The Design of State Variable Feedback Systems

Chapter 11 of Dorf and Bishop (Study unit 1)

Presented by Prof. KR Uren

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Introduction

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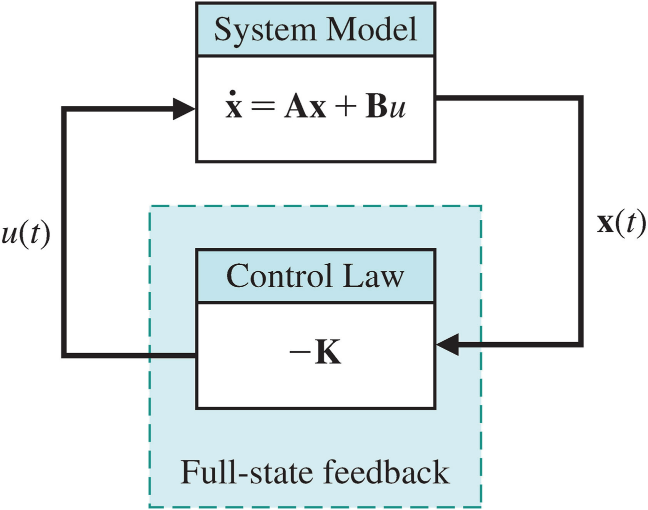
- Time domain method, expressed in state variables
- State variables can also be used for compensation
- We are interested in controlling the system with control signal

$$u(t) = f(\mathbf{x}(t))$$

Introduction



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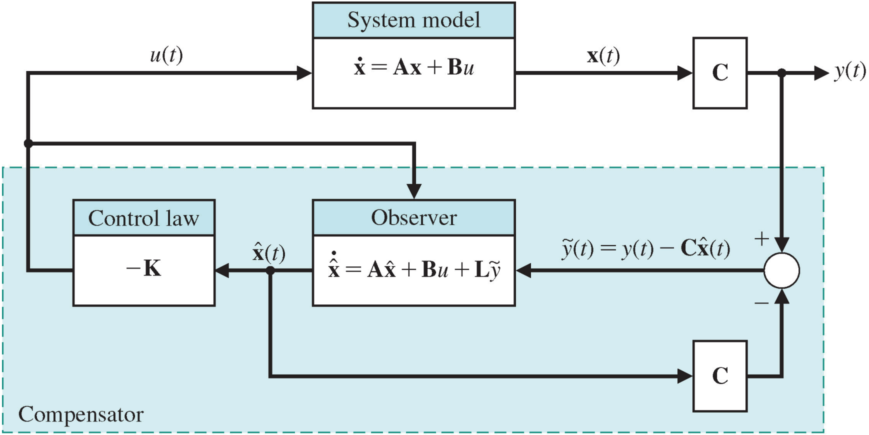
State variable control design

State variable design typically comprises 3 steps:

- Assume all states are measurable (not practical), use them in **full-state feedback control**
- Construct an **observer** to estimate the states
  - Full-state feedback observers
  - Reduced-order observers
- Appropriately connect the observer to the full-state feedback control law

State variable control design



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State variable control design

- The state-variable controller (full-state control law plus the observer) is called a **compensator**
- Additionally it is possible to consider reference inputs to the state variable compensator

A key question:

- Can all the poles of the closed-loop system be arbitrarily placed in the complex plane?
- Remember: the poles of the closed-loop system are equivalent to the **eigenvalues** of the **system matrix** in state variable format

Controllability and Observability

- If as system is **controllable** and **observable**, then we can accomplish the design objective of placing the poles precisely at the desired locations to meet the performance specifications.
- Full-state feedback design commonly relies on **pole-placement** techniques.

Determining Controllability

Definition

A system is **completely controllable** if there exists an unconstrained control input $u(t)$ that can transfer any initial state $\mathbf{x}_0(t)$ to any other desired location $\mathbf{x}$ in a finite time, $t_0 \le t \le T$.

For a system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{1}$$

we can check for controllability by examining either of the following conditions:

- rank $\begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A^2B} & \dots & \mathbf{A^{n-1}B} \end{bmatrix} = n$
- $\det \mathbf{P}_c \neq 0$, where $\mathbf{P}_c = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A^2B} & \dots & \mathbf{A^{n-1}B} \end{bmatrix}$

For a single-input, single-output system, $\mathbf{P}_c$ is called the **controllability matrix**.

Determining Observability

Definition

A system is **completely observable** if and only if there exists a finite time T such that the initial state $\mathbf{x}(0)$ can be determined from the observation history $y(t)$ given the control $u(t)$, $0 \leq t \leq T$.

Determining Observability

For a system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u,\ \ y(t) = \mathbf{C}\mathbf{x}, \tag{2}$$

we can check for observability by considering the following condition:

$$\det \mathbf{P}_o \neq 0 \tag{3}$$

where

$$\mathbf{P}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}, \tag{4}$$

For a single-input, single-output system, $\mathbf{P}_o$ is called the **observability matrix**.

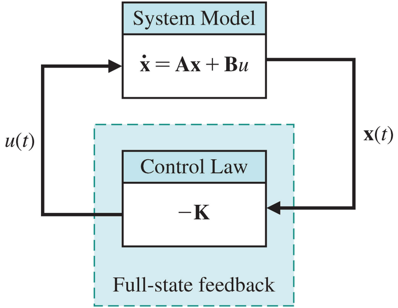
Full-state feedback control design

- First step requires us to assume that **all states are available** for feedback. $\Rightarrow \mathbf{x}(t)$ for all t is available
- The system input is then given by $u = -\mathbf{K}\mathbf{x}$
- The objective is to determine the gain matrix $\mathbf{K}$

Full-state feedback control design

- The system is defined by $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$
- The control feedback is given by $u = -\mathbf{K}\mathbf{x}$
- The closed-loop system is therefore givne by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$$



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- The characteristic equation is given by
$$\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{BK})) = 0$$
- If all the roots of the characteristic equation lie in the left half-plane, then the closed-loop system is stable
$$\mathbf{x}(t) = e^{(\mathbf{A}-\mathbf{BK})t}\mathbf{x}(t_0) \to 0 \text{ as } t \to \infty$$
- The addition of a reference input can be written as
$$u(t) = -\mathbf{K}\mathbf{x}(t) + Nr(t),$$
where $r(t)$ is the reference input.

Ackermann's formula

- For a single input,single-output system, Ackermann's formula is useful for determining the state variable feedback matrix

$$\mathbf{K} = \left[\begin{array}{cccc} k_1 & k_2 & \ldots & k_n \end{array} \right]$$

where

$$u = -\mathbf{K}\mathbf{x}.$$

Ackermann's formula

- Given the desired characteristic equation

$$q(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_0,$$

the state feedback gain matrix is

$$\mathbf{K} = \left[\begin{array}{cccccc} 0 & 0 & \ldots & 0 & 1 \end{array}\right] \mathbf{P}_c^{-1} q(\mathbf{A}),$$

where

$$q(\mathbf{A}) = \mathbf{A}^n + \alpha_{n-1}\mathbf{A}^{n-1} + \ldots + \alpha_1\mathbf{A} + \alpha_0\mathbf{I}.$$

Observers

- In practical systems only a subset of the states are readily measurable and available for feedback
- Cost and complexity of a control system increases as the number of sensor increases. This may be another opportunity to use observers.

Observers

The full-state observer for the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{5}$$

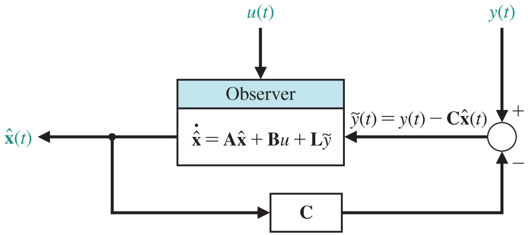
$$y = \mathbf{C}\mathbf{x} \tag{6}$$

is given by

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \mathbf{C}\hat{\mathbf{x}}) \tag{7}$$

where $\hat{\mathbf{x}}$ denotes the estimate of the state $\mathbf{x}$. The matrix $\mathbf{L}$ is the observer gain matrix and is to be determined as part of the observer design.

Observers



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- The observer has two inputs, u and y, and one output, $\hat{\mathbf{x}}$.
- The goal of the observer is to provide an estimate $\hat{\mathbf{x}}$ so that $\hat{\mathbf{x}} \to \mathbf{x}$ as $t \to \infty$

Observers

- Remember we do not know $\mathbf{x}(t_0)$ percisely, therefore we must provide and initial estimate $\hat{\mathbf{x}}(t_0)$ to the observer.
- We define the observer **estimation error** as

$$\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t). \tag{8}$$

- The observer design should produce an observer with the property that $\mathbf{e}(t) \to 0$ as $t \to \infty$
- **If the system is completely observable, we can always find $\mathbf{L}$ so that the tracking error is asymptotically stable**.

Taking the time derivative of the estimation error yields

$$\begin{aligned}
\dot{\mathbf{e}} &= \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} && (9)\\
&= [\mathbf{A}\mathbf{x} + \mathbf{B}u] - [\mathbf{A}\hat{\mathbf{x}} - \mathbf{B}u - \mathbf{L}(y - \mathbf{C}\hat{\mathbf{x}})] && (10)\\
&= \mathbf{A}\mathbf{x} - \mathbf{A}\hat{\mathbf{x}} - \mathbf{L}y + \mathbf{L}\mathbf{C}\hat{\mathbf{x}} && (11)\\
&= (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{x} - (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}} && (12)\\
&= (\mathbf{A} - \mathbf{L}\mathbf{C})(\mathbf{x} - \hat{\mathbf{x}}) && (13)\\
&= (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e} && (14)
\end{aligned}$$

We can guarantee that $\mathbf{e}(t) \rightarrow 0$ as $t \rightarrow \infty$ for any initial tracking error $\mathbf{e}(t_0)$ if the characteristic equation

$$\det(\lambda \mathbf{I} - (\mathbf{A} - \mathbf{LC})) = 0 \tag{15}$$

has all its roots in the **left half-plane**. Therefore, the observer design process reduces to finding the matrix $\mathbf{L}$ such that the roots of the characteristic equation above lie in the left half-plane. This can always be accomplished if the system is completely observable.

Ackermann's formula can also be employed to place the roots of the observer characteristic equation at the desired locations. Consider the observer gain matrix

$$\mathbf{L} = \begin{bmatrix} L_1 & L_2 & \dots & L_n \end{bmatrix}^T \tag{16}$$

and the desired observer characteristic equation

$$p(\lambda) = \lambda^n + \beta_{n-1}\lambda^{n-1} + \ldots + \beta_1\lambda + \beta_0 \tag{17}$$

The β's are selected to meet given performance specifications for the observer. The observer gain matrix is then computed via

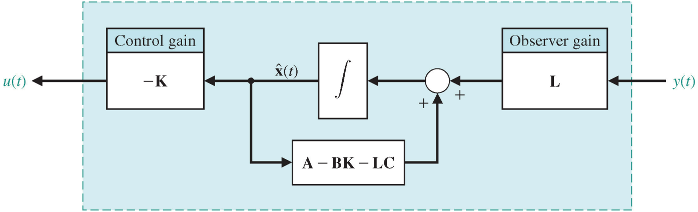
$$\boxed{\mathbf{L} = p(\mathbf{A})\mathbf{P}_o^{-1} \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix}^T,} \tag{18}$$

where $\mathbf{P}_o$ is the observability matrix and

$$p(\mathbf{A}) = \mathbf{A}^n + \beta_{n-1}\mathbf{A}^{n-1} + \ldots + \beta_1\mathbf{A} + \beta_0\mathbf{I} \tag{19}$$

Integrated Full-state feedback and observer

Now we want to connect the full-state feedback control law to the observer. The compensator is shown below



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- We have the control law $u(t) = -\mathbf{K}\mathbf{x}(t)$, but
- we now estimate the state $\hat{\mathbf{x}}(t)$, to obtain the feedback law $u(t) = -\mathbf{K}\hat{\mathbf{x}}(t)$

Is $u(t) = -\mathbf{K}\hat{\mathbf{x}}(t)$ a good idea ?

The question is can the control law with the estimated states retain stability? Consider the observer

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \mathbf{C}\hat{\mathbf{x}}) \tag{20}$$

where $u(t) = -\mathbf{K}\hat{\mathbf{x}}(t)$, then

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}(-\mathbf{K}\hat{\mathbf{x}}) + \mathbf{L}y - \mathbf{L}\mathbf{C}\hat{\mathbf{x}} && (21) \\ &= \mathbf{A}\hat{\mathbf{x}} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} + \mathbf{L}y - \mathbf{L}\mathbf{C}\hat{\mathbf{x}} && (22) \\ &= (\mathbf{A}\hat{\mathbf{x}} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C}) + \mathbf{L}y && (23) \end{aligned}$$

Is $u(t) = -\mathbf{K}\hat{\mathbf{x}}(t)$ a good idea ?

For the underlying system model given by

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad (24) \\ y &= \mathbf{C}\mathbf{x} \qquad (25) \end{aligned}$$

we have $u(t) = -\mathbf{K}\hat{\mathbf{x}}$, resulting in

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} \qquad (26)$$

and since $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$, which implies $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{e}$, we can write

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}(\mathbf{x} - \mathbf{e}) \qquad (27) \\ &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{e} \qquad (28) \end{aligned}$$

Since we obtained previously that

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{LC})\mathbf{e} \tag{29}$$

and

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{BKe} \tag{30}$$

we can write

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A} - \mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} \tag{31}$$

Is $u(t) = -\mathbf{K}\hat{\mathbf{x}}(t)$ a good idea ?

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So the characteristic equation of the compensator consisting of the feedback controller and the observer is given by

$$\Delta(\lambda) = \det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{BK}))\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{LC})) \tag{32}$$

So

- If the roots of $\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{BK})) = 0$ lie in the left half-plain, and
- If the roots of $\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{LC})) = 0$ lie in the left half-plain, then

the overall system is stable. **Therefore employing the strategy of using the state estimates for the feedback is in fact a good strategy.**

1. Determine $\mathbf{K}$ such that the roots of $\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{BK})) = 0$ lie in the left half-plane. Place the poles approximately to meet the design specification.
2. Determine $\mathbf{L}$ such that the roots of $\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{LC})) = 0$ lie in the left half-plane. Place the poles to achieve acceptable observer performance.
3. Connect the observer to the full-state feedback law using $u(t) = -\mathbf{K}\hat{\mathbf{x}}(\mathbf{t})$

Reference inputs

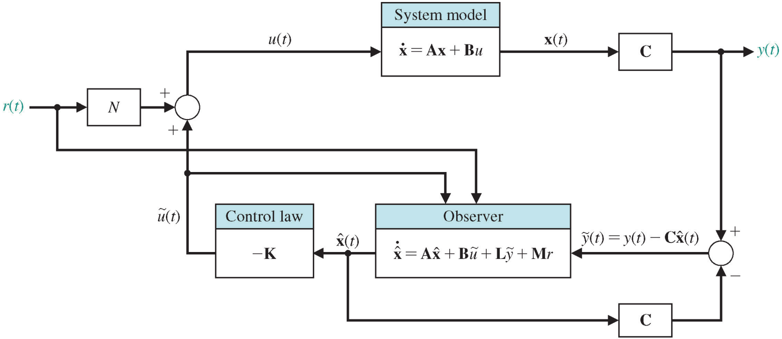
- Previous controller designs were constructed without considering a reference input $\implies r(t) = 0$. These are called **regulators**
- **Command following** is also an important aspect of feedback design.
- Next we are going to consider how we are going to add a reference signal into the state variable feedback compensator.

Reference inputs

Two common methods for adding a reference input are now going to be discussed:

1. Compensator is in the feedback loop
2. Compensator in the forward path

State variable compensator with a reference input



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The general form of the state variable feedback compensator is

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\tilde{u} + \mathbf{L}\tilde{y} + \mathbf{M}r \\ u &= \tilde{u} + Nr = -\mathbf{K}\hat{\mathbf{x}} + Nr \end{aligned} \tag{33}$$

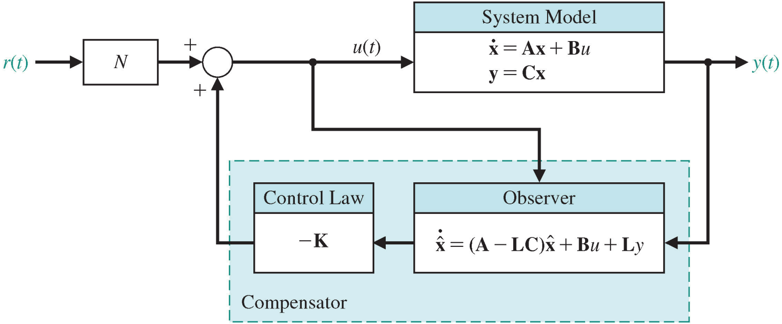
where $\tilde{y} = y - \mathbf{C}\hat{\mathbf{x}}$ and $\tilde{u} = -\mathbf{K}\hat{\mathbf{x}}$. Notice that if $\mathbf{M} = \mathbf{0}$ and $N = 0$ then the compensator is a regulator.

Key design parameters

- The key design parameters for implementing the command tracking of the reference input are $\mathbf{M}$ and N.
- If the reference input is a scalar, the parameter $\mathbf{M}$ is a column vector of length n, where n is the length of the state vector $\mathbf{x}$, and N is a scalar

Controller in the feedback loop

In this case we select $\mathbf{M}$ and N so that the estimation error $\mathbf{e}(t)$ is independent of the ference input $r(t)$



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Controller in the feedback loop

The estimation error is found to be described by the differential equation

$$
\begin{aligned}
\dot{\mathbf{e}} &= \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} && (34)\\
&= \mathbf{A}\mathbf{x} + \mathbf{B}u - \mathbf{A}\hat{\mathbf{x}} - \mathbf{B}\tilde{u} - \mathbf{L}\tilde{y} - \mathbf{M}r && (35)\\
&= (\mathbf{A}\mathbf{x} - \mathbf{A}\hat{\mathbf{x}}) + \mathbf{B}u - \mathbf{B}\tilde{u} - \mathbf{L}(y - \mathbf{C}\hat{\mathbf{x}}) - \mathbf{M}r && (36)\\
&= (\mathbf{A}\mathbf{x} - \mathbf{A}\hat{\mathbf{x}}) + \mathbf{B}(\tilde{u} + Nr) - \mathbf{B}\tilde{u} - \mathbf{L}y + \mathbf{L}\mathbf{C}\hat{\mathbf{x}} && (37)\\
&- \mathbf{M}r\\
&= (\mathbf{A}\mathbf{x} - \mathbf{A}\hat{\mathbf{x}}) + \mathbf{B}Nr - \mathbf{M}r - \mathbf{L}\mathbf{C}(\mathbf{x} - \hat{\mathbf{x}}) && (38)\\
&= \mathbf{A}\mathbf{e} + (\mathbf{B}N - \mathbf{M})r - \mathbf{L}\mathbf{C}(\mathbf{x} - \hat{\mathbf{x}}) && (39)\\
&= (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e} + (\mathbf{B}N - \mathbf{M})r && (40)
\end{aligned}
$$

Suppose we select

$$\mathbf{M} = \mathbf{B}N \tag{41}$$

Then the corresponding estimation error is given by

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e} \tag{42}$$

The remaining task is to determine a suitable value of N, since the value of $\mathbf{M}$ follows from (41). For example, we might choose N to obtain a zero steady-state tracking error to a step input $r(t)$.

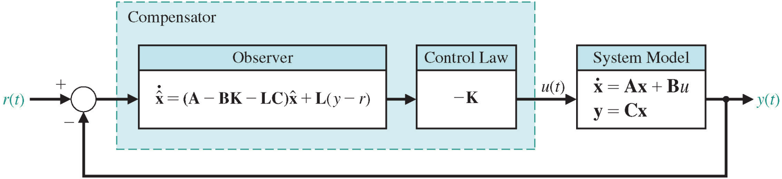
Controller in the feedback loop

With $\mathbf{M} = \mathbf{B}N$, we find that the compensator is given by

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}\tilde{y} && (43) \\ u &= -\mathbf{K}\hat{\mathbf{x}} + Nr && (44) \end{aligned}$$

Controller in the forward path

In this case we select $\mathbf{M}$ and N so that the tracking error $r(t) - y(t)$ is used as input to the compensator.



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As an alternative approach, suppose that we select $N = 0$ and $\mathbf{M} = -\mathbf{L}$. Then, the compensator equation is given by

$$
\begin{aligned}
\dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}\tilde{y} - \mathbf{L}r & (45)\\
&= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}(-\mathbf{K}\hat{\mathbf{x}} + Nr) + \mathbf{L}(y - \mathbf{C}\hat{\mathbf{x}}) - \mathbf{L}r & (46)\\
&= \mathbf{A}\hat{\mathbf{x}} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} - \mathbf{L}\mathbf{C}\hat{\mathbf{x}} + \mathbf{B}Nr + \mathbf{L}(y - r) & (47)\\
&= \mathbf{A}\hat{\mathbf{x}} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} - \mathbf{L}\mathbf{C}\hat{\mathbf{x}} + \mathbf{L}(y - r), \text{since } N = 0 & (48)\\
u &= -\mathbf{K}\hat{\mathbf{x}} + Nr & (49)
\end{aligned}
$$

THE END