In[1]:=
```
q = q0 * Exp[I * (k * x + w * t)];
qjn = q0 * Exp[I * (k * xj + w * tn)];
qjbar = Integrate[q, {x, xj - dx / 2, xj + dx / 2}] / (dx);
qjnbar = qjbar /. t → tn;
MA = qjn / qjnbar;

qntbar = Integrate[q, {t, tn, tn + dt}] / (dt);
qjntbar = qntbar /. x → xj;
MtA = qjntbar / qjn;

qjphn = q0 * Exp[I * (k * (xj + dx / 2) + w * tn)];
RA = Simplify[MA * qjphn / (qjn)];

vmultG = H + H^3 / 3 * k^2;
GnA = -U * RA / vmultG;
GGA = RA / vmultG;
GcA = -U * H / vmultG;

fn1A = H * vh + U * eh;
fn1A = fn1A /. vh → (GGA * Gca + GnA * eca) /. eh → RA * eca;
fn1Gca0A = fn1A /. Gca → 0;
fn1eca0A = fn1A /. eca → 0;
fnnA = Simplify[fn1Gca0A / eca];
fnGA = fn1eca0A / Gca;
fncA = H * GcA;

fG1A = U * Gh + U * H * vh + g * H * eh;
fG1A = fG1A /. vh → (GGA * Gca + GnA * eca) /. eh → RA * eca /. Gh → RA * Gca;
fG1Gca0A = fG1A /. Gca → 0;
fG1eca0A = fG1A /. eca → 0;
fGnA = Simplify[fG1Gca0A / eca];
fGGA = Simplify[fG1eca0A / Gca];
fGcA = U * H * GcA;

FnnA = -MtA * dt / dx * (1 - Exp[-I * k * dx]) * fnnA;
FnGA = -MtA * dt / dx * (1 - Exp[-I * k * dx]) * fnGA;
FGnA = -MtA * dt / dx * (1 - Exp[-I * k * dx]) * fGnA;
FGGA = -MtA * dt / dx * (1 - Exp[-I * k * dx]) * fGGA;

MatA = {{FnnA, FnGA}, {FGnA, FGGA}};
```

$$wAp = U * k + \frac{\sqrt{3}\ k \sqrt{g H \left(3 + H^2 k^2\right)}}{3 + H^2 k^2};$$

$$wAm = U * k - \frac{\sqrt{3}\ k \sqrt{g H \left(3 + H^2 k^2\right)}}{3 + H^2 k^2};$$

In[36]:=
```
M2 = 1
Series[M2 - MA, {dx, 0, 10}]
```

Out[36]= 1

Out[37]= $$-\frac{k^2\,dx^2}{24} - \frac{7\,k^4\,dx^4}{5760} - \frac{31\,k^6\,dx^6}{967\,680} - \frac{127\,k^8\,dx^8}{154\,828\,800} - \frac{73\,k^{10}\,dx^{10}}{3\,503\,554\,560} + O[dx]^{11}$$

In[38]:=
```
Rm = (1 + I * Sin[k * dx] / 2)
Series[Rm - RA, {dx, 0, 10}]
Rp = Exp[I * k * dx] * (1 - I * Sin[k * dx] / 2)
Series[Rp - RA, {dx, 0, 10}]
```

Out[38]= $$1 + \frac{1}{2}\,i\,\text{Sin}[dx\,k]$$

Out[39]= $$\frac{k^2\,dx^2}{12} - \frac{1}{12}\,i\,k^3\,dx^3 + \frac{k^4\,dx^4}{720} + \frac{1}{240}\,i\,k^5\,dx^5 + \frac{k^6\,dx^6}{30\,240} - \frac{i\,k^7\,dx^7}{10\,080} + \frac{k^8\,dx^8}{1\,209\,600} + \frac{i\,k^9\,dx^9}{725\,760} + \frac{k^{10}\,dx^{10}}{47\,900\,160} + O[dx]^{11}$$

Out[40]= $$e^{i\,dx\,k}\left(1 - \frac{1}{2}\,i\,\text{Sin}[dx\,k]\right)$$

Out[41]= $$\frac{k^2\,dx^2}{12} + \frac{1}{6}\,i\,k^3\,dx^3 - \frac{89\,k^4\,dx^4}{720} - \frac{7}{120}\,i\,k^5\,dx^5 + \frac{631\,k^6\,dx^6}{30\,240} + \frac{31\,i\,k^7\,dx^7}{5040} - \frac{1889\,k^8\,dx^8}{1\,209\,600} - \frac{127\,i\,k^9\,dx^9}{362\,880} + \frac{481\,k^{10}\,dx^{10}}{6\,842\,880} + O[dx]^{11}$$

In[42]:=
```
Ru = (1 + Exp[I * k * dx]) / 2
Series[Ru - Exp[I * k * dx / 2], {dx, 0, 10}]
Gold = H - H^3 / 3 * (2 * Cos[k * dx] - 2) / dx^2
GG2 = Ru / Gold
Series[GG2 - GGA, {dx, 0, 5}]

Gn2 = -U * Ru / Gold
Series[Gn2 - GnA, {dx, 0, 5}]
```

Out[42]= $\frac{1}{2}\left(1 + e^{i\, dx\, k}\right)$

Out[43]= $-\frac{k^2\, dx^2}{8} - \frac{1}{16}\, i\, k^3\, dx^3 + \frac{7\, k^4\, dx^4}{384} + \frac{1}{256}\, i\, k^5\, dx^5 -$
$\frac{31\, k^6\, dx^6}{46\,080} - \frac{i\, k^7\, dx^7}{10\,240} + \frac{127\, k^8\, dx^8}{10\,321\,920} + \frac{17\, i\, k^9\, dx^9}{12\,386\,304} - \frac{73\, k^{10}\, dx^{10}}{530\,841\,600} + O[dx]^{11}$

Out[44]= $H - \frac{H^3\, (-2 + 2\, Cos[dx\, k])}{3\, dx^2}$

Out[45]= $\frac{1 + e^{i\, dx\, k}}{2\left(H - \frac{H^3\, (-2+2\, Cos[dx\, k])}{3\, dx^2}\right)}$

Out[46]= $\frac{\left(-6\, k^2 - H^2\, k^4\right) dx^2}{4\, H\left(3 + H^2\, k^2\right)^2} - \frac{i\left(6\, k^3 + H^2\, k^5\right) dx^3}{8\, H\left(3 + H^2\, k^2\right)^2} +$
$\frac{\left(144\, k^4 + 45\, H^2\, k^6 + 4\, H^4\, k^8\right) dx^4}{240\, H\left(3 + H^2\, k^2\right)^3} - \frac{i\left(-54\, k^5 + H^4\, k^9\right) dx^5}{480\, H\left(3 + H^2\, k^2\right)^3} + O[dx]^6$

Out[47]= $-\frac{\left(1 + e^{i\, dx\, k}\right) U}{2\left(H - \frac{H^3\, (-2+2\, Cos[dx\, k])}{3\, dx^2}\right)}$

Out[48]= $\frac{\left(6\, k^2 + H^2\, k^4\right) U\, dx^2}{4\, H\left(3 + H^2\, k^2\right)^2} + \frac{i\left(6\, k^3 + H^2\, k^5\right) U\, dx^3}{8\, H\left(3 + H^2\, k^2\right)^2} -$
$\frac{\left(\left(144\, k^4 + 45\, H^2\, k^6 + 4\, H^4\, k^8\right) U\right) dx^4}{240\left(H\left(3 + H^2\, k^2\right)^3\right)} + \frac{i\left(-54\, k^5 + H^4\, k^9\right) U\, dx^5}{480\, H\left(3 + H^2\, k^2\right)^3} + O[dx]^6$

In[49]:=
```
fnn2 = H * Gn2 + U / 2 * (Rm + Rp) - (Sqrt[g * H]) / (2) * (Rp - Rm);
Fnn2 = -dt * (1 - Exp[-I * k * dx]) / dx * fnn2
Fnn2TA = Series[Fnn2 - FnnA, {dx, 0, 4}, {dt, 0, 3}];
Refine[Fnn2TA , {k > 0, U > 0, H > 0, g > 0}]
fnG2 = H * GG2 ;
FnG2 = -dt * (1 - Exp[-I * k * dx]) / dx * fnG2
FnG2TA = Series[FnG2 - FnGA, {dx, 0, 4}, {dt, 0, 3}];
Refine[FnG2TA , {k > 0, U > 0, H > 0, g > 0}]
```

Out[50]=
$$-\frac{1}{dx}dt\left(1-e^{-i\,dx\,k}\right)\left(-\frac{\left(1+e^{i\,dx\,k}\right)H\,U}{2\left(H-\frac{H^3\,(-2+2\,Cos[dx\,k])}{3\,dx^2}\right)}-\frac{1}{2}\sqrt{g\,H}\left(-1+e^{i\,dx\,k}\left(1-\frac{1}{2}\,i\,Sin[dx\,k]\right)-\frac{1}{2}\,i\,Sin[dx\,k]\right)+\frac{1}{2}\,U\left(1+e^{i\,dx\,k}\left(1-\frac{1}{2}\,i\,Sin[dx\,k]\right)+\frac{1}{2}\,i\,Sin[dx\,k]\right)\right)$$

Out[52]=
$$\left(-\frac{\left(H^2\,k^3\,U\,w\right)dt^2}{2\left(3+H^2\,k^2\right)}-\frac{i\,H^2\,k^3\,U\,w^2\,dt^3}{6\left(3+H^2\,k^2\right)}+O[dt]^4\right)+\left(-\frac{i\left(27\,k^3+9\,H^2\,k^5+H^4\,k^7\right)U\,dt}{12\left(3+H^2\,k^2\right)^2}+O[dt]^4\right)dx^2+\left(-\frac{1}{8}\left(\sqrt{g\,H}\;k^4\right)dt+O[dt]^4\right)dx^3+\left(\frac{i\left(405\,k^5\,U+351\,H^2\,k^7\,U+116\,H^4\,k^9\,U+13\,H^6\,k^{11}\,U\right)dt}{240\left(3+H^2\,k^2\right)^3}+O[dt]^4\right)dx^4+O[dx]^5$$

Out[54]=
$$-\frac{dt\left(1-e^{-i\,dx\,k}\right)\left(1+e^{i\,dx\,k}\right)H}{2\,dx\left(H-\frac{H^3\,(-2+2\,Cos[dx\,k])}{3\,dx^2}\right)}$$

Out[56]=
$$\left(-\frac{3\,(k\,w)\,dt^2}{2\left(3+H^2\,k^2\right)}-\frac{i\,k\,w^2\,dt^3}{2\left(3+H^2\,k^2\right)}+O[dt]^4\right)+\left(\frac{i\left(6\,k^3+H^2\,k^5\right)dt}{4\left(3+H^2\,k^2\right)^2}+O[dt]^4\right)dx^2+\left(\frac{i\left(-54\,k^5+H^4\,k^9\right)dt}{240\left(3+H^2\,k^2\right)^3}+O[dt]^4\right)dx^4+O[dx]^5$$

In[57]:=
```
fGn2 = U * H * Gn2 + g * H * (Rm + Rp) / 2 + (U * Sqrt[ g * H]) / (2) * (Rm - Rp);
FGn2 = -dt * (1 - Exp[-I * k * dx]) / dx * fGn2
FGn2TA = Series[FGn2 - FGnA, {dx, 0, 4}, {dt, 0, 3}];
Refine[FGn2TA , {k > 0, U > 0, H > 0, g > 0}]
fGG2 = U * H * GG2 + U / 2 * (Rm + Rp) - (Sqrt[ g * H]) / (2) * (Rp - Rm);
FGG2 = -dt * (1 - Exp[-I * k * dx]) / dx * fGG2
FGG2TA = Series[FGG2 - FGGA, {dx, 0, 4}, {dt, 0, 3}];
Refine[FGG2TA , {k > 0, U > 0, H > 0, g > 0}]
```

Out[58]=
$$-\frac{1}{dx}dt\left(1-e^{-i\,dx\,k}\right)\left(-\frac{\left(1+e^{i\,dx\,k}\right)H\,U^2}{2\left(H-\frac{H^3\,(-2+2\,Cos[dx\,k])}{3\,dx^2}\right)}+\frac{1}{2}\sqrt{g\,H}\;U\left(1-e^{i\,dx\,k}\left(1-\frac{1}{2}\,i\,Sin[dx\,k]\right)+\frac{1}{2}\,i\,Sin[dx\,k]\right)+\frac{1}{2}\,g\,H\left(1+e^{i\,dx\,k}\left(1-\frac{1}{2}\,i\,Sin[dx\,k]\right)+\frac{1}{2}\,i\,Sin[dx\,k]\right)\right)$$

Out[60]=
$$\left(-\frac{\left(k\left(3\,g\,H+g\,H^3\,k^2-3\,U^2\right)w\right)dt^2}{2\left(3+H^2\,k^2\right)}-\frac{i\,k\left(3\,g\,H+g\,H^3\,k^2-3\,U^2\right)w^2\,dt^3}{6\left(3+H^2\,k^2\right)}+O[dt]^4\right)+\left(-\frac{i\left(9\,g\,H\,k^3+6\,g\,H^3\,k^5+g\,H^5\,k^7+18\,k^3\,U^2+3\,H^2\,k^5\,U^2\right)dt}{12\left(3+H^2\,k^2\right)^2}+O[dt]^4\right)dx^2+\left(-\frac{1}{8}\left(\sqrt{g\,H}\;k^4\,U\right)dt+O[dt]^4\right)dx^3+\left(\left(i\left(351\,g\,H\,k^5+351\,g\,H^3\,k^7+117\,g\,H^5\,k^9+13\,g\,H^7\,k^{11}+54\,k^5\,U^2-H^4\,k^9\,U^2\right)dt\right)/\left(240\left(3+H^2\,k^2\right)^3\right)+O[dt]^4\right)dx^4+O[dx]^5$$

Out[62]=
$$-\frac{1}{dx}dt\left(1-e^{-i\,dx\,k}\right)\left(\frac{\left(1+e^{i\,dx\,k}\right)H\,U}{2\left(H-\frac{H^3\,(-2+2\,Cos[dx\,k])}{3\,dx^2}\right)}-\frac{1}{2}\sqrt{g\,H}\;\left(-1+e^{i\,dx\,k}\left(1-\frac{1}{2}\,i\,Sin[dx\,k]\right)-\frac{1}{2}\,i\,Sin[dx\,k]\right)+\frac{1}{2}\,U\left(1+e^{i\,dx\,k}\left(1-\frac{1}{2}\,i\,Sin[dx\,k]\right)+\frac{1}{2}\,i\,Sin[dx\,k]\right)\right)$$

Out[64]=
$$\left(-\frac{\left(k\left(6+H^2\,k^2\right)U\,w\right)dt^2}{2\left(3+H^2\,k^2\right)}-\frac{i\,k\left(6+H^2\,k^2\right)U\,w^2\,dt^3}{6\left(3+H^2\,k^2\right)}+O[dt]^4\right)+\left(-\frac{i\left(-9\,k^3\,U+3\,H^2\,k^5\,U+H^4\,k^7\,U\right)dt}{12\left(3+H^2\,k^2\right)^2}+O[dt]^4\right)dx^2+\left(-\frac{1}{8}\left(\sqrt{g\,H}\;k^4\right)dt+O[dt]^4\right)dx^3+\left(\frac{i\left(297\,k^5\,U+351\,H^2\,k^7\,U+118\,H^4\,k^9\,U+13\,H^6\,k^{11}\,U\right)dt}{240\left(3+H^2\,k^2\right)^3}+O[dt]^4\right)dx^4+O[dx]^5$$

In[65]:=
```
Fmat2 = {{Fnn2, FnG2}, {FGn2, FGG2}};
EigvFmat2 = Eigenvalues[Fmat2];

RKStep = Log[ 1 + EigvFmat2 + EigvFmat2^2 / 2] / (I * dt);
RKstepTay = Series[RKStep, {dx, 0, 4}, {dt, 0, 4}];
Simplify[-RKstepTay - {wAp, wAm}, {k > 0, H > 0, g > 0, U > 0} ]
```

Out[69]=

$$\left\{\left(\frac{1}{6\left(3+H^2k^2\right)^2}k^3\left(\sqrt{3}\sqrt{gH\left(3+H^2k^2\right)}+\left(3+H^2k^2\right)U\right)\right.$$
$$\left(3gH+U\left(2\sqrt{3}\sqrt{gH\left(3+H^2k^2\right)}+\left(3+H^2k^2\right)U\right)\right)dt^2+\frac{1}{8\left(3+H^2k^2\right)^3}$$
$$ik^4\left(\sqrt{3}\sqrt{gH\left(3+H^2k^2\right)}+\left(3+H^2k^2\right)U\right)\left(3g\left(\sqrt{3}H\sqrt{gH\left(3+H^2k^2\right)}+9HU+3H^3k^2U\right)+\right.$$
$$\left.\left.U^2\left(H^4k^4U+9\left(\sqrt{3}\sqrt{gH\left(3+H^2k^2\right)}+U\right)+3k^2\left(\sqrt{3}\sqrt{gH^5\left(3+H^2k^2\right)}+2H^2U\right)\right)\right)dt^3-$$
$$\frac{1}{20\left(3+H^2k^2\right)^3}\left(k^5\left(\sqrt{3}\sqrt{gH\left(3+H^2k^2\right)}+\left(3+H^2k^2\right)U\right)\right.$$
$$\left(9g^2H^2+6gHU\left(2\sqrt{3}\sqrt{gH\left(3+H^2k^2\right)}+3\left(3+H^2k^2\right)U\right)+\right.$$
$$\left.\left.\left.U^3\left(12\sqrt{3}\sqrt{gH\left(3+H^2k^2\right)}+9U+H^4k^4U+2k^2\left(2\sqrt{3}\sqrt{gH^5\left(3+H^2k^2\right)}+3H^2U\right)\right)\right)\right)$$
$$\left.dt^4+O[dt]^5\right)+\left(\frac{k^3\left(-3\sqrt{3}\sqrt{gH\left(3+H^2k^2\right)}+2\left(3+H^2k^2\right)^2U\right)}{24\left(3+H^2k^2\right)^2}+\frac{1}{48\left(3+H^2k^2\right)^3}\right.$$
$$k^5\left(g\left(-9\sqrt{3}H\sqrt{gH\left(3+H^2k^2\right)}+18H^3k^2U+6H^5k^4U\right)+\right.$$
$$U^2\left(27\sqrt{3}\sqrt{gH\left(3+H^2k^2\right)}+54U+2H^6k^6U+3k^2\left(7\sqrt{3}\sqrt{gH^5\left(3+H^2k^2\right)}+18H^2U\right)+\right.$$
$$\left.\left.2k^4\left(2\sqrt{3}\sqrt{gH^9\left(3+H^2k^2\right)}+9H^4U\right)\right)\right)dt^2+$$
$$\frac{1}{48\left(3+H^2k^2\right)^3}ik^6\left(3gH+U\left(2\sqrt{3}\sqrt{gH\left(3+H^2k^2\right)}+\left(3+H^2k^2\right)U\right)\right)\left(-9gH+\right.$$
$$\left.U\left(3\sqrt{3}\sqrt{gH\left(3+H^2k^2\right)}+18U+2H^4k^4U+2k^2\left(\sqrt{3}\sqrt{gH^5\left(3+H^2k^2\right)}+6H^2U\right)\right)\right)$$
$$dt^3-\frac{1}{96\left(3+H^2k^2\right)^4}\left(k^7\left(-3\sqrt{3}\sqrt{gH\left(3+H^2k^2\right)}+2\left(3+H^2k^2\right)^2U\right)\right.$$
$$\left(9g^2H^2+6gHU\left(2\sqrt{3}\sqrt{gH\left(3+H^2k^2\right)}+3\left(3+H^2k^2\right)U\right)+\right.$$
$$\left.\left.\left.U^3\left(12\sqrt{3}\sqrt{gH\left(3+H^2k^2\right)}+9U+H^4k^4U+2k^2\left(2\sqrt{3}\sqrt{gH^5\left(3+H^2k^2\right)}+3H^2U\right)\right)\right)\right)$$
$$\left.dt^4+O[dt]^5\right)dx^2+\left(-\frac{ik^4\left(2gH\left(3+H^2k^2\right)+\sqrt{3}\sqrt{gH\left(3+H^2k^2\right)}U\right)}{16\sqrt{gH}\left(3+H^2k^2\right)}-\right.$$
$$\left(ik^6\left(gH\left(6\sqrt{gH\left(3+H^2k^2\right)}+\sqrt{3}\left(15+4H^2k^2\right)U\right)+U^2\left(12\sqrt{gH\left(3+H^2k^2\right)}+\right.\right.\right.$$
$$\left.\left.\left.3\sqrt{3}U+k^2\left(2\sqrt{gH^5\left(3+H^2k^2\right)}+\sqrt{3}H^2U\right)\right)\right)dt^2\right)\Big/\left(32\left(3+H^2k^2\right)^{3/2}\right)+$$

$$\left(k^7 \left(3\,g\,H + U\left(2\sqrt{3}\,\sqrt{g\,H\left(3+H^2\,k^2\right)} + \left(3+H^2\,k^2\right)U\right)\right)\left(2\sqrt{3}\,g\,H\left(3+H^2\,k^2\right) + U\left(9\sqrt{g\,H\left(3+H^2\,k^2\right)} + 3\sqrt{3}\,U + k^2\left(2\sqrt{g\,H^5\left(3+H^2\,k^2\right)} + \sqrt{3}\,H^2\,U\right)\right)\right)dt^3\right)\Big/\left(32\left(3+H^2\,k^2\right)^{5/2}\right) + \left(i\,k^8\left(2\,g\,H\left(3+H^2\,k^2\right) + \sqrt{3}\,\sqrt{g\,H\left(3+H^2\,k^2\right)}\,U\right)\left(9\,g^2\,H^2 + 6\,g\,H\,U\left(2\sqrt{3}\,\sqrt{g\,H\left(3+H^2\,k^2\right)} + 3\left(3+H^2\,k^2\right)U\right) + U^3\left(12\sqrt{3}\,\sqrt{g\,H\left(3+H^2\,k^2\right)} + 9\,U + H^4\,k^4\,U + 2\,k^2\left(2\sqrt{3}\,\sqrt{g\,H^5\left(3+H^2\,k^2\right)} + 3\,H^2\,U\right)\right)\right)dt^4\right)\Big/\left(64\sqrt{g\,H}\left(3+H^2\,k^2\right)^3\right) + O[dt]^5\Bigg)dx^3 +$$

$$\Bigg(-\left(\left(k^5\left(3\sqrt{3}\,g\,H\left(177+124\,H^2\,k^2+20\,H^4\,k^4\right) + 104\left(9\sqrt{g\,H\left(3+H^2\,k^2\right)} + 6\,k^2\sqrt{g\,H^5\left(3+H^2\,k^2\right)} + k^4\sqrt{g\,H^9\left(3+H^2\,k^2\right)}\right)U\right)\right)\Big/\left(1920\left(\sqrt{g\,H}\left(3+H^2\,k^2\right)^{5/2}\right)\right)\right) - \frac{1}{11\,520\left(\sqrt{g\,H}\left(3+H^2\,k^2\right)^{7/2}\right)}$$
$$\left(k^7\left(27\sqrt{3}\,g^2\,H^2\left(167+124\,H^2\,k^2+20\,H^4\,k^4\right) + g\,H\,U\left(21\,429\sqrt{3}\,H^2\,k^2\,U + 764\sqrt{3}\,H^6\,k^6\,U + 81\left(232\sqrt{g\,H\left(3+H^2\,k^2\right)} + 267\sqrt{3}\,U\right) + 24\,k^4\left(84\sqrt{g\,H^9\left(3+H^2\,k^2\right)} + 293\sqrt{3}\,H^4\,U\right)\right) + 16\left(459\sqrt{g\,H\left(3+H^2\,k^2\right)}\,U^3 + 153\,k^4\sqrt{g\,H^9\left(3+H^2\,k^2\right)}\,U^3 + 17\,k^6\sqrt{g\,H^{13}\left(3+H^2\,k^2\right)}\,U^3 + 9\,k^2\left(88\sqrt{g^3\,H^7\left(3+H^2\,k^2\right)}\,U + 51\sqrt{g\,H^5\left(3+H^2\,k^2\right)}\,U^3\right)\right)\right)\right)dt^2 -$$
$$\frac{1}{3840\left(3+H^2\,k^2\right)^4}i\,k^8\left(54\,g^2\,H^2\left(81+62\,H^2\,k^2+10\,H^4\,k^4\right) + 84\,H^8\,k^8\,U^4 + 243\,U^3\left(39\sqrt{3}\,\sqrt{g\,H\left(3+H^2\,k^2\right)} + 28\,U\right) + 36\,k^4\,U^3\left(85\sqrt{3}\,\sqrt{g\,H^9\left(3+H^2\,k^2\right)} + 126\,H^4\,U\right) + 4\,k^6\,U^3\left(83\sqrt{3}\,\sqrt{g\,H^{13}\left(3+H^2\,k^2\right)} + 252\,H^6\,U\right) + 9\,k^2\left(600\sqrt{3}\,\sqrt{g^3\,H^7\left(3+H^2\,k^2\right)}\,U + 1039\sqrt{3}\,\sqrt{g\,H^5\left(3+H^2\,k^2\right)}\,U^3 + 1008\,H^2\,U^4\right) + 3\,g\,H\,U\left(13\,500\,H^2\,k^2\,U + 472\,H^6\,k^6\,U + 27\left(97\sqrt{3}\,\sqrt{g\,H\left(3+H^2\,k^2\right)} + 504\,U\right) + 4\,k^4\left(71\sqrt{3}\,\sqrt{g\,H^9\left(3+H^2\,k^2\right)} + 1101\,H^4\,U\right)\right)\right)dt^3 +$$
$$\frac{1}{23\,040\sqrt{g\,H}\left(3+H^2\,k^2\right)^{9/2}}k^9\left(81\sqrt{3}\,g^3\,H^3\left(157+124\,H^2\,k^2+20\,H^4\,k^4\right) + 6\sqrt{3}\,g^2\,H^2\left(32\,535+32\,481\,H^2\,k^2+10\,584\,H^4\,k^4+1124\,H^6\,k^6\right)U^2 + 3\,g\,H\,U^3\left(69\,120\sqrt{g\,H\left(3+H^2\,k^2\right)} + 34\,155\sqrt{3}\,U + 44\,982\sqrt{3}\,H^2\,k^2\,U + 396\sqrt{3}\,H^8\,k^8\,U + 27\,k^4\left(816\sqrt{g\,H^9\left(3+H^2\,k^2\right)} + 821\sqrt{3}\,H^4\,U\right) + 28\,k^6\left(84\sqrt{g\,H^{13}\left(3+H^2\,k^2\right)} + 173\sqrt{3}\,H^6\,U\right)\right) +$$

$$8\,U\left(348\,k^6\sqrt{g\,H^{13}\left(3+H^2\,k^2\right)}\;U^4+29\,k^8\sqrt{g\,H^{17}\left(3+H^2\,k^2\right)}\;U^4+81\left(130\sqrt{g^5\,H^5\left(3+H^2\,k^2\right)}+29\sqrt{g\,H\left(3+H^2\,k^2\right)}\;U^4\right)+54\,k^2\left(137\sqrt{g^5\,H^9\left(3+H^2\,k^2\right)}+472\sqrt{g^3\,H^7\left(3+H^2\,k^2\right)}\;U^2+58\sqrt{g\,H^5\left(3+H^2\,k^2\right)}\;U^4\right)+27\,k^4\left(43\sqrt{g^5\,H^{13}\left(3+H^2\,k^2\right)}+58\sqrt{g\,H^9\left(3+H^2\,k^2\right)}\;U^4\right)\right)\,dt^4+O[dt]^5\Bigg)\,dx^4+$$

$$O[dx]^5,\Bigg(\frac{1}{6\left(3+H^2\,k^2\right)^2}k^3\left(-\sqrt{3}\sqrt{g\,H\left(3+H^2\,k^2\right)}+\left(3+H^2\,k^2\right)U\right)\left(3\,g\,H+U\left(-2\sqrt{3}\sqrt{g\,H\left(3+H^2\,k^2\right)}+\left(3+H^2\,k^2\right)U\right)\right)\,dt^2+\frac{i\,k^4\left(3\,g\,H+U\left(-2\sqrt{3}\sqrt{g\,H\left(3+H^2\,k^2\right)}+\left(3+H^2\,k^2\right)U\right)\right)^2\,dt^3}{8\left(3+H^2\,k^2\right)^2}-\frac{1}{20\left(3+H^2\,k^2\right)^4}\left(k^5\left(-\sqrt{3}\sqrt{g\,H\left(3+H^2\,k^2\right)}+\left(3+H^2\,k^2\right)U\right)^3\left(3\,g\,H+U\left(-2\sqrt{3}\sqrt{g\,H\left(3+H^2\,k^2\right)}+\left(3+H^2\,k^2\right)U\right)\right)\right)\,dt^4+O[dt]^5\Bigg)+$$

$$\Bigg(\frac{k^3\left(3\sqrt{3}\sqrt{g\,H\left(3+H^2\,k^2\right)}+2\left(3+H^2\,k^2\right)^2U\right)}{24\left(3+H^2\,k^2\right)^2}+\frac{1}{48\left(3+H^2\,k^2\right)^3}k^5\left(3\,g\left(3\sqrt{3}\,H\sqrt{g\,H\left(3+H^2\,k^2\right)}+6\,H^3\,k^2\,U+2\,H^5\,k^4\,U\right)+U^2\left(-27\sqrt{3}\sqrt{g\,H\left(3+H^2\,k^2\right)}+54\,U+2\,H^6\,k^6\,U-3\,k^2\left(7\sqrt{3}\sqrt{g\,H^5\left(3+H^2\,k^2\right)}-18\,H^2\,U\right)-2\,k^4\left(2\sqrt{3}\sqrt{g\,H^9\left(3+H^2\,k^2\right)}-9\,H^4\,U\right)\right)\right)\,dt^2+\frac{1}{48\left(3+H^2\,k^2\right)^3}i\,k^6\left(3\,g\,H+U\left(-2\sqrt{3}\sqrt{g\,H\left(3+H^2\,k^2\right)}+\left(3+H^2\,k^2\right)U\right)\right)\left(-9\,g\,H+U\left(-3\sqrt{3}\sqrt{g\,H\left(3+H^2\,k^2\right)}+18\,U+2\,H^4\,k^4\,U-2\,k^2\left(\sqrt{3}\sqrt{g\,H^5\left(3+H^2\,k^2\right)}-6\,H^2\,U\right)\right)\right)\,dt^3-\frac{1}{96\left(3+H^2\,k^2\right)^4}\left(k^7\left(3\sqrt{3}\sqrt{g\,H\left(3+H^2\,k^2\right)}+2\left(3+H^2\,k^2\right)^2U\right)\left(3\,g\,H+U\left(-2\sqrt{3}\sqrt{g\,H\left(3+H^2\,k^2\right)}+\left(3+H^2\,k^2\right)U\right)\right)^2\right)\,dt^4+O[dt]^5\Bigg)$$

$$dx^2 + \left(\frac{1}{16} i \sqrt{g H}\ k^4 \left(-2 + \frac{\sqrt{3}\ U}{\sqrt{g H \left(3 + H^2 k^2\right)}} \right) + \left(i k^6 \left(g \left(-6 H \sqrt{g H \left(3 + H^2 k^2\right)} + 15 \sqrt{3}\ H U + 4 \sqrt{3}\ H^3 k^2 U \right) + U^2 \left(-12 \sqrt{g H \left(3 + H^2 k^2\right)} + 3 \sqrt{3}\ U + k^2 \left(-2 \sqrt{g H^5 \left(3 + H^2 k^2\right)} + \sqrt{3}\ H^2 U \right) \right) \right) dt^2 \right) \Big/ \left(32 \left(3 + H^2 k^2\right)^{3/2} \right) - \left(\left(k^7 \left(3 g H + U \left(-2 \sqrt{3}\ \sqrt{g H \left(3 + H^2 k^2\right)} + \left(3 + H^2 k^2\right) U \right) \right) \left(2 \sqrt{3}\ g H \left(3 + H^2 k^2\right) + U \left(-9 \sqrt{g H \left(3 + H^2 k^2\right)} + 3 \sqrt{3}\ U + k^2 \left(-2 \sqrt{g H^5 \left(3 + H^2 k^2\right)} + \sqrt{3}\ H^2 U \right) \right) \right) \right) dt^3 \right) \Big/ \left(32 \left(3 + H^2 k^2\right)^{5/2} \right) + \left(i k^8 \left(2 g H \left(3 + H^2 k^2\right) - \sqrt{3}\ \sqrt{g H \left(3 + H^2 k^2\right)}\ U \right) \left(3 g H + U \left(-2 \sqrt{3}\ \sqrt{g H \left(3 + H^2 k^2\right)} + \left(3 + H^2 k^2\right) U \right) \right)^2 dt^4 \right) \Big/ \left(64 \sqrt{g H}\ \left(3 + H^2 k^2\right)^3 \right) + O[dt]^5 \right) dx^3 + \left(\left(k^5 \left(3 \sqrt{3}\ g H \left(177 + 124 H^2 k^2 + 20 H^4 k^4\right) - 104 \left(9 \sqrt{g H \left(3 + H^2 k^2\right)} + 6 k^2 \sqrt{g H^5 \left(3 + H^2 k^2\right)} + k^4 \sqrt{g H^9 \left(3 + H^2 k^2\right)} \right) U \right) \right) \Big/ \left(1920 \sqrt{g H}\ \left(3 + H^2 k^2\right)^{5/2} \right) + \frac{1}{11\,520 \sqrt{g H}\ \left(3 + H^2 k^2\right)^{7/2}} k^7 \left(27 \sqrt{3}\ g^2 H^2 \left(167 + 124 H^2 k^2 + 20 H^4 k^4\right) + g H U \left(21\,429 \sqrt{3}\ H^2 k^2 U + 764 \sqrt{3}\ H^6 k^6 U + 81 \left(-232 \sqrt{g H \left(3 + H^2 k^2\right)} + 267 \sqrt{3}\ U \right) - 24 k^4 \left(84 \sqrt{g H^9 \left(3 + H^2 k^2\right)} - 293 \sqrt{3}\ H^4 U \right) \right) - 16 \left(459 \sqrt{g H \left(3 + H^2 k^2\right)}\ U^3 + 153 k^4 \sqrt{g H^9 \left(3 + H^2 k^2\right)}\ U^3 + 17 k^6 \sqrt{g H^{13} \left(3 + H^2 k^2\right)}\ U^3 + 9 k^2 \left(88 \sqrt{g^3 H^7 \left(3 + H^2 k^2\right)}\ U + 51 \sqrt{g H^5 \left(3 + H^2 k^2\right)}\ U^3 \right) \right) \right) dt^2 - \frac{1}{3840 \left(3 + H^2 k^2\right)^4} i k^8 \left(54 g^2 H^2 \left(81 + 62 H^2 k^2 + 10 H^4 k^4\right) + 84 H^8 k^8 U^4 + 243 U^3 \left(-39 \sqrt{3}\ \sqrt{g H \left(3 + H^2 k^2\right)} + 28 U \right) + 36 k^4 U^3 \left(-85 \sqrt{3}\ \sqrt{g H^9 \left(3 + H^2 k^2\right)} + 126 H^4 U \right) + 4 k^6 U^3 \left(-83 \sqrt{3}\ \sqrt{g H^{13} \left(3 + H^2 k^2\right)} + 252 H^6 U \right) - 9 k^2 \left(600 \sqrt{3}\ \sqrt{g^3 H^7 \left(3 + H^2 k^2\right)}\ U + 1039 \sqrt{3}\ \sqrt{g H^5 \left(3 + H^2 k^2\right)}\ U^3 - 1008 H^2 U^4 \right) + 3 g H U \left(13\,500 H^2 k^2 U + 472 H^6 k^6 U + 27 \left(-97 \sqrt{3}\ \sqrt{g H \left(3 + H^2 k^2\right)} + 504 U \right) - 4 k^4 \left(71 \sqrt{3}\ \sqrt{g H^9 \left(3 + H^2 k^2\right)} - 1101 H^4 U \right) \right) \right) dt^3 - \frac{1}{23\,040 \left(\sqrt{g H}\ \left(3 + H^2 k^2\right)^{11/2} \right)} \left(k^9 \left(\sqrt{3}\ \sqrt{g H \left(3 + H^2 k^2\right)} - \left(3 + H^2 k^2\right) U \right) \left(4 k^8 U^3 \left(-239 \sqrt{3}\ g H^9 + 58 \sqrt{g H^{17} \left(3 + H^2 k^2\right)}\ U \right) + \right.$$

$$27\,k^2\left(372\sqrt{g^5\,H^9\left(3+H^2\,k^2\right)}-2703\sqrt{3}\,g^2\,H^4\,U+\right.$$
$$4515\sqrt{g^3\,H^7\left(3+H^2\,k^2\right)}\,U^2-4070\sqrt{3}\,g\,H^3\,U^3+928\sqrt{g\,H^5\left(3+H^2\,k^2\right)}\,U^4\Big)+$$
$$9\,k^4\left(180\sqrt{g^5\,H^{13}\left(3+H^2\,k^2\right)}-2672\sqrt{3}\,g^2\,H^6\,U+1392\sqrt{g\,H^9\left(3+H^2\,k^2\right)}\,U^4+g\,H^5\,U^2\right.$$
$$\left(4384\sqrt{g\,H\left(3+H^2\,k^2\right)}-5997\sqrt{3}\,U\right)\Big)+81\left(157\sqrt{g^5\,H^5\left(3+H^2\,k^2\right)}-883\sqrt{3}\right.$$
$$g^2\,H^2\,U+232\sqrt{g\,H\left(3+H^2\,k^2\right)}\,U^4+g\,H\,U^2\left(1527\sqrt{g\,H\left(3+H^2\,k^2\right)}-1033\sqrt{3}\,U\right)\Big)-$$
$$12\,k^6\,U\left(213\sqrt{3}\,g^2\,H^8-232\sqrt{g\,H^{13}\left(3+H^2\,k^2\right)}\,U^3+\right.$$
$$\left.\left.\left.\left.g\,H^7\,U\left(-349\sqrt{g\,H\left(3+H^2\,k^2\right)}+979\sqrt{3}\,U\right)\right)\right)\right)\,dt^4+O[dt]^5\Bigg)\,dx^4+O[dx]^5\Big\}$$