Serre Notes

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1 Regularised Serre Equation

Clamond and Dutykh[?] derived the following regularised Shallow Water Wave equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{1a}$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \epsilon(x,t)\mathscr{R}h^2\right) = 0 \tag{1b}$$

where

$$\mathscr{R} \stackrel{\text{def}}{=} h\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x \partial t} - u\frac{\partial^2 u}{\partial x^2}\right) - g\left(h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right).$$

In this context, regularisation means adding additional terms to an equation to control or eliminate fluctuations or oscillations in the solution.

If $\epsilon = 0$ the non-linear shallow water wave equation are recovered. For $\epsilon \neq 0$, $\mathscr{R}$ is a regularisation term that prevents the formation of shocks. It consists of dispersive term that characterises the Serre equation and additional regularisation terms.

We want to make the same transformation as usual, but now with ϵ not being constant we will pick up some extra terms to make sure that

We usually solve the equivalent Serre equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{2a}$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left[uG + \frac{gh^2}{2} - \epsilon(x,t)h^2\left(2h\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + gh\frac{\partial^2 h}{\partial x^2} + \frac{g}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)\right] = 0 \tag{2b}$$

where

$$G \stackrel{\text{def}}{=} uh - \epsilon(x,t)\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right)$$

However, (2) is not equivalent to (1), which can be seen since $\frac{\partial G}{\partial t}$ will result in a derivative of η w.r.t time.

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We will settle for modified version of (2) that includes some source terms.

$$
\begin{aligned}
\frac{\partial G}{\partial t} &= \frac{\partial (uh)}{\partial t} - \frac{\partial \epsilon}{\partial t}\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right) - \epsilon\frac{\partial}{\partial x \partial t}\left(h^3\frac{\partial u}{\partial x}\right) \\
&= \frac{\partial (uh)}{\partial t} - \frac{\partial \epsilon}{\partial t}\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right) - \epsilon\frac{\partial}{\partial x}\left(3\frac{\partial h}{\partial t}h^2\frac{\partial u}{\partial x} + h^3\frac{\partial u}{\partial x \partial t}\right) \\
&= \frac{\partial (uh)}{\partial t} - \frac{\partial \epsilon}{\partial t}\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right) - \epsilon\frac{\partial}{\partial x}\left(-3\frac{\partial (uh)}{\partial x}h^2\frac{\partial u}{\partial x} + h^3\frac{\partial u}{\partial x \partial t}\right) \\
&= \frac{\partial (uh)}{\partial t} - \frac{\partial \epsilon}{\partial t}\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right) - \epsilon\frac{\partial}{\partial x}\left(-3h^3\left[\frac{\partial u}{\partial x}\right]^2 - 3uh^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x} + h^3\frac{\partial u}{\partial x \partial t}\right) \\
&= \frac{\partial (uh)}{\partial t} - \frac{\partial \epsilon}{\partial t}\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right) - \epsilon\frac{\partial}{\partial x}\left(-3h^3\left[\frac{\partial u}{\partial x}\right]^2 - 3uh^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x} + h^3\frac{\partial u}{\partial x \partial t}\right) \quad (3)
\end{aligned}
$$

Flux term

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left[uG - 2\epsilon h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right] \\
&= \frac{\partial}{\partial x}\left[u^2h - u\epsilon\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right) - 2\epsilon h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right] \\
&= \frac{\partial}{\partial x}\left[u^2h - u\epsilon\left(3h^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}\right) - u\epsilon\left(h^3\frac{\partial^2 u}{\partial x^2}\right) - 2\epsilon h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right] \quad (4)
\end{aligned}
$$

Add them

$$
\begin{aligned}
&\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left[uG - 2\epsilon h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right] \\
&= \frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left[u^2h\right] - \frac{\partial \epsilon}{\partial t}\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right) - \epsilon\frac{\partial}{\partial x}\left(-3h^3\left[\frac{\partial u}{\partial x}\right]^2 - 3uh^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x} + h^3\frac{\partial u}{\partial x \partial t}\right) \\
&\quad + \frac{\partial}{\partial x}\left[-u\epsilon\left(3h^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}\right) - u\epsilon\left(h^3\frac{\partial^2 u}{\partial x^2}\right) - 2\epsilon h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right] \\
&= \frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left[u^2h\right] - \frac{\partial \epsilon}{\partial t}\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right) - \epsilon\frac{\partial}{\partial x}\left(-3h^3\left[\frac{\partial u}{\partial x}\right]^2 - 3uh^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x} + h^3\frac{\partial u}{\partial x \partial t}\right) \\
&\quad + \frac{\partial \epsilon}{\partial x}\left[-u\left(3h^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}\right) - u\left(h^3\frac{\partial^2 u}{\partial x^2}\right) - 2h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right] + \epsilon\frac{\partial}{\partial x}\left[-u\left(3h^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}\right) - u\left(h^3\frac{\partial^2 u}{\partial x^2}\right) - 2h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right] \\
&= \frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left[u^2h\right] - \frac{\partial \epsilon}{\partial t}\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right) + \frac{\partial \epsilon}{\partial x}\left[-u\left(3h^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}\right) - u\left(h^3\frac{\partial^2 u}{\partial x^2}\right) - 2h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right] \\
&\quad - \epsilon\frac{\partial}{\partial x}\left(-3h^3\left[\frac{\partial u}{\partial x}\right]^2 - 3uh^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x} + h^3\frac{\partial u}{\partial x \partial t}\right) - \epsilon\frac{\partial}{\partial x}\left[u\left(3h^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}\right) + u\left(h^3\frac{\partial^2 u}{\partial x^2}\right) + 2h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right] \\
&= \frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left[u^2h\right] - \frac{\partial \epsilon}{\partial t}\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right) + \frac{\partial \epsilon}{\partial x}\left[-u\left(3h^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}\right) - u\left(h^3\frac{\partial^2 u}{\partial x^2}\right) - 2h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right] \\
&\quad - \epsilon\frac{\partial}{\partial x}\left(-3h^3\left[\frac{\partial u}{\partial x}\right]^2 - 3uh^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x} + h^3\frac{\partial u}{\partial x \partial t} + u\left(3h^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}\right) + u\left(h^3\frac{\partial^2 u}{\partial x^2}\right) + 2h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right) \\
&= \frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left[u^2h\right] - \frac{\partial \epsilon}{\partial t}\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right) + \frac{\partial \epsilon}{\partial x}\left[-u\left(3h^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}\right) - u\left(h^3\frac{\partial^2 u}{\partial x^2}\right) - 2h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right] \\
&\quad + \epsilon\frac{\partial}{\partial x}\left(h^3\left[\frac{\partial u}{\partial x}\right]^2 - h^3\frac{\partial u}{\partial x \partial t} - u\left(h^3\frac{\partial^2 u}{\partial x^2}\right)\right) \quad (5)
\end{aligned}
$$

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Excellent so we get the additional terms

$$S_a = \frac{\partial \epsilon}{\partial t}\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right) - \frac{\partial \epsilon}{\partial x}\left[-u\left(3h^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}\right) - u\left(h^3\frac{\partial^2 u}{\partial x^2}\right) - 2h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right]$$

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{6a}$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left[uG + \frac{gh^2}{2} - \epsilon(x,t)h^2\left(2h\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + gh\frac{\partial^2 h}{\partial x^2} + \frac{g}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)\right] \tag{6b}$$

$$= \frac{\partial \epsilon}{\partial t}\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right) - \frac{\partial \epsilon}{\partial x}\left[-u\left(3h^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}\right) - u\left(h^3\frac{\partial^2 u}{\partial x^2}\right) - 2h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right] \tag{6c}$$

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