1 Second Order Finite Difference Method for u

$$h_i^n u_i^{n+1} - (h_i^n)^2 \left(\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} \right) - \frac{(h_i^n)^3}{3} \left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} \right) = -Y_i^n \tag{1}$$

$$\begin{aligned} Y_i^n = 2\Delta t \Bigg[& u_i^n h_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + g h_i^n \frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{2\Delta x} + (h_i^n)^2 \left(\frac{u_{i+1}^{n-1} - u_{i-1}^{n-1}}{2\Delta x} \right)^2 \\ & + \frac{(h_i^n)^3}{3} \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} - (h_i^n)^2 u_i^n \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \\ & - \frac{(h_i^n)^3}{3} u_i^n \frac{u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n}{2\Delta x^3} \Bigg] \\ & - h_i^n u_i^{n-1} + (h_i^n)^2 \frac{u_{i+1}^{n-1} - u_{i-1}^{n-1}}{2\Delta x} + \frac{(h_i^n)^3}{3} \frac{u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}}{\Delta x^2} \end{aligned} \tag{2}$$

2 Second Order Finite Difference Method for h

$$h_i^{n+1} = h_i^{n-1} - \Delta t \left(u_i^n \frac{h_{i+1}^n - h_{i-1}^n}{\Delta x} + h_i^n \frac{u_{i+1}^n - u_{i-1}^n}{\Delta x} \right) \tag{3}$$

3 Lax Wendroff Method for h

$$h_{i+1/2}^{n+1/2} = \frac{1}{2} \left(h_{i+1}^n + h_i^n \right) - \frac{\Delta t}{2\Delta x} \left(u_{i+1}^n h_{i+1}^n - h_i^n u_i^n \right),$$

$$h_{i-1/2}^{n+1/2} = \frac{1}{2} \left(h_i^n + h_{i-1}^n \right) - \frac{\Delta t}{2\Delta x} \left(u_i^n h_i^n - h_{i-1}^n u_{i-1}^n \right)$$

and

$$h_i^{n+1} = h_i^n - \frac{\Delta t}{\Delta x} \left(u_{i+1/2}^{n+1/2} h_{i+1/2}^{n+1/2} - u_{i-1/2}^{n+1/2} h_{i-1/2}^{n+1/2} \right).$$

$$u_{i+1/2}^{n+1/2} = \frac{u_{i+1}^{n+1} + u_{i+1}^{n} + u_{i}^{n+1} + u_{i}^{n}}{4} \tag{4}$$

and

$$u_{i-1/2}^{n+1/2} = \frac{u_{i}^{n} + u_{i}^{n} + u_{i-1}^{n+1} + u_{i-1}^{n}}{4}. \tag{5}$$

4 Actual Work

We do a Von Neumann stability analysis, we assume two different errors for h and u otherwise everything else is the same. We jsut run the errrors of known structure through the method, for convenience we know use h and u to refer to their respective errors, and we use q top refer to a general quantity (k, a different for u and l and b for h)

$$\begin{aligned}
q_j^n &= e^{at}e^{ikx} \\
q_j^{n+1} &= e^{a\Delta t}q_j^n \\
q_j^{n-1} &= e^{-a\Delta t}q_j^n \\
q_{j+1}^n &= e^{ik\Delta x}q_j^n \\
q_{j+2}^n &= e^{2ik\Delta x}q_j^n \\
q_{j-1}^n &= e^{-ik\Delta x}q_j^n \\
q_{j-2}^n &= e^{-ik\Delta x}q_j^n
\end{aligned}$$

$$\frac{\partial q}{\partial x} = \frac{q_{j+1}^n - q_{j-1}^n}{2\Delta x} = \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x}q_{j+1}^n = \frac{i\sin{(k\Delta x)}}{\Delta x}q_{j+1}^n$$

So we define

$$a_q = \frac{i\sin{(k\Delta x)}}{\Delta x}$$

$$\frac{\partial^2 q}{\partial x^2} = \frac{q_{j+1}^n - 2q_j^n + q_{j-1}^n}{\Delta x^2} = \frac{e^{ik\Delta x} + e^{-ik\Delta x} - 2}{\Delta x^2}q_j^n = \frac{2\cos{(k\Delta x)} - 2}{\Delta x^2}q_j^n$$

$$= -\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)q_j^n$$

So we define

$$b_q = -\frac{4}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)$$

$$\frac{\partial^3 q}{\partial x^2} = \frac{-q^n_{j-2} + 2q^n_{j-1} - 2q^n_{j+1} + q^n_{j+2}}{2\Delta x^3} = \frac{2e^{ik\Delta x} - 2e^{-ik\Delta x} + e^{2ik\Delta x} - e^{-2ik\Delta x}}{2\Delta x^3} q^n_j$$

$$= \frac{4i\sin(k\Delta x) + 2i\sin(2k\Delta x)}{2\Delta x^3} q^n_j$$

$$= i\frac{2\sin(k\Delta x) + \sin(2k\Delta x)}{\Delta x^3} q^n_j$$

$$= i\frac{2\sin(k\Delta x) + 2\sin(k\Delta x)\cos(k\Delta x)}{\Delta x^3} q^n_j$$

$$= 2i\sin(k\Delta x)\,\frac{1+\cos(k\Delta x)}{\Delta x^3} q^n_j$$

$$= 2i\sin(k\Delta x)\,2\cos^2\left(\frac{k\Delta x}{2}\right)\frac{1}{\Delta x^3} q^n_j$$

$$= \frac{4i}{\Delta x^3}\sin(k\Delta x)\,2\cos^2\left(\frac{k\Delta x}{2}\right) q^n_j$$

So we define

$$c_q = \frac{4i}{\Delta x^3}\sin(k\Delta x)\,2\cos^2\left(\frac{k\Delta x}{2}\right)$$

5 2nd FD h

$$e^{b\Delta t} h^n_i = e^{-b\Delta t} h^n_i - \Delta t\left(u^n_i h^n_i 2a_h + h^n_i u^n_i 2a_u\right) \tag{6}$$

$$e^{b\Delta t} = e^{-b\Delta t} - \Delta t\left(u^n_i 2a_h + u^n_i 2a_u\right) \tag{7}$$

$$e^{b\Delta t} = e^{-b\Delta t} - 2\Delta t\left(a_h + a_u\right) u^n_i \tag{8}$$

6 Lax Wendroff Method for h

$$h_{i+1/2}^{n+1/2} = \frac{1}{2}\left(e^{il\Delta x}h_i^n + h_i^n\right) - \frac{\Delta t}{2\Delta x}\left(e^{ik\Delta x}u_i^n e^{il\Delta x}h_i^n - h_i^n u_i^n\right),$$

$$h_{i+1/2}^{n+1/2} = \left[\frac{1}{2}\left(e^{il\Delta x} + 1\right) - \frac{\Delta t}{2\Delta x}\left(e^{ik\Delta x}e^{il\Delta x} - 1\right)u_i^n\right]h_i^n$$

$$h_{i-1/2}^{n+1/2} = \left[\frac{1}{2}\left(1 + e^{-il\Delta x}\right) - \frac{\Delta t}{2\Delta x}\left(1 - e^{-ik\Delta x}e^{-il\Delta x}\right)u_i^n\right]h_i^n$$

$$u_{i+1/2}^{n+1/2} = \frac{e^{ik\Delta x}e^{a\Delta t} + e^{ik\Delta x} + e^{a\Delta t} + 1}{4}u_i^n \tag{9}$$

and

$$u_{i-1/2}^{n+1/2} = \frac{e^{a\Delta t}e^{-ik\Delta x} + e^{-ik\Delta x} + e^{a\Delta t} + 1}{4}u_i^n \tag{10}$$

and

$$h_i^{n+1} = h_i^n - \frac{\Delta t}{\Delta x}\left(u_{i+1/2}^{n+1/2}h_{i+1/2}^{n+1/2} - u_{i-1/2}^{n+1/2}h_{i-1/2}^{n+1/2}\right).$$