Generalised Serre-Green-Naghdi Model

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1 gSGN - generalised Serre-Green-Naghdi equations

Clamond and Dutykh[1] derived the following generalised version of the Serre-Green-Naghdi equations:

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{1a}$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{1}{3} h^2 \Gamma \right) = 0 \tag{1b}$$

$$\frac{\partial}{\partial t}\left[\frac{1}{2} h u^2 + \frac{1}{2}\left(1 + \frac{3}{2}\beta_1\right) h^3 \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{1}{2} g h^2 \left(1 + \frac{1}{2}\beta_2 \frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)\right]$$

$$\frac{\partial}{\partial x}\left[uh\left(\frac{1}{2} u^2 + \frac{1}{2}\left(1 + \frac{3}{2}\beta_1\right) h^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + gh\left(1 + \frac{1}{4}\beta_2 \frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right) + \frac{1}{3} h\Gamma\right) + \frac{1}{2}\beta_2 g h^3 \frac{\partial h}{\partial x}\frac{\partial u}{\partial x}\right] = 0 \tag{1c}$$

where

$$\Gamma = \left(1 + \frac{3}{2}\beta_1\right) h \left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x \partial t} - u\frac{\partial^2 u}{\partial x^2}\right] - \frac{3}{2}\beta_2 g \left[h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right] \tag{1d}$$

These equations have the same order of approximation in the lagrangian density (dispersion properties?) when $\beta_1 = \beta_2$. The interesting thing about the equations though, is that we will conserve mass, momentum and energy for all values of β_j.

From these equations the SWWE, the Serre equations and the regularised SWWE [1] can be recovered for certain values of β_1 and β_2.

1.1 Alternative Conservative Form of the gSGN

A major difficulty with solving the SGN is that the dispersive terms contain a mixed spatial-temporal derivative term which is difficult to handle numerically. This mixed derivative term can be rewritten so that the Serre equations can be expressed in conservation law form, with the water depth and a new quantity as conservative

Resulting Equations	β_1	β_2
Serre Equations	0	0
SWWE Equations	$-\dfrac{2}{3}$	0
Regularised SWWE Equations	free variable	$\beta_1 + \dfrac{2}{3}$
Improved Dispersion Serre Equations	free variable	β_1

Table 1: Showing various combinations of β values and equivalent equations

variables. This reformulation allows standard techniques for solving conservation laws to be applied to the Serre equations, even though the Serre equations are neither hyperbolic nor parabolic.

Consider the Serre equations written for a horizontal bed. The flux term in the momentum equation, (1b) contains a mixed spatial and temporal derivative term which is difficult to treat numerically. It is possible to replace this term by a combination of spatial and temporal derivative terms by making the following observation

$$\frac{\partial^2}{\partial x \partial t}\left(\frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)h^3\frac{\partial u}{\partial x}\right) = \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)\frac{\partial}{\partial t}\left(3h^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}+h^3\frac{\partial^2 u}{\partial x^2}\right)$$

$$= \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)\frac{\partial}{\partial x}\left(3h^2\frac{\partial h}{\partial t}\frac{\partial u}{\partial x}+h^3\frac{\partial^2 u}{\partial x \partial t}\right). \quad (2)$$

Rearranging and making use of the continuity equation, (1a) the momentum equation, (1b) becomes

$$\frac{\partial}{\partial t}\left(uh - \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)\frac{\partial}{\partial x}\left[h^3\frac{\partial u}{\partial x}\right]\right)$$
$$+\frac{\partial}{\partial x}\left(u\left[uh - \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)\frac{\partial}{\partial x}\left[h^3\frac{\partial u}{\partial x}\right]\right] + \frac{gh^2}{2} - \frac{2}{3}\left(1+\frac{3}{2}\beta_1\right)h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{2}\beta_2 gh^2\left[h\frac{\partial^2 h}{\partial x^2}+\frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right]\right) = 0.$$
$$(3)$$

The momentum equation can be written in conservation law form as

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}\left(1+\frac{3}{2}\beta_1\right)h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{2}\beta_2 gh^2\left[h\frac{\partial^2 h}{\partial x^2}+\frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right]\right) = 0. \quad (4)$$

where a new conserved quantity, G is given by

$$G = uh - \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right).$$

This expands the conserved variable introduced by [1], as well as in the Serre equations [].

Thus we have the following conservation equations

$$\frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = 0 \tag{5a}$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}\left(1+\frac{3}{2}\beta_1\right)h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{2}\beta_2 gh^2\left[h\frac{\partial^2 h}{\partial x^2}+\frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right]\right) = 0. \tag{5b}$$

with

$$G = uh - \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right). \tag{5c}$$

1.2 Dispersion Relation of Linearised gSGN

Assuming that

$$h(x,t) = h_0 + \delta\eta(x,t) + O(\delta^2)$$
$$u(x,t) = u_0 + \delta v(x,t) + O(\delta^2)$$

By substituting these forms into the linearised Serre equations and neglecting $O(\delta^2)$ terms, we get the linearised regularised Serre equations. We also substitute η_t using the mass equation into the momentum equation.

$$(\delta\eta)_t + u_0(\delta\eta)_x + h_0(\delta v)_x = 0 \tag{6a}$$

$$h_0(\delta v)_t + gh_0(\delta\eta)_x + h_0 u_0(\delta v)_x - \frac{1}{3}\left(1 + \frac{3}{2}\beta_1\right)h_0^3(\delta v)_{xxt} - \frac{1}{3}\left(1 + \frac{3}{2}\beta_1\right)h_0^3 u_0(\delta v)_{xxx} - \frac{g\beta_2}{2}h_0^3(\delta\eta)_{xxx} = 0 \quad (6b)$$

We can remove the δ term, either by removing the common factor, or absorbing it into η and v to get

$$\eta_t + u_0\eta_x + h_0 v_x = 0 \quad (7a)$$

$$h_0(v)_t + gh_0(\eta)_x + h_0 u_0(v)_x - \frac{1}{3}\left(1 + \frac{3}{2}\beta_1\right)h_0^3(v)_{xxt} - \frac{1}{3}\left(1 + \frac{3}{2}\beta_1\right)h_0^3 u_0(v)_{xxx} - \frac{g\beta_2}{2}h_0^3(\eta)_{xxx} = 0 \quad (7b)$$

We now assume that $\eta(x,t) = H\exp\left(i(kx - \omega t)\right)$, $v(x,t) = U\exp\left(i(kx - \omega t)\right)$

$$\eta(x,t) = H\exp\left(i(kx - \omega t)\right)$$
$$v(x,t) = U\exp\left(i(kx - \omega t)\right)$$

substituting these into the linearised Serre equation we get

$$[Hu_0 k - H\omega + Uh_0 k]\,i\exp\left[i\left(kx - \omega t\right)\right] = 0 \quad (8a)$$

$$\Big[3H\beta_2 gh_0^2 k^3 + 6Hgk - 3U\beta_1\omega h_0^2 k^2 + 3U\beta_1 h_0^2 k^3 u_0$$
$$- 2U\omega h_0^2 k^2 - 6U\omega + 2Uh_0^2 k^3 u_0 + 6Uku_0\Big]i\frac{h_0}{6}\exp\left[i\left(kx - \omega t\right)\right] = 0 \quad (8b)$$

This can be written as

$$\begin{bmatrix} u_0 k - \omega & h_0 k \\ 3\beta_2 h_0^2 k^3 + 6kg & -3\beta_1\omega h_0^2 k^2 + 3\beta_1 h_0^2 k^3 u_0 - 2\omega h_0^2 k^2 - 6\omega + 2h_0^2 k^3 u_0 + 6ku_0 \end{bmatrix}\begin{bmatrix} H \\ U \end{bmatrix} = 0 \quad (9)$$

Which has non-trivial solutons when the determinant is zero.
The determinant of this matrix is

$$\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right)\omega^2 - 2u_0 k\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right)\omega + u_0^2 k^2\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right) - gh_0 k^2\left(3\beta_2 h_0^2 k^2 + 6\right) \quad (10)$$

To get non-trivial solution we have

$$\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right)\omega^2 - 2u_0 k\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right)\omega + u_0^2 k^2\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right) - gh_0 k^2\left(3\beta_2 h_0^2 k^2 + 6\right) = 0 \quad (11)$$

$$\omega^2 - 2u_0 k\omega + u_0^2 k^2 - gh_0 k^2\frac{3\beta_2 h_0^2 k^2 + 6}{\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right)} \quad (12)$$

Using quadratic equation, or another quadratic polynomial solver we get that

$$\omega^\pm = u_0 k \pm k\sqrt{gh_0}\sqrt{\frac{3\beta_2 h_0^2 k^2 + 6}{\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right)}} \quad (13)$$

$$\omega^\pm = u_0 k \pm k\sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\frac{2}{3} + \beta_1\right)h_0^2 k^2 + 2}} \quad (14)$$

Thus we have the following regimes

- SWWE wavespeeds (non-dispersive): occurs when $\beta_2 = \frac{2}{3} + \beta_1$ then $\omega^\pm = k\left(u_0 \pm \sqrt{gh_0}\right)$ ([1]).

- Serre wavespeeds - when $\beta_1 = \beta_2 = 0$ - we get $\omega^\pm = k\left(u_0 \pm \sqrt{gh_0}\sqrt{\frac{3}{3 + h_0^2 k^2}}\right)$

- For other values of β_1 and β_2 we can vary the wavespeeds, in certain situations this will lead to a better approximation of the linear wavespeed of waves than the Serre equations ([1]), in others it will be worse. We show an example below.

Now for the group speed

$$v_g^\pm = \frac{\partial \omega^\pm}{\partial k} = u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm \frac{k\sqrt{gh_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\beta_2 h_0^2 k - \frac{h_0^2 k \left(\beta_1 + \frac{2}{3}\right)\left(\beta_2 h_0^2 k^2 + 2\right)}{h_0^2 k^2 \left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$
(15)

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm$$
$$\frac{k\sqrt{gh_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\frac{\beta_2 h_0^2 k^2 \left[h_0^2 k^2 \left(\beta_1 + \frac{2}{3}\right) + 2\right] - h_0^2 k \left(\beta_1 + \frac{2}{3}\right)\left(\beta_2 h_0^2 k^2 + 2\right)}{h_0^2 k^2 \left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm$$
$$\frac{k\sqrt{gh_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\frac{\beta_2 h_0^4 k^3 \left(\beta_1 + \frac{2}{3}\right) + 2\beta_2 h_0^2 k - h_0^2 k \left(\beta_1 + \frac{2}{3}\right)\left(\beta_2 h_0^2 k^2 + 2\right)}{h_0^2 k^2 \left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm$$
$$\frac{k\sqrt{gh_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\frac{\beta_2 h_0^4 k^3 \left(\beta_1 + \frac{2}{3}\right) + 2\beta_2 h_0^2 k - \beta_2 \left(\beta_1 + \frac{2}{3}\right) h_0^4 k^3 - 2 h_0^2 k \left(\beta_1 + \frac{2}{3}\right)}{h_0^2 k^2 \left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm$$
$$\frac{k\sqrt{gh_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\frac{2\beta_2 h_0^2 k - 2 h_0^2 k \left(\beta_1 + \frac{2}{3}\right)}{h_0^2 k^2 \left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm 2\frac{k^2 h_0^2 \sqrt{gh_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\frac{\beta_2 - \beta_1 - \frac{2}{3}}{h_0^2 k^2 \left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left[1 + \frac{\beta_2 - \beta_1 - \frac{2}{3}}{\left(\frac{1}{2}\beta_2 h_0^2 k^2 + 1\right)\left(\left(\frac{1}{3} + \beta_1\right) h_0^2 k^2 + 1\right)}\right]$$

When $\beta_2 = \frac{2}{3} + \beta_1$ then the group speed is equal to the phase speed of the Shallow Water Wave equations. When $\beta_2 = \beta_1 = 0$ then we get

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{2}{\left(\frac{2}{3} h_0^2 k^2 + 2\right)}}\left[1 + \frac{-\frac{1}{3}}{\left(\frac{1}{3} h_0^2 k^2 + 1\right)}\right]$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{3}{h_0^2 k^2 + 3}}\left[1 - \frac{1}{h_0^2 k^2 + 3}\right]$$

which matches [Zoppo,pitt,paper on FDVM methods for Serre equations / Chris thesis]

1.2.1 Wave Speed Bounds

First we want to show that phase and group speed are bounded, when treated as functions of k. The phase speed is:

$$v_p^\pm = \frac{\omega^\pm}{k} = u_0 \pm \sqrt{gh_0} \sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \tag{16}$$

We can do this by noting that when $\beta_1 \geq -\frac{2}{3}$ and $\beta_2 \geq 0$ we have that

$$\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)} \tag{17}$$

is a montone function in terms of $h_0 k$, it is monotone decreasing if $\beta_2 \leq \frac{2}{3} + \beta_1$ and monotone increasing if $\beta_2 \geq \frac{2}{3} + \beta_1$.

We have the following limits

as $k \to 0$

$$v_p^\pm = u_0 \pm \sqrt{gh_0} \tag{18}$$

as $k \to \infty$

$$v_p^\pm = u_0 \pm \sqrt{gh_0} \sqrt{\frac{\beta_2}{\frac{2}{3} + \beta_1}} \tag{19}$$

When $\dfrac{\beta_2}{\frac{2}{3} + \beta_1} \leq 1$ we have

$$u_0 - \sqrt{gh_0} \leq v_p^- \leq u_0 - \sqrt{gh_0} \sqrt{\frac{\beta_2}{\frac{2}{3} + \beta_1}} \leq u_0 \leq u_0 + \sqrt{gh_0} \sqrt{\frac{\beta_2}{\frac{2}{3} + \beta_1}} \leq v_p^+ \leq u_0 + \sqrt{gh_0} \tag{20}$$

(includes Serre and regularised SWWE). For Serre the $k \to \infty$ bounds are u_0, so inequality simplifies for regularised SWWE, phase speed is fixed as k changes.

when $\dfrac{\beta_2}{\frac{2}{3} + \beta_1} > 1$ we have

$$u_0 - \sqrt{gh_0} \sqrt{\frac{\beta_2}{\frac{2}{3} + \beta_1}} \leq v_p^- \leq u_0 - \sqrt{gh_0} \leq u_0 \leq u_0 + \sqrt{gh_0} \leq v_p^+ \leq u_0 + \sqrt{gh_0} \sqrt{\frac{\beta_2}{\frac{2}{3} + \beta_1}} \tag{21}$$

Region 1 is what we expect, dispersive waves travel behind the shock front, Region 2 is inverted and we begin to see dispersive waves travel in front of the shock.

2 Dambreak Investigation

$$h(x, 0) = h_0 + \frac{h_1 - h_0}{2} \left(1 + \tanh\left(\frac{x}{\alpha}\right)\right) \tag{22}$$

$$u(x, 0) = 0 \tag{23}$$

$$G(x, 0) = 0 \tag{24}$$

We use $h_0 = 1$, $h_1 = 2$ and $\alpha = 0.1$

We will now plot various combinations of β_1 and β_2 values which demonstrate the following chains of inequalities above for the wavespeeds.
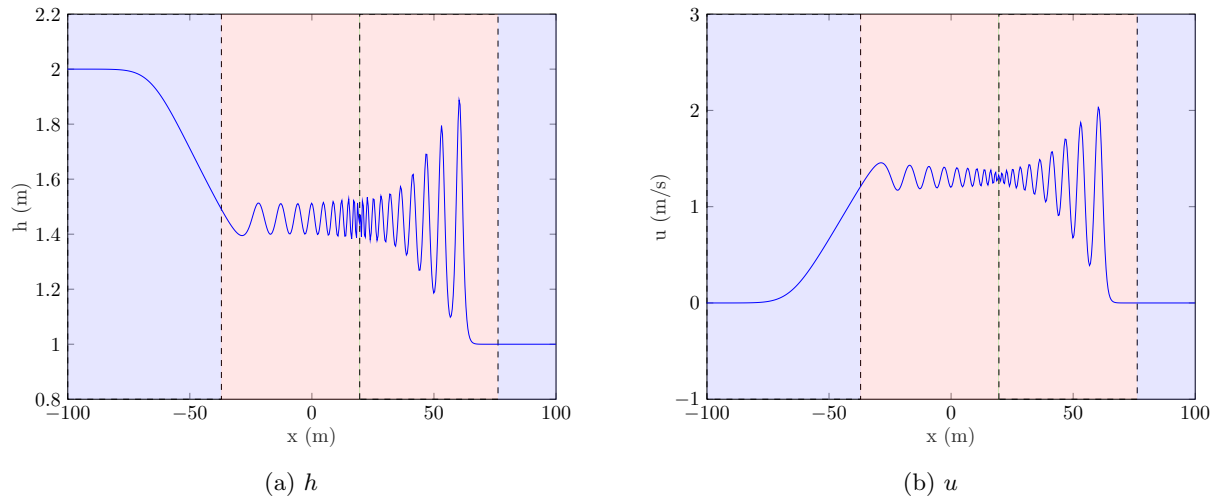
(a) h

(b) u

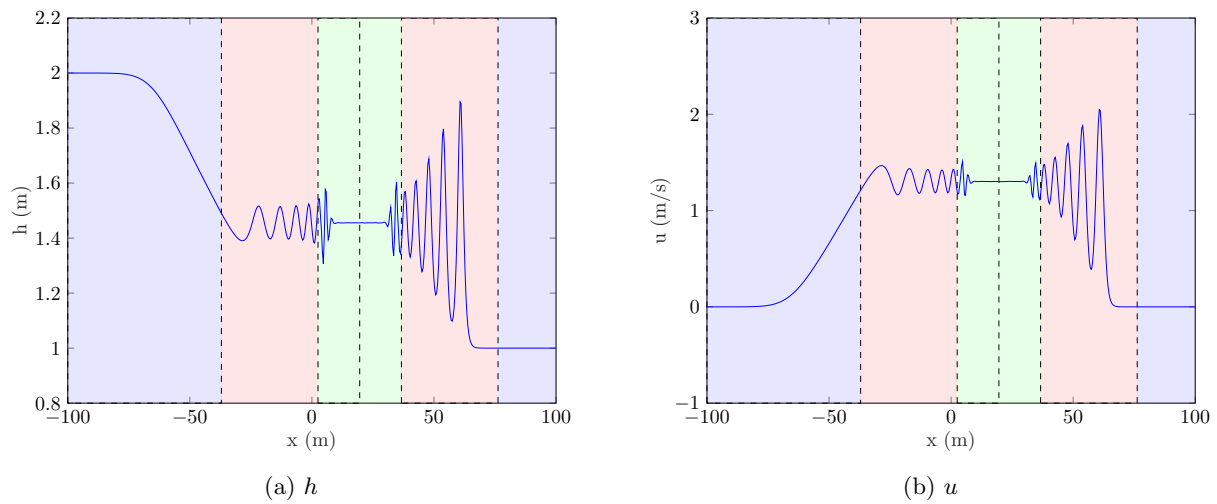Figure 1: Serre $\beta_1 = \beta_2 = 0$, so showing regions of inequality

(a) h

(b) u

Figure 2: Serre $\beta_1 = \beta_2 = \dfrac{1}{15}$, so showing regions of inequality
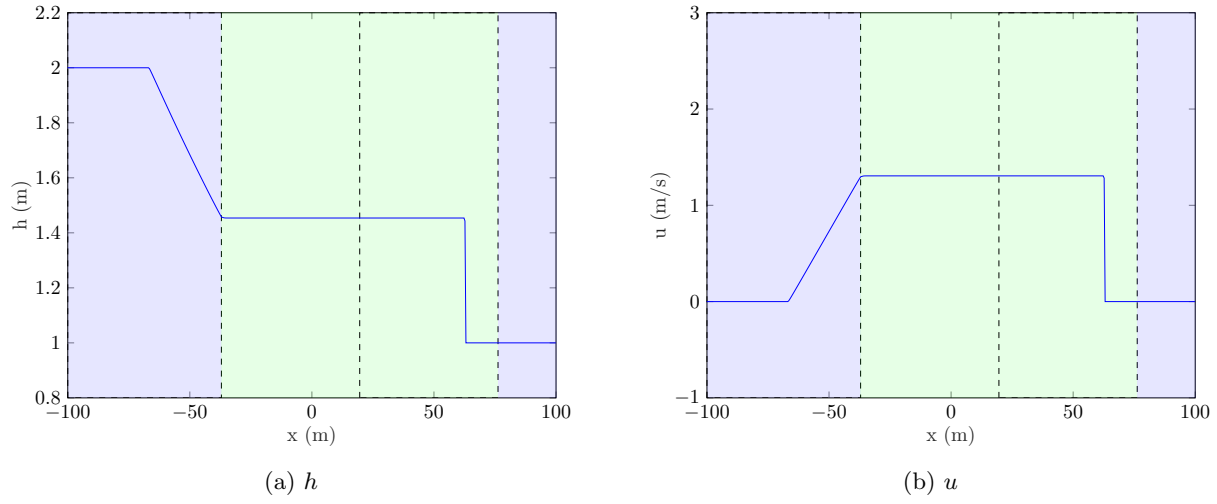
(a) h

(b) u

Figure 3: Regularised SWWE $\beta_1 = -\dfrac{2}{3}$ and $\beta_2 = 0$, so showing regions of inequality
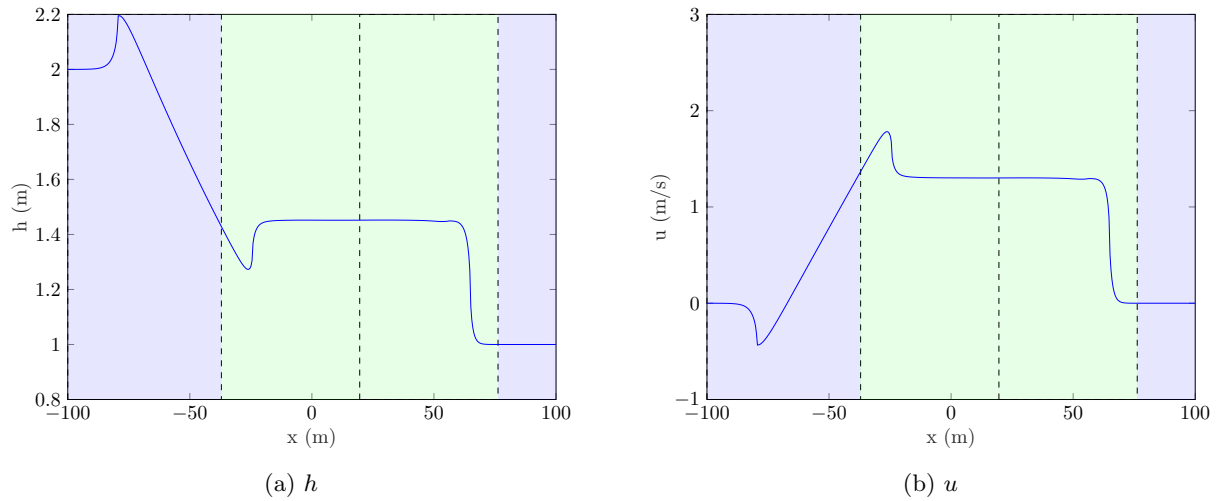

(a) h

(b) u

Figure 4: Regularised SWWE $\beta_1 = 3 - \dfrac{2}{3}$ and $\beta_2 = 3$, so showing regions of inequality
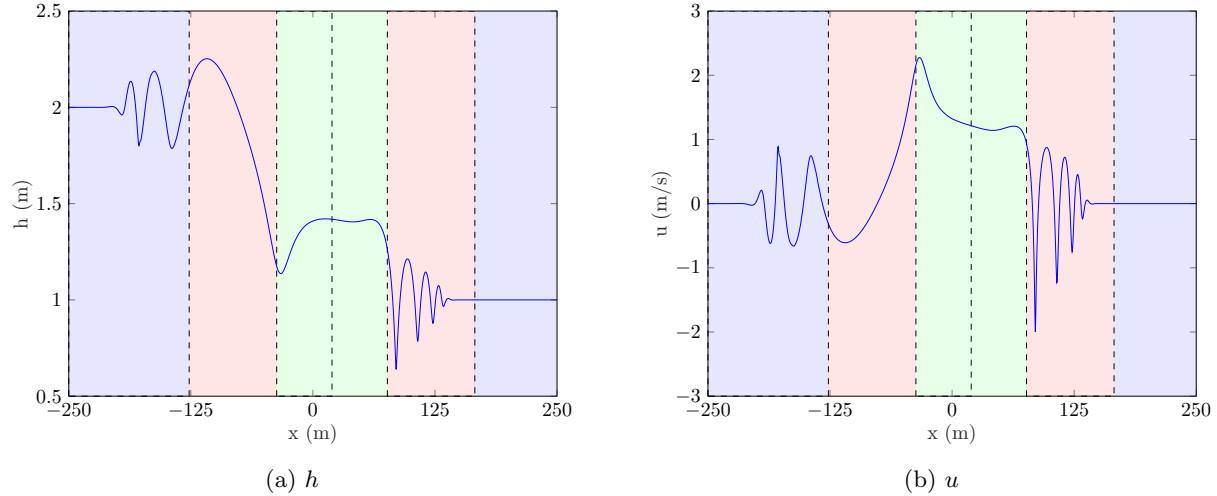
(a) h

(b) u

Figure 5: Region 2 $\beta_1 = 6$ and $\beta_2 = \left(\beta_1 + \dfrac{2}{3}\right)^2$, so showing regions of inequality

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