The Serre equations with surface tension are

$$h_t + (uh)_x = 0 \tag{1a}$$

$$(uh)_t + \left(u^2h + \frac{gh^2}{2} + \frac{h^3}{3}\left[(u_x)^2 - uu_{xx} - u_{xt}\right] - \tau\left(h\frac{\partial^2 h}{\partial x^2} - \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)\right)_x = 0 \tag{1b}$$

rewriting (1b) gives

$$hu_t + uh_t + 2uu_xh + u^2h_x + ghh_x + \left(\frac{h^3}{3}\left[(u_x)^2 - uu_{xx} - u_{xt}\right] - \tau\left(h\frac{\partial^2 h}{\partial x^2} - \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)\right)_x = 0$$

substituting (1a)

$$hu_t - u\left(uh_x + hu_x\right) + 2uu_xh + u^2h_x + ghh_x + \left(\frac{h^3}{3}\left[(u_x)^2 - uu_{xx} - u_{xt}\right] - \tau\left(h\frac{\partial^2 h}{\partial x^2} - \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)\right)_x = 0$$

$$hu_t + uu_xh + ghh_x + \left(\frac{h^3}{3}\left[(u_x)^2 - uu_{xx} - u_{xt}\right] - \tau\left(h\frac{\partial^2 h}{\partial x^2} - \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)\right)_x = 0$$

divide by $h$

$$u_t + uu_x + gh_x + \frac{1}{h}\left(\frac{h^3}{3}\left[(u_x)^2 - uu_{xx} - u_{xt}\right] - \tau\left(h\frac{\partial^2 h}{\partial x^2} - \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)\right)_x = 0$$

Integrating the surface tension term we get

$$\left[\frac{1}{h}\left(-\tau\left(h\frac{\partial^2 h}{\partial x^2} - \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)\right)_x\right] = -\tau h_{xxx}$$

So we get

$$h_t + u_xh + uh_x = 0 \tag{2a}$$

$$u_t + uu_x + gh_x + \frac{1}{3h}\left(h^3\left[(u_x)^2 - uu_{xx} - u_{xt}\right]\right)_x - \tau h_{xxx} = 0 \tag{2b}$$

Want solutions of travelling wave form $h(\xi)$ and $u(\xi)$ where $\xi = x - ct$. For this to be a solution must satisfy (2). First we want to write these equations in terms of $\xi$

## 0.1 Mass

For (2a) using $[q(\xi)]_x = q'(\xi)\xi_x$ and $[q(\xi)]_t = q'(\xi)\xi_t$ we have

$$h'\xi_t + u'h\xi_x + uh'\xi_x = 0$$

since $\xi_x = 1$ and $\xi_t = -c$ then

$$-ch' + u'h + uh' = 0$$

Integrating we get

$$\int -ch' + u'h + uh' d\xi = \int 0 d\xi$$

$$\int -ch' + [uh]' d\xi = \int 0 d\xi$$

Combining the contstants of integration of both integrals into $A$ we get that

$$-ch + uh + A = 0$$

so we get

$$uh = ch - A$$

$$u = c - \frac{A}{h}$$

$$u(\xi) = c - \frac{A}{h(\xi)} \tag{3}$$

## 0.2 Momentum

Now we rewrite (2b) as a function of $\xi$, making use of

$$\begin{aligned}
[q(\xi)]_x &= q'(\xi) \\
[q(\xi)]_{xx} &= q''(\xi) \\
[q(\xi)]_{xxx} &= q'''(\xi) \\
[q(\xi)]_{xt} &= -cq''(\xi) \\
[q(\xi)]_{xxt} &= -cq'''(\xi) \\
[q(\xi)]_t &= -cq'(\xi)
\end{aligned}$$

using $\xi = x - ct$

we get from (2b)

$$-cu' + uu' + gh' + \frac{1}{3h}\left(h^3\left[(u')^2 - uu'' + cu''\right]\right)' - \tau h''' = 0$$

From (3) we have

$$\begin{aligned}
u &= c - \frac{A}{h} \\
u' &= A\frac{h'}{h^2} \\
u'' &= A\frac{hh'' - 2[h']^2}{h^3} \\
u''' &= A\frac{h^2h''' + 6[h']^3 - 6hh'h''}{h^4}
\end{aligned}$$

So we get that

$$-c\left[A\frac{h'}{h^2}\right]+\left[c-\frac{A}{h}\right]\left[A\frac{h'}{h^2}\right]+gh'$$
$$+\frac{1}{3h}\left(h^3\left[\left(A\frac{h'}{h^2}\right)^2-\left[c-\frac{A}{h}\right]\left[A\frac{hh''-2\left[h'\right]^2}{h^3}\right]+c\left[A\frac{hh''-2\left[h'\right]^2}{h^3}\right]\right]\right)'-\tau h'''=0$$

$$-\left[A^2\frac{h'}{h^3}\right]+gh'+\frac{1}{3h}\left(h^3\left[\left(A\frac{h'}{h^2}\right)^2+\left[\frac{A}{h}\right]\left[A\frac{hh''-2\left[h'\right]^2}{h^3}\right]\right]\right)'-\tau h'''=0$$

$$-\left[A^2\frac{h'}{h^3}\right]+gh'+\frac{1}{3h}\left(h^3\left[\left(A^2\frac{\left[h'\right]^2}{h^4}\right)+\left[A^2\frac{hh''-2\left[h'\right]^2}{h^4}\right]\right]\right)'-\tau h'''=0$$

$$-\left[A^2\frac{h'}{h^3}\right]+gh'+\frac{A^2}{3h}\left(h^3\left[\frac{hh''-\left[h'\right]^2}{h^4}\right]\right)'-\tau h'''=0$$

$$-A^2\frac{h'}{h^3}+gh'+\frac{A^2}{3h}\left(\frac{hh''-\left[h'\right]^2}{h}\right)'-\tau h'''=0$$

multiply by $h$

$$-A^2\frac{h'}{h^2}+ghh'+\frac{A^2}{3}\left(\frac{hh''-\left[h'\right]^2}{h}\right)'-\tau h'''=0$$

$$\frac{A^2}{3}\left(\frac{hh''-\left[h'\right]^2}{h}\right)'-\tau h'''=A^2\frac{h'}{h^2}-ghh'$$

Integrating we get

$$\int\frac{A^2}{3}\left(\frac{hh''-\left[h'\right]^2}{h}\right)'d\xi=\int A^2\frac{h'}{h^2}-ghh'+\tau h'''d\xi$$

$C$ constant of integration

$$\frac{A^2}{3}\left(\frac{hh''-\left[h'\right]^2}{h}\right)+C=\int A^2\frac{h'}{h^2}-ghh'+\tau h'''d\xi$$

Absorbing all contstants of intergration into C we get

$$\frac{A^2}{3}\left(\frac{hh''-\left[h'\right]^2}{h}\right)+C=-\frac{A^2}{h}-\frac{gh^2}{2}+\tau h''$$

Thus we have

$$\frac{A^2}{3}\left(hh'' - [h']^2\right) + Ch = -A^2 - \frac{gh^3}{2} + \tau hh''$$

$$\frac{A^2}{3}\left(hh'' - [h']^2\right) + \frac{gh^3}{2} = -A^2 - Ch + \tau hh''$$

divide by $h^2$

$$\frac{A^2}{3}\left(\frac{hh'' - [h']^2}{h^2}\right) + \frac{gh}{2} = \frac{-A^2}{h^2} - \frac{C}{h} + \frac{\tau h''}{h}$$

$$\frac{A^2}{3}\left(\frac{hh'' - [h']^2}{h^2}\right) - \frac{\tau h''}{h} + \frac{gh}{2} = \frac{-A^2}{h^2} - \frac{C}{h}$$

$$-\frac{A^2}{3}\left(\frac{[h']^2}{h^2}\right) + \frac{A^2}{3}\left(\frac{h''}{h}\right) - \frac{\tau h''}{h} + \frac{gh}{2} = \frac{-A^2}{h^2} - \frac{C}{h}$$

$$\frac{A^2 - 3\tau}{3}\left(\frac{h''}{h}\right) - \frac{A^2}{3}\left(\frac{[h']^2}{h^2}\right) + \frac{gh}{2} = \frac{-A^2}{h^2} - \frac{C}{h}$$

So we have two constants of integration which we can set as we like.

In summary we have the following equations that travelling wave solutions must satisfy, for a particular choice of $A$ and $C$.

$$\frac{A^2 - 3\tau}{3}\left(\frac{h''(\xi)}{h(\xi)}\right) - \frac{A^2}{3}\left(\frac{[h'(\xi)]^2}{[h(\xi)]^2}\right) + \frac{gh(\xi)}{2} = -\frac{A^2}{h^2(\xi)} - \frac{C}{h(\xi)} \tag{4a}$$

$$u(\xi) = c - \frac{A}{h(\xi)} \tag{4b}$$

## 0.3 Consistent with soliton

To get $u$ in the same form as the soliton for a solution we must set

$A = ca_0$ where $c = \sqrt{g(a_0 + a_1)}$

Thus we get

$$u(\xi) = c\left[1 - \frac{a_0}{h(\xi)}\right] \tag{5}$$

Thus the equation for $h$ becomes

$$\frac{(ca_0)^2 - 3\tau}{3}\left(\frac{h''(\xi)}{h(\xi)}\right) - \frac{(ca_0)^2}{3}\left(\frac{[h'(\xi)]^2}{[h(\xi)]^2}\right) + \frac{gh(\xi)}{2} = -\frac{(ca_0)^2}{h^2(\xi)} - \frac{C}{h(\xi)} \tag{6}$$

want the peakon solution

$$h(\xi) = a_0 + a_1 \exp\left(-\frac{\sqrt{3}}{a_0}|\xi|\right)$$

The derivatives of this function are

$$h'(\xi) = -\frac{\sqrt{3}a_1}{a_0}\frac{\xi}{|\xi|}\exp\left(-\frac{\sqrt{3}}{a_0}|\xi|\right)$$

Let's assume that $(ca_0)^2 - 3\tau = 0$ thus $(ca_0)^2 = 3\tau$ and $\tau = \frac{c^2 a_0^2}{3}$. So that the second derivative, is not needed, which is a delta function.

So we get

$$-\frac{(ca_0)^2}{3}\left(\frac{[h'(\xi)]^2}{[h(\xi)]^2}\right) + \frac{gh(\xi)}{2} = -\frac{(ca_0)^2}{h^2(\xi)} - \frac{C}{h(\xi)} \tag{7}$$

Multiplying out the $h(\xi)$

$$-\frac{(ca_0)^2}{3}\left(\frac{[h'(\xi)]^2}{[h(\xi)]}\right) + \frac{gh^2(\xi)}{2} = -\frac{(ca_0)^2}{h(\xi)} - C \tag{8}$$

$$C = \frac{(ca_0)^2}{3}\left(\frac{[h'(\xi)]^2}{[h(\xi)]}\right) - \frac{gh^2(\xi)}{2} - \frac{(ca_0)^2}{h(\xi)} \tag{9}$$

$$[h(\xi)]^2 = a_0^2 + 2a_0a_1 + a_1^2\exp\left(-2\frac{\sqrt{3}}{a_0}|\xi|\right)$$

$$[h'(\xi)]^2 = \frac{3a_1^2}{a_0^2}\exp\left(-2\frac{\sqrt{3}}{a_0}|\xi|\right)$$

$$C = \frac{(ca_0)^2}{3}\left(\frac{\frac{3a_1^2}{a_0^2}\exp\left(-2\frac{\sqrt{3}}{a_0}|\xi|\right)}{a_0 + a_1\exp\left(-\frac{\sqrt{3}}{a_0}|\xi|\right)}\right) - \frac{g}{2}\left[a_0^2 + 2a_0a_1 + a_1^2\exp\left(-2\frac{\sqrt{3}}{a_0}|\xi|\right)\right]h^2(\xi) - \frac{(ca_0)^2}{a_0 + a_1\exp\left(-\frac{\sqrt{3}}{a_0}|\xi|\right)} \tag{10}$$