The regularised Serre equations are

$$h_t + (uh)_x = 0 \tag{1a}$$

$$(uh)_t + \left(u^2 h + \frac{gh^2}{2} + \epsilon h^2 \left[h\left[(u_x)^2 - uu_{xx} - u_{xt}\right] - g\left(h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)\right]\right)_x = 0 \tag{1b}$$

rewriting (1b) gives

$$hu_t + uh_t + 2uu_x h + u^2 h_x + ghh_x + \left(\epsilon h^2 \left[h\left[(u_x)^2 - uu_{xx} - u_{xt}\right] - g\left(h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)\right]\right)_x = 0$$

substituting (1a)

$$hu_t - u\left(uh_x + hu_x\right) + 2uu_x h + u^2 h_x + ghh_x + \left(\epsilon h^2 \left[h\left[(u_x)^2 - uu_{xx} - u_{xt}\right] - g\left(h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)\right]\right)_x = 0$$

$$hu_t + uu_x h + ghh_x + \left(\epsilon h^2 \left[h\left[(u_x)^2 - uu_{xx} - u_{xt}\right] - g\left(h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)\right]\right)_x = 0$$

divide by h

$$u_t + uu_x + gh_x + \frac{\epsilon}{h}\left(h^2 \left[h\left[(u_x)^2 - uu_{xx} - u_{xt}\right] - g\left(h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)\right]\right)_x = 0$$

So we get

$$h_t + u_x h + uh_x = 0 \tag{2a}$$

$$u_t + uu_x + gh_x + \frac{\epsilon}{h}\left(h^2 \left[h\left[(u_x)^2 - uu_{xx} - u_{xt}\right] - g\left(h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)\right]\right)_x = 0 \tag{2b}$$

Want solutions of travelling wave form $h(\xi)$ and $u(\xi)$ where $\xi = x - ct$. For this to be a solution must satisfy (2). First we want to write these equations in terms of ξ

For (2a) using $[q(\xi)]_x = q'(\xi)\xi_x$ and $[q(\xi)]_t = q'(\xi)\xi_t$ we have

$$h'\xi_t + u'h\xi_x + uh'\xi_x = 0$$

since $\xi_x = 1$ and $\xi_t = -c$ then

$$-ch' + u'h + uh' = 0$$

Integrating we get

$$\int -ch' + u'h + uh' d\xi = \int 0 d\xi$$

$$\int -ch' + [uh]' \, d\xi = \int 0 d\xi$$

Combining the contstants of integration of both integrals into A we get that

$$-ch + uh + A = 0$$

so we get

$$uh = ch - A$$

$$u = c - \frac{A}{h}$$

$$u(\xi) = c - \frac{A}{h(\xi)} \tag{3}$$

Now we rewrite (2b) as a function of ξ, making use of

$$\begin{aligned}
[q(\xi)]_x &= q'(\xi) \\
[q(\xi)]_{xx} &= q''(\xi) \\
[q(\xi)]_{xxx} &= q'''(\xi) \\
[q(\xi)]_{xt} &= -cq''(\xi) \\
[q(\xi)]_{xxt} &= -cq'''(\xi) \\
[q(\xi)]_t &= -cq'(\xi)
\end{aligned}$$

using $\xi = x - ct$

we get from (2b)

$$-cu' + uu' + gh' + \frac{\epsilon}{h}\left(h^2\left[h\left[(u')^2 - uu'' + cu''\right] - g\left(hh'' + \frac{1}{2}h'h'\right)\right]\right)' = 0$$

$$-cu' + uu' + gh' + \frac{\epsilon}{h}\left(h^3\left[(u')^2 - uu'' + cu''\right] - g\left(h^3h'' + \frac{1}{2}h^2h'h'\right)\right)' = 0$$

From (3) we have

$$\begin{aligned}
u &= c - \frac{A}{h} \\
u' &= A\frac{h'}{h^2} \\
u'' &= A\frac{hh'' - 2\left[h'\right]^2}{h^3} \\
u''' &= A\frac{h^2h''' + 6\left[h'\right]^3 - 6hh'h''}{h^4}
\end{aligned}$$

So we get that

$$-c\left[A\frac{h'}{h^2}\right]+\left[c-\frac{A}{h}\right]\left[A\frac{h'}{h^2}\right]+gh'$$
$$+\frac{\epsilon}{h}\left(h^3\left[\left(A\frac{h'}{h^2}\right)^2-\left[c-\frac{A}{h}\right]\left[A\frac{hh''-2\left[h'\right]^2}{h^3}\right]+c\left[A\frac{hh''-2\left[h'\right]^2}{h^3}\right]\right]-g\left(h^3h''+\frac{1}{2}h^2h'h'\right)\right)'=0$$

$$-\left[A^2\frac{h'}{h^3}\right]+gh'+\frac{\epsilon}{h}\left(h^3\left[\left(A\frac{h'}{h^2}\right)^2+\left[\frac{A}{h}\right]\left[A\frac{hh''-2\left[h'\right]^2}{h^3}\right]\right]-g\left(h^3h''+\frac{1}{2}h^2h'h'\right)\right)'=0$$

$$-\left[A^2\frac{h'}{h^3}\right]+gh'+\frac{\epsilon}{h}\left(h^3\left[\left(A^2\frac{\left[h'\right]^2}{h^4}\right)+\left[A^2\frac{hh''-2\left[h'\right]^2}{h^4}\right]\right]-g\left(h^3h''+\frac{1}{2}h^2h'h'\right)\right)'=0$$

$$-\left[A^2\frac{h'}{h^3}\right]+gh'+\frac{\epsilon}{h}\left(A^2h^3\left[\frac{hh''-\left[h'\right]^2}{h^4}\right]-g\left(h^3h''+\frac{1}{2}h^2h'h'\right)\right)'=0$$

$$-A^2\frac{h'}{h^3}+gh'+\frac{\epsilon}{h}\left(A^2\frac{hh''-\left[h'\right]^2}{h}-gh^2\left(hh''+\frac{1}{2}h'h'\right)\right)'=0$$

multiply by h

$$-A^2\frac{h'}{h^2}+ghh'+\epsilon\left(A^2\frac{hh''-\left[h'\right]^2}{h}-gh^2\left(hh''+\frac{1}{2}h'h'\right)\right)'=0$$

$$\epsilon\left(A^2\frac{hh''-\left[h'\right]^2}{h}-gh^2\left(hh''+\frac{1}{2}h'h'\right)\right)'=A^2\frac{h'}{h^2}-ghh'$$

Integrating we get

$$\int\epsilon\left(A^2\frac{hh''-\left[h'\right]^2}{h}-gh^2\left(hh''+\frac{1}{2}h'h'\right)\right)'\,d\xi=\int A^2\frac{h'}{h^2}-ghh'd\xi$$

C constant of integration

$$\epsilon\left(A^2\frac{hh''-\left[h'\right]^2}{h}-gh^2\left(hh''+\frac{1}{2}h'h'\right)\right)+C=\int A^2\frac{h'}{h^2}-ghh'd\xi$$

Absorbing all contstants of intergration into C we get

$$\epsilon\left(A^2\frac{hh''-[h']^2}{h}-gh^2\left(hh''+\frac{1}{2}h'h'\right)\right)+C=-\frac{A^2}{h}-\frac{gh^2}{2}$$

Thus we have

$$\epsilon\left(A^2\left(hh''-[h']^2\right)-gh^3\left(hh''+\frac{1}{2}h'h'\right)\right)+Ch=-A^2-\frac{gh^3}{2}$$

$$\epsilon\left(A^2\left(hh''-[h']^2\right)-gh^3\left(hh''+\frac{1}{2}h'h'\right)\right)+\frac{gh^3}{2}=-A^2-Ch$$

divide by h^2

$$\epsilon\left(A^2\left(\frac{hh''-[h']^2}{h^2}\right)-gh\left(hh''+\frac{1}{2}h'h'\right)\right)+\frac{gh}{2}=\frac{-A^2}{h^2}-\frac{C}{h}$$

So we have two constants of integration which we can set as we like.

In summary we have the following equations that travelling wave solutions must satisfy, for a particular choice of A and C.

$$\epsilon\left(A^2\left(\frac{h(\xi)h(\xi)''-[h(\xi)']^2}{h(\xi)^2}\right)-gh(\xi)\left(h(\xi)h(\xi)''+\frac{1}{2}h(\xi)'h(\xi)'\right)\right)+\frac{gh(\xi)}{2}=\frac{-A^2}{h(\xi)^2}-\frac{C}{h(\xi)} \tag{4a}$$

$$u=c-\frac{A}{h} \tag{4b}$$

1 SWWE $\epsilon = 0$

When $\epsilon=0$ we get that

$$\frac{gh}{2}=\frac{-A^2}{h^2}-\frac{C}{h}$$
$$u(\xi)=c-\frac{A}{h(\xi)}$$

Which is equivalent to

$$h\left[\frac{gh^2}{2}+C\right]=-A^2$$
$$u(\xi)=c-\frac{A}{h(\xi)}$$

So we must have that $h\left[\frac{gh^2}{2}+C\right]$ is constant over ξ.

Assuming smooth functions, the only function that satisfied this condition are those where $h(\xi)$ is constant.