

The Serre equations with surface tension are

$$h_t + (uh)_x = 0 \quad (1a)$$

$$(uh)_t + \left(u^2 h + \frac{gh^2}{2} + \frac{h^3}{3} [(u_x)^2 - uu_{xx} - u_{xt}] - \tau \left(h \frac{\partial^2 h}{\partial x^2} - \frac{1}{2} \frac{\partial h}{\partial x} \frac{\partial h}{\partial x} \right) \right)_x = 0 \quad (1b)$$

rewriting (1b) gives

$$hu_t + uh_t + 2uu_x h + u^2 h_x + gh h_x + \left(\frac{h^3}{3} [(u_x)^2 - uu_{xx} - u_{xt}] - \tau \left(h \frac{\partial^2 h}{\partial x^2} - \frac{1}{2} \frac{\partial h}{\partial x} \frac{\partial h}{\partial x} \right) \right)_x = 0$$

substituting (1a)

$$hu_t - u(uh_x + hu_x) + 2uu_x h + u^2 h_x + gh h_x + \left(\frac{h^3}{3} [(u_x)^2 - uu_{xx} - u_{xt}] - \tau \left(h \frac{\partial^2 h}{\partial x^2} - \frac{1}{2} \frac{\partial h}{\partial x} \frac{\partial h}{\partial x} \right) \right)_x = 0$$

$$hu_t + uu_x h + gh h_x + \left(\frac{h^3}{3} [(u_x)^2 - uu_{xx} - u_{xt}] - \tau \left(h \frac{\partial^2 h}{\partial x^2} - \frac{1}{2} \frac{\partial h}{\partial x} \frac{\partial h}{\partial x} \right) \right)_x = 0$$

divide by h

$$u_t + uu_x + gh_x + \frac{1}{h} \left(\frac{h^3}{3} [(u_x)^2 - uu_{xx} - u_{xt}] - \tau \left(h \frac{\partial^2 h}{\partial x^2} - \frac{1}{2} \frac{\partial h}{\partial x} \frac{\partial h}{\partial x} \right) \right)_x = 0$$

Integrating the surface tension term we get

$$\left[\frac{1}{h} \left(-\tau \left(h \frac{\partial^2 h}{\partial x^2} - \frac{1}{2} \frac{\partial h}{\partial x} \frac{\partial h}{\partial x} \right) \right) \right]_x = -\tau h_{xxx}$$

So we get

$$h_t + u_x h + u h_x = 0 \quad (2a)$$

$$u_t + uu_x + gh_x + \frac{1}{3h} \left(h^3 [(u_x)^2 - uu_{xx} - u_{xt}] \right)_x - \tau h_{xxx} = 0 \quad (2b)$$

Want solutions of travelling wave form $h(\xi)$ and $u(\xi)$ where $\xi = x - ct$. For this to be a solution must satisfy (2). First we want to write these equations in terms of ξ

0.1 Mass

For (2a) using $[q(\xi)]_x = q'(\xi)\xi_x$ and $[q(\xi)]_t = q'(\xi)\xi_t$ we have

$$h'\xi_t + u'h\xi_x + uh'\xi_x = 0$$

since $\xi_x = 1$ and $\xi_t = -c$ then

$$-ch' + u'h + uh' = 0$$

Integrating we get

$$\int -ch' + u'h + uh'd\xi = \int 0d\xi$$

$$\int -ch' + [uh]' d\xi = \int 0d\xi$$

Combining the constants of integration of both integrals into A we get that

$$-ch + uh + A = 0$$

so we get

$$uh = ch - A$$

$$u = c - \frac{A}{h}$$

$$u(\xi) = c - \frac{A}{h(\xi)} \quad (3)$$

0.2 Momentum

Now we rewrite (2b) as a function of ξ , making use of

$$\begin{aligned} [q(\xi)]_x &= q'(\xi) \\ [q(\xi)]_{xx} &= q''(\xi) \\ [q(\xi)]_{xxx} &= q'''(\xi) \\ [q(\xi)]_{xt} &= -cq''(\xi) \\ [q(\xi)]_{xxt} &= -cq'''(\xi) \\ [q(\xi)]_t &= -cq'(\xi) \end{aligned}$$

using $\xi = x - ct$

we get from (2b)

$$-cu' + uu' + gh' + \frac{1}{3h} \left(h^3 \left[(u')^2 - uu'' + cu'' \right] \right)' - \tau h''' = 0$$

From (3) we have

$$\begin{aligned} u &= c - \frac{A}{h} \\ u' &= A \frac{h'}{h^2} \\ u'' &= A \frac{hh'' - 2[h']^2}{h^3} \\ u''' &= A \frac{h^2h''' + 6[h']^3 - 6hh'h''}{h^4} \end{aligned}$$

So we get that

$$-c \left[A \frac{h'}{h^2} \right] + \left[c - \frac{A}{h} \right] \left[A \frac{h'}{h^2} \right] + gh' + \frac{1}{3h} \left(h^3 \left[\left(A \frac{h'}{h^2} \right)^2 - \left[c - \frac{A}{h} \right] \left[A \frac{hh'' - 2[h']^2}{h^3} \right] + c \left[A \frac{hh'' - 2[h']^2}{h^3} \right] \right] \right)' - \tau h''' = 0$$

$$- \left[A^2 \frac{h'}{h^3} \right] + gh' + \frac{1}{3h} \left(h^3 \left[\left(A \frac{h'}{h^2} \right)^2 + \left[\frac{A}{h} \right] \left[A \frac{hh'' - 2[h']^2}{h^3} \right] \right] \right)' - \tau h''' = 0$$

$$- \left[A^2 \frac{h'}{h^3} \right] + gh' + \frac{1}{3h} \left(h^3 \left[\left(A^2 \frac{[h']^2}{h^4} \right) + \left[A^2 \frac{hh'' - 2[h']^2}{h^4} \right] \right] \right)' - \tau h''' = 0$$

$$- \left[A^2 \frac{h'}{h^3} \right] + gh' + \frac{A^2}{3h} \left(h^3 \left[\frac{hh'' - [h']^2}{h^4} \right] \right)' - \tau h''' = 0$$

$$-A^2 \frac{h'}{h^3} + gh' + \frac{A^2}{3h} \left(\frac{hh'' - [h']^2}{h} \right)' - \tau h''' = 0$$

multiply by h

$$-A^2 \frac{h'}{h^2} + gh'h' + \frac{A^2}{3} \left(\frac{hh'' - [h']^2}{h} \right)' - \tau h''' = 0$$

$$\frac{A^2}{3} \left(\frac{hh'' - [h']^2}{h} \right)' - \tau h''' = A^2 \frac{h'}{h^2} - gh'h'$$

Integrating we get

$$\int \frac{A^2}{3} \left(\frac{hh'' - [h']^2}{h} \right)' d\xi = \int A^2 \frac{h'}{h^2} - gh'h' + \tau h''' d\xi$$

C constant of integration

$$\frac{A^2}{3} \left(\frac{hh'' - [h']^2}{h} \right) + C = \int A^2 \frac{h'}{h^2} - gh'h' + \tau h''' d\xi$$

Absorbing all constants of integration into C we get

$$\frac{A^2}{3} \left(\frac{hh'' - [h']^2}{h} \right) + C = -\frac{A^2}{h} - \frac{gh^2}{2} + \tau h''$$

Thus we have

$$\begin{aligned}\frac{A^2}{3} (hh'' - [h']^2) + Ch &= -A^2 - \frac{gh^3}{2} + \tau hh'' \\ \frac{A^2}{3} (hh'' - [h']^2) + \frac{gh^3}{2} &= -A^2 - Ch + \tau hh''\end{aligned}$$

divide by h^2

$$\begin{aligned}\frac{A^2}{3} \left(\frac{hh'' - [h']^2}{h^2} \right) + \frac{gh}{2} &= \frac{-A^2}{h^2} - \frac{C}{h} + \frac{\tau h''}{h} \\ \frac{A^2}{3} \left(\frac{hh'' - [h']^2}{h^2} \right) - \frac{\tau h''}{h} + \frac{gh}{2} &= \frac{-A^2}{h^2} - \frac{C}{h} \\ -\frac{A^2}{3} \left(\frac{[h']^2}{h^2} \right) + \frac{A^2}{3} \left(\frac{h''}{h} \right) - \frac{\tau h''}{h} + \frac{gh}{2} &= \frac{-A^2}{h^2} - \frac{C}{h} \\ \frac{A^2 - 3\tau}{3} \left(\frac{h''}{h} \right) - \frac{A^2}{3} \left(\frac{[h']^2}{h^2} \right) + \frac{gh}{2} &= \frac{-A^2}{h^2} - \frac{C}{h}\end{aligned}$$

So we have two constants of integration which we can set as we like.

In summary we have the following equations that travelling wave solutions must satisfy, for a particular choice of A and C .

$$\frac{A^2 - 3\tau}{3} \left(\frac{h''(\xi)}{h(\xi)} \right) - \frac{A^2}{3} \left(\frac{[h'(\xi)]^2}{[h(\xi)]^2} \right) + \frac{gh(\xi)}{2} = -\frac{A^2}{h^2(\xi)} - \frac{C}{h(\xi)} \quad (4a)$$

$$u(\xi) = c - \frac{A}{h(\xi)} \quad (4b)$$

0.3 Consistent with soliton

To get u in the same form as the soliton for a solution we must set

$$A = ca_0 \text{ where } c = \sqrt{g(a_0 + a_1)}$$

Thus we get

$$u(\xi) = c \left[1 - \frac{a_0}{h(\xi)} \right] \quad (5)$$

Thus the equation for h becomes

$$\frac{(ca_0)^2 - 3\tau}{3} \left(\frac{h''(\xi)}{h(\xi)} \right) - \frac{(ca_0)^2}{3} \left(\frac{[h'(\xi)]^2}{[h(\xi)]^2} \right) + \frac{gh(\xi)}{2} = -\frac{(ca_0)^2}{h^2(\xi)} - \frac{C}{h(\xi)} \quad (6)$$

want the peakon solution

$$h(\xi) = a_0 + a_1 \exp\left(-\frac{\sqrt{3}}{a_0} |\xi|\right)$$

The derivatives of this function are

$$h'(\xi) = -\frac{\sqrt{3}a_1}{a_0} \frac{\xi}{|\xi|} \exp\left(-\frac{\sqrt{3}}{a_0} |\xi|\right)$$

Let's assume that $(ca_0)^2 - 3\tau = 0$ thus $(ca_0)^2 = 3\tau$ and $\tau = \frac{c^2 a_0^2}{3}$. So that the second derivative, is not needed, which is a delta function.

So we get

$$-\frac{(ca_0)^2}{3} \left(\frac{[h'(\xi)]^2}{[h(\xi)]^2} \right) + \frac{gh(\xi)}{2} = -\frac{(ca_0)^2}{h^2(\xi)} - \frac{C}{h(\xi)} \quad (7)$$

Multiplying out the $h(\xi)$

$$-\frac{(ca_0)^2}{3} \left(\frac{[h'(\xi)]^2}{[h(\xi)]} \right) + \frac{gh^2(\xi)}{2} = -\frac{(ca_0)^2}{h(\xi)} - C \quad (8)$$

$$C = \frac{(ca_0)^2}{3} \left(\frac{[h'(\xi)]^2}{[h(\xi)]} \right) - \frac{gh^2(\xi)}{2} - \frac{(ca_0)^2}{h(\xi)} \quad (9)$$

$$[h(\xi)]^2 = a_0^2 + 2a_0a_1 + a_1^2 \exp\left(-2\frac{\sqrt{3}}{a_0} |\xi|\right)$$

$$[h'(\xi)]^2 = \frac{3a_1^2}{a_0^2} \exp\left(-2\frac{\sqrt{3}}{a_0} |\xi|\right)$$

$$C = \frac{(ca_0)^2}{3} \left(\frac{\frac{3a_1^2}{a_0^2} \exp\left(-2\frac{\sqrt{3}}{a_0} |\xi|\right)}{a_0 + a_1 \exp\left(-\frac{\sqrt{3}}{a_0} |\xi|\right)} \right) - \frac{g}{2} \left[a_0^2 + 2a_0a_1 + a_1^2 \exp\left(-2\frac{\sqrt{3}}{a_0} |\xi|\right) \right] h^2(\xi) - \frac{(ca_0)^2}{a_0 + a_1 \exp\left(-\frac{\sqrt{3}}{a_0} |\xi|\right)} \quad (10)$$