Laboratory 2: The Pendulum

Jordan Ahern

21363697

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1. Introduction

The pendulum is an easily set up experiment, with applications in both everyday life and in all areas of science. An examination of the motion of a classical pendulum was instrumental in gaining an understanding of the natural world. Today, pendulum motion has applications in virology and other health sciences. The motion of pendulums can give an accurate measure of bacterial growth for instance.

These uses are of great importance to our modern lives, clearly, but fully understanding the pendulum is a much more complex task than setting them up. In an ideal world an analytical solution to pendulum motion can be found, to a good approximation, by those with a rudimentary understanding of mathematical physics. Such solutions become greatly more difficult to calculate when non-conservative, dampening and driving, forces are considered. Giving the differential equation:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin(\theta) - k\frac{d\theta}{dt} + Acos(\phi t)$$

Where l=constant length, A=driving amplitude, k=dampening constant.

This equation must be solved accurately and quickly all over the world every day but there is no analytical solution capable of achieving this, instead we must use numerical methods. This report will investigate the use of the trapezoid rule and Runge-Kutta algorithms to gauge their effectiveness in solving the pendulum equation numerically. Both methods work off the principle of approximating the area under a graph by summing the areas of infinitesimally thin rectangles over a certain interval. The Runge-Kutta method is theoretically more accurate, Δt^2 vs Δt.

These tests were run in Python 3, with the assistance of the NumPy and Matplotlib libraries.

2. Methodology

The investigation was run in five parts, the first two concerning the trapezoid rule. Exercise three compared the trapezoid and Runge-Kutta algorithms. The final two parts applied the Runge-Kutta method to the complete pendulum equation.

The pendulum equation was defined within two functions one was the exact, non-linear equation as above, with $\frac{d\theta}{dt}$ defined as ω. The other used the small angle approximation, $\sin\theta \approx \theta$, to define a linear equation. The constants k, A, dt, ϕ, ω, θ were initialised as required for each exercise, g/L was set to one.

The numerical algorithms were implemented as a set of variables which infinitesimally changed the theta and omega and found the average change between the function before and after this infinitesimal step. Theta and omega were then changed by this average. This process was repeatedly iterated over, using for loops over a predefined range.

The values for omega and theta at on each iteration were stored in lists and plotted against steps or time as most suitable.

3. Results

Exercise 1: Solving the linear pendulum equation

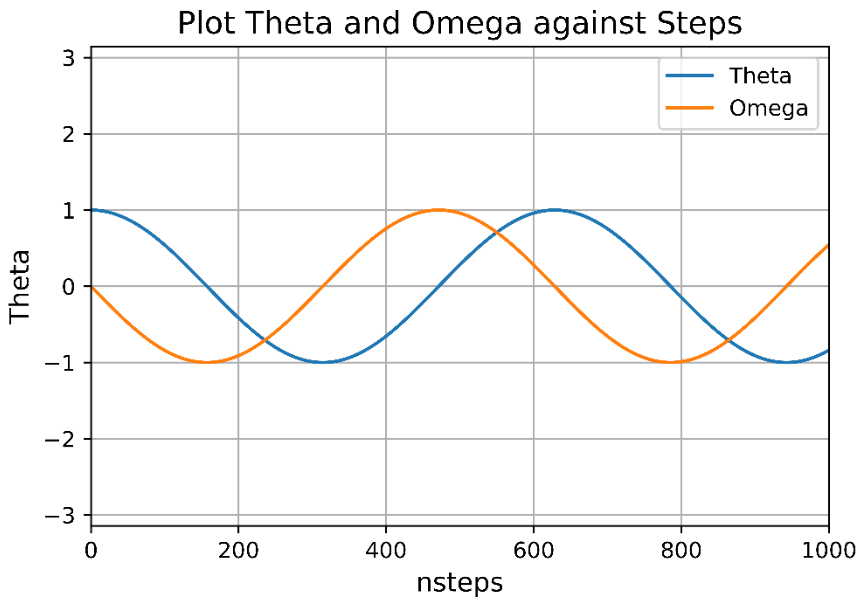

Figure 1

This graph was used to ensure that the trapezoid algorithm could be used to numerically approximate the solutions to the pendulum equation. This goal was clearly achieved as both theta and omega follow sinusoidal curves, as they should, even though the equation is using the linear approximation.

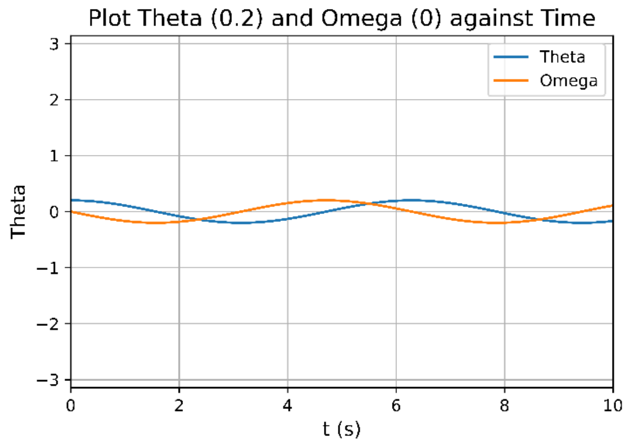

Figure 2

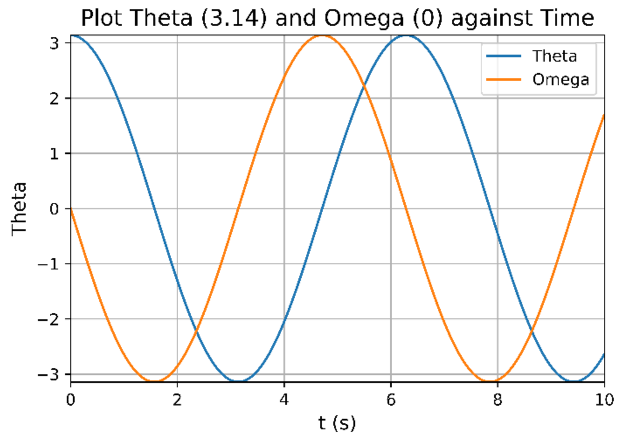

Figure 3

The above graphs demonstrated the linkages between theta and omega. It was evident that the initial amplitude of theta remained as the maximum amplitude, thereafter, as expected based on the lack of any dampening. Omega was also shown to be intrinsically linked to omega as it gained the same maximum amplitude despite starting from zero and being out of phase with theta by $\frac{\pi}{2}$ radians.

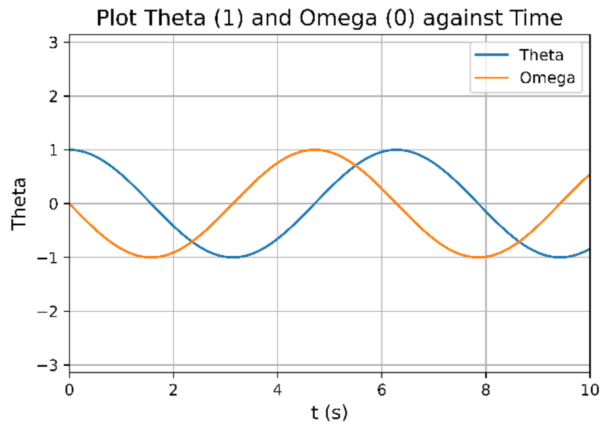

Figure 4

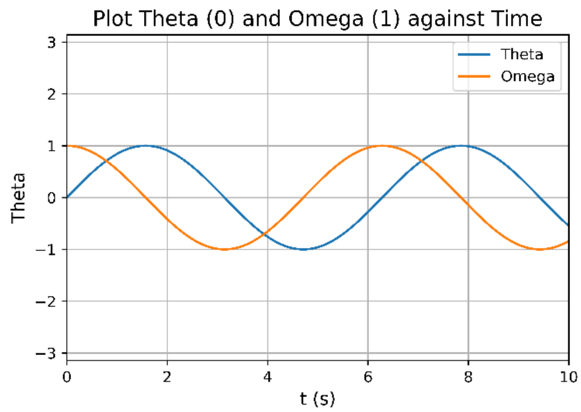

Figure 5

The next two graphs provided a more complete description of the linkage between theta and omega. The wave forms in both graphs were clearly seen to describe the same wave, just shifted to a different point in time. This raises the question in Fig.5 as to why the initial value of theta, zero, does not remain as the maximum amplitude as in the previous three graphs. The answer to said question was that omega is not just dependent on theta, as postulated, but that the reverse is also true and that the two are linked. This was the expected conclusion as omega is the time derivative of theta.

The phase shift between theta and omega was also caused by this relation and the physical reasoning can be understood by observing the graphs. At all points where theta was at its maximum or minimum, omega was zero and crossing the x-axis. This was where the velocity changed sign to move the pendulum away from the extremum. When omega was at an extreme, theta was at its equilibrium position, this is the basis behind all simple harmonic motion as the velocity reaching a maximum will always cause the pendulum to overshoot the equilibrium position.

Exercise 2: Solving the nonlinear pendulum equation

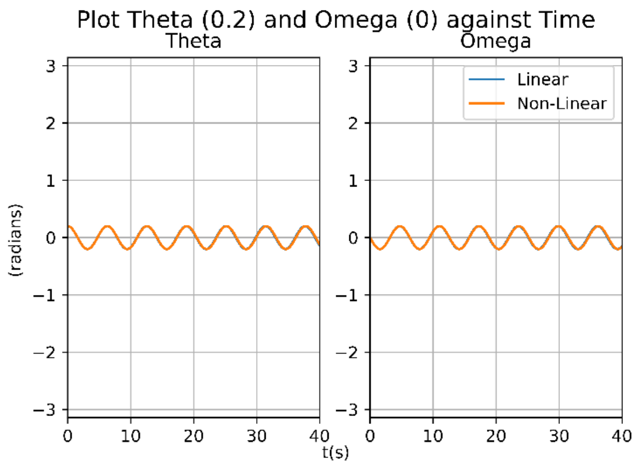

Figure 6

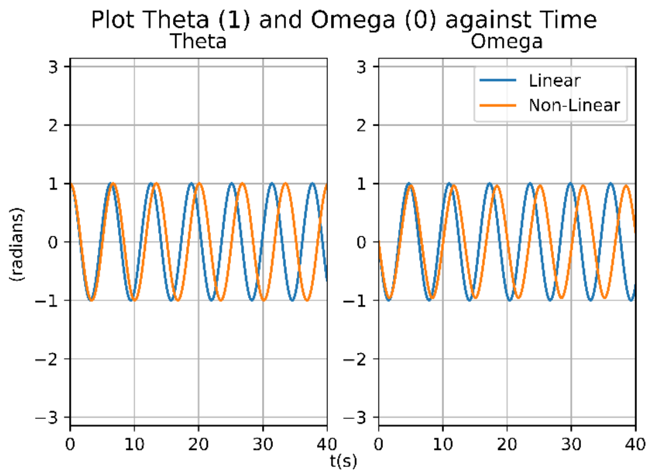

Figure 7

Figure 6 demonstrated motion in line with Ex1. Both the linear and non-linear pendulums were in phase for theta and omega, with the expected phase difference between theta and omega maintaining. The next graph was the first example of a difference between the two equations. The non-linear pendulum was clearly seen to have a marginally larger period than the linear equation, this resulted in the two pendulums becoming gradually more out of sync as time went on despite beginning in sync. Both non-linear theta and omega diverged from the linear values at the same rate. Comparing figures 6 and 7 gives a good example of the accuracy of the linear approximation for small angles and inaccuracy for larger angles.

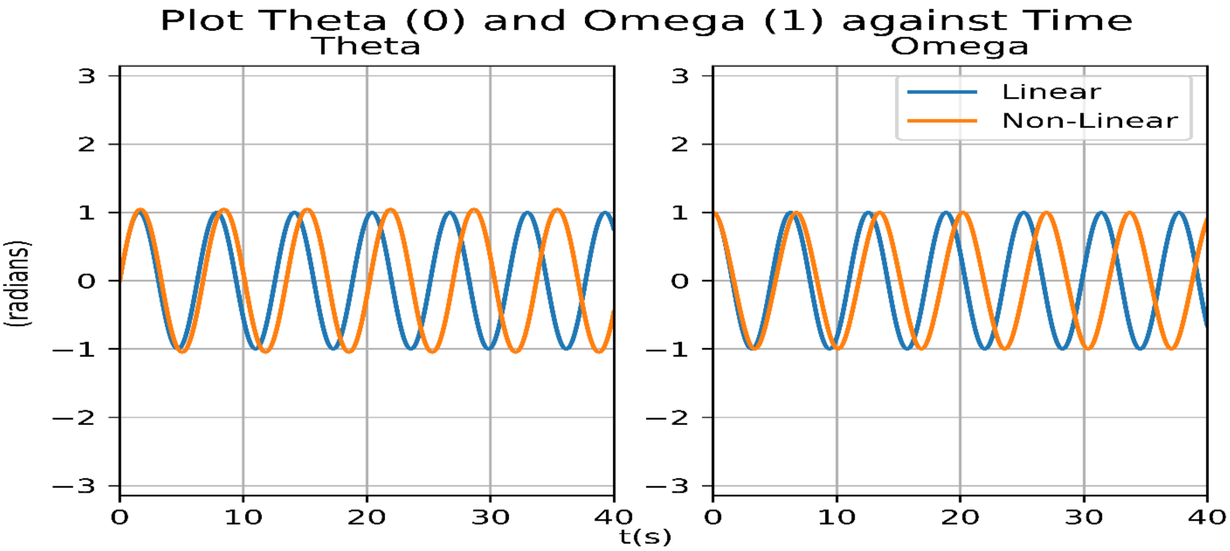

Figure 8

Figures 7 and 8 were directly compared, similarly to in Ex1. They confirmed that even in the non-linear case theta and omega remained directly linked, such that the differing initial conditions describe the same waves with a time shift. The differences between linear and non-linear motion did not affect this relationship as both theta and omega maintained the same period as each other.

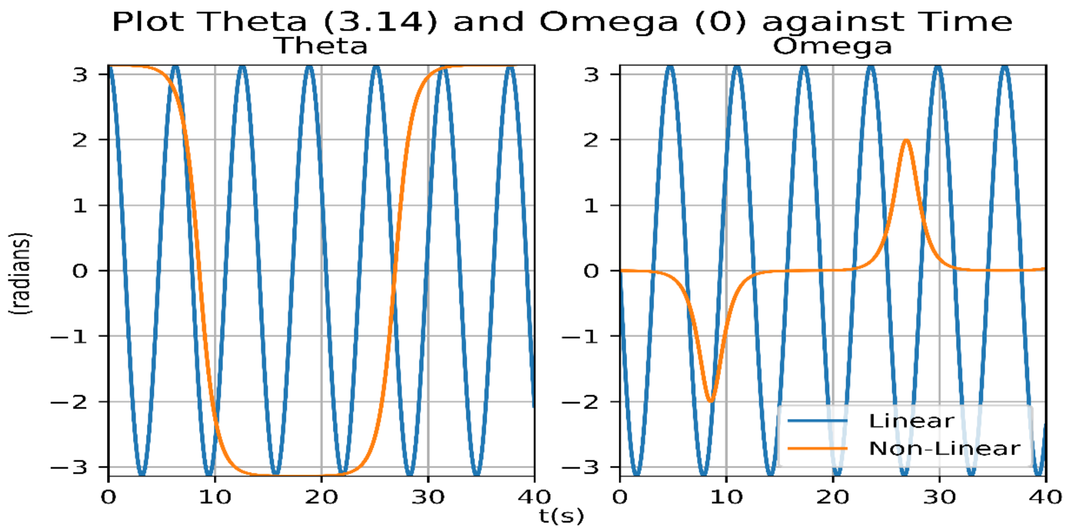

Figure 9

In figure 9, a much greater discrepancy between the linear and non-linear equations and all previous graphs was observed. It was noted that theta was no longer oscillating sinusoidally, and that omega now had a different maximum amplitude to theta.

This motion was likely due to the existence of the pendulum's unstable equilibrium at π radians. The pendulum was never able to reach this point but an equilibrium like effect was observed each time it approached π; the graph levelled off at this point with the pendulum holding position until omega was able to catch back up and pull theta back from its near equilibrium point. This secondary equilibrium caused the pendulum to oscillate in a non-simple harmonic motion.

The existence of this secondary equilibrium can be shown by moving the initial angle closer to π radians.

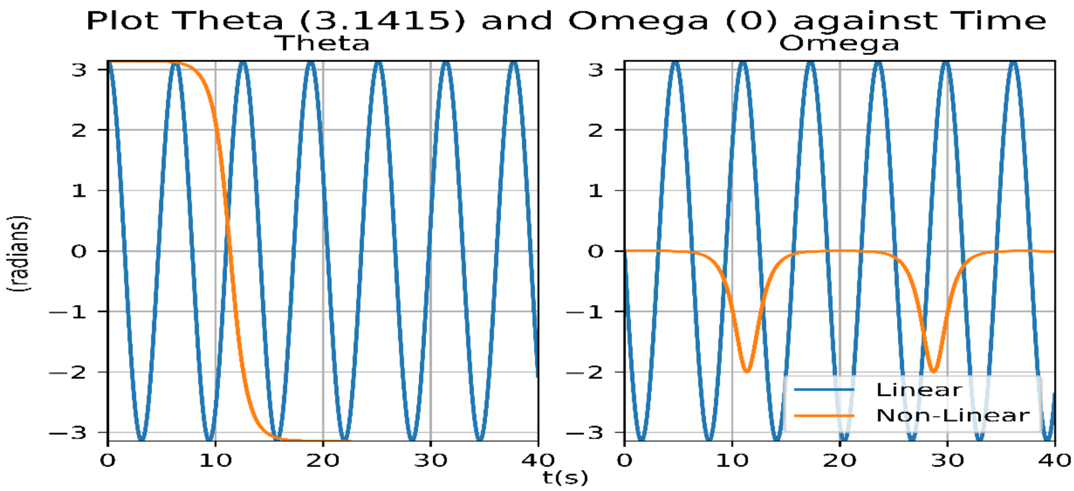

Figure 10

At this amplitude the trapezoid method was not accurate enough and it shot theta passed the equilibrium at π. The graph of omega however described this motion well as we could see that the pendulum was now believed to be moving in a purely circular manner, without oscillation.

Exercise 3: Comparison between trapezoid and Runge-Kutta

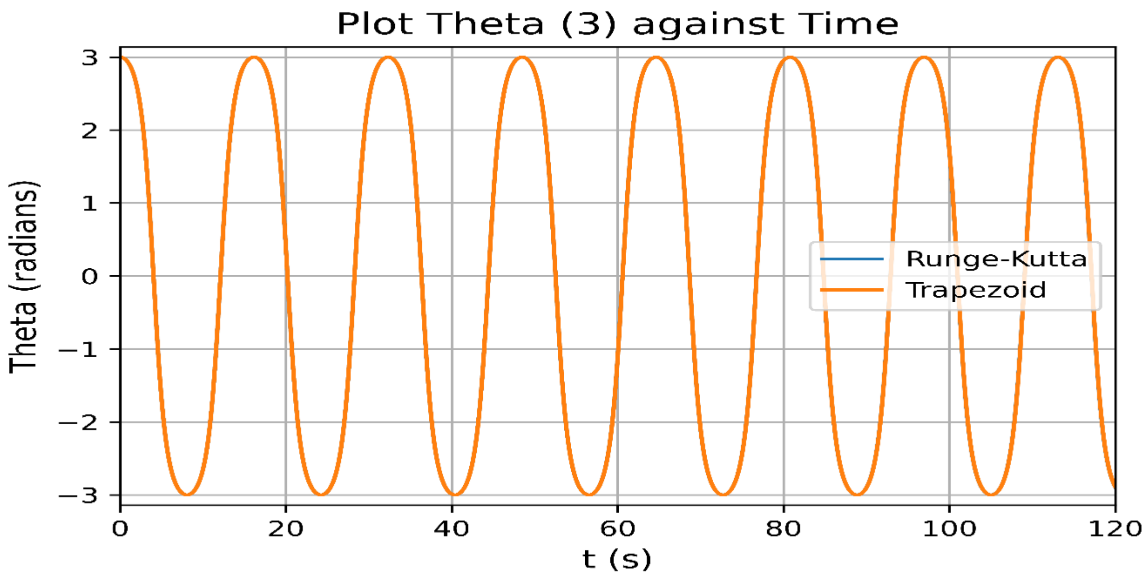

Figure 11

For the reasonably large angle of three radians there was no clear benefit to the use of Runge-Kutta over the more simply implemented trapezoid algorithm. Both methods produced the exact same curve to describe the motion of a non-linear pendulum up to three radians, perfectly overlapping said curves.

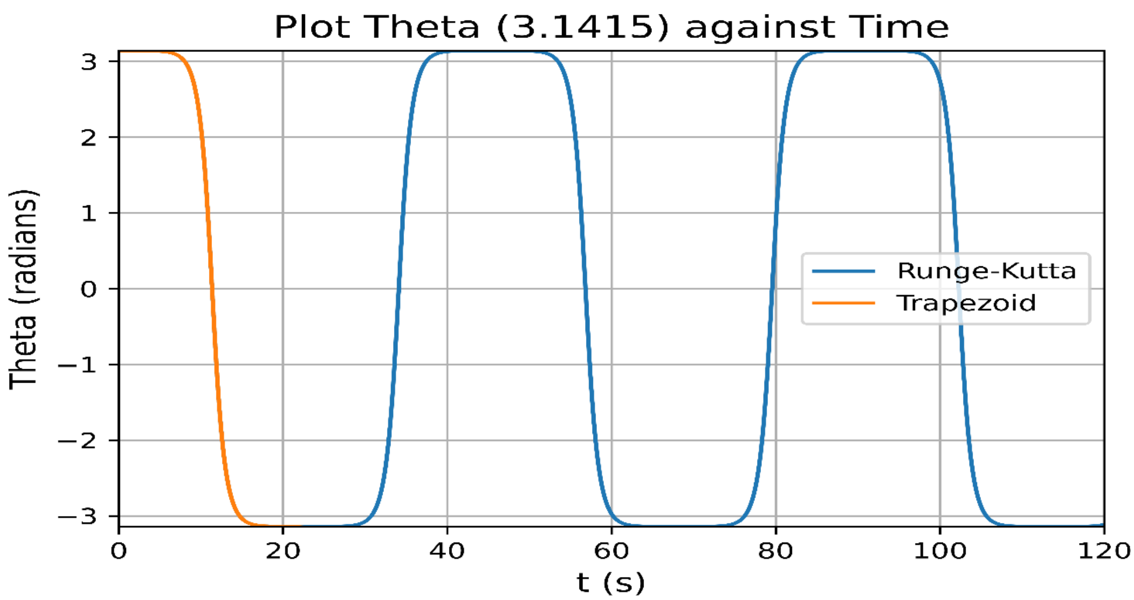

Figure 12

When the angle was brought very close to π the advantage of the RK method became very clear. As stated, at this point trapezoid overshoots the equilibrium at π and starts to incorrectly describe circular motion. RK did not have this issue, due to being accurate to greater order, correctly describing the pendulum approaching, but never reaching, the unstable equilibrium. RK was used from here on.

Exercise 4: The damped non-linear pendulum

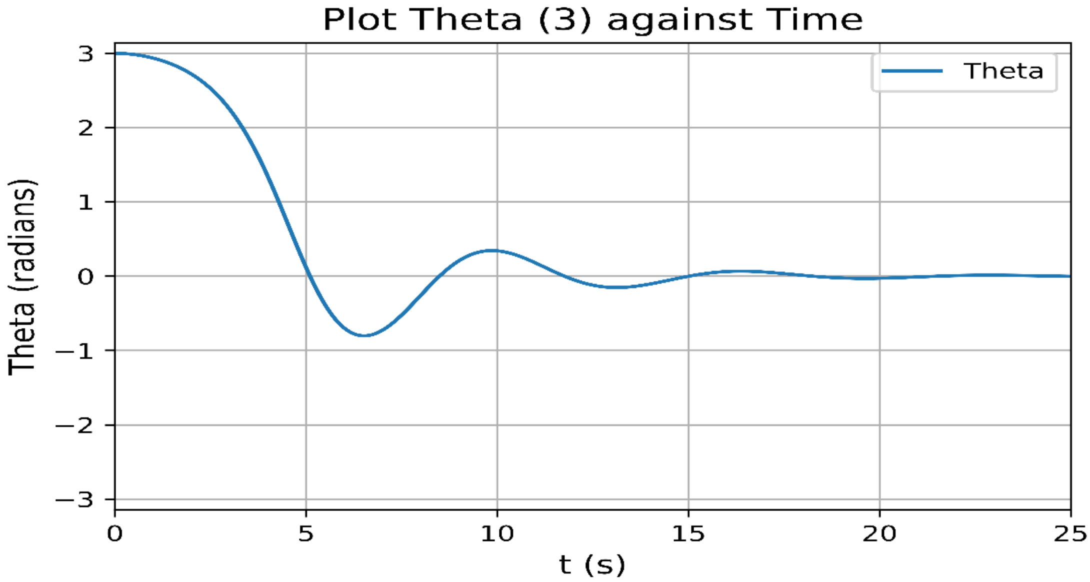

Figure 13

With the dampening constant a graph consistent with the motion of a critically dampened pendulum was produced. This was seen in how the pendulum only went through roughly two periods before the motion went to zero. This level of dampening would be seen when a pendulum interacts with a more viscous substance than air, say a water bath.

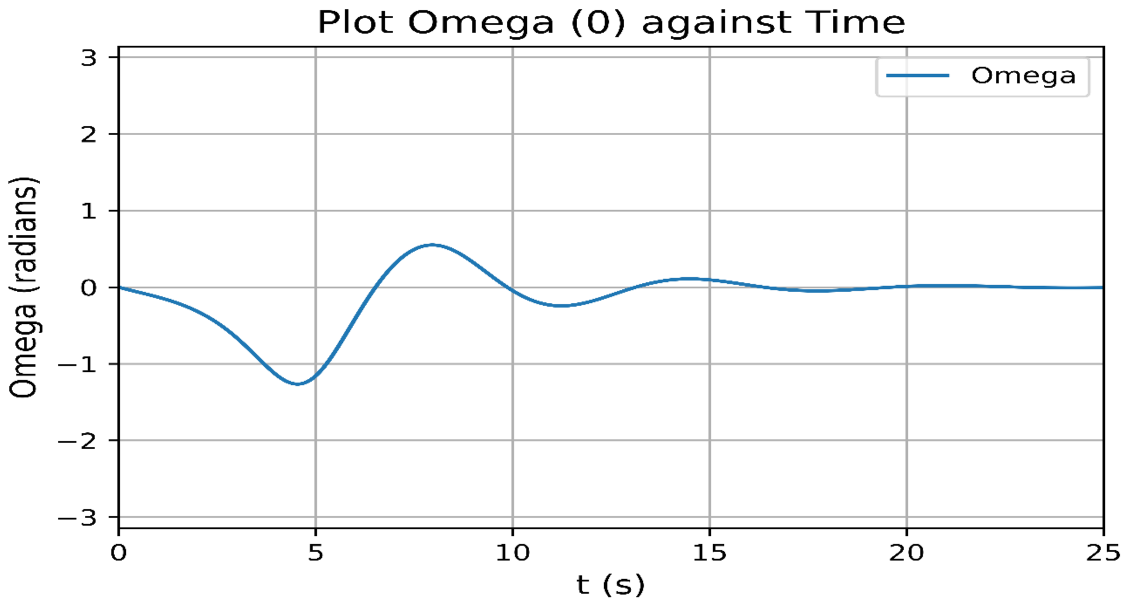

Figure 14

It was seen that dampening changes the phase between that and omega. Focusing around 5 seconds it was noted that omega reached a turning point before theta was at the equilibrium point. This graphically demonstrated dampening in action as with each oscillation theta overshot equilibrium with less and less velocity.

Exercise 5: The damped, driven non-linear pendulum

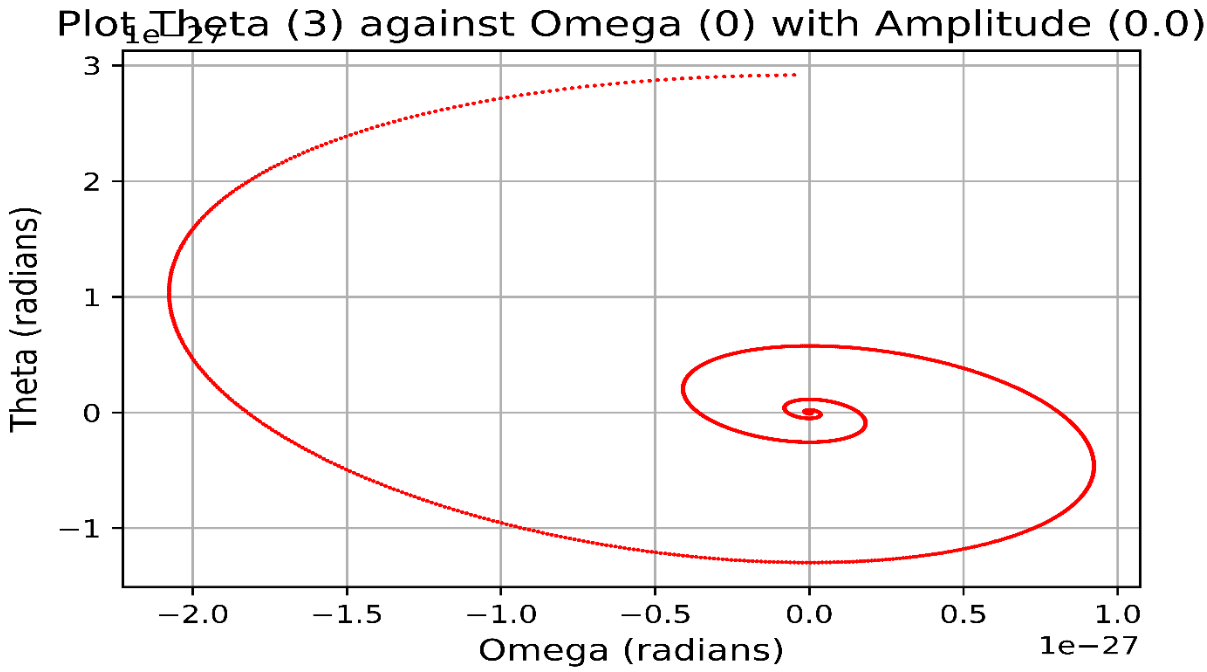

Figure 15

This was the case for a dampened but not driven pendulum. The motion clearly spirals down to noting as there is no input force to counteract the loss to conservative factors.

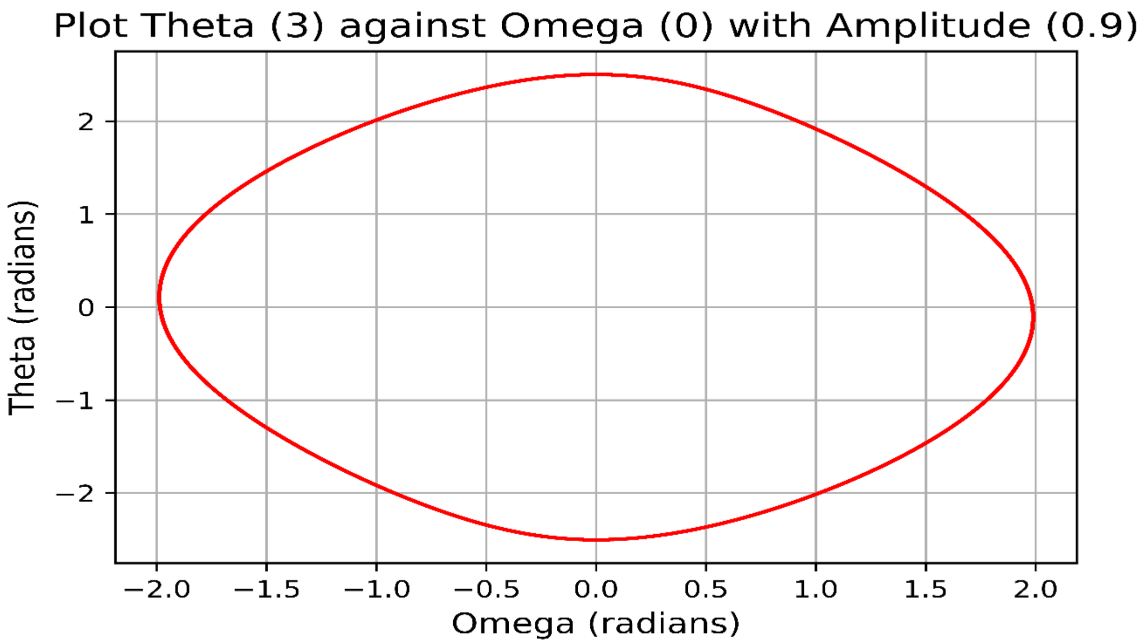

Figure 16

Figure 16, would occur in the ideal case where the driving frequency was in period with the frequency of oscillations, giving stable, periodic motion.

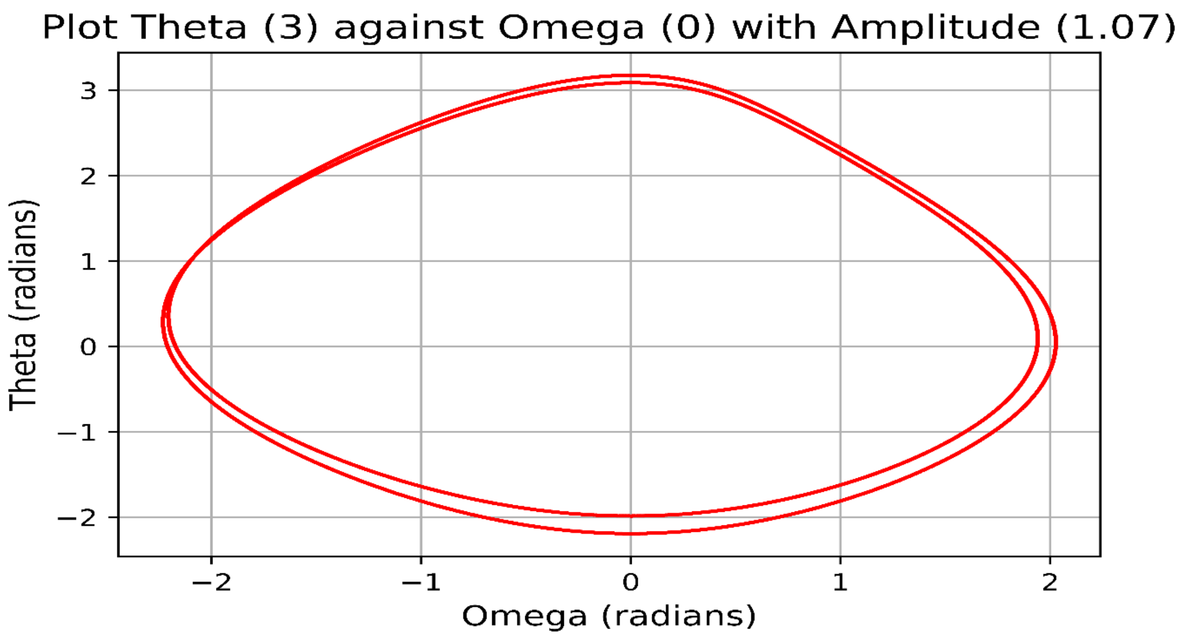

Figure 17

It was seen that a slight increase in amplitude away from the steady state at 0.9 could have large effects on the motion of the pendulum. At 1.07, the motion is seen to be closed again but with a doubled period.

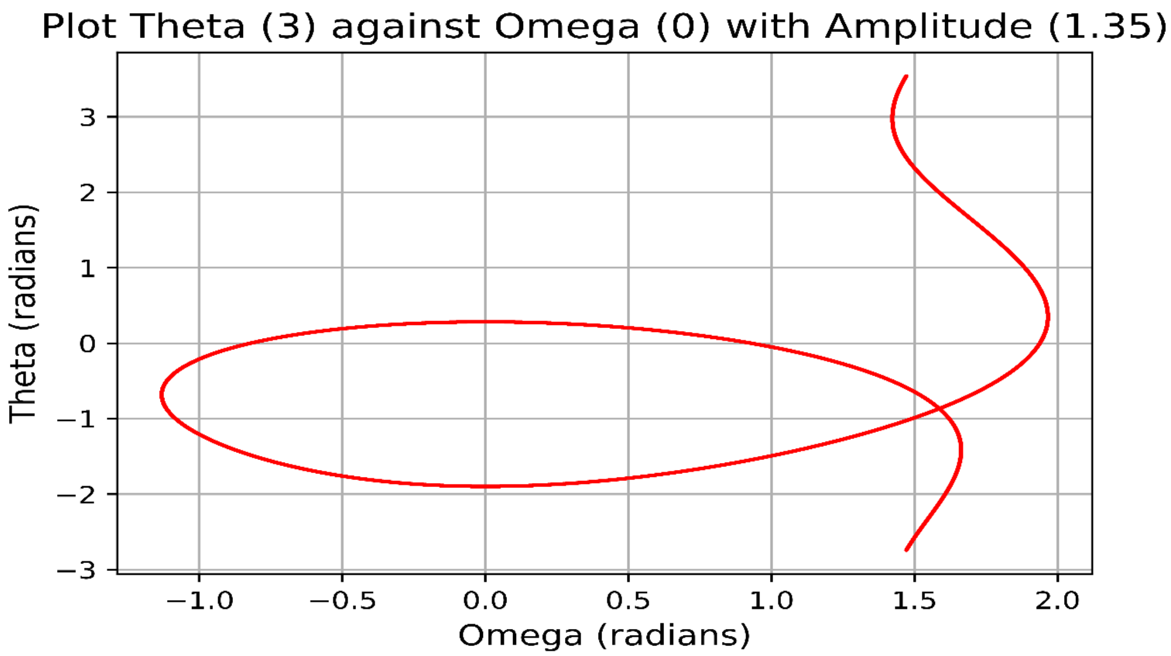

Figure 18

When amplitude increased to 1.35, a new graph was produced, containing a closed loop but what appeared to be two discontinuous tails. These discontinuities would be consistent with the driving force pushing the pendulum past the unstable equilibrium, causing the non-oscillatory motion described in previous parts. The loop would be the pendulum oscillating slightly between impulses.

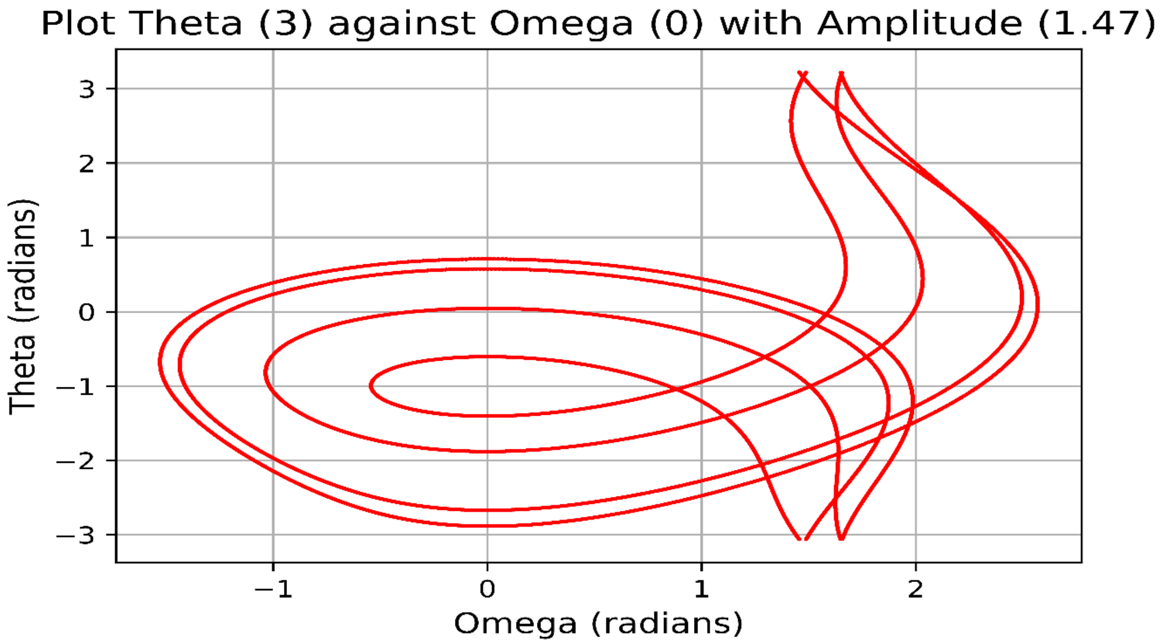

Figure 19

Here it was seen that even the motion described by figure 18 could undergo period multiplication. A four period was shown in this example, to be precise.

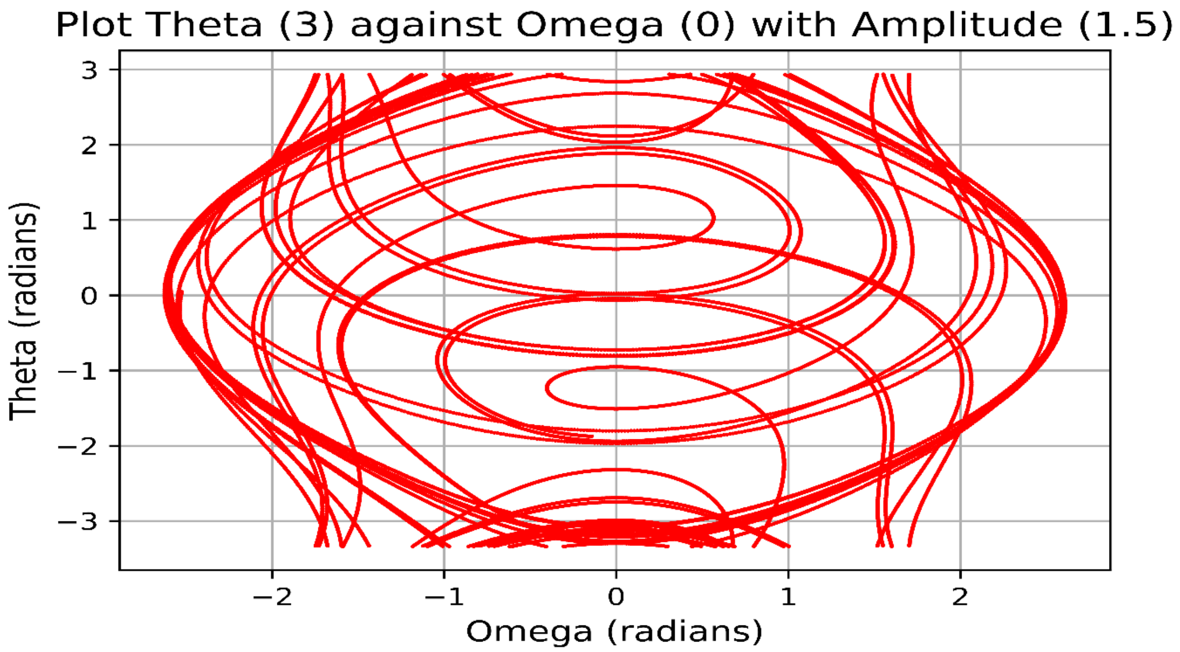

Figure 10

When the amplitude reached 1.5, purely aperiodic and motion was seen to occur. Motion could not be said to be periodic or predictable for any such motion. It is very interesting that an increase of 0.03 was capable of causing such a breakdown in what was still a relatively predictable motion in figure 19.

4. Conclusions

This report began with the goal of investigating the trapezoid and Runge-Kutta algorithms for numerically approximating pendulum motion, to determine which was most subtitle, if either. This can be said to be more that achieved. It was shown that both methods were suitably effective at recreating the expected pendulum motion for undampened and undriven pendulums. The Runge-Kutta method was, however, shown to be more accurate in edge cases, such as when the initial angle was approaching π, as such it was recommended and examined further.

The chosen algorithm was able to handle dampened motion and indicate the factors behind such motion. Likewise, the addition of a driving force was not an issue for Runge-Kutta, and it was used to graph and analyse the affects that such a driving force has on the motion of a pendulum at different amplitudes.

Based on all presented evidence, this report supports the use of numerical algorithms, particularly Runge-Kutta, in the effective analysis of the motion of pendulums in real world environments.

Figures