The Serre equations are

$$h_t + (uh)_x = 0 \tag{1a}$$

$$(uh)_t + \left(u^2h + \frac{gh^2}{2} + \frac{h^3}{3}\left[(u_x)^2 - uu_{xx} - u_{xt}\right]\right)_x = 0 \tag{1b}$$

rewriting (1b) gives

$$hu_t + uh_t + 2uu_xh + u^2h_x + ghh_x + \left(\frac{h^3}{3}\left[(u_x)^2 - uu_{xx} - u_{xt}\right]\right)_x = 0$$

substituting (1a)

$$hu_t - u\left(uh_x + hu_x\right) + 2uu_xh + u^2h_x + ghh_x + \left(\frac{h^3}{3}\left[(u_x)^2 - uu_{xx} - u_{xt}\right]\right)_x = 0$$

$$hu_t + uu_xh + ghh_x + \left(\frac{h^3}{3}\left[(u_x)^2 - uu_{xx} - u_{xt}\right]\right)_x = 0$$

divide by h

$$u_t + uu_x + gh_x + \frac{1}{h}\left(\frac{h^3}{3}\left[(u_x)^2 - uu_{xx} - u_{xt}\right]\right)_x = 0$$

So we get

$$h_t + u_xh + uh_x = 0 \tag{2a}$$

$$u_t + uu_x + gh_x + \frac{1}{3h}\left(h^3\left[(u_x)^2 - uu_{xx} - u_{xt}\right]\right)_x = 0 \tag{2b}$$

Want solutions of travelling wave form $h(\xi)$ and $u(\xi)$ where $\xi = x - ct$. For this to be a solution must satisfy (2). First we want to write these equations in terms of ξ

For (2a) using $[q(\xi)]_x = q'(\xi)\xi_x$ and $[q(\xi)]_t = q'(\xi)\xi_t$ we have

$$h'\xi_t + u'h\xi_x + uh'\xi_x = 0$$

since $\xi_x = 1$ and $\xi_t = -c$ then

$$-ch' + u'h + uh' = 0$$

Integrating we get

$$\int -ch' + u'h + uh' d\xi = \int 0 d\xi$$

$$\int -ch' + [uh]' \, d\xi = \int 0 d\xi$$

Combining the contstants of integration of both integrals into A we get that

$$-ch + uh + A = 0$$

so we get

$$uh = ch - A$$

$$u = c - \frac{A}{h}$$

$$u(\xi) = c - \frac{A}{h(\xi)} \tag{3}$$

Now we rewrite (2b) as a function of ξ, making use of

$$\begin{aligned}
[q(\xi)]_x &= q'(\xi) \\
[q(\xi)]_{xx} &= q''(\xi) \\
[q(\xi)]_{xxx} &= q'''(\xi) \\
[q(\xi)]_{xt} &= -cq''(\xi) \\
[q(\xi)]_{xxt} &= -cq'''(\xi) \\
[q(\xi)]_t &= -cq'(\xi)
\end{aligned}$$

using $\xi = x - ct$

we get from (2b)

$$-cu' + uu' + gh' + \frac{1}{3h}\left(h^3\left[(u')^2 - uu'' + cu''\right]\right)' = 0$$

From (3) we have

$$\begin{aligned}
u &= c - \frac{A}{h} \\
u' &= A\frac{h'}{h^2} \\
u'' &= A\frac{hh'' - 2\,[h']^2}{h^3} \\
u''' &= A\frac{h^2h''' + 6\,[h']^3 - 6hh'h''}{h^4}
\end{aligned}$$

So we get that

$$\begin{aligned}
&-c\left[A\frac{h'}{h^2}\right] + \left[c - \frac{A}{h}\right]\left[A\frac{h'}{h^2}\right] + gh' \\
&+\frac{1}{3h}\left(h^3\left[\left(A\frac{h'}{h^2}\right)^2 - \left[c - \frac{A}{h}\right]\left[A\frac{hh'' - 2\,[h']^2}{h^3}\right] + c\left[A\frac{hh'' - 2\,[h']^2}{h^3}\right]\right]\right)' = 0
\end{aligned}$$

$$-\left[A^2\frac{h'}{h^3}\right] + gh' + \frac{1}{3h}\left(h^3\left[\left(A\frac{h'}{h^2}\right)^2 + \left[\frac{A}{h}\right]\left[A\frac{hh''-2\left[h'\right]^2}{h^3}\right]\right]\right)' = 0$$

$$-\left[A^2\frac{h'}{h^3}\right] + gh' + \frac{1}{3h}\left(h^3\left[\left(A^2\frac{\left[h'\right]^2}{h^4}\right) + \left[A^2\frac{hh''-2\left[h'\right]^2}{h^4}\right]\right]\right)' = 0$$

$$-\left[A^2\frac{h'}{h^3}\right] + gh' + \frac{A^2}{3h}\left(h^3\left[\frac{hh''-\left[h'\right]^2}{h^4}\right]\right)' = 0$$

$$-A^2\frac{h'}{h^3} + gh' + \frac{A^2}{3h}\left(\frac{hh''-\left[h'\right]^2}{h}\right)' = 0$$

multiply by h

$$-A^2\frac{h'}{h^2} + ghh' + \frac{A^2}{3}\left(\frac{hh''-\left[h'\right]^2}{h}\right)' = 0$$

$$\frac{A^2}{3}\left(\frac{hh''-\left[h'\right]^2}{h}\right)' = A^2\frac{h'}{h^2} - ghh'$$

Integrating we get

$$\int \frac{A^2}{3}\left(\frac{hh''-\left[h'\right]^2}{h}\right)' d\xi = \int A^2\frac{h'}{h^2} - ghh' d\xi$$

C constant of integration

$$\frac{A^2}{3}\left(\frac{hh''-\left[h'\right]^2}{h}\right) + C = \int A^2\frac{h'}{h^2} - ghh' d\xi$$

Absorbing all contstants of intergration into C we get

$$\frac{A^2}{3}\left(\frac{hh''-\left[h'\right]^2}{h}\right) + C = -\frac{A^2}{h} - \frac{gh^2}{2}$$

Thus we have

$$\frac{A^2}{3}\left(hh''-\left[h'\right]^2\right) + Ch = -A^2 - \frac{gh^3}{2}$$

$$\frac{A^2}{3}\left(hh''-\left[h'\right]^2\right) + \frac{gh^3}{2} = -A^2 - Ch$$

divide by h^2

$$\frac{A^2}{3}\left(\frac{hh'' - [h']^2}{h^2}\right) + \frac{gh}{2} = \frac{-A^2}{h^2} - \frac{C}{h}$$

$$\frac{A^2}{3}\left(\frac{h'}{h}\right)' + \frac{gh}{2} = \frac{-A^2}{h^2} - \frac{C}{h}$$

$$\frac{A^2}{3}\left(\frac{h'}{h}\right)' + \frac{gh}{2} = -\frac{A^2}{h^2} - \frac{C}{h}$$

So we have two constants of integration which we can set as we like.

In summary we have the following equations that travelling wave solutions must satisfy, for a particular choice of A and C.

$$\frac{A^2}{3}\left(\frac{h'(\xi)}{h(\xi)}\right)' + \frac{gh(\xi)}{2} = -\frac{A^2}{h^2(\xi)} - \frac{C}{h(\xi)} \tag{4a}$$

$$u(\xi) = c - \frac{A}{h(\xi)} \tag{4b}$$

In Dimitri's case we have $A = 1$ and $C = -\left(c^2 + \frac{1}{2}\right)$ Resulting in:

$$\frac{c^2}{3}\left(\frac{h'(\xi)}{h(\xi)}\right)' + \frac{gh(\xi)}{2} = -\frac{c^2}{h^2(\xi)} + \frac{1}{h(\xi)}\left(c^2 + \frac{1}{2}\right)$$

0.1 Validation with soliton

The equations for a soliton are:

$$h(\xi) = a_0 + a_1 \operatorname{sech}^2(\kappa\xi) \tag{5a}$$

$$u(\xi) = c\left(1 - \frac{a_0}{h(x,t)}\right) \tag{5b}$$

where

$$c = \sqrt{g(a_0 + a_1)} \tag{5c}$$

$$\kappa = \frac{\sqrt{3a_1}}{2a_0\sqrt{a_0 + a_1}} \tag{5d}$$

(5b) satisfies (4b) for constant of integration $A = a_0 c$. Using this value we have

$$\frac{a_0^2 c^2}{3}\left(\frac{h'(\xi)}{h(\xi)}\right)' + \frac{gh(\xi)}{2} = -\frac{a_0^2 c^2}{h^2(\xi)} - \frac{C}{h(\xi)}$$

$$h(\xi) = a_0 + a_1 \operatorname{sech}^2(\kappa\xi)$$
$$h'(\xi) = -2a_1\kappa \tanh(\kappa\xi) \operatorname{sech}^2(\kappa\xi)$$

$$\frac{h'(\xi)}{h(\xi)} = \frac{-2a_1\kappa \tanh(\kappa\xi) \operatorname{sech}^2(\kappa\xi)}{a_0 + a_1 \operatorname{sech}^2(\kappa\xi)}$$
$$\left(\frac{h'(\xi)}{h(\xi)}\right)' = -\frac{2a_1\kappa^2 sech^2(\kappa x)\left(a_0 sech^2(\kappa x) + a_1 sech^4(\kappa x) - 2a_0 tanh^2(\kappa x)\right)}{(a_0 + a_1 sech^2(\kappa x))^2}$$

substituting into (4a) we get

$$-\frac{a_0^2 c^2}{3} \frac{2a_1\kappa^2 sech^2(\kappa\xi)\left(a_0 sech^2(\kappa\xi) + a_1 sech^4(\kappa\xi) - 2a_0 tanh^2(\kappa\xi)\right)}{(a_0 + a_1 sech^2(\kappa\xi))^2} = -\frac{g}{2}\left(a_0 + a_1 \operatorname{sech}^2(\kappa\xi)\right) - \frac{a_0^2 c^2}{\left(a_0 + a_1 \operatorname{sech}^2(\kappa\xi)\right)^2} - \frac{C}{\left(a_0 + a_1 \operatorname{sech}^2(\kappa\xi)\right)}$$

$$-\frac{a_0^2 c^2}{3} \frac{2a_1\kappa^2 sech^2(\kappa\xi)\left(a_0 sech^2(\kappa\xi) + a_1 sech^4(\kappa\xi) - 2a_0 tanh^2(\kappa\xi)\right)}{(a_0 + a_1 sech^2(\kappa\xi))^2} = -\frac{g}{2} \frac{\left(a_0 + a_1 \operatorname{sech}^2(\kappa\xi)\right)^3}{\left(a_0 + a_1 \operatorname{sech}^2(\kappa\xi)\right)^2} - \frac{a_0^2 c^2}{\left(a_0 + a_1 \operatorname{sech}^2(\kappa\xi)\right)^2} - \frac{C\left(a_0 + a_1 \operatorname{sech}^2(\kappa\xi)\right)}{\left(a_0 + a_1 \operatorname{sech}^2(\kappa\xi)\right)^2}$$

$$-\frac{a_0^2 c^2}{3} \frac{2a_1\kappa^2 sech^2(\kappa\xi)\left(a_0 sech^2(\kappa\xi) + a_1 sech^4(\kappa\xi) - 2a_0 tanh^2(\kappa\xi)\right)}{(a_0 + a_1 sech^2(\kappa\xi))^2} = \frac{-\frac{g}{2}\left(a_0 + a_1 \operatorname{sech}^2(\kappa\xi)\right)^3 - a_0^2 c^2 - C\left(a_0 + a_1 \operatorname{sech}^2(\kappa\xi)\right)}{\left(a_0 + a_1 \operatorname{sech}^2(\kappa\xi)\right)^2}$$

multiply both sides by $\left(a_0 + a_1 \operatorname{sech}^2(\kappa\xi)\right)^2$

$$-\frac{a_0^2 c^2}{3} 2a_1\kappa^2 sech^2(\kappa\xi)\left(a_0 sech^2(\kappa\xi) + a_1 sech^4(\kappa\xi) - 2a_0 tanh^2(\kappa\xi)\right) = -\frac{g}{2}\left(a_0 + a_1 \operatorname{sech}^2(\kappa\xi)\right)^3 - a_0^2 c^2 - C\left(a_0 + a_1 \operatorname{sech}^2(\kappa\xi)\right)$$

using $\tanh^2(x) = 1 - \operatorname{sech}^2(x)$

$$
-\frac{a_0^2c^2}{3}2a_1\kappa^2 sech^2(\kappa\xi)\left(a_0 sech^2(\kappa\xi)+a_1 sech^4(\kappa\xi)-2a_0\left(1-sech^2(\kappa\xi)\right)\right)
$$
$$
=-\frac{g}{2}\left(a_0+a_1\text{sech}^2\left(\kappa\xi\right)\right)^3-a_0^2c^2-C\left(a_0+a_1\text{sech}^2\left(\kappa\xi\right)\right)
$$

$$
-\frac{a_0^2c^2}{3}2a_1\kappa^2 sech^2(\kappa\xi)\left(3a_0 sech^2(\kappa\xi)+a_1 sech^4(\kappa\xi)-2a_0\right)
$$
$$
=-\frac{g}{2}\left(a_0+a_1\text{sech}^2\left(\kappa\xi\right)\right)^3-a_0^2c^2-C\left(a_0+a_1\text{sech}^2\left(\kappa\xi\right)\right)
$$

$$
-\frac{a_0^2c^2}{3}2a_1\kappa^2 sech^2(\kappa\xi)\left(3a_0 sech^2(\kappa\xi)+a_1 sech^4(\kappa\xi)-2a_0\right)
$$
$$
=-\frac{g}{2}\left(a_0+a_1\text{sech}^2\left(\kappa\xi\right)\right)^3-a_0^2c^2-C\left(a_0+a_1\text{sech}^2\left(\kappa\xi\right)\right)
$$

Just expanding the RHS now

$$
-\frac{a_0^2c^2}{3}2a_1\kappa^2 sech^2(\kappa\xi)\left(3a_0 sech^2(\kappa\xi)+a_1 sech^4(\kappa\xi)-2a_0\right)
$$
$$
=-\frac{g}{2}\left[a_0^3+3a_0a_1^2\text{sech}^4\left(\kappa\xi\right)+3a_0^2a_1\text{sech}^2\left(\kappa\xi\right)+a_1^3\text{sech}^6\left(\kappa\xi\right)\right]-a_0^2c^2-C\left(a_0+a_1\text{sech}^2\left(\kappa\xi\right)\right)
$$

making use of (5c) and (5d) we have $c^2=g(a_0+a_1)$ and $\kappa^2=\frac{3a_1}{4a_0^2(a_0+a_1)}$, first we try the RHS

$$
-\frac{a_0^2g\left(a_0+a_1\right)}{3}2a_1\frac{3a_1}{4a_0^2(a_0+a_1)}sech^2(\kappa\xi)\left(3a_0 sech^2(\kappa\xi)+a_1 sech^4(\kappa\xi)-2a_0\right)
$$
$$
=-\frac{g}{2}\left[a_0^3+3a_0a_1^2\text{sech}^4\left(\kappa\xi\right)+3a_0^2a_1\text{sech}^2\left(\kappa\xi\right)+a_1^3\text{sech}^6\left(\kappa\xi\right)\right]-a_0^2g\left(a_0+a_1\right)-C\left(a_0+a_1\text{sech}^2\left(\kappa\xi\right)\right)
$$

$$
-\frac{ga_1^2}{2}sech^2(\kappa\xi)\left(3a_0 sech^2(\kappa\xi)+a_1 sech^4(\kappa\xi)-2a_0\right)
$$
$$
=-\frac{g}{2}\left[a_0^3+3a_0a_1^2\text{sech}^4\left(\kappa\xi\right)+3a_0^2a_1\text{sech}^2\left(\kappa\xi\right)+a_1^3\text{sech}^6\left(\kappa\xi\right)\right]-a_0^2g\left(a_0+a_1\right)-C\left(a_0+a_1\text{sech}^2\left(\kappa\xi\right)\right)
$$

$$
-\frac{ga_1^2}{2}sech^2(\kappa\xi)\left(3a_0 sech^2(\kappa\xi)\right)
$$
$$
-\frac{ga_1^2}{2}sech^2(\kappa\xi)\left(a_1 sech^4(\kappa\xi)\right)
$$
$$
-\frac{ga_1^2}{2}sech^2(\kappa\xi)\left(-2a_0\right)
$$
$$
=-\frac{g}{2}\left[a_0^3+3a_0a_1^2\text{sech}^4\left(\kappa\xi\right)+3a_0^2a_1\text{sech}^2\left(\kappa\xi\right)+a_1^3\text{sech}^6\left(\kappa\xi\right)\right]-a_0^2g\left(a_0+a_1\right)-C\left(a_0+a_1\text{sech}^2\left(\kappa\xi\right)\right)
$$

$$
\begin{aligned}
&-\frac{3ga_0a_1^2}{2}sech^4(\kappa\xi)-\frac{ga_1^3}{2}sech^6(\kappa\xi)+ga_0a_1^2sech^2(\kappa\xi)\left(a_0\right)\\
&=-\frac{g}{2}\left[a_0^3+3a_0a_1^2\mathrm{sech}^4\left(\kappa\xi\right)+3a_0^2a_1\mathrm{sech}^2\left(\kappa\xi\right)+a_1^3\mathrm{sech}^6\left(\kappa\xi\right)\right]-a_0^2g\left(a_0+a_1\right)-C\left(a_0+a_1\mathrm{sech}^2\left(\kappa\xi\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
-\frac{3ga_0a_1^2}{2}sech^4(\kappa\xi)-\frac{ga_1^3}{2}sech^6(\kappa\xi)+ga_0a_1^2sech^2(\kappa\xi)&\\
=-\frac{g}{2}\left[a_0^3\right]&\\
-\frac{g}{2}\left[3a_0a_1^2\mathrm{sech}^4\left(\kappa\xi\right)\right]&\\
-\frac{g}{2}\left[3a_0^2a_1\mathrm{sech}^2\left(\kappa\xi\right)\right]&\\
-\frac{g}{2}\left[a_1^3\mathrm{sech}^6\left(\kappa\xi\right)\right]&\\
-a_0^2g\left(a_0+a_1\right)-C\left(a_0+a_1\mathrm{sech}^2\left(\kappa\xi\right)\right)&
\end{aligned}
$$

cancelling like terms we get

$$
\begin{aligned}
ga_0a_1^2sech^2(\kappa\xi)=-\frac{3g}{2}\left[a_0^3\right]&\\
-\frac{g}{2}\left[3a_0^2a_1\mathrm{sech}^2\left(\kappa\xi\right)\right]&\\
-a_0^2ga_1-C\left(a_0+a_1\mathrm{sech}^2\left(\kappa\xi\right)\right)&
\end{aligned}
$$

$$
\begin{aligned}
ga_0a_1^2sech^2(\kappa\xi)+\frac{3g}{2}\left[a_0^3\right]&\\
+\frac{g}{2}\left[3a_0^2a_1\mathrm{sech}^2\left(\kappa\xi\right)\right]&\\
+a_0^2ga_1=C\left(a_0+a_1\mathrm{sech}^2\left(\kappa\xi\right)\right)&
\end{aligned}
$$

$$
\begin{aligned}
ga_0a_1^2sech^2(\kappa\xi)+\frac{g}{2}\left[3a_0^2a_1\mathrm{sech}^2\left(\kappa\xi\right)\right]+\frac{3g}{2}\left[a_0^3\right]+a_0^2ga_1&\\
=C\left(a_0+a_1\mathrm{sech}^2\left(\kappa\xi\right)\right)&
\end{aligned}
$$

$$
\begin{aligned}
\left[ga_0a_1+\frac{3g}{2}a_0^2\right]a_1\mathrm{sech}^2\left(\kappa\xi\right)+\left[\frac{3g}{2}a_0^2+a_0ga_1\right]a_0&\\
=C\left(a_0+a_1\mathrm{sech}^2\left(\kappa\xi\right)\right)&
\end{aligned}
$$

So when $C = ga_0a_1 + \dfrac{3ga_0^2}{2}$ then (5) satisfies (4). Thus as we would expect, the soliton analytic solution equations satisfies the travelling wave solution equations (4).