Numerical Scheme for Generalised Serre Green Naghdi Equations

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Outline

- Motivation
- Equations
- Scheme
- Validation

Motivation - Water Waves

We require accurate models of water waves to understand natural hazards in particular

- Tsunamis
- Storm Surges

Motivation

(a) Sulawesi Tsunami (Indonesia, 2018).

(b) Hurricane Florence (U.S.A, 2018)

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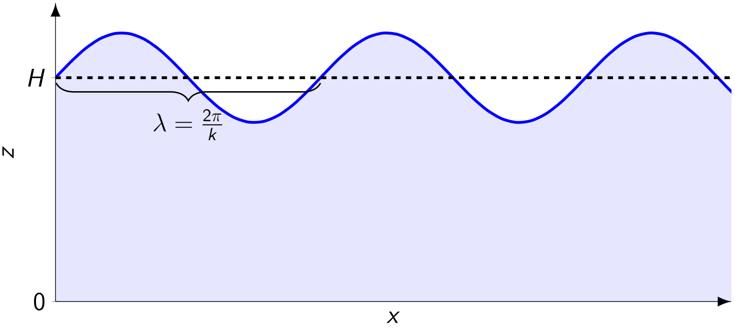
What's the effect of wave frequency effects on these natural hazards?

Dispersion Relationship of Water Waves (Linear Theory)



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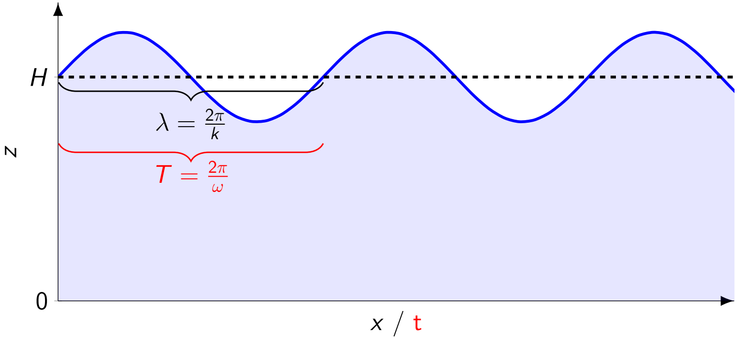

Dispersion Relationship of Water Waves (Linear Theory)

$$\omega^2 = gk \tanh(kH)$$

Gives the angular frequency (ω) as a function of gravitational acceleration (g), wave number (k) and mean background water depth (H).

Phase speed (c)

$$c^2 = \frac{\omega^2}{k^2} = \frac{g}{k} \tanh(kH)$$

Taylor Expansion of Phase Speed in k

$$c^2 = \frac{g}{k}\tanh\left(kH\right) = gH - \frac{g}{3}H^3k^2 + \frac{2}{15}gH^5k^4 + \mathcal{O}\left(k^5\right)$$

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Resulting non-linear equations

- Non-dispersive Shallow Water Wave Equations
- Serre Equations
- Improved Dispersion Serre Equations

Set Up

Equations for conservation of mass and momentum written in terms of the water depth $h(x,t)$, the depth average horizontal velocity $u(x,t)$ and acceleration due to gravity g.

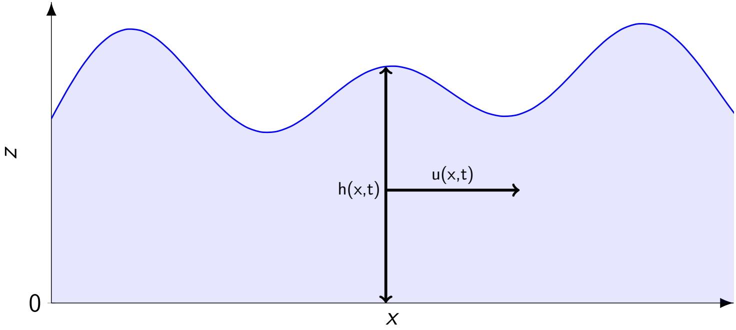


Figure: Relevant Quantities.

Generalised Dispersive Equations

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0$$

$$\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2 + \frac{h^3}{2}\beta_1 \Phi - \frac{gh^2}{2}\beta_2 \Psi\right) = 0$$

where

$$\Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x \partial t} - u\frac{\partial^2 u}{\partial x^2}$$

$$\Psi = h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}$$

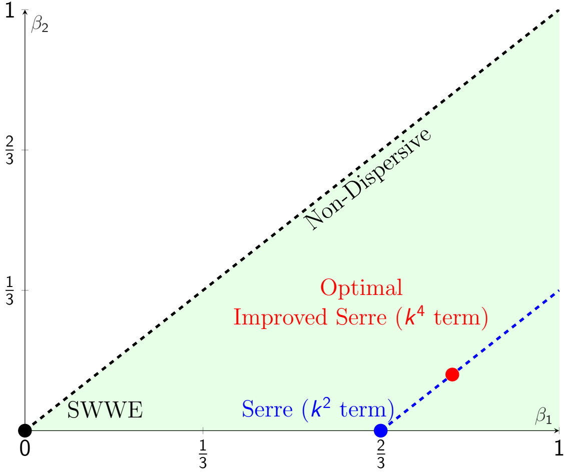
Dispersion Relation of Linearised Equations

$$c^2_{\text{model}} = gH\frac{\beta_2 H^2 k^2 + 2}{\beta_1 H^2 k^2 + 2}$$
$$= gH - \frac{1}{2} gH^3 \left(\beta_1 - \beta_2\right) k^2 + \frac{1}{4}\beta_1 H^5 k^4 \left(\beta_1 - \beta_2\right) + \mathcal{O}\left(k^5\right)$$

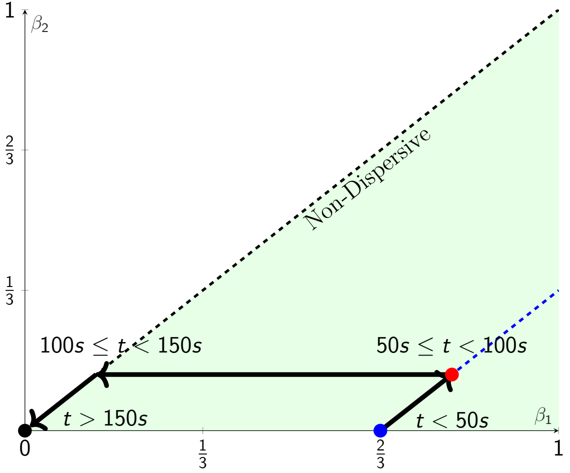
Want

$$c^2 = \frac{g}{k}\tanh\left(kH\right)$$
$$= gH - \frac{g}{3} H^3 k^2 + \frac{2}{15} gH^5 k^4 + \mathcal{O}\left(k^5\right)$$

Dispersion Regions



Dispersion Regions - First Animation



Model - Conclusion

- We now have one set of equations we can solve that includes
  - Non-dispersive wave models
  - k^2 accurate dispersive wave models
  - k^4 accurate dispersive wave models

We can thus implement one numerical scheme and investigate the impact of different β values to investigate effect of dispersion on natural hazards.

Equations

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0$$
$$\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2 + \frac{h^3}{2}\beta_1 \Phi - \frac{gh^2}{2}\beta_2 \Psi\right) = 0$$

where

$$\Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x \partial t} - u\frac{\partial^2 u}{\partial x^2}$$
$$\Psi = h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}$$

Reformulation

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}\beta_1 h^3 \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{2}\beta_2 g h^2 \left[h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x} \right] \right) = 0.$$

where

$$G = uh - \frac{1}{2}\beta_1 \frac{\partial}{\partial x}\left(h^3 \frac{\partial u}{\partial x} \right).$$

Central Idea - Starting with h and G

Solve

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Then solve

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}\beta_1 h^3 \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{2}\beta_2 g h^2 \left[h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right]\right) = 0.$$

with a finite volume method to update h and G to next time.

First Example - Extension of Previous Methods for Serre

- Solve u given h and G, using a second-order finite difference approximation.
- Update h and G using a second-order finite volume approximation.

Why Finite Volume?

The central reason is robustness.

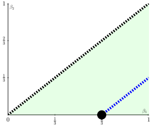
- Equations possess weak solutions with discontinuities
  - Shallow Water Wave Equations (Shocks - Jump Discontinuity)
  - Certain Parameter Combinations (Weak Discontinuities - Continuous with discontinuous derivative)
- Reproduction of those underlying conservation properties.

We were able to produce the first well validated method this way.

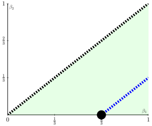
Validation

Analytic solutions only known for some β values.

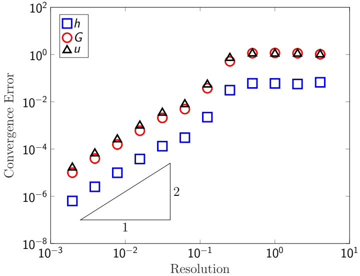
Just k^2 accurate Serre today.

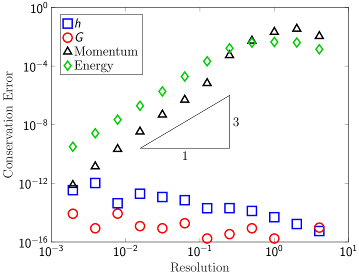


Serre (k^2 accurate) Analytic Solution



Serre (k^2 accurate) Error





Conclusions

- Can solve these equations with our scheme
- First well validated robust method

Good progress towards addressing the motivation.

Thanks!