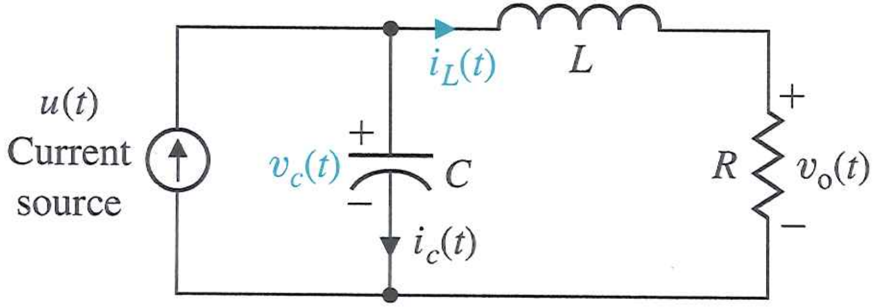
Exercise 1

Derive the state space model and transfer function of the following system: $x_1(t) = v_C(t)$, $x_2(t) = i_L(t)$ where $u(t)$ is the input and $v_0(t)$ is the output.



State-space model. $x_1 = V_C \;, x_2 = i_L$ $y = v_0 = i_L R = x_2 R$

Remember:

$V_L = L\dfrac{di_L}{dt}$

$i_L = \dfrac{1}{L}\int_0^t V_L\,dt$

$V_C = \dfrac{1}{C}\int_0^t i_c\,dt$

$i_c = C\dfrac{dV_c}{dt}$

$\Sigma\, i = 0$

$\Rightarrow \quad u(t) - i_c(t) - i_L(t) = 0$

$u(t) - C\dot{x}_1 - x_2 = 0$

$C\dot{x}_1 = -x_2 + u(t)$

$\dot{x}_1 = -\frac{1}{C}x_2 + \frac{1}{C}u(t)$

$\Sigma\, v = 0$

$V_c - V_L - V_0 = 0$

$x_1 - L\dot{x}_2 - x_2 R = 0$

$-L\dot{x}_2 = x_2 R - x_1$

$\dot{x}_2 = \frac{1}{L}x_1 - \frac{R}{L}x_2$

$$\therefore \dot{\mathbb{X}} = \begin{bmatrix} 0 & -1/C \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix}\mathbb{X} + \begin{bmatrix} 1/C \\ 0 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 0 & R \end{bmatrix}\mathbb{X} + [0]\, u(t)$$

Fast way to determine transfer function from state space model

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$$G(s) = \frac{Y(s)}{U(s)} \quad \Rightarrow \quad G(s) = \mathbb{C}\,\Phi(s)\,\mathbb{B} + \mathbb{D}$$

$$\Phi(s) = [s\mathbb{I} - \mathbb{A}]^{-1}$$

$$= \left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & -1/C \\ \frac{1}{L} & -R/L \end{pmatrix}\right]^{-1} = \begin{bmatrix} s & \frac{1}{C} \\ -1/L & (s+R/L) \end{bmatrix}^{-1}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\therefore \left[s\mathbb{I} - A \right]^{-1} = \frac{1}{s\left(s+\frac{R}{L}\right)+\frac{1}{LC}} \begin{bmatrix} \left(s+\frac{R}{L}\right) & -1/C \\ \frac{1}{L} & s \end{bmatrix}$$

Let $\Delta(s) = s^2 + \frac{R}{L}s + \frac{1}{LC}$.

$$\therefore G(s) = \overset{1\times 2}{\begin{bmatrix} 0 & R \end{bmatrix}} \overset{2\times 2}{\begin{bmatrix} \frac{s+\frac{R}{L}}{\Delta(s)} & \frac{-1}{C\,\Delta(s)} \\ \frac{1}{L\,\Delta(s)} & \frac{s}{\Delta(s)} \end{bmatrix}} \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$$

$$= \overset{1\times 2}{\begin{bmatrix} \frac{R}{L\,\Delta(s)} & \frac{Rs}{\Delta s} \end{bmatrix}} \overset{2\times 1}{\begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}}$$

$$= \frac{R}{L\Delta s} \cdot \frac{1}{C}$$

$$= \frac{R/LC}{\Delta(s)}$$

$$= \frac{R/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \quad \square$$