$$\mathcal{C}_2 = \frac{2\cos\left(k\Delta x\right) - 2}{\Delta x^2}$$

$$\mathcal{C}_4 = \frac{-2\cos\left(2k\Delta x\right) + 32\cos\left(k\Delta x\right) - 30}{12\Delta x^2}$$

$$\mathcal{G} = \left[H - \frac{H^3}{3}\mathcal{C}\right]$$

$$\mathcal{M}_3 = \frac{24}{26 - 2\cos\left(k\Delta x\right)}$$

$\mathcal{M}_1 = \mathcal{M}_2 = 1$. $\mathcal{R}_1^+ = e^{ik\Delta x}$ and $\mathcal{R}_1^- = 1$.

$$\mathcal{R}_2^- = 1 + \frac{i\sin\left(k\Delta x\right)}{2}$$

$$\mathcal{R}_2^+ = e^{ik\Delta x}\left(1 - \frac{i\sin\left(k\Delta x\right)}{2}\right)$$

$$R_3^- = \frac{\mathcal{M}_3}{6}\left[5 + -e^{-ik\Delta x} + 2e^{ik\Delta x}\right]$$

$$R_3^+ = \frac{\mathcal{M}_3 e^{ik\Delta x}}{6}\left[5 + 2e^{-ik\Delta x} - e^{ik\Delta x}\right]$$

$$R_2^u = \frac{e^{ik\Delta x} + 1}{2}$$

$$R_3^u = \frac{-3e^{2ik\Delta x} + 27e^{ik\Delta x} + 27 - 3e^{-ik\Delta x}}{48}$$

$$\mathcal{F}^{h,u} = H\mathcal{R}^u$$

$$\mathcal{F}^{h,h} = -\frac{\sqrt{gH}}{2}\left[\mathcal{R}^+ - \mathcal{R}^-\right]$$

$$\mathcal{F}^{u,u} = -\frac{\sqrt{gH}}{2}\mathcal{G}\left[\mathcal{R}^+ \ - \mathcal{R}^-\right]$$

$$\mathcal{F}^{u,h} = \frac{gH\mathcal{R}^- + gH\mathcal{R}^+}{2}$$

$\mathcal{D} = 1 - e^{-ik\Delta x}$

1 Equations

We have

$$h_t + Hu_x = 0$$
$$G_t + gHh_x = 0$$

We can exactly calculate them to find

$$\omega h + kHu = 0$$
$$\omega G + gHh = 0$$

Going through our FVM approximation exactly we have for the first equation

$$\bar{h}_j^{n+1} = \bar{h}_j^n - \frac{\Delta t}{\Delta x}\left[\frac{1}{\Delta t}\int_n^{n+1} Hu_{j+1/2}\,dt - \frac{1}{\Delta t}\int_n^{n+1} Hu_{j-1/2}\,dt\right]$$

$$\mathcal{A}h_j^{n+1} = \mathcal{A}h_j^n - \frac{H}{\Delta x}\left[\int_n^{n+1} u_{j+1/2}\,dt - \int_n^{n+1} u_{j-1/2}\,dt\right]$$

$$\mathcal{A}h_j^{n+1} = \mathcal{A}h_j^n - \frac{H}{\Delta x}\left[e^{i\frac{k\Delta x}{2}}\int_n^{n+1} u_j\,dt - e^{-i\frac{k\Delta x}{2}}\int_n^{n+1} u_j\,dt\right]$$

$$\mathcal{A}h_j^{n+1} - \mathcal{A}h_j^n = -\frac{H}{\Delta x}\left[e^{i\frac{k\Delta x}{2}} - e^{-i\frac{k\Delta x}{2}}\right]\int_n^{n+1} u_j\,dt$$

$$\mathcal{A}\left[e^{i\omega\Delta t} - 1\right]h_j^n = -\frac{H}{\Delta x}\left[2i\sin\left(\frac{\Delta xk}{2}\right)\right]\left[\frac{-i}{\omega}u_j\right]_n^{n+1}$$

$$\omega\mathcal{A}\left[e^{i\omega\Delta t} - 1\right]h_j^n = -\frac{2H}{\Delta x}\left[\sin\left(\frac{\Delta xk}{2}\right)\right]\left(e^{i\omega\Delta t} - 1\right)u_j$$

$$\omega h_j^n = -\frac{2H}{\mathcal{A}\Delta x}\left[\sin\left(\frac{\Delta xk}{2}\right)\right]u_j$$

$$\omega h_j^n = -\frac{2H}{\frac{2}{\Delta xk}\sin\left(\Delta xk/2\right)\Delta x}\left[\sin\left(\frac{\Delta xk}{2}\right)\right]u_j$$

$$\omega h_j^n = -\frac{2H}{\frac{2}{k}}u_j$$

$$\omega h_j^n = -kHu_j$$

So we recover this exactly. Now we replace things with their approximations.

$$\mathcal{M}h_j^{n+1} = \mathcal{M}h_j^n - \frac{\Delta t}{\Delta x}\left[\frac{1}{\Delta t}\int_n^{n+1} Hu_{j+1/2}\,dt - \frac{1}{\Delta t}\int_n^{n+1} Hu_{j-1/2}\,dt\right]$$

$$\mathcal{M}h_j^{n+1} = \mathcal{M}h_j^n - \frac{\Delta t}{\Delta x}\left[\mathcal{DF}^{(h,u)}u_j + \mathcal{DF}^{(h,h)}h_j\right]$$

$$\mathcal{M}\left(e^{i\omega\Delta t} - 1\right)h_j^{n+1} = -\frac{\Delta t}{\Delta x}\left[\mathcal{DF}^{(h,u)}u_j + \mathcal{DF}^{(h,h)}h_j\right]$$

$$\frac{e^{i\omega\Delta t} - 1}{\Delta t}h_j^n = -\frac{1}{\mathcal{M}\Delta x}\left[\mathcal{DF}^{(h,u)}u_j + \mathcal{DF}^{(h,h)}h_j\right]$$

We don't care about time accuracy so we can just replace the LHS coefficient with $i\omega$, so it is correct, then it must be that

$$\frac{1}{\mathcal{M}\Delta x}\left[\mathcal{DF}^{(h,u)}u_j + \mathcal{DF}^{(h,h)}h_j\right] \approx ikHu_j$$

2 Taylor Expansions

Around 0 we have

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + O(x^{12})$$

$$\sin(x) = \frac{1}{1!}x^1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + O(x^{11})$$

$$e^{ix} = \cos(x) + i\sin(x)$$

$$e^{ix} = 1+ix-\frac{1}{2}x^2-\frac{i}{3!}x^3+\frac{1}{4!}x^4+\frac{i}{5!}x^5-\frac{1}{6!}x^6-\frac{i}{7!}x^7+\frac{1}{8!}x^8+\frac{i}{9!}x^9-\frac{1}{10!}x^{10}+O(x^{11})$$

3 Going Down List

So

$$\mathcal{G}_a = \left[H + \frac{H^3}{3}k^2\right]$$

$$\mathcal{C}_2 = \frac{2\left(-\frac{1}{2!}(k\Delta x)^2 + \frac{1}{4!}(k\Delta x)^4 - \frac{1}{6!}(k\Delta x)^6 + \frac{1}{8!}(k\Delta x)^8 - \frac{1}{10!}(k\Delta x)^{10} + O(x^{12})\right)}{\Delta x^2}$$

$$\mathcal{C}_2 = \frac{-(k\Delta x)^2 + \frac{2}{4!}(k\Delta x)^4 - \frac{2}{6!}(k\Delta x)^6 + \frac{2}{8!}(k\Delta x)^8 - \frac{2}{10!}(k\Delta x)^{10} + O(x^{12})}{\Delta x^2}$$

$$\mathcal{C}_2 = -k^2 + \frac{2}{4!}k^4(\Delta x)^2 - \frac{2}{6!}k^6(\Delta x)^4 + \frac{2}{8!}k^8(\Delta x)^6 - \frac{2}{10!}k^{10}(\Delta x)^8 + O(x^{10})$$

$$\mathcal{C}_2 = -k^2 + \frac{1}{12}k^4(\Delta x)^2 + O(x^4)$$

$$\mathcal{C}_4 = \frac{-2\cos(2k\Delta x) + 32\cos(k\Delta x) - 30}{12\Delta x^2}$$

$$-2\cos(2k\Delta x) = -2\left[1 - \frac{1}{2}2^2k^2\Delta x^2 + \frac{1}{24}2^4k^4\Delta x^4 - \frac{1}{720}2^6k^6\Delta x^6 + \frac{1}{40320}2^8k^8\Delta x^8 + O(\Delta x^{10})\right]$$

$$= -2 + \frac{1}{2}2^3k^2\Delta x^2 - \frac{1}{24}2^5k^4\Delta x^4 + \frac{1}{720}2^7k^6\Delta x^6 - \frac{1}{40320}2^9k^8\Delta x^8 + O(\Delta x^{10})$$

$$= -2 + \frac{8}{2}k^2\Delta x^2 - \frac{32}{24}k^4\Delta x^4 + \frac{128}{720}k^6\Delta x^6 - \frac{512}{40320}k^8\Delta x^8 + O(\Delta x^{10})$$

$$32\cos(k\Delta x) = 32 - 16(k\Delta x)^2 + \frac{32}{24}(k\Delta x)^4 - \frac{32}{720}(k\Delta x)^6 + \frac{32}{40320}(k\Delta x)^8 + O((k\Delta x)^{10})$$

$$32\cos(k\Delta x) - 2\cos(2k\Delta x) = 30 - 12(k\Delta x)^2 + \frac{96}{720}(k\Delta x)^6 - \frac{480}{40320}(k\Delta x)^8 + O((k\Delta x)^{10})$$

$$\mathcal{C}_4 = \frac{1}{12\Delta x^2}\left[-12(k\Delta x)^2 + \frac{96}{720}(k\Delta x)^6 - \frac{480}{40320}(k\Delta x)^8 + O((k\Delta x)^{10})\right]$$

$$\mathcal{C}_4 = \left[-(k)^2 + \frac{96}{8640}k^6(\Delta x)^4 - \frac{480}{483840}k^8(\Delta x)^6 + O((k\Delta x)^8)\right]$$

$$\mathcal{C}_4 = \left[-(k)^2 + \frac{1}{90}k^6(\Delta x)^4 + O(\Delta x^6)\right]$$

Analytically

$$G = uh - h^3/3(u_{xx})$$

For the fourier nodes we have

$$G = uH - (-1)(k^2)H^3/3(u)$$

$$G = uH + (k^2)H^3/3u$$

Using above derivative approximations

$$\mathcal{G}_2 = H - \frac{H^3}{3}\left[-k^2 + \frac{1}{12}k^4(\Delta x)^2 + O(x^4)\right]$$

$$\mathcal{G}_2 = H + \frac{H^3}{3}k^2 - \frac{H^3k^4(\Delta x)^2}{36} + O(x^4)$$

$$\mathcal{G}_4 = H - \frac{H^3}{3}\left[-(k)^2 + \frac{1}{90}k^6(\Delta x)^4 + O(\Delta x^6)\right]$$

$$\mathcal{G}_4 = H + \frac{H^3k^2}{3} - \frac{H^3k^6(\Delta x)^4}{270} + O(\Delta x^6)$$

Analtyically we have

$$\bar{u}_j = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u_j dx$$

$$\bar{u}_j = \frac{1}{\Delta x}\frac{-i}{k}u_j\left[e^{ik\Delta x/2} - e^{-ik\Delta x/2}\right]dx$$

$$\bar{u}_j = \frac{1}{\Delta x}\frac{-i}{k}u_j\left[2i\sin\left(ik\Delta x/2\right)\right]$$

$$\bar{u}_j = \frac{1}{\Delta x}\frac{2}{k}\left[\sin\left(ik\Delta x/2\right)\right]u_j$$

So

$$u_j = k\frac{\Delta x}{2}\left[\csc\left(ik\Delta x/2\right)\right]\bar{u}_j$$

$$\mathcal{M}_a = k\frac{\Delta x}{2}\left[\csc\left(ik\Delta x/2\right)\right]$$

The taylor expansion of this is

$$\mathcal{M}_a = 1 + \frac{1}{24}(k\Delta x)^2 + \frac{7}{5760}(k\Delta x)^4 + O(x^6)$$

$$\mathcal{M}_3 = \frac{26 - 2\left(1 - \frac{1}{2!}(k\Delta x)^2 + \frac{1}{4!}(k\Delta x)^4 - \frac{1}{6!}(k\Delta x)^6 + \frac{1}{8!}(k\Delta x)^8 - \frac{1}{10!}(k\Delta x)^{10} + O(x^{12})\right)}{24}$$

$$\mathcal{M}_3 = \frac{12 - \left(-\frac{1}{2!}(k\Delta x)^2 + \frac{1}{4!}(k\Delta x)^4 - \frac{1}{6!}(k\Delta x)^6 + \frac{1}{8!}(k\Delta x)^8 - \frac{1}{10!}(k\Delta x)^{10} + O(x^{12})\right)}{12}$$

$$\mathcal{M}_3 = \frac{12 + \frac{1}{2!}(k\Delta x)^2 - \frac{1}{4!}(k\Delta x)^4 + \frac{1}{6!}(k\Delta x)^6 - \frac{1}{8!}(k\Delta x)^8 + \frac{1}{10!}(k\Delta x)^{10} + O(x^{12})}{12}$$

$$\mathcal{M}_3 = 1 + \frac{1}{24}(k\Delta x)^2 - \frac{1}{288}(k\Delta x)^4 + \frac{1}{8640}(k\Delta x)^6 + O(x^8)$$

$$\mathcal{M}_3 = 1 + \frac{1}{24}(k\Delta x)^2 - \frac{1}{288}(k\Delta x)^4 + O(x^6)$$

So we see that $\mathcal{M}_3$ has a fourth order error compared to the analytic solution.

Reconstruction, analytically we have

$$u_{j+1/2} = u_j e^{ik\Delta x/2}$$

So

$$\mathcal{R}_a = \mathcal{R}_a^+ = \mathcal{R}_a^- = e^{ik\Delta x/2}$$

So we have the Taylor expansion

$$\mathcal{R}_a = 1+i(k\Delta x/2)-\frac{1}{2}(k\Delta x/2)^2-\frac{i}{3!}(k\Delta x/2)^3+\frac{1}{4!}(k\Delta x/2)^4+\frac{i}{5!}(k\Delta x/2)^5-\frac{1}{6!}(k\Delta x/2)^6+O(x^7)$$

$$\mathcal{R}_a = 1+i\frac{1}{2}(k\Delta x)-\frac{1}{2}\frac{1}{4}(k\Delta x)^2-\frac{i}{3!}\frac{1}{8}(k\Delta x)^3+\frac{1}{4!}\frac{1}{16}(k\Delta x)^4+\frac{i}{5!}\frac{1}{32}(k\Delta x)^5-\frac{1}{6!}\frac{1}{64}(k\Delta x)^6+O(x^7)$$

$$\mathcal{R}_a = 1+\frac{i}{2}k\Delta x-\frac{1}{8}k^2\Delta x^2-\frac{i}{48}k^3\Delta x^3+O(x^4)$$

$\mathcal{R}_1^+ = e^{ik\Delta x}$ and $\mathcal{R}_1^- = 1$.

$$\mathcal{R}_1^+ = 1+ik\Delta x-\frac{1}{2}(k\Delta x)^2-\frac{i}{3!}(k\Delta x)^3+\frac{1}{4!}(k\Delta x)^4+O((k\Delta x)^5)$$

So both have first order errors.

$$\mathcal{R}_2^- = 1+\frac{i\sin(k\Delta x)}{2}$$

$$\mathcal{R}_2^- = 1+\frac{i}{2}\left[\frac{1}{1!}(k\Delta x)^1-\frac{1}{3!}(k\Delta x)^3+\frac{1}{5!}(k\Delta x)^5-\frac{1}{7!}(k\Delta x)^7+\frac{1}{9!}(k\Delta x)^9+O(x^{11})\right]$$

$$\mathcal{R}_2^- = 1+\frac{i}{2}(k\Delta x)-\frac{i}{2}\frac{1}{3!}(k\Delta x)^3+\frac{i}{2}\frac{1}{5!}(k\Delta x)^5-\frac{i}{2}\frac{1}{7!}(k\Delta x)^7+\frac{i}{2}\frac{1}{9!}(k\Delta x)^9+O(x^{11})$$

$$\mathcal{R}_2^+ = e^{ik\Delta x}\left(1-\frac{i\sin(k\Delta x)}{2}\right)$$

$$\mathcal{R}_2^+ = e^{ik\Delta x}\left(1-\frac{i}{2}(k\Delta x)+\frac{i}{2}\frac{1}{3!}(k\Delta x)^3-\frac{i}{2}\frac{1}{5!}(k\Delta x)^5+\frac{i}{2}\frac{1}{7!}(k\Delta x)^7-\frac{i}{2}\frac{1}{9!}(k\Delta x)^9+O(x^{11})\right)$$

$$\mathcal{R}_2^+ = \left(1 + i(k\Delta x) - \frac{1}{2}(k\Delta x)^2 - \frac{i}{3!}(k\Delta x)^3 + \frac{1}{4!}(k\Delta x)^4 + \frac{i}{5!}(k\Delta x)^5 + O\big((\Delta x^2)\big)\right)$$
$$\times \left(1 - \frac{i}{2}(k\Delta x) + \frac{i}{2}\frac{1}{3!}(k\Delta x)^3 - \frac{i}{2}\frac{1}{5!}(k\Delta x)^5 + O(x^7)\right) \quad (1)$$

$$\mathcal{R}_2^+ = 1 + \frac{ik\Delta x}{2} + \frac{i}{6}k^3\Delta x^3 + O\left(\Delta x^4\right)$$

So for both second order reconstructions we have a second order error.

$$R_3^- = \frac{\mathcal{M}_3}{6}\left[5 + -e^{-ik\Delta x} + 2e^{ik\Delta x}\right]$$

$$R_3^- = \frac{1 + \frac{1}{24}(k\Delta x)^2 - \frac{1}{288}(k\Delta x)^4 + O(x^6)}{6}$$
$$\left[5 - \left[1 - ik\Delta x - \frac{1}{2}(k\Delta x)^2 + \frac{i}{3!}(k\Delta x)^3 + \frac{1}{4!}(k\Delta x)^4 - \frac{i}{5!}(k\Delta x)^5 - O(\Delta x^6)\right]\right.$$
$$\left.+2\left[1 + i(k\Delta x) - \frac{1}{2}(k\Delta x)^2 - \frac{i}{3!}(k\Delta x)^3 + \frac{1}{4!}(k\Delta x)^4 + \frac{i}{5!}(k\Delta x)^5 + O(\Delta x^6)\right]\right] \quad (2)$$

$$R_3^- = \frac{1 + \frac{1}{24}(k\Delta x)^2 - \frac{1}{288}(k\Delta x)^4 + O(x^6)}{6}$$
$$\left[6 + 3ik\Delta x - \frac{1}{2}(k\Delta x)^2 - \frac{3i}{3!}(k\Delta x)^3 + \frac{1}{4!}(k\Delta x)^4 O(\Delta x^5)\right] \quad (3)$$

$$R_3^- = \frac{1}{6}\left[6 + 3ik\Delta x - \frac{1}{2}(k\Delta x)^2 - \frac{3i}{3!}(k\Delta x)^3 + \frac{1}{4!}(k\Delta x)^4 O(\Delta x^5)\right]$$
$$+ \frac{\frac{1}{24}(k\Delta x)^2}{6}\left[6 + 3ik\Delta x - \frac{1}{2}(k\Delta x)^2 - \frac{3i}{3!}(k\Delta x)^3 + \frac{1}{4!}(k\Delta x)^4 O(\Delta x^5)\right]$$
$$+ \frac{-\frac{1}{288}(k\Delta x)^4}{6}\left[6 + 3ik\Delta x - \frac{1}{2}(k\Delta x)^2 - \frac{3i}{3!}(k\Delta x)^3 + \frac{1}{4!}(k\Delta x)^4 O(\Delta x^5)\right]$$
$$+ O(x^6) \quad (4)$$

$$
\begin{aligned}
R_3^- = 1 &+ \frac{i}{2}k\Delta x \\
&+ \frac{1}{6}\Bigg[-\frac{1}{2}(k\Delta x)^2 - \frac{3i}{3!}(k\Delta x)^3 + \frac{1}{4!}(k\Delta x)^4 O(\Delta x^5)\Bigg] \\
&+ \frac{\frac{1}{24}(k\Delta x)^2}{6}\Bigg[6 + 3ik\Delta x - \frac{1}{2}(k\Delta x)^2 - \frac{3i}{3!}(k\Delta x)^3 + \frac{1}{4!}(k\Delta x)^4 O(\Delta x^5)\Bigg] \\
&+ \frac{-\frac{1}{288}(k\Delta x)^4}{6}\Bigg[6 + 3ik\Delta x - \frac{1}{2}(k\Delta x)^2 - \frac{3i}{3!}(k\Delta x)^3 + \frac{1}{4!}(k\Delta x)^4 O(\Delta x^5)\Bigg] \\
&+ O(x^6) \qquad (5)
\end{aligned}
$$

$$
R_3^- = 1 + \frac{i}{2}k\Delta x - \frac{1}{24}(k\Delta x)^2 - \frac{1}{16}(k\Delta x)^3 + O(\Delta x^4) \qquad (6)
$$

So our reconstruction is under order...

$$
R_3^+ = \frac{\mathcal{M}_3 e^{ik\Delta x}}{6}\left[5 + 2e^{-ik\Delta x} - e^{ik\Delta x}\right]
$$

from wolfram

$$
R_3^+ = 1 + \frac{i}{2}k\Delta x - \frac{1}{24}(k\Delta x)^2 - \frac{5}{48}(k\Delta x)^3 + O(\Delta x^4)
$$

Again we have second order errors. But they are the same... Hopefully some cancellation works out.

$$
R_2^u = \frac{e^{ik\Delta x} + 1}{2}
$$

$$
R_2^u = 1 + \frac{i}{2}k\Delta x - \frac{1}{4}(k\Delta x)^2 - \frac{i}{12}(k\Delta x)^3 + O(\Delta x^4)
$$

$$
R_3^u = \frac{-3e^{2ik\Delta x} + 27e^{ik\Delta x} + 27 - 3e^{-ik\Delta x}}{48}
$$

$$
R_3^u = 1 + \frac{i}{2}k\Delta x - \frac{1}{8}(k\Delta x)^2 - \frac{i}{48}(k\Delta x)^3 + O(\Delta x^4)
$$

So here we get correct order of errors. In fact the reconstruction is higher than 3rd.

Ok so for flux we have to get the analytic result. Is

$$F_1^h(u_j, h_j) = kiHu_j$$

$$F_a^h(u_j, h_j) = kigHh_j$$

$\mathcal{D} = 1 - e^{-ik\Delta x}$

$$\mathcal{D} = ik\Delta x + \frac{(k\Delta x)^2}{2} - \frac{i(k\Delta x)^3}{6} + O(\Delta x^4)$$

$$\mathcal{F}_1^{h,u} = H\mathcal{R}_2^u$$

$$\mathcal{F}_1^{h,u} = H\left[1 + \frac{i}{2}k\Delta x - \frac{1}{4}(k\Delta x)^2 - \frac{i}{12}(k\Delta x)^3 + O(\Delta x^4)\right]$$

$$\mathcal{D}\mathcal{F}_1^{h,u} = H\left[ik\Delta x - \frac{i}{6}(k\Delta x)^3 + O(\Delta x^4)\right]$$

$$\mathcal{F}_1^{h,h} = -\frac{\sqrt{gH}}{2}\left[\mathcal{R}_1^+ - \mathcal{R}_1^-\right]$$

$$\mathcal{F}_1^{h,h} = -\frac{\sqrt{gH}}{2}\left[ik\Delta x - \frac{(k\Delta x)^2}{2} - \frac{i(k\Delta x)^3}{6} + O(\Delta x^4)\right]$$

$$\mathcal{D}\mathcal{F}_1^{h,h} = -\frac{\sqrt{gH}}{2}\left[(k\Delta x)^2 + O(\Delta x^4)\right]$$

$$F_1^h = \frac{1}{\Delta x}\left[\mathcal{D}\mathcal{F}_1^{h,u}u_j + \mathcal{D}\mathcal{F}_1^{h,h}h_j\right]$$

$$\begin{aligned} F_1^h = H\left[ik - \frac{i}{6}k^3(\Delta x)^2 + O(\Delta x^3)\right]u_j \\ + -\frac{\sqrt{gH}}{2}\left[k^2(\Delta x) + O(\Delta x^3)\right]h_j \end{aligned} \tag{7}$$

So first order error is completely on h interesting.

$$\mathcal{F}_1^{u,u} = -\frac{\sqrt{gH}}{2}\mathcal{G}_1\left[\mathcal{R}_1^+ \; - \mathcal{R}_1^-\right]$$

$$\mathcal{F}_1^{u,u} = -\frac{\sqrt{gH}}{2}\left[\frac{1}{3}iH(H^2+3)(k\Delta x) - \frac{1}{6}iH(H^2+3)(k\Delta x)^2 - \frac{1}{12}iH(H^2+3)(k\Delta x)^3 + O(\Delta x^4)\right]$$

$$\mathcal{D}\mathcal{F}_1^{u,u} = -\frac{\sqrt{gH}}{2}\left[-\frac{1}{3}iH(H^2+3)(k\Delta x)^2 + \frac{1}{36}H(2H^2+3)(k\Delta x)^4 + O(\Delta x^6)\right]$$

$$\mathcal{F}_1^{u,h} = \frac{gH}{2}[\mathcal{R}_1^- + \mathcal{R}_1^+]$$

$$\mathcal{F}_1^{u,h} = \frac{gH}{2}\left[2 + ik\Delta x - \frac{(k\Delta x)^2}{2} - \frac{i(k\Delta x)^3}{6} + O(\Delta x^4)\right]$$

$$\mathcal{D}\mathcal{F}_1^{u,h} = \frac{gH}{2}\left[2ik\Delta x - \frac{i(k\Delta x)^3}{3} + O(\Delta x^4)\right]$$

So

$$F_1^u = -\frac{\sqrt{gH}}{2}\left[-\frac{1}{3}iH(H^2+3)k^2(\Delta x) + \frac{1}{36}H(2H^2+3)k^4(\Delta x)^3 + O(\Delta x^5)\right]u_j + \frac{gH}{2}\left[2ik - \frac{ik^3(\Delta x)^2}{3} + O(\Delta x^3)\right]h_j \quad (8)$$

$$F_1^u = -\frac{\sqrt{gH}}{2}\left[-\frac{1}{3}iH(H^2+3)k^2(\Delta x) + \frac{1}{36}H(2H^2+3)k^4(\Delta x)^3 + O(\Delta x^5)\right]u_j + gH\left[ik - \frac{ik^3(\Delta x)^2}{6} + O(\Delta x^3)\right]h_j \quad (9)$$

OK Seeing a pattern, but we have to make sure this holds as we go, anyway we now claculate just the D F together always to make it simpler.

$$\mathcal{D}\mathcal{F}_2^{h,u} = H\mathcal{D}\mathcal{R}_2^u$$

$$\mathcal{DF}_2^{h,u} = H\left[i(k\Delta x) - \frac{i}{6}(k\Delta x)^3 + \frac{i}{120}(k\Delta x)^5 + O(\Delta x^6)\right]$$

$$\mathcal{DF}^{h,h} = -\frac{\sqrt{gH}}{2}\mathcal{D}\left[\mathcal{R}^+ - \mathcal{R}^-\right]$$

$$\mathcal{DF}^{h,h} = -\frac{\sqrt{gH}}{2}\left[-\frac{1}{4}(k\Delta x)^4 + \frac{1}{24}(k\Delta x)^6 + O(\Delta x^8)\right]$$

$$F_2^h = H\left[2ik - \frac{i}{3}k^3(\Delta x)^2 + \frac{i}{60}k^5(\Delta x)^4 + O(\Delta x^5)\right]u_j \\ + -\frac{\sqrt{gH}}{2}\left[-\frac{1}{4}k^4(\Delta x)^3 + \frac{1}{24}k^6(\Delta x)^5 + O(\Delta x^7)\right]h_j \quad (10)$$

$$\mathcal{DF}^{u,u} = -\frac{\sqrt{gH}}{2}\mathcal{DG}\left[\mathcal{R}^+ \ - \mathcal{R}^-\right]$$

$$\mathcal{DF}^{u,u} = -\frac{\sqrt{gH}}{2}\left[-\frac{1}{12}H(H^2+3)k^4\Delta x^4 + \frac{1}{48}H(H^2+2)k^6\Delta x^6 + O(\Delta x^7)\right]$$

$$\mathcal{DF}^{u,h} = \frac{gH}{2}\mathcal{D}[\mathcal{R}^- + \mathcal{R}^+]$$

$$\mathcal{DF}^{u,h} = \frac{gH}{2}\left[2ik\Delta x + \frac{i}{6}k^3\Delta x^3 - \frac{13i}{120}k^5\Delta x^5 + O(\Delta x^7)\right]$$

$$F_2^u = -\frac{\sqrt{gH}}{2}\left[-\frac{1}{12}H(H^2+3)k^4\Delta x^3 + \frac{1}{48}H(H^2+2)k^6\Delta x^5 + O(\Delta x^6)\right]u_j \\ + \frac{gH}{2}\left[2ik + \frac{i}{6}k^3\Delta x^2 - \frac{13i}{120}k^5\Delta x^4 + O(\Delta x^6)\right]h_j \quad (11)$$

So everyhting is the correct order for the second order scheme as well.

$$\mathcal{DF}_3^{h,u} = H\mathcal{DR}_3^u$$

$$\mathcal{DF}_3^{h,u} = H\left[ik\Delta x - \frac{i}{24}k^3\Delta x^3 - \frac{11i}{480}k^5\Delta x^5 + O(\Delta x^7)\right]$$

$$\mathcal{DF}_3^{h,h} = -\frac{\sqrt{gH}}{2}\mathcal{D}\left[\mathcal{R}_3^+ - \mathcal{R}_3^-\right]$$

$$\mathcal{DF}_3^{h,h} = -\frac{\sqrt{gH}}{2}\left[-\frac{1}{6}k^4\Delta x^4 + \frac{1}{48}k^6\Delta x^6 - \frac{1}{2880}k^8\Delta x^8 + O(\Delta x^9)\right]$$

$$\begin{aligned} F_3^h = H &\left[ik - \frac{i}{24}k^3\Delta x^2 - \frac{11i}{480}k^5\Delta x^4 + O(\Delta x^6)\right]u_j \\ &+ -\frac{\sqrt{gH}}{2}\left[-\frac{1}{6}k^4\Delta x^3 + \frac{1}{48}k^6\Delta x^5 - \frac{1}{2880}k^8\Delta x^7 + O(\Delta x^8)\right]h_j \end{aligned} \tag{12}$$

$$\mathcal{DF}_3^{u,u} = -\frac{\sqrt{gH}}{2}\mathcal{D}\mathcal{G}_3\left[\mathcal{R}_3^+ \ - \mathcal{R}_3^-\right]$$

$$\begin{aligned} \mathcal{DF}_3^{u,u} = -\frac{\sqrt{gH}}{2}&\left[-\frac{1}{6}k^4\Delta x^4 + \frac{1}{48}k^6\Delta x^6 - \frac{1}{2880}k^8\Delta x^8 + O(\Delta x^9)\right] \\ &\left[(\frac{H^3}{3} + H) - \frac{H^3k^4\Delta x^4}{270} + \frac{H^3k^6\Delta x^6}{3024} - \frac{H^3k^8\Delta x^8}{64800} + O(\Delta x^9)\right] \end{aligned} \tag{13}$$

$$\mathcal{DF}_3^{u,u} = -\frac{\sqrt{gH}}{2}\left[-(\frac{H^3}{3} + H)\frac{1}{6}k^4\Delta x^4 + \frac{1}{48}(\frac{H^3}{3} + H)k^6\Delta x^6 + O(\Delta x^8)\right] \tag{14}$$

$$\mathcal{DF}_3^{u,h} = \frac{gH}{2}\mathcal{D}[\mathcal{R}_3^- + \mathcal{R}_3^+]$$

$$\mathcal{DF}_3^{u,h} = \frac{gH}{2}\left[2ik\Delta x + \frac{i}{12}k^3\Delta x^3 - \frac{-53i\Delta x^5}{720} + O(\Delta x^7)\right]$$

$$\begin{aligned} F_3^u = \frac{\sqrt{gH}}{2}&\left[-(\frac{H^3}{3} + H)\frac{1}{6}k^4\Delta x^3 + \frac{1}{48}(\frac{H^3}{3} + H)k^6\Delta x^5 + O(\Delta x^7)\right]u_j \\ &+ \frac{gH}{2}\left[2ik + \frac{i}{12}k^3\Delta x^2 - \frac{-53ik^5\Delta x^4}{720} + O(\Delta x^6)\right]h_j \end{aligned} \tag{15}$$

So yes our second order error in the flux continues over, it does however occur only in the imaginary direction, does that matter?

4 3rd

M was defined wrong, also what terms correspond to was a little out, so we have an error throughout, only effects third order.

$$R_3^- = 1 + \frac{i}{2}k\Delta x - \frac{1}{8}(k\Delta x)^2 - \frac{5}{48}(k\Delta x)^3 + O(\Delta x^4)$$

$$R_3^+ = 1 + \frac{i}{2}k\Delta x - \frac{1}{8}(k\Delta x)^2 + \frac{1}{16}(k\Delta x)^3 + O(\Delta x^4)$$

Now we report

$$\frac{1}{\mathcal{M}}\mathcal{D}\mathcal{F}_3^{h,u} = H\frac{\mathcal{D}}{\mathcal{M}}\mathcal{R}_3^u$$

$$\frac{\mathcal{D}}{\mathcal{M}}\mathcal{F}_3^{h,u} = H\left[ik\Delta x - \frac{9i}{320}k^5\Delta x^5 - \frac{i}{448}k^7\Delta x^7 + O(\Delta x^9)\right]$$

$$\frac{\mathcal{D}}{\mathcal{M}}\mathcal{F}_3^{h,h} = -\frac{\sqrt{gH}}{2}\frac{\mathcal{D}}{\mathcal{M}}\left[\mathcal{R}_3^+ - \mathcal{R}_3^-\right]$$

$$\frac{\mathcal{D}}{\mathcal{M}}\mathcal{F}_3^{h,h} = -\frac{\sqrt{gH}}{2}\left[-\frac{1}{6}k^4\Delta x^4 + \frac{1}{36}k^6\Delta x^6 - \frac{1}{480}k^8\Delta x^8 + O(\Delta x^{10})\right]$$

$$\frac{\mathcal{D}}{\mathcal{M}}\mathcal{F}_3^{u,u} = -\frac{\sqrt{gH}}{2}\frac{\mathcal{D}}{\mathcal{M}}\mathcal{G}_3\left[\mathcal{R}_3^+ \ - \mathcal{R}_3^-\right]$$

$$\frac{\mathcal{D}}{\mathcal{M}}\mathcal{F}_3^{u,u} = -\frac{\sqrt{gH}}{2}$$
$$\left[\frac{iH}{3}(H^2+3)k\Delta x + \frac{H}{6}(H^2+3)k^2\Delta x^2 - \frac{iH}{24}(H^2+3)k^3\Delta x^3 - \frac{H}{144}(H^2+3)k^4\Delta x^4\right]$$
$$\left[\frac{i}{6}k^3\Delta x^3 - \frac{1}{24}k^4\Delta x^4 - \frac{7i}{144}k^5\Delta x^5\right] \quad (16)$$

$$\frac{\mathcal{D}}{\mathcal{M}}\mathcal{F}_3^{u,u} = -\frac{\sqrt{gH}}{2}\left[-\frac{H}{18}(H^2+3)k^4\Delta x^4 + \frac{iH}{36}(H^2+3)k^5\Delta x^5 - \frac{iH}{72}(H^2+3)k^5\Delta x^5 + O(\Delta x^6)\right.$$

$$\frac{\mathcal{D}}{\mathcal{M}}\mathcal{F}_3^{u,u} = -\frac{\sqrt{gH}}{2}\left[-\frac{H}{18}(H^2+3)k^4\Delta x^4 + \frac{iH}{72}(H^2+3)k^5\Delta x^5 + O(\Delta x^6)\right] \tag{18}$$

$$\frac{\mathcal{D}}{\mathcal{M}}\mathcal{F}_3^{u,h} = \frac{gH}{2}\frac{\mathcal{D}}{\mathcal{M}}[\mathcal{R}_3^- + \mathcal{R}_3^+]$$

$$\frac{\mathcal{D}}{\mathcal{M}}\mathcal{F}_3^{u,h} = \frac{gH}{2}\left[2ik\Delta x - \frac{i}{15}k^5\Delta x^5 + O(\Delta x^7)\right]$$