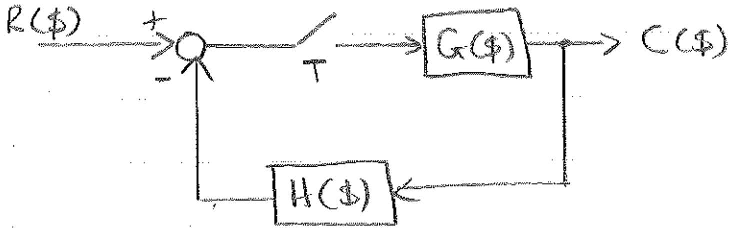
Chapter 7 - STABILITY ANALYSIS TECHNIQUES

- In general, the stability analysis techniques applicable to LTI continuous-time systems may also be applied to the analysis of LTI discrete systems, if certain modifications are made.
- Techniques: Routh-Hurwitz
  root-locus
  Bode diagrams
- Jury stability is specifically developed for discrete systems.

STABILITY



sampled-data system

$$C(z) = \frac{G(z)\,R(z)}{1 + \overline{GH}(z)}$$

$$= \frac{K \prod^{m} (z - z_i)}{\prod^{n} (z - p_i)} R(z)$$

z_i and p_i are the zeros and poles of the system transfer function

- Use partial fraction expansion.
  (for the case that the transfer-function pole are distinct)

∴ no repeated poles

$$C(z) = \frac{k_1 z}{z - p_1} + \cdots + \frac{k_n z}{z - p_n} + C_R(z)$$

- $C_R(z)$ contains terms of $C(z)$ which originate in the poles of $R(z)$

- The first n terms are the natural response terms of $C(z)$.

- If the inverse z-transform of these terms tend to zero as time increases, the system is stable, and these terms are called the <u>transient response</u>.

- The inverse z-transform of the ith term is

$$\mathcal{Z}^{-1}\left[\frac{k_i z}{z - p_i}\right] = k_i (p_i)^k \qquad (1)$$

- If the magnitude of p_i is less than 1, this term approaches zero as k approaches ∞.

- Note that the factors $(z - p_i)$ originate in the characteristic eq.

$$1 + \overline{GH}(z) = 0 \qquad (2)$$

- The system is stable provided that all the roots of (2) lie inside the unit circle.

- can also use

$$1 + \overline{GH}^*(s) = 0 \qquad (3)$$

- roots of (3) must lie in the LH of the s-plane.

- For the case that a root of the characteristic equation is unity in magnitude

  (e.g. $p_i = 1 \angle \theta$

$\therefore$ (1) is constant magnitude.

$\therefore$ Natural response has a term that neither dies out nore becom unbounded as k approaches ∞.

= System is marginally stable.

NB MARGINALLY STABLE SYSTEM:

The Characteristic eq has at least 1 zero on the unit circle, with no zeros outside the unit circle.

Bilinear Transformation

- Routh-Hurwitz criterion is based on the property that in the s-plane the stability boundary is the imaginary axis.

- cannot be applied to LTI discrete-time systems in the z-plane, since the stability boundary is the unit circle.

- Use a transform:

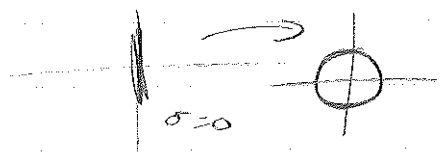
$$z = \frac{1 + (T/2)w}{1 - (T/2)w}$$

or solving for w

$$w = \frac{2}{T}\frac{z-1}{z+1}$$

=> unit circle in the z-plane transforms into the imaginary axis of the w-plane.

$$z = e^{sT} = e^{j\omega T}\Big|_{\sigma=0}$$



$$w = \frac{2}{T}\frac{z-1}{z+1}\Bigg|_{z=e^{j\omega T}}$$

$$= \frac{2}{T}\frac{e^{j\omega T}-1}{e^{j\omega T}+1} \times \frac{e^{j\omega T}+1}{e^{j\omega T}+1}$$

$$= \frac{2}{T}\frac{e^{2j\omega T}-1}{e^{j2\omega T}+2e^{j\omega T}+1} = \frac{2}{T}\frac{e^{j\omega T/2}-e^{-j\omega T/2}}{e^{j\omega T/2}+e^{-j\omega T/2}}$$

$$w = j\frac{2}{T}\tan\frac{\omega T}{2}$$

Thus it can be seen that the unit circle of the z-plane transforms into the imaginary axis of the w-plane.

Let $j\omega_w$ be the imaginary part of w

ω_w is the w-plane frequency

$$\omega_w = \frac{2}{T}\tan\frac{\omega T}{2}$$

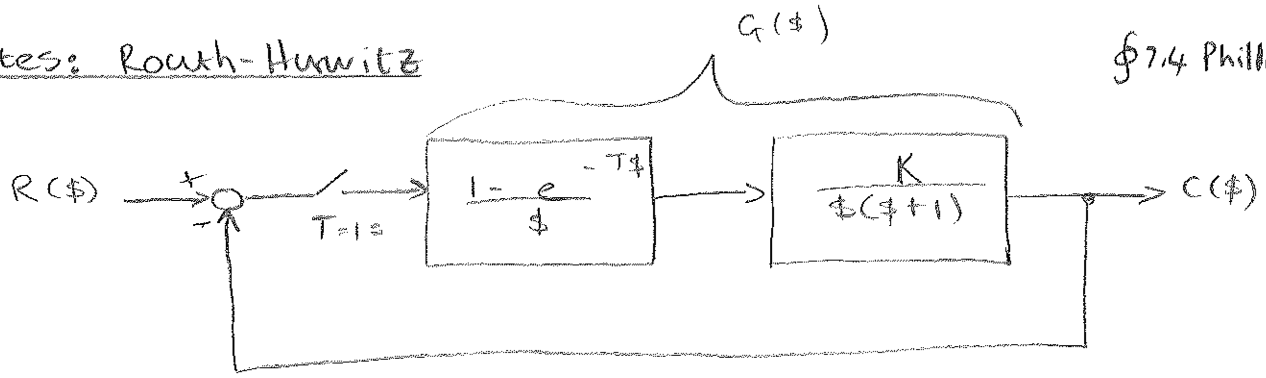
Gives the frequency relationship between frequencies in the s-plane and frequencies in the w-plane.

$$\omega_w = \frac{2}{T}\tan\frac{\omega T}{2} \approx \frac{2}{T}\left(\frac{\omega T}{2}\right) = \omega$$

For $\frac{\omega T}{2} \leq \frac{\pi}{10}$, $\omega \leq \frac{2\pi}{10T} = \frac{\omega_s}{10}$

Notes: Routh-Hurwitz

§7.4 Phillips



$K = 1$

$$G(s) = \frac{1-e^{-Ts}}{s}\ \frac{1}{s(s+1)}$$

$$G(z) = \frac{z-1}{z}\ \mathcal{Z}\left[\frac{1}{s^2(s+1)}\right]$$

$$= \frac{z-1}{z}\left[\frac{(aT-1+e^{-aT})z^2 + (1-e^{-aT}-aTe^{-aT})z}{a(z-1)^2\,(z-e^{-aT})}\right]_{\substack{a=1\\T=1}}$$

$$= \frac{z-1}{z}\left[\frac{(e^{-1})z^2 + (1-e^{-1}-1e^{-1})z}{(z-1)^2\,(z-e^{-1})}\right]$$

$$= \frac{z-1}{z}\left[\frac{0{,}368\,z^2 + 0{,}264\,z}{(z-1)^2\,(z-0{,}368)}\right]$$

$$= \frac{0{,}368\,z + 0{,}264}{(z-1)(z-0{,}368)} = \frac{0{,}368\,z + 0{,}264}{z^2 - 1{,}368\,z + 0{,}368}$$

Determine $G(w)$

$$G(w) = G(z)\Big|_{z = \frac{1+0{,}5w}{1-0{,}5w}}$$

$$z = \frac{1+(T/2)w}{1-(T/2)w}$$

Bilinear Transform.

$$G(W) = \frac{0,368\left(\frac{1+0,5W}{1-0,5W}\right) + 0,264}{\left(\frac{1+0,5W}{1-0,5W}\right)^2 - 1,368\left(\frac{1+0,5W}{1-0,5W}\right) + 0,368}$$

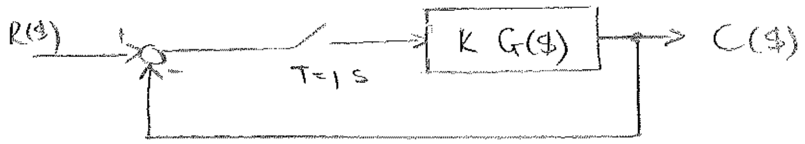
$$= \frac{0,368(1+0,5W)(1-0,5W) + 0,264(1-0,5W)^2}{(1+0,5W)^2 - 1,368(1+0,5W)(1-0,5W) + 0,368(1-0,5W)^2}$$

$$= \frac{0,368(1-0,25W^2) + 0,264(1 - 1W + 0,25W^2)}{(1 + 1W + 0,25W^2) - 1,368(1-0,25W^2) + 0,368(1-W+0,25W^2)}$$

$$= \frac{0,368 - 0,092W^2 + 0,264 - 0,264W + 0,066W^2}{1 + W + 0,25W^2 - 1,368 + 0,342W^2 + 0,368 - 0,368W + 0,092W^2}$$

$$= \frac{-0,026W^2 - 0,264W + 0,632}{0,684W^2 + 0,632W}$$

Lets add a gain factor K to the plant



$\therefore$ System characteristic eq.

$$1 + KG(W) = 0$$

$$1 + K\left(\frac{-0,026W^2 - 0,264W + 0,632}{0,684W^2 + 0,632W}\right) = 0$$

$$0,684W^2 + 0,632W + K(-0,026W^2 - 0,264W + 0,632) = 0$$

$$(0,684 - 0,026K)W^2 + (0,632 - 0,264K)W + 0,632K = 0$$

Routh Array:

$$\begin{array}{lll} w^2 & 0{,}684 - 0{,}026K & 0{,}632K \\ w^1 & 0{,}632 - 0{,}264K & 0 \\ w^0 & a & b = 0 \end{array}$$

$$a = - \frac{\begin{vmatrix} (0{,}684 - 0{,}026K) & 0{,}632K \\ (0{,}632 - 0{,}264K) & 0 \end{vmatrix}}{(0{,}632 - 0{,}264K)}$$

$$= - \frac{\left[0 - (0{,}632K)(0{,}632 - 0{,}264K) \right]}{(0{,}632 - 0{,}264K)}$$

$$= 0{,}632K$$

① $0{,}632K > 0$

$K > 0$

② $0{,}632 - 0{,}264K > 0$

$-0{,}264K > -0{,}632$

$K < 2{,}39$

③ $0{,}684 - 0{,}026K > 0$

$-0{,}026K > -0{,}684$

$K < 26{,}3$

$\therefore \quad 0 < K < 2{,}39$

- We can use the Rout-Hurwitz criterion to determine the value of K at which the root locus crosses into the right half-plane. (The value of K at which the system becomes unstable

- That value of K is the gain at which the system is <u>marginally stable</u>

- This info can be used to determine the resultant frequency of oscillation

$\therefore$ At $K = 2,39$ = gain at which the system is marginally stable.

Auxiliary eq: $Q(w) = (0,684 - 0,026\,K)\,w^2 + 0,632\,K = 0$

For $K = 2,39$: $Q(w) = 0,622\,w^2 + 1,51 = 0$

$$\therefore \quad w^2 = -1,51 / 0,622$$
$$= -2,428$$
$$w = \pm j\sqrt{2,428}$$
$$= \pm j\,1,558$$

$$w = j\,\omega_w \quad \Rightarrow \quad \omega_w = 1,558$$

$$\omega = \frac{2}{T}\tan^{-1}\left(\frac{\omega_w T}{2}\right)$$
$$= \frac{2}{1}\tan^{-1}\left(\frac{1,558\,(1)}{2}\right)$$

$= 1,32$ rad/s = s-plane real frequency at which the system will oscillate wit $K = 2,39$.