In[811]:=
```
MA = k * x / (2 * Sin[k * x / 2])
RA = Exp[I * k * x / 2] * k * x / (2 * Sin[k * x / 2])
GA = k * x / ((H + H^3 / 3 * k^2) * Exp[-I * k * x / 2] * (2 * Sin[k * x / 2]))
FnnA = 0
FnGA = I * k / (1 + H^2 * k^2 / 3)
FGnA = g * H * I * k
FGGA = 0
FmatA = {{FnnA, FnGA}, {FGnA, FGGA}}
wAp = Sqrt[g * H] * k * Sqrt[3 / (3 + H^2 * k^2)]
wAm = -Sqrt[g * H] * k * Sqrt[3 / (3 + H^2 * k^2)]
Eigenvalues[FmatA]
```

Out[811]= $\frac{1}{2} k x \text{Csc}\left[\frac{k x}{2}\right]$

Out[812]= $\frac{1}{2} e^{\frac{i k x}{2}} k x \text{Csc}\left[\frac{k x}{2}\right]$

Out[813]= $\frac{e^{\frac{i k x}{2}} k x \text{Csc}\left[\frac{k x}{2}\right]}{2\left(H + \frac{H^3 k^2}{3}\right)}$

Out[814]= 0

Out[815]= $\frac{i k}{1 + \frac{H^2 k^2}{3}}$

Out[816]= $i g H k$

Out[817]= 0

Out[818]= $\left\{\left\{0, \frac{i k}{1 + \frac{H^2 k^2}{3}}\right\}, \{i g H k, 0\}\right\}$

Out[819]= $\sqrt{3}\ \sqrt{g H}\ k \sqrt{\frac{1}{3 + H^2 k^2}}$

Out[820]= $-\sqrt{3}\ \sqrt{g H}\ k \sqrt{\frac{1}{3 + H^2 k^2}}$

Out[821]= $\left\{-\frac{i \sqrt{3}\ k \sqrt{3 g H + g H^3 k^2}}{3 + H^2 k^2}, \frac{i \sqrt{3}\ k \sqrt{3 g H + g H^3 k^2}}{3 + H^2 k^2}\right\}$

In[822]:=
```
M = (26 - 2 * Cos[k * x]) / 24
Series[M - MA, {x, 0, 10}]
```

Out[822]= $\frac{1}{24} (26 - 2 \text{Cos}[k x])$

Out[823]= $-\frac{3 k^4 x^4}{640} + \frac{3 k^6 x^6}{35\,840} - \frac{149 k^8 x^8}{51\,609\,600} + \frac{29 k^{10} x^{10}}{13\,624\,934\,400} + O[x]^{11}$

In[824]:=
```
Rm = (5 - Exp[-I * k * x] + 2 * Exp[I * k * x]) / 6
Series[Rm - RA, {x, 0, 10}]
Rp = Exp[I * k * x] * (5 + 2 * Exp[-I * k * x] - Exp[I * k * x]) / 6
Series[Rp - RA, {x, 0, 10}]
Ru = (-Exp[-I * k * x] + 9 * Exp[I * k * x] - Exp[2 * I * k * x] + 9) / 16
Series[Ru - Exp[I * k * x / 2], {x, 0, 10}]
```

Out[824]= $\frac{1}{6}\left(5 - e^{-i\,k\,x} + 2\,e^{i\,k\,x}\right)$

Out[825]= $-\frac{1}{12}\,i\,k^3\,x^3 + \frac{k^4\,x^4}{120} + \frac{1}{240}\,i\,k^5\,x^5 - \frac{k^6\,x^6}{5040} - \frac{i\,k^7\,x^7}{10\,080} + \frac{k^8\,x^8}{201\,600} + \frac{i\,k^9\,x^9}{725\,760} - \frac{k^{10}\,x^{10}}{39\,916\,800} + O[x]^{11}$

Out[826]= $\frac{1}{6}\,e^{i\,k\,x}\left(5 + 2\,e^{-i\,k\,x} - e^{i\,k\,x}\right)$

Out[827]= $\frac{1}{12}\,i\,k^3\,x^3 - \frac{3\,k^4\,x^4}{40} - \frac{3}{80}\,i\,k^5\,x^5 + \frac{23\,k^6\,x^6}{1680} + \frac{41\,i\,k^7\,x^7}{10\,080} - \frac{209\,k^8\,x^8}{201\,600} - \frac{169\,i\,k^9\,x^9}{725\,760} + \frac{89\,k^{10}\,x^{10}}{1\,900\,800} + O[x]^{11}$

Out[828]= $\frac{1}{16}\left(9 - e^{-i\,k\,x} + 9\,e^{i\,k\,x} - e^{2\,i\,k\,x}\right)$

Out[829]= $-\frac{3\,k^4\,x^4}{128} - \frac{3}{256}\,i\,k^5\,x^5 + \frac{5\,k^6\,x^6}{1024} + \frac{3\,i\,k^7\,x^7}{2048} - \frac{63\,k^8\,x^8}{163\,840} - \frac{17\,i\,k^9\,x^9}{196\,608} + \frac{289\,k^{10}\,x^{10}}{16\,515\,072} + O[x]^{11}$

In[830]:=
```
Gold = H - H^3 / 3 * (32 * Cos[k * x] - 2 * Cos[2 * k * x] - 30) / (12 * x^2)
G = M * Ru / (Gold)
Series[G, {x, 0, 3} ]
Series[GA, {x, 0, 3} ]
Series[G - GA, {x, 0, 5}]
```

Out[830]= $H - \frac{H^3\,(-30 + 32\,\text{Cos}[k\,x] - 2\,\text{Cos}[2\,k\,x])}{36\,x^2}$

Out[831]= $\frac{\left(9 - e^{-i\,k\,x} + 9\,e^{i\,k\,x} - e^{2\,i\,k\,x}\right)\,(26 - 2\,\text{Cos}[k\,x])}{384\left(H - \frac{H^3\,(-30+32\,\text{Cos}[k\,x]-2\,\text{Cos}[2\,k\,x])}{36\,x^2}\right)}$

Out[832]= $\frac{1}{H + \frac{H^3\,k^2}{3}} + \frac{i\,k\,x}{2\left(H + \frac{H^3\,k^2}{3}\right)} - \frac{k^2\,x^2}{12\left(H + \frac{H^3\,k^2}{3}\right)} + O[x]^4$

Out[833]= $\frac{1}{H + \frac{H^3\,k^2}{3}} + \frac{i\,k\,x}{2\left(H + \frac{H^3\,k^2}{3}\right)} - \frac{k^2\,x^2}{12\left(H + \frac{H^3\,k^2}{3}\right)} + O[x]^4$

Out[834]= $\frac{\left(-243\,k^4 - 49\,H^2\,k^6\right)\,x^4}{960\,H\left(3 + H^2\,k^2\right)^2} - \frac{i\left(243\,k^5 + 49\,H^2\,k^7\right)\,x^5}{1920\,H\left(3 + H^2\,k^2\right)^2} + O[x]^6$

In[977]:=
```
fnn = - Sqrt[g * H] / 2 * (Rp - Rm);
fng = H * G;
fgg = - Sqrt[g * H] / 2 * (Rp - Rm);
fgn = g * H * (Rp + Rm) / 2;

Fnn = (1 - Exp[-I * k * x]) / x * fnn
Series[Fnn - FnnA, {x, 0, 5}]
Fng = (1 - Exp[-I * k * x]) / x * fng
Series[Fng - FnGA, {x, 0, 5}]
Fgg = (1 - Exp[-I * k * x]) / x * fgg
Series[Fgg - FGGA, {x, 0, 5}]
Fgn = (1 - Exp[-I * k * x]) / x * fgn
Series[Fgn - FGnA, {x, 0, 5}]

Fmat = {{Fnn , Fng}, {Fgn , Fgg}}
EigvFmat = Eigenvalues[Fmat];
 Simplify[Series[EigvFmat , {x, 0, 5}]]
t =  x / (2 * Sqrt[g * H])
RKStep = Log[ 1 - t * EigvFmat + (t * EigvFmat)^2 / 2 -  (t * EigvFmat)^3 / 6 ] / (I * t);
RKstepTay = Series[RKStep, {x, 0, 5}]
Simplify[RKstepTay, k * H > 0 ]
Simplify[RKstepTay - {wAp, wAm}, k * H > 0]
```

Out[981]=
$$-\frac{1}{2\,x}\left(1-e^{-i\,k\,x}\right)\left(\frac{1}{6}\left(-5+e^{-i\,k\,x}-2\,e^{i\,k\,x}\right)+\frac{1}{6}\,e^{i\,k\,x}\left(5+2\,e^{-i\,k\,x}-e^{i\,k\,x}\right)\right)\sqrt{g\,H}$$

Out[982]=
$$\frac{1}{12}\sqrt{g\,H}\;k^4\,x^3-\frac{1}{72}\left(\sqrt{g\,H}\;k^6\right)x^5+O[x]^6$$

Out[983]=
$$\frac{\left(1-e^{-i\,k\,x}\right)\left(9-e^{-i\,k\,x}+9\,e^{i\,k\,x}-e^{2\,i\,k\,x}\right)H\,(26-2\,\text{Cos}[k\,x])}{384\,x\left(H-\frac{H^3\,(-30+32\,\text{Cos}[k\,x]-2\,\text{Cos}[2\,k\,x])}{36\,x^2}\right)}$$

Out[984]=
$$-\frac{i\left(243\,k^5+49\,H^2\,k^7\right)x^4}{960\left(3+H^2\,k^2\right)^2}+O[x]^6$$

Out[985]=
$$-\frac{1}{2\,x}\left(1-e^{-i\,k\,x}\right)\left(\frac{1}{6}\left(-5+e^{-i\,k\,x}-2\,e^{i\,k\,x}\right)+\frac{1}{6}\,e^{i\,k\,x}\left(5+2\,e^{-i\,k\,x}-e^{i\,k\,x}\right)\right)\sqrt{g\,H}$$

Out[986]=
$$\frac{1}{12}\sqrt{g\,H}\;k^4\,x^3-\frac{1}{72}\left(\sqrt{g\,H}\;k^6\right)x^5+O[x]^6$$

Out[987]=
$$\frac{\left(1-e^{-i\,k\,x}\right)\left(\frac{1}{6}\,e^{i\,k\,x}\left(5+2\,e^{-i\,k\,x}-e^{i\,k\,x}\right)+\frac{1}{6}\left(5-e^{-i\,k\,x}+2\,e^{i\,k\,x}\right)\right)g\,H}{2\,x}$$

Out[988]=
$$-\frac{1}{30}\,i\,g\,H\,k^5\,x^4+O[x]^6$$

Out[989]=
$$\left\{\left\{-\frac{\left(1-e^{-i\,k\,x}\right)\left(\frac{1}{6}\left(-5+e^{-i\,k\,x}-2\,e^{i\,k\,x}\right)+\frac{1}{6}\,e^{i\,k\,x}\left(5+2\,e^{-i\,k\,x}-e^{i\,k\,x}\right)\right)\sqrt{g\,H}}{2\,x},\right.\right.$$
$$\left.\frac{\left(1-e^{-i\,k\,x}\right)\left(9-e^{-i\,k\,x}+9\,e^{i\,k\,x}-e^{2\,i\,k\,x}\right)H\,(26-2\,\text{Cos}[k\,x])}{384\,x\left(H-\frac{H^3\,(-30+32\,\text{Cos}[k\,x]-2\,\text{Cos}[2\,k\,x])}{36\,x^2}\right)}\right\},$$
$$\left\{\frac{\left(1-e^{-i\,k\,x}\right)\left(\frac{1}{6}\,e^{i\,k\,x}\left(5+2\,e^{-i\,k\,x}-e^{i\,k\,x}\right)+\frac{1}{6}\left(5-e^{-i\,k\,x}+2\,e^{i\,k\,x}\right)\right)g\,H}{2\,x},\right.$$
$$\left.\left.-\frac{\left(1-e^{-i\,k\,x}\right)\left(\frac{1}{6}\left(-5+e^{-i\,k\,x}-2\,e^{i\,k\,x}\right)+\frac{1}{6}\,e^{i\,k\,x}\left(5+2\,e^{-i\,k\,x}-e^{i\,k\,x}\right)\right)\sqrt{g\,H}}{2\,x}\right\}\right\}$$

Out[991]=
$$\left\{-\frac{i\sqrt{3}\,g\,H\,k}{\sqrt{g\,H\left(3+H^2\,k^2\right)}}+\frac{1}{12}\sqrt{g\,H}\,k^4\,x^3+\frac{i\,g^2\,H^2\,k^5\left(531+145\,H^2\,k^2\right)x^4}{1920\sqrt{3}\left(g\,H\left(3+H^2\,k^2\right)\right)^{3/2}}-\frac{1}{72}\left(\sqrt{g\,H}\,k^6\right)x^5+O[x]^6,\right.$$
$$\left.\frac{i\sqrt{3}\,g\,H\,k}{\sqrt{g\,H\left(3+H^2\,k^2\right)}}+\frac{1}{12}\sqrt{g\,H}\,k^4\,x^3-\frac{i\,g^2\,H^2\,k^5\left(531+145\,H^2\,k^2\right)x^4}{1920\sqrt{3}\left(g\,H\left(3+H^2\,k^2\right)\right)^{3/2}}-\frac{1}{72}\left(\sqrt{g\,H}\,k^6\right)x^5+O[x]^6\right\}$$

Out[992]=
$$\frac{x}{2\sqrt{g\,H}}$$

Out[994]=
$$\left\{\frac{\sqrt{3}\,k\sqrt{g\,H\left(3+H^2\,k^2\right)}}{3+H^2\,k^2}-\frac{i\sqrt{g\,H}\left(-153\,k^4-96\,H^2\,k^6-16\,H^4\,k^8\right)x^3}{192\left(3+H^2\,k^2\right)^2}-\right.$$
$$\frac{\left(g\,H\left(1485\sqrt{3}\,k^5+966\sqrt{3}\,H^2\,k^7+145\sqrt{3}\,H^4\,k^9\right)\right)x^4}{5760\left(\left(3+H^2\,k^2\right)^2\sqrt{g\,H\left(3+H^2\,k^2\right)}\right)}-$$
$$\frac{i\sqrt{g\,H}\left(891\,k^6+864\,H^2\,k^8+288\,H^4\,k^{10}+32\,H^6\,k^{12}\right)x^5}{2304\left(3+H^2\,k^2\right)^3}+O[x]^6,$$
$$-\frac{\sqrt{3}\,k\sqrt{g\,H\left(3+H^2\,k^2\right)}}{3+H^2\,k^2}-\frac{i\sqrt{g\,H}\left(-153\,k^4-96\,H^2\,k^6-16\,H^4\,k^8\right)x^3}{192\left(3+H^2\,k^2\right)^2}+$$
$$\frac{g\,H\left(1485\sqrt{3}\,k^5+966\sqrt{3}\,H^2\,k^7+145\sqrt{3}\,H^4\,k^9\right)x^4}{5760\left(3+H^2\,k^2\right)^2\sqrt{g\,H\left(3+H^2\,k^2\right)}}-$$
$$\left.\frac{i\sqrt{g\,H}\left(891\,k^6+864\,H^2\,k^8+288\,H^4\,k^{10}+32\,H^6\,k^{12}\right)x^5}{2304\left(3+H^2\,k^2\right)^3}+O[x]^6\right\}$$

Out[995]=
$$\left\{\frac{\sqrt{3}\ g\,H\,k}{\sqrt{g\,H\left(3+H^2\,k^2\right)}}+\frac{i\sqrt{g\,H}\ k^4\left(153+96\,H^2\,k^2+16\,H^4\,k^4\right)x^3}{192\left(3+H^2\,k^2\right)^2}-\right.$$
$$\frac{\left(\sqrt{g\,H}\ k^5\left(1485+966\,H^2\,k^2+145\,H^4\,k^4\right)\right)x^4}{1920\left(\sqrt{3}\ \left(3+H^2\,k^2\right)^{5/2}\right)}-$$
$$\frac{i\sqrt{g\,H}\ k^6\left(891+864\,H^2\,k^2+288\,H^4\,k^4+32\,H^6\,k^6\right)x^5}{2304\left(3+H^2\,k^2\right)^3}+O[x]^6,\ -\frac{\sqrt{3}\ g\,H\,k}{\sqrt{g\,H\left(3+H^2\,k^2\right)}}+$$
$$\frac{i\sqrt{g\,H}\ k^4\left(153+96\,H^2\,k^2+16\,H^4\,k^4\right)x^3}{192\left(3+H^2\,k^2\right)^2}+\frac{\sqrt{g\,H}\ k^5\left(1485+966\,H^2\,k^2+145\,H^4\,k^4\right)x^4}{1920\sqrt{3}\ \left(3+H^2\,k^2\right)^{5/2}}-$$
$$\left.\frac{i\sqrt{g\,H}\ k^6\left(891+864\,H^2\,k^2+288\,H^4\,k^4+32\,H^6\,k^6\right)x^5}{2304\left(3+H^2\,k^2\right)^3}+O[x]^6\right\}$$

Out[996]=
$$\left\{\frac{i\sqrt{g\,H}\ k^4\left(153+96\,H^2\,k^2+16\,H^4\,k^4\right)x^3}{192\left(3+H^2\,k^2\right)^2}-\frac{\left(\sqrt{g\,H}\ k^5\left(1485+966\,H^2\,k^2+145\,H^4\,k^4\right)\right)x^4}{1920\left(\sqrt{3}\ \left(3+H^2\,k^2\right)^{5/2}\right)}-\right.$$
$$\frac{i\sqrt{g\,H}\ k^6\left(891+864\,H^2\,k^2+288\,H^4\,k^4+32\,H^6\,k^6\right)x^5}{2304\left(3+H^2\,k^2\right)^3}+O[x]^6,$$
$$\frac{i\sqrt{g\,H}\ k^4\left(153+96\,H^2\,k^2+16\,H^4\,k^4\right)x^3}{192\left(3+H^2\,k^2\right)^2}+\frac{\sqrt{g\,H}\ k^5\left(1485+966\,H^2\,k^2+145\,H^4\,k^4\right)x^4}{1920\sqrt{3}\ \left(3+H^2\,k^2\right)^{5/2}}-$$
$$\left.\frac{i\sqrt{g\,H}\ k^6\left(891+864\,H^2\,k^2+288\,H^4\,k^4+32\,H^6\,k^6\right)x^5}{2304\left(3+H^2\,k^2\right)^3}+O[x]^6\right\}$$

In[972]:=
```
t =  x / (4 * Sqrt[g * H])
RKStep = Log[ 1 - t * EigvFmat + (t * EigvFmat)^2 / 2 -  (t * EigvFmat)^3 / 6 ] / (I * t);
RKstepTay = Series[RKStep, {x, 0, 5}]
Simplify[RKstepTay, k * H > 0 ]
Simplify[RKstepTay - {wAp, wAm}, k * H > 0]
```

Out[972]=
$$\frac{x}{4\sqrt{g\,H}}$$

Out[974]=
$$\left\{\frac{\sqrt{3}\ k\sqrt{g H\left(3+H^2 k^2\right)}}{3+H^2 k^2}-i\sqrt{g H}\left(-\frac{k^4}{12}-\frac{3 k^4}{512\left(3+H^2 k^2\right)^2}\right)x^3-\right.$$
$$\frac{\left(g H\left(6345\sqrt{3}\ k^5+3864\sqrt{3}\ H^2 k^7+580\sqrt{3}\ H^4 k^9\right)\right)x^4}{23040\left(\left(3+H^2 k^2\right)^2\sqrt{g H\left(3+H^2 k^2\right)}\right)}-$$
$$\frac{i\sqrt{g H}\left(27675 k^6+27648 H^2 k^8+9216 H^4 k^{10}+1024 H^6 k^{12}\right)x^5}{73728\left(3+H^2 k^2\right)^3}+O[x]^6,$$
$$-\frac{\sqrt{3}\ k\sqrt{g H\left(3+H^2 k^2\right)}}{3+H^2 k^2}-i\sqrt{g H}\left(-\frac{k^4}{12}-\frac{3 k^4}{512\left(3+H^2 k^2\right)^2}\right)x^3+$$
$$\frac{g H\left(6345\sqrt{3}\ k^5+3864\sqrt{3}\ H^2 k^7+580\sqrt{3}\ H^4 k^9\right)x^4}{23040\left(3+H^2 k^2\right)^2\sqrt{g H\left(3+H^2 k^2\right)}}-$$
$$\left.\frac{i\sqrt{g H}\left(27675 k^6+27648 H^2 k^8+9216 H^4 k^{10}+1024 H^6 k^{12}\right)x^5}{73728\left(3+H^2 k^2\right)^3}+O[x]^6\right\}$$

Out[975]=
$$\left\{\frac{\sqrt{3}\ g H k}{\sqrt{g H\left(3+H^2 k^2\right)}}-\frac{i\sqrt{g H}\ k^4\left(-128-\frac{9}{\left(3+H^2 k^2\right)^2}\right)x^3}{1536}-\right.$$
$$\frac{\left(\sqrt{g H}\ k^5\left(6345+3864 H^2 k^2+580 H^4 k^4\right)\right)x^4}{7680\left(\sqrt{3}\ \left(3+H^2 k^2\right)^{5/2}\right)}-$$
$$\frac{i\sqrt{g H}\ k^6\left(27675+27648 H^2 k^2+9216 H^4 k^4+1024 H^6 k^6\right)x^5}{73728\left(3+H^2 k^2\right)^3}+O[x]^6,-\frac{\sqrt{3}\ g H k}{\sqrt{g H\left(3+H^2 k^2\right)}}-$$
$$\frac{i\sqrt{g H}\ k^4\left(-128-\frac{9}{\left(3+H^2 k^2\right)^2}\right)x^3}{1536}+\frac{\sqrt{g H}\ k^5\left(6345+3864 H^2 k^2+580 H^4 k^4\right)x^4}{7680\sqrt{3}\ \left(3+H^2 k^2\right)^{5/2}}-$$
$$\left.\frac{i\sqrt{g H}\ k^6\left(27675+27648 H^2 k^2+9216 H^4 k^4+1024 H^6 k^6\right)x^5}{73728\left(3+H^2 k^2\right)^3}+O[x]^6\right\}$$

Out[976]=
$$\left\{-\frac{i\sqrt{g H}\ k^4\left(-128-\frac{9}{\left(3+H^2 k^2\right)^2}\right)x^3}{1536}-\frac{\left(\sqrt{g H}\ k^5\left(6345+3864 H^2 k^2+580 H^4 k^4\right)\right)x^4}{7680\left(\sqrt{3}\ \left(3+H^2 k^2\right)^{5/2}\right)}-\right.$$
$$\frac{i\sqrt{g H}\ k^6\left(27675+27648 H^2 k^2+9216 H^4 k^4+1024 H^6 k^6\right)x^5}{73728\left(3+H^2 k^2\right)^3}+O[x]^6,$$
$$-\frac{i\sqrt{g H}\ k^4\left(-128-\frac{9}{\left(3+H^2 k^2\right)^2}\right)x^3}{1536}+\frac{\sqrt{g H}\ k^5\left(6345+3864 H^2 k^2+580 H^4 k^4\right)x^4}{7680\sqrt{3}\ \left(3+H^2 k^2\right)^{5/2}}-$$
$$\left.\frac{i\sqrt{g H}\ k^6\left(27675+27648 H^2 k^2+9216 H^4 k^4+1024 H^6 k^6\right)x^5}{73728\left(3+H^2 k^2\right)^3}+O[x]^6\right\}$$