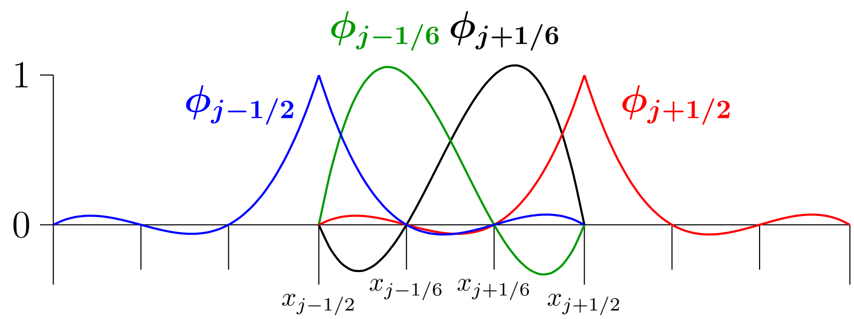

Figure 1: Basis functions

The basis functions space $= V_j = \left[\phi_{j-1/2}, \phi_{j-1/6}, \phi_{j+1/6}, \phi_{j+1/2}\right]$ over a cell and then $V = \cup_j V_j$

Want to approximate

$$G = uh \tag{1}$$

This becomes in weak form where

$$\int_\Omega Gvdx = \int_\Omega uhvdx \tag{2}$$

We reduce to finding solutions to

$$\int_\Omega vdx = \int_\Omega uhvdx \tag{3}$$

where $v \in V$. Using the partitioning of domain into cells we have

$$\sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} Gvdx = \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} uhvdx \tag{4}$$

Can write G, u and h on basis functions

$$q = \sum_j \left(q_{j-1/2}\phi_{j-1/2} + q_{j-1/6}\phi_{j-1/6} + q_{j+1/6}\phi_{j+1/6} + q_{j+1/2}\phi_{j+1/2}\right)$$

$$q = \boldsymbol{q} \cdot \boldsymbol{\phi}$$

where $\boldsymbol{q}$ is the vector of all the nodal values and $\boldsymbol{\phi}$ is the vector of all basis functions.

$$\int_{x_{j-1/2}}^{x_{j+1/2}} Gvdx = \int_{x_{j-1/2}}^{x_{j+1/2}} \left(G_{j-1/2}\phi_{j-1/2} + G_{j-1/6}\phi_{j-1/6} + G_{j+1/6}\phi_{j+1/6} + G_{j+1/2}\phi_{j+1/2}\right) \begin{bmatrix} \phi_{j-1/2} \\ \phi_{j-1/6} \\ \phi_{j+1/6} \\ \phi_{j+1/2} \end{bmatrix} \tag{5}$$

$$= \int_{x_{j-1/2}}^{x_{j+1/2}} \begin{bmatrix} G_{j-1/2}\phi_{j-1/2}\phi_{j-1/2} + G_{j-1/6}\phi_{j-1/6}\phi_{j-1/2} + G_{j+1/6}\phi_{j+1/6}\phi_{j-1/2} + G_{j+1/2}\phi_{j+1/2}\phi_{j-1/2} \\ G_{j-1/2}\phi_{j-1/2}\phi_{j-1/6} + G_{j-1/6}\phi_{j-1/6}\phi_{j-1/6} + G_{j+1/6}\phi_{j+1/6}\phi_{j-1/6} + G_{j+1/2}\phi_{j+1/2}\phi_{j-1/6} \\ G_{j-1/2}\phi_{j-1/2}\phi_{j+1/6} + G_{j-1/6}\phi_{j-1/6}\phi_{j+1/6} + G_{j+1/6}\phi_{j+1/6}\phi_{j+1/6} + G_{j+1/2}\phi_{j+1/2}\phi_{j+1/6} \\ G_{j-1/2}\phi_{j-1/2}\phi_{j+1/2} + G_{j-1/6}\phi_{j-1/6}\phi_{j+1/2} + G_{j+1/6}\phi_{j+1/6}\phi_{j+1/2} + G_{j+1/2}\phi_{j+1/2}\phi_{j+1/2} \end{bmatrix} \tag{6}$$

$$= \int_{x_{j-1/2}}^{x_{j+1/2}} \begin{bmatrix} \phi_{j-1/2}\phi_{j-1/2} & \phi_{j-1/6}\phi_{j-1/2} & \phi_{j+1/6}\phi_{j-1/2} & \phi_{j+1/2}\phi_{j-1/2} \\ \phi_{j-1/2}\phi_{j-1/6} & \phi_{j-1/6}\phi_{j-1/6} & \phi_{j+1/6}\phi_{j-1/6} & \phi_{j+1/2}\phi_{j-1/6} \\ \phi_{j-1/2}\phi_{j+1/6} & \phi_{j-1/6}\phi_{j+1/6} & \phi_{j+1/6}\phi_{j+1/6} & \phi_{j+1/2}\phi_{j+1/6} \\ \phi_{j-1/2}\phi_{j+1/2} & \phi_{j-1/6}\phi_{j+1/2} & \phi_{j+1/6}\phi_{j+1/2} & \phi_{j+1/2}\phi_{j+1/2} \end{bmatrix} \begin{bmatrix} G_{j-1/2} \\ G_{j-1/6} \\ G_{j+1/6} \\ G_{j+1/2} \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_{j-1/2}\phi_{j-1/2}dx & \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_{j-1/6}\phi_{j-1/2}dx & \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_{j+1/6}\phi_{j-1/2}dx & \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_{j+1/2}\phi_{j-1/2}dx \\ \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_{j-1/2}\phi_{j-1/6}dx & \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_{j-1/6}\phi_{j-1/6}dx & \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_{j+1/6}\phi_{j-1/6}dx & \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_{j+1/2}\phi_{j-1/6}dx \\ \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_{j-1/2}\phi_{j+1/6}dx & \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_{j-1/6}\phi_{j+1/6}dx & \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_{j+1/6}\phi_{j+1/6}dx & \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_{j+1/2}\phi_{j+1/6}dx \\ \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_{j-1/2}\phi_{j+1/2}dx & \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_{j-1/6}\phi_{j+1/2}dx & \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_{j+1/6}\phi_{j+1/2}dx & \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_{j+1/2}\phi_{j+1/2}dx \end{bmatrix} \begin{bmatrix} G_{j-1/2} \\ G_{j-1/6} \\ G_{j+1/6} \\ G_{j+1/2} \end{bmatrix} \tag{8}$$

for the uh term

$$\int_{x_{j-1/2}}^{x_{j+1/2}} uhvdx = \int_{x_{j-1/2}}^{x_{j+1/2}} \left(u_{j-1/2}\phi_{j-1/2} + u_{j-1/6}\phi_{j-1/6} + u_{j+1/6}\phi_{j+1/6} + u_{j+1/2}\phi_{j+1/2}\right) \left(h_{j-1/2}\phi_{j-1/2} + h_{j-1/6}\phi_{j-1/6} + h_{j+1/6}\phi_{j+1/6} + h_{j+1/2}\phi_{j+1/2}\right) \begin{bmatrix} \phi_{j-1/2} \\ \phi_{j-1/6} \\ \phi_{j+1/6} \\ \phi_{j+1/2} \end{bmatrix} \tag{9}$$

$$\int_{x_{j-1/2}}^{x_{j+1/2}} uhvdx = \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\boldsymbol{u}_j \cdot \boldsymbol{\phi}_j\right) \left(\boldsymbol{h}_j \cdot \boldsymbol{\phi}_j\right) \begin{bmatrix} \phi_{j-1/2} \\ \phi_{j-1/6} \\ \phi_{j+1/6} \\ \phi_{j+1/2} \end{bmatrix} \tag{10}$$

$$\int_{x_{j-1/2}}^{x_{j+1/2}} uhvdx = \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\boldsymbol{u}_j \cdot \boldsymbol{\phi}_j\right) \boldsymbol{\phi}_j \left(\boldsymbol{h}_j \cdot \boldsymbol{\phi}_j\right) \tag{11}$$

$$\int_{x_{j-1/2}}^{x_{j+1/2}} uhvdx = \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\boldsymbol{u}_j^T \boldsymbol{\phi}_j\right) \boldsymbol{\phi}_j \left(\boldsymbol{h}_j^T \boldsymbol{\phi}_j\right) \tag{12}$$

$$\int_{x_{j-1/2}}^{x_{j+1/2}} uhvdx = \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\boldsymbol{h}_j^T \boldsymbol{\phi}_j\right) \boldsymbol{\phi}_j \left(\boldsymbol{\phi}_j^T \boldsymbol{u}_j\right) \tag{13}$$

$$\int_{x_{j-1/2}}^{x_{j+1/2}} uhvdx = \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\boldsymbol{h}_j^T \boldsymbol{\phi}_j\right) \left(\boldsymbol{\phi}_j \boldsymbol{\phi}_j^T\right) \boldsymbol{u}_j \tag{14}$$

$$\int_{x_{j-1/2}}^{x_{j+1/2}} uhvdx = \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\boldsymbol{h}_j^T \boldsymbol{\phi}_j\right) \left(\boldsymbol{\phi}_j \boldsymbol{\phi}_j^T\right) \boldsymbol{u}_j \tag{15}$$

$$\int_{x_{j-1/2}}^{x_{j+1/2}} uhvdx = \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\boldsymbol{h}_j^T \boldsymbol{\phi}_j^2 \boldsymbol{\phi}_j^T\right) \boldsymbol{u}_j \tag{16}$$