TCD TP Chaos Assignment March 2023

Assignment 1: Lotka-Volterra

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The Lotka-Volterra Equations

$$\begin{aligned}\frac{dx}{dt} &= k_1 x - k_2 xy \\ \frac{dy}{dt} &= k_3 xy - k_4 y\end{aligned} \tag{1}$$

All graphs use dt=0.01, Steps=10000 unless stated otherwise.
Code available on GitHub[1].

Question 1

(a) The Euler method was used to solve the Lotka-Volterra equations. This allowed the populations X and Y to be plotted over time.

Figure 1, allows one to estimate the periodicity of these populations. By reading the graph one can see that both populations have a period of just over 60 s.

(b) The same method and solutions can give the phase space diagram for the model. Figure 2 plotted the phase portraits for a number of initial coordinates, along with the locations of the fixed points.

[1] github.com/JPAhern/SF-TP-Projects

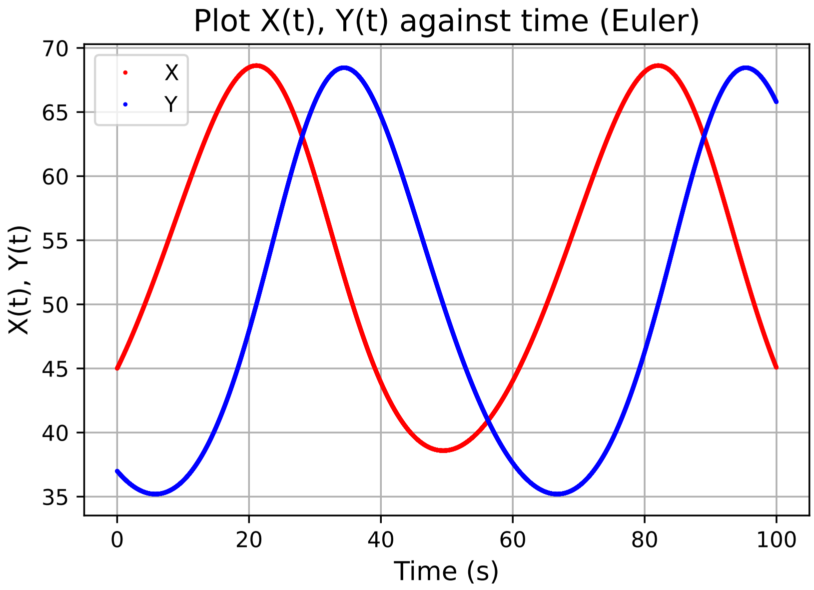

Figure 1: Graph of x(t), y(t) against time (s). Using Euler Method. Initial conditions x(0)=45, y(0)=37, $k_1 = 0.09$, $k_2 = 0.0018$, $k_3 = 0.0023$, $k_4 = 0.12$.

It can be seen, in Figure 2, that the phase trajectory depends on the position of the initial coordinates. Close to Fixed point 2, a much more regular elliptical shape can be seen. This shape becomes almost triangular when moving close to Fixed point 1.

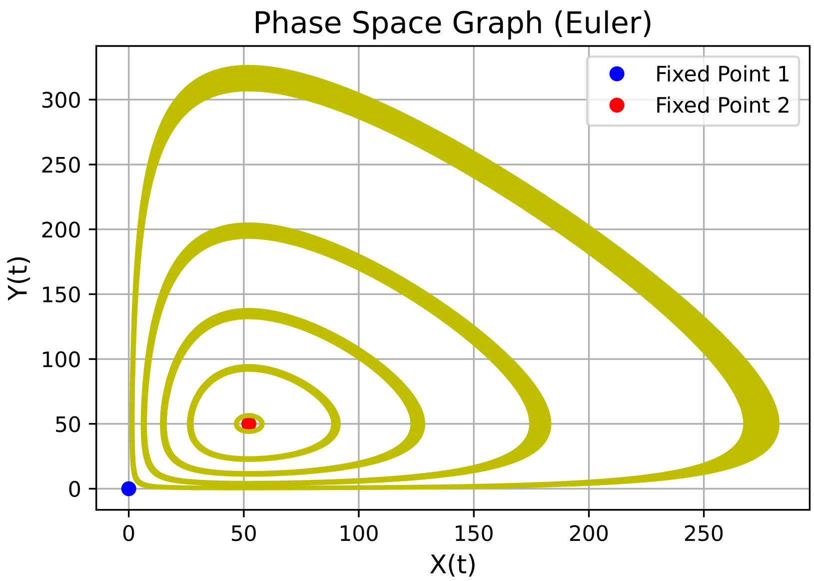

Figure 2: Graph of y(t) against x(t). Using Euler Method. Initial conditions $k_1 = 0.09$, $k_2 = 0.0018$, $k_3 = 0.0023$, $k_4 = 0.12$, (5,5), (15,10), (20,24), (30,35), (55,45).

It is notable however, in Fig 2, that the trajectories form bands of variable width, suggesting a not quite constant trajectory. This does not fully agree with expectations.

We can further examine this trend by graphing the populations over a very long time, close to Fixed point 2. Fig. 3 further demonstrates the issue with the Euler method.

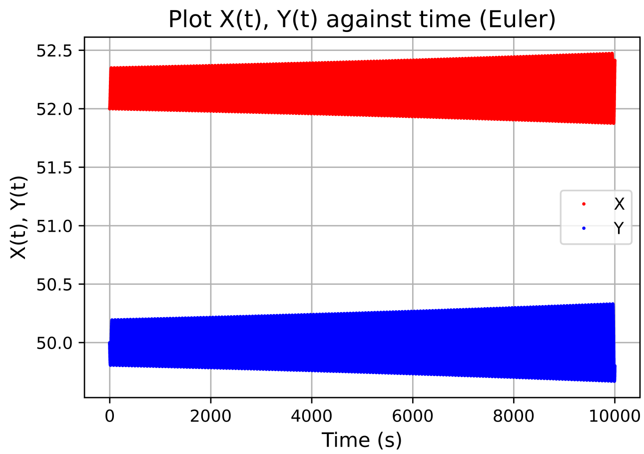

Figure 3: Graph of x(t), y(t) against time (s). Using Euler Method. Initial conditions x(0)=52, y(0)=50, $k_1 = 0.09$, $k_2 = 0.0018$, $k_3 = 0.0023$, $k_4 = 0.12$.

About fixed point 2, we would expect steady oscillations but here we saw a gradual growth of both populations.

Question 2

The Lotka-Volterra equations can also be solved using an "Improved" Euler method, first the Trapezoid rule. The previous phase graph was reproduced to test this method.

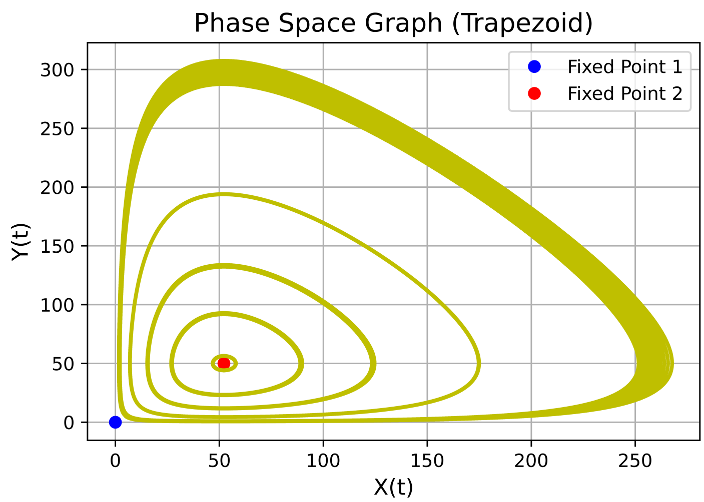

Figure 4: Graph of y(t) against x(t). Using Trapezoid Method. ICs $k_1 = 0.09$, $k_2 = 0.0018$, $k_3 = 0.0023$, $k_4 = 0.12$, (5,5), (15,10), (20,24), (30,35), (55,45).

It is clear looking at Fig. 4 that the Trapezoid rule is an improvement on the standard Euler method. For the most part, trajectories appear too be more stable, bringing them loser to expectations. This method still exhibits issues close to fixed points though, as clearly seen in the outermost phase portrait.

As a further improvement the Runge-Kutta method was implemented.

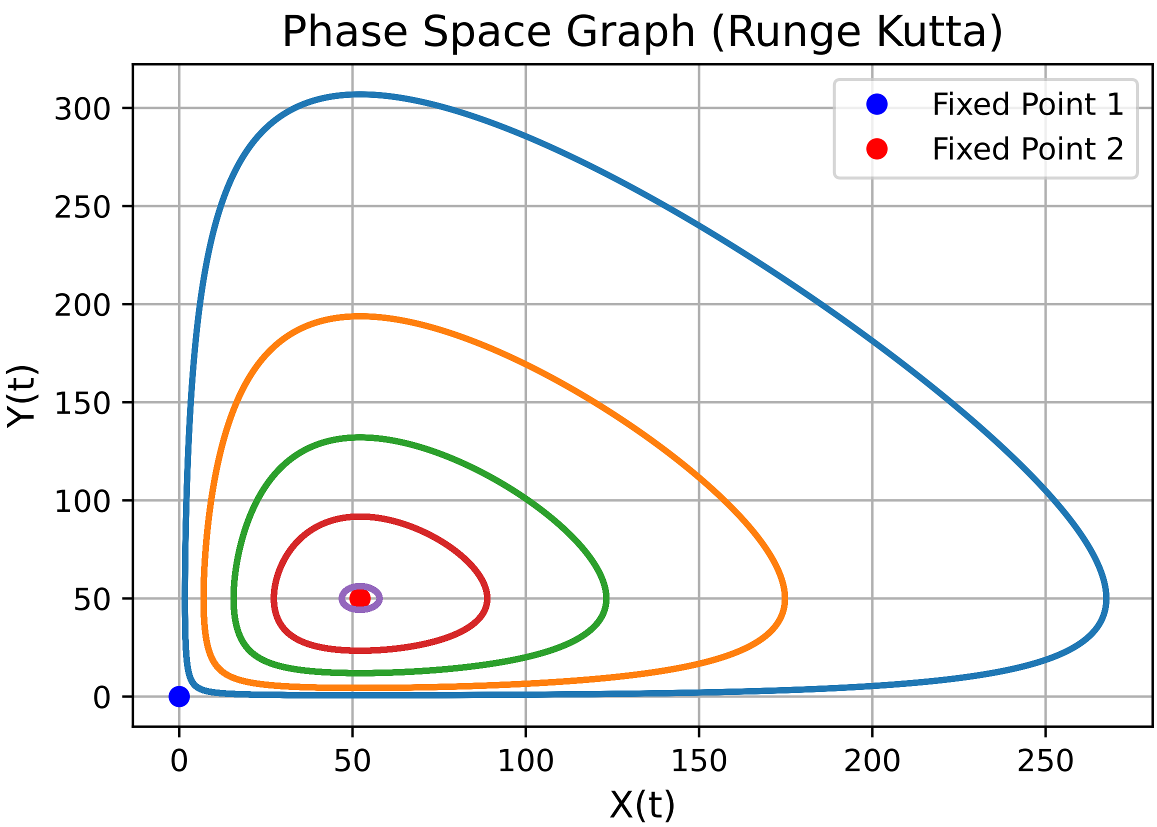

Figure 5: Graph of y(t) against x(t). Using Runge-Kutta Method. ICs $k_1 = 0.09$, $k_2 = 0.0018$, $k_3 = 0.0023$, $k_4 = 0.12$, (5,5), (15,10), (20,24), (30,35), (55,45).

The Runge-Kutta method is a clear improvement on both previous methods. All trajectories can now be said to act as expected according to the analysis from our tutorial. Fixed point 1 displays the behaviour of a saddle point, expected as the eigenvalues at this point are one attractor and one repeller. Fixed point 2 has two imaginary eigenvalues and so oscillations would be expected about it and this is exactly what was observed.

It is noted that reproducing the conditions for Fig 3 results does not reproduce the same graph using this method, instead the amplitude of oscillations remains consistent as expected.

Question 3

(a) Analytical integration of the Lotka-Volterra equation via separation of variables.

$$\begin{aligned}
\frac{\frac{dx}{dt}}{\frac{dy}{dt}} &= \frac{dx}{dy} = \frac{x\,(k_1 - k_2 y)}{(k_3 x - k_4)\,y} \\
\int \frac{(k_3 x - k_4)}{x}\,dx &= \int \frac{(k_1 - k_2 y)}{y}\,dy \\
\int k_3\,dx - k_4 \int \frac{1}{x}\,dx &= k_1 \int \frac{1}{y}\,dy - \int k_4\,dy \\
k_3 x - k_4 \ln x + c_1 &= k_1 \ln y - k_2 y + c_2
\end{aligned} \tag{2}$$

Which of course gives the required equation when constants are combined.

$$C(x(t), y(t)) = k_3 x(t) - k_4 \ln x(t) + k_2 y(t) - k_1 \ln y(t) \tag{3}$$

Figure 6, gives a phase space diagram for the Lotka-Volterra equations. Initial conditions x(0)=54, y(0)=50, $k_1 = 0.09$, $k_2 = 0.0018$, $k_3 = 0.0023$, $k_4 = 0.12$. Using these initial values, Equation 3 gives the exact value $C_A = -0.6165601561$.

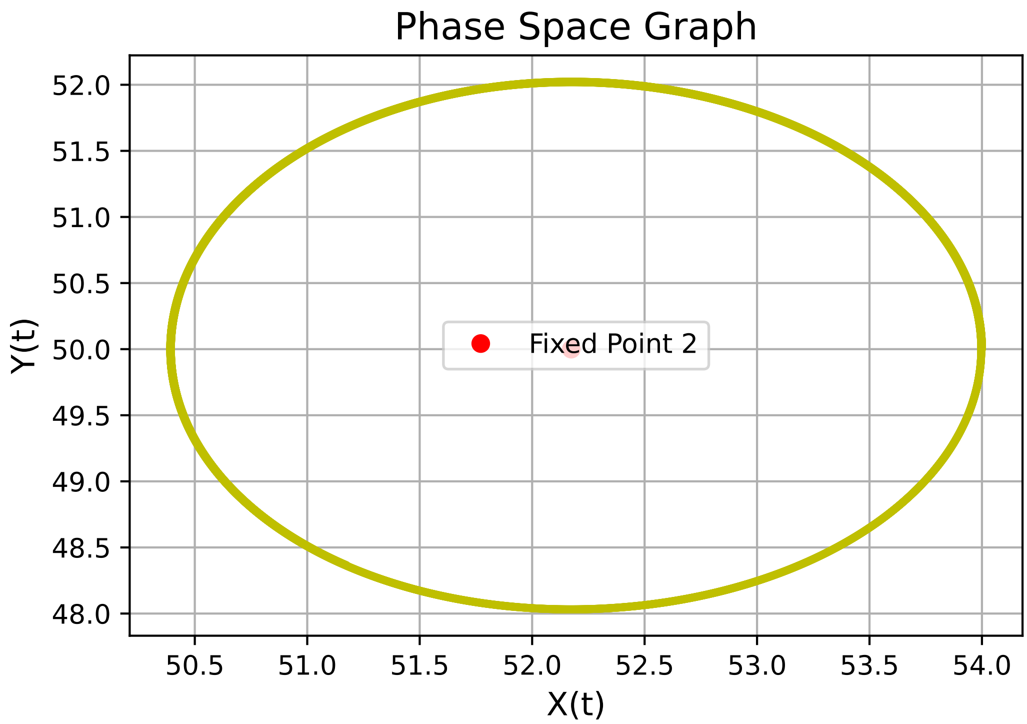

Figure 6: Graph of y(t) against x(t), using improved Eulers Method, $k_1 = 0.09$, $k_2 = 0.0018$, $k_3 = 0.0023$, $k_4 = 0.12$, x(0)=54, y(0)=50, Fixed point 2 = $(\frac{0.12}{0.0023}, 50)$

(b) A selection of coordinates are read directly from the phase portrait, Figure 6 and these values are substituted into Equation 3.

Table 1: Table of numerical values for C

x(t)	**y(t)**	**C(x(t),y(t))**
52	52	-0.6165611809
52.5	52	-0.6165611809
51	48.5	-0.6165598177
51	51.5	-0.6165614386
	$\bar{C}$ (mean)	-0.6165604881

The average of the numerical values for C ($\bar{C}$ in Table 1) perfectly matches the analytically predicted value up to the sixth decimal place, which is well within expected deviation. Frankly, considering the smallest input value only goes to the fourth decimal place it is reasonable to say $C_A = \bar{C}$. This result shows that C is constant in time. As such it is a conserved quantity, in fact it can be identified as a Hamiltonian for the system [1].

Question 4

The period of oscillations about a fixed point in phase space is predicted to be analytically determined by corresponding eigenvalues.

$$T = \frac{2\pi}{\sqrt{\lambda_+ \lambda_-}} \tag{4}$$

The eigenvalues for the second fixed point are known from the Tutorial to be $\lambda_+ = i\sqrt{k_1 k_2}$ and $\lambda_- = -i\sqrt{k_1 k_2}$. Sine both eigenvalues are imaginary and are of opposing sign we do not need to worry about the period being imaginary.

Using the same initial conditions as in Question 3, the period can be analytically calculated to be $T = 60.46$ s, Equation 4. Figure 7, plots the populations of x(t) and y(t) over a period of 100 s and in this graph a period of ≈ 60 s between the peaks of each population can be noted. This period is further verified in Figure 8, which plots the phase spaces for the same conditions but for 60 s and 60.5 s. It is clearly seen that the ellipse is not yet complete at 60 s but is complete after 60.5 s. This is in line with the analytical prediction for the period under these initial conditions ($T = 60.46$ s).

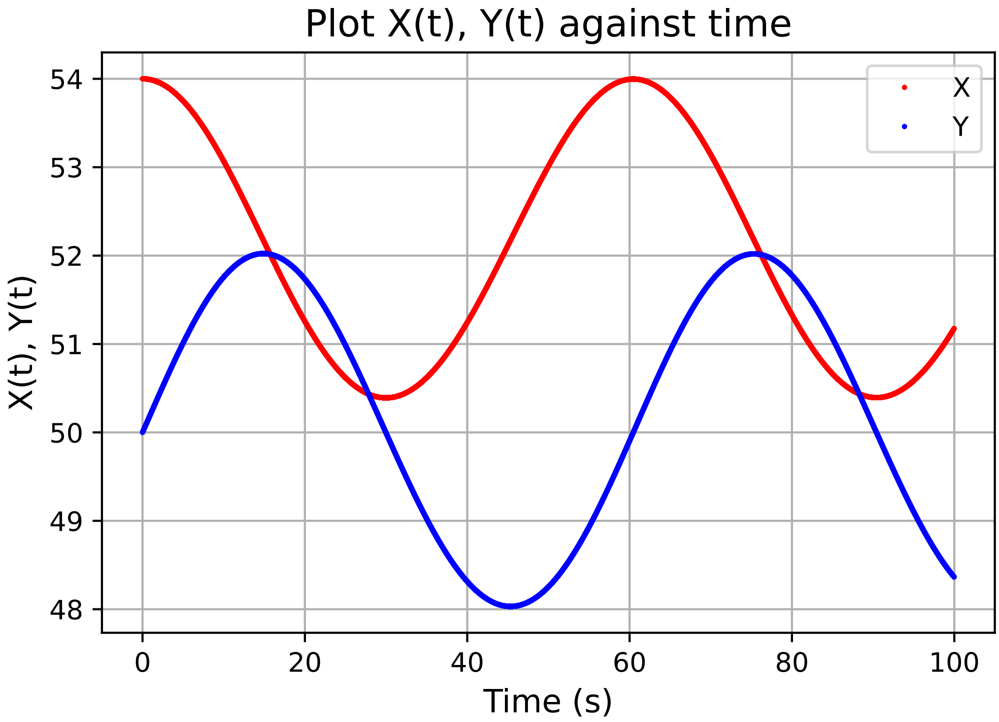

Figure 7: Graph of X(t),Y(t) against time (s), using improved Euler Method, $k_1 = 0.09$, $k_2 = 0.0018$, $k_3 = 0.0023$, $k_4 = 0.12$, x(0)=54, y(0)=50

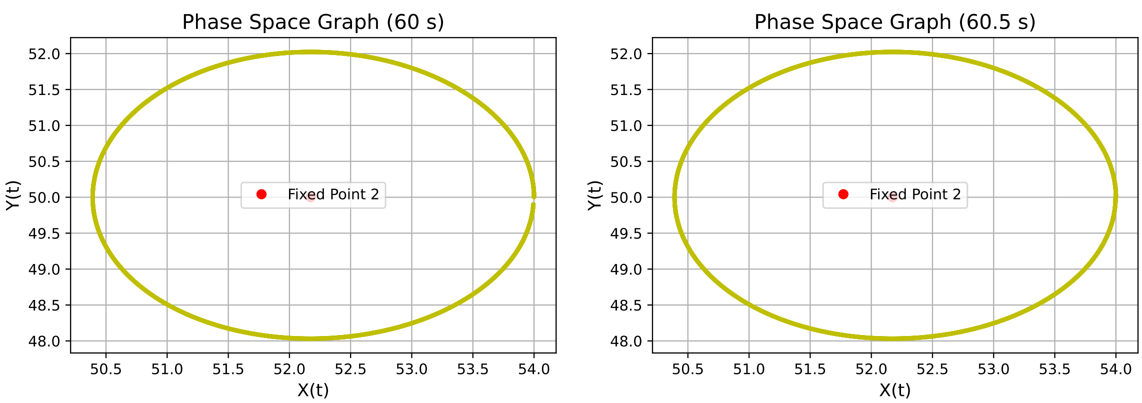

Figure 8: Phase Space graphs over 60s and 60.5s.

References

[1]P. A. Damianou and F. Petalidou, "On the liouville intergrability of lotka-volterra systems", Frontiers in Physics **2**, 50 (2014).

List of Tables

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