Open-loop discrete-time systems

Chapter 4 of Phillips, Nagle and Chakrabortty (Study unit 5)

Presented by Prof. KR Uren

It all starts here ®

NORTH-WEST UNIVERSITY ®
YUNIBESITI YA BOKONE-BOPHIRIMA
NOORDWES-UNIVERSITEIT
POTCHEFSTROOM CAMPUS

The relationship between $E(z)$ and $E^*(s)$

The z-transform of a number sequence $\{e(k)\}$ is

$$\mathcal{Z}[\{e(k)\}] = E(z) = e(0) + e(T)z^{-1} + e(2T)z^{-2} + \dots \quad (1)$$

$$= \sum_{k=0}^{\infty} e(kT)z^{-k} \quad (2)$$

and the starred transform for the time function $e(t)$ is

$$E^*(s) = e(0) + e(T)\epsilon^{-Ts} + e(2T)\epsilon^{-2Ts} + \dots \quad (3)$$

$$= e(0) + e(T)\left(\epsilon^{Ts}\right)^{-1} + e(2T)\left(\epsilon^{Ts}\right)^{-2} \quad (4)$$

$$= \sum_{k=0}^{\infty} e(kT)\epsilon^{-Ts} \quad (5)$$

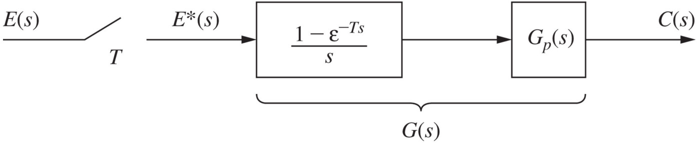
$$\boxed{E(z) = E^*(s)|_{\epsilon^{sT} = z}} \quad (6)$$

Determine the starred transform Ex. 1

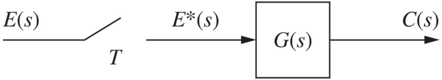
We can see that the z-transform can be considered to be a special case of the Laplace transform for our purposes.
To determine the starred transform we will first determine the z-transform from the tables to find $E(z)$, and then use inverse of (6) to find $E^*(s)$.
Determine $E^*(s)$ if

$$E(s) = \frac{1}{(s+1)(s+2)}$$

Pulse transfer function



(a)



(b)

The product of the plant transfer function $G_p(s)$ and th zero-order hold transfer function is defined as $G(s)$, such that

$$G(s) = \frac{1 - \epsilon^{-Ts}}{s} G_p(s) \tag{7}$$

Pulse transfer function

$$
\begin{aligned}
C(s) &= G(s)E^*(s) \qquad (8)\\
C^*(s) &= [G(s)E^*(s)]^* = G^*(s)E^*(s) \qquad (9)\\
C(z) &= G(z)E(z) \qquad (10)
\end{aligned}
$$

$G(z)$ is called the *pulse transfer function* and is the transfer function between the sampled input and the output at the sampling instants. [1]

[1] $[\cdot]^*$ denotes the starred transform

General procedure

The procedure helps deriving the z-transform of systems containing a sampler: Let the a given function be presented by

$$A(s) = B(s)F^*(s) \tag{11}$$

where $F^*(s)$ can be presented as

$$F^*(s) = f_0 + f_1\epsilon^{-Ts} + f_2\epsilon^{-2Ts} + \dots \tag{12}$$

then

$$A^*(s) = B^*(s)F^*(s) \tag{13}$$

and

$$A(z) = B(z)F(z) \tag{14}$$

where $B(s)$ is a function of s only and $F^*(s)$ is a function of ϵ^{Ts} only, that is s appears only in the form ϵ^{Ts} in $F^*(s)$.

So

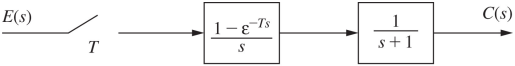
$$\begin{aligned} B(z) &= \mathcal{Z}[B(s)] && (15) \\ F(z) &= F^{*}(s)|_{\epsilon^{Ts}=z} && (16) \end{aligned}$$

Determine the z-transform of

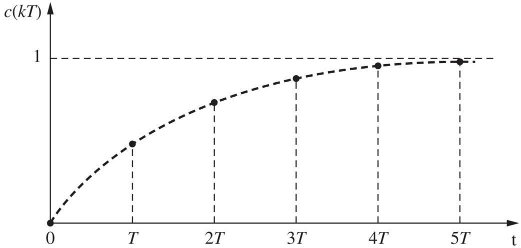
$$A(s) = \frac{1-\epsilon^{-Ts}}{s(s+1)} \quad (17)$$

Example 4.3

Given the system shown in the Figure, with input $e(t)$ a **unit step function**, determine the output function $c(t)$



(a)



(b)

Some important remarks

Given $c(t)$, we can find $c(kT)$ by replacing t with kT. However, given $c(kT)$ from a z-transform analysis, we cannot replace kT with t and have the correct expression for $c(t)$.

The steady-state output of the system, $c_{ss}(k)$, for a unity step input $E(z) = z/(z-1)$ is

$$\begin{aligned} c_{ss}(k) &= \lim_{z\to 1}[(z-1)C(z)] \\ &= \lim_{z\to 1}[(z-1)G(z)E(z)] \\ &= \lim_{z\to 1}\left[(z-1)G(z)\frac{z}{(z-1)}\right] = G(1) \end{aligned}$$

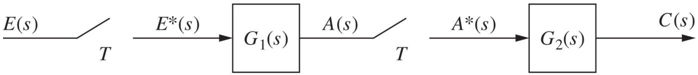
Definition of dc gain

$$\boxed{\text{dc gain} = \lim_{z\to 1} G(z) = \lim_{s\to 0} G_p(s)} \tag{18}$$

$$\lim_{z\to 1} G(z) = \lim_{z\to 1}\left(\frac{1-\epsilon^{-T}}{z-\epsilon^{-T}}\right) = 1; \tag{19}$$

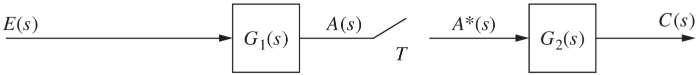
$$\lim_{z\to 0} G_p(s) = \lim_{s\to 0}\left(\frac{1}{s+1}\right) = 1 \tag{20}$$

Special cases



(a)

(b)



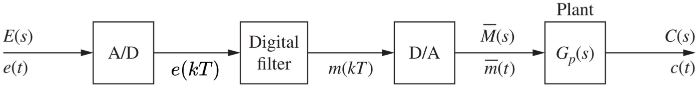
(c)

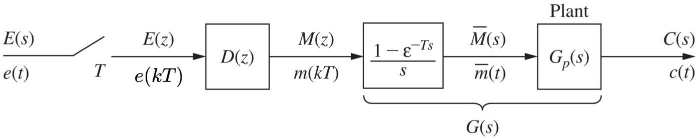
Very important

$$\overline{G_1G_2}(z) = \mathcal{Z}[G_1(s)G_2(s)]$$

$$\boxed{\overline{G_1G_2}(z) \neq G_1(z)G_2(z)}$$

Open-loop systems with digital filters





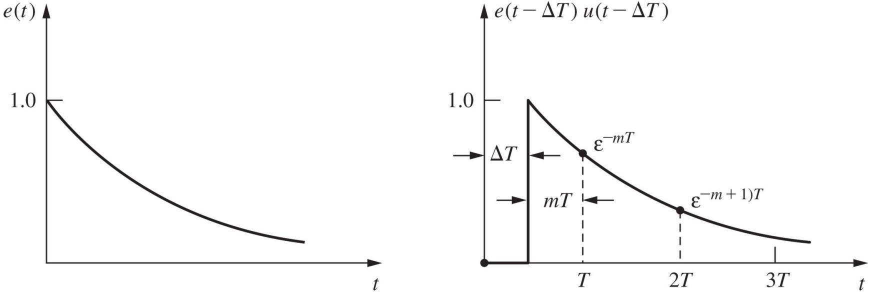
The modified z-transform

Consider Ex.4.5 as a means of explaining this concept. The delayed z-transform is firstly defined: as:

$$E(z,\Delta) = \mathcal{Z}[e(t-\Delta T)u(t-\Delta T)] = \mathcal{Z}[E(s)\epsilon^{-\Delta T s}] \tag{21}$$

$$E(z,\Delta) = \sum_{n=1}^{\infty} e(nT - \Delta T)z^{-n} \tag{22}$$

Ex 4.5



Properties of the modified z-transform

$$E(z,m) = E(z,\Delta)|_{\Delta=1-m} = \mathcal{Z}[E(s)\epsilon^{-\Delta T s}]|_{\Delta=1-m} \tag{23}$$

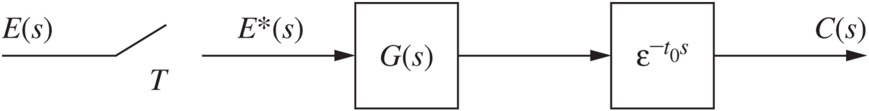
Property 1

$$E(z,1) = E(z,m)|_{m=1} = E(z) - e(0) \tag{24}$$

Property 2

$$E(z,0) = E(z,m)|_{m=0} = z^{-1}E(z) \tag{25}$$

Systems with time delays



$$C(z) = z^{-k}\mathcal{Z}[G(s)\epsilon^{-\Delta T s}]E(z) = z^{-k}G(z,m)E(z) \qquad (26)$$

where $m = 1 - \Delta$

THE END