Generalised Serre-Green-Naghdi Model

May 13, 2020

1 gSGN - generalised Serre-Green-Naghdi equations

Clamond and Dutykh[1] derived the following generalised version of the Serre-Green-Naghdi equations:

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{1a}$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{1}{3}h^2\Gamma\right) = 0 \tag{1b}$$

$$\begin{aligned}\frac{\partial}{\partial t}&\left[\frac{1}{2}hu^2 + \frac{1}{2}\left(1 + \frac{3}{2}\beta_1\right)h^3 \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{1}{2}gh^2\left(1 + \frac{1}{2}\beta_2 \frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)\right] \\ &\frac{\partial}{\partial x}\left[uh\left(\frac{1}{2}u^2 + \frac{1}{2}\left(1 + \frac{3}{2}\beta_1\right)h^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + gh\left(1 + \frac{1}{4}\beta_2 \frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right) + \frac{1}{3}h\Gamma\right) + \frac{1}{2}\beta_2 g h^3 \frac{\partial h}{\partial x}\frac{\partial u}{\partial x}\right] = 0\end{aligned} \tag{1c}$$

where

$$\Gamma = \left(1 + \frac{3}{2}\beta_1\right) h \left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x \partial t} - u\frac{\partial^2 u}{\partial x^2}\right] - \frac{3}{2}\beta_2 g \left[h \frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right] \tag{1d}$$

These equations have the same order of approximation in the lagrangian density (dispersion properties?) when $\beta_1 = \beta_2$. The interesting thing about the equations though, is that we will conserve mass, momentum and energy for all values of β_j.

From these equations the SWWE, the Serre equations and the regularised SWWE [1] can be recovered for certain values of β_1 and β_2.

1.1 Alternative Conservative Form of the gSGN

A major difficulty with solving the SGN is that the dispersive terms contain a mixed spatial-temporal derivative term which is difficult to handle numerically. This mixed derivative term can be rewritten so that the Serre equations can be expressed in conservation law form, with the water depth and a new quantity as conservative

Resulting Equations	β_1	β_2
Serre Equations	0	0
SWWE Equations	$-\frac{2}{3}$	0
Regularised SWWE Equations	free variable	$\beta_1 + \frac{2}{3}$
Improved Dispersion Serre Equations	free variable	β_1

Table 1: Showing various combinations of β values and equivalent equations

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variables. This reformulation allows standard techniques for solving conservation laws to be applied to the Serre equations, even though the Serre equations are neither hyperbolic nor parabolic.

Consider the Serre equations written for a horizontal bed. The flux term in the momentum equation, (1b) contains a mixed spatial and temporal derivative term which is difficult to treat numerically. It is possible to replace this term by a combination of spatial and temporal derivative terms by making the following observation

$$\frac{\partial^2}{\partial x \partial t}\left(\frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)h^3\frac{\partial u}{\partial x}\right) = \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)\frac{\partial}{\partial t}\left(3h^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}+h^3\frac{\partial^2 u}{\partial x^2}\right)$$
$$= \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)\frac{\partial}{\partial x}\left(3h^2\frac{\partial h}{\partial t}\frac{\partial u}{\partial x}+h^3\frac{\partial^2 u}{\partial x \partial t}\right). \quad (2)$$

Rearranging and making use of the continuity equation, (1a) the momentum equation, (1b) becomes

$$\frac{\partial}{\partial t}\left(uh - \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)\frac{\partial}{\partial x}\left[h^3\frac{\partial u}{\partial x}\right]\right)$$
$$+\frac{\partial}{\partial x}\left(u\left[uh - \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)\frac{\partial}{\partial x}\left[h^3\frac{\partial u}{\partial x}\right]\right] + \frac{gh^2}{2} - \frac{2}{3}\left(1+\frac{3}{2}\beta_1\right)h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{2}\beta_2 g h^2\left[h\frac{\partial^2 h}{\partial x^2}+\frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right]\right) = 0. \quad (3)$$

The momentum equation can be written in conservation law form as

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}\left(1+\frac{3}{2}\beta_1\right)h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{2}\beta_2 g h^2\left[h\frac{\partial^2 h}{\partial x^2}+\frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right]\right) = 0. \quad (4)$$

where a new conserved quantity, G is given by

$$G = uh - \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right).$$

This expands the conserved variable introduced by [1], as well as in the Serre equations [].

Thus we have the following conservation equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \quad (5a)$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}\left(1+\frac{3}{2}\beta_1\right)h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{2}\beta_2 g h^2\left[h\frac{\partial^2 h}{\partial x^2}+\frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right]\right) = 0. \quad (5b)$$

with

$$G = uh - \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right). \quad (5c)$$

1.2 Dispersion Relation of Linearised gSGN

Assuming that

$$h(x,t) = h_0 + \delta\eta(x,t) + O(\delta^2)$$
$$u(x,t) = u_0 + \delta v(x,t) + O(\delta^2)$$

By substituting these forms into the linearised Serre equations and neglecting $O(\delta^2)$ terms, we get the linearised regularised Serre equations. We also substitute η_t using the mass equation into the momentum equation.

$$(\delta\eta)_t + u_0(\delta\eta)_x + h_0(\delta v)_x = 0 \quad (6a)$$

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$$h_0(\delta v)_t+gh_0(\delta\eta)_x+h_0u_0(\delta v)_x-\frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)h_0^3(\delta v)_{xxt}-\frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)h_0^3u_0(\delta v)_{xxx}-\frac{g\beta_2}{2}h_0^3(\delta\eta)_{xxx}=0 \quad (6b)$$

We can remove the δ term, either by removing the common factor, or absorbing it into η and v to get

$$\eta_t+u_0\eta_x+h_0v_x=0 \quad (7a)$$

$$h_0(v)_t+gh_0(\eta)_x+h_0u_0(v)_x-\frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)h_0^3(v)_{xxt}-\frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)h_0^3u_0(v)_{xxx}-\frac{g\beta_2}{2}h_0^3(\eta)_{xxx}=0 \quad (7b)$$

We now assume that $\eta(x,t)=H\exp\left(i(kx-\omega t)\right)$, $v(x,t)=U\exp\left(i(kx-\omega t)\right)$

$$\eta(x,t)=H\exp\left(i(kx-\omega t)\right)$$
$$v(x,t)=U\exp\left(i(kx-\omega t)\right)$$

substituting these into the linearised Serre equation we get

$$\left[Hu_0k-H\omega+Uh_0k\right]i\exp\left[i\left(kx-\omega t\right)\right]=0 \quad (8a)$$

$$\Big[3H\beta_2gh_0^2k^3+6Hgk-3U\beta_1\omega h_0^2k^2+3U\beta_1h_0^2k^3u_0$$
$$-2U\omega h_0^2k^2-6U\omega+2Uh_0^2k^3u_0+6Uku_0\Big]i\frac{h_0}{6}\exp\left[i\left(kx-\omega t\right)\right]=0 \quad (8b)$$

This can be written as

$$\begin{bmatrix} u_0k-\omega & h_0k \\ 3\beta_2h_0^2k^3+6kg & -3\beta_1\omega h_0^2k^2+3\beta_1h_0^2k^3u_0-2\omega h_0^2k^2-6\omega+2h_0^2k^3u_0+6ku_0\end{bmatrix}\begin{bmatrix}H\\U\end{bmatrix}=0 \quad (9)$$

Which has non-trivial solutons when the determinant is zero.
The determinant of this matrix is

$$\left(3\beta_1h_0^2k^2+2h_0^2k^2+6\right)\omega^2-2u_0k\left(3\beta_1h_0^2k^2+2h_0^2k^2+6\right)\omega+u_0^2k^2\left(3\beta_1h_0^2k^2+2h_0^2k^2+6\right)-gh_0k^2\left(3\beta_2h_0^2k^2+6\right) \quad (10)$$

To get non-trivial solution we have

$$\left(3\beta_1h_0^2k^2+2h_0^2k^2+6\right)\omega^2-2u_0k\left(3\beta_1h_0^2k^2+2h_0^2k^2+6\right)\omega+u_0^2k^2\left(3\beta_1h_0^2k^2+2h_0^2k^2+6\right)-gh_0k^2\left(3\beta_2h_0^2k^2+6\right)=0 \quad (11)$$

$$\omega^2-2u_0k\omega+u_0^2k^2-gh_0k^2\frac{3\beta_2h_0^2k^2+6}{\left(3\beta_1h_0^2k^2+2h_0^2k^2+6\right)} \quad (12)$$

Using quadratic equation, or another quadratic polynomial solver we get that

$$\omega^{\pm}=u_0k\pm k\sqrt{gh_0}\sqrt{\frac{3\beta_2h_0^2k^2+6}{\left(3\beta_1h_0^2k^2+2h_0^2k^2+6\right)}} \quad (13)$$

$$\omega^{\pm}=u_0k\pm k\sqrt{gh_0}\sqrt{\frac{\beta_2h_0^2k^2+2}{\left(\frac{2}{3}+\beta_1\right)h_0^2k^2+2}} \quad (14)$$

Thus we have the following regimes

- SWWE wavespeeds (non-dispersive): occurs when $\beta_2=\frac{2}{3}+\beta_1$ then $\omega^{\pm}=k\left(u_0\pm\sqrt{gh_0}\right)$ ([1]).
- Serre wavespeeds - when $\beta_1=\beta_2=0$ - we get $\omega^{\pm}=k\left(u_0\pm\sqrt{gh_0}\sqrt{\frac{3}{3+h_0^2k^2}}\right)$

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- For other values of β_1 and β_2 we can vary the wavespeeds, in certain situations this will lead to a better approximation of the linear wavespeed of waves than the Serre equations ([1]), in others it will be worse. We show an example below.

Now for the group speed

$$v_g^{\pm} = \frac{\partial \omega^{\pm}}{\partial k} = u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm \frac{k\sqrt{gh_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\beta_2 h_0^2 k - \frac{h_0^2 k \left(\beta_1 + \frac{2}{3}\right)\left(\beta_2 h_0^2 k^2 + 2\right)}{h_0^2 k^2 \left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right] \tag{15}$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm \frac{k\sqrt{gh_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\frac{\beta_2 h_0^2 k^2 \left[h_0^2 k^2 \left(\beta_1 + \frac{2}{3}\right) + 2\right] - h_0^2 k \left(\beta_1 + \frac{2}{3}\right)\left(\beta_2 h_0^2 k^2 + 2\right)}{h_0^2 k^2 \left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm \frac{k\sqrt{gh_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\frac{\beta_2 h_0^4 k^3 \left(\beta_1 + \frac{2}{3}\right) + 2\beta_2 h_0^2 k - h_0^2 k \left(\beta_1 + \frac{2}{3}\right)\left(\beta_2 h_0^2 k^2 + 2\right)}{h_0^2 k^2 \left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm \frac{k\sqrt{gh_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\frac{\beta_2 h_0^4 k^3 \left(\beta_1 + \frac{2}{3}\right) + 2\beta_2 h_0^2 k - \beta_2 \left(\beta_1 + \frac{2}{3}\right) h_0^4 k^3 - 2 h_0^2 k \left(\beta_1 + \frac{2}{3}\right)}{h_0^2 k^2 \left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm \frac{k\sqrt{gh_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\frac{2\beta_2 h_0^2 k - 2 h_0^2 k \left(\beta_1 + \frac{2}{3}\right)}{h_0^2 k^2 \left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm 2\frac{k^2 h_0^2\sqrt{gh_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\frac{\beta_2 - \beta_1 - \frac{2}{3}}{h_0^2 k^2 \left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left[1 + \frac{\beta_2 - \beta_1 - \frac{2}{3}}{\left(\frac{1}{2}\beta_2 h_0^2 k^2 + 1\right)\left(\left(\frac{1}{3} + \beta_1\right) h_0^2 k^2 + 1\right)}\right]$$

When $\beta_2 = \frac{2}{3} + \beta_1$ then the group speed is equal to the phase speed of the Shallow Water Wave equations. When $\beta_2 = \beta_1 = 0$ then we get

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{2}{\left(\frac{2}{3} h_0^2 k^2 + 2\right)}}\left[1 + \frac{-\frac{1}{3}}{\left(\frac{1}{3} h_0^2 k^2 + 1\right)}\right]$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{3}{h_0^2 k^2 + 3}}\left[1 - \frac{1}{h_0^2 k^2 + 3}\right]$$

which matches [Zoppo,pitt,paper on FDVM methods for Serre equations / Chris thesis]

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1.2.1 Wave Speed Bounds

First we want to show that phase and group speed are bounded, when treated as functions of k. The phase speed is:

$$v_p^{\pm} = \frac{\omega^{\pm}}{k} = u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \tag{16}$$

So we need to demonstrate that $\exists \alpha$ s.t $\forall k$

$$\frac{\beta_2 h_0^2 k^2 + 2}{\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2} \leq \alpha \tag{17}$$

This is equivalent to showing that $\exists \alpha$ s.t $\forall \mu$

$$\frac{a\mu^2 + 1}{b\mu^2 + 1} \leq \alpha \tag{18}$$

One such bound is simple enough to get and that is when $\alpha = \max\left\{1, \frac{a}{b}\right\}$ when $b \neq 0$ and $\alpha = \max\left\{1, a\mu^2 + 1\right\}$ when $b = 0$. So if $b = 0$ and $a \neq 0$ then the wavespeeds are no longer bounded. Thus we must restrict our numerical method to only allow $\beta_1 = -\frac{2}{3}$ only when $\beta_2 = 0$. Note that the regularised SWWE and the Serre equations fall under this condition (all our problems of interest) so its ok that our method is not appropriate for the condition.

2 Validation

2.1 Forced Solution

To demonstate the validity and versatility of our method to solve the gSGN whilst allowing varying β_i values, we make use of forced solutions. To generate a forced solution we consider the modified gSGN equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = \frac{\partial h^*}{\partial t} + \frac{\partial (u^* h^*)}{\partial x} \tag{19a}$$

$$\begin{aligned}\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}\left(1 + \frac{3}{2}\beta_1\right) h^3 \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{2}\beta_2 g h^2 \left[h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right]\right) = \\ \frac{\partial G^*}{\partial t} + \frac{\partial}{\partial x}\left(u^* G^* + \frac{g\left(h^*\right)^2}{2} - \frac{2}{3}\left(1 + \frac{3}{2}\beta_1\right)\left(h^*\right)^3 \frac{\partial u^*}{\partial x}\frac{\partial u^*}{\partial x} - \frac{1}{2}\beta_2 g\left(h^*\right)^2\left[h^*\frac{\partial^2 h^*}{\partial x^2} + \frac{1}{2}\frac{\partial h^*}{\partial x}\frac{\partial h^*}{\partial x}\right]\right)\end{aligned} \tag{19b}$$

which admit the solutions h^*, u^* and G^* assuming G^* appropriately satisfies (5c). Since these equations are satisfied for any chosen h^*, u^* and G^*, we can generated any desired solution. Since the left hand-side of these modified equations are approximated by our numerical method, if we combine the numerical method with analytic expressions for the right handside, we have a method that approximates the forced gSGN equation with the same convergence properties as the underlying numerical method for the gSGN equations.

Since we are free to choose h^*, u^* and G^* we can generate solutions and thus test our convergence properties for situations for which no analytic solution to the equations exist, in particular in this paper we are interested in solutions where the β values vary in space and time.

I have used this technique to investigate the method for the following forced solutions

$$h^*(x,t) = a_0 + a_1 \exp\left(\frac{\left(x - a_2 t\right)^2}{2a_3}\right) \tag{20a}$$

$$u^*(x,t) = a_4 \exp\left(\frac{\left(x - a_2 t\right)^2}{2a_3}\right) \tag{20b}$$

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$$\beta_1(x,t) = a_6(x - a_5 t) + a_7 \tag{20c}$$

$$\beta_2(x,t) = a_8(x - a_5 t) + a_9 \tag{20d}$$

where G^* is determined by (5c) with the above values.

2.2 Analytic Solution

2.2.1 Solitary Travelling Wave - Serre ($\beta_1 = \beta_2 = 0$)

When $\beta_1 = \beta_2 = 0$ the gSGN are equivalent to the SGN equations which admit the following travelling wave solution

$$h(x,t) = a_0 + a_1 \text{sech}^2\left(\kappa(x - ct)\right) \tag{21a}$$

$$u(x,t) = c\left(1 - \frac{a_0}{h(x,t)}\right) \tag{21b}$$

where

$$\kappa = \frac{\sqrt{3a_1}}{2a_0\sqrt{a_0 + a_1}} \tag{21c}$$

$$c = \sqrt{g\left(a_0 + a_1\right)} \tag{21d}$$

References

[1] Clamond, D. and D. Dutykh, Non-dispersive conservative regularisation of nonlinear shallow water (and isentropic Euler equations), Communications in Nonlinear Science and Numerical Simulation, 55(42), 237-247.

[2] Dutykh, D., M. Hoefer and D. Mitsotakis, Soliary wave solutions and their interactions for fully nonlinear water waves with surface tension in the generalized Serre equations, Theoretical and Computational Fluid Dynamics, 32(3), 371-397.

[3] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Volume 19, American Mathematical Society, New York, (1997).

[4] A. Harten, High resolution schemes for hyperbolic conservation laws, Journal of Computational Physics, 49 (3) (1983) 357-393.

[5] A. Kurganov, S. Noelle, G. Petrova, Semidiscrete central-upwind schemes for hyperbolic conservation laws and Hamilton-Jacobi equations, Journal of Scientific Computing, Society for Industrial and Applied Mathematics, 23 (3) (2002) 707-740.

[6] B. van Leer, Towards the ultimate conservative difference scheme, V. A second-order sequel to Godunov's method, Journal of Computational Physics, 32 (1) (1979) 101-136.

[7] C.W. Shu, S. Osher, Efficient implementation of essentially non-oscillatory shock-capturing schemes, Journal of Computational Physics, 77 (2) (1988) 439-471.

[8] F. Serre, Contribution à l'étude des écoulements permanents et variables dans les canaux, La Houille Blanche, 6 (1953) 830-872.

[9] C. Zoppou, Numerical solution of the one-dimensional and cylindrical Serre equations for rapidly varying free surface flows, Ph.D., Mathematical Sciences Institute, College of Physical and Mathematical Sciences, The Australian National University, (2014).

[10] C. Zoppou, J. Pitt and S.G. Roberts, Numerical solution of the fully non-linear weakly dispersive Serre equations for steep gradient flows, Applied Mathematical Modelling, 48, 70-95.

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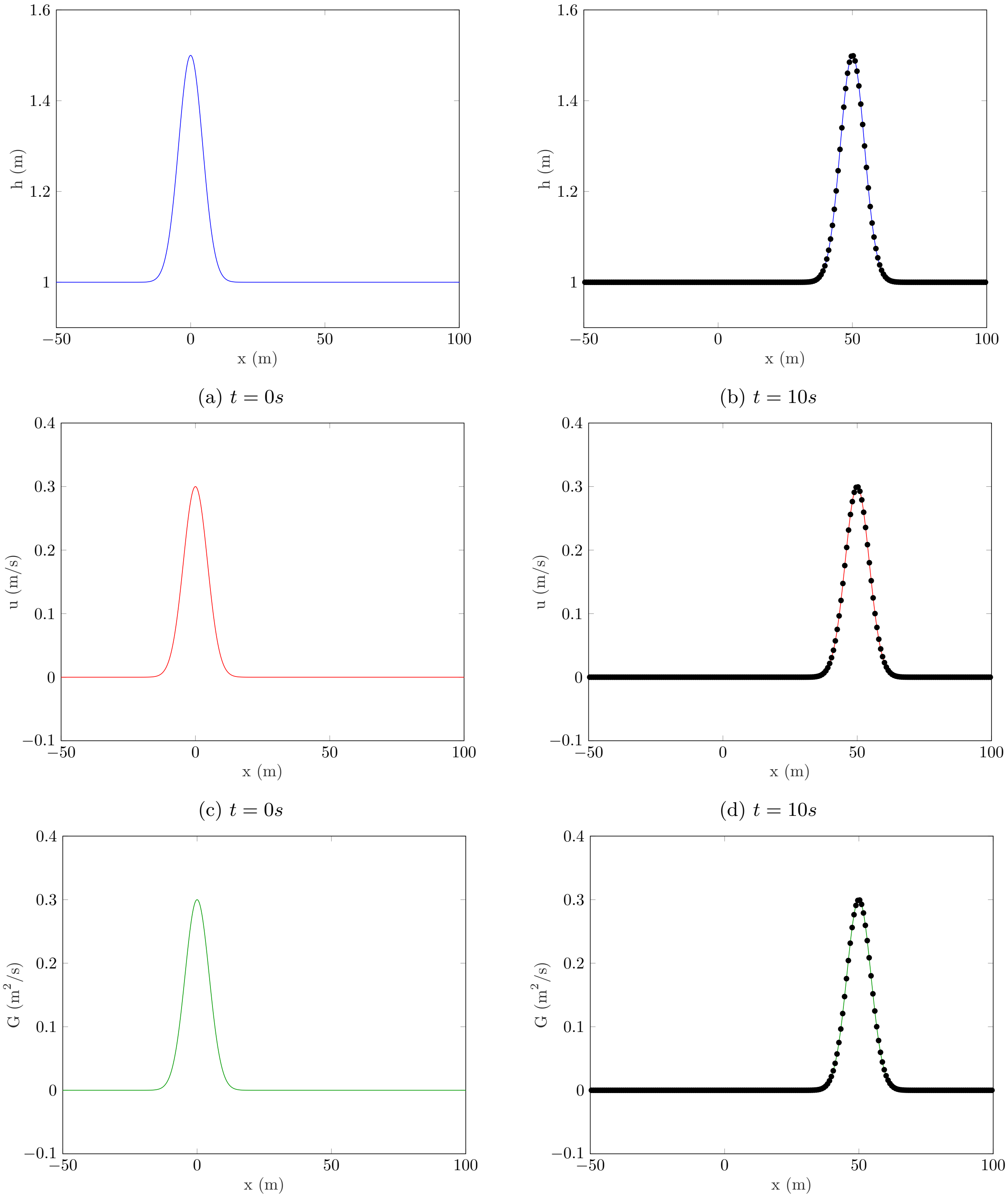

(a) $t = 0s$

(b) $t = 10s$

(c) $t = 0s$

(d) $t = 10s$

(e) $t = 0s$

(f) $t = 10s$

Figure 1: $\Delta x = 0.0469m$.

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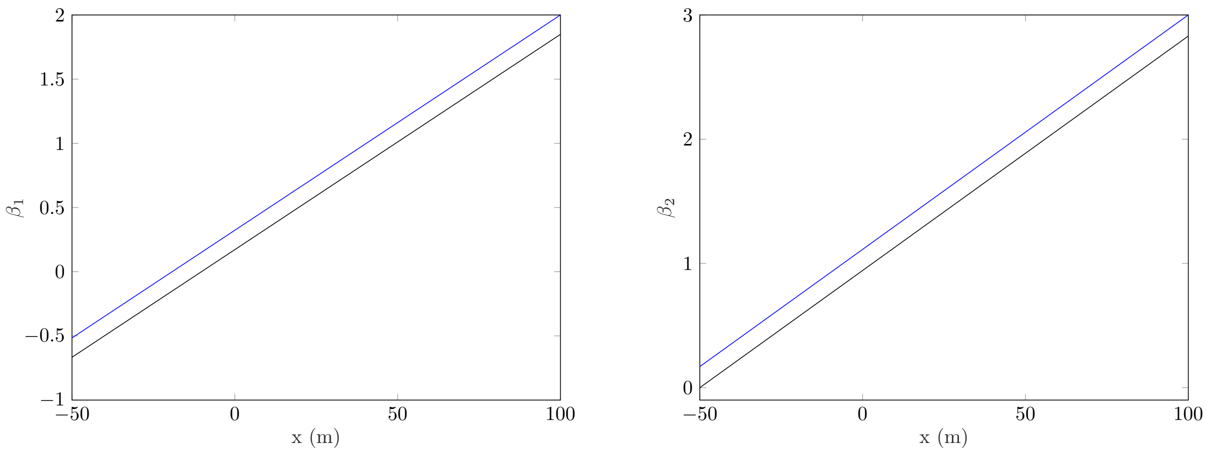

(a) β_1

(b) β_2

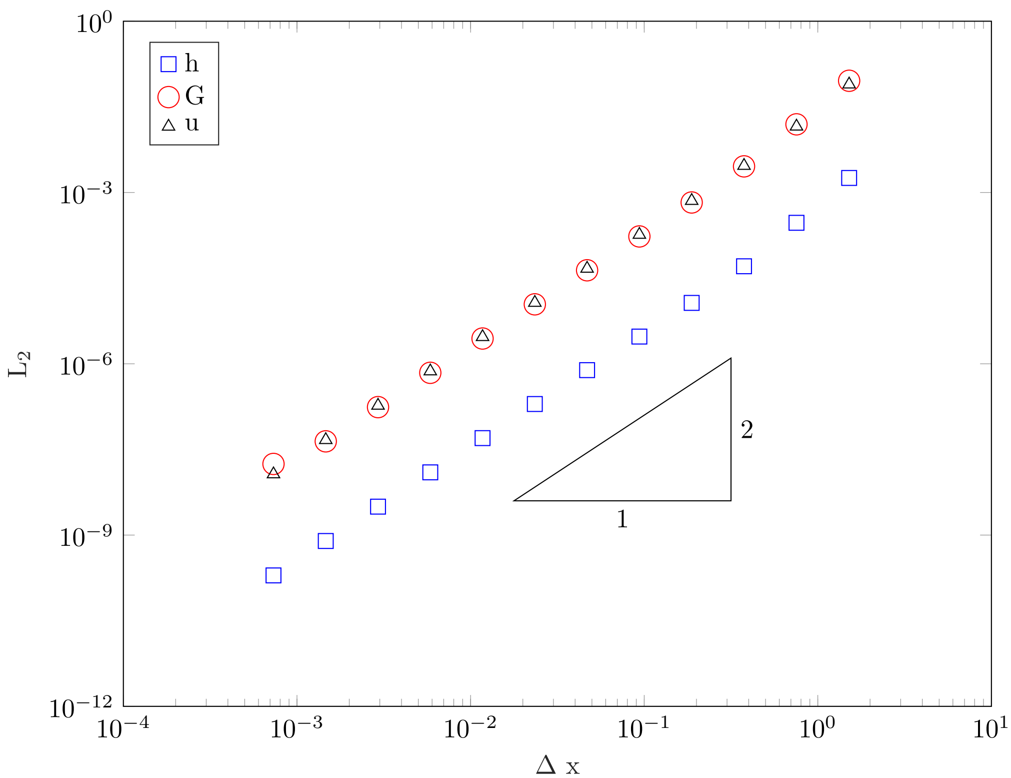

Figure 3: L_2 comparing numerical solution and forced soluiton

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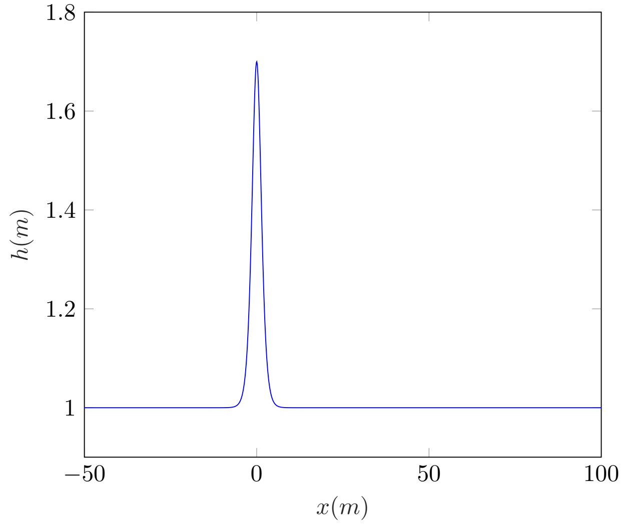
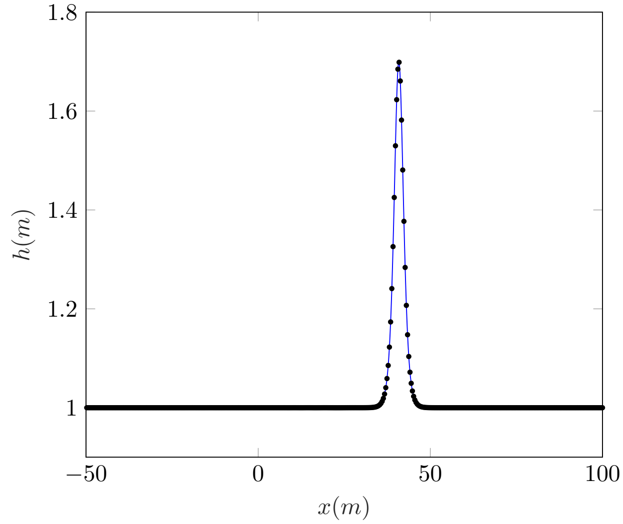
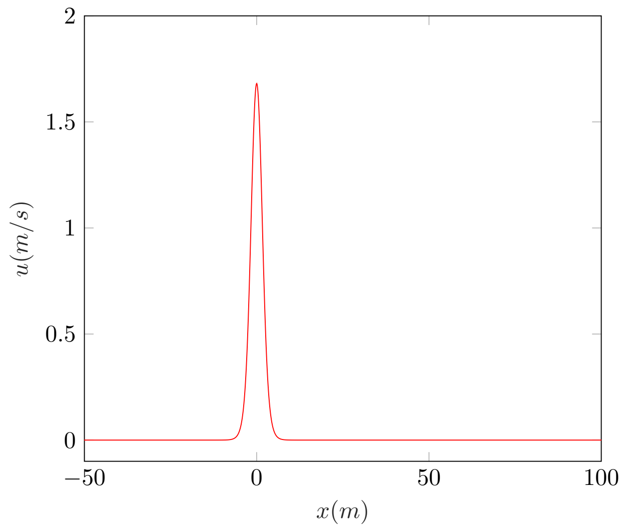


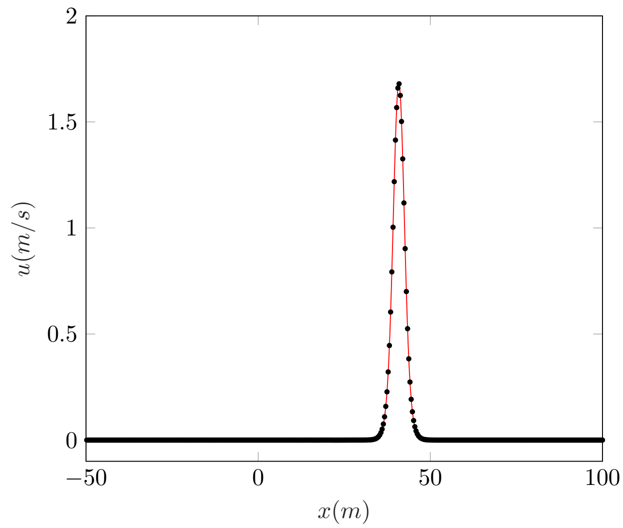
(a) $t = 0s$



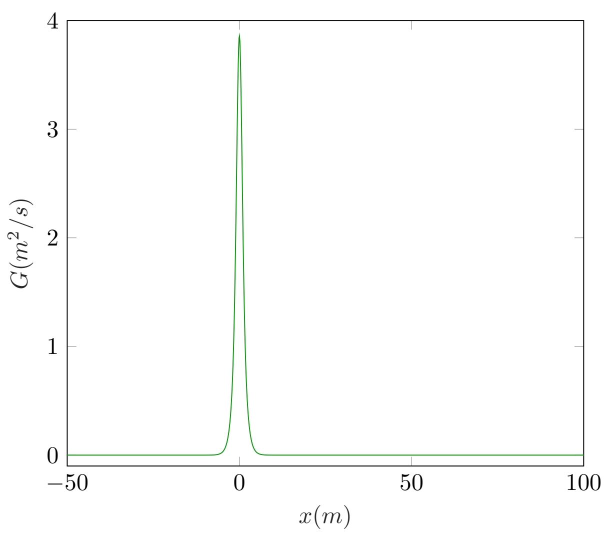
(b) $t = 10s$



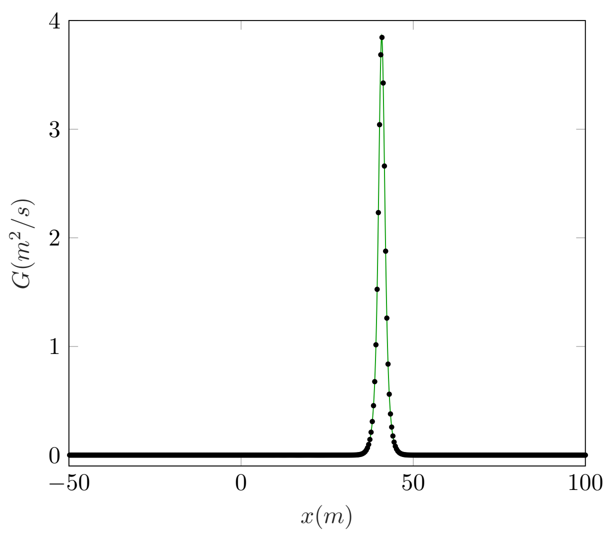
(c) $t = 0s$



(d) $t = 10s$



(e) $t = 0s$



(f) $t = 10s$

Figure 4: $\Delta x = 0.0234m$.

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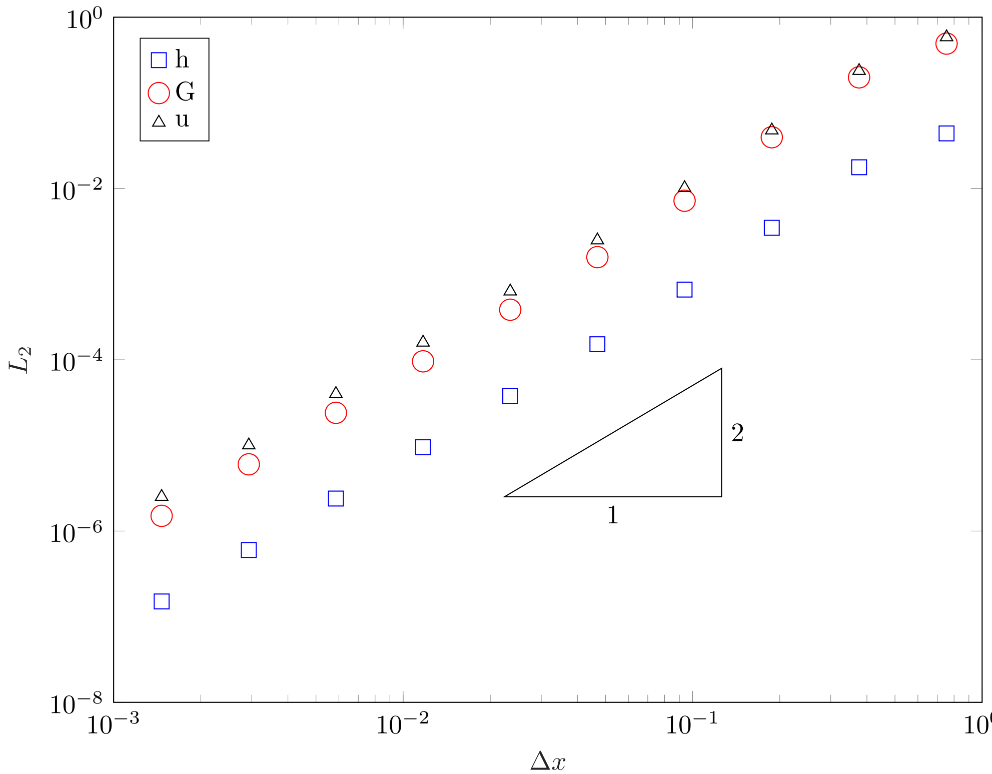

Figure 5: L_2 norm comparing numerical solution and analytic solution for travelling wave solution with $\beta_1 = \beta_2 = 0$.

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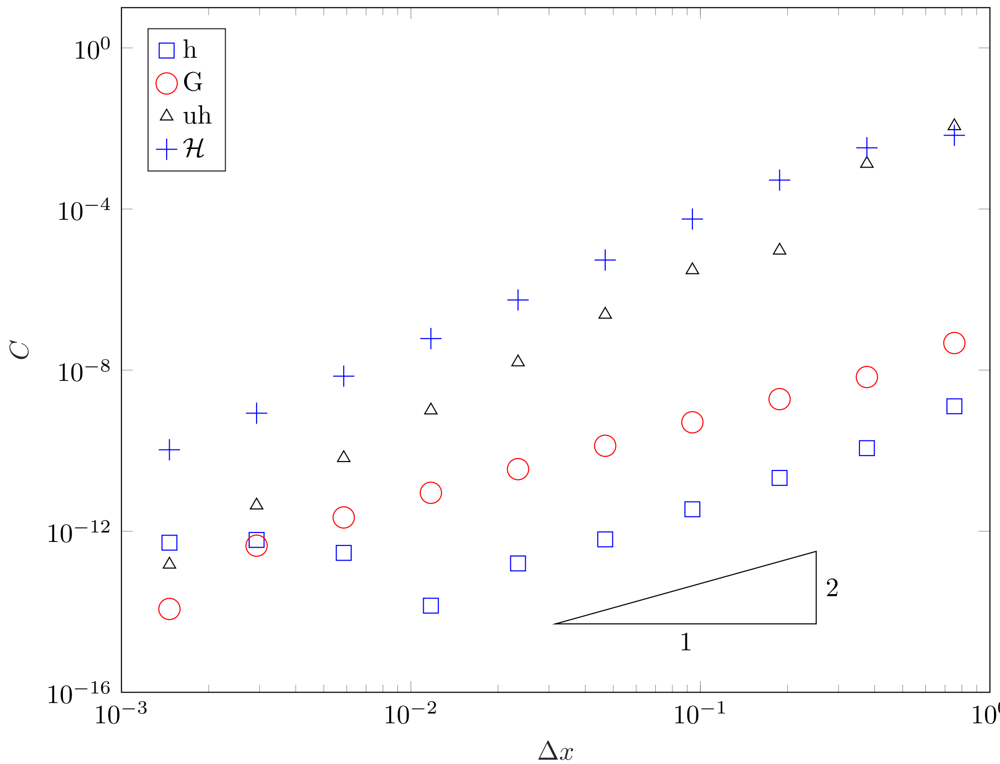

Figure 6: C comparing total amount of conserved quantities in initial conditions and numerical solution for travelling wave solution with $\beta_1 = \beta_2 = 0$.

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