Regularised Serre

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1 Regularised Serre Equation

Clamond and Dutykh[1] derived the following regularised Shallow Water Wave equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{1a}$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \epsilon \mathscr{R} h^2\right) = 0 \tag{1b}$$

where

$$\mathscr{R} \stackrel{\text{def}}{=} h\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x \partial t} - u\frac{\partial^2 u}{\partial x^2}\right) - g\left(h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right).$$

In this context, regularisation means adding additional terms to an equation to control or eliminate fluctuations or oscillations in the solution.

If $\epsilon = 0$ the non-linear shallow water wave equation are recovered. For $\epsilon \neq 0$, $\mathscr{R}$ is a regularisation term that prevents the formation of shocks. It consists of dispersive term that characterises the Serre equation and additional regularisation terms.

Unlike other regularisations, this regularisation conserves mass, momentum and energy[1]. Currently ϵ allows us to switch between the SWWE when $\epsilon = 0$, when $\epsilon \neq 0$ then we get the a regularised version of the SWWE.

We want to allow another parameter that allows us to include all three cases: SWWE, Serre equations and regularised SWWE. To do this we add a new parameter α that leads to the [] equations:

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{2a}$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} - g\alpha\epsilon h^2 \mathscr{R} + \epsilon h^3 \mathscr{S}\right) = 0 \tag{2b}$$

where

$$\mathscr{R} = h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x} \tag{3}$$

$$\mathscr{S} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x \partial t} - u\frac{\partial^2 u}{\partial x^2} \tag{4}$$

Thus we get

- SWWE - when $\epsilon = 0$ (α can be anything is free)
- Regularised SWWE - when $\alpha = 1$ and $\epsilon \neq 0$

- Serre equations - when $\alpha = 0$ and $\epsilon = \frac{1}{3}$

Using our standard techniques we can write this in conservation law form:

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{5a}$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left[uG + \frac{gh^2}{2} - 2\epsilon h^3 \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - g\epsilon\alpha h^2\left(h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)\right] = 0 \tag{5b}$$

where

$$G \stackrel{\text{def}}{=} uh - \epsilon\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right)$$

which is identical to (1).

Written in conservative form (**??**) where

$$\mathbf{q} = \begin{bmatrix} h \\ G \end{bmatrix}, \tag{6}$$

and the flux vector

$$\mathbf{F}(\mathbf{q}) = \begin{bmatrix} uh \\ uG + \frac{gh^2}{2} - 2\epsilon h^3 \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - g\epsilon\alpha h^2\left(h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right) \end{bmatrix}. \tag{7}$$

1.1 Wave Speeds

Assuming that

$$\begin{aligned} h(x,t) &= h_0 + \delta\eta(x,t) + O(\delta^2) \\ u(x,t) &= u_0 + \delta v(x,t) + O(\delta^2) \end{aligned}$$

By substituting these forms into the linearised Serre equations and neglecting $O(\delta^2)$ terms, we get the linearised regularised Serre equations. We also substitute η_t using the mass equation into the momentum equation.

$$(\delta\eta)_t + u_0(\delta\eta)_x + h_0(\delta v)_x = 0 \tag{8a}$$

$$h_0(\delta v)_t + gh_0(\delta\eta)_x + h_0u_0(\delta v)_x - \epsilon h_0^3(\delta v)_{xxt} - g\alpha\epsilon h_0^3(\delta\eta)_{xxx} - \epsilon h_0^3 u_0(\delta v)_{xxx} = 0 \tag{8b}$$

We can remove the δ term, either by removing the common factor, or absorbing it into η and v to get

$$\eta_t + u_0\eta_x + h_0 v_x = 0 \tag{9a}$$

$$h_0 v_t + gh_0\eta_x + h_0u_0v_x - \epsilon h_0^3 v_{xxt} - g\alpha\epsilon h_0^3\eta_{xxx} - \epsilon h_0^3 u_0 v_{xxx} = 0 \tag{9b}$$

We now assume that $\eta(x,t) = H\exp\left(i(kx-\omega t)\right)$, $v(x,t) = U\exp\left(i(kx-\omega t)\right)$

$$\begin{aligned} \eta(x,t) &= H\exp\left(i(kx-\omega t)\right) \\ v(x,t) &= U\exp\left(i(kx-\omega t)\right) \end{aligned}$$

substituting these into the linearised Serre equation we get

$$\left[Hu_0k - H\omega + Uh_0k\right] i\exp\left[i\left(kx-\omega t\right)\right] = 0 \tag{10a}$$

$$\left[g\alpha\epsilon Hh_0^2k^3 + gkH + Uh_0^2u_0\epsilon k^3 - Uh_0^2\epsilon k^2\omega + Uu_0k - U\omega\right] ih_0\exp\left[i\left(kx-\omega t\right)\right] = 0 \tag{10b}$$

This can be written as

$$\begin{bmatrix} u_0 k - \omega & h_0 k \\ g\alpha\epsilon h_0^2 k^3 + kg & h_0^2 u_0 \epsilon k^3 - h_0^2 \epsilon k^2 \omega + u_0 k - \omega \end{bmatrix} \begin{bmatrix} H \\ U \end{bmatrix} = 0 \tag{11}$$

Which has non-trivial solutons when the determinant is zero.
The determinant of this matrix is

$$\left(\epsilon h_0^2 k^2 + 1\right)\omega^2 - 2Uk\left(\epsilon h_0^2 k^2 + 1\right)\omega + u_0^2 k^2\left(\epsilon h_0^2 k^2 + 1\right) - h_0 k^2 g\left(\epsilon\alpha k^2 h_0^2 + 1\right) \tag{12}$$

To get non-trivial solution we have

$$\left(\epsilon h_0^2 k^2 + 1\right)\omega^2 - 2Uk\left(\epsilon h_0^2 k^2 + 1\right)\omega + u_0^2 k^2\left(\epsilon h_0^2 k^2 + 1\right) - h_0 k^2 g\left(\epsilon\alpha k^2 h_0^2 + 1\right) = 0 \tag{13}$$

$$\omega^2 - 2Uk\omega + u_0^2 k^2 - h_0 k^2 g \frac{\epsilon\alpha k^2 h_0^2 + 1}{\epsilon k^2 h_0^2 + 1} = 0 \tag{14}$$

Using quadratic equation, or another quadratic polynomial solver we get that

$$\omega = u_0 k \pm k\sqrt{gh_0}\sqrt{\frac{\left(\alpha\epsilon k^2 h_0^2 + 1\right)}{\left(\epsilon k^2 h_0^2 + 1\right)}} \tag{15}$$

$$\omega = u_0 k \pm k\sqrt{gh_0}\sqrt{\frac{\left(\epsilon k^2 h_0^2 + 1\right) + [\alpha - 1]\,\epsilon k^2 h_0^2}{\left(\epsilon k^2 h_0^2 + 1\right)}} \tag{16}$$

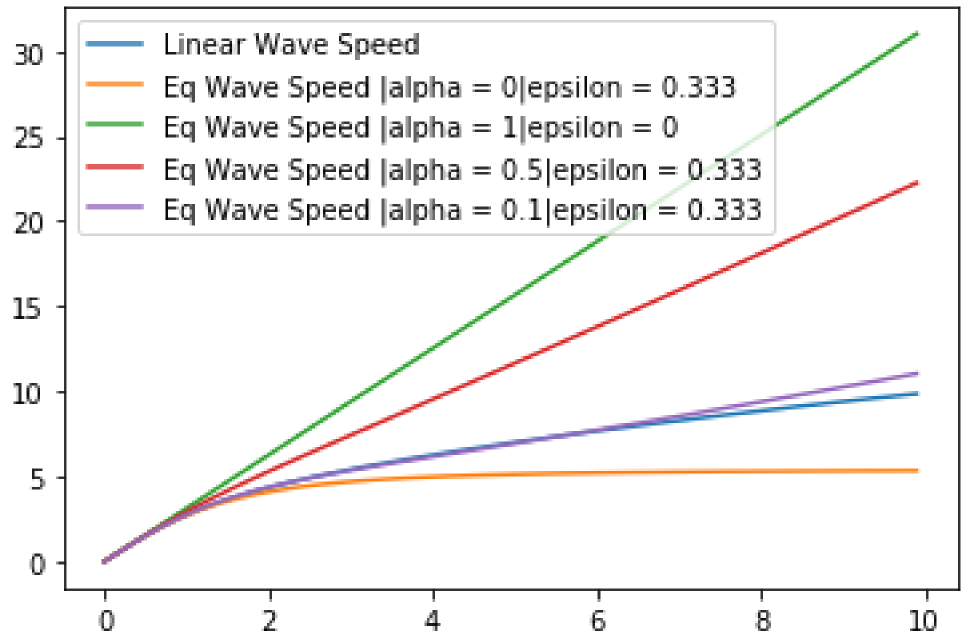
$$\omega = u_0 k \pm k\sqrt{gh_0}\sqrt{1 + (\alpha - 1)\,\epsilon \frac{k^2 h_0^2}{\left(\epsilon k^2 h_0^2 + 1\right)}} \tag{17}$$

Thus we have the following regimes

- SWWE wavespeeds: occurs when $\epsilon = 0$ or when $\alpha = 1$ - thus for the SWWE and regularised SWWE cases we wanted we get the correct wave speed.
- Serre wavespeeds - when $\alpha = 0$ and $\epsilon = \frac{1}{3}$ - we get $\omega = k\left(u_0 \pm \sqrt{gh_0}\sqrt{\frac{3}{3 + h_0^2 k^2}}\right)$
- For other values of $\alpha \in [0, 1]$ and $\epsilon \in \left[0, \frac{1}{3}\right]$ we can vary the wavespeeds, in certain situations this will lead to a better approximation of the linear wavespeed of waves than the Serre equations, in others it will be worse. We show an example below.

I gave some example plots, just varying k with various values of α and ϵ. The results suggest that we can in addition to obtaining the above examples, we could also choose α and ϵ to better match the linear wavespeed for waves given by

$$\omega = u_0 k \pm \sqrt{gk \tanh\left(kh_0\right)}$$



In the above picture $u_0 = 0$, $h_0 = 1$, $g = 9.81$. While the x axis is k and the y axis is ω.

1.1.1 Phase Speed Bounds

First we want to show that phase and group speed is bounded.

$$v_p = \frac{\omega}{k} = u_0 \pm \sqrt{gh_0}\sqrt{1 + (\alpha - 1)\,\epsilon \frac{k^2 h_0^2}{(\epsilon k^2 h_0^2 + 1)}} \tag{18}$$

As $k \to 0$ then $v_p \to u_0 \pm \sqrt{gh_0}$. Meanwhile as $k \to \infty$ then $v_p \to u_0 \pm \sqrt{\alpha g h_0}$. For our purposes we will have $\epsilon \in [0, 1]$ and $\alpha \in [0, 1]$. Thus we have for these restrictions that

$$u - \sqrt{gh_0} \leq v_p \leq u + \sqrt{gh_0} \tag{19}$$

as desired. This is only the case though because $\alpha \leq 1$.

References

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