1 Numerical Method Break Down

Our conservative update is, for our equations is

$$\bar{q}_j^{n+1} = \bar{q}_j^n - \frac{\Delta t}{\Delta x}\left[F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right]$$

This converts to (both analytical and numerical)

$$\mathcal{M}q_j^{n+1} = \mathcal{M}q_j^n - \frac{\Delta t}{\Delta x}\left[\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j - \mathcal{F}^{q,v}v_{j-1} - \mathcal{F}^{q,q}q_{j-1}\right]$$

$$\mathcal{M}q_j^{n+1} = \mathcal{M}q_j^n - \frac{\Delta t}{\Delta x}\left[\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j - \mathcal{F}^{q,v}e^{-ik\Delta x}v_j - \mathcal{F}^{q,q}e^{-ik\Delta x}q_j\right]$$

Defining $\mathcal{D}_x = 1 - e^{-ik\Delta x}$

$$\mathcal{M}q_j^{n+1} = \mathcal{M}q_j^n - \frac{\Delta t}{\Delta x}\left[\mathcal{D}_x\mathcal{F}^{q,v}v_j + \mathcal{D}_x\mathcal{F}^{q,q}q_j\right]$$

So we have

$$q_j^{n+1} = q_j^n - \frac{\mathcal{D}_x\Delta t}{\mathcal{M}\Delta x}\left[\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j\right]$$

Thus we have

$$\begin{bmatrix} h \\ \mathcal{G}u \end{bmatrix}_j^{n+1} = \begin{bmatrix} h \\ \mathcal{G}u \end{bmatrix}_j^{n} - \frac{\mathcal{D}_x\Delta t}{\mathcal{M}\Delta x}\begin{bmatrix} \mathcal{F}^{h,h} & \mathcal{F}^{h,u} \\ \mathcal{F}^{u,h} & \mathcal{F}^{u,u} \end{bmatrix}\begin{bmatrix} h \\ u \end{bmatrix}_j^{n}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \begin{bmatrix} h \\ u \end{bmatrix}_j^{n} - \frac{\mathcal{D}_x\Delta t}{\mathcal{M}\Delta x}\begin{bmatrix} \mathcal{F}^{h,h} & \mathcal{F}^{h,u} \\ \frac{1}{\mathcal{G}}\mathcal{F}^{u,h} & \frac{1}{\mathcal{G}}\mathcal{F}^{u,u} \end{bmatrix}\begin{bmatrix} h \\ u \end{bmatrix}_j^{n}$$

Lets define

$$\boldsymbol{F} = \frac{\mathcal{D}_x}{\mathcal{M}\Delta x}\begin{bmatrix} \mathcal{F}^{h,h} & \mathcal{F}^{h,u} \\ \frac{1}{\mathcal{G}}\mathcal{F}^{u,h} & \frac{1}{\mathcal{G}}\mathcal{F}^{u,u} \end{bmatrix}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \begin{bmatrix} h \\ u \end{bmatrix}_j^{n} - \Delta t\boldsymbol{F}\begin{bmatrix} h \\ u \end{bmatrix}_j^{n}$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = (\boldsymbol{I} - \Delta t \boldsymbol{F}) \left[\begin{array}{c} h \\ u \end{array}\right]_j^n$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} - \left[\begin{array}{c} h \\ u \end{array}\right]_j^n = -\Delta t \boldsymbol{F} \left[\begin{array}{c} h \\ u \end{array}\right]_j^n$$

$$e^{i\omega \Delta t} \left[\begin{array}{c} h \\ u \end{array}\right]_j^n - \left[\begin{array}{c} h \\ u \end{array}\right]_j^n = -\Delta t \boldsymbol{F} \left[\begin{array}{c} h \\ u \end{array}\right]_j^n$$

$$\left(e^{i\omega \Delta t} - 1\right) \left[\begin{array}{c} h \\ u \end{array}\right]_j^n = -\Delta t \boldsymbol{F} \left[\begin{array}{c} h \\ u \end{array}\right]_j^n$$

$$\frac{e^{i\omega \Delta t} - 1}{\Delta t} \left[\begin{array}{c} h \\ u \end{array}\right]_j^n = -\boldsymbol{F} \left[\begin{array}{c} h \\ u \end{array}\right]_j^n$$