P11.6 - Ackermann's Formula

A dynamic system is represented by

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x} + [0]\, u.$$

We want to place the closed-loop poles at $s = -2 \pm j2$. Determine the required state variable feedback using Ackermann's formula. Assume that the complete state vector is available for feedback.

We require the desired characteristic equation to be

$$q(\lambda) = (\lambda + 2 + j2)(\lambda + 2 - j2)$$
$$= \lambda^2 + 2\lambda - 2j\lambda + 4 + 2\lambda - j4 + 2j\lambda + 4j - j^2 4$$
$$= \lambda^2 + 4\lambda + 8$$

$$q(\lambda) = \lambda^2 + \alpha_1 \lambda + \alpha_0 \quad \Rightarrow \alpha_1 = 4 \quad \alpha_0 = 8$$

$\mathbb{P}_c$:

$$\mathbb{P}_c = \begin{bmatrix} \mathbb{B} & \mathbb{A}\mathbb{B} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\mathbb{A}\mathbb{B} = \underset{2\times 2}{\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\det \mathbb{P}_c = 1 - 0$$
$$= 1 \neq 0 \quad \therefore \text{ controllable.}$$

$$\mathbb{K} = [0 \ 0 \ \dots \ 0 \ 1] \mathbb{P}_c^{-1} q(\mathbb{A})$$

For $n=2$

$$\Rightarrow \mathbb{K} = [0 \quad 1] \mathbb{P}_c^{-1} q(\mathbb{A})$$

$$= [0 \quad 1] \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}^{-1} q(\mathbb{A})$$

$$\mathbb{P}_c^{-1} = \frac{1}{|\mathbb{P}_c|} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{1 \cdot 1 - (-2 \cdot 0)} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{1} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$q(\mathbb{A}) = \mathbb{A}^2 + \alpha_1 \mathbb{A} + \alpha_0 \mathbb{I}$$

$$= \overset{2\times 2}{\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}} \overset{2\times 2}{\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}} + 4 \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} + 8 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \cdot -2 + 0 \cdot 1 & -2 \cdot 0 + 0 \cdot 0 \\ 1 \cdot -2 + 0 \cdot 1 & 1 \cdot 0 + 0 \cdot 0 \end{bmatrix} + \begin{bmatrix} -8 & 0 \\ 4 & 0 \end{bmatrix} + \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} +4 & 0 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} -8 & 0 \\ 4 & 0 \end{bmatrix} + \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 2 & 8 \end{bmatrix}$$

$$\mathbb{K} = \overset{1\times 2}{[0 \quad 1]} \overset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 4 & 0 \\ 2 & 8 \end{bmatrix}$$

$$= [0 \quad 1] \begin{bmatrix} 4 & 0 \\ 2 & 8 \end{bmatrix}$$

$$= [2 \quad 8] \Rightarrow K_1 = 2, \ K_2 = 8$$