Ex 11.5 - Design of a third-order system

Let us consider the third-order system with the differential equation

$$\frac{d^3y}{dt^3} + 5\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = u. \tag{1}$$

Design a state variable feedback controller with a minimal overshoot, implying $\zeta = 0.8$ and settling time (with a 2 % criterion) equal to 1 second.

Select state variable as: $x_1 = y$ $\quad x_2 = \frac{dy}{dt}$ $\quad x_3 = \frac{d^2y}{dt^2}$

$\therefore \quad \dot{x}_3 + 5x_3 + 3x_2 + 2x_1 = u$

$$\dot{x}_3 = -2x_1 - 3x_2 - 5x_3 + u$$

$$\dot{x}_1 = \frac{dy}{dt} = x_2$$

$$\dot{x}_2 = \frac{d^2y}{dt^2} = x_3$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad \text{and} \quad y = x_1$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

And the characteristic equation is:

$$\Delta(\lambda) = \det\left(\lambda \mathbb{I} - (\mathbb{A} - \mathbb{B}\mathbb{K})\right) = 0$$

$$= \det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2-k_1 & -3-k_2 & -5-k_3 \end{bmatrix}\right)$$

$$= \det\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 2+k_1 & 3+k_2 & \lambda+5+k_3 \end{bmatrix}$$

$$= \lambda \begin{vmatrix} \lambda & -1 \\ 3+k_2 & \lambda+5+k_3 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 \\ 2+k_1 & (\lambda+5+k_3) \end{vmatrix} + 0 \begin{vmatrix} 0 & \lambda \\ 2+k_1 & 3+k_2 \end{vmatrix}$$

$$= \lambda\left[(\lambda^2 + 5\lambda + k_3\lambda) - (-3-k_2)\right] + 1\left[0 - (-2-k_1)\right] + 0$$

$$= \lambda\left[\lambda^2 + 5\lambda + k_3\lambda + 3 + k_2\right] + 2 + k_1$$

$$= \lambda^3 + 5\lambda^2 + k_3\lambda^2 + 3\lambda + k_2\lambda + 2 + k_1$$

$$= \lambda^3 + (5+k_3)\lambda^2 + (3+k_2)\lambda + (2+k_1) = 0$$

If the state variable feedback matrix is

$$\mathbb{K} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

and $$u = -\mathbb{K}\mathbf{x}$$

Then the closed-loop system is

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + \mathbb{B}u \quad , \text{ where } u = -\mathbb{K}\mathbf{x} \\ &= A\mathbf{x} + \mathbb{B}(-\mathbb{K}\mathbf{x}) \\ &= A\mathbf{x} - \mathbb{B}\mathbb{K}\mathbf{x} \\ &= (A - \mathbb{B}\mathbb{K})\mathbf{x} \end{aligned}$$

$3\times1 \quad 1\times3$

$\therefore$ The state feedback matrix is

$$\begin{aligned} [A - \mathbb{B}\mathbb{K}] &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -5 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -5 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_1 & k_2 & k_3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2-k_1 & -3-k_2 & -5-k_3 \end{bmatrix} \end{aligned}$$

The desired characteristic eq. for a 3rd-order system

$$\Delta(\lambda) = (\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2)(\lambda + \zeta\omega_n)$$

$\zeta = 0{,}8$

$$T_s = \frac{4}{\zeta\omega_n} = 1$$

$$\Rightarrow \frac{4}{0{,}8\,\omega_n} = 1$$

$$4 = 0{,}8\,\omega_n$$

$$\omega_n = 4/0{,}8 = 5$$

handboek kies $\omega_n = 6$

$$\therefore (\lambda^2 + 2(0{,}8)(5)\lambda + (5)^2)(\lambda + (0{,}8)(5)) = 0$$

$$(\lambda^2 + 8\lambda + 25)(\lambda + 4) = \lambda^3 + 8\lambda^2 + 25\lambda + 4\lambda^2 + 32\lambda + 100$$

$$= \lambda^3 + 12\lambda^2 + 57\lambda + 100$$

$\therefore$

$5 + k_3 = 12 \Rightarrow k_3 = 7$

$3 + k_2 = 57 \Rightarrow k_2 = 54$

$2 + k_1 = 100 \Rightarrow k_1 = 98$