NORTH-WEST UNIVERSITY
YUNIBESITI YA BOKONE-BOPHIRIMA
NOORDWES-UNIVERSITEIT
POTCHEFSTROOMKAMPUS

Benodigdhede vir hierdie vraestel/Requirements for this paper:			
Antwoordskrifte/ Answer scripts:	[X]	Multikeusekaarte (A5)/ Multi-choice cards (A5):	[ ]
Presensiestrokies (Invulvraestel)/ Attendance slips (Fill-in paper):	[ ]	Multikeusekaarte (A4)/ Multi-choice cards (A4):	[ ]
Rofwerkpapier/ Scrap paper:	[ ]	Grafiekpapier/ Graph paper:	[ ]

Sakrekenaars / Calculators: Ja/Yes
Ander hulpmiddels/ Other resources:

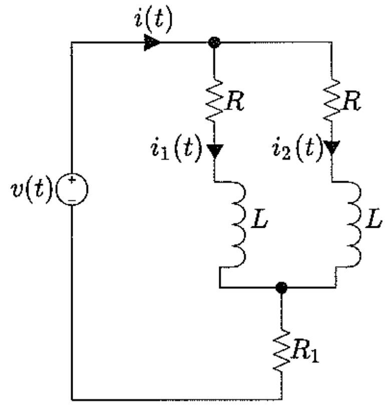
Tipe Assessering/ Type of Assessment:	**Semester toets 1 / Semester test 1**	Kwalifikasie/ Qualification:	**B.ING**
Modulekode/ Module code:	**EERI 418**	Tydsduur/ Duration:	**2 ure/hours**
Module beskrywing/ Module description:	**Beheerteorie II / Control theory II**	Maks/ Max:	**43**
Eksaminator(e)/ Examiner(s):	**PROF. K.R. UREN**	Datum/	**10/03/2017**
Moderator(s):	**PROF. G. VAN SCHOOR**	Tyd/Time	**09:00**

Inhandiging van antwoordskrifte/Submission of answer scripts: **Gewoon/Ordinary**

VRAAG 1 / QUESTION 1 [15]

Beskou die elektriese stroombaan gegee in Figuur 1. Die inset van die stelsel is die spanning $v(t)$, en die uitset die stroom deur R_1. Die toestande van die stelsel is $x_1(t) = i_1(t)$ en $x_2(t) = i_2(t)$. Lei die toestandsruimte model af vir die stelsel en bepaal dan of die stelsel beheerbaar en waarneembaar is. /

Consider the electrical circuit shown in Figure 1. The input to the system is the voltage $v(t)$, and the output is the current through R_1. The states of the system is $x_1(t) = i_1(t)$ and $x_2(t) = i_2(t)$. Derive the state space model of the sytem and determine if the system is controllable and observable.



Figuur 1 / Figure 1

VRAAG 2/ QUESTION 2 [10]

Gegee die volgende toestandsruimte model van 'n stelsel en die spesifikasies, ontwerp 'n toestandsruimte terugvoer beheerder deur van Ackermann se formule gebruik te maak. /

Given the following state variable model of a system and the specifications, design a state-variable feedback controller using Ackermann's formula.

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \\ y(t) &= \mathbf{C}\mathbf{x}(t), \end{aligned}$$

waar / *where*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -10.5 & -11.3 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 \\ 0.55 \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Die ontwerp spesifikasies word as volg gegee: / *The design specifications are as follows:*

(a) Die vestigingtyd is: / *The settling time is:* $T_s = 1$ s.

(b) Die persentasie verbyskiet is: / *The percentage overshoot is:* P.O. $= 5$ %.

Vraag 3 / Question 3 [18]

'n Stelsel word deur die volgende verskilvergelyking gemodelleer: / *A system is modelled by the following difference equation:*

$$x(k+3) - 2.2x(k+2) + 1.57x(k+1) - 0.36x(k) = e(k)$$

(3.1) Bepaal die oordragsfunksie van die stelsel $\left(\frac{X(z)}{E(z)}\right)$. / *Determine the transfer function of the system* $\left(\frac{X(z)}{E(z)}\right)$. (2)

(3.2) Bepaal $x(k)$ vir die stesel in (3.1) vir 'n eenheidstrapinset deur van magreeksuitbreiding gebruik te maak. Aanvaar alle begintoestande as nul en bereken tot die vyfde term ($x(4)$). / *Determine $x(k)$ for the system in* (3.1) *for a unit step input. Use the power series method and determine up to the fifth term ($x(4)$). All initial conditions can be taken as zero.* (5)

(3.3) Gestel dat die uitdrukking vir $X(z)$ gegee word deur: / *Let the expression for $X(z)$ be given as:*

$$X(z) = \frac{1}{(z-0.5)(z-0.8)(z-0.9)}E(z)$$

Bepaal $x(k)$ vir die stelsel in (3.1) in geslote vorm vir 'n eenheidstrapinset deur van parsiële breuk uitbreiding gebruik te maak. / *Determine $x(k)$ for the system in* (3.1) *in closed form for a unit step input using partial fraction expansion.* (6)

(3.4) Bepaal die waarnemer-kanoniese toestandsruimtevoorstelling vir die stelsel. / *Determine the observer canonical state space representation of the sytem.* (5)

Tabelle / Tables

TABLE 2-2 Properties of the *z*-Transform

Sequence	Transform
$e(k)$	$E(z) = \sum_{k=0}^{\infty} e(k)z^{-k}$
$a_1e_1(k) + a_2e_2(k)$	$a_1E_1(z) + a_2E_2(z)$
$e(k-n)u(k-n);\quad n \geq 0$	$z^{-n}E(z)$
$e(k+n)u(k);\quad n \geq 1$	$z^n\left[E(z) - \sum_{k=0}^{n-1} e(k)z^{-k}\right]$
$\varepsilon^{akT}e(k)$	$E(z\varepsilon^{-aT})$
$ke(k)$	$-z\dfrac{dE(z)}{dz}$
$e_1(k) * e_2(k)$	$E_1(z)E_2(z)$
$e_1(k) = \sum_{n=0}^{k} e(n)$	$E_1(z) = \dfrac{z}{z-1}E(z)$
Initial value: $e(0) = \lim_{z\to\infty} E(z)$	
Final value: $e(\infty) = \lim_{z\to 1}(z-1)E(z)$, if $e(\infty)$ exists	

TABLE 2-3 ***z*-Transforms**

Sequence	Transform
$\delta(k-n)$	z^{-n}
1	$\dfrac{z}{z-1}$
k	$\dfrac{z}{(z-1)^2}$
k^2	$\dfrac{z(z+1)}{(z-1)^3}$
a^k	$\dfrac{z}{z-a}$
ka^k	$\dfrac{az}{(z-a)^2}$
$\sin ak$	$\dfrac{z\sin a}{z^2-2z\cos a+1}$
$\cos ak$	$\dfrac{z(z-\cos a)}{z^2-2z\cos a+1}$
$a^k \sin bk$	$\dfrac{az\sin b}{z^2-2az\cos b+a^2}$
$a^k \cos bk$	$\dfrac{z^2-az\cos b}{z^2-2az\cos b+a^2}$

Question 1 [15]

The voltage loop equations can be expressed as follows:

$$\text{Loop 1} : v(t) - R i_1(t) - L\frac{di_1(t)}{dt} - R_1 (i_1(t) + i_2(t)) = 0$$

$$\text{Loop 2} : v(t) - R i_2(t) - L\frac{di_2(t)}{dt} - R_1 (i_1(t) + i_2(t)) = 0$$

$x_1 = i_1(t)$
$x_2 = i_2(t)$

$$\therefore \quad L\dot{x}_1(t) = -R x_1(t) - R_1 [x_1(t) + x_2(t)] + v(t)$$
$$L\dot{x}_2(t) = -R x_2(t) - R_1 [x_1(t) + x_2(t)] + v(t)$$

$$\therefore \quad \dot{x}_1(t) = -\frac{R}{L}x_1(t) - \frac{R_1}{L}x_1(t) - \frac{R_1}{L}x_2(t) + \frac{1}{L}v(t)$$
$$= -\frac{(R+R_1)}{L}x_1(t) - \frac{R_1}{L}x_2(t) + \frac{1}{L}v(t)$$

$$\dot{x}_2(t) = -\frac{R}{L}x_2(t) - \frac{R_1}{L}x_1(t) - \frac{R_1}{L}x_2(t) + v(t)$$
$$= -\frac{(R+R_1)}{L}x_2(t) - \frac{R_1}{L}x_1 + v(t)$$

$$\dot{\mathbb{X}} = \underbrace{\begin{bmatrix} -\frac{(R+R_1)}{L} & -\frac{R_1}{L} \\ -\frac{R_1}{L} & -\frac{(R+R_1)}{L} \end{bmatrix}}_{\mathbb{A}} \mathbb{X} + \underbrace{\begin{bmatrix} 1/L \\ 1/L \end{bmatrix}}_{\mathbb{B}} u(t)$$

$y = x_1 + x_2$ since the current through R_1 is $i_1(t) + i_2(t)$

$$y = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{\mathbb{C}} \mathbb{X}$$

Controllability: For controllability $\det \mathbb{P}_c \neq 0$

for this system $\mathbb{P}_c = [\mathbb{B} \quad \mathbb{A}\mathbb{B}]$

$$\mathbb{A}\mathbb{B} = \begin{bmatrix} -\frac{(R+R_1)}{L} & -\frac{R_1}{L} \\ -\frac{R_1}{L} & -\frac{(R+R_1)}{L} \end{bmatrix} \times \begin{bmatrix} 1/L \\ 1/L \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{(R+R_1)}{L^2} - \frac{R_1}{L^2} \\ -\frac{R_1}{L^2} - \frac{(R+R_1)}{L^2} \end{bmatrix} = \begin{bmatrix} -\frac{R+2R_1}{L^2} \\ -\frac{R+2R_1}{L^2} \end{bmatrix}$$

$$\mathbb{P}_c = \begin{bmatrix} -\frac{1}{L} & -\frac{R+2R_1}{L^2} \\ -\frac{1}{L} & -\frac{R+2R_1}{L^2} \end{bmatrix}$$ ✓✓

$$\det \mathbb{P}_c = \frac{R+2R_1}{L^3} - \frac{R+2R_1}{L^3} = 0$$

∴ Not controllable ✓

Observability $\mathbb{P}_o \neq 0$ wher $\mathbb{P}_o = \begin{bmatrix} \mathbb{C} \\ \mathbb{C}\mathbb{A} \end{bmatrix}$

$$\mathbb{C}\mathbb{A} = \overset{1\times 2}{[1 \quad 1]} \overset{2\times 2}{\begin{bmatrix} -\frac{(R+R_1)}{L} & -\frac{R_1}{L} \\ -\frac{R_1}{L} & -\frac{(R+R_1)}{L} \end{bmatrix}}$$

$$= \left[-\frac{R+R_1}{L} - \frac{R_1}{L} \quad -\frac{R_1}{L} - \frac{(R+R_1)}{L} \right]$$

$$= \left[-\frac{R+2R_1}{L} \quad -\frac{R+2R_1}{L} \right]$$

$$\mathbb{P}_o = \begin{bmatrix} 1 & 1 \\ -\frac{R+2R_1}{L} & -\frac{R+2R_1}{L} \end{bmatrix}$$

$$\det \mathbb{P}_o = -\frac{R+2R_1}{L} + \frac{R+2R_1}{L}$$

$$= 0$$

$\therefore$ Not observable.

Question 2 [10]

$$\dot{\mathbb{X}}(t) = \begin{bmatrix} 0 & 1 \\ -10{,}5 & -11{,}3 \end{bmatrix} \mathbb{X} + \begin{bmatrix} 0 \\ 0{,}55 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbb{X}(t)$$

Ackermann's Formula: $\mathbb{K} = [0 \quad 1] \mathbb{P}_c^{-1} q(A)$

Desired characteristic equation

$$q(\lambda) = \lambda^2 + \alpha_1 \lambda + \alpha_0$$

$T_s = \frac{4}{\zeta \omega_n}$ and $\zeta = \sqrt{\frac{\ln^2\left(\frac{P.O}{100}\right)}{\ln^2\left(\frac{P.O}{100}\right) + \pi^2}}$

$e_{ss} = 0$, P.O. = 5%

$T_s = 1\,s \Rightarrow \frac{4}{\zeta \omega_n} = 1$

$$\zeta = \sqrt{\frac{\ln^2(0{,}05)}{\ln^2(0{,}05) + \pi^2}} = \sqrt{\frac{8{,}974}{8{,}974 + 9{,}8696}}$$

$$= 0{,}69 \approx 0{,}7 \quad ✓$$

$4 = \zeta \omega_n$

$\omega_n = 4 / 0{,}7 = 5{,}714$ rad/s ✓

$$q(\lambda) = \lambda^2 + 2\zeta\omega_n \lambda + \omega_n^2$$
$$= \lambda^2 + 2(0{,}7)(5{,}714)\lambda + (5{,}714)^2$$
$$= \lambda^2 + 8\lambda + 32{,}65 \quad ✓$$

(5)

$\therefore \quad \alpha_1 = 8 \quad \text{and} \quad \alpha_0 = 32,65$

$$q(A) = A^2 + \alpha_1 A + \alpha_0 I \quad ✓$$

$$= \begin{bmatrix} 0 & 1 \\ -10,5 & -11,3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -10,5 & -11,3 \end{bmatrix} + 8\begin{bmatrix} 0 & 1 \\ -10,5 & -11,3 \end{bmatrix}$$

$$+ 32,65 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (0 \; -10,5) & (0 - 11,3) \\ 0 \; +(11,3 \cdot 10,5) & (-10,5 + 11,3^2) \end{bmatrix} + \begin{bmatrix} 0 & 8 \\ -84 & -90,4 \end{bmatrix}$$

$$+ \begin{bmatrix} 32,65 & 0 \\ 0 & 32,65 \end{bmatrix}$$

$$= \begin{bmatrix} -10,5 & -11,3 \\ 118,65 & 117,19 \end{bmatrix} + \begin{bmatrix} 0 & 8 \\ -84 & -90,4 \end{bmatrix} + \begin{bmatrix} 32,65 & 0 \\ 0 & 32,65 \end{bmatrix}$$

$$= \begin{bmatrix} 22,15 & -3,3 \\ 34,65 & 59,44 \end{bmatrix} \quad ✓$$

$$P_c = [B \quad AB] = \begin{bmatrix} 0 & 0,55 \\ 0,55 & -6,215 \end{bmatrix} \quad ✓$$

$$P_c^{-1} = \begin{bmatrix} 20,5455 & 1,8182 \\ 1,8182 & 0 \end{bmatrix} \quad ✓$$

$$K = [0 \quad 1]\begin{bmatrix} 20,5455 & 1,8182 \\ 1,8182 & 0 \end{bmatrix}\begin{bmatrix} 22,15 & -3,3 \\ 34,65 & 59,44 \end{bmatrix} \quad ✓$$

$$= [40,2731 \quad -6,001]$$

$K_1 = 40,2731$ ✓

$K_2 = -6,001$ ✓ □

Question 3

(3.1) $x(k+3) - 2,2\,x(k+2) + 1,57\,x(k+1) - 0,36\,x(k) = e(k)$

$$X(z)z^3 - 2,2X(z)z^2 + 1,57X(z)z - 0,36\,X(z) = E(z)$$ ✓

$$X(z)\left[z^3 - 2.2z^2 + 1,57z - 0,36\right] = E(z)$$ ✓

$$\frac{X(z)}{E(z)} = \frac{1}{z^3 - 2.2z^2 + 1,57z - 0,36} \qquad (2)$$

3.2 $$X(z) = \frac{1}{z^3 - 2.2z^2 + 1,57z - 0,36}\,E(z)$$

$$= \frac{1}{z^3 - 2.2z^2 + 1,57z - 0,36}\,\frac{z}{z-1}$$ ✓

$$= \frac{z}{z^4 - 3.2z^3 + 3.77z^2 - 1,93z + 0,36}$$

✓

$$\begin{array}{r|l} & z^{-3} + 3,2z^{-4} \\ z^4 - 3,2z^3 + 3,77z^2 - 1,93z + 0,36 & z \\ & \underline{z - 3,2 + 3,77z^{-1} - 1,93z^{-2} + 0,36z^{-3}} \\ & 3,2 - 3,77z^{-1} + 1,93z^{-2} - 0,36z^{-3} \\ & 3,2 - 10,24z^{-1} + 12,064z^{-2} - 6,176z^{-3} \\ & + 1,152z^{-4} \end{array}$$

✓

$\therefore$ $x(0) = 0$

$x(1) = 0$

$x(2) = 0$

$x(3) = 1$ ✓

$x(4) = 3,2$ ✓

(5)

(3.3) $$X(z) = \frac{z}{(z-0,5)(z-0,8)(z-0,9)(z-1)}$$

$$\frac{X(z)}{z} = \frac{A}{(z-0,5)} + \frac{B}{(z-0,8)} + \frac{C}{(z-0,9)} + \frac{D}{(z-1)}$$

$A = 1/(z-0,8)(z-0,9)(z-1)\big|_{z=0,5}$

$= -16,67$

$B = 1/(z-0,5)(z-0,9)(z-1)\big|_{z=0,8}$ ✓✓

$= 166,67$

$C = 1/(z-0,5)(z-0,8)(z-1)\big|_{z=0,9}$

$= -250$

$D = 1/(z-0,5)(z-0,8)(z-0,9)\big|_{z=1}$

$= 100$

$$X(z) = -16,67\frac{z}{z-0,5} + 166,67\frac{z}{z-0,8} - 250\frac{z}{z-0,9} + 100\frac{z}{z-1}$$

✓ ✓

$$x(k) = 100 - 16{,}67\,(0{,}5)^k + 166{,}67\,(0{,}8)^k - 250\,(0{,}9)^k$$

$x(0) = 0$

$x(1) = 0{,}001 \approx 0$

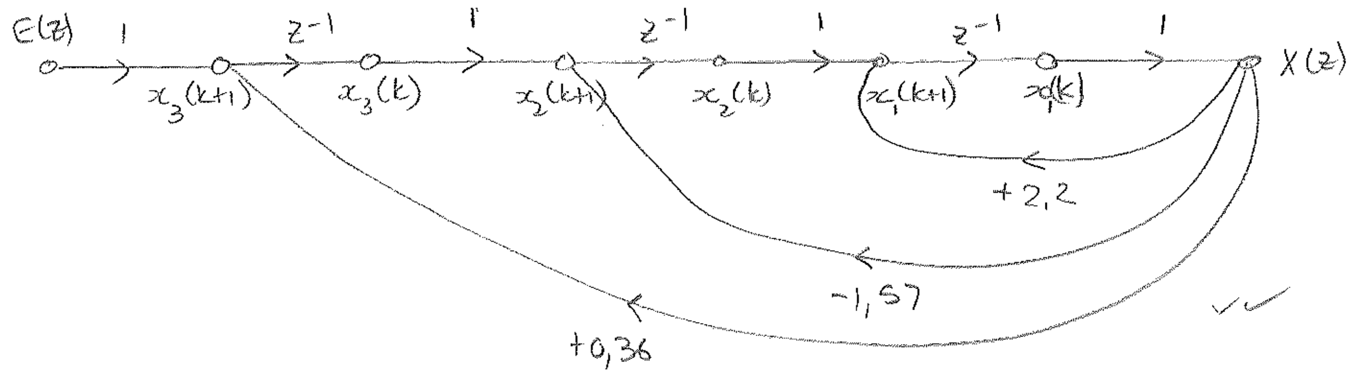
$x(2) = 0{,}0013 \approx 0$ To check

$x(3) = 1{,}00129 \approx 1$ (6)

$x(4) = 3{,}2$

3.4
$$\frac{X(z)}{E(z)} = \frac{1}{z^3 - 2.2z^2 + 1{,}57z - 0{,}36}$$

$$= \frac{z^{-3}}{1 - 2{,}2z^{-1} + 1{,}57z^{-2} - 0{,}36z^{-3}}$$



$x_1(k+1) = x_2(k) + 2{,}2\,x_1(k)$ $\quad y(k) = x_1(k)$

$x_2(k+1) = x_3(k) - 1{,}57\,x_1(k)$

$x_3(k+1) = 0{,}36\,x_1(k) + e(k)$

$$X(k+1) = \begin{bmatrix} 2{,}2 & 1 & 0 \\ -1{,}57 & 0 & 1 \\ 0{,}36 & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e(k)$$

(5)

$$y(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(k)$$