Serre Equations with Weak Surface Tension

Contents

1. Equations

The Serre equations with surface tension were derived by [1]

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0 \tag{1a}$$

$$\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2 + \frac{1}{3}h^3\Gamma - \tau\mathcal{T}\right) = 0 \tag{1b}$$

$$\frac{\partial \left(\mathcal{E}\right)}{\partial t} + \frac{\partial}{\partial x}\left[hu\left(\frac{u^2}{2} + \frac{h^2}{6}\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + gh + \frac{h^2}{3}\Gamma - \tau\frac{\partial^2 h}{\partial x^2}\right) + \tau\frac{\partial h}{\partial x}\frac{\partial}{\partial x}\left(uh\right)\right] = 0 \tag{1c}$$

where

$$\Gamma = \left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x \partial t} - u\frac{\partial^2 u}{\partial x^2}\right] \tag{1d}$$

$$\mathcal{T} = h\frac{\partial^2 h}{\partial x^2} - \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x} \tag{1e}$$

$$\mathcal{H} = \frac{1}{2}\left[uh^2 + \frac{h^3}{3}\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + gh^2 + \tau\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right] \tag{1f}$$

Can be rearranged to be

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0 \tag{2a}$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \tau\mathcal{T}\right) = 0 \tag{2b}$$

$$\tag{2c}$$

where

$$G = hu - \frac{1}{2}\left(\frac{2}{3} + \beta_1\right)\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right) \tag{2d}$$

2. Peakon

The peakon solution of the serre equations with surface tension are

$$h(x,0) = a_0 + a_1 \exp\left(-\frac{\sqrt{3}}{a_0}\left|x - ct\right|\right) \tag{3a}$$

$$u(x,0) = c\left(\frac{h(x,0) - a_0}{h(x,0)}\right) \tag{3b}$$

where

$$c = \sqrt{1 + \frac{a_1}{a_0}} \tag{3c}$$

$$g = 1 \tag{3d}$$

$$G = uh - h^2 h_x u_x - h^3 u_{xx}/3 \tag{4}$$

$$u_x = ca_0 \frac{h_x}{h^2} u_{xx} \qquad = ca_0 \frac{hh_{xx} - 2h_x^2}{h^3} \tag{5}$$

$$h_x = h_{xx} \qquad = \tag{6}$$

3. References

[1] D. Mitsotakis, D. Dutykh, A. Assylbekuly, and D. Zhakebayev. On weakly singular and fully nonlinear travelling shallow capillarygravity waves in the critical regime. *Physics Letters A*, 381(20):1719 – 1726, 2017. ISSN 0375-9601.