Non-linear dispersive water wave models

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Outline

- Motivation
- Equations
- Linear Theory
- Comparison To Numerical Solutions

Motivation - Water Waves

We require accurate models of water waves to understand natural hazards in particular

- Tsunamis
- Storm Surges

Motivation

(a) Sulawesi Tsunami (Indonesia, 2018).

(b) Hurricane Florence (U.S.A, 2018)

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What's the effect of dispersion on these natural hazards?

Today's Focus

- Compare linear theory and numerical solutions for the models of interest.

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Which family of equations?

Depth Averaged Set Up

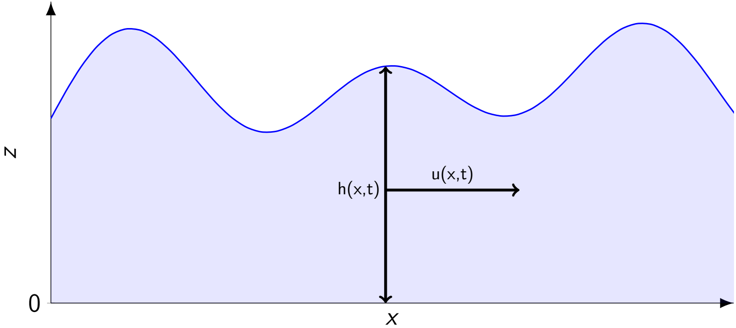


Figure: Relevant Quantities.

Generalised Serre-Green-Naghdi Equations

$$
\begin{aligned}
&\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0 \\
&\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2 + \beta_1 \Phi - \beta_2 \Psi\right) = 0
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi &= \frac{h^3}{2}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x \partial t} - u\frac{\partial^2 u}{\partial x^2}\right) \\
\Psi &= \frac{gh^2}{2}\left(h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)
\end{aligned}
$$

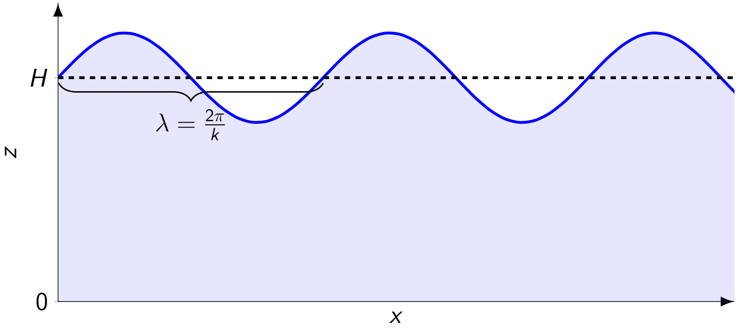
Generalised Serre-Green-Naghdi Equations - Properties

- Conserves mass, momentum and energy
- Reduces to well known equations for particular β values
- Depth averaged equations have been very successful for large scale models
- Nice linear dispersion properties as well

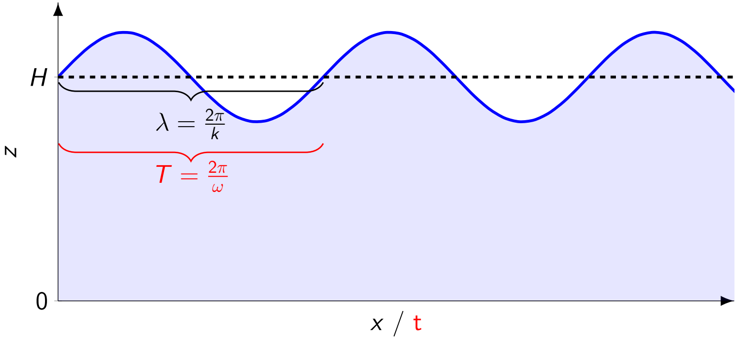
Linear Dispersion Relationship

- Linearise equations - considering waves of small amplitude
- Relate angular frequency (ω) to wave number (k)

Linearise equations (Space)



Linearise equations (Space/ Time)



Dispersion Relations of generalised Serre-Green-Naghdi and Water (linearised)

$$\omega_{\text{gSGN}} = k\sqrt{gH}\sqrt{\frac{\beta_2 H^2 k^2 + 2}{\beta_1 H^2 k^2 + 2}}$$

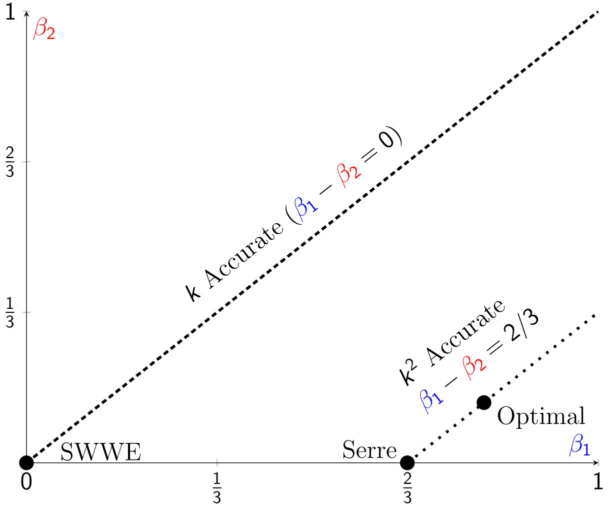
$$\omega_{\text{water}} = \sqrt{gk\tanh\left(kH\right)}$$

Taylor Series Expansions

$$\omega_{\text{gSGN}} = \sqrt{gH}k \quad - \sqrt{\frac{\beta_1 - \beta_2}{2}}\sqrt{gH}Hk^2 \quad + \sqrt{\frac{\beta_1\left(\beta_1 - \beta_2\right)}{4}}\sqrt{gH}H^2k^3 \quad + \mathcal{O}\left(k^4\right)$$

$$\omega_{\text{water}} = \sqrt{gH}k \quad - \sqrt{\frac{1}{3}}\sqrt{gH}Hk^2 \quad + \sqrt{\frac{2}{15}}\sqrt{gH}H^2k^3 \quad + \mathcal{O}\left(k^4\right)$$

Accuracy Summary Plot



Phase Speed (c)

Looking at the phase speed $c_{\text{gSGN}} = \dfrac{\omega_{\text{gSGN}}}{k}$ we get

$$c_{\text{gSGN}} = \sqrt{gH}\sqrt{\frac{\beta_2 H^2 k^2 + 2}{\beta_1 H^2 k^2 + 2}}$$

Phase Speed (c) Bounds

The phase speed can be bounded in the following way with three cases

- When $\beta_1 > \beta_2$ we have

$$0 \leq \sqrt{\frac{\beta_2}{\beta_1}}\sqrt{gH} \leq c_{\text{gSGN}} \leq \sqrt{gH}$$

- When $\beta_1 = \beta_2$ we have

$$c_{\text{gSGN}} = \sqrt{gH}$$

- When $\beta_1 < \beta_2$ we have

$$0 \leq \sqrt{gH} \leq c_{\text{gSGN}} \leq \sqrt{\frac{\beta_2}{\beta_1}}\sqrt{gH}$$

Phase Speed Regions

Because $\sqrt{gH}$ controls the speed of shocks these chains of inequalities lead to the following behaviours:

- When $\beta_1 > \beta_2$ we have dispersive waves form behind shocks

$$c_{\text{gSGN}} \leq \sqrt{gH}$$

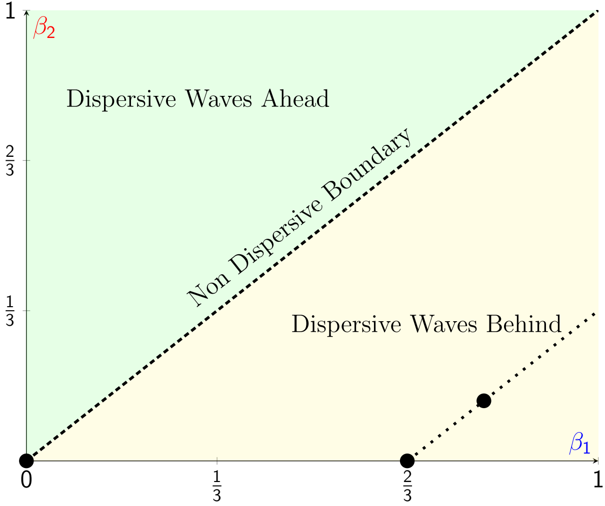
- When $\beta_1 = \beta_2$ we have no dispersive waves

$$c_{\text{gSGN}} = \sqrt{gH}$$

- When $\beta_1 < \beta_2$ we have dispersive waves form ahead of shocks

$$c_{\text{gSGN}} \geq \sqrt{gH}$$

Accuracy Summary Plot

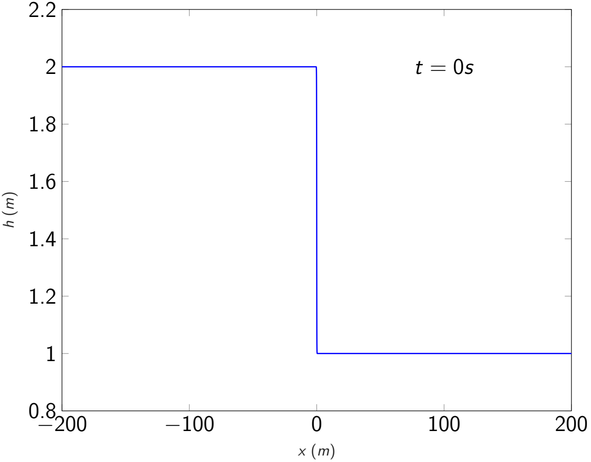


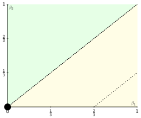
Comparsion Between Linear Theory and Numerical Solutions

Are these widths replicated in the numerical solutions of the non-linear dispersive wave equations?

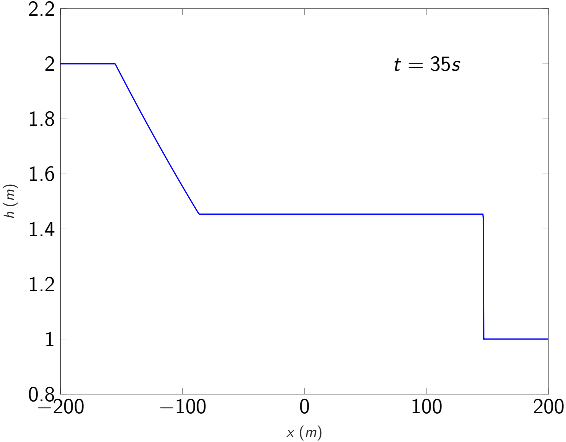
We answer this by comparing to numerical solutions to the dam-break initial condition problem.

Dambreak Problem Initial Conditions

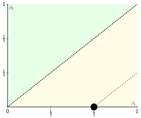


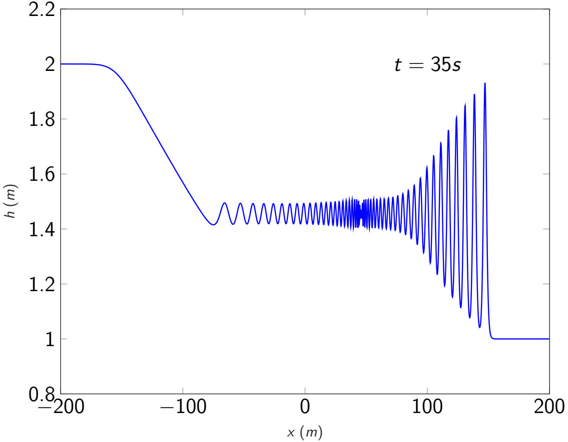


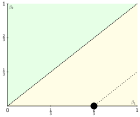
Shallow Water Wave Equations



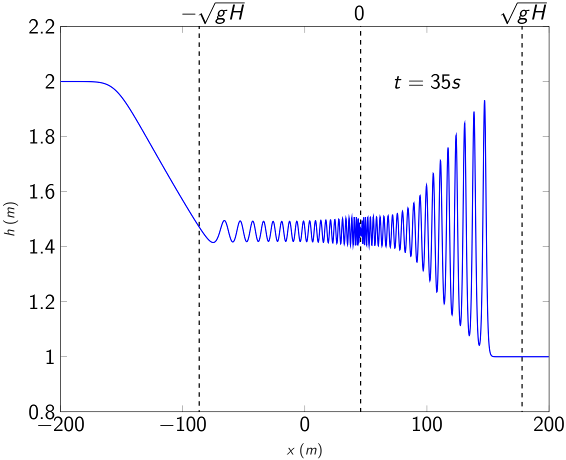
Serre Equations

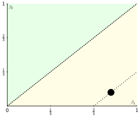




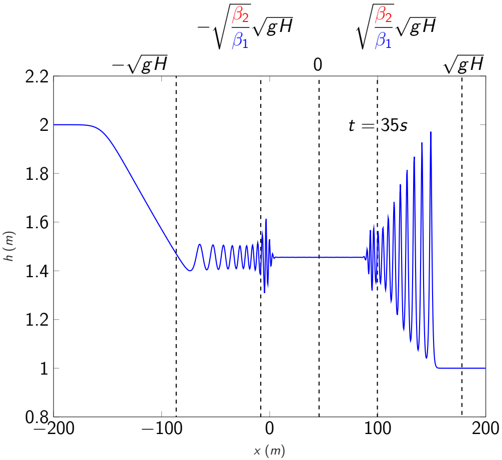


Serre Equations

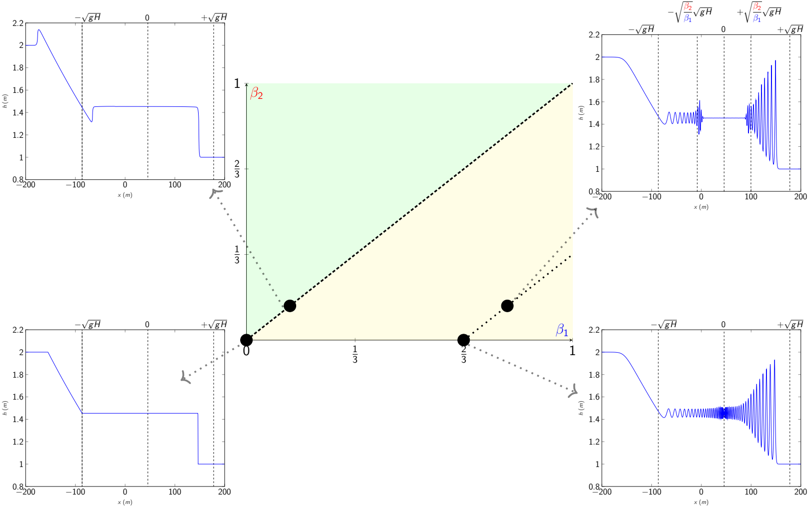




Optimal Dispersion Equations



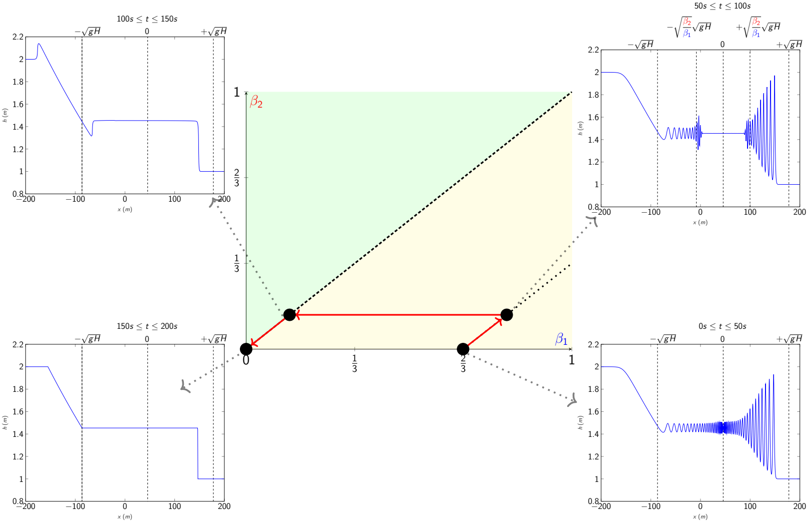
Conclusion


β_2
β_1
0
$\frac{1}{3}$
$\frac{2}{3}$
1
$-\sqrt{gH}$
$+\sqrt{gH}$
$-\sqrt{\frac{\beta_2}{\beta_1}}\sqrt{gH}$
$+\sqrt{\frac{\beta_2}{\beta_1}}\sqrt{gH}$
$h\ (m)$
$x\ (m)$

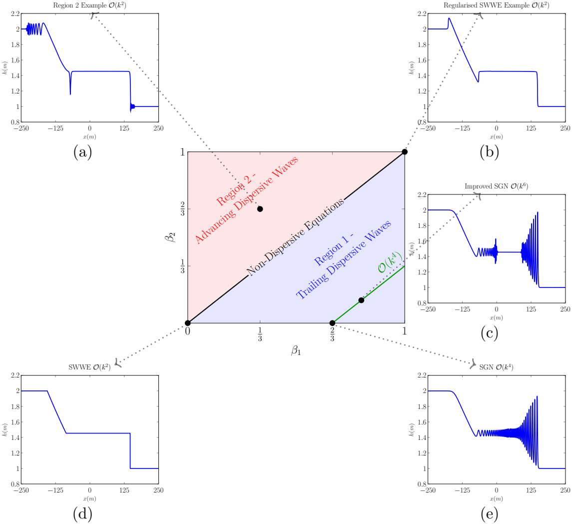
Changing β values

Now that we have gone through in depth each of the particular β values, we can begin to understand what happens if we change β values over time. Coming back to the original video at the beginning of the talk.

Changing β values Outline



Expanded Grid



Ahead Of Shock