1 Linearisation

So we do a linearisation for a pertubation on still water so (keep $O(\epsilon)$ terms)

$$h(x,t) = H + \epsilon\eta(x,t)$$
$$h = H + \epsilon\eta$$

$$u(x,t) = 0 + \epsilon v(x,t)$$

$$u = \epsilon v$$

We start with G

$$G = uh - \left(\frac{h^3}{3}u_x\right)_x$$

$$G = (\epsilon v)\left(H + \epsilon\eta\right) - h^2 h_x u_x - \left(\frac{h^3}{3}u_{xx}\right)$$

$$G = \epsilon vH + \epsilon^2 v\eta - \left(H + \epsilon\eta\right)^2 \epsilon\eta_x \epsilon v_x - \left(\frac{H + \epsilon\eta^3}{3}\right)\epsilon v_{xx}$$

$$G = \epsilon vH - \left(\frac{H + \epsilon\eta}{3}\right)^3 \epsilon v_{xx}$$

$$G = \epsilon vH - \frac{H^3}{3}\epsilon v_{xx}$$

Mass equation

$$h_t + uh_x + u_x h = 0$$

$$\left(H + \epsilon\eta\right)_t + \epsilon v\left(H + \epsilon\eta\right)_x + \epsilon v_x\left(H + \epsilon\eta\right) = 0$$

$$\epsilon\eta_t + \epsilon H v_x = 0$$

Momentum equation:

$$G_t + \left(Gu + \frac{gh^2}{2} - \frac{2h^3}{3}u_x u_x\right)_x = 0$$

$$\left(uh-\left(\frac{h^3}{3}u_x\right)_x\right)_t+\left(\left(uh-\left(\frac{h^3}{3}u_x\right)_x\right)u+\frac{gh^2}{2}-\frac{2h^3}{3}u_xu_x\right)_x=0$$

$$(uh)_t-\left(\frac{h^3}{3}u_x\right)_{xt}+\left(u^2h-u\left(\frac{h^3}{3}u_x\right)_x+\frac{gh^2}{2}-\frac{2h^3}{3}u_xu_x\right)_x=0$$

$$u_th+uh_t-\left(\frac{h^3}{3}u_x\right)_{xt}+\left(u^2h\right)_x+\left(-u\left(\frac{h^3}{3}u_x\right)_x\right)_x+\left(\frac{gh^2}{2}\right)_x+\left(-\frac{2h^3}{3}u_xu_x\right)_x=0$$

$$H\epsilon v_t-\left(h^2h_xu_x+\frac{h^3}{3}u_{xx}\right)_t+\left(u^2h\right)_x+\left(-u\left(\frac{h^3}{3}u_x\right)_x\right)_x+\left(\frac{gh^2}{2}\right)_x+\left(-\frac{2h^3}{3}u_xu_x\right)_x=0$$

$$H\epsilon v_t-\frac{H^3}{3}\epsilon v_{xxt}+\left(u^2h\right)_x+\left(-u\left(\frac{h^3}{3}u_x\right)_x\right)_x+\left(\frac{gh^2}{2}\right)_x+\left(-\frac{2h^3}{3}u_xu_x\right)_x=0$$

$$H\epsilon v_t-\frac{H^3}{3}\epsilon v_{xxt}+2uu_xh+u^2h_x+\left(-u\left(\frac{h^3}{3}u_x\right)_x\right)_x+\left(\frac{gh^2}{2}\right)_x+\left(-\frac{2h^3}{3}u_xu_x\right)_x=0$$

$$H\epsilon v_t-\frac{H^3}{3}\epsilon v_{xxt}+\left(-u\left(\frac{h^3}{3}u_x\right)_x\right)_x+\left(\frac{gh^2}{2}\right)_x+\left(-\frac{2h^3}{3}u_xu_x\right)_x=0$$

$$H\epsilon v_t-\frac{H^3}{3}\epsilon v_{xxt}+\left(-u\left(h^2h_xu_x+\frac{h^3}{3}u_{xx}\right)\right)_x+\left(\frac{gh^2}{2}\right)_x+\left(-\frac{2h^3}{3}u_xu_x\right)_x=0$$

$$H\epsilon v_t-\frac{H^3}{3}\epsilon v_{xxt}+\left(-uh^2h_xu_x-u\frac{h^3}{3}u_{xx}\right)_x+\left(\frac{gh^2}{2}\right)_x+\left(-\frac{2h^3}{3}u_xu_x\right)_x=0$$

$$H\epsilon v_t-\frac{H^3}{3}\epsilon v_{xxt}+\left(-uh^2h_xu_x\right)_x+\left(-u\frac{h^3}{3}u_{xx}\right)_x+\left(\frac{gh^2}{2}\right)_x+\left(-\frac{2h^3}{3}u_xu_x\right)_x=0$$

$$H\epsilon v_t - \frac{H^3}{3}\epsilon v_{xxt} + (-u_x h^2 h_x u_x) + (-uhh_xh_xu_x)_x + (-uh^2h_{xx}u_x) + (-uh^2h_xu_{xx}) + \left(-u\frac{h^3}{3}u_{xx}\right)_x + \Bigg($$

$$H\epsilon v_t - \frac{H^3}{3}\epsilon v_{xxt} + \left(-u\frac{h^3}{3}u_{xx}\right)_x + \left(\frac{gh^2}{2}\right)_x + \left(-\frac{2h^3}{3}u_xu_x\right)_x = 0$$

$$H\epsilon v_t - \frac{H^3}{3}\epsilon v_{xxt} - \epsilon v\frac{(h+\epsilon\eta)^3}{3}\epsilon v_{xxx} + \left(\frac{gh^2}{2}\right)_x + \left(-\frac{2h^3}{3}u_xu_x\right)_x = 0$$

$$H\epsilon v_t - \frac{H^3}{3}\epsilon v_{xxt} + \left(\frac{gh^2}{2}\right)_x = 0$$

$$H\epsilon v_t - \frac{H^3}{3}\epsilon v_{xxt} + gH\epsilon\eta_x = 0$$

So we have

$$G = \epsilon vH - \frac{H^3}{3}\epsilon v_{xx}$$

$$\epsilon\eta_t + \epsilon Hv_x = 0$$

$$H\epsilon v_t - \frac{H^3}{3}\epsilon v_{xxt} + gH\epsilon\eta_x = 0$$

using u's instead of v's, incorporating epsilon into its relevant term and using h instead of H we get.

$$G = uh - \frac{h^3}{3}u_{xx}$$

$$\eta_t + hu_x = 0$$

$$hu_t - \frac{h^3}{3}u_{xxt} + gh\eta_x = 0$$