# Notes : Jury's stability test

§7.5 Phillips

- To be able to use the Routh-Hurwitz criterion, we need to transform the characteristic equation in the z-domain to the W-domain using the bilinear transform.

- The advantage of the Jury stability test is the fact that no transform is required. The test is applied directly to the characteristic equation written as a function of z.

Let the characteristic equation of a discrete-time system be given by

$$Q(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 = 0,$$

$a_n > 0$ ∴ $a_n$ may not be negative
∴ The constant parameter associated with the highest order of z must be positive

JURY TABLE

| | $z^0$ | $z^1$ | $z^2$ | ... | $z^{n-k}$ | ... | $z^{n-1}$ | $z^n$ |
|---|---|---|---|---|---|---|---|---|
| Row 1 | $a_0$ | $a_1$ | $a_2$ | ... | $a_{n-k}$ | ... | $a_{n-1}$ | $a_n$ |
| Row 2 | $a_n$ | $a_{n-1}$ | $a_{n-2}$ | ... | $a_k$ | ... | $a_1$ | $a_0$ |
| Row 3 | $b_0$ | $b_1$ | $b_2$ | ... | $b_{n-k}$ | ... | $b_{n-1}$ | |
| Row 4 | $b_{n-1}$ | $b_{n-2}$ | $b_{n-3}$ | ... | $b_{k-1}$ | ... | $b_0$ | |
| Row 5 | $c_0$ | $c_1$ | $c_2$ | ... | $c_{n-k}$ | ... | | |
| Row 6 | $c_{n-2}$ | $c_{n-3}$ | $c_{n-4}$ | ... | $c_{k-2}$ | ... | | |

etc.

EVEN - Considering the table
The elements of each of the even-numbered rows are the elements of the preceding row in reverse order

ODD - The elements of the odd-numbered rows are

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}$$

$$c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix}$$

$$d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix}$$

etc.

- The necessary and sufficient conditions for the polynomial $Q(z)$ to have no roots outside or on the unit circle, with $a_n > 0$ are as follows

$$Q(1) > 0$$
$$(-1)^n Q(-1) > 0$$
$$|a_0| < a_n$$
$$|b_0| > |b_{n-1}|$$
$$|c_0| > |c_{n-2}|$$
$$|d_0| > |d_{n-3}|$$

etc.

- For a $2^{nd}$-order system, the array contains only one row.

- For each additional order, two additional rows are added to the array.

- Constraints: For an $n^{th}$-order system, there are a total of $n+1$ constraints.

APPLICATION OF THE JURY TEST

1) CHECK THE 3 CONDITIONS $Q(1) > 0$

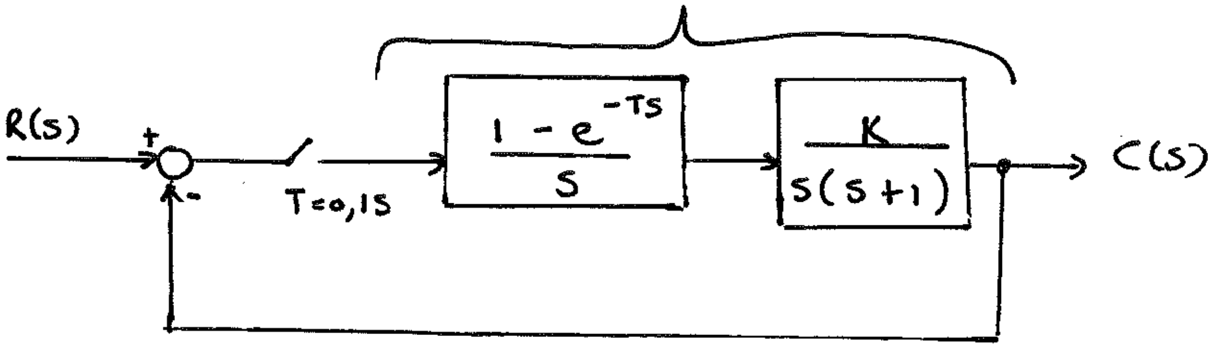
$$(-1)^n Q(-1) > 0$$

$$|a_0| < a_n$$

STOP IF ANY OF THE CONDITIONS ARE NOT SATISFIED.

2) CONSTRUCT THE ARRAY, CHECK THE CONDITIONS AS EACH ROW IS CALCULATED.
STOP IF ANY CONDITION IS NOT SATISFIED

<u>Example 1 (Jury test)</u>: $= \underbrace{K\,G(s)}_{}$ 2nd-order system




$$G(s) = \frac{1-e^{-Ts}}{s}\left[\frac{\overset{1}{\cancel{K}}}{s(s+1)}\right]$$

$$G(z) = \frac{z-1}{z}\left[\frac{(e^{-T}+T-1)z^2 + (1-e^{-T}-Te^{-T})z}{(z-1)^2(z-e^{-T})}\right]$$

$$= \left.\frac{0{,}00484\,z + 0{,}00468}{(z-1)(z-0{,}905)}\right|_{T=0,1}$$

$$G(z)\Big|_{T=1} = \frac{z-1}{z}\left[\frac{(0{,}368)z^2 + (0{,}264)z}{(z-1)^2(z-0{,}368)}\right]$$

$$= \frac{0{,}368\,z + 0{,}264}{(z-1)(z-0{,}368)} = \frac{0{,}368\,z + 0{,}264}{z^2 - 1{,}368\,z + 0{,}368}$$

CHARACTERISTIC EQ: $1 + K\,G(z) = 0$

$$\therefore\quad 1 + \frac{K(0{,}368\,z + 0{,}264)}{z^2 - 1{,}368\,z + 0{,}368} = 0$$

$$z^2 + (0{,}368\,K - 1{,}368)z + (0{,}368 + 0{,}264\,K) = 0$$

Jury array :

| $z^0$ | $z^1$ | $z^2$ |
|---|---|---|
| $(0,368 + 0,264K)$ | $(0,368K - 1,368)$ | $1$ |

CHECK CONDITIONS:

① $Q(1) > 0$

$$Q(z) = z^2 + (0,368K - 1,368)z + (0,368 + 0,264K)$$
$$Q(1) = 1^2 + 0,368K - 1,368 + 0,368 + 0,264K > 0$$

$$\therefore \quad 0,632K > 0$$
$$\Rightarrow \quad \underline{K > 0}$$

② $(-1)^2 Q(-1) > 0$

$$(-1)^2 + (-0,368K) + 1,368 + 0,368 + 0,264K > 0$$
$$1 + 1,368 + 0,368 - 0,368K + 0,264K > 0$$
$$2,736 - 0,104K > 0$$
$$\cancel{K < \frac{0,104}{2,}}$$
$$K < \frac{-2,736}{-0,104}$$
$$\Rightarrow \underline{K < 26,3}$$

③ $|a_0| < a_2$

$$(0,368 + 0,264K) < 1$$
$$0,264K < 1 - 0,368$$
$$\Rightarrow \quad K < (1 - 0,368)/0,264 = \underline{2,39}.$$

$$\therefore \boxed{0 < K < 2,39}$$

Test :

$$\underbrace{Q(z)\Big|_{K=2,39}}_{\text{Marginally stable}} = z^2 + [0,368(2,39) - 1,368]z + 0,368 + 0,264(2,39)$$

$$= z^2 - 0,488z + 1 \qquad = 0$$

Determine the roots of $z^2 - 0,488z + 1$

$$z = 0,244 \pm j0,96978 = 1\,\angle \pm 75,9^\circ$$
$$= 1\,\angle \pm 1,32 \text{ rad}$$
$$= 1\,\angle \omega T$$

Since $T = 1\,s$ $\therefore$ $\omega = 1,32$ rad/s.

$$\omega = \tan^{-1}\left(\frac{Im}{Re}\right)$$
$$= \tan^{-1}\left(\frac{0,96978}{0,244}\right)$$
$$= 1,32 \text{ rad/s}$$

## Example 2 (Jury test, 3rd-order system)

CHARACTERISTIC EQ:

$$Q(z) = z^3 - 1,8z^2 + 1,05z - 0,20 = 0$$

$\therefore$ $a_3 = 1$ $a_2 = -1,8$ $a_1 = 1,05$ $a_0 = -0,2$.

TEST CONDITIONS:

① $Q(1) = 1 - 1,8 + 1,05 - 0,2 > 0$

$= 0,05 > 0$ ✓

② $(-1)^3 Q(-1) = -1\left[(-1)^3 - 1,8 + 1,05(-1) - 0,2\right] > 0$

$(-1)(-4,05) > 0$

$4,05 > 0$ ✓

③ $|a_0| < a_n$

$|-0,20| < 1 = a_3$

$0,2 < 1$ ✓

n+1 conditions need to satisfied $\therefore$ $3+1=4$

since the first 3 are satisfied we can continue with Jury array.

| $z^0$ | $z^1$ | $z^2$ | $z^3$ |
|---|---|---|---|
| $-0,2$ | $1,05$ | $-1,8$ | $1$ |
| $1$ | $-1,8$ | $1,05$ | $-0,2$ |
| $b_0 = -0,96$ | $b_1$ | $b_2$ | |

$$b_0 = \begin{vmatrix} -0,2 & 1 \\ 1 & -0,2 \end{vmatrix} = \begin{vmatrix} a_0 & a_{3-0} \\ a_3 & a_3 \end{vmatrix} = (-0,2)(-0,2) - (1)(1) = -0,96$$

$$b_1 = \begin{vmatrix} a_0 & a_{3-1} \\ a_3 & a_1 \end{vmatrix} = \begin{vmatrix} -0{,}2 & -1{,}8 \\ 1 & 1{,}05 \end{vmatrix} = (-0{,}2)(1{,}05) - (1)(-1{,}8) = 1{,}59$$

$$b_2 = \begin{vmatrix} a_0 & a_{3-2} \\ a_3 & a_{32} \end{vmatrix} = \begin{vmatrix} -0{,}2 & 1{,}05 \\ 1 & -1{,}8 \end{vmatrix} = -0{,}69$$

$$|b_0| > |b_{n-1}|$$

$$|b_0| > |b_2|$$

$$|-0{,}96| > |-0{,}69| \quad \checkmark$$

All conditions are satisfied $\therefore$ system is stable.