The regularised Serre equations are

$$h_t + (uh)_x = 0 \tag{1a}$$

$$(uh)_t + \left(u^2 h + \frac{gh^2}{2} + \epsilon h^2 \left[h\left[(u_x)^2 - uu_{xx} - u_{xt}\right] - g\left(h\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial x}\right)\right]\right)_x = 0 \tag{1b}$$

Assuming that

$$h(x,t) = h_0 + \delta\eta(x,t) + O(\delta^2)$$
$$u(x,t) = u_0 + \delta v(x,t) + O(\delta^2)$$

By substituting these forms into the linearised Serre equations and neglecting $O(\delta^2)$ terms, we get the linearised regularised Serre equations. We also substitute $\eta_t$ using the mass equation into the momentum equation.

$$(\delta\eta)_t + u_0(\delta\eta)_x + h_0(\delta v)_x = 0 \tag{2a}$$

$$h_0(\delta v)_t + gh_0(\delta\eta)_x + h_0u_0(\delta v)_x - \epsilon h_0^3(\delta v)_{xxt} - g\epsilon h_0^3(\delta\eta)_{xxx} - \epsilon h_0^3 u_0(\delta v)_{xxx} = 0 \tag{2b}$$

We can remove the $\delta$ term, either by removing the common factor, or absorbing it into $\eta$ and $v$ to get

$$\eta_t + u_0\eta_x + h_0v_x = 0 \tag{3a}$$

$$h_0v_t + gh_0\eta_x + h_0u_0v_x - \epsilon h_0^3 v_{xxt} - g\epsilon h_0^3\eta_{xxx} - \epsilon h_0^3u_0v_{xxx} = 0 \tag{3b}$$

We now assume that $\eta(x,t) = H\exp\left(i(\kappa x - \omega t)\right)$, $v(x,t) = U\exp\left(i(\kappa x - \omega t)\right)$

$$\eta(x,t) = H\exp\left(i(\kappa x - \omega t)\right)$$
$$v(x,t) = U\exp\left(i(\kappa x - \omega t)\right)$$

substituting these into the linearised Serre equation we get

$$\left[Hu_0\kappa - H\omega + Uh_0\kappa\right] i\exp\left[i\left(\kappa x - \omega t\right)\right] = 0 \tag{4a}$$

$$\left[g\epsilon Hh_0^2\kappa^3 + g\kappa H + Uh_0^2u_0\epsilon\kappa^3 - Uh_0^2\epsilon\kappa^2\omega + Uu_0\kappa - U\omega\right] ih_0\exp\left[i\left(\kappa x - \omega t\right)\right] = 0 \tag{4b}$$

This can be written as

$$\begin{bmatrix} u_0\kappa - \omega & h_0\kappa \\ g\epsilon h_0^2\kappa^3 + \kappa g & h_0^2u_0\epsilon\kappa^3 - h_0^2\epsilon\kappa^2\omega + u_0\kappa - \omega \end{bmatrix}\begin{bmatrix} H \\ U \end{bmatrix} = 0 \tag{5}$$

Which has non-trivial solutons when the determinant is zero.
The determinant of this matrix is

$$\left(\epsilon h_0^2\kappa^2 + 1\right)\left[\omega^2 - 2u_0\kappa\omega + u_0^2\kappa^2 - gh_0\kappa^2\right] \tag{6}$$

To solve for 0 we get

$$\omega^2 - 2u_0\kappa\omega + u_0\kappa^2 - gh_0\kappa^2 = 0 \tag{7}$$

by quadratic equation we have

$$\omega = \frac{2u_0\kappa \pm \sqrt{4u_0^2\kappa^2 - 4(u_0^2\kappa^2 - gh_0\kappa^2)}}{2} \tag{8}$$

$$\omega = (u_0 \pm \sqrt{gh_0})\kappa \tag{9}$$

so wave speed and phase speed are equal,and also independent of $\epsilon$