Non-linear dispersive water wave models

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Outline

- Motivation
- Equations
- Linear Theory
- Comparison To Numerical Solutions

Motivation - Water Waves

We require accurate models of water waves to understand natural hazards in particular

- Tsunamis
- Storm Surges

Motivation

(a) Sulawesi Tsunami (Indonesia, 2018).

(b) Hurricane Florence (U.S.A, 2018)

Motivation

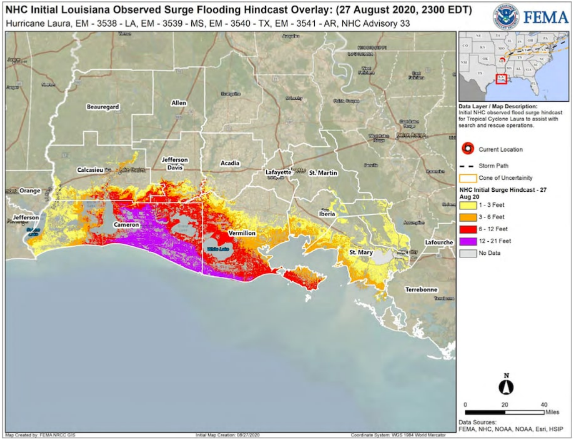

Figure: Hurricane Laura (U.S.A, 2020).

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Which family of equations?

Depth Averaged Set Up

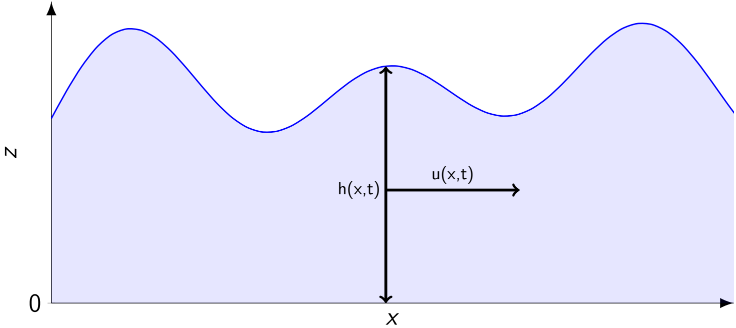


Figure: Relevant Quantities.

Generalised Serre-Green-Naghdi Equations

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0$$

$$\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2 + \beta_1 \Phi - \beta_2 \Psi\right) = 0$$

where

$$\Phi = \frac{h^3}{2}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x \partial t} - u\frac{\partial^2 u}{\partial x^2}\right)$$

$$\Psi = \frac{gh^2}{2}\left(h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)$$

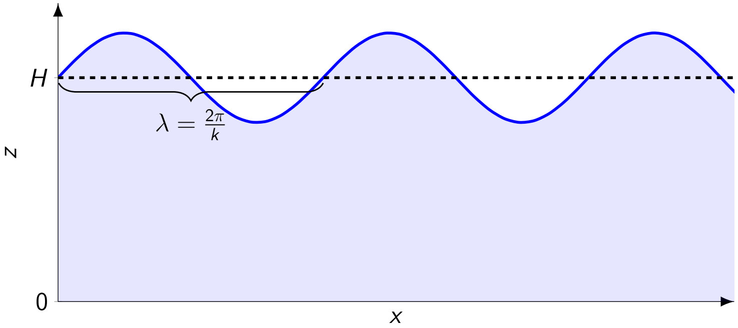
Generalised Serre-Green-Naghdi Equations - Properties

- Conserves mass, momentum and energy
- Reduces to well known equations for particular β values
- Depth averaged equations have been very successful for large scale models
- Nice linear dispersion properties as well

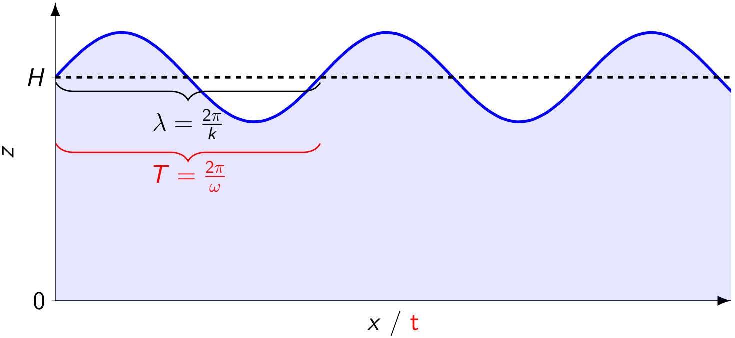
Linear Dispersion Relationship

- Linearise equations - considering waves of small amplitude (we are going to neglect background velocity (U) at first)
- Relate angular frequency (ω) to wave number (k)

Linearise equations (Space)



Linearise equations (Space/ Time)



Dispersion Relations of generalised Serre-Green-Naghdi and Water (linearised)

$$\omega^2_{\text{gSGN}} = gHk^2 \frac{\beta_2 H^2 k^2 + 2}{\beta_1 H^2 k^2 + 2}$$

$$\omega^2_{\text{water}} = gk \tanh(kH)$$

Taylor Series Expansions

$$\omega^2_{\text{gSGN}} = gHk^2 \quad - \frac{\beta_1 - \beta_2}{2} gH^3 k^4 \quad + \frac{\beta_1\left(\beta_1 - \beta_2\right)}{4} gH^5 k^6 \quad + \mathcal{O}\left(k^8\right)$$

$$\omega^2_{\text{water}} = gHk^2 \quad - \frac{1}{3} gH^3 k^4 \quad + \frac{2}{15} gH^5 k^6 \quad + \mathcal{O}\left(k^8\right)$$

Taylor Series Expansions

$$\omega^2_{\text{gSGN}} = gHk^2 - \frac{\beta_1 - \beta_2}{2} gH^3k^4 + \frac{\beta_1\left(\beta_1 - \beta_2\right)}{4} gH^5k^6 + \mathcal{O}\left(k^8\right)$$

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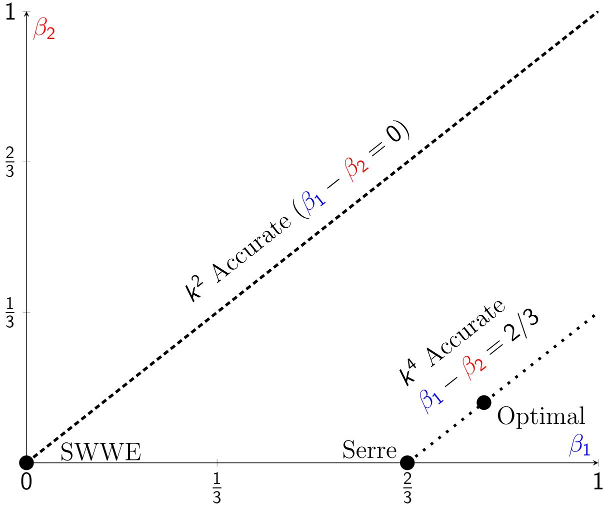
So

$\beta_1 - \beta_2 = 0$	$\mathcal{O}\left(k^2\right)$ Accurate (Non-dispersive)
$\beta_1 - \beta_2 = 2/3$	$\mathcal{O}\left(k^4\right)$ Accurate
$\beta_1 - \beta_2 = 2/3$ and $\beta_1 = \frac{4}{5}$	$\mathcal{O}\left(k^6\right)$ Accurate

Accuracy Summary Plot



Phase Speed (c)

$$c_{\text{gSGN}}^2 = \frac{\omega_{\text{gSGN}}^2}{k^2} = gH \frac{\beta_2 H^2 k^2 + 2}{\beta_1 H^2 k^2 + 2}$$

Phase Speed (c)

$$c_{\text{gSGN}}^2 = \frac{\omega_{\text{gSGN}}^2}{k^2} = gH \frac{\beta_2 H^2 k^2 + 2}{\beta_1 H^2 k^2 + 2}$$

We will now allow non-zero background velocities (U) and no longer look at the square of the phase speed (resulting in negative and positive branches). Thus we get

$$c_{\text{gSGN}}^{+} = U + \sqrt{gH}\sqrt{\frac{\beta_2 H^2 k^2 + 2}{\beta_1 H^2 k^2 + 2}}$$

$$c_{\text{gSGN}}^{-} = U - \sqrt{gH}\sqrt{\frac{\beta_2 H^2 k^2 + 2}{\beta_1 H^2 k^2 + 2}}$$

Phase Speed (c)

The important term here in terms of dispersive properties and the effect of β values is

$$\sqrt{\frac{\beta_2 H^2 k^2 + 2}{\beta_1 H^2 k^2 + 2}}$$

In particular we have that

- this term is monotone for $k \geq 0$
- $\sqrt{\frac{\beta_2 H^2 k^2+2}{\beta_1 H^2 k^2+2}} \to 1$ as $k \to 0$
- $\sqrt{\frac{\beta_2 H^2 k^2+2}{\beta_1 H^2 k^2+2}} \to \sqrt{\frac{\beta_2}{\beta_1}}$ as $k \to \infty$
- $\sqrt{\frac{\beta_2}{\beta_1}} \leq \sqrt{\frac{\beta_2 H^2 k^2+2}{\beta_1 H^2 k^2+2}} \leq 1$ when $\beta_1 > \beta_2$
- $1 \leq \sqrt{\frac{\beta_2 H^2 k^2+2}{\beta_1 H^2 k^2+2}} \leq \sqrt{\frac{\beta_2}{\beta_1}}$ when $\beta_1 < \beta_2$

Phase Speed Regions

These properties result in the following cases and associated chain of inequalities

- When $\beta_1 > \beta_2$ we have

$$U - \sqrt{gH} \le c^{-}_{\text{gSGN}} \le U - \sqrt{\frac{\beta_2}{\beta_1}}\sqrt{gH} \le 0 \le U + \sqrt{\frac{\beta_2}{\beta_1}}\sqrt{gH} \le c^{+}_{\text{gSGN}} \le U + \sqrt{gH}$$

- When $\beta_1 = \beta_2$ we have

$$U - \sqrt{gH} = c^{-}_{\text{gSGN}} \quad \text{and} \quad U + \sqrt{gH} = c^{+}_{\text{gSGN}}$$

- When $\beta_1 < \beta_2$ we have

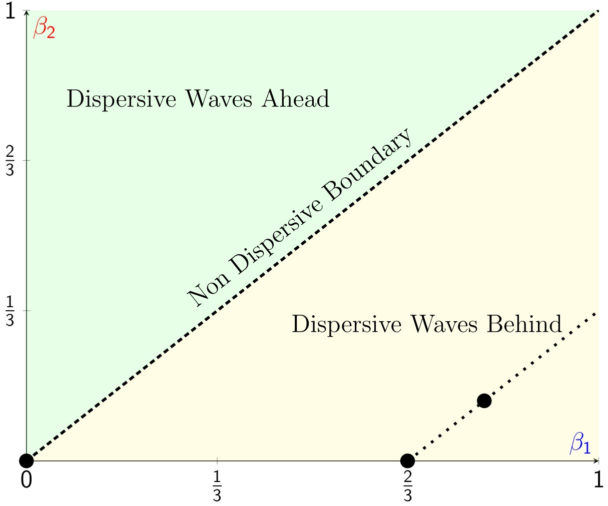
$$U - \sqrt{\frac{\beta_2}{\beta_1}}\sqrt{gH} \le c^{-}_{\text{gSGN}} \le U - \sqrt{gH} \le 0 \le U + \sqrt{gH} \le c^{+}_{\text{gSGN}} \le U + \sqrt{\frac{\beta_2}{\beta_1}}\sqrt{gH}$$

Phase Speed Regions

Since $U - \sqrt{gH}$ and $U + \sqrt{gH}$ will bound location of the shocks we have that

- When $\beta_1 > \beta_2$ we have dispersive waves form behind shocks
- When $\beta_1 = \beta_2$ we have no dispersive waves
- When $\beta_1 < \beta_2$ we have dispersive waves form ahead of shocks

Accuracy Summary Plot

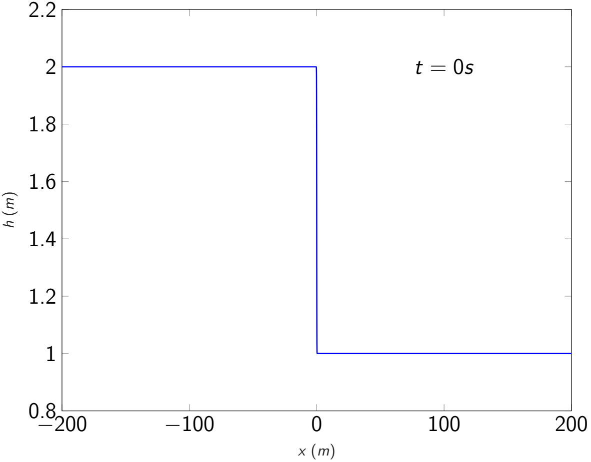


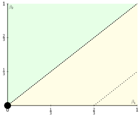
Comparsion Between Linear Theory and Numerical Solutions

Look at a solutions of the Dam break problem initial condition for

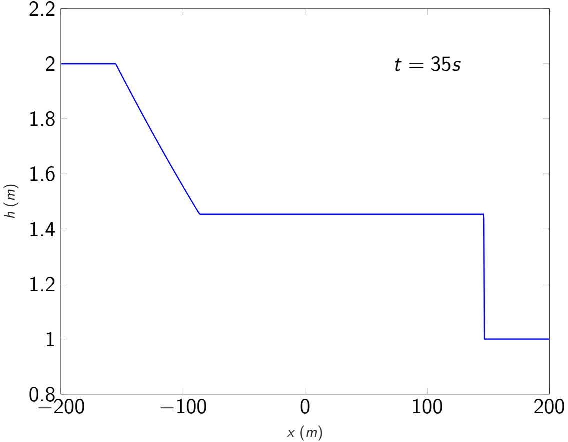
- Fixed β values
- Changing β values

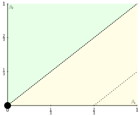
Dambreak Problem Initial Conditions





Shallow Water Wave Equations





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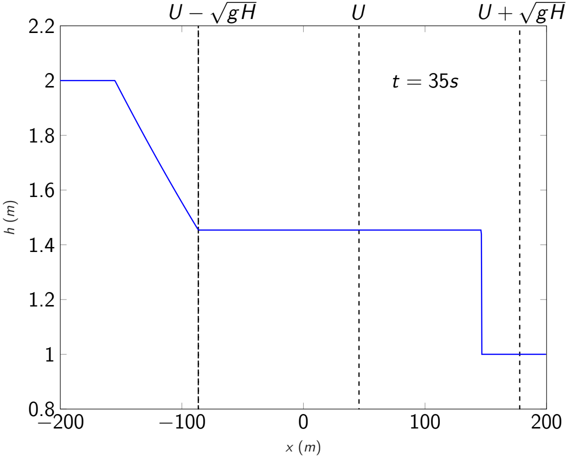
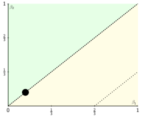
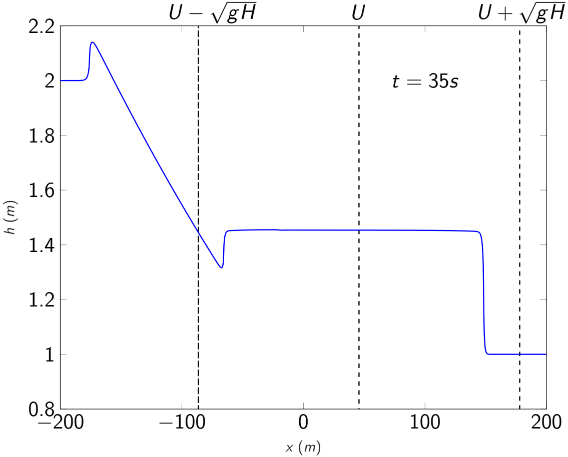
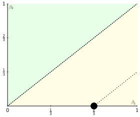


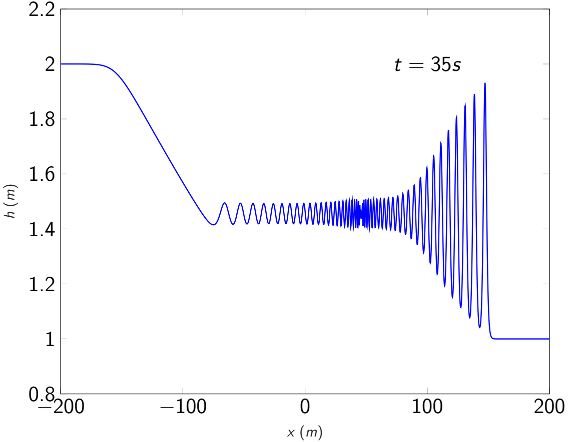


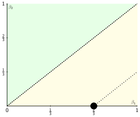
Other Non-Dispersive Equations



Serre Equations







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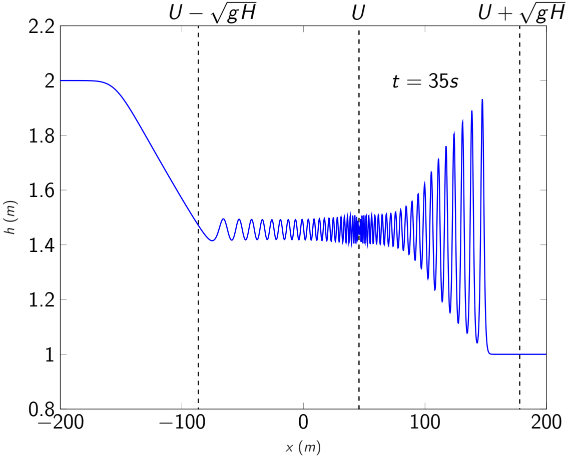
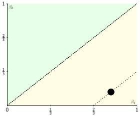
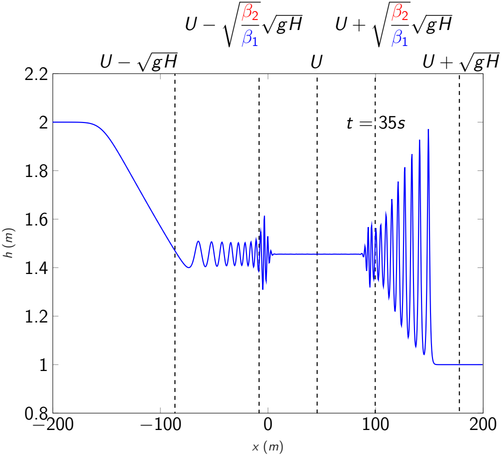




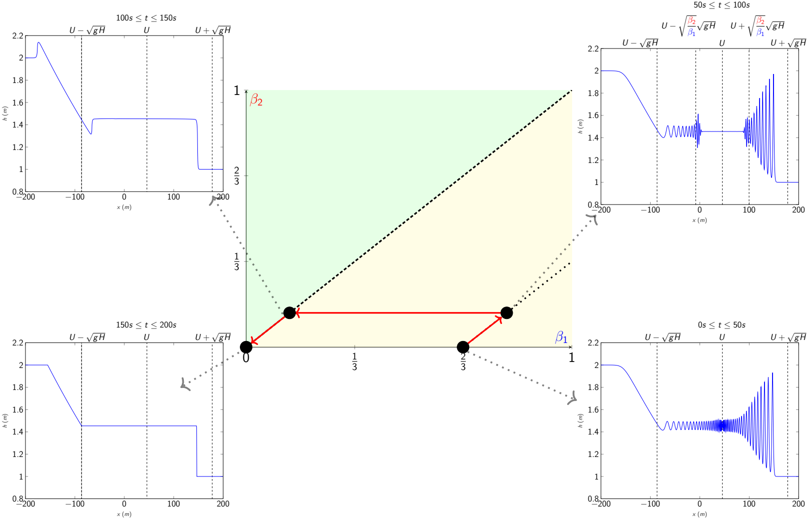
Optimal Dispersion Equations



Changing β values

Now that we have gone through in depth each of the particular β values, we can begin to understand what happens if we change β values over time. Coming back to the original video at the beginning of the talk.

Changing β values Outline



Thanks