Tutorial 2 Memo - 20 February 2017 [Total = 20]

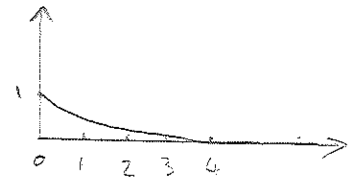
Question 1 - Problem 2.3-3 (b) [7]

Find the z-transform of the number sequences generated by sampling the following time functions every T seconds, beginning at $t = 0$. Express these transforms in closed form.

(b) $e(t) = \epsilon^{-(t-T)}u(t-T)$

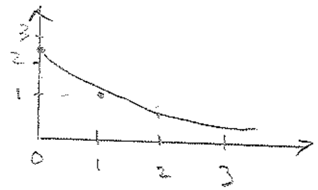
Answer:

Consider $e(t) = \varepsilon^{-t}$;

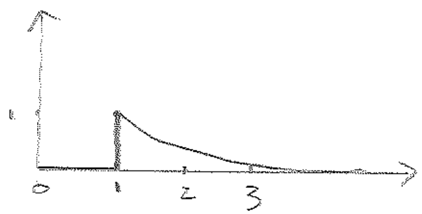


$e(t-1) = \varepsilon^{-(t-1)}$

$= \varepsilon^{-t+1}$



$e(t-1)\,u(t-1)$ where $u(t)$ = unity step function.



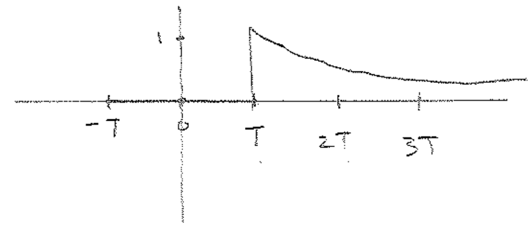
method 1 :

So we have $e(t) = \varepsilon^{-(t-T)} u(t-T)$

In discrete form $e(k) = \varepsilon^{-(kT-T)} u(kT-T)$

So $k=0$ $\quad e(0) = \varepsilon^{-(-T)} \cdot 0$ — step is zero at $k=0$

$= 0$



$k=1$ $\quad e(1) = \varepsilon^{-(T-T)} \cdot 1$ — step is one at $k=1$

$= e^{-0} = 1$

$k=2$ $\quad e(2) = \varepsilon^{-(2T-T)} \cdot 1$ — step is one at $k=2$

$= \varepsilon^{-T}$

$k=3$ $\quad e(3) = \varepsilon^{-(3T-T)} \cdot 1$ — step is one at $k=3$

$= \varepsilon^{-2T}$

$$F(z) = \sum_{k=0}^{\infty} e(k) z^{-k}$$

$$= e(0) z^0 + e(1) z^{-1} + e(2) z^{-2} + \cdots$$

$$= 0 + 1 z^{-1} + \varepsilon^{-T} z^{-2} + \varepsilon^{-2T} z^{-3} + \cdots$$

$$= z^{-1} \left[1 + \varepsilon^{-T} z^{-1} + \varepsilon^{-2T} z^{-2} + \cdots \right]$$

$$= z^{-1} \left[1 + (\varepsilon^{-T} z^{-1})^1 + (\varepsilon^{-T} + z^{-1})^2 + \cdots \right]$$

$$= z^{-1} \left[\frac{1}{1 - (\varepsilon^{-T} z^{-1})} \right] = z^{-1} \left[\frac{z}{z - \varepsilon^{-T}} \right] = \frac{1}{z - \varepsilon^{-T}} \quad \square$$

(7)

Method 2 :

Use property of tab 2-2 : $e(k-n)\,u(k-n);\ n \geq 0 \overset{z}{\Longleftrightarrow} z^{-n} E(z)$

Consider $e(t) = \varepsilon^{-(t-T)} u(t-T)$ without delay.

then $e(t) = \varepsilon^{-t} u(t)$

$\therefore e(k) = e^{-kT} u(kT)$

$e(0) = e^{-0} \cdot 1$

$= 1$

$e(1) = e^{-T}$

$e(2) = e^{-2T}$

$$E(z) = \sum_{k=0}^{\infty} e(k) z^{-k}$$

$$= e(0) z^{0} + e(1) z^{-1} + e(2) z^{-2} + \dots$$

$$= 1 + e^{-T} z^{-1} + e^{-2T} z^{-2} + \dots$$

$$= \frac{1}{1 - (e^{-T} z^{-1})}$$

✓✓✓✓
✓✓✓ (7)

$$= \frac{z}{z - e^{-T}}$$

so $e(k-1)T = e^{-(k-1)T} u(k-1)T \Rightarrow z^{-1} E(z) = z^{-1} \dfrac{z}{z - e^{-T}}$

$$= \frac{1}{z - e^{-T}} \quad \square$$

Question 2 - Problem 2.6-1 (b) and (c) [13]

Solve the given difference equation for $x(k)$ using:

(b) The z-transform.

(c) Will the final-value theorem give the correct value of $x(k)$ as $k \to \infty$?

$$x(k) - 5x(k-1) + 6x(k-2) = e(k)$$

where

$$e(k) = \begin{cases} 1, & k = 0, 1 \\ 0, & k \geq 2 \end{cases}$$

$$x(-1) = x(-2) = 0$$

Answer:

(b) Take the z-transform of the difference equation

$$X(z) - 5z^{-1}X(z) + 6z^{-2}X(z) = E(z)$$

$$X(z)\left[1 - 5z^{-1} + 6z^{-2}\right] = E(z) \quad ✓$$

$$X(z) = \frac{1}{1 - 5z^{-1} + 6z^{-2}} E(z)$$

$$= \left[\frac{z^2}{z^2 - 5z + 6}\right] E(z) \quad ✓$$

Determine $E(z)$

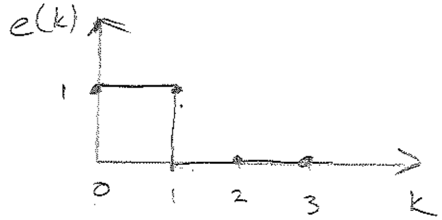
$e(0) = 1$

$e(1) = 1$

$e(2) = 0$ ✓

$e(3) = 0$

$\Rightarrow$



$$E(z) = \sum_{k=0}^{\infty} e(k) z^{-k} = e(0)z^{-0} + e(1)z^{-1} + e(2)z^{-2} + e(3)z^{-3} + \ldots \quad ✓$$

$$= 1 + 1z^{-1} + 0 + 0 \ldots \quad ✓$$

$$= 1 + \frac{1}{z}$$

$$= \frac{z+1}{z} \quad ✓$$

So $$X(z) = \left[\frac{z^2}{z^2 - 5z + 6}\right] \frac{z+1}{z}$$

$$= \frac{(z+1)z}{z^2 - 5z + 6}$$ ✓

To determine $x(k)$ we use the partial fraction approach

$$\therefore \quad \frac{X(z)}{z} = \frac{(z+1)}{z^2 - 5z + 6} = \frac{A}{(z-2)} + \frac{B}{(z-3)}$$ ✓

So $$A = \left.\frac{z+1}{z-3}\right|_{z=2} = \frac{2+1}{2-3} = \frac{3}{-1} = -3$$

$$B = \left.\frac{z+1}{z-2}\right|_{z=3} = \frac{3+1}{3-2} = \frac{4}{1} = 4$$

$$X(z) = \frac{-3z}{(z-2)} + \frac{4z}{(z-3)}$$ ✓

$$x(k) = -3(2)^k + 4(3)^k$$ ✓

Final value $$\lim_{k \to \infty} e(k) = \lim_{z \to 1} (z-1) E(z)$$ ✓ so

$$\therefore \quad \lim_{k \to \infty} x(k) = \lim_{z \to 1} (z-1) X(z)$$

$$\lim_{z \to 1} (z-1)X(z) = \frac{(z-1)(z+1)z}{(z-2)(z-3)}$$

$$= \frac{(1-1)(1+1)1}{(1-2)(1-3)}$$

$$= 0$$

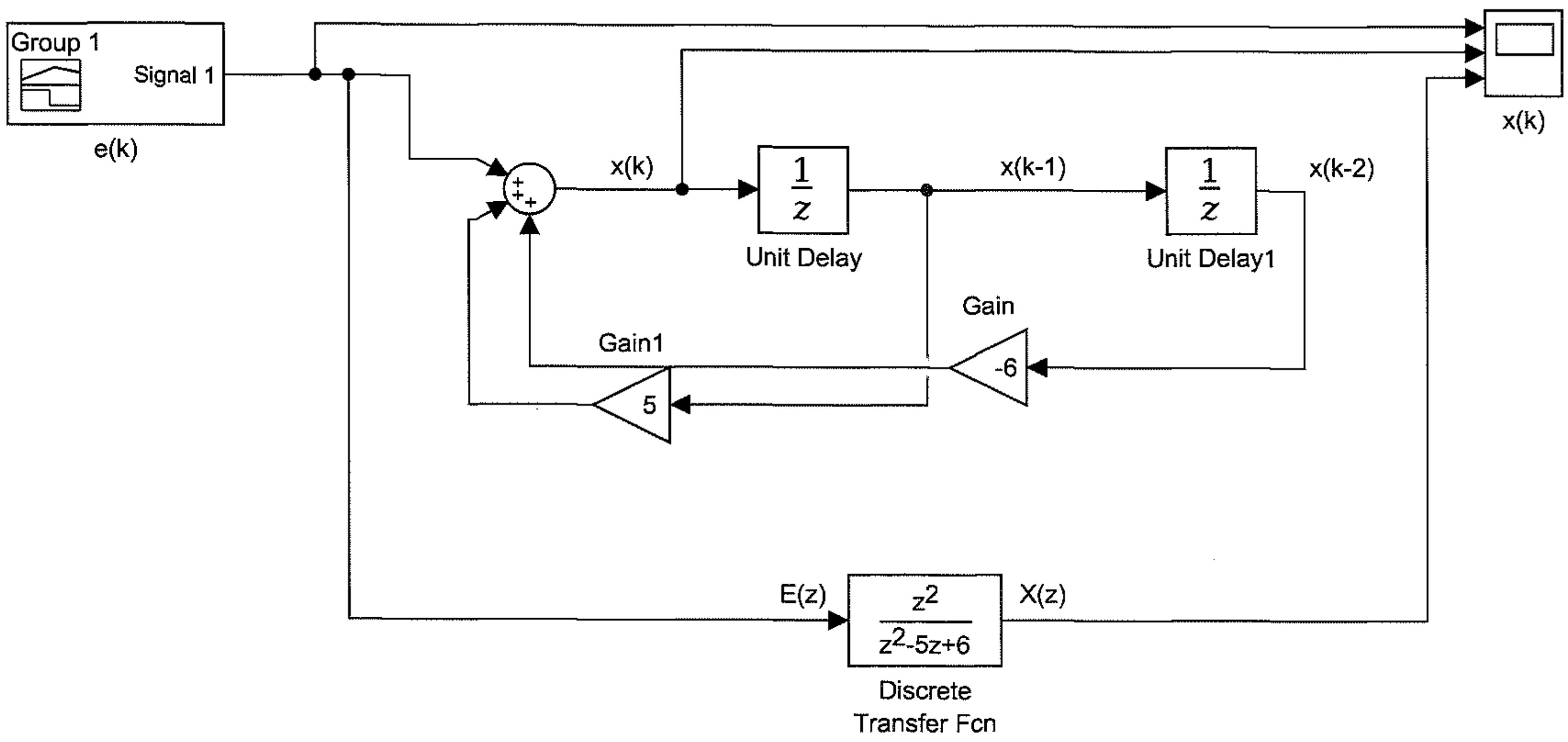
$$\lim_{k \to \infty} x(k) = -3(2)^k + 4(3)^k \Big|_{k=\infty}$$

$$= -3\infty + 4\infty$$

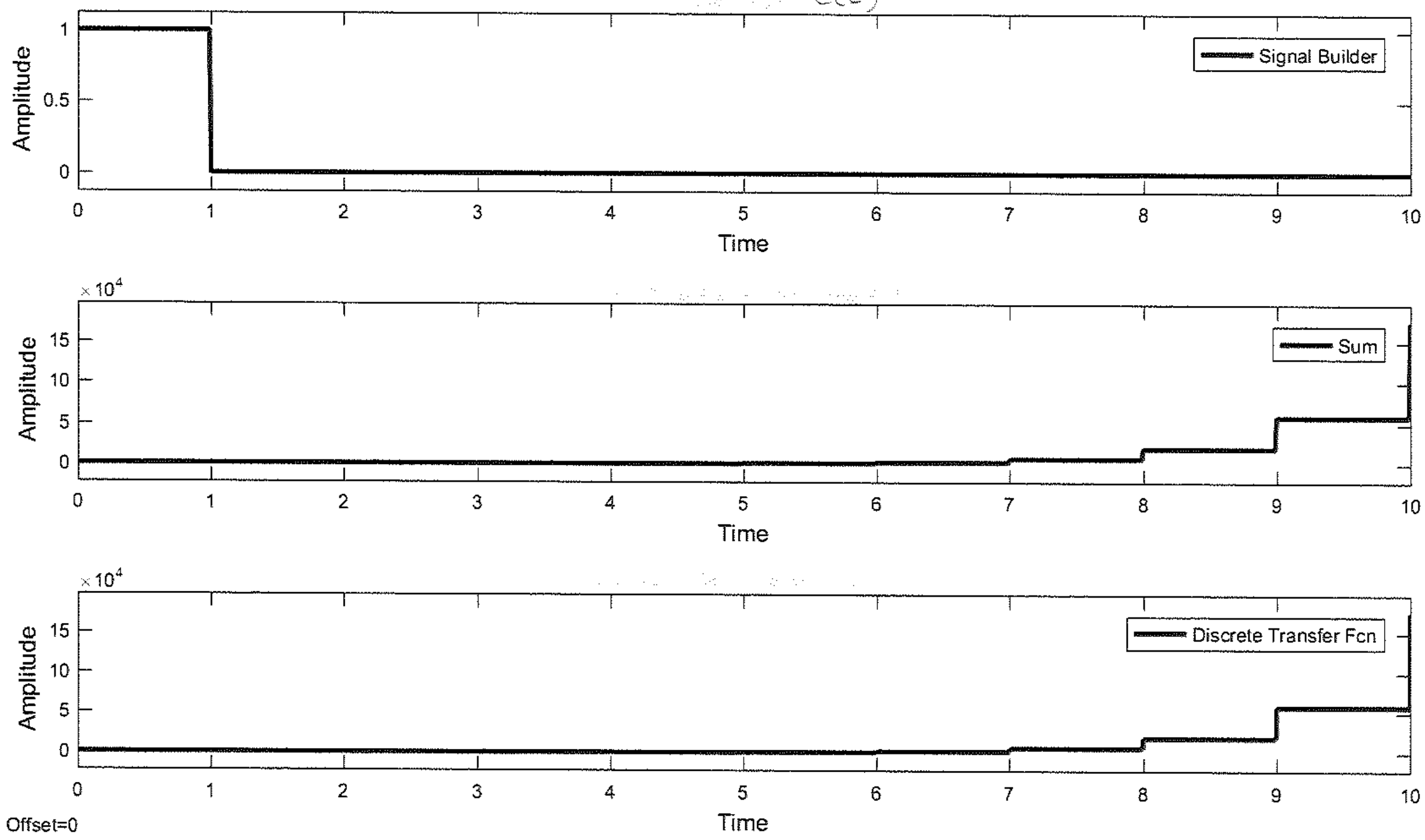
$$= \text{undefined}.$$

This means the left side and the right side of the final value theorem are not the same

$\Rightarrow$ Final value does not exist.


Group 1
Signal 1
e(k)
x(k)
Unit Delay
x(k-1)
Unit Delay1
x(k-2)
Gain
-6
Gain1
5
x(k)
E(z)
z2
z2-5z+6
X(z)
Discrete
Transfer Fcn

e(k)



It is clear that the system becomes unstable