P 11.11 - Full-state variable control

The state variable model of a plant to be controlled is

$$\dot{\mathbf{x}} = \begin{bmatrix} -5 & -2 \\ 2 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u$$

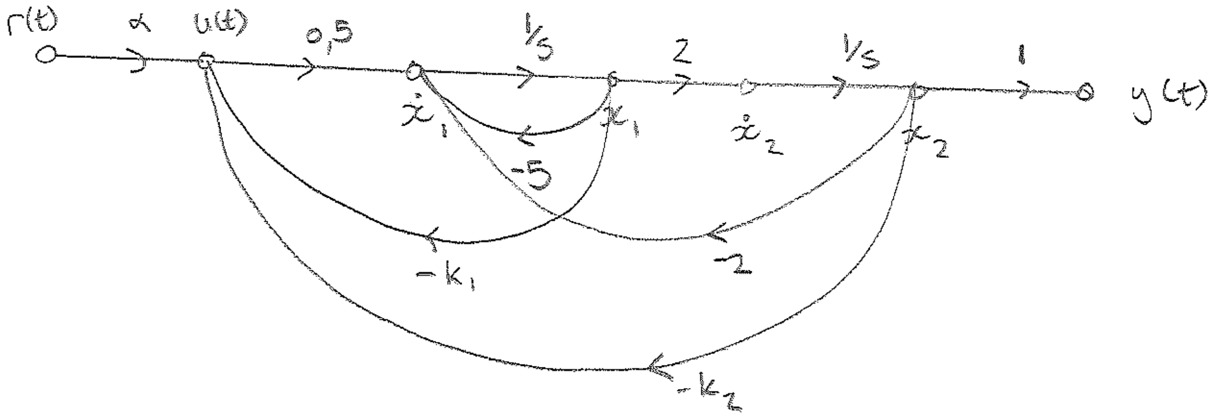
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x} + [0]\, u.$$

Use state variable feedback and incorporate a command input $u = -\mathbf{K}\mathbf{x} + \alpha r$. Select the gains $\mathbf{K}$ and α so that the system has a rapid response with an overshoot of approximately 1 %, a settling time (with a 2 % criterion) less than 1 second, and a zero steady-state error to a unit step input.

Determine the closed loop transfer function:

Draw a signal-flow diagram

$$u = -\mathbb{K}\,\mathbb{X} + \alpha r(t)$$
$$= \begin{bmatrix} -k_1 & -k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \alpha r(t) = -k_1 x_1 - k_2 x_2 + \alpha r(t)$$



Mason's Rule: $T(s) = \dfrac{P_k \Delta_k}{\Delta} = \dfrac{P_1 \Delta_1}{\Delta}$

$P_1 = \dfrac{2(0{,}5)\alpha}{s^2}$

$L_1 = \dfrac{-5}{s}$ $\qquad L_3 = \dfrac{-0{,}5k_1}{s}$

$L_2 = \dfrac{-4}{s^2}$ $\qquad L_4 = \dfrac{-0{,}5\times 2\, k_2}{s^2}$

$\Delta_1 = 1$

$$T(s) = \frac{\frac{\alpha}{s^2}}{1 - \left(\frac{-5}{s} - \frac{4}{s^2} - \frac{0{,}5k_1}{s} - \frac{k_2}{s^2}\right)}$$
$$= \frac{\alpha / s^2}{1 + \left(\frac{5 + 0{,}5k_1}{s}\right) + \left(\frac{4 + k_2}{s^2}\right)}$$
$$= \frac{\alpha}{s^2 + (5 + 0{,}5k_1)s + (4 + k_2)}$$

Design for P.O. $= 1\%$ $\quad T_s < 1 \quad e_{ss} = 0$

For P.O. of 1% $\Rightarrow$

$$\zeta = \sqrt{\frac{\ln^2\left(\frac{P.O.}{100}\right)}{\ln^2\left(\frac{P.O}{100}\right) + \pi^2}}$$

$$= \sqrt{\frac{\ln^2(0{,}01)}{\ln^2(0{,}01) + \pi^2}}$$

$$= \sqrt{0{,}68} = 0{,}824$$

$$T_s < 1$$

$$\frac{4}{\zeta w_n} < 1$$

$$\frac{4}{(0{,}824)\, w_n} < 1$$

$$4 < 0{,}824\, w_n$$

$$4/0{,}824 < w_n$$

$$4{,}85 < w_n \quad \Rightarrow \text{choose } w_n = 4.9 \text{ rad/s}$$

So the desired characteristic eq is:

$$q(s) = s^2 + 2\zeta w_n s + w_n^2$$

$$= s^2 + 2(0{,}824)(4{,}9)\, s + (4{,}9)^2$$

$$= s^2 + 8{,}1 s + 24$$

$\therefore$

$$5 + 0{,}5k_1 = 8{,}1 \quad \Rightarrow \quad 0{,}5k_1 = 8{,}1 - 5$$

$$k_1 = (8{,}1-5)/0{,}5 = 6{,}2$$

$$4 + k_2 = 24 \quad \Rightarrow \quad k_2 = 24 - 4 = 20$$

Use the steady-state requirement to determine α

only for a unit step

$$y_{ss} = \lim_{s \to 0} s\,Y(s) = 1$$

$Y(s) = T(s)\,R(s)$ where $R(s)$ is the input

For a step input $R(s) = 1/s$

$\therefore \quad Y(s) = T(s)/s$

$$= \lim_{s \to 0} \frac{s\,T(s)}{s}$$

$$= \lim_{s \to 0} T(s)$$

$$= \lim_{s \to 0} \frac{\alpha}{s^2 + (5 + 0{,}5k_1)s + (4 + k_2)}$$

$$\Rightarrow \quad \frac{\alpha}{4 + k_2} = 1$$

$$\alpha = 4 + k_2$$

$$= 4 + 20 = 24$$