Arbitrarily High Order TVD Reconstruction Polynomials

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1 Goal

Suppose we are given a domain $[a, b]$ discretised into n cells of fixed width Δx so that we have the i^{th} cell with midpoint x_i and defined by $\left[x_{i-1/2}, x_{i+1/2}\right]$ where $x_{i\pm1/2} = x_i \pm \Delta x/2$. We are then given a vector of length n $\bar{\boldsymbol{q}} = [\bar{q}_0, \dots, \bar{q}_{n-1}]$ where each element represents the cell average of some function $q(x)$.

Such that

$$\bar{q}_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i-1/2}} q(x) dx$$

The task then is to for each cell in the domain to find some polynomial approximation to $q(x)$ over each cell, $P_i^r(x) : \left[x_{i-1/2}, x_{i+1/2}\right] \to \Re$ where r is the order of the polynomial.

These polynomial approximations should follow these rules:

1. Conserve the cell average in that cell so that

$$\int_{x_{i-1/2}}^{x_{i+1/2}} P_i^r(x) dx = \Delta x \bar{q}_i$$

2. Achieve desired r theoretical order of accuracy over the associated cell (when q is smooth). Let's restrict ourselves to L^p so we would want

$$\left(\int_{x_{i-1/2}}^{x_{i+1/2}} \left| P_i^r(x) - q(x) \right|^p dx \right)^{1/p} = \mathcal{O}\left(\Delta x^{r+1}\right)$$

3. Be total variation diminishing so that

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \left| \frac{\partial P_i^r(x)}{\partial x} \right| dx \leq \int_{x_{i-1/2}}^{x_{i+1/2}} \left| \frac{\partial q(x)}{\partial x} \right| dx$$

Satisfying these assumptions allows the reconstruction to be utilised in MUSCL type schemes.

The development of these reconstructions has faded away as other schemes have taken root. WENO- TVB (so generates oscillations). MUSCL - Serre - Need derivative approximations as well - hence the need to think about polynomials instead of just pointwise values as done previously. Deeply sceptical about peoples claims about TVD being only first order, especially because any reconstruction that is situation agnostic (i.e doesnt have detailed knowledge of the space the function lives in) will be quite poor.

etc.

2 Notation

We are going to use the following notation for all polynomials

$$P_i^r(x) = a_r^{(r)}(x - x_i)^r + a_r^{(r-1)}(x - x_i)^{r-1} + \dots + a_r^{(0)}$$

3 Implications

3.1 Cell Averages

If we use an equally spaced cell then we have the following property, that the cell average value of $P_i^r(x)$ only depends on the even powers in the following way

$$\int_{x_{i-1/2}}^{x_{i+1/2}} P_i^r(x)dx = \int_{x_{i-1/2}}^{x_{i+1/2}} a_r^{(r)}(x-x_i)^r + a_r^{(r-1)}(x-x_i)^{r-1} + \dots + a_r^{(0)} dx =$$

$$= \left[\frac{1}{r+1}a_r^{(r)}(x-x_i)^{r+1} + \frac{1}{r}a_r^{(r-1)}(x-x_i)^r + \dots + a_r^{(0)}\left(x-x_i\right)\right]_{x_{i-1/2}}^{x_{i+1/2}}$$

All the even powers in the difference cancel, lets define $\phi(n) = 0$ if n even and 1 is n is odd

$$= \frac{2\phi\left(r+1\right)}{r+1}a_r^{(r)}\left(\frac{\Delta x}{2}\right)^{r+1} + \frac{2\phi\left(r\right)}{r}a_r^{(r-1)}\left(\frac{\Delta x}{2}\right)^{r} + \dots + 2a_r^{(0)}\left(\frac{\Delta x}{2}\right)$$

For our ones of interest $r < 3$

$$\int_{x_{i-1/2}}^{x_{i+1/2}} P_i^0(x)dx = 2a_0^{(0)}\left(\frac{\Delta x}{2}\right) = \Delta x\bar{q}_i$$

$$\int_{x_{i-1/2}}^{x_{i+1/2}} P_i^1(x)dx = 2a_1^{(0)}\left(\frac{\Delta x}{2}\right) = \Delta x\bar{q}_i$$

$$\int_{x_{i-1/2}}^{x_{i+1/2}} P_i^2(x)dx = \frac{2}{3}a_2^{(2)}\left(\frac{\Delta x}{2}\right)^3 + 2a_2^{(0)}\left(\frac{\Delta x}{2}\right) = \Delta x\bar{q}_i$$

$$\int_{x_{i-1/2}}^{x_{i+1/2}} P_i^3(x)dx = \frac{2}{3}a_3^{(2)}\left(\frac{\Delta x}{2}\right)^3 + 2a_3^{(0)}\left(\frac{\Delta x}{2}\right) = \Delta x\bar{q}_i$$

So we can maintain conservation if we just alter the odd coefficients.

3.2 TVD

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \left|\frac{\partial P_i^r(x)}{\partial x}\right| dx = \int_{x_{i-1/2}}^{x_{i+1/2}} \left|\frac{\partial}{\partial x}\left(a_r^{(r)}(x-x_i)^r + a_r^{(r-1)}(x-x_i)^{r-1} + \dots + a_r^{(0)}\right)\right| dx$$

$$= \int_{x_{i-1/2}}^{x_{i+1/2}} \left|\left(r\right)a_r^{(r)}(x-x_i)^{(r-1)} + \left(r-1\right)a_r^{(r-1)}(x-x_i)^{r-2} + \dots + a_r^{(1)}\right| dx$$

Now we do have that

$$\left|\left(r\right)a_r^{(r)}(x-x_i)^{(r-1)} + \left(r-1\right)a_r^{(r-1)}(x-x_i)^{r-2} + \dots + a_r^{(1)}\right| \le \left|\left(r\right)a_r^{(r)}(x-x_i)^{(r-1)}\right| + \left|\left(r-1\right)a_r^{(r-1)}(x-x_i)^{r-2}\right| + \dots + \left|a_r^{(1)}\right|$$

Assuming r is even (otherwise its odd we have)

$$\left|\left(r\right)a_r^{(r)}(x-x_i)^{(r-1)} + \left(r-1\right)a_r^{(r-1)}(x-x_i)^{r-2} + \dots + a_r^{(1)}\right| \le$$

$$r\left|\left(r\right)a_r^{(r)}(x-x_i)^{(r-1)}\right| + \left(r-1\right)\left(x-x_i\right)^{r-2}\left|a_r^{(r-1)}\right| + \dots + 3\left|a_r^{(3)}\right|\left(x-x_i\right)^2 + 2\left|a_r^{(2)}(x-x_i)^1\right| + \left|a_r^{(1)}\right|$$

For third order we have

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \left|\frac{\partial P_i^3(x)}{\partial x}\right| dx = \int_{x_{i-1/2}}^{x_{i+1/2}} \left|\frac{\partial}{\partial x}\left(a_3^{(3)}(x-x_i)^3 + a_3^{(2)}(x-x_i)^2 + a_3^{(1)}(x-x_i) + a_3^{(0)}\right)\right| dx$$

$$= \int_{x_{i-1/2}}^{x_{i+1/2}} \left|3a_3^{(3)}(x-x_i)^2 + 2a_3^{(2)}(x-x_i) + a_3^{(1)}\right| dx$$

By triangle inequality

$$\leq \int_{x_{i-1/2}}^{x_{i+1/2}} \left|3a_3^{(3)}(x-x_i)^2\right| dx + \int_{x_{i-1/2}}^{x_{i+1/2}} \left|2a_3^{(2)}(x-x_i)\right| dx + \int_{x_{i-1/2}}^{x_{i+1/2}} \left|a_3^{(1)}\right| dx$$

$$\leq 3\int_{x_{i-1/2}}^{x_{i+1/2}} \left|a_3^{(3)}\right|(x-x_i)^2 dx + 2\int_{x_{i-1/2}}^{x_{i+1/2}} \left|a_3^{(2)}(x-x_i)\right| dx + \int_{x_{i-1/2}}^{x_{i+1/2}} \left|a_3^{(1)}\right| dx$$

$$\leq 3\left|a_3^{(3)}\right| \int_{x_{i-1/2}}^{x_{i+1/2}} (x-x_i)^2 dx + 2\left|a_3^{(2)}\right| \int_{x_{i-1/2}}^{x_{i+1/2}} |(x-x_i)|\, dx + \int_{x_{i-1/2}}^{x_{i+1/2}} \left|a_3^{(1)}\right| dx$$

$$\leq 3\left|a_3^{(3)}\right| \left[\frac{1}{3}(x-x_i)^3\right]_{x_{i-1/2}}^{x_{i+1/2}} + 2\left|a_3^{(2)}\right| \int_{x_{i-1/2}}^{x_{i+1/2}} |(x-x_i)|\, dx + \int_{x_{i-1/2}}^{x_{i+1/2}} \left|a_3^{(1)}\right| dx$$

$$\leq 2\left|a_3^{(3)}\right| \left(\frac{\Delta x}{2}\right)^3 + 2\left|a_3^{(2)}\right| \int_{x_{i-1/2}}^{x_{i+1/2}} |(x-x_i)|\, dx + \left|a_3^{(1)}\right| \Delta x$$

$$\leq 2\left|a_3^{(3)}\right| \left(\frac{\Delta x}{2}\right)^3 + 2\left|a_3^{(2)}\right| \Delta x^2 + \left|a_3^{(1)}\right| \Delta x$$

What this means is that if we want to build a system to go from high-order polynomial approximations to low order ones to maintain cell averages we can get away with only altering the odd degree coefficients which can be used to alter the TV over the interval. However, to get it to 0 will require altering all coefficients, except the constant term.

4 Suggestions

4.1 Increasing Powers

4.1.1 Examples

Use $P_i^r(x)$

$$P_i^0(x) = \bar{q}_i$$

$$P_i^1(x) = w_1 a_1^{(1)}(x-x_i)^1 - \frac{w_1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} a_1^{(1)}(x-x_i)^1 dx + \bar{q}_i$$

$$P_i^1(x) = w_1 a_1^{(1)}(x-x_i)^1 + \frac{w_1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \frac{a_1^{(1)}}{2}(x-x_i)^2 dx + \bar{q}_i$$

$$P_i^1(x) = w_1 a_1^{(1)}(x-x_i)^1 - \frac{w_1}{\Delta x}\frac{a_1^{(1)}}{2}\left[(x-x_i)^2\right]_{x_{j-1/2}}^{x_{j+1/2}} + \bar{q}_i$$

which of course is $\left[(x-x_i)^2\right]_{x_{j-1/2}}^{x_{j+1/2}} = 0$

$$P_i^1(x) = w_1 a_1^{(1)}(x-x_i)^1 + \bar{q}_i$$

Where $1 \geq w_1 \geq 0$ and $w_1 \to 0$ in regions where $q(x)$ is non-smooth. This allows TVD to be maintained as we can take $w_1 = 0$ and will ensure that the cell average is maintained for all w_1 values as

$$\int_{x_{j-1/2}}^{x_{j+1/2}} P_i^1(x)dx = w_1 a_1^{(1)}\int_{x_{j-1/2}}^{x_{j+1/2}} (x-x_i)^1 dx + \left[-w_1\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} a_1^{(1)}(x-x_i)^1 dx + \bar{q}_i\right]\int_{x_{j-1/2}}^{x_{j+1/2}} 1dx$$

$$\int_{x_{j-1/2}}^{x_{j+1/2}} P_i^1(x)dx = w_1 a_1^{(1)}\int_{x_{j-1/2}}^{x_{j+1/2}} (x-x_i)^1 dx + \left[-w_1\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} a_1^{(1)}(x-x_i)^1 dx + \bar{q}_i\right]\Delta x$$

$$\int_{x_{j-1/2}}^{x_{j+1/2}} P_i^1(x)dx = \bar{q}_i \Delta x$$

As desired.

For $P_i^2(x)$ we have

$$\begin{aligned} P_i^2(x) =& w_2 a_2^{(2)} (x - x_i)^2 \\ &+ \left[w_2 \left(a_2^{(1)} - w_1 a_1^{(1)} \right) + w_1 a_1^{(1)} \right] (x - x_i)^1 \\ &+ \bar{q}_i - \frac{w_2}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} a_2^{(2)} (x - x_i)^2 dx \end{aligned}$$

$$\begin{aligned} P_i^2(x) =& w_2 a_2^{(2)} (x - x_i)^2 \\ &+ \left[w_2 \left(a_2^{(1)} - w_1 a_1^{(1)} \right) + w_1 a_1^{(1)} \right] (x - x_i)^1 \\ &+ \bar{q}_i - \frac{w_2}{\Delta x} \frac{a_2^{(2)}}{3} \left[(x - x_i)^3 \right]_{x_{j-1/2}}^{x_{j+1/2}} \end{aligned}$$

$$\begin{aligned} P_i^2(x) =& w_2 a_2^{(2)} (x - x_i)^2 \\ &+ \left[w_2 \left(a_2^{(1)} - w_1 a_1^{(1)} \right) + w_1 a_1^{(1)} \right] (x - x_i)^1 \\ &+ \bar{q}_i - \frac{w_2}{\Delta x} \frac{2a_2^{(2)}}{3} \left(\frac{\Delta x}{2} \right)^3 \end{aligned}$$

$$\begin{aligned} P_i^2(x) =& w_2 a_2^{(2)} (x - x_i)^2 - \frac{w_2 a_2^{(2)}}{3} \left(\frac{\Delta x}{2} \right)^2 \\ &+ \left[w_2 \left(a_2^{(1)} - w_1 a_1^{(1)} \right) + w_1 a_1^{(1)} \right] (x - x_i)^1 \\ &+ \bar{q}_i \end{aligned}$$

Where $1 \geq w_1 \geq 0$,$w_1 \to 0$ and $1 \geq w_2 \geq 0$ and $w_2 \to 0$ in regions where $q(x)$ is non-smooth. This allows TVD to be maintained as we can take $w_1 = w_2 = 0$ and will ensure that the cell average is maintained for all w_1 values as. Also allows us to Recover P_i^1 when $w_2 = 0$ and $w_1 = 1$. We also again see that the cell average is maintained as we have

$$\begin{aligned} \int_{x_{j-1/2}}^{x_{j+1/2}} P_i^2(x)dx =& \int_{x_{j-1/2}}^{x_{j+1/2}} w_2 a_2^{(2)} (x - x_i)^2 dx - \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{w_2}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} a_2^{(2)} (x - x_i)^2 dx dx \\ & \int_{x_{j-1/2}}^{x_{j+1/2}} a_2^{(2)} (x - x_i)^2 dx \left[w_2 \left(a_2^{(1)} - w_1 a_1^{(1)} \right) + w_1 a_1^{(1)} \right] (x - x_i)^1 dx \\ &+ \int_{x_{j-1/2}}^{x_{j+1/2}} \bar{q}_i dx \end{aligned}$$

$$\begin{aligned} \int_{x_{j-1/2}}^{x_{j+1/2}} P_i^2(x)dx =& \int_{x_{j-1/2}}^{x_{j+1/2}} w_2 a_2^{(2)} (x - x_i)^2 dx - \frac{w_2}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} a_2^{(2)} (x - x_i)^2 dx \int_{x_{j-1/2}}^{x_{j+1/2}} 1 dx \\ &+ a_2^{(2)} (x - x_i)^2 dx \left[w_2 \left(a_2^{(1)} - w_1 a_1^{(1)} \right) + w_1 a_1^{(1)} \right] \int_{x_{j-1/2}}^{x_{j+1/2}} (x - x_i)^1 dx \\ &+ \bar{q}_i \int_{x_{j-1/2}}^{x_{j+1/2}} 1 dx \end{aligned}$$

$$
\begin{aligned}
\int_{x_{j-1/2}}^{x_{j+1/2}} P_i^2(x)dx = &\int_{x_{j-1/2}}^{x_{j+1/2}} w_2 a_2^{(2)}(x-x_i)^2 dx - w_2 \int_{x_{j-1/2}}^{x_{j+1/2}} a_2^{(2)}(x-x_i)^2 dx \\
&+ 0 \\
&+ \bar{q}_i \Delta x
\end{aligned}
$$

therefore

$$
\int_{x_{j-1/2}}^{x_{j+1/2}} P_i^2(x)dx = \bar{q}_i \Delta x
$$

As desired.

Finally for the cubics we have

$$
\begin{aligned}
P_i^3(x) =& w_3 a_3^{(3)}(x-x_i)^3 \\
&+ \left[w_3\left(a_2^{(3)} - w_2 a_2^{(2)}\right) + w_2 a_2^{(2)}\right](x-x_i)^2 - \frac{\left[w_3\left(a_2^{(3)} - w_2 a_2^{(2)}\right) + w_2 a_2^{(2)}\right]}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} (x-x_i)^2 dx \\
&+ \left[w_3\left(a_3^{(1)} - \left[w_2\left(a_2^{(1)} - w_1 a_1^{(1)}\right) + w_1 a_1^{(1)}\right]\right) + \left[w_2\left(a_2^{(1)} - w_1 a_1^{(1)}\right) + w_1 a_1^{(1)}\right]\right](x-x_i)^1 \\
&+ \bar{q}_i
\end{aligned}
$$

So we can recover TVD, and lower order polynomial approximations when w_i's are chosen correctly. We also maintain our cell average because

$$
\begin{aligned}
\int_{x_{j-1/2}}^{x_{j+1/2}} P_i^3(x)dx =& w_3 a_3^{(3)} \int_{x_{j-1/2}}^{x_{j+1/2}} (x-x_i)^3 dx \\
&+ \int_{x_{j-1/2}}^{x_{j+1/2}} \left[w_3\left(a_2^{(3)} - w_2 a_2^{(2)}\right) + w_2 a_2^{(2)}\right](x-x_i)^2 dx \\
&- \frac{\left[w_3\left(a_2^{(3)} - w_2 a_2^{(2)}\right) + w_2 a_2^{(2)}\right]}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} (x-x_i)^2 dx \int_{x_{j-1/2}}^{x_{j+1/2}} 1 dx \\
&+ \int_{x_{j-1/2}}^{x_{j+1/2}} \left[w_3\left(a_3^{(1)} - \left[w_2\left(a_2^{(1)} - w_1 a_1^{(1)}\right) + w_1 a_1^{(1)}\right]\right) + \left[w_2\left(a_2^{(1)} - w_1 a_1^{(1)}\right)\right]\right](x-x_i)^1 dx \\
&+ \bar{q}_i \int_{x_{j-1/2}}^{x_{j+1/2}} 1 dx
\end{aligned}
$$

$$
\begin{aligned}
\int_{x_{j-1/2}}^{x_{j+1/2}} P_i^3(x)dx =& 0 \\
&+ \left[w_3\left(a_2^{(3)} - w_2 a_2^{(2)}\right) + w_2 a_2^{(2)}\right] \int_{x_{j-1/2}}^{x_{j+1/2}} (x-x_i)^2 dx \\
&- \left[w_3\left(a_2^{(3)} - w_2 a_2^{(2)}\right) + w_2 a_2^{(2)}\right] \int_{x_{j-1/2}}^{x_{j+1/2}} (x-x_i)^2 dx \\
&+ 0 \\
&+ \Delta x \bar{q}_i
\end{aligned}
$$

$$
\int_{x_{j-1/2}}^{x_{j+1/2}} P_i^3(x)dx = \Delta x \bar{q}_i
$$

$$P_i^r = \bar{q}_i + \sum_{n=1}^{r} a_r^{(n)} \left(x - x_i\right)^n - \frac{a_r^{(n)}}{x_{j+1/2} - x_{j-1/2}} \int_{x_{j-1/2}}^{x_{j+1/2}} (x - x_i)^n dx$$

4.1.2 Properties

We can see that all of these maintain the conservation property over the i^{th} cell, indeed I have directly shown it for these polynomials.

To obtain accuracy, we can use any already well established method to find the coefficients a, I will just take the simplest one, and pick a fixed stencil approximation, that finds coefficients that ensure that when $w_i = 1$ we get a fixed stencil approximation that agrees with neighbouring cell averages. I will have to do this for all polynomials of lower order to get it back. Sympy.

Choices: $P_i^1(x)$ slope passing through q_{i+1} and $qi - 1$ - weird poly, centered (1 unknown) $P_i^2(x)$ agree with cell averages $\bar{q}_{i+1}$ and $qi - 1$ - weird poly, centered (2 unknowns) $P_i^3(x)$ agree with cell averages $\bar{q}_{i+1}$ and $qi - 1$ - weird poly, centered (3 unknown)

To keep TVD we need an indicator function which is $w_i = 1$ when smooth and $w_i = 0$ when non-smooth.