The Serre equations are

$$h_t + [uh]_x = 0 \tag{1a}$$

$$u_t + h_x + uu_x - \frac{1}{3h}\frac{\partial}{\partial x}\left[h^3(u_{xt} + uu_{xx} - (u_x)^2)\right] = 0. \tag{1b}$$

A horizontal bottom is located at $z = -d_0$ below the still water surface, at $z = 0$ and the free surface is located $\eta(x,t)$ above the still water level. Therefore, the water depth is $h(x,t) = \eta(x,t) + d_0$. The depth average fluid velocity is $u(x,t)$.

Assuming that the solution has the form of a travelling wave solution, where $h(x,t) = h(\xi)$, $u(x,t) = u(\xi)$ and $\xi = x - ct$, where c is a constant.

Substituting into the (1) to obtain a relationship between, $\eta(\xi)$ and $u(\xi)$, then

$$h'(\xi)\xi_t + u(\xi)h(\xi)'\xi_x + h(\xi)u(\xi)'\xi_x = 0 \tag{2a}$$

$$u'(\xi)\xi_t + h(\xi)'\xi_x + u(\xi)u(\xi)'\xi_x \tag{2b}$$

$$-\frac{1}{3h(\xi)}\left[h(\xi)^3(u(\xi)''\xi_x\xi_t + u(\xi)u(\xi)''\xi_x\xi_x - (u(\xi)'\xi_x)^2)\right]' = 0. \tag{2c}$$

Now $\partial\xi//\partial x = 1$ and $\partial\xi/\partial t = -c$, then (2) become

$$-ch' + uh' + hu' = 0 \tag{3a}$$

$$-cu(\xi)' + h(\xi)' + u(\xi)u(\xi)' - \frac{1}{3h(\xi)}\left[h^3(-cu(\xi)'' + u(\xi)u(\xi)'' - (u(\xi)')^2)\right]' = 0. \tag{3b}$$

which are functions of ξ only.

Integrating (3a)

$$\int -ch(\xi)' + u(\xi)h(\xi)' + h(\xi)u(\xi)'\,d\xi = 0$$

and substituting $h = \eta + 1$, where we have chosen $d_0 = 1$ then

$$-ch(\xi) + \int u(\xi)\eta(\xi)' + (\eta(\xi) + 1)u(\xi)'\,d\xi = 0$$

or

$$-ch(\xi) + u(\xi) + \int u(\xi)\eta(\xi)' + \eta(\xi)u(\xi)'\,d\xi = 0. \tag{4}$$

Now using the integration by parts on the first term in the integrand

$$\int u(\xi)\eta(\xi)'\,d\xi = u(\xi)\eta(\xi) - \int u(\xi)'\eta(\xi)'\,d\xi.$$

Substituting into (4), then

$$-ch(\xi) + u(\xi) + u(\xi)\eta(\xi) = 0.$$

Rearranging, then

$$u(\xi) = \frac{c\eta(\xi)}{1+\eta}$$

or

$$u(\xi) = \frac{c(h(\xi)-1)}{h(\xi)} \tag{5}$$

which is a relationship between $u(\xi)$ and $h(\xi)$.

Returning to the momentum equation, (2), which will provide travelling wave solution. The relationships, $u(\xi)'$ and $u(\xi)''$ are required, which are

$$u(\xi)' = \frac{ch(\xi)'}{h(\xi)^2}$$

and

$$u(\xi)'' = \frac{c(h(\xi)''h(\xi) - 2h(\xi)^2)}{h(\xi)^3}.$$

Substituting into (2), then

$$-\frac{c^2h(\xi)'}{h(\xi)^2} + \frac{cu(\xi)h(\xi)'}{h(\xi)^2} + h(\xi)'$$

$$= -\frac{1}{3h(\xi)}\left(\frac{1}{h(\xi)}\left[-c^2h(\xi)\left(h(\xi)''h(\xi) - 2(h(\xi)')^2\right) + u(\xi)ch(\xi)\left(h(\xi)''h(\xi) - 2(h(\xi)')^2\right) - c^2(h(\xi)')^2\right]\right)'$$

Making use of (5), then

$$-\frac{c^2h(\xi)'}{h(\xi)^2} + \frac{c^2(h(\xi)-1)h(\xi)'}{h(\xi)^3} + h(\xi)' = -\frac{c^2}{3h(\xi)}\left(\frac{h(\xi)''h(\xi) - (h(\xi)')^2}{h(\xi)}\right)'.$$

Simplifying

$$-\frac{c^2h(\xi)'}{h(\xi)^3} + h(\xi)' = -\frac{c^2}{3h(\xi)}\left(\frac{h(\xi)''h(\xi) - (h(\xi)')^2}{h(\xi)}\right)'.$$

Integrating

$$\frac{c^2}{2h(\xi)^2} + h(\xi) = ??????????$$

The above is what I get. I am not sure how to integrate the last term in the above equation, hence ?????? above.

This is what Demitri has:

Making use of (5), then

$$-c\frac{h(\xi)'}{h(\xi)^2} + c^2\frac{(h(\xi)-1)h(\xi)'}{h(\xi)^2} + h(\xi)' = -c^2\frac{1}{3h(\xi)}\left(\frac{h(\xi)''h(\xi) - (h(\xi)')^2}{h(\xi)}\right)'.$$

After simplificationn of same terms this gives

$$-c^2 \frac{h(\xi)'}{h(\xi)^2} + h\xi h(\xi)' = -c^2 \frac{1}{3h(\xi)} \left(\frac{h(\xi)'' h(\xi) - (h(\xi)')^2}{h(\xi)} \right)'.$$

which after Integrating and division by h becomes

$$c^2 \frac{1}{h(\xi)^2} + \frac{h(\xi)}{2} = -\frac{c^2}{3} \left(\frac{h(\xi)'}{h(\xi)} \right) + \left(c^2 + \frac{1}{2} \right) \frac{1}{h(\xi)}$$

I am not sure how he got this, and I am not sure that it is correct.