1 Numerical Method for dispersion error Break Down

To do the first analysis we first construct the update matrix F, which plays a similar role to λ I showed you in the space continuos case. We then diagonalise this so that we get back to just factors (basically). This also means I would like to update what we present in the table, I think we should present all the elements of the matrices, for $\mathcal{F}^{h,u}$ and $\mathcal{F}^{h,h}$ there is no change but for $\mathcal{F}^{u,u}$ and $\mathcal{F}^{u,h}$ this means also dividing it by $\mathcal{G}$ as well as $\mathcal{M}$ and Δx. Anyway onto the method.

1.1 $\boldsymbol{F}$

$\boldsymbol{F}$ comes from the FVM update scheme which for us is

$$\bar{q}_j^{n+1} = \bar{q}_j^n - \frac{\Delta t}{\Delta x}\left[F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right]$$

This converts to (both analytical and numerical)

$$\mathcal{M}q_j^{n+1} = \mathcal{M}q_j^n - \frac{\Delta t}{\Delta x}\left[\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j - \mathcal{F}^{q,v}v_{j-1} - \mathcal{F}^{q,q}q_{j-1}\right]$$

$$\mathcal{M}q_j^{n+1} = \mathcal{M}q_j^n - \frac{\Delta t}{\Delta x}\left[\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j - \mathcal{F}^{q,v}e^{-ik\Delta x}v_j - \mathcal{F}^{q,q}e^{-ik\Delta x}q_j\right]$$

Defining $\mathcal{D}_x = 1 - e^{-ik\Delta x}$

$$\mathcal{M}q_j^{n+1} = \mathcal{M}q_j^n - \frac{\Delta t}{\Delta x}\left[\mathcal{D}_x\mathcal{F}^{q,v}v_j + \mathcal{D}_x\mathcal{F}^{q,q}q_j\right]$$

So we have

$$q_j^{n+1} = q_j^n - \frac{\mathcal{D}_x\Delta t}{\mathcal{M}\Delta x}\left[\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j\right]$$

Thus we have

$$\begin{bmatrix} h \\ \mathcal{G}u \end{bmatrix}_j^{n+1} = \begin{bmatrix} h \\ \mathcal{G}u \end{bmatrix}_j^{n} - \frac{\mathcal{D}_x\Delta t}{\mathcal{M}\Delta x}\begin{bmatrix} \mathcal{F}^{h,h} & \mathcal{F}^{h,u} \\ \mathcal{F}^{u,h} & \mathcal{F}^{u,u} \end{bmatrix}\begin{bmatrix} h \\ u \end{bmatrix}_j^{n}$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = \left[\begin{array}{c} h \\ u \end{array}\right]_j^{n} - \frac{\mathcal{D}_x \Delta t}{\mathcal{M}\Delta x}\left[\begin{array}{cc} \mathcal{F}^{h,h} & \mathcal{F}^{h,u} \\ \frac{1}{\mathcal{G}}\mathcal{F}^{u,h} & \frac{1}{\mathcal{G}}\mathcal{F}^{u,u} \end{array}\right]\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n}$$

Lets define

$$\boldsymbol{F} = \frac{\mathcal{D}_x}{\mathcal{M}\Delta x}\left[\begin{array}{cc} \mathcal{F}^{h,h} & \mathcal{F}^{h,u} \\ \frac{1}{\mathcal{G}}\mathcal{F}^{u,h} & \frac{1}{\mathcal{G}}\mathcal{F}^{u,u} \end{array}\right]$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = \left[\begin{array}{c} h \\ u \end{array}\right]_j^{n} - \Delta t \boldsymbol{F}\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n}$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = (\boldsymbol{I} - \Delta t \boldsymbol{F})\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n}$$

Thats our Euler Step, the difference between this and the previous version is we didn't divide that bottom Row by $\mathcal{G}$ So we have to change our approximation stuff. Also we would like the know what the analytic value of $\boldsymbol{F}$ is and approximations to it.

1.2 Analytic

$$\frac{\mathcal{D}_a}{\Delta x \mathcal{M}_a}\mathcal{F}_a^{h,u} = ikH$$

$$\frac{\mathcal{D}_a}{\Delta x \mathcal{M}_a}\mathcal{F}_a^{h,h} = 0$$

$$\frac{\mathcal{D}_a}{\mathcal{G}_a \Delta x \mathcal{M}_a}\mathcal{F}_a^{u,h} = \frac{ikgH}{H + \frac{H^3}{3}k^2} = i\frac{k}{H}gH\frac{3}{3 + H^2k^2}$$

using $\omega = \pm k\sqrt{gH}\sqrt{\frac{3}{H^2k^2+3}}$, $\omega^2 = k^2 gH\frac{3}{H^2k^2+3}$

$$\frac{\mathcal{D}_a}{\mathcal{G}\Delta x \mathcal{M}_a}\mathcal{F}_a^{u,h} = i\frac{k}{H}\frac{\omega^2}{k^2} = -i\frac{\omega^2}{kH}$$

$$\frac{\mathcal{D}_a}{\mathcal{G}\Delta x \mathcal{M}_a}\mathcal{F}_a^{u,u} = 0$$

So we have

$$\boldsymbol{F} = \left[\begin{array}{cc} 0 & ikH \\ \frac{\omega^2}{ikH} & 0 \end{array}\right] = \frac{1}{ikH}\left[\begin{array}{cc} 0 & -k^2H^2 \\ \omega^2 & 0 \end{array}\right]$$

We can diagonalise this ($A = SDS^{-1}$) with the follwing matrices

$$\boldsymbol{F} = \frac{1}{ikH} \begin{bmatrix} -\frac{ikH}{\omega} & \frac{ikH}{\omega} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -ikH\omega & 0 \\ 0 & ikH\omega \end{bmatrix} \begin{bmatrix} -\frac{ikH}{\omega} & \frac{ikH}{\omega} \\ 1 & 1 \end{bmatrix}^{-1}$$

$$\boldsymbol{F} = \begin{bmatrix} -\frac{ikH}{\omega} & \frac{ikH}{\omega} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -\omega & 0 \\ 0 & \omega \end{bmatrix} \begin{bmatrix} -\frac{ikH}{\omega} & \frac{ikH}{\omega} \\ 1 & 1 \end{bmatrix}^{-1}$$

We will use the following notation

$$\boldsymbol{W}_a = \begin{bmatrix} -\omega & 0 \\ 0 & \omega \end{bmatrix}$$

$$\boldsymbol{S}_a = \begin{bmatrix} -\frac{ikH}{\omega} & \frac{ikH}{\omega} \\ 1 & 1 \end{bmatrix}$$

So we have

$$\boldsymbol{F}_a = \boldsymbol{S}_a \boldsymbol{W}_a \boldsymbol{S}_a^{-1}$$

1.3 Third Order

$$\begin{bmatrix} h \\ u \end{bmatrix}^1 = (\boldsymbol{I} - \Delta t \boldsymbol{F}_3) \begin{bmatrix} h \\ u \end{bmatrix}^n$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^2 = (\boldsymbol{I} - \Delta t \boldsymbol{F}_3) \begin{bmatrix} h \\ u \end{bmatrix}^1$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^3 = \frac{3}{4} \begin{bmatrix} h \\ u \end{bmatrix}^n + \frac{1}{4} \begin{bmatrix} h \\ u \end{bmatrix}^2$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^4 = (\boldsymbol{I} - \Delta t \boldsymbol{F}_3) \begin{bmatrix} h \\ u \end{bmatrix}^3$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^{n+1} = \frac{1}{3} \begin{bmatrix} h \\ u \end{bmatrix}^n + \frac{2}{3} \begin{bmatrix} h \\ u \end{bmatrix}^4$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^3 = \frac{3}{4} \begin{bmatrix} h \\ u \end{bmatrix}^n + \frac{1}{4} (\boldsymbol{I} - \Delta t \boldsymbol{F}_3)^2 \begin{bmatrix} h \\ u \end{bmatrix}^n$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^3 = \left(\frac{3}{4}\boldsymbol{I} + \frac{1}{4}(\boldsymbol{I} - \Delta t \boldsymbol{F}_3)^2\right)\left[\begin{array}{c} h \\ u \end{array}\right]^n$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \frac{1}{3}\left[\begin{array}{c} h \\ u \end{array}\right]^n + \frac{2}{3}(\boldsymbol{I} - \Delta t \boldsymbol{F}_3)\left(\frac{3}{4}\boldsymbol{I} + \frac{1}{4}\left(\boldsymbol{I} - \Delta t \boldsymbol{F}_3\right)^2\right)\left[\begin{array}{c} h \\ u \end{array}\right]^n$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\frac{1}{3}\boldsymbol{I} + \frac{2}{3}(\boldsymbol{I} - \Delta t \boldsymbol{F}_3)\left(\frac{3}{4}\boldsymbol{I} + \frac{1}{4}\left(\boldsymbol{I} - \Delta t \boldsymbol{F}_3\right)^2\right)\right)\left[\begin{array}{c} h \\ u \end{array}\right]^n$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\frac{1}{3}\boldsymbol{I} + (\boldsymbol{I} - \Delta t \boldsymbol{F}_3)\left(\frac{1}{2}\boldsymbol{I} + \frac{1}{6}\left(\boldsymbol{I} - 2\Delta t \boldsymbol{F}_3 + \Delta t^2 \boldsymbol{F}_3^2\right)\right)\right)\left[\begin{array}{c} h \\ u \end{array}\right]^n$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\frac{1}{3}\boldsymbol{I} + (\boldsymbol{I} - \Delta t \boldsymbol{F}_3)\left(\frac{2}{3}\boldsymbol{I} - \frac{1}{3}\Delta t \boldsymbol{F}_3 + \frac{1}{6}\Delta t^2 \boldsymbol{F}_3^2\right)\right)\left[\begin{array}{c} h \\ u \end{array}\right]^n$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\frac{1}{3}\boldsymbol{I} + \frac{2}{3}\boldsymbol{I} - \frac{1}{3}\Delta t \boldsymbol{F}_3 + \frac{1}{6}\Delta t^2 \boldsymbol{F}_3^2 + (-\Delta t \boldsymbol{F}_3)\left(\frac{2}{3}\boldsymbol{I} - \frac{1}{3}\Delta t \boldsymbol{F}_3 + \frac{1}{6}\Delta t^2 \boldsymbol{F}_3^2\right)\right)\left[\begin{array}{c} h \\ u \end{array}\right]^n$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\boldsymbol{I} - \frac{1}{3}\Delta t \boldsymbol{F}_3 + \frac{1}{6}\Delta t^2 \boldsymbol{F}_3^2 - \frac{2}{3}\Delta t \boldsymbol{F}_3 + \frac{1}{3}\Delta t \boldsymbol{F}_3 \Delta t \boldsymbol{F}_3 - \frac{1}{6}\Delta t^2 \boldsymbol{F}_3^2 \Delta t \boldsymbol{F}_3\right)\left[\begin{array}{c} h \\ u \end{array}\right]^n$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\boldsymbol{I} - \boldsymbol{F}_3 + \frac{1}{6}\Delta t^2 \boldsymbol{F}_3^2 + \frac{1}{3}\Delta t^2 \boldsymbol{F}_3^2 - \frac{1}{6}\Delta t^3 \boldsymbol{F}_3^3\right)\left[\begin{array}{c} h \\ u \end{array}\right]^n$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\boldsymbol{I} - \Delta t \boldsymbol{F}_3 + \frac{1}{2}\Delta t^2 \boldsymbol{F}_3^2 - \frac{1}{6}\Delta t^3 \boldsymbol{F}_3^3\right)\left[\begin{array}{c} h \\ u \end{array}\right]^n$$

lets say we have

$$\boldsymbol{F}_3 = \boldsymbol{S}_3 \boldsymbol{D}_3 \boldsymbol{S}_3^{-1}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^{n+1} = \left(\boldsymbol{I} - \Delta t \boldsymbol{S}_3 \boldsymbol{D}_3 \boldsymbol{S}_3^{-1} + \frac{1}{2}\Delta t^2 \boldsymbol{S}_3 \boldsymbol{D}_3^2 \boldsymbol{S}_3^{-1} - \frac{1}{6}\Delta t^3 \boldsymbol{S}_3 \boldsymbol{D}_3^3 \boldsymbol{S}_3^{-1} \right) \begin{bmatrix} h \\ u \end{bmatrix}^{n}$$

$$\boldsymbol{S}_3^{-1} \begin{bmatrix} h \\ u \end{bmatrix}^{n+1} = \left(\boldsymbol{S}_3^{-1} - \Delta t \boldsymbol{D}_3 \boldsymbol{S}_3^{-1} + \frac{1}{2}\Delta t^2 \boldsymbol{D}_3^2 \boldsymbol{S}_3^{-1} - \frac{1}{6}\Delta t^3 \boldsymbol{D}_3^3 \boldsymbol{S}_3^{-1} \right) \begin{bmatrix} h \\ u \end{bmatrix}^{n}$$

$$\boldsymbol{S}_3^{-1} e^{i\omega\Delta t} \begin{bmatrix} h \\ u \end{bmatrix}^{n} = \left(\boldsymbol{I} - \Delta t \boldsymbol{D}_3 + \frac{1}{2}\Delta t^2 \boldsymbol{D}_3^2 - \frac{1}{6}\Delta t^3 \boldsymbol{D}_3^3 \right) \boldsymbol{S}_3^{-1} \begin{bmatrix} h \\ u \end{bmatrix}^{n}$$

$$e^{i\omega\Delta t} \boldsymbol{S}_3^{-1} \begin{bmatrix} h \\ u \end{bmatrix}^{n} = \left(\boldsymbol{I} - \Delta t \boldsymbol{D}_3 + \frac{1}{2}\Delta t^2 \boldsymbol{D}_3^2 - \frac{1}{6}\Delta t^3 \boldsymbol{D}_3^3 \right) \boldsymbol{S}_3^{-1} \begin{bmatrix} h \\ u \end{bmatrix}^{n}$$

$$\begin{gathered} e^{i\omega\Delta t} \boldsymbol{S}_3^{-1} \begin{bmatrix} h \\ u \end{bmatrix}^{n} = \\ \begin{bmatrix} 1 - \Delta t \lambda_{3,-} + \frac{\Delta t^2}{2}\lambda_{3,-}^2 - \frac{\Delta t^3}{6}\lambda_{3,-}^3 & 0 \\ 0 & 1 - \Delta t \lambda_{3,+} + \frac{\Delta t^2}{2}\lambda_{3,+}^2 - \frac{\Delta t^3}{6}\lambda_{3,+}^3 \end{bmatrix} \boldsymbol{S}_3^{-1} \begin{bmatrix} h \\ u \end{bmatrix}^{n} \end{gathered} \tag{1}$$

So $e^{i\omega\Delta t}$ an eigenvalue of this diagonal matrix, so it must be that

$$e^{i\omega\Delta t} = 1 - \Delta t \lambda_{3,\pm} + \frac{\Delta t^2}{2}\lambda_{3,\pm}^2 - \frac{\Delta t^3}{6}\lambda_{3,\pm}^3$$

$$i\omega\Delta t = \ln\left(1 - \Delta t \lambda_{3,\pm} + \frac{\Delta t^2}{2}\lambda_{3,\pm}^2 - \frac{\Delta t^3}{6}\lambda_{3,\pm}^3\right)$$

$$\omega = \frac{1}{i\Delta t}\ln\left(1 - \Delta t \lambda_{3,\pm} + \frac{\Delta t^2}{2}\lambda_{3,\pm}^2 - \frac{\Delta t^3}{6}\lambda_{3,\pm}^3\right)$$

Yes so its possible, now for the other methods. Where $\boldsymbol{F}_3$ is given by the following

1.4 Second Order

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = (\boldsymbol{I} - \Delta t \boldsymbol{F_1}) \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

lets say we have

$$\boldsymbol{F}_1 = \boldsymbol{S}_1 \begin{bmatrix} \lambda_{1,-} & 0 \\ 0 & \lambda_{1,+} \end{bmatrix} \boldsymbol{S}_1^{-1}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \left(\boldsymbol{I} - \Delta t \boldsymbol{S}_1 \begin{bmatrix} \lambda_{1,-} & 0 \\ 0 & \lambda_{1,+} \end{bmatrix} \boldsymbol{S}_1^{-1} \right) \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$\boldsymbol{S}_1^{-1} \begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \left(\boldsymbol{S}_1^{-1} - \Delta t \begin{bmatrix} \lambda_{1,-} & 0 \\ 0 & \lambda_{1,+} \end{bmatrix} \boldsymbol{S}_1^{-1} \right) \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$e^{i\omega \Delta t} \boldsymbol{S}_1^{-1} \begin{bmatrix} h \\ u \end{bmatrix}_j^{n} = \left(\boldsymbol{I} - \Delta t \begin{bmatrix} \lambda_{1,-} & 0 \\ 0 & \lambda_{1,+} \end{bmatrix} \right) \boldsymbol{S}_1^{-1} \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$e^{i\omega \Delta t} \boldsymbol{S}_1^{-1} \begin{bmatrix} h \\ u \end{bmatrix}_j^{n} = \begin{bmatrix} 1 - \Delta t \lambda_{1,-} & 0 \\ 0 & 1 - \Delta t \lambda_{1,+} \end{bmatrix} \boldsymbol{S}_1^{-1} \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

So we have

$$e^{i\omega \Delta t} = 1 - \Delta t \lambda_{1,-}$$

$$\omega = \frac{1}{i\Delta t} \ln \left(1 - \Delta t \lambda_{1,-} \right)$$

1.5 Second Order

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{1} = (\boldsymbol{I} - \Delta t \boldsymbol{F}) \begin{bmatrix} h \\ u \end{bmatrix}_j^n$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{2} = (\boldsymbol{I} - \Delta t \boldsymbol{F}) \begin{bmatrix} h \\ u \end{bmatrix}_j^1$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \frac{1}{2} \left(\begin{bmatrix} h \\ u \end{bmatrix}_j^n + \begin{bmatrix} h \\ u \end{bmatrix}_j^2 \right)$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = \frac{1}{2}\left(\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n} + (\boldsymbol{I} - \Delta t \boldsymbol{F})^2 \left[\begin{array}{c} h \\ u \end{array}\right]_j^{n}\right)$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = \frac{1}{2}\left(\boldsymbol{I} + (\boldsymbol{I} - \Delta t \boldsymbol{F})^2\right) \left[\begin{array}{c} h \\ u \end{array}\right]_j^{n}$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = \frac{1}{2}\left(\boldsymbol{I} + \boldsymbol{I} - 2\Delta t \boldsymbol{F} + \Delta t^2 \boldsymbol{F}^2\right) \left[\begin{array}{c} h \\ u \end{array}\right]_j^{n}$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = \frac{1}{2}\left(2\boldsymbol{I} - 2\Delta t \boldsymbol{F} + \Delta t^2 \boldsymbol{F}^2\right) \left[\begin{array}{c} h \\ u \end{array}\right]_j^{n}$$

lets say we have

$$\boldsymbol{F}_2 = \boldsymbol{S}_2 \left[\begin{array}{cc} \lambda_{2,-} & 0 \\ 0 & \lambda_{2,+} \end{array}\right] \boldsymbol{S}_2^{-1}$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = \frac{1}{2}\left(2\boldsymbol{I} - 2\Delta t \boldsymbol{S}_2 \left[\begin{array}{cc} \lambda_{2,-} & 0 \\ 0 & \lambda_{2,+} \end{array}\right] \boldsymbol{S}_2^{-1} + \Delta t^2 \boldsymbol{S}_2 \left[\begin{array}{cc} \lambda_{2,-}^2 & 0 \\ 0 & \lambda_{2,+}^2 \end{array}\right] \boldsymbol{S}_2^{-1}\right) \left[\begin{array}{c} h \\ u \end{array}\right]_j^{n}$$

$$\boldsymbol{S}_2^{-1}\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = \frac{1}{2}\left(2\boldsymbol{S}_2^{-1} - 2\Delta t \left[\begin{array}{cc} \lambda_{2,-} & 0 \\ 0 & \lambda_{2,+} \end{array}\right] \boldsymbol{S}_2^{-1} + \Delta t^2 \left[\begin{array}{cc} \lambda_{2,-}^2 & 0 \\ 0 & \lambda_{2,+}^2 \end{array}\right] \boldsymbol{S}_2^{-1}\right) \left[\begin{array}{c} h \\ u \end{array}\right]_j^{n}$$

$$\boldsymbol{S}_2^{-1}\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = \frac{1}{2}\left(2\boldsymbol{S}_2^{-1} + \left[\begin{array}{cc} \Delta t^2 \lambda_{2,-}^2 - 2\Delta t \lambda_{2,-} & 0 \\ 0 & \Delta t^2 \lambda_{2,+}^2 - 2\Delta t \lambda_{2,-} \end{array}\right] \boldsymbol{S}_2^{-1}\right) \left[\begin{array}{c} h \\ u \end{array}\right]_j^{n}$$

$$e^{i\omega\Delta t}\left(\boldsymbol{S}_2^{-1}\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n}\right) = \frac{1}{2}\left(2 + \left[\begin{array}{cc} \Delta t^2 \lambda_{2,-}^2 - 2\Delta t \lambda_{2,-} & 0 \\ 0 & \Delta t^2 \lambda_{2,+}^2 - 2\Delta t \lambda_{2,-} \end{array}\right]\right)\left(\boldsymbol{S}_2^{-1}\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n}\right)$$

$$e^{i\omega\Delta t}\left(\boldsymbol{S}_2^{-1}\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n}\right) = \frac{1}{2}\left[\begin{array}{cc} 2 + \Delta t^2 \lambda_{2,-}^2 - 2\Delta t \lambda_{2,-} & 0 \\ 0 & 2 + \Delta t^2 \lambda_{2,+}^2 - 2\Delta t \lambda_{2,-} \end{array}\right]\left(\boldsymbol{S}_2^{-1}\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n}\right)$$

So we have

$$e^{i\omega\Delta t} = 1 + \frac{1}{2}\Delta t^2 \lambda_{2,\pm}^2 - \Delta t \lambda_{2,\pm}$$

$$\omega = \frac{1}{i\Delta t}\ln\left(1 + \frac{1}{2}\Delta t^2 \lambda_{2,\pm}^2 - \Delta t \lambda_{2,\pm}\right)$$