + 2u_j^{n-1} - u_{j-1}^{n-1}}{2\Delta x^2\Delta t} \\
= 0 \quad (3)
\end{aligned}
$$

$$
\begin{aligned}
\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t} + g\frac{i\sin(l\Delta x)}{\Delta x}h_j^n + u_0\frac{i\sin(k\Delta x)}{\Delta x}u_j^n \\
+ \frac{h_0^2}{3}\left(4iu_0\sin(k\Delta x)\frac{\sin^2\left(\frac{k\Delta x}{2}\right)}{\Delta x^3}\right)u_j^n \\
- \frac{h_0^2}{6\Delta t}\left(\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} - \frac{u_{j+1}^{n-1} - 2u_j^{n-1} + u_{j-1}^{n-1}}{\Delta x^2}\right) \\
= 0 \quad (4)
\end{aligned}
$$

$$
\begin{aligned}
\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t} &+ g\frac{i\sin\left(l\Delta x\right)}{\Delta x}h_j^n + u_0\frac{i\sin\left(k\Delta x\right)}{\Delta x}u_j^n \\
&\qquad + \frac{h_0^2}{3}\left(4iu_0\sin\left(k\Delta x\right)\frac{\sin^2\left(\frac{k\Delta x}{2}\right)}{\Delta x^3}\right)u_j^n \\
&- \frac{h_0^2}{6\Delta t}\left(-\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)u_j^{n+1} - -\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)u_j^{n-1}\right) \\
&\qquad\qquad = 0
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
&\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t} - \frac{h_0^2}{6\Delta t}\left(-\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)u_j^{n+1} - -\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)u_j^{n-1}\right) \\
&= +\frac{h_0^2}{3}\left(4iu_0\sin\left(k\Delta x\right)\frac{\sin^2\left(\frac{k\Delta x}{2}\right)}{\Delta x^3}\right)u_j^n - g\frac{i\sin\left(l\Delta x\right)}{\Delta x}h_j^n + u_0\frac{i\sin\left(k\Delta x\right)}{\Delta x}u_j^n
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
&\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t} - \frac{h_0^2}{6\Delta t}\left(-\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)u_j^{n+1} - -\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)u_j^{n-1}\right) \\
&= \left[\frac{h_0^2}{3}\left(4iu_0\sin\left(k\Delta x\right)\frac{\sin^2\left(\frac{k\Delta x}{2}\right)}{\Delta x^3}\right) + u_0\frac{i\sin\left(k\Delta x\right)}{\Delta x}\right]u_j^n - g\frac{i\sin\left(l\Delta x\right)}{\Delta x}h_j^n
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&u_j^{n+1} - u_j^{n-1} + \frac{h_0^2}{3}\left(\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)u_j^{n+1} - \frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)u_j^{n-1}\right) \\
&= 2\Delta t\left(u_0\frac{i\sin\left(k\Delta x\right)}{\Delta x}\left[\frac{4h_0^2}{3\Delta x^2}\left(\sin^2\left(\frac{k\Delta x}{2}\right)\right) + 1\right]u_j^n - g\frac{i\sin\left(l\Delta x\right)}{\Delta x}h_j^n\right)
\end{aligned}
\tag{8}
$$

$$u_j^{n+1}\left[1+\frac{h_0^2}{3}\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)\right]$$

$$= u_j^{n-1}\left[1+\frac{h_0^2}{3}\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)\right]+$$

$$2\Delta t\left(u_0\frac{i\sin\left(k\Delta x\right)}{\Delta x}\left[\frac{4h_0^2}{3\Delta x^2}\left(\sin^2\left(\frac{k\Delta x}{2}\right)\right)+1\right]u_j^n - g\frac{i\sin\left(l\Delta x\right)}{\Delta x}h_j^n\right) \quad (9)$$

$$u_j^{n+1}$$

$$= u_j^{n-1}$$

$$+2\Delta t\left(u_0\frac{i\sin\left(k\Delta x\right)}{\Delta x}u_j^n - g\frac{1}{\left[1+\frac{h_0^2}{3}\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)\right]}\frac{i\sin\left(l\Delta x\right)}{\Delta x}h_j^n\right) \quad (10)$$

2.2 FD for h

$$\frac{h_j^{n+1}-h_j^{n-1}}{2\Delta t}+h_0\frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x}+u_0\frac{h_{j+1}^n-h_{j-1}^n}{2\Delta x}=0$$

$$\frac{h_j^{n+1}-h_j^{n-1}}{2\Delta t}+h_0\frac{i\sin\left(k\Delta x\right)}{\Delta x}u_j^n+u_0\frac{i\sin\left(l\Delta x\right)}{\Delta x}h_j^n=0$$

$$h_j^{n+1}-h_j^{n-1}=-2\Delta t\left[h_0\frac{i\sin\left(k\Delta x\right)}{\Delta x}u_j^n+u_0\frac{i\sin\left(l\Delta x\right)}{\Delta x}h_j^n\right]=0$$

$$h_j^{n+1}=h_j^{n-1}-\frac{2i\Delta t}{\Delta x}\left[h_0\sin\left(k\Delta x\right)u_j^n+u_0\sin\left(l\Delta x\right)h_j^n\right]=0$$

2.2.1 Together

We can formulate these schemes together to get

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \begin{bmatrix} h \\ u \end{bmatrix}_j^{n-1} + \begin{bmatrix} -\frac{2i\Delta t}{\Delta x}u_0\sin\left(l\Delta x\right) & -\frac{2i\Delta t}{\Delta x}h_0\sin\left(k\Delta x\right) \\ -\frac{2\Delta t}{1+\frac{h_0^2}{3}\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)}g\frac{i\sin(l\Delta x)}{\Delta x} & \frac{2i\Delta t}{\Delta x}u_0\sin\left(k\Delta x\right) \end{bmatrix}\begin{bmatrix} h \\ u \end{bmatrix}_j^n \quad (11)$$

Also we now have $k=l$

Let $A = \begin{bmatrix} -\frac{2i\Delta t}{\Delta x} u_0 \sin\left(k\Delta x\right) & -\frac{2i\Delta t}{\Delta x} h_0 \sin\left(k\Delta x\right) \\ -\frac{2\Delta t}{1+\frac{h_0^2}{3}\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)} g\frac{i\sin(k\Delta x)}{\Delta x} & \frac{2i\Delta t}{\Delta x} u_0 \sin\left(k\Delta x\right) \end{bmatrix}$

Then

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \begin{bmatrix} h \\ u \end{bmatrix}_j^{n-1} + A \begin{bmatrix} h \\ u \end{bmatrix}_j^{n}$$

Assume $\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = G \begin{bmatrix} h \\ u \end{bmatrix}_j^{n}$ So $\begin{bmatrix} h \\ u \end{bmatrix}_j^{n-1} = \frac{1}{G} \begin{bmatrix} h \\ u \end{bmatrix}_j^{n}$

Then

$$G \begin{bmatrix} h \\ u \end{bmatrix}_j^{n} = \frac{1}{G} \begin{bmatrix} h \\ u \end{bmatrix}_j^{n} + A \begin{bmatrix} h \\ u \end{bmatrix}_j^{n}$$

$$0 = \left[\frac{1}{G} I - GI + A\right] \begin{bmatrix} h \\ u \end{bmatrix}_j^{n}$$

$$0 = \left[A - \frac{G^2-1}{G} I\right] \begin{bmatrix} h \\ u \end{bmatrix}_j^{n}$$

So we have that $\frac{G^2-1}{G} = \lambda_{A,1}, \lambda_{A,2}$

$$G^2 - 1 = \lambda_{A,1} G$$

$$G^2 - \lambda_{A,1} G - 1 = 0$$

Thus

$$G = \frac{1}{2}\left(\lambda_{A,1} \pm \sqrt{\lambda_{A,1}^2 + 4}\right)$$

Also

$$G = \frac{1}{2}\left(\lambda_{A,2} \pm \sqrt{\lambda_{A,2}^2 + 4}\right)$$

Want $\max\{|G|\} \leq 1$

This is the equivalent to writing it like so:

$$
\begin{bmatrix} h_j^{n+1} \\ u_j^{n+1} \\ h_j^n \\ u_j^n \end{bmatrix} = \begin{bmatrix} -\frac{2i\Delta t}{\Delta x} u_0 \sin(k\Delta x) & -\frac{2i\Delta t}{\Delta x} h_0 \sin(k\Delta x) & 1 & 0 \\ -\frac{2\Delta t}{1+\frac{h_0^2}{3}\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)} g \frac{i\sin(k\Delta x)}{\Delta x} & \frac{2i\Delta t}{\Delta x} u_0 \sin(k\Delta x) & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_j^n \\ u_j^n \\ h_j^{n-1} \\ u_j^{n-1} \end{bmatrix} \tag{12}
$$

and having the eigenvalues of this less than 1 because if we let

$$
A = \begin{bmatrix} -\frac{2i\Delta t}{\Delta x} u_0 \sin(k\Delta x) & -\frac{2i\Delta t}{\Delta x} h_0 \sin(k\Delta x) & 1 & 0 \\ -\frac{2\Delta t}{1+\frac{h_0^2}{3}\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)} g \frac{i\sin(k\Delta x)}{\Delta x} & \frac{2i\Delta t}{\Delta x} u_0 \sin(k\Delta x) & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
$$

and we have a grwoth factor G then

$$
G \begin{bmatrix} h_j^n \\ u_j^n \\ h_j^{n-1} \\ u_j^{n-1} \end{bmatrix} = A \begin{bmatrix} h_j^n \\ u_j^n \\ h_j^{n-1} \\ u_j^{n-1} \end{bmatrix}
$$

Then

$$
(A - GI) \begin{bmatrix} h_j^n \\ u_j^n \\ h_j^{n-1} \\ u_j^{n-1} \end{bmatrix} = 0
$$

So G are the eigenvalues of A and so as long as $\rho(A) \leq 1$ then $G \leq 1$ as well.

3 Lax Wendroff Nonlinear

We have

$$u_j^{n+1}$$

$$= u_j^{n-1}$$

$$+ 2\Delta t \left(u_0 \frac{i\sin(k\Delta x)}{\Delta x} u_j^n - g \frac{1}{\left[1 + \frac{h_0^2}{3}\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)\right]} \frac{i\sin(k\Delta x)}{\Delta x} h_j^n \right) \tag{13}$$

lets define

$$c = -g \frac{2\Delta t}{\left[1 + \frac{h_0^2}{3}\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)\right]} \frac{i\sin(k\Delta x)}{\Delta x}$$

$$d = u_0 \frac{2i\Delta t \sin(k\Delta x)}{\Delta x}$$

Then

$$u_j^{n+1} = u_j^{n-1} + du_j^n + ch_j^n$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h_0 u + u_0 h) = 0$$

Using the two step richtemyer LW method

$$h_{j+1/2}^{n+1/2} = \frac{1}{2}\left(h_{j+1}^n + h_j^n\right) - \frac{\Delta t}{2\Delta x}\left[H(u_{j+1}^n - u_j^n) + U(h_{j+1}^n - h_j^n)\right]$$

$$h_{j-1/2}^{n+1/2} = \frac{1}{2}\left(h_j^n + h_{j-1}^n\right) - \frac{\Delta t}{2\Delta x}\left[H(u_j^n - u_{j-1}^n) + U(h_j^n - h_{j-1}^n)\right]$$

$$h_j^{n+1} = h_j^n - \frac{\Delta t}{\Delta x}\left[H(u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2}) + U(h_{j+1/2}^{n+1/2} - h_{j-1/2}^{n+1/2})\right]$$

We calculate $u_{j+1/2}^{n+1/2}$ by taking the average over the updated u and the current u. So we have

$$u_{j+1/2}^{n+1/2} = \frac{u_{j+1}^{n+1} + u_j^{n+1} + u_{j+1}^n + u_j^n}{4}$$

$$u_{j-1/2}^{n+1/2} = \frac{u_j^{n+1} + u_{j-1}^{n+1} + u_j^n + u_{j-1}^n}{4}$$

Using Fourier Nodes

$$u_{j+1/2}^{n+1/2} = \frac{e^{ik\Delta x} u_j^{n+1} + u_j^{n+1} + e^{ik\Delta x} u_j^n + u_j^n}{4}$$

$$u_{j+1/2}^{n+1/2} = \frac{\left(e^{ik\Delta x} + 1\right)\left(u_j^{n+1} + u_j^n\right)}{4}$$

$$u_{j-1/2}^{n+1/2} = \frac{\left(e^{-ik\Delta x} + 1\right)\left(u_j^{n+1} + u_j^n\right)}{4}$$

$$u_{j-1/2}^{n+1/2} = e^{-ik\Delta x} \frac{\left(1 + e^{ik\Delta x}\right)\left(u_j^{n+1} + u_j^n\right)}{4}$$

$$u_{j-1/2}^{n+1/2} = e^{-ik\Delta x} u_{j+1/2}^{n+1/2}$$

$$h_{j+1/2}^{n+1/2} = \frac{1}{2}\left(e^{ik\Delta x} + 1\right) h_j^n - \frac{\Delta t}{2\Delta x}\left[H\left(e^{ik\Delta x} - 1\right) u_j^n + U\left(e^{ik\Delta x} - 1\right) h_j^n\right]$$

$$h_{j-1/2}^{n+1/2} = \frac{1}{2}\left(1 + e^{-ik\Delta x}\right) h_j^n - \frac{\Delta t}{2\Delta x}\left[H\left(1 - e^{-ik\Delta x}\right) u_j^n + U\left(1 - e^{-ik\Delta x}\right) h_j^n\right]$$

$$h_{j-1/2}^{n+1/2} = e^{-ik\Delta x}\left[\frac{1}{2}\left(e^{ik\Delta x} + 1\right) h_j^n - \frac{\Delta t}{2\Delta x}\left[H\left(e^{ik\Delta x} - 1\right) u_j^n + U\left(e^{ik\Delta x} - 1\right) h_j^n\right]\right]$$

$$h_{j-1/2}^{n+1/2} = e^{-ik\Delta x} h_{j+1/2}^{n+1/2}$$

$$h_{j+1/2}^{n+1/2} = \left(\frac{1}{2}\left(e^{ik\Delta x} + 1\right) - \frac{\Delta t}{2\Delta x} U\left(e^{ik\Delta x} - 1\right)\right) h_j^n - \left(\frac{\Delta t}{2\Delta x} H\left(e^{ik\Delta x} - 1\right)\right) u_j^n$$

$$u_{j+1/2}^{n+1/2} = \frac{\left(e^{ik\Delta x} + 1\right)\left(u_j^{n+1} + u_j^n\right)}{4}$$

$$u_{j+1/2}^{n+1/2} = \frac{\left(e^{ik\Delta x}+1\right)\left(u_j^{n-1}+du_j^n+ch_j^n+u_j^n\right)}{4}$$

$$u_{j+1/2}^{n+1/2} = \frac{e^{ik\Delta x}+1}{4}\left(u_j^{n-1}+(d+1)u_j^n+ch_j^n\right)$$

So we have

$$h_j^{n+1} = h_j^n - \frac{\Delta t}{\Delta x}\left[H(1-e^{-ik\Delta x})u_{j+1/2}^{n+1/2} + U(1-e^{-ik\Delta x})h_{j+1/2}^{n+1/2}\right]$$

$$h_j^{n+1} = h_j^n - \frac{\Delta t}{\Delta x}H(1-e^{-ik\Delta x})\frac{e^{ik\Delta x}+1}{4}\left(u_j^{n-1}+(d+1)u_j^n+ch_j^n\right) \\ -\frac{\Delta t}{\Delta x}U(1-e^{-ik\Delta x})\left[\left(\frac{1}{2}\left(e^{ik\Delta x}+1\right)-\frac{\Delta t}{2\Delta x}U\left(e^{ik\Delta x}-1\right)\right)h_j^n - \left(\frac{\Delta t}{2\Delta x}H\left(e^{ik\Delta x}-1\right)\right)u_j^n\right] \tag{14}$$

$$h_j^{n+1} = \\ \left[1-\frac{\Delta t}{\Delta x}cH\frac{2i\sin(k\Delta x)}{4}-\frac{\Delta t}{\Delta x}U(1-e^{-ik\Delta x})\left(\frac{1}{2}\left(e^{ik\Delta x}+1\right)-\frac{\Delta t}{2\Delta x}U\left(e^{ik\Delta x}-1\right)\right)\right]h_j^n \\ -\frac{\Delta t}{\Delta x}H(1-e^{-ik\Delta x})\frac{e^{ik\Delta x}+1}{4}\left(u_j^{n-1}+(d+1)u_j^n\right) \\ -\frac{\Delta t}{\Delta x}U(1-e^{-ik\Delta x})\left[-\left(\frac{\Delta t}{2\Delta x}H\left(e^{ik\Delta x}-1\right)\right)u_j^n\right] \tag{15}$$

$$h_j^{n+1} = \\ \left[1-\frac{\Delta t}{\Delta x}cH\frac{i\sin(k\Delta x)}{2}-\frac{\Delta t}{\Delta x}U\left(\frac{1}{2}\left(2i\sin(k\Delta x)\right)-\frac{\Delta t}{2\Delta x}U\left(2\cos(ik\Delta x)-2\right)\right)\right]h_j^n \\ -\frac{\Delta t}{\Delta x}H\frac{i\sin(k\Delta x)}{2}\left(u_j^{n-1}+(d+1)u_j^n\right) \\ -\frac{\Delta t}{\Delta x}U\left[-\left(\frac{\Delta t}{2\Delta x}H\left(2\cos(ik\Delta x)-2\right)\right)u_j^n\right] \tag{16}$$

$$
\begin{aligned}
&h_j^{n+1} = \\
&\left[1 - \frac{\Delta t}{\Delta x} cH \frac{i\sin(k\Delta x)}{2} - \frac{\Delta t}{\Delta x} U \left((i\sin(k\Delta x)) - \frac{\Delta t}{\Delta x} U (\cos(ik\Delta x) - 1) \right) \right] h_j^n \\
&\qquad - \frac{\Delta t}{\Delta x} H \frac{i\sin(k\Delta x)}{2} \left(u_j^{n-1} + (d+1) u_j^n \right) \\
&\qquad - \frac{\Delta t}{\Delta x} U \left[- \left(\frac{\Delta t}{\Delta x} H (\cos(ik\Delta x) - 1) \right) u_j^n \right]
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
&h_j^{n+1} = \\
&\left[1 - \frac{\Delta t}{\Delta x} cH \frac{i\sin(k\Delta x)}{2} - \frac{\Delta t}{\Delta x} U \left((i\sin(k\Delta x)) - \frac{\Delta t}{\Delta x} U (\cos(ik\Delta x) - 1) \right) \right] h_j^n \\
&\qquad - \frac{\Delta t}{\Delta x} H \frac{i\sin(k\Delta x)}{2} u_j^{n-1} \\
&\qquad - \frac{\Delta t}{\Delta x} H \frac{i\sin(k\Delta x)}{2} (d+1) u_j^n - \frac{\Delta t}{\Delta x} U \left[- \left(\frac{\Delta t}{\Delta x} H (\cos(ik\Delta x) - 1) \right) u_j^n \right]
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
&h_j^{n+1} = \\
&\left[1 - \frac{\Delta t}{\Delta x} cH \frac{i\sin(k\Delta x)}{2} - \frac{\Delta t}{\Delta x} U \left((i\sin(k\Delta x)) - \frac{\Delta t}{\Delta x} U (\cos(ik\Delta x) - 1) \right) \right] h_j^n \\
&\qquad - \frac{\Delta t}{\Delta x} H \frac{i\sin(k\Delta x)}{2} u_j^{n-1} \\
&\qquad - \frac{\Delta t}{\Delta x} \left[H \frac{i\sin(k\Delta x)}{2} (d+1) + U \left[- \left(\frac{\Delta t}{\Delta x} H (\cos(ik\Delta x) - 1) \right) \right] \right] u_j^n
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
&h_j^{n+1} = \\
&\left[1 - \frac{\Delta t}{\Delta x} cH \frac{i\sin(k\Delta x)}{2} - \frac{\Delta t}{\Delta x} U \left((i\sin(k\Delta x)) - \frac{\Delta t}{\Delta x} U (\cos(ik\Delta x) - 1) \right) \right] h_j^n \\
&\qquad - \frac{\Delta t}{\Delta x} H \frac{i\sin(k\Delta x)}{2} u_j^{n-1} \\
&\qquad - \frac{\Delta t}{\Delta x} \left[H \frac{i\sin(k\Delta x)}{2} (d+1) - U \left(\frac{\Delta t}{\Delta x} H (\cos(ik\Delta x) - 1) \right) \right] u_j^n
\end{aligned} \tag{20}
$$

defining $a = \left[1 - \frac{\Delta t}{\Delta x} cH \frac{i \sin(k\Delta x)}{2} - \frac{\Delta t}{\Delta x} U \left((i \sin(k\Delta x)) - \frac{\Delta t}{\Delta x} U (\cos(ik\Delta x) - 1) \right) \right]$
$b = -\frac{\Delta t}{\Delta x} \left[H \frac{i \sin(k\Delta x)}{2} (d+1) - U \left(\frac{\Delta t}{\Delta x} H (\cos(ik\Delta x) - 1) \right) \right]$
$e = -\frac{\Delta t}{\Delta x} H \frac{i \sin(k\Delta x)}{2}$
Then we have

$$\begin{bmatrix} h_j^{n+1} \\ u_j^{n+1} \\ h_j^n \\ u_j^n \end{bmatrix} = \begin{bmatrix} a & b & 0 & e \\ c & d & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_j^n \\ u_j^n \\ h_j^{n-1} \\ u_j^{n-1} \end{bmatrix}$$

equivalently we can write this as

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \begin{bmatrix} 0 & e \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^{n-1} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

Taking out the growth factors we have

$$G \begin{bmatrix} h \\ u \end{bmatrix}_j^n = \frac{1}{G} \begin{bmatrix} 0 & e \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^n + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$\left(\begin{bmatrix} aG & bG+e \\ cG & dG+1 \end{bmatrix} - G^2 I \right) \begin{bmatrix} h \\ u \end{bmatrix}_j^n = 0$$

So G^2 is the eigenvalues of $\begin{bmatrix} aG & bG+e \\ cG & dG+1 \end{bmatrix}$ we have that the eigenvalues of this matrix are

$$\lambda = \frac{1}{2} \left(\pm \sqrt{(d-a)^2 G^2 + 4bcG^2 + 2(d-a)G + 4ceG + 1} + (d+a)G + 1 \right)$$

Therefore

$$G^2 = \frac{1}{2} \left(\pm \sqrt{(d-a)^2 G^2 + 4bcG^2 + 2(d-a)G + 4ceG + 1} + (d+a)G + 1 \right)$$

$$2G^2 - (d+a)G - 1 = \pm \sqrt{(d-a)^2 G^2 + 4bcG^2 + 2(d-a)G + 4ceG + 1}$$

$$\left(2G^2-(d+a)G-1\right)^2=(d-a)^2\,G^2+4bcG^2+2(d-a)G+4ceG+1$$

$$4G^4-4(a+d)G^3+(a+d)^2G^2-4G^2+2(a+d)G+1=(d-a)^2\,G^2+4bcG^2+2(d-a)G+4ceG+1$$

$$4G^4-4(a+d)G^3+4adG^2-4G^2+2(a)G=+4bcG^2+2(-a)G+4ceG$$

$$4G^4-4(a+d)G^3+4(ad-bc)G^2-4G^2+4(a-ce)G=0$$

$$G^4-(a+d)G^3+(ad-bc)G^2-G^2+(a-ce)G=0$$
$$G\left[G^3-(a+d)G^2+(ad-bc-1)G+(a-ce)\right]=0$$

4 LWL

we have

$$u_j^{n+1}$$

$$= u_j^{n-1}$$

$$+ 2\Delta t \left(u_0 \frac{i\sin(k\Delta x)}{\Delta x} u_j^n - g \frac{1}{\left[1 + \frac{h_0^2}{3}\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)\right]} \frac{i\sin(k\Delta x)}{\Delta x} h_j^n \right) \quad (21)$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h_0 u + u_0 h) = 0$$

Since h equation is linear e can use the linear LW method

$$h_j^{n+1} = h_j^n - \frac{\Delta t}{2\Delta x}\left[H\left(u_{j+1}^n - u_{j-1}^n\right) + U\left(h_{j+1}^n - h_{j-1}^n\right)\right] + \frac{\Delta t^2}{2\Delta x^2}\left[H^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) + U^2\left(h_{j+1}^n - 2h_j^n + h_{j-1}^n\right)\right]$$

$$h_j^{n+1} = h_j^n - \frac{\Delta t}{2\Delta x}\left[H\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)u_j^n + U\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)h_j^n\right] + \frac{\Delta t^2}{2\Delta x^2}\left[H^2\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)u_j^n + U^2\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)h_j^n\right]$$

$$h_j^{n+1} = h_j^n - \frac{\Delta t}{\Delta x}\left[Hi\sin(k\Delta x)\,u_j^n + Ui\sin(k\Delta x)\,h_j^n\right] + \frac{\Delta t^2}{\Delta x^2}\left[H^2(\cos(k\Delta x) - 1)\,u_j^n + U^2(\cos(k\Delta x) - 1)\,h_j^n\right]$$

$$h_j^{n+1} = h_j^n - \frac{\Delta t}{\Delta x}\left[Hi\sin(k\Delta x)\,u_j^n + Ui\sin(k\Delta x)\,h_j^n\right] - \frac{\Delta t^2}{\Delta x^2}\left[2H^2\sin^2\left(\frac{k\Delta x}{2}\right)u_j^n + 2U^2\sin^2\left(\frac{k\Delta x}{2}\right)h_j^n\right]$$

$$h_j^{n+1} = \left[1 - \frac{\Delta t}{\Delta x} U i \sin\left(k\Delta x\right) + \frac{\Delta t^2}{\Delta x^2} 2U^2 \sin^2\left(\frac{k\Delta x}{2}\right)\right] h_j^n$$
$$- \left[-\frac{\Delta t}{\Delta x} H i \sin\left(k\Delta x\right) + \frac{\Delta t^2}{\Delta x^2} 2H^2 \sin^2\left(\frac{k\Delta x}{2}\right)\right] u_j^n$$

So we have

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^{n-1}$$
$$+ \begin{bmatrix} 1 - \frac{\Delta t}{\Delta x} U i \sin\left(k\Delta x\right) - \frac{\Delta t^2}{\Delta x^2} 2U^2 \sin^2\left(\frac{k\Delta x}{2}\right) & -\frac{\Delta t}{\Delta x} H i \sin\left(k\Delta x\right) - \frac{\Delta t^2}{\Delta x^2} 2H^2 \sin^2\left(\frac{k\Delta x}{2}\right) \\ -\frac{2\Delta t}{1 + \frac{h_0^2}{3}\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)} g \frac{i \sin\left(l\Delta x\right)}{\Delta x} & \frac{2i\Delta t}{\Delta x} u_0 \sin\left(k\Delta x\right) \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^n \tag{22}$$

Lets define

$$a = 1 - \frac{\Delta t}{\Delta x} U i \sin\left(k\Delta x\right) - \frac{\Delta t^2}{\Delta x^2} 2U^2 \sin^2\left(\frac{k\Delta x}{2}\right)$$

$$b = -\frac{\Delta t}{\Delta x} H i \sin\left(k\Delta x\right) - \frac{\Delta t^2}{\Delta x^2} 2H^2 \sin^2\left(\frac{k\Delta x}{2}\right)$$

$$c = -\frac{2\Delta t}{1 + \frac{h_0^2}{3}\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)} g \frac{i \sin\left(l\Delta x\right)}{\Delta x}$$

$$d = \frac{2i\Delta t}{\Delta x} u_0 \sin\left(k\Delta x\right)$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^{n-1} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^n \tag{23}$$

As before we have a growth factor G

$$G \begin{bmatrix} h \\ u \end{bmatrix}_j^n = \frac{1}{G} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^n + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_j^n \tag{24}$$

$$\left(-GI + \frac{1}{G}\left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right] + \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]\right)\left[\begin{array}{c} h \\ u \end{array}\right]_j^n = 0 \quad (25)$$

$$\left(-G^2 I + \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right] + \left[\begin{array}{cc} aG & bG \\ cG & dG \end{array}\right]\right)\left[\begin{array}{c} h \\ u \end{array}\right]_j^n = 0 \quad (26)$$

$$\left(\left[\begin{array}{cc} aG & bG \\ cG & dG+1 \end{array}\right] - G^2 I\right)\left[\begin{array}{c} h \\ u \end{array}\right]_j^n = 0 \quad (27)$$

So G^2 are the eigenvalues of $\left[\begin{array}{cc} aG & bG \\ cG & dG+1 \end{array}\right]$
We have

$$\lambda_1 = \frac{1}{2}\left(-\sqrt{a^2G^2 - 2adG^2 - 2aG + 4bcG^2 + d^2G^2 + 2dG + 1} + aG + dG + 1\right)$$

$$\lambda_2 = \frac{1}{2}\left(\sqrt{a^2G^2 - 2adG^2 - 2aG + 4bcG^2 + d^2G^2 + 2dG + 1} + aG + dG + 1\right)$$

$$G^2 = \frac{1}{2}\left(\pm\sqrt{\left(a^2 - 2ad + d^2 + 4bc\right)G^2 + \left(2d - 2a\right)G + 1} + aG + dG + 1\right)$$

$$2G^2 = \pm\sqrt{\left(a^2 - 2ad + d^2 + 4bc\right)G^2 + \left(2d - 2a\right)G + 1} + aG + dG + 1$$

$$2G^2 - (a+d)G - 1 = \pm\sqrt{\left(a^2 - 2ad + d^2 + 4bc\right)G^2 + \left(2d - 2a\right)G + 1}$$

$$\left(2G^2 - (a+d)G - 1\right)^2 = \left(a^2 - 2ad + d^2 + 4bc\right)G^2 + \left(2d - 2a\right)G + 1$$

$$4G^4 - 4aG^3 - 4dG^3 + a^2G^2 + 2adG^2 + d^2G^2 - 4G^2 + 2aG + 2dG + 1$$
$$= \left(a^2 - 2ad + d^2 + 4bc\right) G^2 + (2d - 2a)\, G + 1$$

$$4G^4 - 4\,(a + d)\, G^3 + 4adG^2 - 4G^2 + 2aG = 4bcG^2 - 2aG$$

$$4G^4 - 4\,(a + d)\, G^3 + 4(ad - bc - 1)G^2 + 4aG = 0$$

$$G^4 - (a + d)\, G^3 + (ad - bc - 1)G^2 + aG = 0$$

$$G\left(G^3 - (a + d)\, G^2 + (ad - bc - 1)G + a\right) = 0$$