1 Elliptic Equation

The linearised elliptic equation is

$$G = Hu - \frac{H^3}{3}u_{xx}$$

Want to find out the FEM approximation to this G' such that

$$G' = \mathcal{G}_{FE_1}u$$

for P^1 FEM

$$G' = \mathcal{G}_{FE_2}u$$

for P^2 FEM.

To do so we begin by first multiplying by an arbitrary test function v so that

$$Gv = Huv - \frac{H^3}{3}u_{xx}v$$

and then we integrate over the entire domain to get

$$\int_\Omega Gvdx = \int_\Omega Huvdx - \int_\Omega \frac{H^3}{3}u_{xx}vdx$$

for all v

We then make use of integration by parts, with Dirchlet boundaries to get

$$\int_\Omega Gvdx = \int_\Omega Huvdx + \int_\Omega \frac{H^3}{3}u_x v_x dx$$

Our FVM discretisation already has a natrual structure with intervals of like $[x_{j-1/2}, x_{j+1/2}]$

So we can reformulate this as

$$\sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} Gvdx = \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} Huvdx + \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{H^3}{3}u_{xx}vdx$$

or more aptly

$$\sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} Gvdx - \int_{x_{j-1/2}}^{x_{j+1/2}} Huvdx - \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{H^3}{3} u_{xx} vdx = 0$$

for all v

$$\sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} Gvdx - H \int_{x_{j-1/2}}^{x_{j+1/2}} uvdx - \frac{H^3}{3} \int_{x_{j-1/2}}^{x_{j+1/2}} u_{xx} vdx = 0$$