Generalised Serre-Green-Naghdi Model

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1 gSGN - generalised Serre-Green-Naghdi equations

Clamond and Dutykh[1] derived the following generalised version of the Serre-Green-Naghdi equations:

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{1a}$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{1}{3}h^2\Gamma\right) = 0 \tag{1b}$$

$$\begin{aligned}\frac{\partial}{\partial t}&\left[\frac{1}{2}hu^2 + \left(\frac{1}{6}+\frac{1}{4}\beta_1\right)h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{1}{2}gh^2\left(1+\frac{1}{2}\beta_2\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)\right] \\ &\frac{\partial}{\partial x}\left[uh\left(\frac{1}{2}u^2 + \left(\frac{1}{6}+\frac{1}{4}\beta_1\right)h^2\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + gh\left(1+\frac{1}{4}\beta_2\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right) + \frac{1}{3}h\Gamma\right) + \frac{1}{2}\beta_2 gh^3\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}\right] = 0\end{aligned} \tag{1c}$$

where

$$\Gamma = \left(1+\frac{3}{2}\beta_1\right)h\left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x\partial t} - u\frac{\partial^2 u}{\partial x^2}\right] - \frac{3}{2}\beta_2 g\left[h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right] \tag{1d}$$

These equations have the same order of approximation in the lagrangian density (dispersion properties?) when $\beta_1 = \beta_2$. The interesting thing about the equations though, is that we will conserve mass, momentum and energy for all values of β_j.

From these equations the SWWE, the Serre equations and the regularised SWWE [1] can be recovered for certain values of β_1 and β_2.

- When $\beta_1 = 2\epsilon - \frac{2}{3}$ and $\beta_2 = 2\epsilon$ we get the regularised Saint Venant equations of [1]. Which admit the SWWE when $\epsilon = 0$.
- When $\beta_1 = \beta_2$ we get the improved Serre-Green-Naghdi equations, which when $\beta_1 = \beta_2 = 0$ are the Serre-Green-Naghdi equations.

1.1 Alternative Conservative Form of the gSGN

A major difficulty with solving the SGN is that the dispersive terms contain a mixed spatial-temporal derivative term which is difficult to handle numerically. This mixed derivative term can be rewritten so that the Serre equations can be expressed in conservation law form, with the water depth and a new quantity as conservative variables. This reformulation allows standard techniques for solving conservation laws to be applied to the Serre equations, even though the Serre equations are neither hyperbolic nor parabolic.

Consider the Serre equations written for a horizontal bed. The flux term in the momentum equation, (1b) contains a mixed spatial and temporal derivative term which is difficult to treat numerically. It is possible to replace this term by a combination of spatial and temporal derivative terms by making the following observation

$$\frac{\partial^2}{\partial x \partial t}\left(\frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)h^3\frac{\partial u}{\partial x}\right) = \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)\frac{\partial}{\partial t}\left(3h^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x} + h^3\frac{\partial^2 u}{\partial x^2}\right)$$
$$= \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)\frac{\partial}{\partial x}\left(3h^2\frac{\partial h}{\partial t}\frac{\partial u}{\partial x} + h^3\frac{\partial^2 u}{\partial x \partial t}\right). \quad (2)$$

Rearranging and making use of the continuity equation, (1a) the momentum equation, (1b) becomes

$$\frac{\partial}{\partial t}\left(uh - \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)\frac{\partial}{\partial x}\left[h^3\frac{\partial u}{\partial x}\right]\right)$$
$$+\frac{\partial}{\partial x}\left(u\left[uh - \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)\frac{\partial}{\partial x}\left[h^3\frac{\partial u}{\partial x}\right]\right] + \frac{gh^2}{2} - \frac{2}{3}\left(1+\frac{3}{2}\beta_1\right)h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{2}\beta_2 g h^2\left[h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right]\right) = 0. \quad (3)$$

The momentum equation can be written in conservation law form as

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}\left(1+\frac{3}{2}\beta_1\right)h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{2}\beta_2 g h^2\left[h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right]\right) = 0. \quad (4)$$

where a new conserved quantity, G is given by

$$G = uh - \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right).$$

This expands the conserved variable introduced by [1], as well as in the Serre equations [].

Thus we have the following conservation equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \quad (5a)$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}\left(1+\frac{3}{2}\beta_1\right)h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{2}\beta_2 g h^2\left[h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right]\right) = 0. \quad (5b)$$

with

$$G = uh - \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right). \quad (5c)$$

1.2 Wave Speeds

Assuming that

$$h(x,t) = h_0 + \delta\eta(x,t) + O(\delta^2)$$
$$u(x,t) = u_0 + \delta v(x,t) + O(\delta^2)$$

By substituting these forms into the linearised Serre equations and neglecting $O(\delta^2)$ terms, we get the linearised regularised Serre equations. We also substitute η_t using the mass equation into the momentum equation.

$$(\delta\eta)_t + u_0(\delta\eta)_x + h_0(\delta v)_x = 0 \quad (6a)$$
$$h_0(\delta v)_t + g h_0(\delta\eta)_x + h_0 u_0(\delta v)_x - \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)h_0^3(\delta v)_{xxt} - \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)h_0^3 u_0(\delta v)_{xxx} - \frac{g\beta_2}{2}h_0^3(\delta\eta)_{xxx} = 0 \quad (6b)$$

We can remove the δ term, either by removing the common factor, or absorbing it into η and v to get

$$\eta_t + u_0 \eta_x + h_0 v_x = 0 \tag{7a}$$

$$h_0 (v)_t + g h_0 (\eta)_x + h_0 u_0 (v)_x - \frac{1}{3}\left(1 + \frac{3}{2}\beta_1\right) h_0^3 (v)_{xxt} - \frac{1}{3}\left(1 + \frac{3}{2}\beta_1\right) h_0^3 u_0 (v)_{xxx} - \frac{g\beta_2}{2} h_0^3 (\eta)_{xxx} = 0 \tag{7b}$$

We now assume that $\eta(x,t) = H \exp\left(i(kx - \omega t)\right)$, $v(x,t) = U \exp\left(i(kx - \omega t)\right)$

$$\eta(x,t) = H \exp\left(i(kx - \omega t)\right)$$
$$v(x,t) = U \exp\left(i(kx - \omega t)\right)$$

substituting these into the linearised Serre equation we get

$$\left[H u_0 k - H\omega + U h_0 k\right] i \exp\left[i\left(kx - \omega t\right)\right] = 0 \tag{8a}$$

$$\Big[3H\beta_2 g h_0^2 k^3 + 6Hgk - 3U\beta_1 \omega h_0^2 k^2 + 3U\beta_1 h_0^2 k^3 u_0$$
$$- 2U\omega h_0^2 k^2 - 6U\omega + 2U h_0^2 k^3 u_0 + 6Uku_0\Big] i \frac{h_0}{6} \exp\left[i\left(kx - \omega t\right)\right] = 0 \tag{8b}$$

This can be written as

$$\begin{bmatrix} u_0 k - \omega & h_0 k \\ 3\beta_2 h_0^2 k^3 + 6kg & -3\beta_1 \omega h_0^2 k^2 + 3\beta_1 h_0^2 k^3 u_0 - 2\omega h_0^2 k^2 - 6\omega + 2h_0^2 k^3 u_0 + 6ku_0 \end{bmatrix} \begin{bmatrix} H \\ U \end{bmatrix} = 0 \tag{9}$$

Which has non-trivial solutons when the determinant is zero.
The determinant of this matrix is

$$\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right)\omega^2 - 2u_0 k\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right)\omega + u_0^2 k^2\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right) - g h_0 k^2\left(3\beta_2 h_0^2 k^2 + 6\right) \tag{10}$$

To get non-trivial solution we have

$$\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right)\omega^2 - 2u_0 k\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right)\omega + u_0^2 k^2\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right) - g h_0 k^2\left(3\beta_2 h_0^2 k^2 + 6\right) = 0 \tag{11}$$

$$\omega^2 - 2u_0 k\omega + u_0^2 k^2 - g h_0 k^2 \frac{3\beta_2 h_0^2 k^2 + 6}{\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right)} \tag{12}$$

Using quadratic equation, or another quadratic polynomial solver we get that

$$\omega^{\pm} = u_0 k \pm k\sqrt{g h_0}\sqrt{\frac{3\beta_2 h_0^2 k^2 + 6}{\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right)}} \tag{13}$$

$$\omega^{\pm} = u_0 k \pm k\sqrt{g h_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2}} \tag{14}$$

Thus we have the following regimes

- SWWE wavespeeds (non-dispersive): occurs when $\beta_2 = \frac{2}{3} + \beta_1$ then $\omega^{\pm} = k\left(u_0 \pm \sqrt{g h_0}\right)$ ([1]).
- Serre wavespeeds - when $\beta_1 = \beta_2 = 0$ - we get $\omega^{\pm} = k\left(u_0 \pm \sqrt{g h_0}\sqrt{\frac{3}{3 + h_0^2 k^2}}\right)$
- For other values of β_1 and β_2 we can vary the wavespeeds, in certain situations this will lead to a better approximation of the linear wavespeed of waves than the Serre equations ([1]), in others it will be worse. We show an example below.

1.2.1 Wave Speed Bounds

First we want to show that phase and group speed are bounded, when treated as functions of k. The phase speed is:

$$v_p^{\pm} = \frac{\omega^{\pm}}{k} = u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \tag{15}$$

So we need to demonstrate that $\exists \alpha$ s.t $\forall k$

$$\frac{\beta_2 h_0^2 k^2 + 2}{\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2} \leq \alpha \tag{16}$$

This is equivalent to showing that $\exists \alpha$ s.t $\forall \mu$

$$\frac{a\mu^2 + 1}{b\mu^2 + 1} \leq \alpha \tag{17}$$

One such bound is simple enough to get and that is when $\alpha = \max\left\{1, \frac{a}{b}\right\}$ when $b \neq 0$ and $\alpha = \max\left\{1, a\mu^2 + 1\right\}$ when $b = 0$. So if $b = 0$ and $a \neq 0$ then the wavespeeds are no longer bounded. Thus we must restrict our numerical method to only allow $\beta_1 = -\frac{2}{3}$ only when $\beta_2 = 0$. Note that the regularised SWWE and the Serre equations fall under this condition (all our problems of interest) so its ok that our method is not appropriate for the condition.

Now for the group speed

$$v_g^{\pm} = \frac{\partial \omega^{\pm}}{\partial k} = u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm \frac{k\sqrt{gh_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\beta_2 h_0^2 k - \frac{h_0^2 k \left(\beta_1 + \frac{2}{3}\right)\left(\beta_2 h_0^2 k^2 + 2\right)}{h_0^2 k^2 \left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right] \tag{18}$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm \frac{k\sqrt{gh_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\frac{\beta_2 h_0^2 k^2 \left[h_0^2 k^2 \left(\beta_1 + \frac{2}{3}\right) + 2\right] - h_0^2 k \left(\beta_1 + \frac{2}{3}\right)\left(\beta_2 h_0^2 k^2 + 2\right)}{h_0^2 k^2 \left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm \frac{k\sqrt{gh_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\frac{\beta_2 h_0^4 k^3 \left(\beta_1 + \frac{2}{3}\right) + 2\beta_2 h_0^2 k - h_0^2 k \left(\beta_1 + \frac{2}{3}\right)\left(\beta_2 h_0^2 k^2 + 2\right)}{h_0^2 k^2 \left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm \frac{k\sqrt{gh_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\frac{\beta_2 h_0^4 k^3 \left(\beta_1 + \frac{2}{3}\right) + 2\beta_2 h_0^2 k - \beta_2 \left(\beta_1 + \frac{2}{3}\right) h_0^4 k^3 - 2 h_0^2 k \left(\beta_1 + \frac{2}{3}\right)}{h_0^2 k^2 \left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm \frac{k\sqrt{gh_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\frac{2\beta_2 h_0^2 k - 2 h_0^2 k \left(\beta_1 + \frac{2}{3}\right)}{h_0^2 k^2 \left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$

$$= \quad u_0 \;\pm\; \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3}+\beta_1\right) h_0^2 k^2 + 2\right)}} \;\pm\; 2\frac{k^2 h_0^2 \sqrt{gh_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3}+\beta_1\right) h_0^2 k^2 + 2\right)}}\left(\frac{\beta_2 - \beta_1 - \frac{2}{3}}{h_0^2 k^2\left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3}+\beta_1\right) h_0^2 k^2 + 2\right)}}\left[1 + \frac{\beta_2 - \beta_1 - \frac{2}{3}}{\left(\frac{1}{2}\beta_2 h_0^2 k^2 + 1\right)\left(\left(\frac{1}{3}+\beta_1\right) h_0^2 k^2 + 1\right)}\right]$$

When $\beta_2 = \frac{2}{3} + \beta_1$ then the group speed is equal to the phase speed of the Shallow Water Wave equations. When $\beta_2 = \beta_1 = 0$ then we get

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{2}{\left(\frac{2}{3} h_0^2 k^2 + 2\right)}}\left[1 + \frac{-\frac{1}{3}}{\left(\frac{1}{3} h_0^2 k^2 + 1\right)}\right]$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{3}{h_0^2 k^2 + 3}}\left[1 - \frac{1}{h_0^2 k^2 + 3}\right]$$

which matches [Zoppo,pitt,paper on FDVM methods for Serre equations / Chris thesis]

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