EERI 418
CONTROL THEORY II

Multivariable and digital control systems theory

SU 3 Discrete-time Systems and the Z-transform

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Contents of this study unit

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Discrete-time systems

It all starts here

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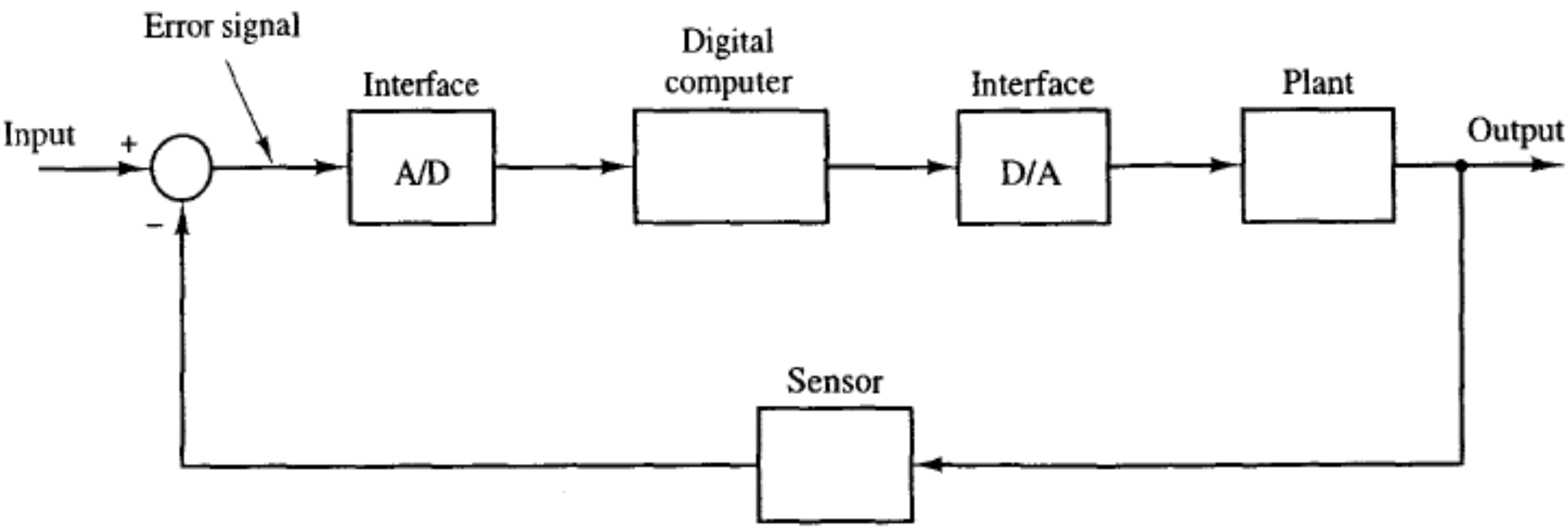
Modelling of discrete-time systems

- Continuous-time systems are modeled by a set of differential equations.
- Discrete-time systems are modeled by a set of difference equations.
- Transforms:
  - Continuous-time linear time-invariant => Laplace Transform
  - Discrete-time linear time-invariant => Z-transform
- IN THIS UNIT WE WILL BE LOOKING AT THE MODELLING OF DISCRETE TIME SYSTEMS BY MEANS OF
  - DIFFERENCE EQUATIONS
  - TRANSFER FUNCTIONS
  - STATE VARIABLE EQUATIONS

Digital control system



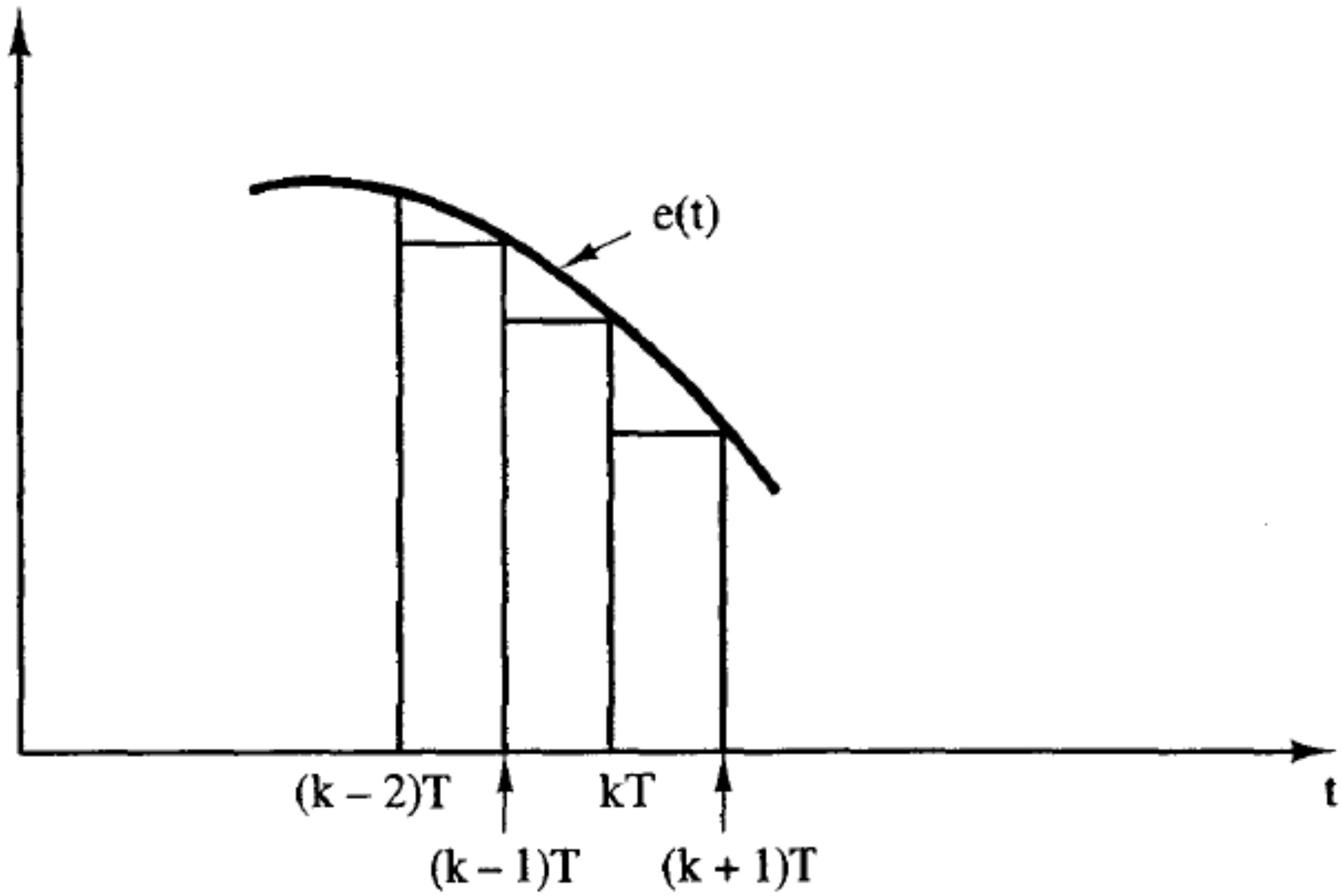
Digital PI controller

- The analog controller output is given by

$$m(t) = K_P e(t) + K_I \int_0^t e(\tau)\, d\tau$$

- The digital computer can be programmed to multiply, add, and integrate numerically.
- Therefore the controller equation can be realized using the digital computer

Rectangular rule for numerical integration



General form of linear difference equations

$$x(k) = b_n e(k) + b_{n-1} e(k-1) + \cdots + b_0 e(k-n) - a_{n-1} x(k-1) - \cdots - a_0 x(k-n)$$

$$y(t) = \beta_n \frac{d^n e(t)}{dt^n} + \cdots + \beta_1 \frac{de(t)}{dt} + \beta_0 e(t) - \alpha_n \frac{d^n y(t)}{dt^n} - \cdots - \alpha_1 \frac{dy(t)}{dt}$$

Two approaches towards the design of digital compensators

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1) An analog compensator may be designed and then converted by some approximate procedure to a digital compensator.
2) Exact methods of designing digital compensators
   - The linear difference equation described a digital filter to be solved by the digital computer.
   - Therefore one need to determine
     - T, the sampling period
     - n, the order of the difference equation
     - a_i and b_i , the filter coefficients
   - Need also consider the word length of the computer (round-off errors)

The Z-transform

Z-transform

- Need a transform to be utilised in the analysis of discrete-time systems modeled by difference equations
- Transform is defined for number sequences as follows:
  - The function *E(z)* is defined as a power series in z^{-k}
  - With coefficients equal to the values of the number sequence { *e(k)* }

$$E(z) = \mathfrak{z}[\{e(k)\}] = e(0) + e(1)z^{-1} + e(2)z^{-2} + \cdots$$

$$e(k) = \mathfrak{z}^{-1}[E(z)] = \frac{1}{2\pi j}\oint_{\Gamma} E(z)z^{k-1}\,dz, \qquad j = \sqrt{-1}$$

where $\mathfrak{z}(\cdot)$ indicates the z-transform operation and $\mathfrak{z}^{-1}(\cdot)$ indicates the inverse

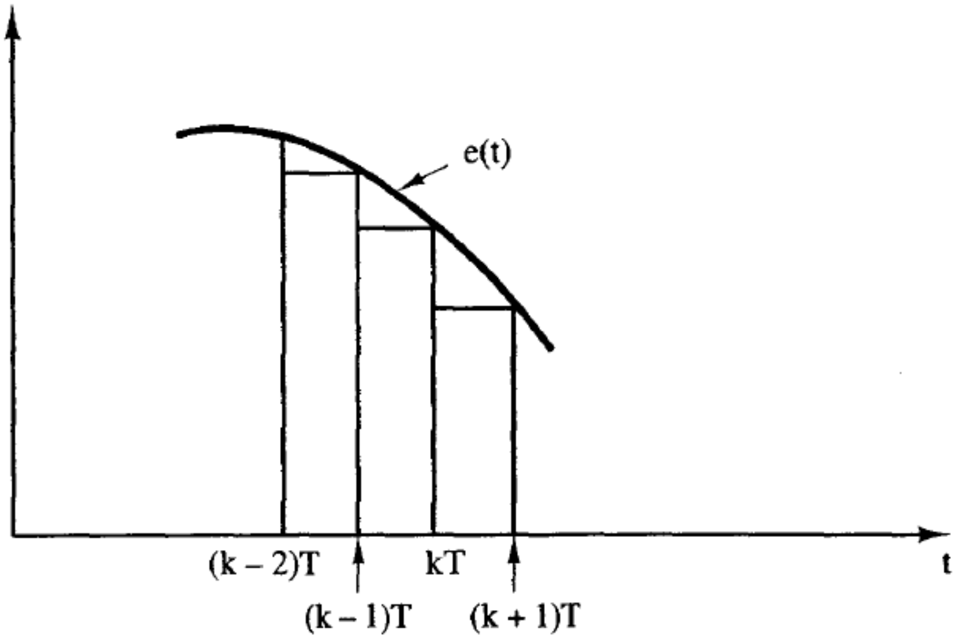
Z-transform in compact form

$$E(z) = \mathfrak{z}[\{e(k)\}] = \sum_{k=0}^{\infty} e(k)z^{-k}$$

For convenience, we often omit the braces and express $\mathfrak{z}[\{e(k)\}]$ as $\mathfrak{z}[e(k)]$. However, it should be remembered that the z-transform applies to a sequence.

- May be used by any type of system described by linear time-invariant difference equations.
- The equation above is a single-sided Z-transform
  - Called and ORDINARY Z-TRANSFORM

Z-Transform as a generating function



- *e(kT)* is a sequence generated from a time function *e(t)* by sampling every *T* seconds.
- *NB!! e(k)* is understood to be *e(kT),* the *T* is dropped for convenience

Properties of the Z-transform

- Addition and subtraction

$$\mathfrak{z}[e_1(k) \pm e_2(k)] = E_1(z) \pm E_2(z)$$

- Proof

$$\mathfrak{z}[e_1(k) \pm e_2(k)] = \sum_{k=0}^{\infty} [e_1(k) \pm e_2(k)]z^{-k}$$

$$= \sum_{k=0}^{\infty} e_1(k)z^{-k} \pm \sum_{k=0}^{\infty} e_2(k)z^{-k} = E_1(z) \pm E_2(z)$$

Properties of the Z-transform

- Multiplying by a constant

$$\mathfrak{z}[ae(k)] = a\mathfrak{z}[e(k)] = aE(z)$$

- Proof

$$\mathfrak{z}[ae(k)] = \sum_{k=0}^{\infty} ae(k)z^{-k} = a\sum_{k=0}^{\infty} e(k)z^{-k} = aE(z)$$

Properties of the Z-transform

- Real translation

$$\mathfrak{z}[e(k-n)u(k-n)] = z^{-n}E(z)$$

$$\mathfrak{z}[e(k+n)u(k)] = z^{n}\left[E(z) - \sum_{k=0}^{n-1} e(k)z^{-k}\right]$$

Properties of the Z-transform

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- Complex translation

$$\mathfrak{z}[\epsilon^{ak} e(k)] = E(z\epsilon^{-a})$$

- Proof

$$\begin{aligned}\mathfrak{z}[\epsilon^{ak} e(k)] &= e(0) + \epsilon^{a} e(1) z^{-1} + \epsilon^{2a} e(2) z^{-2} + \cdots \\ &= e(0) + e(1)(z\epsilon^{-a})^{-1} + e(2)(z\epsilon^{-a})^{-2} + \cdots\end{aligned}$$

or

$$\mathfrak{z}[\epsilon^{ak} e(k)] = E(z)\big|_{z \leftarrow z\epsilon^{-a}} = E(z\epsilon^{-a})$$

Properties of the Z-transform

- Initial value

Property. Given that the z-transform of $e(k)$ is $E(z)$. Then

$$e(0) = \lim_{z \to \infty} E(z)$$

Proof. Since

$$E(z) = e(0) + e(1)z^{-1} + e(2)z^{-2} + \cdots$$

Properties of the Z-transform

- Final value

Property. Given that the z-transform of $e(k)$ is $E(z)$. Then

$$\lim_{n \to \infty} e(n) = \lim_{z \to 1} (z - 1)E(z)$$

provided that the left-side limit exists.

Properties of the Z-transform

TABLE 2-2 PROPERTIES OF THE z-TRANSFORM

Sequence	Transform
$e(k)$	$E(z) = \sum_{k=0}^{\infty} e(k) z^{-k}$
$a_1 e_1(k) + a_2 e_2(k)$	$a_1 E_1(z) + a_2 E_2(z)$
$e(k-n)u(k-n); \quad n \geq 0$	$z^{-n} E(z)$
$e(k+n)u(k); \quad n \geq 1$	$z^n \left[E(z) - \sum_{k=0}^{n-1} e(k) z^{-k} \right]$
$\epsilon^{ak} e(k)$	$E(z\epsilon^{-a})$
$ke(k)$	$-z \dfrac{dE(z)}{dz}$
$e_1(k) * e_2(k)$	$E_1(z) E_2(z)$
$e_1(k) = \sum_{n=0}^{k} e(n)$	$E_1(z) = \dfrac{z}{z-1} E(z)$

Initial value: $e(0) = \lim_{z \to \infty} E(z)$

Final value: $e(\infty) = \lim_{z \to 1} (z-1)E(z)$, if $e(\infty)$ exists

Solution of difference equations

Solution of difference equations

- Consider the following *n^th^*-order difference equation, where it is assumed that { *e(k)* } is known.

$$m(k) + a_{n-1}m(k-1) + \cdots + a_0 m(k-n) = b_n e(k) + b_{n-1}e(k-1) + \cdots + b_0 e(k-n)$$

- By using the real translation property, the Z-transform is given by

$$M(z) + a_{n-1}z^{-1}M(z) + \cdots + a_0 z^{-n}M(z) = b_n E(z) + b_{n-1}z^{-1}E(z) + \cdots + b_0 z^{-n}E(z)$$

- Solving for *M(z),* we get

$$M(z) = \frac{b_n + b_{n-1}z^{-1} + \cdots + b_0 z^{-n}}{1 + a_{n-1}z^{-1} + \cdots + a_0 z^{-n}} E(z)$$

Inverse Z-transform

- Power series method (Long division)
- Partial fraction method and Z-transform tables
- Inversion formula method
- Discrete convolution

Power series method

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- The inverse z-transform of a function *E(z)* which is expressed as the ratio of two polynomials in z involves dividing the denominator of E(z) into the numerator such that a power series of the following form is obtained:

$$E(z) = e_0 + e_1 z^{-1} + e_2 z^{-2} + \cdots$$

Example

It is desired to find the values of $e(k)$ for $E(z)$ given by the expression

$$E(z) = \frac{z}{z^2 - 3z + 2}$$

Example

Using long division, we obtain

$$
\begin{array}{r|l}
 & z^{-1} + 3z^{-2} + 7z^{-3} + 15z^{-4} + \cdots \\
\hline
z^2 - 3z + 2 & z \\
 & \underline{z - 3 + 2z^{-1}} \\
 & \quad 3 - 2z^{-1} \\
 & \quad \underline{3 - 9z^{-1} + 6z^{-2}} \\
 & \qquad 7z^{-1} - 6z^{-2} \\
 & \qquad \underline{7z^{-1} - 21z^{-2} + 14z^{-3}} \\
 & \qquad\qquad 15z^{-2} - 14z^{-3} + \cdots \\
 & \qquad\qquad\quad \cdots\cdots\cdots
\end{array}
$$

and therefore

$$
\begin{array}{ll}
e(0) = 0 & e(4) = 15 \\
e(1) = 1 & \cdots \\
e(2) = 3 & e(k) = 2^k - 1 \\
e(3) = 7 & \cdots
\end{array}
$$

Partial fraction expansion method

TABLE 2-3 z-TRANSFORMS

Sequence	z-Transform
$\delta(k-n)$	z^{-n}
1	$\dfrac{z}{z-1}$
k	$\dfrac{z}{(z-1)^2}$
k^2	$\dfrac{z(z+1)}{(z-1)^3}$
a^k	$\dfrac{z}{z-a}$
ka^k	$\dfrac{az}{(z-a)^2}$
$\sin ak$	$\dfrac{z \sin a}{z^2 - 2z \cos a + 1}$
$\cos ak$	$\dfrac{z(z - \cos a)}{z^2 - 2z \cos a + 1}$
$a^k \sin bk$	$\dfrac{az \sin b}{z^2 - 2az \cos b + a^2}$
$a^k \cos bk$	$\dfrac{z^2 - az \cos b}{z^2 - 2az \cos b + a^2}$

Example

$$E(z) = \frac{z}{(z-1)(z-2)}$$

$$\frac{E(z)}{z} = \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\mathcal{Z}^{-1}[E(z)] = \mathcal{Z}^{-1}\left[\frac{-z}{z-1}\right] + \mathcal{Z}^{-1}\left[\frac{z}{z-2}\right]$$

$$e(k) = -1 + 2^k$$

Simulation diagrams, signal flow diagrams and state models

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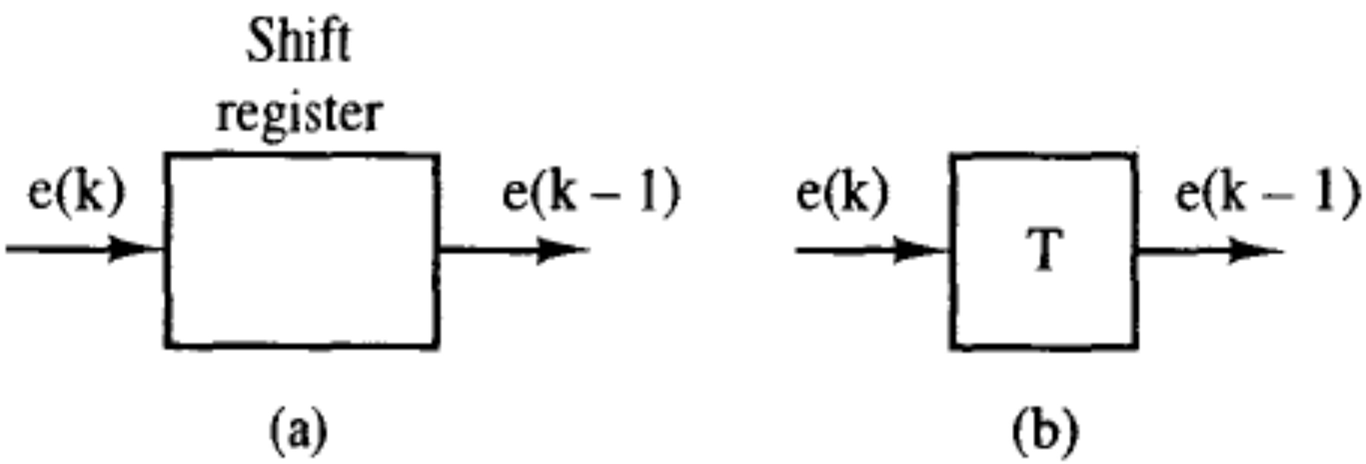
Simulation diagram

- How do we represent a linear time-invariant discrete system?
  - Difference equation
  - Transfer function
  - Simulation diagram

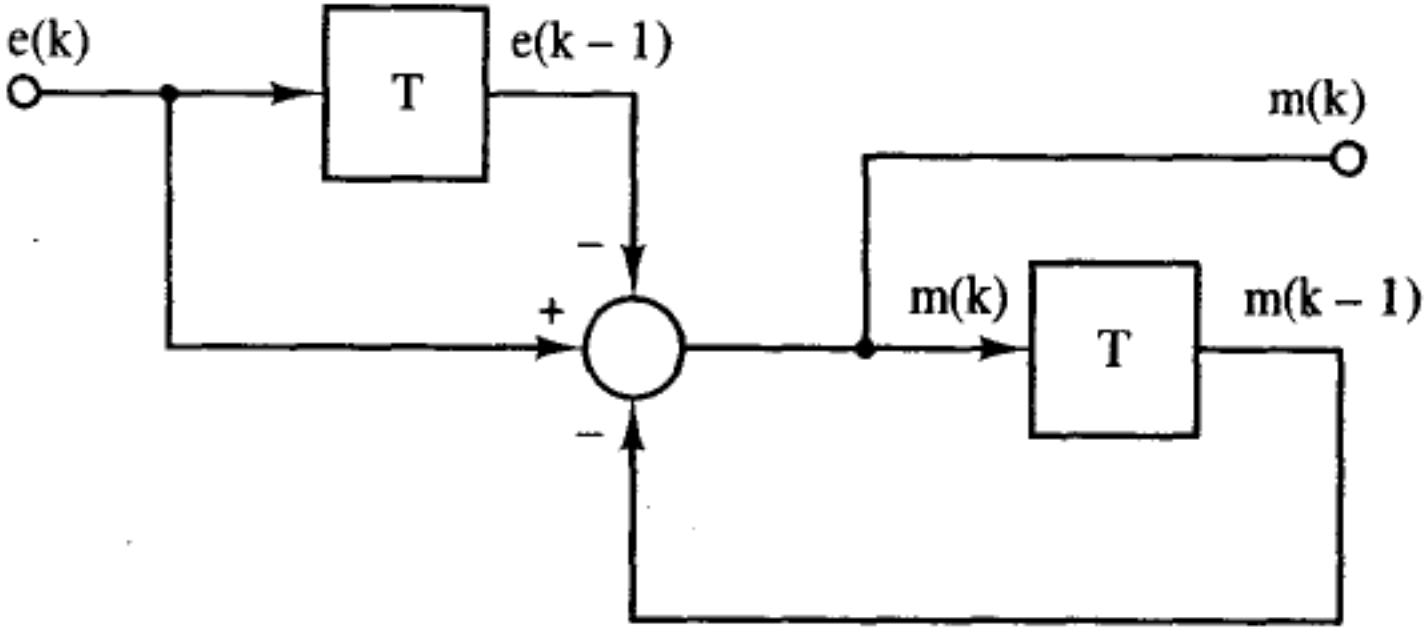
Ideal time-delay

- Block = shift register
- Every *T* seconds, a number is shifted into the register
- At that instant the number that was stored in the register is shifted out
- *e(k)* is the number shifted into register at *t=kT*
- *e(k-1)* is the number shifted out



Simulation diagram for a difference equation

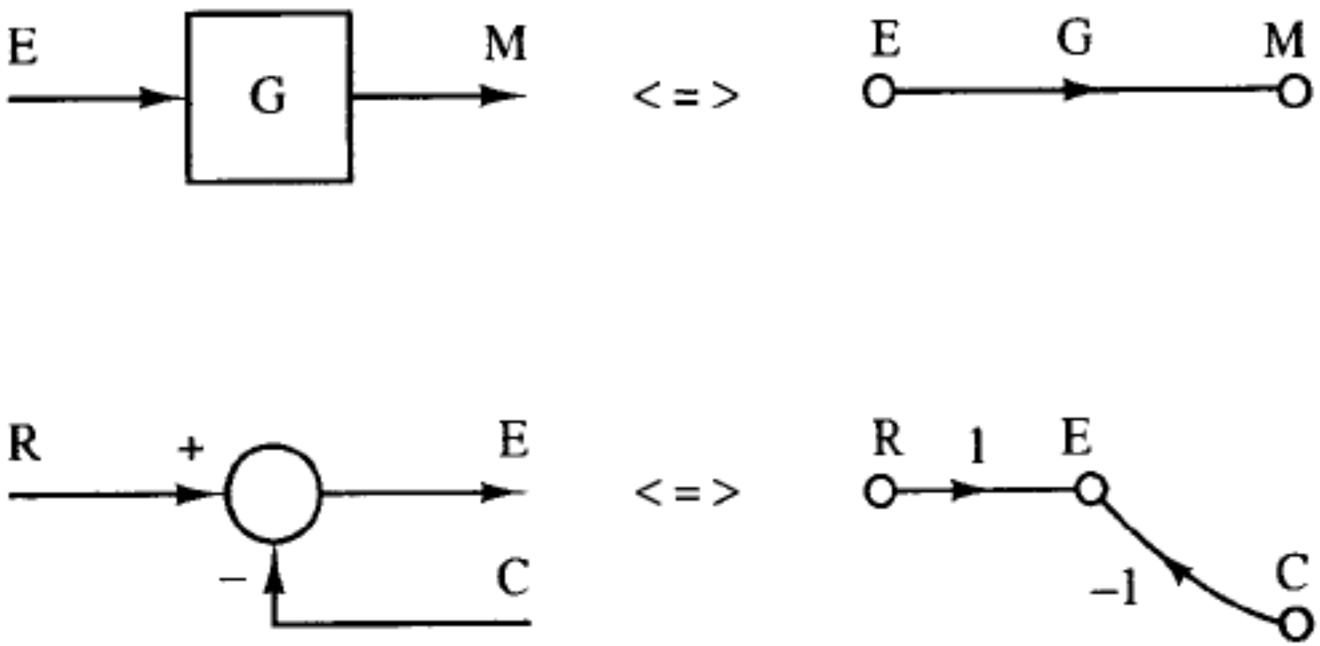
$$m(k) = e(k) - e(k-1) - m(k-1)$$



Relation between continuous and discrete

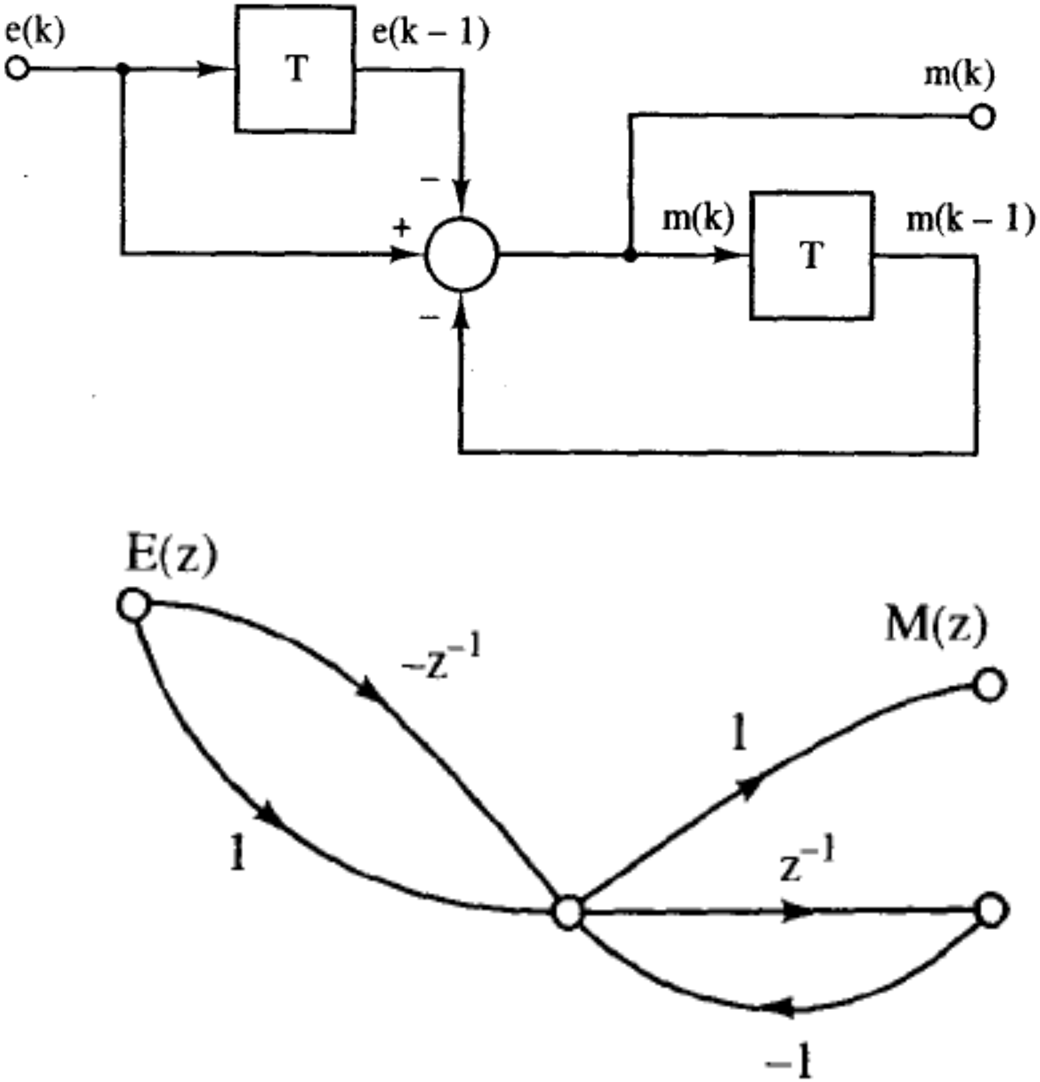
Recall that in the analog simulation of continuous systems, the basic element is the integrator. In the simulation of discrete systems, the basic element is the time delay (or memory) of T seconds.

Block diagram / Flow diagram



(a) (b)

Flow diagram



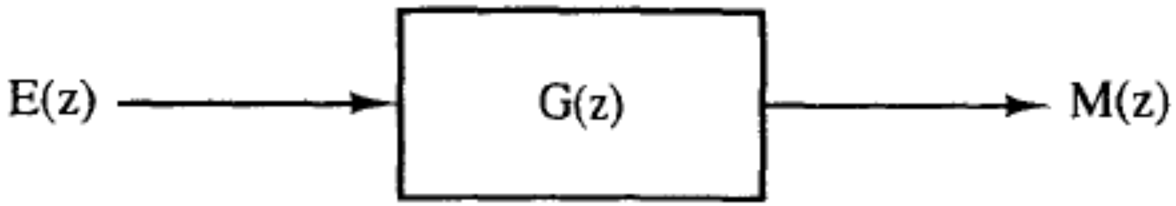
Mason's rule

$$\frac{M(z)}{E(z)} = \frac{1 - z^{-1}}{1 + z^{-1}} = \frac{z - 1}{z + 1}$$

State variables

- For a linear time-invariant discrete-time system with input *E(z)*, output *M(z)*, and the transfer function *G(z)*, we can write

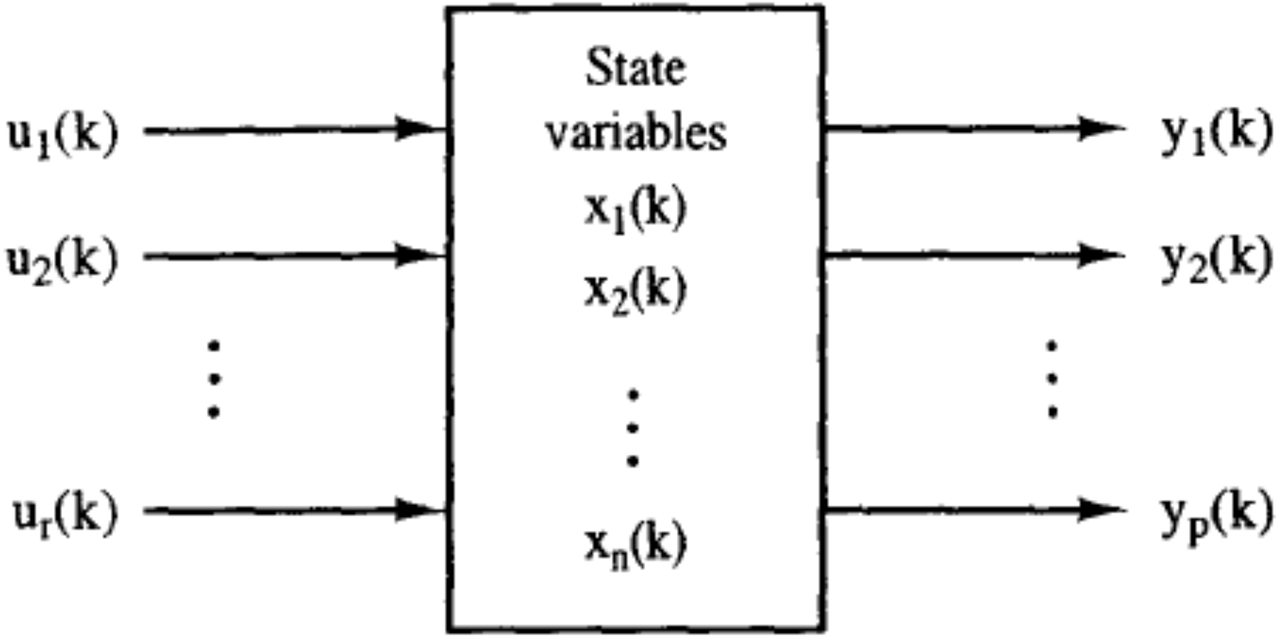
$$M(z) = G(z)E(z)$$



(a)

State variable representation

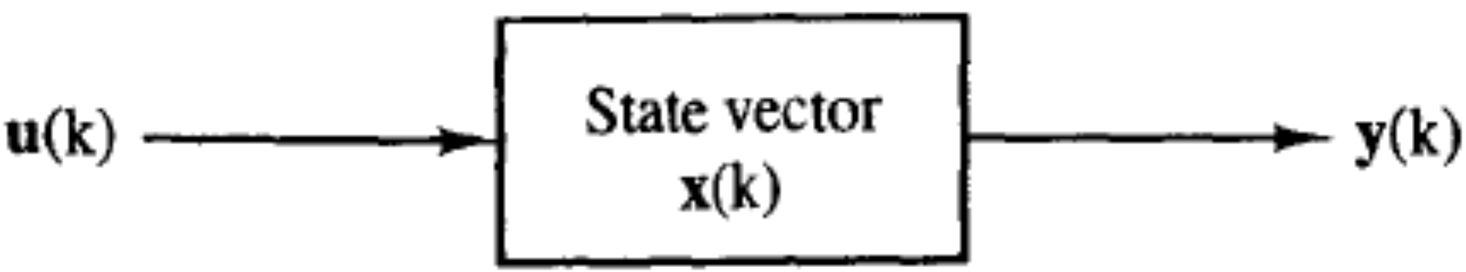
- $u_i(k),\ i=1,\ldots,r$ are the external inputs which drive the system
- $y_i(k),\ i=1,\ldots,p$ represent the system outputs or the system responses
- $x_i(k),\ i=1,\ldots,n$ are the internal or state variables of the system



(b)

State vector representation

$$\mathbf{u}(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \\ \vdots \\ u_r(k) \end{bmatrix}, \qquad \mathbf{y}(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_p(k) \end{bmatrix}, \qquad \mathbf{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}$$



General nonlinear state space equations

$$\mathbf{x}(k+1) = \mathbf{f}[\mathbf{x}(k), \mathbf{u}(k)]$$

$$\mathbf{y}(k) = \mathbf{g}[\mathbf{x}(k), \mathbf{u}(k)]$$

As stated earlier, when we say time k, we actually mean time kT.

Linear time-varying state space equation

$$\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k)$$

$$\mathbf{y}(k) = \mathbf{C}(k)\mathbf{x}(k) + \mathbf{D}(k)\mathbf{u}(k)$$

Linear time invariant

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$$

Difference equation to a state space eq.

$$y(k+2) = u(k) + 1.7y(k+1) - 0.72y(k)$$

States

Let

$$x_1(k) = y(k)$$

$$x_2(k) = x_1(k+1) = y(k+1)$$

Then

$$x_2(k+1) = y(k+2) = u(k) + 1.7x_2(k) - 0.72x_1(k)$$

States

Then

$$x_2(k+1) = y(k+2) = u(k) + 1.7x_2(k) - 0.72x_1(k)$$

or, from these equations, we write

$$x_1(k+1) = x_2(k)$$

$$x_2(k+1) = -0.72x_1(k) + 1.7x_2(k) + u(k)$$

$$y(k) = x_1(k)$$

State equation

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -0.72 & 1.7 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

Z-transfer function to state space model

$$G(z) = \frac{Y(z)}{U(z)} = \frac{z^2 + 2z + 1}{z^3 + 2z^2 + z + \frac{1}{2}}$$

Z-transfer function to state space model

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

General transfer function

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$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_{n-1}z^{-1} + b_{n-2}z^{-2} + \cdots + b_1 z^{1-n} + b_0 z^{-n}}{1 + a_{n-1}z^{-1} + \cdots + a_1 z^{1-n} + a_0 z^{-n}} \frac{E(z)}{E(z)} \qquad (2\text{-}61)$$

State space model formats

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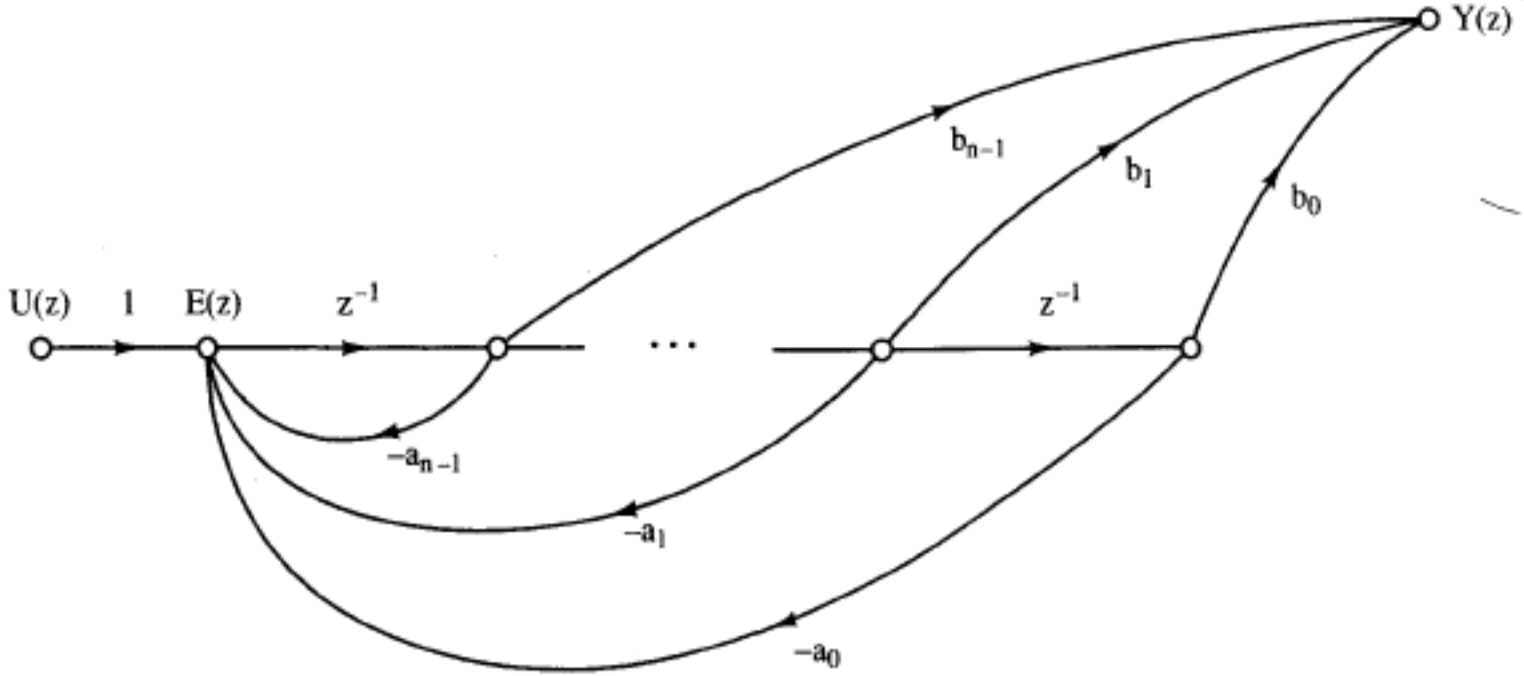
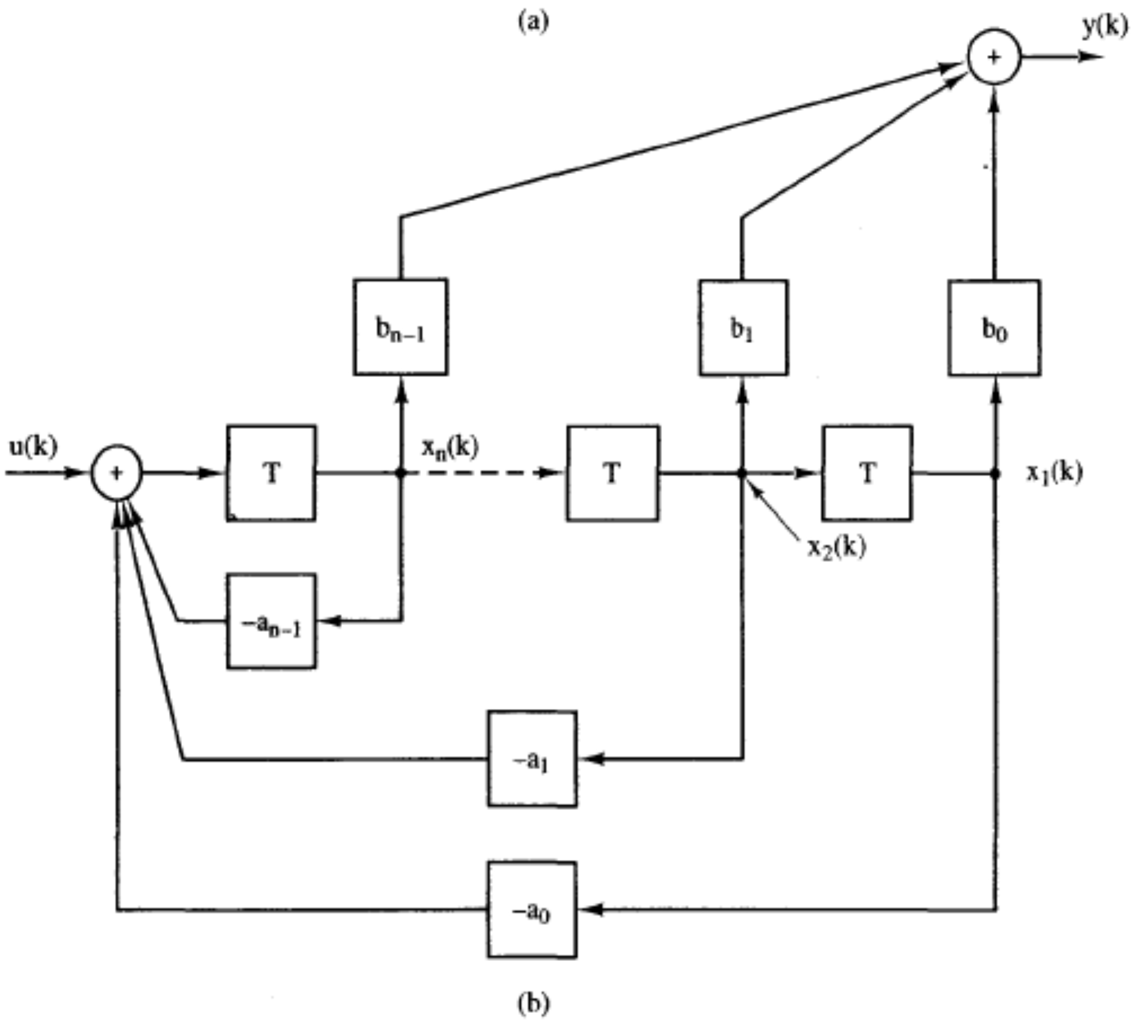
- Control canonical form = phase variable canonical form



State space model formats



State space model formats

- Observer canonical form = Input Feed-forward

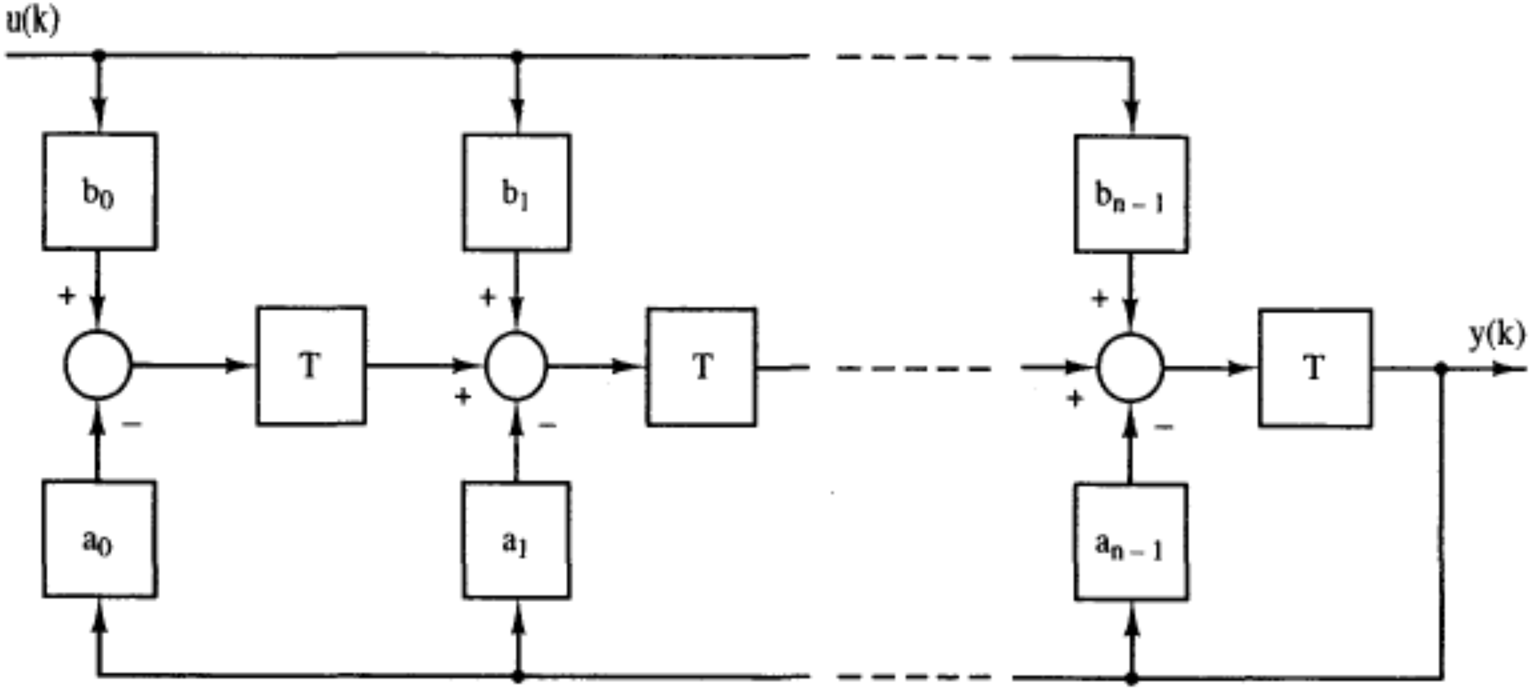


Figure 2-10 Observer canonical form.

Transfer functions

- Derive a transfer function from a state space model.

END