

1 Linearised Equations

$$G = uh - \frac{h^3}{3}u_{xx}$$

$$\eta_t + hu_x = 0$$

$$hu_t - \frac{h^3}{3}u_{xxt} + gh\eta_x = 0$$

$$(G)_t + gh\eta_x = 0$$

2 Numerical Approximation

We investigate our numerical technique by adding in a fourier mode so $W_j = W_0 e^{i(vt+kx_j)}$, and rewriting the equations using our spatial discretisation

2.1 G

Analytic:

$$G_j = u_j h_j - \left(\frac{h_j^3}{3}u_{xx}\right)_j$$

Numerical approximation, we used second order central differences so we replace the second derivative of u with this approximation to it So we get

$$G_j = u_j h_j - \frac{h_j^3}{3} \left(\frac{-u_{j+2} + 16u_{j+1} - 30u_j + 16u_{j-1} - u_{j-2}}{12\Delta x^2} \right)$$

$$G_j = u_j h_j - \frac{h_j^3}{3} u_j \left(\frac{-e^{2k\Delta x} + 16e^{k\Delta x} - 30 + 16e^{-k\Delta x} - e^{-2k\Delta x}}{12\Delta x^2} \right)$$

$$G_j = u_j h_0 - \frac{h_0^3}{3} u_j \left(\frac{-2 \cos(2k\Delta x) + 32 \cos(k\Delta x) - 30}{12\Delta x^2} \right)$$

$$G_j = u_j \left(h_0 - \frac{h_0^3}{3} \left(\frac{-2 \cos(2k\Delta x) + 32 \cos(k\Delta x) - 30}{12\Delta x^2} \right) \right)$$

We are dealing with time continuous variables so, we first take the derivative in time exactly for the Fourier nodes so that:

So what we have is something that depends on the order used to approximate $u_x x$, lets call it \mathcal{C}_3 Thus:

$$\mathcal{C}_3 = \frac{-2 \cos(2k\Delta x) + 32 \cos(k\Delta x) - 30}{12\Delta x^2}$$

$$G_j = u_j \left(h_0 - \frac{h_0^3}{3} \mathcal{C}_3 \right)$$

Furthermore we will call this whole thing \mathcal{G}_2 So we have

$$\mathcal{G}_3 = \left(h_0 - \frac{h_0^3}{3} \mathcal{C}_3 \right)$$

then

$$G_j = u_j \mathcal{G}_3$$

Now we move on to

$$\eta_t + hu_x = 0$$

our equations are time continuous so that:

$$\eta_t + hu_x = 0$$

$$iv\eta + hu_x = 0$$

next we approximate

our conservation equations of the form

$$q_t + [f(q)]_x = 0$$

by

$$q_t + \frac{1}{\Delta x} [F_{j+1/2} - F_{j-1/2}] = 0$$

where $F_{j\pm 1/2}$ given by Kurganovs method. In this equation h is constant so $f(\eta, u) = hu$. We start Kurganovs method by doing a reconstruction, we start by doing a central differencing approximation to obtain that

we note that the result is something like

$$q_{j+1/2}^- = \bar{q}_j + \frac{1}{6} (\bar{q}_j - \bar{q}_{j-1}) + \frac{1}{3} (\bar{q}_{j+1} - \bar{q}_j)$$

$$q_{j+1/2}^+ = \bar{q}_{j+1} - \frac{1}{3}(\bar{q}_{j+1} - \bar{q}_j) - \frac{1}{6}(\bar{q}_{j+2} - \bar{q}_{j+1})$$

we need to convert aberages to midpoints which we do by the following formula:

$$q_j = \frac{-\bar{q}_{j+1} + 26\bar{q}_j - \bar{q}_{j-1}}{24}$$

$$q_j = \bar{q}_j \frac{-e^{ik\Delta x} + 26 - e^{-ik\Delta x}}{24}$$

$$q_j = \bar{q}_j \frac{26 - 2\cos(k\Delta x)}{24}$$

Defining

$$\mathcal{M}_3 = \frac{26 - 2\cos(k\Delta x)}{24}$$

$$q_j = \bar{q}_j \mathcal{M}_3$$

So we have

$$q_{j+1/2}^- = \mathcal{M}_3 \left[q_j + \frac{1}{6}(q_j - q_{j-1}) + \frac{1}{3}(q_{j+1} - q_j) \right]$$

$$q_{j+1/2}^- = \mathcal{M}_3 q_j \left[1 + \frac{1}{6}(1 - e^{-ik\Delta x}) + \frac{1}{3}(e^{ik\Delta x} - 1) \right]$$

$$q_{j+1/2}^- = \mathcal{M}_3 q_j \left[\frac{5}{6} + \frac{1}{6}(-e^{-ik\Delta x}) + \frac{1}{3}(e^{ik\Delta x}) \right]$$

$$q_{j+1/2}^- = \frac{\mathcal{M}_3}{6} [5 + -e^{-ik\Delta x} + 2e^{ik\Delta x}] q_j$$

So deifning

$$R_3^- = \frac{\mathcal{M}_3}{6} [5 + -e^{-ik\Delta x} + 2e^{ik\Delta x}]$$

$$q_{j+1/2}^- = R_3^- q_j$$

for plus

$$q_{j+1/2}^+ = \mathcal{M}_3 \left[q_{j+1} - \frac{1}{3}(q_{j+1} - q_j) - \frac{1}{6}(q_{j+2} - q_{j+1}) \right]$$

$$q_{j+1/2}^+ = \mathcal{M}_3 \left[1 - \frac{1}{3} (1 - e^{-k\Delta x}) - \frac{1}{6} (e^{k\Delta x} - 1) \right] q_{j+1}$$

$$q_{j+1/2}^+ = \mathcal{M}_3 \left[\frac{5}{6} + \frac{1}{3} e^{-k\Delta x} - \frac{1}{6} e^{k\Delta x} \right] q_{j+1}$$

$$q_{j+1/2}^+ = \frac{\mathcal{M}_3}{6} [5 + 2e^{-k\Delta x} - e^{k\Delta x}] q_{j+1}$$

$$q_{j+1/2}^+ = \frac{\mathcal{M}_3 e^{ik\Delta x}}{6} [5 + 2e^{-ik\Delta x} - e^{ik\Delta x}] q_j$$

Defining

$$R_3^+ = \frac{\mathcal{M}_3 e^{ik\Delta x}}{6} [5 + 2e^{-ik\Delta x} - e^{ik\Delta x}]$$

$$q_{j+1/2}^+ = R_3^+ q_j$$

THIS IS ALL THE SAME NOW?

Next we have to use the wavespeeds, up to our linearisation assuming still water the velocities are zero so

$$a_{j+1/2}^- = -\sqrt{gh_{j+1/2}^-}$$

$$a_{j+1/2}^+ = +\sqrt{gh_{j+1/2}^-}$$

We have that

$$F_{i+\frac{1}{2}} = \frac{a_{i+\frac{1}{2}}^+ f(q_{i+\frac{1}{2}}^-) - a_{i+\frac{1}{2}}^- f(q_{i+\frac{1}{2}}^+)}{a_{i+\frac{1}{2}}^+ - a_{i+\frac{1}{2}}^-} + \frac{a_{i+\frac{1}{2}}^+ a_{i+\frac{1}{2}}^-}{a_{i+\frac{1}{2}}^+ - a_{i+\frac{1}{2}}^-} [q_{i+\frac{1}{2}}^+ - q_{i+\frac{1}{2}}^-] \quad (1)$$

$$F_{i+\frac{1}{2}} = \frac{\left(\sqrt{gh_{j+1/2}^-}\right) f(q_{i+\frac{1}{2}}^-) - \left(-\sqrt{gh_{j+1/2}^-}\right) f(q_{i+\frac{1}{2}}^+)}{\left(\sqrt{gh_{j+1/2}^-}\right) - \left(-\sqrt{gh_{j+1/2}^-}\right)} + \frac{\left(\sqrt{gh_{j+1/2}^-}\right) \left(-\sqrt{gh_{j+1/2}^-}\right)}{\left(+\sqrt{gh_{j+1/2}^-}\right) - \left(-\sqrt{gh_{j+1/2}^-}\right)} [q_{i+\frac{1}{2}}^+ - q_{i+\frac{1}{2}}^-] \quad (2)$$

$$F_{i+\frac{1}{2}} = \frac{\left(+\sqrt{gh_{j+1/2}^-}\right) f\left(q_{i+\frac{1}{2}}^-\right) - \left(-\sqrt{gh_{j+1/2}^-}\right) f\left(q_{i+\frac{1}{2}}^+\right)}{2\sqrt{gh_{j+1/2}^-}} + \frac{\left(+\sqrt{gh_{j+1/2}^-}\right) \left(-\sqrt{gh_{j+1/2}^-}\right)}{2\sqrt{gh_{j+1/2}^-}} \left[q_{i+\frac{1}{2}}^+ - q_{i+\frac{1}{2}}^-\right] \quad (3)$$

$$F_{i+\frac{1}{2}} = \frac{\left(+\sqrt{gh_{j+1/2}^-}\right) f\left(q_{i+\frac{1}{2}}^-\right) - \left(-\sqrt{gh_{j+1/2}^-}\right) f\left(q_{i+\frac{1}{2}}^+\right)}{2\sqrt{gh_{j+1/2}^-}} + \frac{-gh_{j+1/2}^-}{2\sqrt{gh_{j+1/2}^-}} \left[q_{i+\frac{1}{2}}^+ - q_{i+\frac{1}{2}}^-\right] \quad (4)$$

$$F_{i+\frac{1}{2}} = \frac{f\left(q_{i+\frac{1}{2}}^-\right) + f\left(q_{i+\frac{1}{2}}^+\right)}{2} - \frac{\sqrt{gh_{j+1/2}^-}}{2} \left[q_{i+\frac{1}{2}}^+ - q_{i+\frac{1}{2}}^-\right] \quad (5)$$

for eta this becomes

$$F_{i+\frac{1}{2}}(\eta) = \frac{hu_{i+\frac{1}{2}}^- + hu_{i+\frac{1}{2}}^+}{2} - \frac{\sqrt{gh_{j+1/2}^-}}{2} \left[\eta_{i+\frac{1}{2}}^+ - \eta_{i+\frac{1}{2}}^-\right]$$

up to order the last term becomes

$$\begin{aligned} F_{i+\frac{1}{2}}(\eta) &= \frac{hu_{i+\frac{1}{2}}^- + hu_{i+\frac{1}{2}}^+}{2} - \frac{\sqrt{gh}}{2} \left[\eta_{i+\frac{1}{2}}^+ - \eta_{i+\frac{1}{2}}^-\right] \\ F_{i+\frac{1}{2}}(\eta) &= \frac{h\mathcal{R}^-u_j + h\mathcal{R}^+u_j}{2} - \frac{\sqrt{gh}}{2} \left[\mathcal{R}^+\eta_j - \mathcal{R}^-\eta_j\right] \\ F_{i+\frac{1}{2}}(\eta) &= \frac{h\mathcal{R}^- + h\mathcal{R}^+}{2}u_j - \frac{\sqrt{gh}}{2} \left[\mathcal{R}^+ - \mathcal{R}^-\right]\eta_j \end{aligned}$$

$$F_{i+\frac{1}{2}}(\eta) = F_2^{\eta,u}u_j + F_2^{\eta,\eta}\eta_j$$

where

$$F_2^{\eta,u} = \frac{h\mathcal{R}^- + h\mathcal{R}^+}{2}$$

$$F_2^{\eta,\eta} = -\frac{\sqrt{gh}}{2} [\mathcal{R}^+ - \mathcal{R}^-]$$

For G this becomes

$$F_{i+\frac{1}{2}}(G) = \frac{gh\eta_{i+\frac{1}{2}}^- + gh\eta_{i+\frac{1}{2}}^+}{2} - \frac{\sqrt{gh_{j+1/2}^-}}{2} [\mathcal{G}u_{i+\frac{1}{2}}^+ - \mathcal{G}u_{i+\frac{1}{2}}^-] \quad (6)$$

$$F_{i+\frac{1}{2}}(G) = \frac{gh\mathcal{R}^-\eta_j + gh\mathcal{R}^+\eta_j}{2} - \frac{\sqrt{gh_{j+1/2}^-}}{2} [\mathcal{G}\mathcal{R}^+u_j - \mathcal{G}\mathcal{R}^-u_j] \quad (7)$$

up to order the last term becomes

$$F_{i+\frac{1}{2}}(G) = \frac{gh\mathcal{R}^-\eta_j + gh\mathcal{R}^+\eta_j}{2} - \frac{\sqrt{gh}}{2} [\mathcal{G}\mathcal{R}^+u_j - \mathcal{G}\mathcal{R}^-u_j] \quad (8)$$

$$F_{i+\frac{1}{2}}(G) = \frac{gh\mathcal{R}^- + gh\mathcal{R}^+}{2} \eta_j - \frac{\sqrt{gh}}{2} [\mathcal{G}\mathcal{R}^+ - \mathcal{G}\mathcal{R}^-] u_j \quad (9)$$

$$F_{i+\frac{1}{2}}(G) = F_2^{G,u}u_j + F_2^{G,\eta}\eta_j$$

where

$$F_2^{G,u} = -\frac{\sqrt{gh}}{2} [\mathcal{G}\mathcal{R}^+ - \mathcal{G}\mathcal{R}^-]$$

$$F_2^{G,\eta} = \frac{gh\mathcal{R}^- + gh\mathcal{R}^+}{2}$$

So we have

$$ivG_j + gh\eta_x = 0$$

$$iv\eta + hu_x = 0$$

become

$$iv\eta + \frac{1}{\Delta x} [(1 - e^{-ik\Delta x})F_2^{\eta,u}u_j + (1 - e^{-ik\Delta x})F_2^{\eta,\eta}\eta_j] = 0$$

We let $\mathcal{D} = (1 - e^{-ik\Delta x})$

$$iv\eta_j + \frac{1}{\Delta x} [\mathcal{D}F_2^{\eta,u}u_j + \mathcal{D}F_2^{\eta,\eta}\eta_j] = 0$$

$$\left[iv + \frac{1}{\Delta x} \mathcal{D}F_2^{\eta,\eta} \right] \eta_j + \frac{1}{\Delta x} [\mathcal{D}F_2^{\eta,u}] u_j = 0$$

Similarly for G

$$\mathcal{G} \left[iv + \frac{1}{\Delta x} \mathcal{D}F_2^{G,u} \right] u_j + \frac{1}{\Delta x} [\mathcal{D}F_2^{G,\eta}] \eta_j = 0$$

$$\begin{bmatrix} iv + \frac{1}{\Delta x} \mathcal{D}\mathcal{F}_2^{(\eta,\eta)} & \frac{1}{\Delta x} \mathcal{D}\mathcal{F}_2^{(\eta,u)} \\ \frac{1}{\Delta x} \mathcal{D}\mathcal{F}_2^{(G,\eta)} & iv\mathcal{G} + \mathcal{G}\frac{1}{\Delta x} \mathcal{D}\mathcal{F}_2^{(G,u)} \end{bmatrix} \begin{bmatrix} \eta_j \\ u_j \end{bmatrix} = 0$$

for a nontrivial solution

$$\left[iv + \frac{1}{\Delta x} \mathcal{D}\mathcal{F}_2^{(\eta,\eta)} \right] \left[iv\mathcal{G} + \mathcal{G}\frac{1}{\Delta x} \mathcal{D}\mathcal{F}_2^{(G,u)} \right] - \frac{1}{\Delta x} \mathcal{D}\mathcal{F}_2^{(\eta,u)} \frac{1}{\Delta x} \mathcal{D}\mathcal{F}_2^{(G,\eta)}$$

$$\left[iv + \frac{1}{\Delta x} \mathcal{D}\mathcal{F}_2^{(\eta,\eta)} \right] \left[iv\mathcal{G} + \mathcal{G}\frac{1}{\Delta x} \mathcal{D}\mathcal{F}_2^{(G,u)} \right] - \frac{1}{\Delta x^2} \mathcal{D}^2 \mathcal{F}_2^{(\eta,u)} \mathcal{F}_2^{(G,\eta)}$$

$$-v^2\mathcal{G} + iv\mathcal{G}\frac{1}{\Delta x} \mathcal{D}\mathcal{F}_2^{(G,u)} + iv\mathcal{G}\frac{1}{\Delta x} \mathcal{D}\mathcal{F}_2^{(\eta,\eta)} + \frac{1}{\Delta x} \mathcal{D}\mathcal{F}_2^{(\eta,\eta)} \mathcal{G}\frac{1}{\Delta x} \mathcal{D}\mathcal{F}_2^{(G,u)} - \frac{1}{\Delta x^2} \mathcal{D}^2 \mathcal{F}_2^{(\eta,u)} \mathcal{F}_2^{(G,\eta)} = 0$$

$$-\mathcal{G}v^2 + i \left[\mathcal{G}\frac{1}{\Delta x} \mathcal{D}\mathcal{F}_2^{(G,u)} + \mathcal{G}\frac{1}{\Delta x} \mathcal{D}\mathcal{F}_2^{(\eta,\eta)} \right] v + \frac{1}{\Delta x^2} \mathcal{D}^2 \mathcal{G} \mathcal{F}_2^{(\eta,\eta)} \mathcal{F}_2^{(G,u)} - \frac{1}{\Delta x^2} \mathcal{D}^2 \mathcal{F}_2^{(\eta,u)} \mathcal{F}_2^{(G,\eta)} = 0$$

$$-\mathcal{G}v^2 + i \left[\mathcal{G}\frac{1}{\Delta x} \mathcal{D}\mathcal{F}_2^{(G,u)} + \mathcal{G}\frac{1}{\Delta x} \mathcal{D}\mathcal{F}_2^{(\eta,\eta)} \right] v + \frac{1}{\Delta x^2} \mathcal{D}^2 \left[\mathcal{G} \mathcal{F}_2^{(\eta,\eta)} \mathcal{F}_2^{(G,u)} - \mathcal{F}_2^{(\eta,u)} \mathcal{F}_2^{(G,\eta)} \right] = 0$$