

Numerical Study of The Generalised Serre-Green-Naghdi Model

1. Lagrangian

These were derived to reduce to SWWE when bed varies, and allow the dispersion regularisation. For this purpose the lagrangian works, but it may not reduce to previous definition of SGN equations with a varying bed. In fact, it seems like it doesnt.

The lagrangian for the gSGN was derived as

$$\mathcal{L}_{gSGN} = \frac{1}{2}hu^2 - \frac{1}{2}gh^2 + \left(\frac{1}{6} + \frac{1}{4}\beta_1\right)h^3u_x^2 - \frac{1}{4}\beta_2gh^2h_x^2 + [h_t + (uh)_x]\phi$$

The lagrangian derived by [] for the rSV with a varying bottom was

$$\mathcal{L}_{rSVB} = \frac{1}{2}hu^2 - \frac{1}{2}g\eta^2 + \epsilon\frac{1}{2}h^3u_x^2 - \frac{1}{2}\epsilon gh^2\eta_x^2 + [h_t + (uh)_x]\phi$$

Clearly, to ensure that the rSV with a bed remains consistent with the lagrangian for the gSGN with a flat bed (where $h = \eta$), we can separete these two ϵ terms into ϵ_1 which maps to β_1 and ϵ_2 which maps to β_2 to recover the information lost when performing the transformation from gSGN to rSV in the first paper, defined by $\beta_1 = 2\epsilon - \frac{2}{3}$ and $\beta_2 = 2\epsilon$. Thus for the dispersive terms we can transfer $\epsilon \mapsto \frac{1}{2}(\beta_1 + \frac{2}{3})$ and for the surface tension regularisation we have $\epsilon \mapsto \frac{1}{2}\beta_2$. However, we can only recover the gSGN if we are careful about which epsilon maps to which β , since we have lost information in the gSGN to rSWWE transition.

This leads to

$$\mathcal{L}_{gSGNB} = \frac{1}{2}hu^2 - \frac{1}{2}g\eta^2 + \epsilon_1\frac{1}{2}h^3u_x^2 - \frac{1}{2}\epsilon_2gh^2\eta_x^2 + [h_t + (uh)_x]\phi$$

which reduces to \mathcal{L}_{gSGN} when $\epsilon_1 \mapsto \frac{1}{2}(\beta_1 + \frac{2}{3})$ and $\epsilon_2 \mapsto \frac{1}{2}\beta_2$ and the bed is flat and thus $h = \eta$.

Following [] we thus obtain

$$\frac{\partial \mathcal{L}}{\partial \phi} : h_t + [uh]_x = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial u} : hu - h\phi_x - \epsilon_1 [h^3u_x] = 0 \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \eta} : \frac{1}{2}u^2 - g\eta - \phi_t - u\phi_x + \frac{3}{2}\epsilon_1h^2u_x^2 + \epsilon_2g [h^2\eta_x]_x - \epsilon_2gh\eta_x^2 = 0 \quad (3)$$

2. Equations

With a flat bed

$$\frac{\partial h}{\partial t} + \frac{\partial(hu)}{\partial x} = 0 \quad (4a)$$

$$\frac{\partial(hu)}{\partial t} + \frac{\partial}{\partial x} \left(hu^2 + \frac{1}{2}gh^2 + \frac{1}{3}h^2\Gamma \right) = 0 \quad (4b)$$

$$\frac{\partial(\mathcal{E})}{\partial t} + \frac{\partial}{\partial x} \left[hu \left(\frac{1}{2}u^2 + \frac{1}{4} \left(\frac{2}{3} + \beta_1 \right) h^2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + gh \left(1 + \frac{1}{4}\beta_2 \frac{\partial h}{\partial x} \frac{\partial h}{\partial x} \right) + \frac{1}{3}h\Gamma \right) + \frac{1}{2}\beta_2gh^3 \frac{\partial h}{\partial x} \frac{\partial u}{\partial x} \right] = 0 \quad (4c)$$

where

$$\Gamma = \frac{3}{2} \left(\frac{2}{3} + \beta_1 \right) h \left[\frac{\partial u}{\partial x} \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x \partial t} - u \frac{\partial^2 u}{\partial x^2} \right] - \frac{3}{2} \beta_2 g \left[h \frac{\partial^2 h}{\partial x^2} + \frac{1}{2} \frac{\partial h}{\partial x} \frac{\partial h}{\partial x} \right] \quad (4d)$$

$$\mathcal{E} = \frac{1}{2} h u^2 + \frac{1}{4} \left(\frac{2}{3} + \beta_1 \right) h^3 \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + \frac{1}{2} g h^2 \left(1 + \frac{1}{2} \beta_2 \frac{\partial h}{\partial x} \frac{\partial h}{\partial x} \right) \quad (4e)$$

To get rSWWE with a flat bed we set $\beta_1 = 2\epsilon - \frac{2}{3}$ and $\beta_2 = 2\epsilon$. Thus for the dispersive terms we can transfer $\epsilon \mapsto \frac{1}{2} (\beta_1 + \frac{2}{3})$ and for the surface tension regularisation we have $\epsilon \mapsto \frac{1}{2} \beta_2$. However, we can only recover the gSGN if we are careful about which epsilon maps to which β , since we have lost information in the gSGN to rSWWE transition.

The Clamond rSWWE with an uneven bathymetry, where ϵ subscripts are used to denote either dispersive (1) or surface tension (2) term are

$$\begin{aligned} h_t + [hu]_x &= 0, \\ G_t + \left[uG + \frac{1}{2} g h^2 - 2\epsilon_1 h^3 u_x^2 - \epsilon_2 \left(g h^3 \eta_{xx} + \frac{1}{2} g h^2 \eta_x^2 + g h^2 \eta_x d_x \right) \right]_x &= g h d_x + \epsilon_2 g h^2 \eta_x d_{xx} \\ G &= u h - \epsilon_1 [h^3 u_x]_x \end{aligned}$$

The SGN equations with a bottom by

$$\begin{aligned} h_t + [hu]_x &= 0, \\ G_t + \left[uG + \frac{1}{2} g h^2 - 2\epsilon_1 h^3 u_x^2 - \epsilon_2 \left(g h^3 \eta_{xx} + \frac{1}{2} g h^2 \eta_x^2 + g h^2 \eta_x d_x \right) \right]_x &= g h d_x + \epsilon_2 g h^2 \eta_x d_{xx} \end{aligned}$$