Modified Dispersive Wave Equations

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Outline

- Motivation
- Equations
- Scheme
- Validation

Motivation - Water Waves

We require accurate models of water waves to understand natural hazards in particular

- Tsunamis
- Storm Surges

Motivation

(a) Sulawesi Tsunami (Indonesia, 2018).

(b) Hurricane Florence (U.S.A, 2018)

Motivation

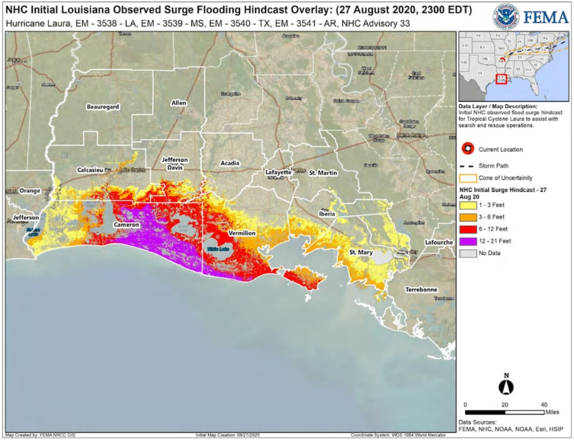

Figure: Hurricane Laura (U.S.A, 2020).

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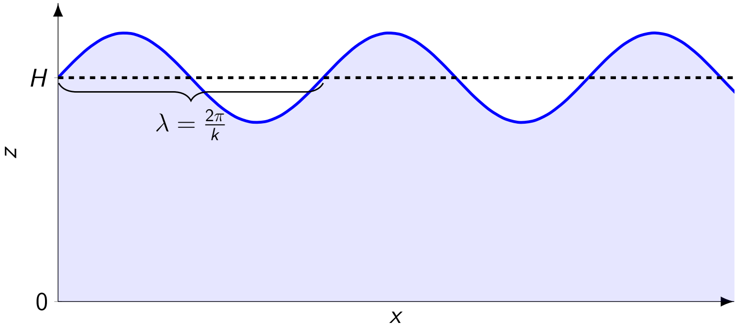
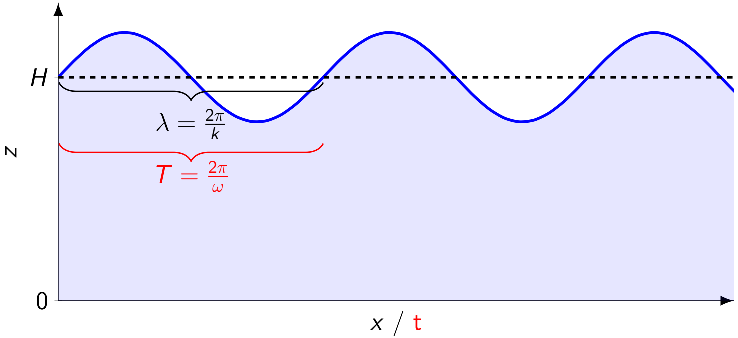
What's the effect of wave frequency effects on these natural hazards?

Dispersion Relationship of Water Waves (Linear Theory)



Dispersion Relationship of Water Waves (Linear Theory)



Dispersion Relationship of Water Waves (Linear Theory)

$$\omega^2 = gk \tanh(kH)$$

Gives the angular frequency (ω) as a function of gravitational acceleration (g), wave number (k) and mean background water depth (H).

Phase speed (c)

$$c^2 = \frac{\omega^2}{k^2} = \frac{g}{k} \tanh(kH)$$

Taylor Expansion of Phase Speed in k

$$c^2 = \frac{g}{k}\tanh\left(kH\right) = gH - \frac{1}{3}gH^3k^2 + \frac{2}{15}gH^5k^4 + \mathcal{O}\left(k^5\right)$$

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Resulting non-linear equations

- Non-dispersive Shallow Water Wave Equations
- Serre Equations
- Improved Dispersion Serre Equations

Modified Dispersive Wave Equations generalise all these.

Set Up

Equations for conservation of mass and momentum written in terms of the water depth $h(x,t)$, the depth average horizontal velocity $u(x,t)$ and acceleration due to gravity g.

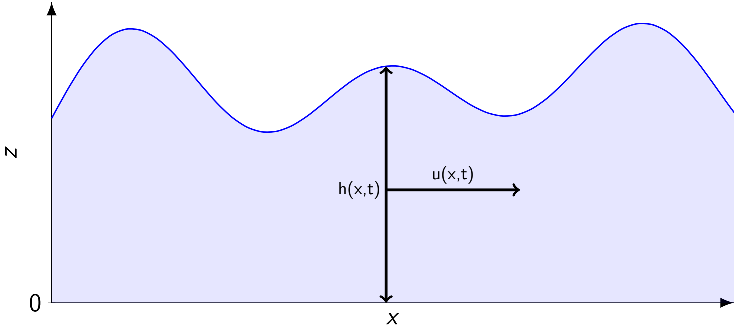

Figure: Relevant Quantities.

Modified Dispersive Wave Equations

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0$$
$$\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2 + \frac{1}{2}h^3\beta_1\Phi - \frac{1}{2}gh^2\beta_2\Psi\right) = 0$$

where

$$\Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x \partial t} - u\frac{\partial^2 u}{\partial x^2}$$
$$\Psi = h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}$$

Dispersion Relation of Linearised Equations

$$
\begin{aligned}
c_{\text{model}}^2 &= gH\frac{\beta_2 H^2 k^2 + 2}{\beta_1 H^2 k^2 + 2} \\
&= gH - \frac{(\beta_1 - \beta_2)}{2} gH^3 k^2 + \frac{\beta_1 (\beta_1 - \beta_2)}{4} gH^5 k^4 + \mathcal{O}\left(k^5\right)
\end{aligned}
$$

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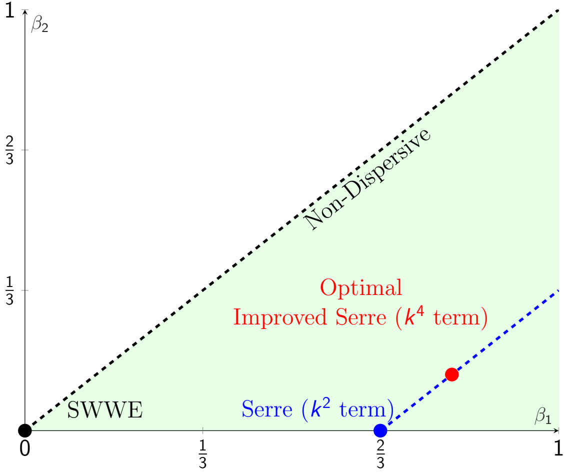
\end{aligned}
$$

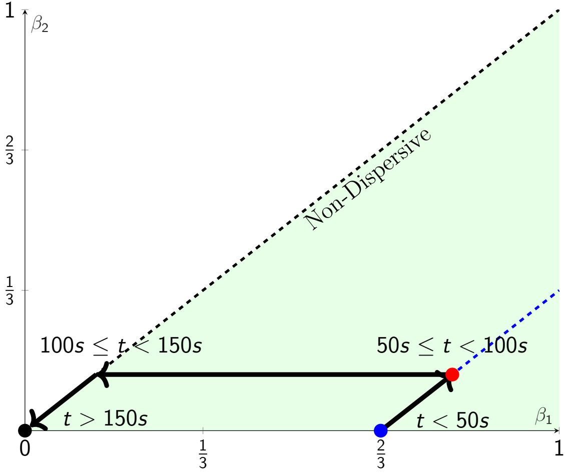
Want

$$
\begin{aligned}
c^2 &= \frac{g}{k} \tanh\left(kH\right) \\
&= gH - \frac{1}{3} gH^3 k^2 + \frac{2}{15} gH^5 k^4 + \mathcal{O}\left(k^5\right)
\end{aligned}
$$

Dispersion Regions



Dispersion Regions - First Animation



Model - Conclusion

- We now have one set of equations we can solve that includes
  - Non-dispersive wave models
  - k^2 accurate dispersive wave models
  - k^4 accurate dispersive wave models

We can thus implement one numerical scheme and investigate the impact of different β values to investigate effect of dispersion on natural hazards.

Equations

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0$$
$$\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2 + \frac{h^3}{2}\beta_1 \Phi - \frac{gh^2}{2}\beta_2 \Psi\right) = 0$$

where

$$\Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x \partial t} - u\frac{\partial^2 u}{\partial x^2}$$
$$\Psi = h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}$$

Reformulation

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \beta_1 h^3 \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{2}\beta_2 g h^2 \left[h \frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x} \right] \right) = 0.$$

where

$$G = uh - \frac{1}{2}\beta_1 \frac{\partial}{\partial x}\left(h^3 \frac{\partial u}{\partial x} \right).$$

Central Idea - Starting with h and G

Solve

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to obtain u.

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Then solve

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \beta_1 h^3 \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{2}\beta_2 g h^2 \left[h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right]\right) = 0.$$

with a finite volume method to update h and G to next time.

First Example - Extension of Previous Methods for Serre

- Solve u given h and G, using a second-order finite difference approximation.
- Update h and G using a second-order finite volume approximation.

Why Finite Volume?

The central reason is robustness.

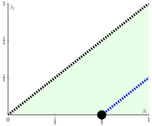
- Equations possess weak solutions with discontinuities
  - Shallow Water Wave Equations (Shocks - Jump Discontinuity)
  - Certain Parameter Combinations (Weak Discontinuities - Continuous with discontinuous derivative)
- Reproduction of those underlying conservation properties.

We were able to produce the first well validated method this way.

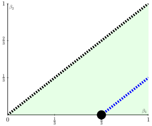
Validation

Analytic solutions only known for some β values.

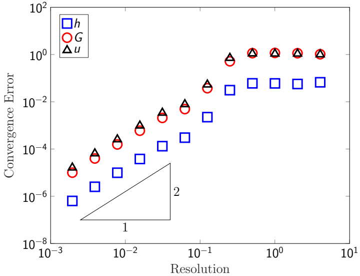
Just k^2 accurate Serre today.

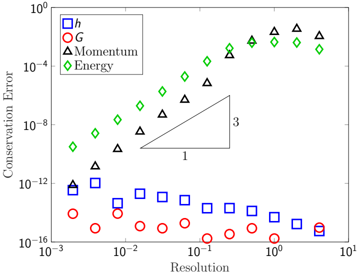


Serre (k^2 accurate) Analytic Solution



Serre (k^2 accurate) Error





Conclusions

- Can solve these equations with our scheme
- First well validated robust method

Good progress towards addressing the motivation.

Thanks!