1 Linearised Equations

$$G = uh - \frac{h^3}{3} u_{xx}$$

$$\eta_t + hu_x = 0$$

$$hu_t - \frac{h^3}{3} u_{xxt} + gh\eta_x = 0$$

$$(G)_t + gh\eta_x = 0$$

2 Numerical Approximation

We investigate our numerical technique by adding in a fourier mode so $W_j = W_0 e^{i(vt+kx_j)}$, and rewriting the equations using our spatial discretisation

2.1 G

Analytic:

$$G_j = u_j h_j - (\frac{h_j^3}{3} u_{xx})_j$$

Numerical approximation, we used second order central differences so we replace the second derivative of u with this approximation to it So we get

$$G_j = u_j h_j - \frac{h_j^3}{3} \left(\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} \right)$$

$$G_j = u_0 e^{i(vt+kx_j)} h_0 - \frac{h_0}{3} u_0 \left(\frac{e^{i(vt+kx_{j+1})} - 2e^{i(vt+kx_j)} + e^{i(vt+kx_{j-1})}}{\Delta x^2} \right)$$

$$G_j = u_0 e^{i(vt+kx_j)} h_0 - \frac{h_0}{3} u_0 \left(\frac{e^{i(vt+kx_j)+ik\Delta x} - 2e^{i(vt+kx_j)} + e^{i(vt+kx_j)+ik\Delta x}}{\Delta x^2} \right)$$

$$G_j = u_j h_0 - \frac{h_0}{3} u_j \left(\frac{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}{\Delta x^2} \right)$$

$$G_j = u_j \left(h_0 - \frac{h_0^3}{3} \left(\frac{2\cos\left(k\Delta x\right) - 2}{\Delta x^2} \right) \right)$$

We are dealing with time continuous variables so, we first take the derivative in time exactly for the Fourier nodes so that:

$$iv\eta_j + \left(h_j u_x\right)_j = 0$$

$$\left(G_t\right)_j + \left(gh\eta_x\right)_j = 0$$

$$\left(u_j \left(h_0 - \frac{h_0^3}{3} \left(\frac{2\cos\left(k\Delta x\right) - 2}{\Delta x^2} \right) \right) \right)_t + \left(gh\eta_x\right)_j = 0$$

$$ivG_j + \left(gh\eta_x\right)_j = 0$$

So we have

$$iv\eta_j + \left(hu_x\right)_j = 0$$

$$ivG_j + \left(gh\eta_x\right)_j = 0$$

Four our first step we need to preform the reconstruction to obtain the flux values at the edges., the only values we need to reconstruct are u and η (h fixed), we will just use centered differencing so that

$$u^+_{j+1/2} = u_{j+1} - \frac{u_{j+2} - u_j}{4}$$

$$u^-_{j+1/2} = u_j + \frac{u_{j+1} - u_{j-1}}{4}$$

We use our shift operators to get it in terms of $j+1$ so that

$$u^+_{j+1/2} = u_{j+1} - \frac{u_{j+1}e^{ik\Delta x} - u_{j+1}e^{-ik\Delta x}}{4}$$

$$u^+_{j+1/2} = u_{j+1} \left(1 - \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{4} \right)$$

$$u^+_{j+1/2} = u_j e^{ik\Delta x} \left(1 - \frac{2i\sin\left(ik\Delta x\right)}{4} \right)$$

$$u^+_{j+1/2} = u_j e^{ik\Delta x}\left(1 - \frac{i\sin\left(ik\Delta x\right)}{2}\right)$$

Now for the other side

$$u^-_{j+1/2} = u_j + \frac{u_j e^{ik\Delta x} - u_j e^{-ik\Delta x}}{4}$$

$$u^-_{j+1/2} = u_j\left(1 + \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{4}\right)$$

$$u^-_{j+1/2} = u_j\left(1 + \frac{2i\sin\left(ik\Delta x\right)}{4}\right)$$

$$u^-_{j+1/2} = u_j\left(1 + \frac{i\sin\left(ik\Delta x\right)}{2}\right)$$

So we have:

$$u^+_{j+1/2} = u_j e^{ik\Delta x}\left(1 - \frac{i\sin\left(ik\Delta x\right)}{2}\right)$$

$$u^-_{j+1/2} = u_j\left(1 + \frac{i\sin\left(ik\Delta x\right)}{2}\right)$$

Similarly

$$\eta^+_{j+1/2} = \eta_j e^{ik\Delta x}\left(1 - \frac{i\sin\left(ik\Delta x\right)}{2}\right)$$

$$\eta^-_{j+1/2} = \eta_j\left(1 + \frac{i\sin\left(ik\Delta x\right)}{2}\right)$$

For G we pick up the factor for u but then it becomes the same process:

$$G^+_{j+1/2} = u^+_{j+1/2}\left(h_0 - \frac{h_0^3}{3}\left(\frac{2\cos\left(k\Delta x\right) - 2}{\Delta x^2}\right)\right)$$

$$G^-_{j+1/2} = u^+_{j+1/2}\left(h_0 - \frac{h_0^3}{3}\left(\frac{2\cos\left(k\Delta x\right) - 2}{\Delta x^2}\right)\right)$$

Thus

$$G^+_{j+1/2} = u_j e^{ik\Delta x}\left(h_0 - \frac{h_0^3}{3}\left(\frac{2\cos\left(k\Delta x\right) - 2}{\Delta x^2}\right)\right)\left(1 - \frac{i\sin\left(ik\Delta x\right)}{2}\right)$$

$$G^-_{j+1/2} = u_j \left(h_0 - \frac{h_0^3}{3}\left(\frac{2\cos\left(k\Delta x\right) - 2}{\Delta x^2}\right)\right)\left(1 + \frac{i\sin\left(ik\Delta x\right)}{2}\right)$$

We will use the follow to denote this recontructioin part:

$$\mathcal{S}^+ = e^{ik\Delta x}\left(1 - \frac{i\sin\left(ik\Delta x\right)}{2}\right)$$

$$\mathcal{S}^- = \left(1 + \frac{i\sin\left(ik\Delta x\right)}{2}\right)$$

From these we calculate $F^+_{j+1/2}$, $F^-_{j+1/2}$, $a^+_{j+1/2}$ and $a^-_{j+1/2}$.
We first will just choose the minus point for our wave speeds so that

$$a^-_{j+1/2} = u_j\mathcal{S}^- - \sqrt{g}\sqrt{h_0 + \eta_j\mathcal{S}^-}$$

$$a^-_{j+1/2} = u_j\mathcal{S}^- - \sqrt{gh_0}\sqrt{1 + \frac{\eta_j}{h_0}\mathcal{S}^-}$$

$$a^-_{j+1/2} = u_j\mathcal{S}^- - \sqrt{gh_0}\left(1 + \frac{1}{2}\frac{\eta_j}{h_0}\mathcal{S}^- + \left(\frac{1}{2}\frac{\eta_j}{h_0}\mathcal{S}^-\right)^2 + \cdots\right)$$

but we dropped the ϵ^2 terms before so lets do it again

$$a^-_{j+1/2} = u_j\mathcal{S}^- - \sqrt{gh_0}\left(1 + \frac{1}{2}\frac{\eta_j}{h_0}\mathcal{S}^-\right)$$

Similarly

$$a^+_{j+1/2} = u_j\mathcal{S}^+ + \sqrt{gh_0}\left(1 + \frac{1}{2}\frac{\eta_j}{h_0}\mathcal{S}^+\right)$$

We'll just use the placeholders to make writing a formula for the matrix determinant easier.

Our flux approximations are claculated like so

$$F_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}} f\left(q^-_{j+\frac{1}{2}}\right) - a^-_{j+\frac{1}{2}} f\left(q^+_{j+\frac{1}{2}}\right)}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \frac{a^+_{j+\frac{1}{2}}\, a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} \left[q^+_{j+\frac{1}{2}} - q^-_{j+\frac{1}{2}}\right] \tag{1}$$

For mass

$$F_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}} h_0 u^-_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}} h_0 u^+_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \frac{a^+_{j+\frac{1}{2}}\, a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} \left[\eta^+_{j+\frac{1}{2}} - \eta^-_{j+\frac{1}{2}}\right] \tag{2}$$

$$F_{j+\frac{1}{2}} = \frac{1}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} \left(a^+_{j+\frac{1}{2}} h_0 u^-_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}} h_0 u^+_{j+\frac{1}{2}} + a^+_{j+\frac{1}{2}}\, a^-_{j+\frac{1}{2}} \left[\eta^+_{j+\frac{1}{2}} - \eta^-_{j+\frac{1}{2}}\right]\right) \tag{3}$$

$$F_{j+\frac{1}{2}} = \frac{1}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} \left(a^+_{j+\frac{1}{2}} h_0 u_j \mathcal{S}^- - a^-_{j+\frac{1}{2}} h_0 u_j \mathcal{S}^+ + a^+_{j+\frac{1}{2}}\, a^-_{j+\frac{1}{2}} \left[\eta_j \mathcal{S}^+ - \eta_j \mathcal{S}^-\right]\right) \tag{4}$$

$$F_{j+\frac{1}{2}} = \frac{1}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} \left(h_0 u_j \left(a^+_{j+\frac{1}{2}} \mathcal{S}^- - a^-_{j+\frac{1}{2}} \mathcal{S}^+\right) + \eta_j a^+_{j+\frac{1}{2}}\, a^-_{j+\frac{1}{2}} \left[\mathcal{S}^+ - \mathcal{S}^-\right]\right) \tag{5}$$

$$F_{j+\frac{1}{2}} = h_0 u_j \left(\frac{a^+_{j+\frac{1}{2}} \mathcal{S}^- - a^-_{j+\frac{1}{2}} \mathcal{S}^+}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\right) + \eta_j \left(\frac{a^+_{j+\frac{1}{2}}\, a^-_{j+\frac{1}{2}} \left[\mathcal{S}^+ - \mathcal{S}^-\right]}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\right) \tag{6}$$

We can just apply the shift operator to do this but for $j-1$ to get $F_{j-\frac{1}{2}}$

$$F_{j-\frac{1}{2}} = h_0 u_{j-1} \left(\frac{a^+_{j-\frac{1}{2}} \mathcal{S}^- - a^-_{j-\frac{1}{2}} \mathcal{S}^+}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\right) + \eta_{j-1} \left(\frac{a^+_{j-\frac{1}{2}}\, a^-_{j-\frac{1}{2}} \left[\mathcal{S}^+ - \mathcal{S}^-\right]}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\right) \tag{7}$$

$$F_{j-\frac{1}{2}} = h_0 u_j e^{-ik\Delta x}\left(\frac{a^+_{j-\frac{1}{2}}\mathcal{S}^- - a^-_{j-\frac{1}{2}}\mathcal{S}^+}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\right) + \eta_j e^{-ik\Delta x}\left(\frac{a^+_{j-\frac{1}{2}}\,a^-_{j-\frac{1}{2}}\,[\mathcal{S}^+ - \mathcal{S}^-]}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\right) \tag{8}$$

$$F_{j-\frac{1}{2}} = e^{-ik\Delta x}\left(h_0 u_j\left(\frac{a^+_{j-\frac{1}{2}}\mathcal{S}^- - a^-_{j-\frac{1}{2}}\mathcal{S}^+}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\right) + \eta_j\left(\frac{a^+_{j-\frac{1}{2}}\,a^-_{j-\frac{1}{2}}\,[\mathcal{S}^+ - \mathcal{S}^-]}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\right)\right) \tag{9}$$

So our approximation for momentum becomes:

$$iv\eta_j + \frac{1}{\Delta x}\left(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right) = 0$$

$$\begin{aligned} iv\eta_j + \frac{1}{\Delta x}[h_0 u_j\left(\frac{a^+_{j+\frac{1}{2}}\mathcal{S}^- - a^-_{j+\frac{1}{2}}\mathcal{S}^+}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\right) + \eta_j\left(\frac{a^+_{j+\frac{1}{2}}\,a^-_{j+\frac{1}{2}}\,[\mathcal{S}^+ - \mathcal{S}^-]}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\right) \\ - e^{-ik\Delta x}\left(h_0 u_j\left(\frac{a^+_{j-\frac{1}{2}}\mathcal{S}^- - a^-_{j-\frac{1}{2}}\mathcal{S}^+}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\right) + \eta_j\left(\frac{a^+_{j-\frac{1}{2}}\,a^-_{j-\frac{1}{2}}\,[\mathcal{S}^+ - \mathcal{S}^-]}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\right)\right)] = 0 \end{aligned} \tag{10}$$

$$\begin{aligned} iv\eta_j + \frac{1}{\Delta x}[h_0 u_j\left(\frac{a^+_{j+\frac{1}{2}}\mathcal{S}^- - a^-_{j+\frac{1}{2}}\mathcal{S}^+}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\right) - e^{-ik\Delta x}\left(h_0 u_j\left(\frac{a^+_{j-\frac{1}{2}}\mathcal{S}^- - a^-_{j-\frac{1}{2}}\mathcal{S}^+}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\right)\right)] \\ + \frac{1}{\Delta x}[\eta_j\left(\frac{a^+_{j+\frac{1}{2}}\,a^-_{j+\frac{1}{2}}\,[\mathcal{S}^+ - \mathcal{S}^-]}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\right) - e^{-ik\Delta x}\left(\eta_j\left(\frac{a^+_{j-\frac{1}{2}}\,a^-_{j-\frac{1}{2}}\,[\mathcal{S}^+ - \mathcal{S}^-]}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\right)\right)] = 0 \end{aligned} \tag{11}$$

$$\begin{aligned} iv\eta_j + \frac{h_0 u_j}{\Delta x}\left[\left(\frac{a^+_{j+\frac{1}{2}}\mathcal{S}^- - a^-_{j+\frac{1}{2}}\mathcal{S}^+}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\right) - e^{-ik\Delta x}\left(\frac{a^+_{j-\frac{1}{2}}\mathcal{S}^- - a^-_{j-\frac{1}{2}}\mathcal{S}^+}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\right)\right] \\ + \frac{\eta_j}{\Delta x}\left[\left(\frac{a^+_{j+\frac{1}{2}}\,a^-_{j+\frac{1}{2}}\,[\mathcal{S}^+ - \mathcal{S}^-]}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\right) - e^{-ik\Delta x}\left(\frac{a^+_{j-\frac{1}{2}}\,a^-_{j-\frac{1}{2}}\,[\mathcal{S}^+ - \mathcal{S}^-]}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\right)\right] = 0 \end{aligned} \tag{12}$$

I will now introduce $\mathcal{F}$ like so

$$\mathcal{F} = \left[\left(\frac{a^+_{j+\frac{1}{2}}\mathcal{S}^- - a^-_{j+\frac{1}{2}}\mathcal{S}^+}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\right) - e^{-ik\Delta x}\left(\frac{a^+_{j-\frac{1}{2}}\mathcal{S}^- - a^-_{j-\frac{1}{2}}\mathcal{S}^+}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\right)\right]$$

thus

$$iv\eta_j + \frac{h_0 u_j}{\Delta x}\mathcal{F} + \frac{\eta_j}{\Delta x}\mathcal{F} = 0$$

$$\eta_j\left(iv + \frac{\mathcal{F}}{\Delta x}\right) + \frac{h_0\mathcal{F}}{\Delta x}u_j = 0$$

Now for momentum we pick up that factor

$$\mathcal{G} = \left(h_0 - \frac{h_0^3}{3}\left(\frac{2\cos\left(k\Delta x\right) - 2}{\Delta x^2}\right)\right)$$

on u

thus

$$ivu_j\mathcal{G} + \frac{1}{\Delta x}\left(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right) = 0$$

where

$$F_{j+\frac{1}{2}} = gh_0\eta_j\left(\frac{a^+_{j+\frac{1}{2}}\mathcal{S}^- - a^-_{j+\frac{1}{2}}\mathcal{S}^+}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\right) + u_j\mathcal{G}\left(\frac{a^+_{j+\frac{1}{2}}\,a^-_{j+\frac{1}{2}}\left[\mathcal{S}^+ - \mathcal{S}^-\right]}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\right) \quad (13)$$

$$F_{j-\frac{1}{2}} = e^{-ik\Delta x}\left(gh_0\eta_j\left(\frac{a^+_{j-\frac{1}{2}}\mathcal{S}^- - a^-_{j-\frac{1}{2}}\mathcal{S}^+}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\right) + u_j\mathcal{G}\left(\frac{a^+_{j-\frac{1}{2}}\,a^-_{j-\frac{1}{2}}\left[\mathcal{S}^+ - \mathcal{S}^-\right]}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\right)\right) \quad (14)$$

so

$$
\begin{aligned}
&iv u_j\mathcal{G}+\frac{1}{\Delta x}\left(gh_0\eta_j\left(\frac{a^+_{j+\frac{1}{2}}\mathcal{S}^- - a^-_{j+\frac{1}{2}}\mathcal{S}^+}{a^+_{j+\frac{1}{2}}-a^-_{j+\frac{1}{2}}}\right)+u_j\mathcal{G}\left(\frac{a^+_{j+\frac{1}{2}}\,a^-_{j+\frac{1}{2}}\left[\mathcal{S}^+-\mathcal{S}^-\right]}{a^+_{j+\frac{1}{2}}-a^-_{j+\frac{1}{2}}}\right)\right)\\
&-\frac{1}{\Delta x}\left(e^{-ik\Delta x}\left(gh_0\eta_j\left(\frac{a^+_{j-\frac{1}{2}}\mathcal{S}^- - a^-_{j-\frac{1}{2}}\mathcal{S}^+}{a^+_{j-\frac{1}{2}}-a^-_{j-\frac{1}{2}}}\right)+u_j\mathcal{G}\left(\frac{a^+_{j-\frac{1}{2}}\,a^-_{j-\frac{1}{2}}\left[\mathcal{S}^+-\mathcal{S}^-\right]}{a^+_{j-\frac{1}{2}}-a^-_{j-\frac{1}{2}}}\right)\right)\right)=0
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
iv u_j\mathcal{G}+\frac{1}{\Delta x}&\left(gh_0\eta_j\left(\frac{a^+_{j+\frac{1}{2}}\mathcal{S}^- - a^-_{j+\frac{1}{2}}\mathcal{S}^+}{a^+_{j+\frac{1}{2}}-a^-_{j+\frac{1}{2}}}\right)\right)\\
&-\frac{1}{\Delta x}\left(e^{-ik\Delta x}\left(gh_0\eta_j\left(\frac{a^+_{j-\frac{1}{2}}\mathcal{S}^- - a^-_{j-\frac{1}{2}}\mathcal{S}^+}{a^+_{j-\frac{1}{2}}-a^-_{j-\frac{1}{2}}}\right)\right)\right)\\
&\quad+\frac{1}{\Delta x}\left(u_j\mathcal{G}\left(\frac{a^+_{j+\frac{1}{2}}\,a^-_{j+\frac{1}{2}}\left[\mathcal{S}^+-\mathcal{S}^-\right]}{a^+_{j+\frac{1}{2}}-a^-_{j+\frac{1}{2}}}\right)\right)\\
&\quad-\frac{1}{\Delta x}\left(e^{-ik\Delta x}\left(u_j\mathcal{G}\left(\frac{a^+_{j-\frac{1}{2}}\,a^-_{j-\frac{1}{2}}\left[\mathcal{S}^+-\mathcal{S}^-\right]}{a^+_{j-\frac{1}{2}}-a^-_{j-\frac{1}{2}}}\right)\right)\right)=0
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
&iv u_j\mathcal{G}+\frac{1}{\Delta x}\left(gh_0\eta_j\left(\frac{a^+_{j+\frac{1}{2}}\mathcal{S}^- - a^-_{j+\frac{1}{2}}\mathcal{S}^+}{a^+_{j+\frac{1}{2}}-a^-_{j+\frac{1}{2}}}\right)\right)-\frac{1}{\Delta x}\left(e^{-ik\Delta x}\left(gh_0\eta_j\left(\frac{a^+_{j-\frac{1}{2}}\mathcal{S}^- - a^-_{j-\frac{1}{2}}\mathcal{S}^+}{a^+_{j-\frac{1}{2}}-a^-_{j-\frac{1}{2}}}\right)\right)\right)\\
&+\frac{1}{\Delta x}\left(u_j\mathcal{G}\left(\frac{a^+_{j+\frac{1}{2}}\,a^-_{j+\frac{1}{2}}\left[\mathcal{S}^+-\mathcal{S}^-\right]}{a^+_{j+\frac{1}{2}}-a^-_{j+\frac{1}{2}}}\right)\right)-\frac{1}{\Delta x}\left(e^{-ik\Delta x}\left(u_j\mathcal{G}\left(\frac{a^+_{j-\frac{1}{2}}\,a^-_{j-\frac{1}{2}}\left[\mathcal{S}^+-\mathcal{S}^-\right]}{a^+_{j-\frac{1}{2}}-a^-_{j-\frac{1}{2}}}\right)\right)\right)=0
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
&iv u_j\mathcal{G}+gh_0\eta_j\frac{1}{\Delta x}\left(\left(\frac{a^+_{j+\frac{1}{2}}\mathcal{S}^- - a^-_{j+\frac{1}{2}}\mathcal{S}^+}{a^+_{j+\frac{1}{2}}-a^-_{j+\frac{1}{2}}}\right)-e^{-ik\Delta x}\left(\frac{a^+_{j-\frac{1}{2}}\mathcal{S}^- - a^-_{j-\frac{1}{2}}\mathcal{S}^+}{a^+_{j-\frac{1}{2}}-a^-_{j-\frac{1}{2}}}\right)\right)\\
&+u_j\mathcal{G}\frac{1}{\Delta x}\left(\left(\frac{a^+_{j+\frac{1}{2}}\,a^-_{j+\frac{1}{2}}\left[\mathcal{S}^+-\mathcal{S}^-\right]}{a^+_{j+\frac{1}{2}}-a^-_{j+\frac{1}{2}}}\right)\right)-e^{-ik\Delta x}\left(\frac{a^+_{j-\frac{1}{2}}\,a^-_{j-\frac{1}{2}}\left[\mathcal{S}^+-\mathcal{S}^-\right]}{a^+_{j-\frac{1}{2}}-a^-_{j-\frac{1}{2}}}\right)=0
\end{aligned}
\tag{18}
$$

$$ivu_j\mathcal{G}+gh_0\eta_j\frac{1}{\Delta x}\left(\left(\frac{a^+_{j+\frac{1}{2}}\mathcal{S}^- - a^-_{j+\frac{1}{2}}\mathcal{S}^+}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\right) - e^{-ik\Delta x}\left(\frac{a^+_{j-\frac{1}{2}}\mathcal{S}^- - a^-_{j-\frac{1}{2}}\mathcal{S}^+}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\right)\right)$$
$$+u_j\mathcal{G}\frac{1}{\Delta x}\left(\left(\frac{a^+_{j+\frac{1}{2}}\,a^-_{j+\frac{1}{2}}\left[\mathcal{S}^+ - \mathcal{S}^-\right]}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\right)\right) - e^{-ik\Delta x}\left(\frac{a^+_{j-\frac{1}{2}}\,a^-_{j-\frac{1}{2}}\left[\mathcal{S}^+ - \mathcal{S}^-\right]}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\right) = 0 \tag{19}$$

$$ivu_j\mathcal{G} + gh_0\eta_j\frac{1}{\Delta x}\left(\mathcal{F}\right) + u_j\mathcal{G}\frac{1}{\Delta x}\left(\mathcal{F}\right) = 0$$

$$u_j\left(iv\mathcal{G}\frac{1}{\Delta x}\mathcal{F}\right) + \eta_j\left(gh_0\frac{1}{\Delta x}\mathcal{F}\right) = 0$$

So we have

$$\eta_j\left(iv + \frac{\mathcal{F}}{\Delta x}\right) + \frac{h_0\mathcal{F}}{\Delta x}u_j = 0$$

$$u_j\left(iv\mathcal{G}\frac{1}{\Delta x}\mathcal{F}\right) + \eta_j\left(gh_0\frac{1}{\Delta x}\mathcal{F}\right) = 0$$

In matrix form we have

$$\begin{bmatrix} iv + \frac{1}{\Delta x}\mathcal{F} & h_0\frac{1}{\Delta x}\mathcal{F} \\ gh_0\frac{1}{\Delta x}\mathcal{F} & iv\mathcal{G}\frac{1}{\Delta x}\mathcal{F} \end{bmatrix}\begin{bmatrix} \eta_j \\ u_j \end{bmatrix} = 0$$

This admits a nontrivial solution when

$$\left(iv + \frac{1}{\Delta x}\mathcal{F}\right)\left(iv\mathcal{G}\frac{1}{\Delta x}\mathcal{F}\right) - \left(h_0\frac{1}{\Delta x}\mathcal{F}\right)\left(gh_0\frac{1}{\Delta x}\mathcal{F}\right) = 0$$

$$\left(-v^2\mathcal{G}\frac{1}{\Delta x}\mathcal{F}\right) + \left(iv\mathcal{G}\frac{1}{\Delta x^2}\mathcal{F}^2\right) - \frac{gh_0^2}{\Delta x^2}\mathcal{F}^2 = 0$$

This is a quadratic in v with the following solutions

$$v = -\frac{i\mathcal{G}\frac{1}{\Delta x^2}\mathcal{F}^2 \pm \sqrt{\left(i\mathcal{G}\frac{1}{\Delta x^2}\mathcal{F}^2\right)^2 - 4\left(\mathcal{G}\frac{1}{\Delta x}\mathcal{F}\frac{gh_0^2}{\Delta x^2}\mathcal{F}^2\right)}}{-2\mathcal{G}\frac{1}{\Delta x}\mathcal{F}}$$

$$v = -\frac{i\mathcal{G}\frac{1}{\Delta x^2}\mathcal{F}^2 \pm \sqrt{\left(-\mathcal{G}^2\frac{1}{\Delta x^4}\mathcal{F}^4\right) - 4\left(\mathcal{G}\mathcal{F}^3\frac{gh_0^2}{\Delta x^3}\right)}}{-2\mathcal{G}\frac{1}{\Delta x}\mathcal{F}}$$

$$v = -\frac{i\mathcal{G}\frac{1}{\Delta x^2}\mathcal{F}^2 \pm \frac{1}{\Delta x}\mathcal{F}\sqrt{\left(-\mathcal{G}^2\frac{1}{\Delta x^2}\mathcal{F}^2\right) - 4\left(\mathcal{G}\mathcal{F}\frac{gh_0^2}{\Delta x}\right)}}{-2\mathcal{G}\frac{1}{\Delta x}\mathcal{F}}$$

$$v = -\frac{i\mathcal{G}\frac{1}{\Delta x}\mathcal{F} \pm \sqrt{\left(-\mathcal{G}^2\frac{1}{\Delta x^2}\mathcal{F}^2\right) - 4\left(\mathcal{G}\mathcal{F}\frac{gh_0^2}{\Delta x}\right)}}{-2\mathcal{G}}$$

$$v = \frac{i\mathcal{G}\frac{1}{\Delta x}\mathcal{F} \pm i\sqrt{\mathcal{G}^2\frac{1}{\Delta x^2}\mathcal{F}^2 + 4\mathcal{G}\mathcal{F}\frac{gh_0^2}{\Delta x}}}{2\mathcal{G}}$$

method fix k, then vary Delta x to produce plots for different schemes, only difference should be $\mathcal{G}$ and $\mathcal{F}$, and the only difference there should be $\mathcal{G}$ and $\mathcal{S}^{\pm}$, and a's should be different as well. Could do these numerically as well. Write up code to calculate v.