Numerical Study of The Generalised Serre-Green-Naghdi Model

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1 Introduction

2 Generalised Serre-Green-Naghdi Equations

Clamond and Dutykh[**?**] derived the generalised Serre-Green-Naghdi (gSGN) equations:

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{1a}$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{1}{3}h^2\Gamma\right) = 0 \tag{1b}$$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}hu^2 + \frac{1}{4}\left(\frac{2}{3}+\beta_1\right)h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{1}{2}gh^2\left(1+\frac{1}{2}\beta_2\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)\right]$$
$$\frac{\partial}{\partial x}\left[uh\left(\frac{1}{2}u^2 + \frac{1}{4}\left(\frac{2}{3}+\beta_1\right)h^2\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + gh\left(1+\frac{1}{4}\beta_2\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right) + \frac{1}{3}h\Gamma\right) + \frac{1}{2}\beta_2 gh^3\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}\right] = 0 \tag{1c}$$

Resulting Equations	β_1	β_2
Serre Equations	0	0
SWWE Equations	$-\dfrac{2}{3}$	0
Regularised SWWE Equation Family	free variable	$\beta_1 + \dfrac{2}{3}$
Modified Dispersion Serre Equation Family	free variable	β_1

Table 1: Showing various combinations of β values and equivalent equations

where

$$\Gamma = \frac{3}{2}\left(\frac{2}{3} + \beta_1\right) h \left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x \partial t} - u\frac{\partial^2 u}{\partial x^2}\right] - \frac{3}{2}\beta_2 g \left[h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right] \tag{1d}$$

These equations have the same order of approximation in the lagrangian density (dispersion properties?) when $\beta_1 = \beta_2$. The interesting thing about the equations though, is that we will conserve mass, momentum and energy for all values of β_j.

From these equations the SWWE, the Serre equations and the regularised SWWE [?] can be recovered for certain values of β_1 and β_2.

2.1 Alternative Conservative Form of the gSGN

A major difficulty with solving the SGN is that the dispersive terms contain a mixed spatial-temporal derivative term which is difficult to handle numerically. This mixed derivative term can be rewritten so that the Serre equations can be expressed in conservation law form, with the water depth and a new quantity as conservative variables. This reformulation allows standard techniques for solving conservation laws to be applied to the Serre equations, even though the Serre equations are neither hyperbolic nor parabolic.

Consider the Serre equations written for a horizontal bed. The flux term in the momentum equation, (1b) contains a mixed spatial and temporal derivative term which is difficult to treat numerically. It is possible to replace this term by a combination of spatial and temporal derivative terms by making the following observation

$$\begin{aligned}\frac{\partial^2}{\partial x \partial t}\left(\frac{1}{3}\left(1 + \frac{3}{2}\beta_1\right) h^3 \frac{\partial u}{\partial x}\right) &= \frac{1}{3}\left(1 + \frac{3}{2}\beta_1\right)\frac{\partial}{\partial t}\left(3h^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x} + h^3\frac{\partial^2 u}{\partial x^2}\right)\\ &= \frac{1}{3}\left(1 + \frac{3}{2}\beta_1\right)\frac{\partial}{\partial x}\left(3h^2\frac{\partial h}{\partial t}\frac{\partial u}{\partial x} + h^3\frac{\partial^2 u}{\partial x \partial t}\right).\end{aligned} \tag{2}$$

Rearranging and making use of the continuity equation, (1a) the momentum equation, (1b) becomes

$$\begin{aligned}&\frac{\partial}{\partial t}\left(uh - \frac{1}{3}\left(1 + \frac{3}{2}\beta_1\right)\frac{\partial}{\partial x}\left[h^3\frac{\partial u}{\partial x}\right]\right)\\ &+\frac{\partial}{\partial x}\left(u\left[uh - \frac{1}{3}\left(1 + \frac{3}{2}\beta_1\right)\frac{\partial}{\partial x}\left[h^3\frac{\partial u}{\partial x}\right]\right] + \frac{gh^2}{2} - \frac{2}{3}\left(1 + \frac{3}{2}\beta_1\right)h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{2}\beta_2 g h^2\left[h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right]\right) = 0.\end{aligned} \tag{3}$$

The momentum equation can be written in conservation law form as

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}\left(1 + \frac{3}{2}\beta_1\right)h^3\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{2}\beta_2 g h^2\left[h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right]\right) = 0. \tag{4}$$

where a new conserved quantity, G is given by

$$G = uh - \frac{1}{3}\left(1 + \frac{3}{2}\beta_1\right)\frac{\partial}{\partial x}\left(h^3\frac{\partial u}{\partial x}\right).$$

This expands the conserved variable introduced by [?], as well as in the Serre equations [].

Thus we have the following conservation equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{5a}$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}\left(1 + \frac{3}{2}\beta_1\right)h^3 \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{2}\beta_2 g h^2 \left[h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right]\right) = 0. \tag{5b}$$

with

$$G = uh - \frac{1}{3}\left(1 + \frac{3}{2}\beta_1\right)\frac{\partial}{\partial x}\left(h^3 \frac{\partial u}{\partial x}\right). \tag{5c}$$

2.2 Dispersion Relation of Linearised gSGN

Assuming that

$$h(x,t) = h_0 + \delta\eta(x,t) + O(\delta^2)$$
$$u(x,t) = u_0 + \delta v(x,t) + O(\delta^2)$$

By substituting these forms into the linearised Serre equations and neglecting $O(\delta^2)$ terms, we get the linearised regularised Serre equations. We also substitute η_t using the mass equation into the momentum equation.

$$(\delta\eta)_t + u_0(\delta\eta)_x + h_0(\delta v)_x = 0 \tag{6a}$$

$$h_0(\delta v)_t + gh_0(\delta\eta)_x + h_0u_0(\delta v)_x - \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)h_0^3(\delta v)_{xxt} - \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)h_0^3u_0(\delta v)_{xxx} - \frac{g\beta_2}{2}h_0^3(\delta\eta)_{xxx} = 0 \tag{6b}$$

We can remove the δ term, either by removing the common factor, or absorbing it into η and v to get

$$\eta_t + u_0\eta_x + h_0v_x = 0 \tag{7a}$$

$$h_0(v)_t + gh_0(\eta)_x + h_0u_0(v)_x - \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)h_0^3(v)_{xxt} - \frac{1}{3}\left(1+\frac{3}{2}\beta_1\right)h_0^3u_0(v)_{xxx} - \frac{g\beta_2}{2}h_0^3(\eta)_{xxx} = 0 \tag{7b}$$

We now assume that $\eta(x,t) = H\exp(i(kx-\omega t))$, $v(x,t) = U\exp(i(kx-\omega t))$

$$\eta(x,t) = H\exp(i(kx-\omega t))$$
$$v(x,t) = U\exp(i(kx-\omega t))$$

substituting these into the linearised Serre equation we get

$$[Hu_0k - H\omega + Uh_0k]\, i \exp[i(kx-\omega t)] = 0 \tag{8a}$$

$$\Big[3H\beta_2 g h_0^2 k^3 + 6Hgk - 3U\beta_1\omega h_0^2k^2 + 3U\beta_1 h_0^2k^3u_0$$
$$- 2U\omega h_0^2k^2 - 6U\omega + 2Uh_0^2k^3u_0 + 6Uku_0\Big] i\frac{h_0}{6}\exp[i(kx-\omega t)] = 0 \tag{8b}$$

This can be written as

$$\begin{bmatrix} u_0k - \omega & h_0k \\ 3\beta_2h_0^2k^3 + 6kg & -3\beta_1\omega h_0^2k^2 + 3\beta_1h_0^2k^3u_0 - 2\omega h_0^2k^2 - 6\omega + 2h_0^2k^3u_0 + 6ku_0 \end{bmatrix}\begin{bmatrix} H \\ U \end{bmatrix} = 0 \tag{9}$$

Which has non-trivial solutions when the determinant is zero.

The determinant of this matrix is

$$\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right)\omega^2 - 2u_0 k\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right)\omega + u_0^2 k^2\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right) - g h_0 k^2\left(3\beta_2 h_0^2 k^2 + 6\right) \tag{10}$$

To get non-trivial solution we have

$$\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right)\omega^2 - 2u_0 k\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right)\omega + u_0^2 k^2\left(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6\right) - g h_0 k^2\left(3\beta_2 h_0^2 k^2 + 6\right) = 0 \tag{11}$$

$$\omega^2 - 2u_0 k\omega + u_0^2 k^2 - g h_0 k^2 \frac{3\beta_2 h_0^2 k^2 + 6}{(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6)} \tag{12}$$

Using quadratic equation, or another quadratic polynomial solver we get that

$$\omega^{\pm} = u_0 k \pm k\sqrt{g h_0}\sqrt{\frac{3\beta_2 h_0^2 k^2 + 6}{(3\beta_1 h_0^2 k^2 + 2h_0^2 k^2 + 6)}} \tag{13}$$

$$\omega^{\pm} = u_0 k \pm k\sqrt{g h_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2}} \tag{14}$$

Thus we have the following regimes

- SWWE wavespeeds (non-dispersive): occurs when $\beta_2 = \frac{2}{3} + \beta_1$ then $\omega^{\pm} = k\left(u_0 \pm \sqrt{g h_0}\right)$ ([?]).
- Serre wavespeeds - when $\beta_1 = \beta_2 = 0$ - we get $\omega^{\pm} = k\left(u_0 \pm \sqrt{g h_0}\sqrt{\dfrac{3}{3 + h_0^2 k^2}}\right)$
- For other values of β_1 and β_2 we can vary the wavespeeds, in certain situations this will lead to a better approximation of the linear wavespeed of waves than the Serre equations ([?]), in others it will be worse. We show an example below.

Now for the group speed

$$v_g^{\pm} = \frac{\partial \omega^{\pm}}{\partial k} = u_0 \pm \sqrt{g h_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm \frac{k\sqrt{g h_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\beta_2 h_0^2 k - \frac{h_0^2 k\left(\beta_1 + \frac{2}{3}\right)\left(\beta_2 h_0^2 k^2 + 2\right)}{h_0^2 k^2\left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right] \tag{15}$$

$$= u_0 \pm \sqrt{g h_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm \frac{k\sqrt{g h_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\frac{\beta_2 h_0^2 k^2\left[h_0^2 k^2\left(\beta_1 + \frac{2}{3}\right) + 2\right] - h_0^2 k\left(\beta_1 + \frac{2}{3}\right)\left(\beta_2 h_0^2 k^2 + 2\right)}{h_0^2 k^2\left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$

$$= u_0 \pm \sqrt{g h_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm \frac{k\sqrt{g h_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\frac{\beta_2 h_0^4 k^3\left(\beta_1 + \frac{2}{3}\right) + 2\beta_2 h_0^2 k - h_0^2 k\left(\beta_1 + \frac{2}{3}\right)\left(\beta_2 h_0^2 k^2 + 2\right)}{h_0^2 k^2\left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$

$$= u_0 \pm \sqrt{g h_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm \frac{k\sqrt{g h_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\frac{\beta_2 h_0^4 k^3\left(\beta_1 + \frac{2}{3}\right) + 2\beta_2 h_0^2 k - \beta_2\left(\beta_1 + \frac{2}{3}\right) h_0^4 k^3 - 2h_0^2 k\left(\beta_1 + \frac{2}{3}\right)}{h_0^2 k^2\left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \pm$$

$$\frac{k\sqrt{gh_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\frac{2\beta_2 h_0^2 k - 2h_0^2 k\left(\beta_1 + \frac{2}{3}\right)}{h_0^2 k^2\left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$

$$= \quad u_0 \;\pm\; \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \;\pm\; 2\frac{k^2 h_0^2\sqrt{gh_0}}{\beta_2 h_0^2 k^2 + 2}\left[\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left(\frac{\beta_2 - \beta_1 - \frac{2}{3}}{h_0^2 k^2\left(\beta_1 + \frac{2}{3}\right) + 2}\right)\right]$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}}\left[1 + \frac{\beta_2 - \beta_1 - \frac{2}{3}}{\left(\frac{1}{2}\beta_2 h_0^2 k^2 + 1\right)\left(\left(\frac{1}{3} + \beta_1\right) h_0^2 k^2 + 1\right)}\right]$$

When $\beta_2 = \frac{2}{3} + \beta_1$ then the group speed is equal to the phase speed of the Shallow Water Wave equations. When $\beta_2 = \beta_1 = 0$ then we get

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{2}{\left(\frac{2}{3} h_0^2 k^2 + 2\right)}}\left[1 + \frac{-\frac{1}{3}}{\left(\frac{1}{3} h_0^2 k^2 + 1\right)}\right]$$

$$= u_0 \pm \sqrt{gh_0}\sqrt{\frac{3}{h_0^2 k^2 + 3}}\left[1 - \frac{1}{h_0^2 k^2 + 3}\right]$$

which matches [Zoppo,pitt,paper on FDVM methods for Serre equations / Chris thesis]

2.2.1 Wave Speed Bounds

First we want to show that phase and group speed are bounded, when treated as functions of k. The phase speed is:

$$v_p^{\pm} = \frac{\omega^{\pm}}{k} = u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)}} \tag{16}$$

For $\beta_1 \geq -\frac{2}{3}$ and $\beta_2 \geq 0$ we have that

$$\frac{\beta_2 h_0^2 k^2 + 2}{\left(\left(\frac{2}{3} + \beta_1\right) h_0^2 k^2 + 2\right)} \tag{17}$$

is a montone function in terms of $h_0 k$, it is monotone decreasing if $\beta_2 \leq \frac{2}{3} + \beta_1$ and monotone increasing if $\beta_2 \geq \frac{2}{3} + \beta_1$.

We have the following limits

as $k \to 0$

$$v_p^{\pm} = u_0 \pm \sqrt{gh_0} \tag{18}$$

as $k \to \infty$

$$v_p^{\pm} = u_0 \pm \sqrt{gh_0}\sqrt{\frac{\beta_2}{\frac{2}{3} + \beta_1}} \tag{19}$$

When $\dfrac{\beta_2}{\frac{2}{3} + \beta_1} \leq 1$ we have

$$u_0 - \sqrt{gh_0} \leq v_p^- \leq u_0 - \sqrt{gh_0}\sqrt{\frac{\beta_2}{\frac{2}{3} + \beta_1}} \leq u_0 \leq u_0 + \sqrt{gh_0}\sqrt{\frac{\beta_2}{\frac{2}{3} + \beta_1}} \leq v_p^+ \leq u_0 + \sqrt{gh_0} \tag{20}$$

(includes Serre and regularised SWWE). For Serre the $k \to \infty$ bounds are u_0, so inequality simplifies for regularised SWWE, phase speed is fixed as k changes.

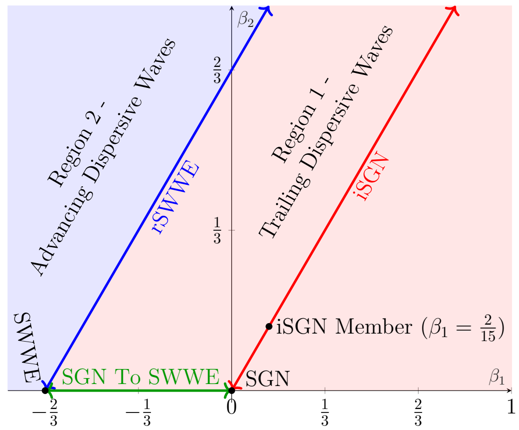


Figure 1: Wave speed region plots showing important families of equations and particular members of these families.

when $\dfrac{\beta_2}{\frac{2}{3}+\beta_1} > 1$ we have

$$u_0 - \sqrt{gh_0}\sqrt{\frac{\beta_2}{\frac{2}{3}+\beta_1}} \le v_p^- \le u_0 - \sqrt{gh_0} \le u_0 \le u_0 + \sqrt{gh_0} \le v_p^+ \le u_0 + \sqrt{gh_0}\sqrt{\frac{\beta_2}{\frac{2}{3}+\beta_1}} \tag{21}$$

Region 1 is what we expect, dispersive waves travel behind the shock front, Region 2 is inverted and we begin to see dispersive waves travel in front of the shock.

We thus have the following plots

3 Numerical Method

4 Validation

4.1 Analytic Solutions

4.1.1 Serre Equations ($\beta_1 = \beta_2 = 0$) - Solitary Travelling Wave Solution

When $\beta_1 = \beta_2 = 0$ the gSGN are equivalent to the SGN equations which admit the following travelling wave solution

$$h(x,t) = a_0 + a_1 \text{sech}^2\left(\kappa(x-ct)\right) \tag{22a}$$

$$u(x,t) = c\left(1 - \frac{a_0}{h(x,t)}\right) \tag{22b}$$

where

$$\kappa = \frac{\sqrt{3a_1}}{2a_0\sqrt{a_0+a_1}} \tag{22c}$$

$$c = \sqrt{g\left(a_0+a_1\right)} \tag{22d}$$

Results - Example,Convergence and Conservation Plot

4.1.2 SWWE ($\beta_1 = -\frac{2}{3}$ and $\beta_2 = 0$) - Dambreak Solution

$$h(x,0) = \begin{cases} h_0 & x < 0 \\ h_1 & x \geq 0 \end{cases} \tag{23}$$

$$u(x,0) = 0 \tag{24}$$

$$G(x,0) = 0 \tag{25}$$

We have an analytic solution for the SWWE for the discontinuous limit of these equations as $\alpha \to 0$. It is 3 constant states $(h_0, 0)$, (h_s, u_s) and $(h_1, 0)$ where

$$h_s = \frac{h_0}{2}\sqrt{1 + 8\left(\frac{2h_s}{h_s - h_0}\left(\frac{\sqrt{gh_1} - \sqrt{gh_s}}{\sqrt{gh_0}}\right)\right)^2 - 1} \tag{26}$$

$$u_s = 2\left(\sqrt{gh_1} - \sqrt{gh_s}\right). \tag{27}$$

Where $(h_0, 0)$ and (h_s, u_s) are joined by a

Results - Example and Conservation Table

4.2 Forced Solutions

To demonstate the validity and versatility of our method to solve the gSGN whilst allowing varying β_i values, we make use of forced solutions. To generate a forced solution we consider the modified gSGN equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = \frac{\partial h^*}{\partial t} + \frac{\partial (u^* h^*)}{\partial x} \tag{28a}$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}\left(1 + \frac{3}{2}\beta_1\right)h^3 \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{2}\beta_2 g h^2 \left[h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right]\right) =$$
$$\frac{\partial G^*}{\partial t} + \frac{\partial}{\partial x}\left(u^* G^* + \frac{g\left(h^*\right)^2}{2} - \frac{2}{3}\left(1 + \frac{3}{2}\beta_1\right)\left(h^*\right)^3 \frac{\partial u^*}{\partial x}\frac{\partial u^*}{\partial x} - \frac{1}{2}\beta_2 g\left(h^*\right)^2\left[h^*\frac{\partial^2 h^*}{\partial x^2} + \frac{1}{2}\frac{\partial h^*}{\partial x}\frac{\partial h^*}{\partial x}\right]\right) \tag{28b}$$

which admit the solutions h^*, u^* and G^* assuming G^* appropriately satisfies (5c). Since these equations are satisfied for any chosen h^*, u^* and G^*, we can generated any desired solution. Since the left hand-side of these modified equations are approximated by our numerical method, if we combine the numerical method with analytic expressions for the right handside, we have a method that approximates the forced gSGN equation with the same convergence properties as the underlying numerical method for the gSGN equations.

Since we are free to choose h^*, u^* and G^* we can generate solutions and thus test our convergence properties for situations for which no analytic solution to the equations exist, in particular in this paper we are interested in solutions where the β values vary in space and time.

I have used this technique to investigate the method for the following forced solutions

$$h^*(x,t) = a_0 + a_1 \exp\left(\frac{\left(x - a_2 t\right)^2}{2a_3}\right) \tag{29a}$$

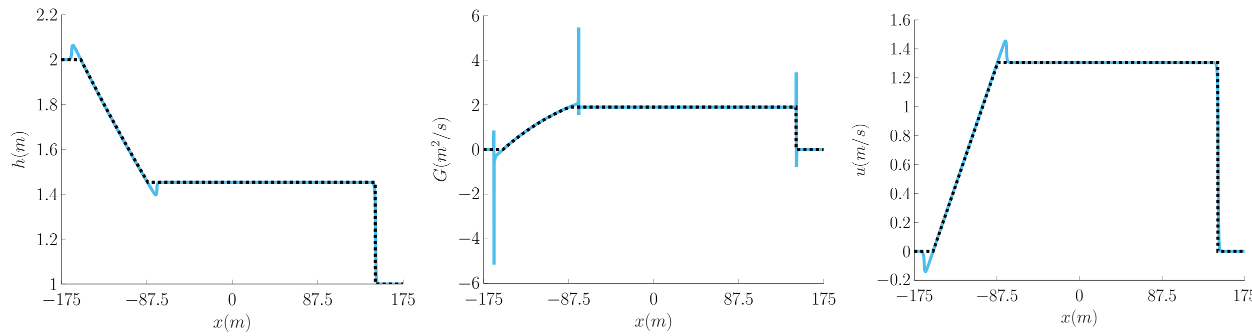
$$u^*(x,t) = a_4 \exp\left(\frac{\left(x - a_2 t\right)^2}{2a_3}\right) \tag{29b}$$

$$\beta_1(x,t) = a_6 \tag{29c}$$

$$\beta_2(x,t) = a_7 \tag{29d}$$

where G^* is determined by (5c) with the above values.

Results - Example plot and L1 convergence, comment that this worked for all beta values, and we only show one for conciseness



(a) h (b) G (c) u

Figure 2: Plot of example numerical solution for representative memeber of regularised SWWE family to smooth dambreak problem at $t = 35s$.

Colour region	Condition
first blue	$\frac{x}{t} \le u_s - \sqrt{gh_s}$
first red	$u_s - \sqrt{gh_s} \le \frac{x}{t} \le u_s - \sqrt{gh_s}\sqrt{\frac{\beta_2}{\frac{2}{3}+\beta_1}}$
first green	$u_s - \sqrt{gh_s}\sqrt{\frac{\beta_2}{\frac{2}{3}+\beta_1}} \le \frac{x}{t} \le u_s$
second green	$u_s \le \frac{x}{t} \le u_s + \sqrt{gh_s}\sqrt{\frac{\beta_2}{\frac{2}{3}+\beta_1}}$
second red	$u_s + \sqrt{gh_s}\sqrt{\frac{\beta_2}{\frac{2}{3}+\beta_1}} \le \frac{x}{t} \le u_s + \sqrt{gh_s}$
second blue	$u_s + \sqrt{gh_s} \le \frac{x}{t}$

Table 2: Regions for plots.

5 Smooth Dambreak Study

$$h(x,0) = h_0 + \frac{h_1 - h_0}{2}\left(1 + \tanh\left(\frac{x}{\alpha}\right)\right) \tag{30}$$

$$u(x,0) = 0 \tag{31}$$

$$G(x,0) = 0 \tag{32}$$

5.1 Regularised SWWE Family $\beta_2 = \frac{2}{3} + \beta_1$

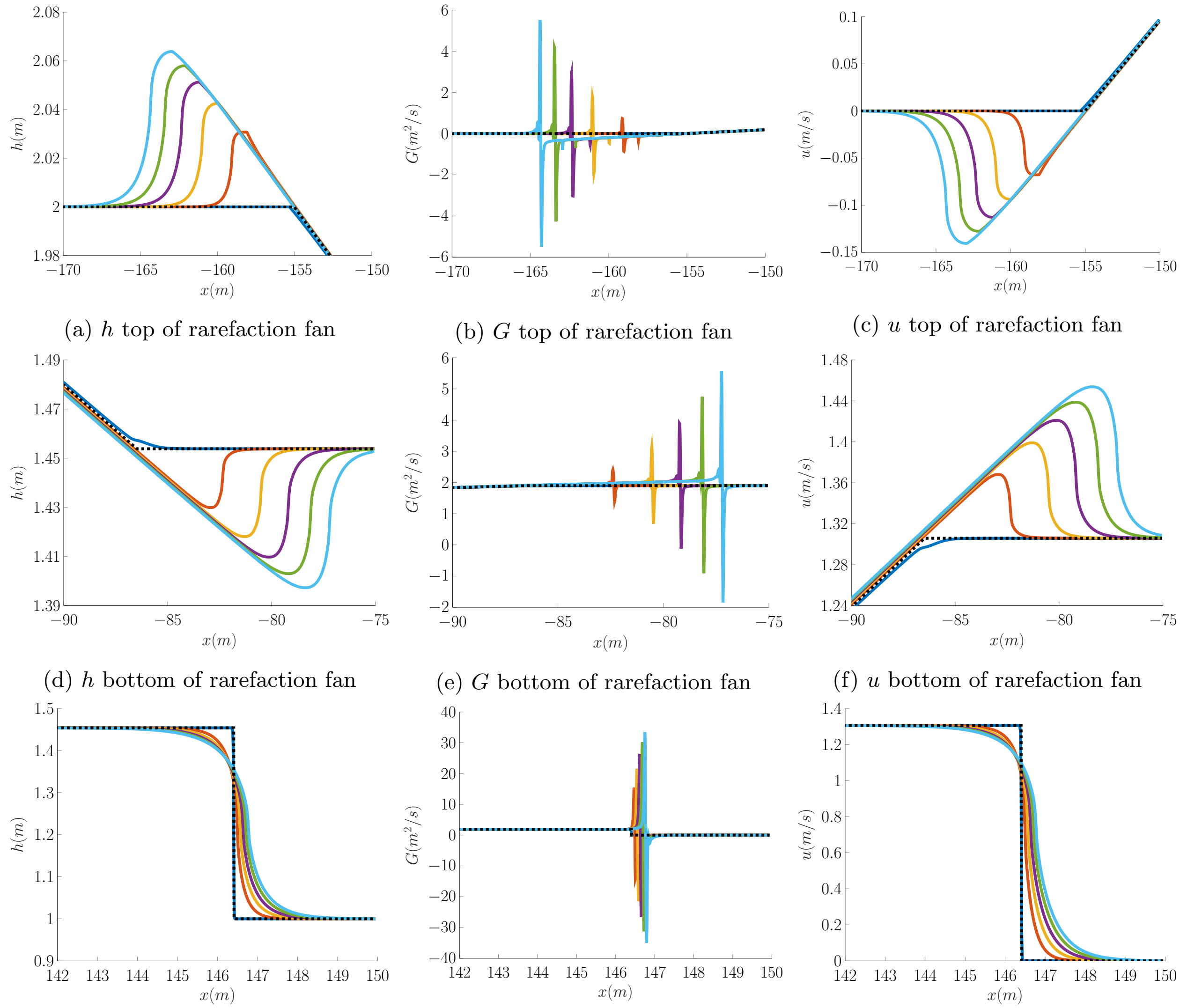
Justify interest, cite denys paper on regularised SWWE

- we do get nice well behaved transitions between behaviours when varying β values we get nice convergence.
- Although solutions look nice in h, in G there can be quite large values in the smoothed regions. However, u still looks quite nice.

5.2 Modified Dispersion Serre Family $\beta_2 = \beta_1$

Justify interest, cite denys paper on dispersion modified equations

Points:



(a) h top of rarefaction fan (b) G top of rarefaction fan (c) u top of rarefaction fan

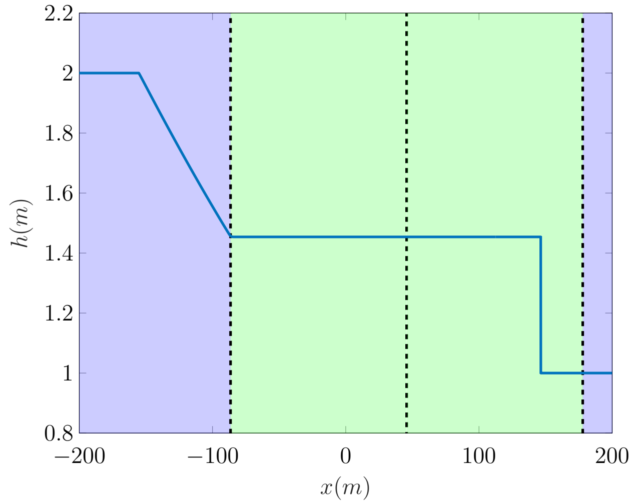
(d) h bottom of rarefaction fan (e) G bottom of rarefaction fan (f) u bottom of rarefaction fan

(g) h shock front (h) G shock front (i) u shock front

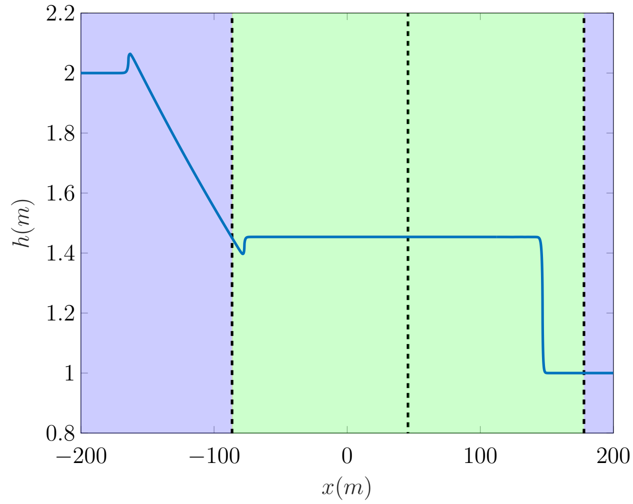
Figure 3: Plot of multiple smooth dambreak numerical solutions at $t = 35s$. $\beta_2 = 0$ (—), $\beta_2 = 0.1$ (—), $\beta_2 = 0.2$ (—), $\beta_2 = 0.3$ (—), $\beta_2 = 0.4$ (—), $\beta_2 = 0.5$ (—).

β_2	h	G	uh	$\mathcal{H}$
0	5.8×10^{-13}	6.3×10^{-13}	6.3×10^{-9}	3.7×10^{-3}
0.1	6.4×10^{-13}	6.4×10^{-13}	9.8×10^{-6}	3.9×10^{-3}
0.2	6.7×10^{-13}	6.4×10^{-13}	1.1×10^{-5}	4.1×10^{-3}
0.3	6.9×10^{-13}	6.4×10^{-13}	1.2×10^{-5}	4.3×10^{-3}
0.4	7.1×10^{-13}	6.6×10^{-13}	1.2×10^{-5}	4.4×10^{-3}
0.5	7.1×10^{-13}	6.5×10^{-13}	1.3×10^{-5}	4.6×10^{-3}

Table 3: Conservation errors for Regularised SWWE for the solutions provided above with $\beta_1 = \beta_2 - \frac{2}{3}$.



(a) $\beta_2 = 0$



(b) $\beta_2 = 0.5$

Figure 4: Regions of wave speeds for the smooth dambreak numerical solution at $t = 35s$.

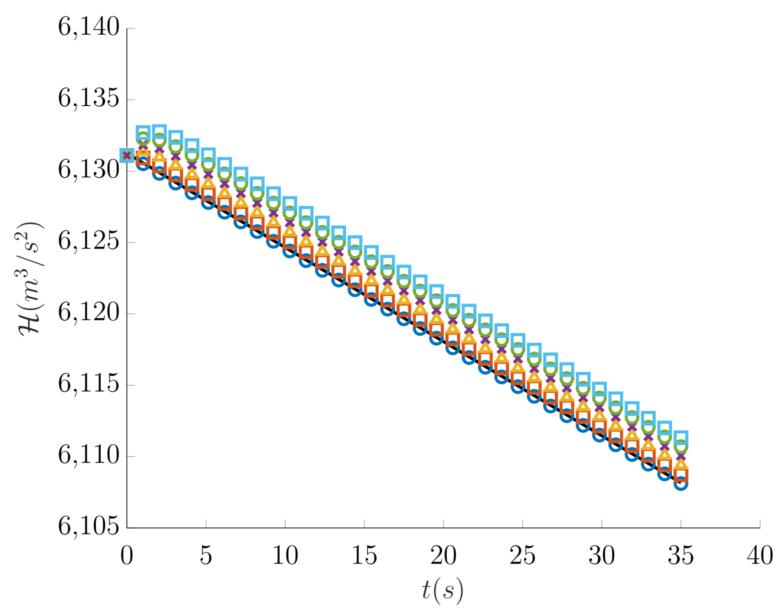
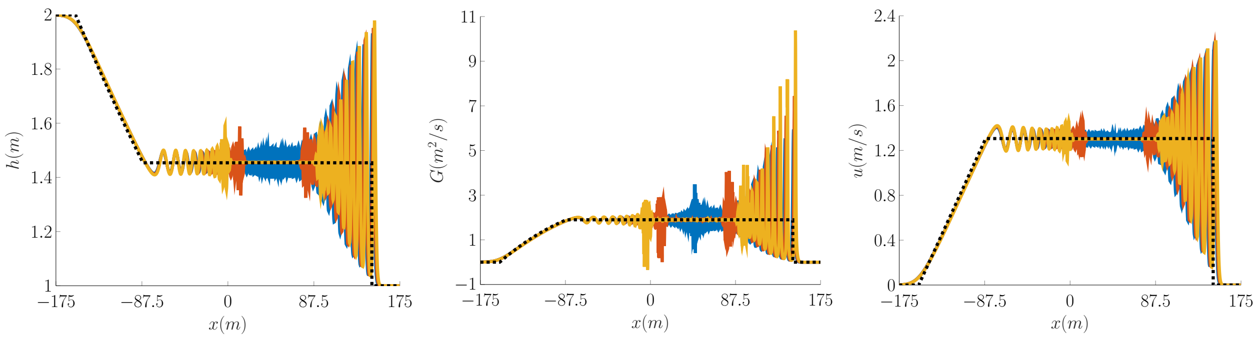


Figure 5: Energy over time for the numerical solutions, for the β values in the Regularised SWWE family.



(a) h (b) G (c) u

Figure 6: Plot of multiple smooth dambreak numerical solutions at $t = 35s$.

β_1	h	G	uh	$\mathcal{H}$
0	8.0×10^{-13}	6.3×10^{-13}	3.3×10^{-7}	3.8×10^{-6}
$\frac{1}{30}$	8.1×10^{-13}	6.4×10^{-13}	3.5×10^{-7}	1.9×10^{-5}
$\frac{1}{15}$	8.2×10^{-13}	6.3×10^{-13}	5.8×10^{-7}	1.1×10^{-4}

Table 4: Conservation errors for Modified Dispersion Serre Equations for the solutions provided above with $\beta_2 = \beta_1$.

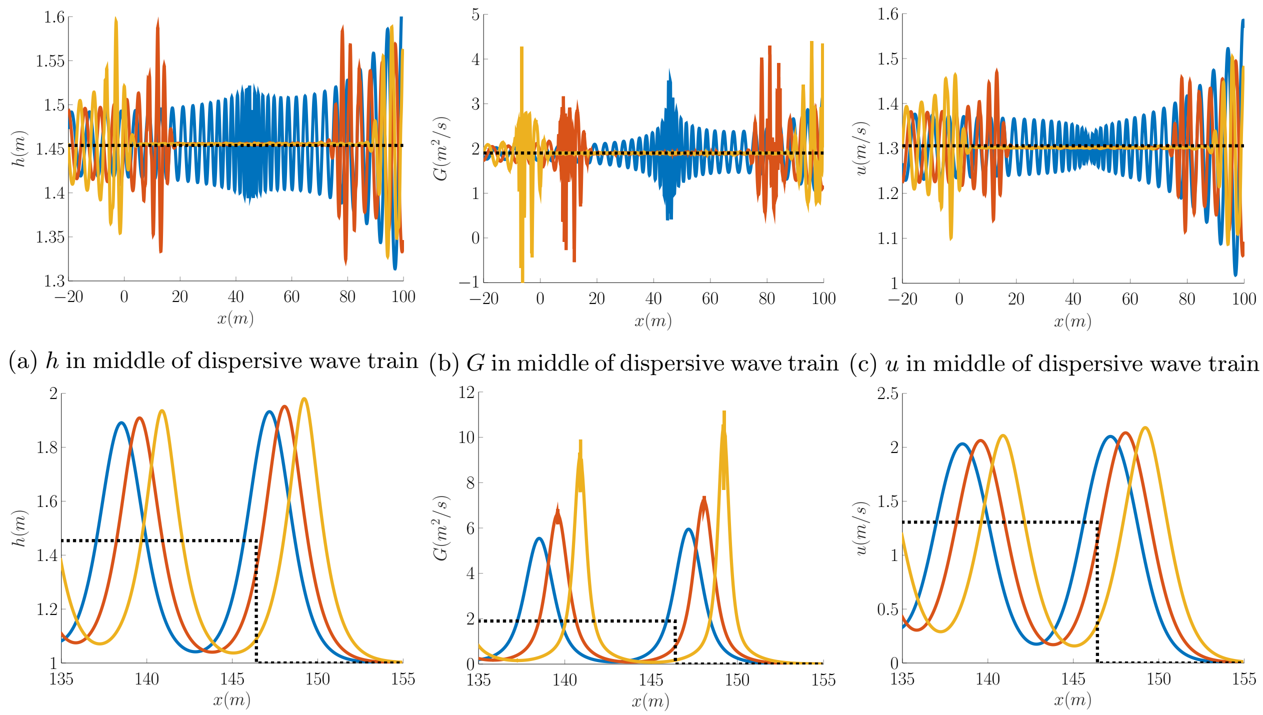
- we do get nice well behaved transitions between behaviours when varying β
- Dispersive wave train location well approximated by using the linear wave speeds. Although the bump around these bounds suggests non-linear effects are important as well, particularly close to transitions across the wavespeed boundaries.
- Improved dispersion may better approximate the dispsersion relation when $k \ll 1$, but at the cost of the middle of the dispersive wave train
- We get the bump behaviour observed in the db paper, but now dispersive wave trains are separate.
- Conservation is pretty similar, except for energy where it gets worse - this suggests that we will need finer grids to get good solutions particularly when gradients are large.

5.3 Serre To SWWE Family $\beta_2 = 0$ and $-\frac{2}{3} \leq \beta_1 \leq 0$

Justify interest, switching dispersion on and off, effect and giving structure/understanding to various schemes that turn it on and off.

Points:

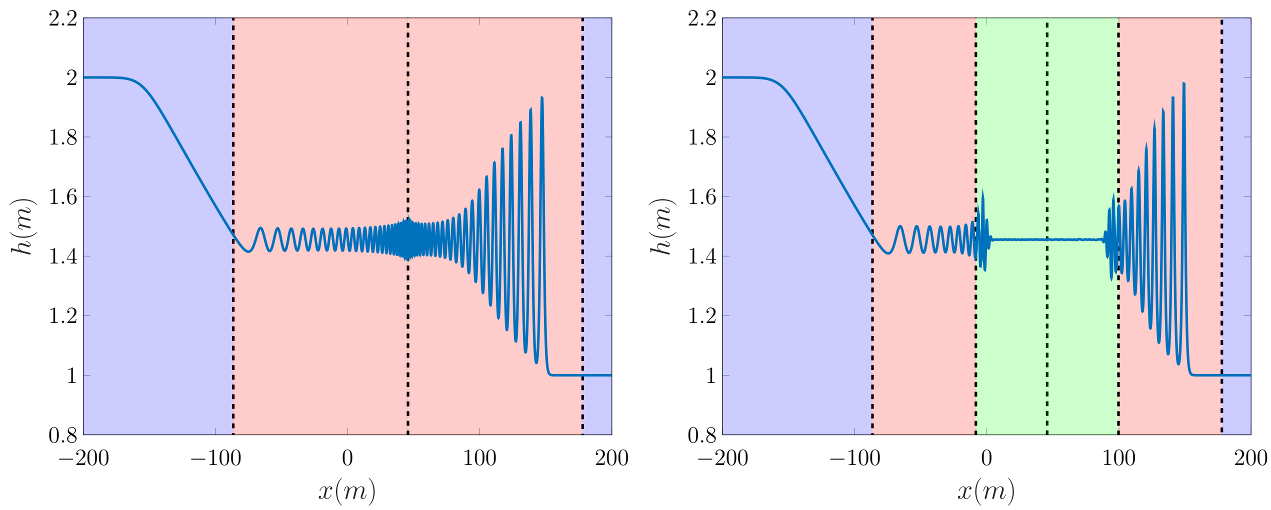
- we do get convergence, but most of the convergent behaviour occurs very close to the critical SWWE value. This is because gradients are large in the initial conditions
- Middle of dispersive wave train, and base and top of rarefaction fan most affected (closer to Serre)
- Still quite a large dispersive wave train even for β_1 values close to $-2/3$. (This appears to justify switching)
- the front of the shock became larger for some intermediate values.



(a) h in middle of dispersive wave train (b) G in middle of dispersive wave train (c) u in middle of dispersive wave train

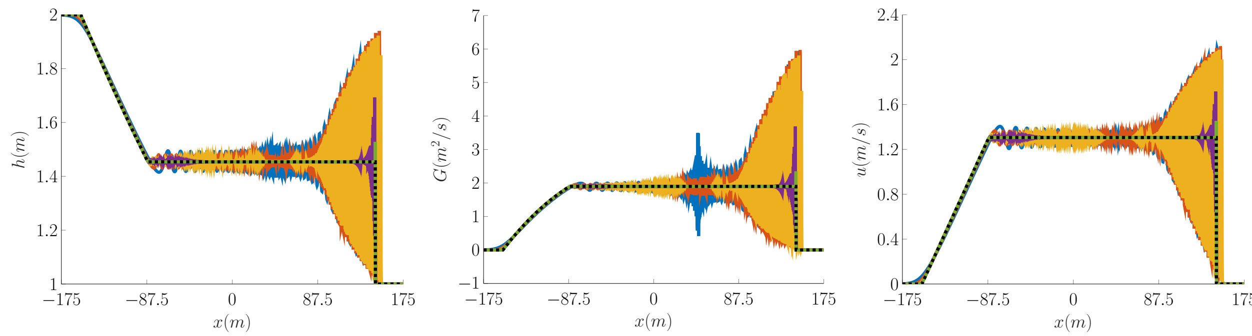
(d) h Front of dispersive wave train (e) G Front of dispersive wave train (f) u Front of dispersive wave train

Figure 7: Plot of multiple smooth dambreak numerical solutions at important locations at $t = 35s$.



(a) $\beta_1 = \beta_2 = 0$ (b) $\beta_1 = \beta_2 = \frac{1}{15}$

Figure 8: Regions of wave speeds for the smooth dambreak numerical solution at $t = 35s$.



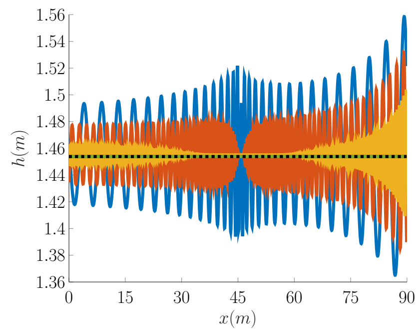
(a) h (b) G (c) u

Figure 9: Plot of multiple smooth dambreak numerical solutions at $t = 35s$.

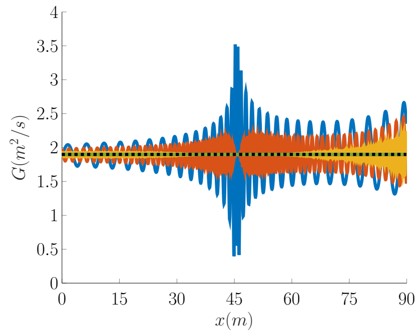
β_1	h	G	uh	$\mathcal{H}$
0	8.0×10^{-13}	6.3×10^{-13}	3.3×10^{-7}	3.8×10^{-6}
$-\frac{2}{3} + 10^{-1}$	7.2×10^{-13}	6.3×10^{-13}	3.0×10^{-6}	6.3×10^{-6}
$-\frac{2}{3} + 10^{-2}$	6.6×10^{-13}	6.2×10^{-13}	3.2×10^{-5}	3.7×10^{-4}
$-\frac{2}{3} + 10^{-3}$	6.0×10^{-13}	6.3×10^{-13}	1.2×10^{-5}	3.7×10^{-3}
$-\frac{2}{3} + 10^{-4}$	5.9×10^{-13}	6.2×10^{-13}	1.2×10^{-6}	3.7×10^{-3}

Table 5: Conservation errors for Serre To SWWE Faimly of Equations for the solutions provided above with $\beta_2 = 0$ and $-\frac{2}{3} \leq \beta_1 \leq 0$.

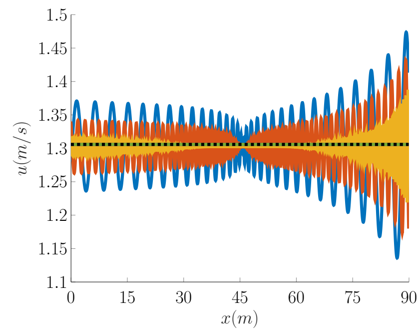
- Trade-off between momentum convergence (since G = uh for SWWE) and energy convergence. Dispersive models conserve energy better due to smooth solutions, but conserve momentum worse because conserved quantitiy is not uh.



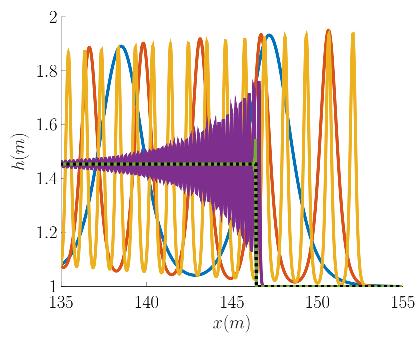
(a) h middle of dispersive wave train



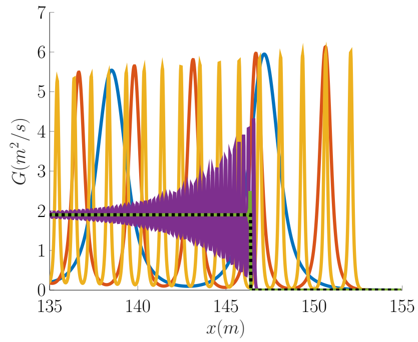
(b) G middle of dispersive wave train



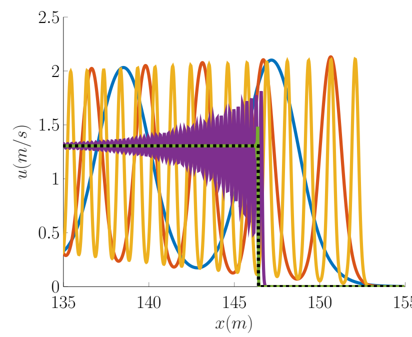
(c) u middle of dispersive wave train



(d) h front of dispersive wave train

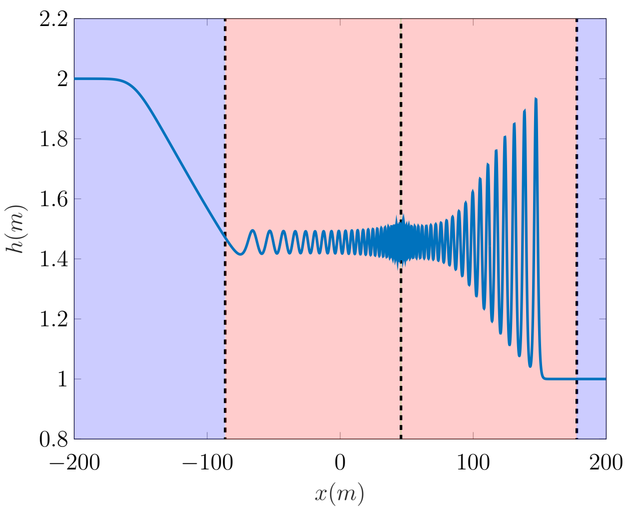


(e) G front of dispersive wave train

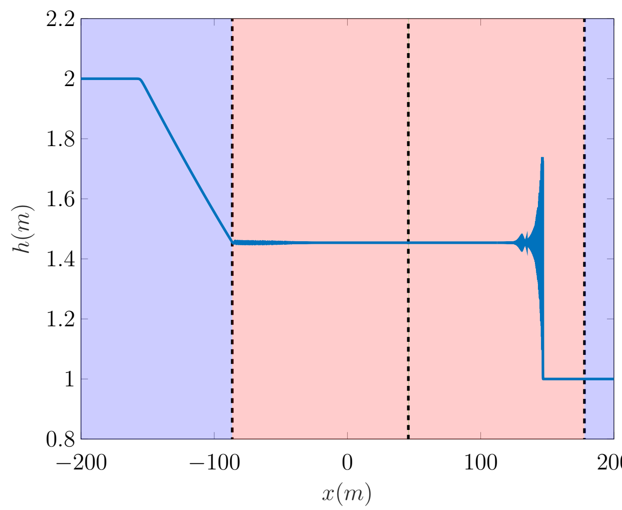


(f) u front of dispersive wave train

Figure 10: Plot of multiple smooth dambreak numerical solutions at $t = 35s$.



(a) $\beta_1 = \beta_2 = 0$



(b) $\beta_1 = -\frac{2}{3} + 10^{-3}$ and $\beta_2 = 0$

Figure 11: Regions of wave speeds for the smooth dambreak numerical solution at $t = 35s$.