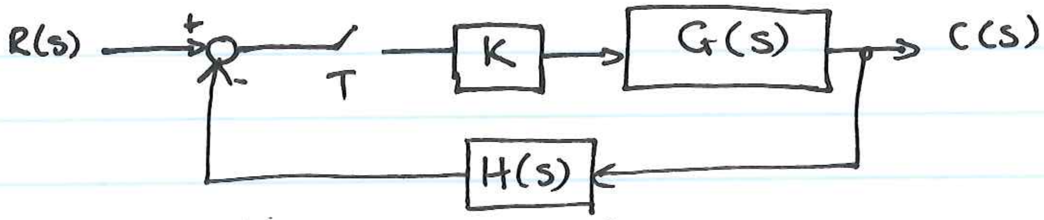
Notes: Root-locus §7.6 Phillips

- Consider a sampled data system



- $T(z) = \dfrac{C(z)}{R(z)} = \dfrac{KG(z)}{1 + K\overline{GH}(z)}$

- CHARACTERISTIC EQ: $1 + K\overline{GH}(z) = 0$

- The root locus of the system above is a plot of the locus of the roots of the characteristic eq. in the z-plane as a function of K.

- Rules for the root-locus construction are identical for both discrete-time and continuous-time systems.

- Reason: Roots of an equation are only dependent on the coefficients of the eq.

RULES FOR ROOT-LOCUS CONSTRUCTION

$$1 + K\overline{GH}(z) = 0$$

1. Loci originate on poles of $\overline{GH}(z)$ and terminate on zero of $\overline{GH}(z)$.

2. The root locus on the real axis always lies in the section of the real axis to the left of an odd number of poles and zeros on the real axis

3. The root locus is symmetrical with respect to the real axis.

4. Number of asymptotes: $n_p - n_z$
   where n_p = poles of $\overline{GH}(z)$, the number of poles
   n_z = number of zeros of $\overline{GH}(z)$.

   Angles of the asymptotes:

   From dorf

$$\phi_A = \frac{(2k+1)}{n_p - n_z}\pi \qquad k = 0, 1, \ldots, (n_p - n_z - 1)$$

   In radians.

5. Asymptotes intersect the real axis at σ_A

$$\sigma_A = \frac{[\Sigma \text{ poles of } \overline{GH}(z)] - [\Sigma \text{ zeros of } \overline{GH}(z)]}{n_p - n_z}$$

6. Breakaway points are given by the roots of

$$\frac{d[\overline{GH}(z)]}{dz} = 0 \qquad \text{and let } \overline{GH}(z) = \frac{N(z)}{D(z)}$$

Equivalently

$$D(z)\frac{dN(z)}{dz} - N(z)\frac{dD(z)}{dz} = 0 \qquad ?$$

since

$$\frac{d}{dz}\left[N(z)\,D^{-1}(z)\right] = \frac{d}{dz}N(z)\cdot D^{-1}(z) + (-1)\,D^{-2}(z)N(z)\frac{d}{dz}D(z)$$

$$= \frac{\frac{dN(z)}{dz}}{D(z)} - \frac{\frac{d}{dz}D(z)\,N(z)}{D^2(z)}$$

$$= \frac{D(z)\frac{dN(z)}{dz} - N(z)\frac{dD(z)}{dz}}{D^2(z)}$$

$$\therefore \quad \frac{D(z)\frac{dN(z)}{dz} - N(z)\frac{dD(z)}{dz}}{D^2(z)} = 0$$

$$\therefore \quad D(z)\frac{dN(z)}{dz} - N(z)\frac{dD(z)}{dz} = 0 \qquad \square$$

ALTERNATIVELY:

$$1 + K\overline{GH}(z) = 0$$

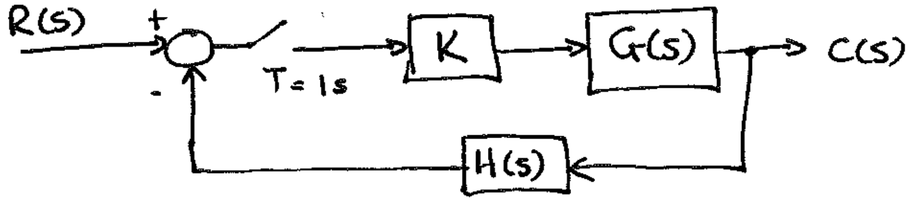
$$K\overline{GH}(z) = -1$$

$$K = -\left(1/\overline{GH}(z)\right) = p(z)$$

setup a table of z vs $p(z)$ and determine the maximum point of $p(z)$.

Example 1 (Root locus)

Consider the following system



In this case $H(s) = 1$, $G_p(s) = \frac{1}{s(s+1)}$

$$G(s) = \frac{1-e^{-Ts}}{s}\left[\frac{1}{s(s+1)}\right]$$

$$G(z) = \frac{z-1}{z}\left[\frac{z\left[(1-1+e^{-T})z + (1-e^{-T}-e^{-T})\right]}{(z-1)^2\,(z-e^{-T})}\right]$$

$$= \frac{(z-1)}{z}\left[\frac{z\left[0,368\,z + 0,264\right]}{(z-1)^2\,(z-0,368)}\right]$$

$$= \frac{0,368\,(z+0,717)}{(z-1)(z-0,368)}$$

$$\therefore \quad K\,G(z) = \frac{0,368\,K\,(z+0,717)}{(z-1)(z-0,368)}$$

$\therefore \quad n_p = 2 \quad : \quad z = 1$ and $z = 0,368$

$n_z = 1 \quad : \quad z = -0,717$

- The loci originate at $z = 1$ and $z = 0,368$ and terminate at $z = -0,717$ and $z = \infty$

- Number of asymptotes $= n_p - n_z = 2 - 1 = 1$

- The angle of the asymptote:

$$\phi_A = \frac{(2k+1)\pi}{n_p - n_z} \qquad k = 0, 1, \ldots, n_p - n_z - 1$$
$$= 0$$

$$= \frac{1}{2-1} \cdot \pi$$

$$= \pi = 180°$$

- Asymptotes intersect : $\sigma_A = \dfrac{[(1) + (0,368)] - [0,717]}{2-1}$

$$= 0,651$$

① - Breakaway point:

$$p(z) = -\frac{1}{G(z)}$$
$$= -\frac{(z-1)(z-0,368)}{0,368\,(z+0,717)} = K = 0,196.$$

z	$0,368$	$\ldots\ldots$	$0,65$	$\ldots\ldots$	1
$p(z)$	0		$0,196$		0

② - Breakaway $\dfrac{d}{dz} G(z) = 0$

$$G(z) = \frac{[0,368\,z + 0,264]}{[z^2 - 1,368\,z + 0,368]} = \frac{N(z)}{D(z)}$$

$$D(z)\frac{dN(z)}{dz} - N(z)\frac{dD(z)}{dz} = 0$$

$$\therefore \ [(z-1)(z-0,368)] \cdot \frac{d}{dz}[0,368z + 0,264]$$

$$- [0,368z + 0,264] \frac{d}{dz}[z^2 - 1,368z + 0,368] = 0$$

$$= [(z-1)(z-0,368)] \cdot (0,368)$$

$$- [0,368z + 0,264][2z - 1,368] = 0$$

$$= 0,368z^2 - 0,503z + 0,135$$
$$- [0,736z^2 + 0,0246z - 0,361] = 0$$

$$= -0,368z^2 - 0,528z + 0,496 = 0$$

$$\therefore \quad 0,368z^2 + 0,528z - 0,496 = 0$$

$$z = 0,6\overset{5}{4} \qquad z = -2,08$$

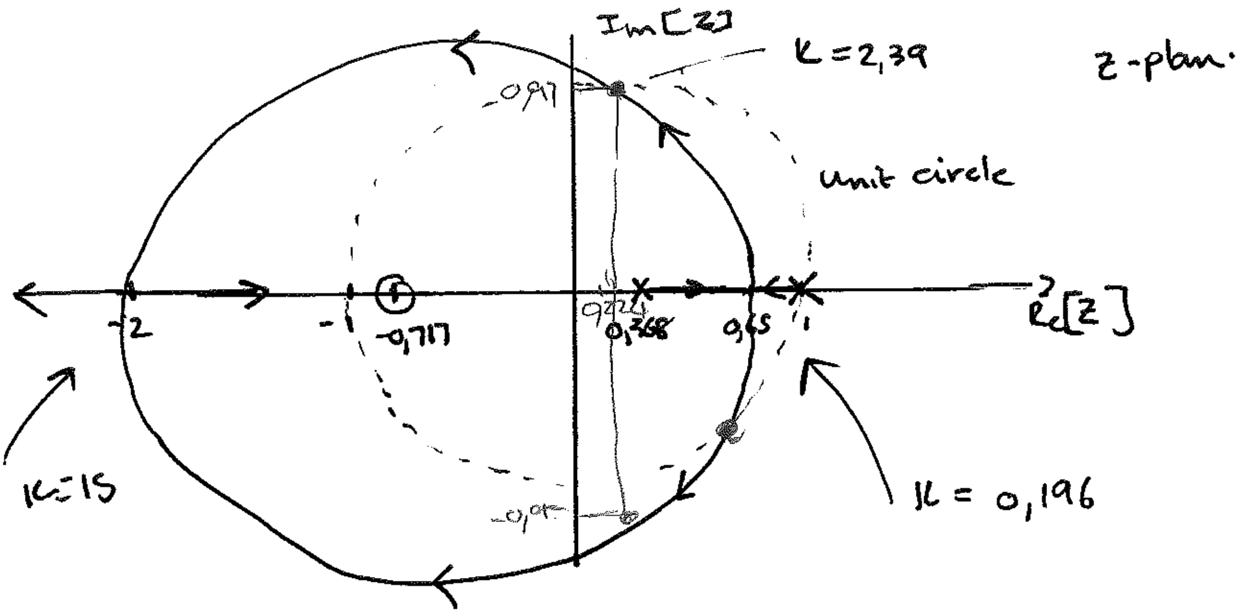
First break away point $\quad z = 0,65 \quad \therefore K = 0,196$ ✓

For the second breakaway:

$$z = -2,08 \quad \therefore \quad K = p(z) = -\frac{1}{G(z)}$$

$$= \left.\frac{-(z-1)(z-0,368)}{0,368(z+0,717)}\right|_{z=-2,08}$$

$$= 15$$



- Determine the points of intersection of the root loci with the unit circle by using the Jury stability test.

- The characteristic equation is :

$$1 + K\overline{GH(z)} = 0$$

In this case $\quad 1 + KG(z) = 0$

$$1 + \frac{(0,368z + 0,264)K}{z^2 - 1,368z + 0,368} = 0$$

or $\quad z^2 + (0,368K - 1,368)z + (0,368 + 0,264K) = 0$

Jury	z^0	z^1	z^2
	$(0,368 + 0,264K)$	$(0,368K - 1,368)$	1

For $\quad Q(1) > 0 \quad \Rightarrow \quad K > 0$

$(-1)^2 Q(-1) > 0 \quad \Rightarrow \quad \cancel{K = 26} \quad K < 26,3$

$|a_0| < a_2 \quad \Rightarrow \quad K < 2,39.$

$\therefore 0 < K < 2,39 \qquad \therefore$ For $K = 2,39$ the system is marginally stable.

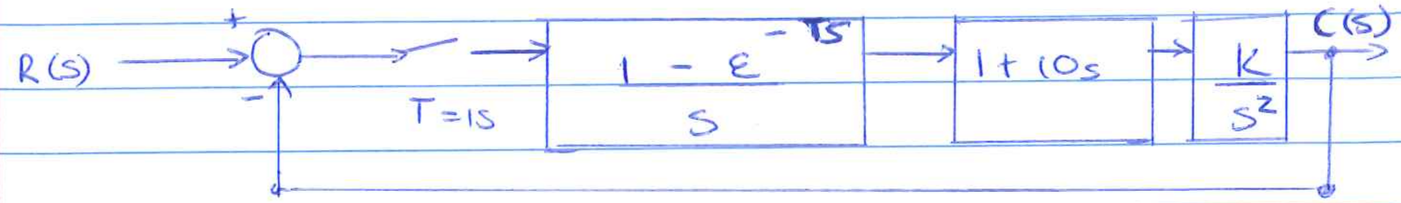
$$z^2 + (0{,}368K - 1{,}368)z + (0{,}368 + 0{,}264K)\Big|_{K=2{,}39}$$

$$= z^2 - 0{,}488z + 1 = 0$$

$$\therefore\ z = 0{,}244 \pm j0{,}970 = 1\ \angle \pm 75{,}9^\circ$$
$$= 1\ \angle \pm 1{,}32 \text{ rad}$$
$$= 1\ \angle \pm \omega T$$

points where the root locus crosses the unit circle.

Example Rootlocus (7.8)



$$KG(s) = 1 + 10s \times \frac{K}{s^2}$$

$$= \frac{K}{s^2} + \frac{10K}{s} = \frac{K + 10Ks}{s^2} = \frac{K(1+10s)}{s^2}$$

$$KG(z) = \mathcal{Z}\left[\frac{1-\varepsilon^{-Ts}}{s}\right]\mathcal{Z}\left[\frac{K(1+10s)}{s^2}\right]$$

$$= \frac{z-1}{z} \times \mathcal{Z}\left[\frac{K(1+10s)}{s^3}\right]$$

$$= \frac{z-1}{z} \times \left[\frac{KT^2(z+1)z}{2(z-1)^3} + \frac{10KTz}{(z-1)^2}\right] \quad \frac{K}{s^3} + \frac{10K}{s^2}$$

$$= \frac{KT^2(z+1)}{2(z-1)^2} + \frac{10KT}{(z-1)}$$

$$= \left.\frac{\frac{K}{2}T^2(z+1) + 10KT(z-1)}{(z-1)^2}\right| T=1$$

$$= \frac{0{,}5K(1)z + 0{,}5K + 10Kz - 10K}{(z-1)^2}$$

$$= \frac{10{,}5Kz - 9{,}5K}{(z-1)^2}$$

$$= \frac{10{,}5K(z - 0{,}9048)}{(z-1)^2}$$

$n_p = 2 \quad : \quad z = 1 \quad , \quad z = 1$

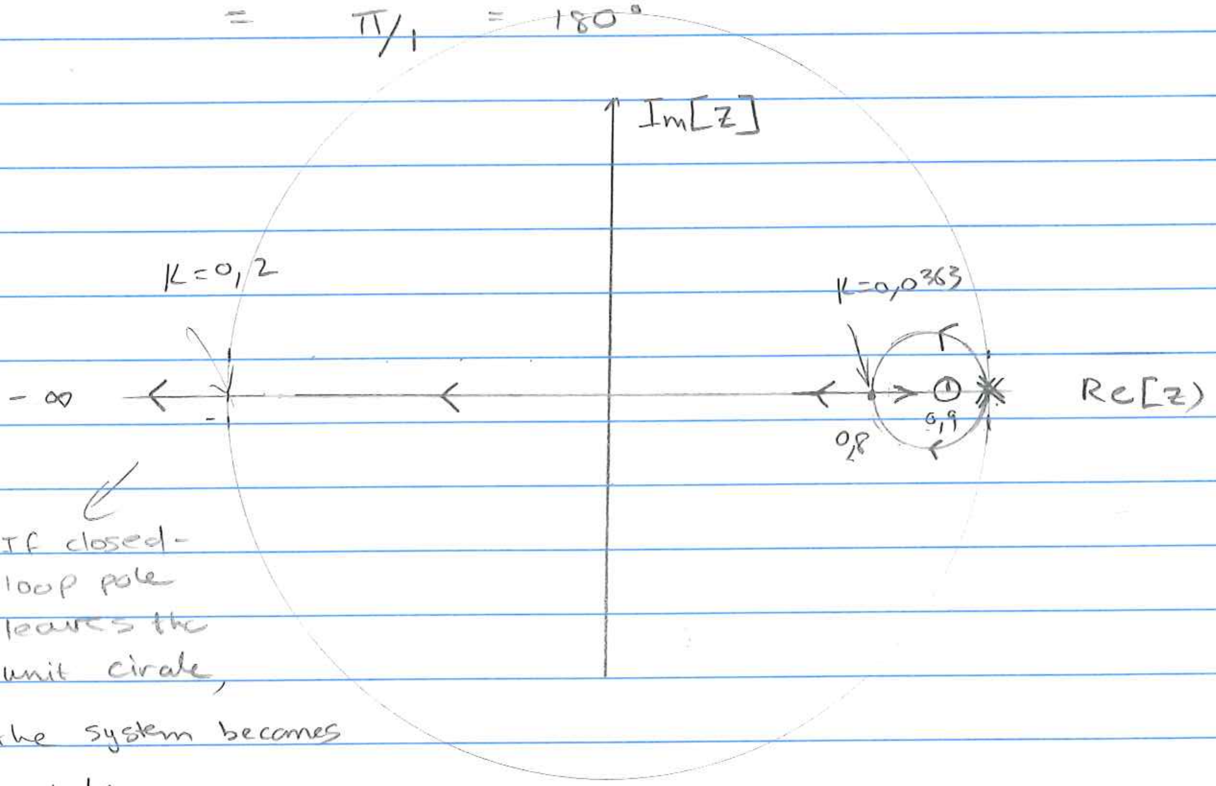
$n_z = 1 \quad : \quad z = 0{,}9048$

$\therefore$ Loci originate at $z = 1$ and terminate at $z = 0{,}9048$ and $z = -\infty$.

Number of asymptotes $\Rightarrow n_p - n_z = 2 - 1 = 1$

$$\phi_A = \frac{(2k+1)\pi}{n_p - n_z} \qquad k = 0, \ldots, n_p - n_z - 1 = 0$$

$$= \pi/1 = 180°$$



If closed-loop pole leaves the unit circle, the system becomes unstable.

Breakaway: $\frac{d}{dz} G(z) = 0$

$$\therefore \quad D(z)\frac{dN(z)}{dz} - N(z)\frac{dD(z)}{dz} = 0$$

$$(z-1)(z-1)$$
$$= z^2 - z - z + 1$$
$$= z^2 - 2z + 1$$

$$G(z) = \frac{10{,}5z - 9{,}5}{z^2 - 2z + 1} = \frac{N(z)}{D(z)}$$

$$(z^2 - 2z + 1)\frac{d}{dz}[10{,}5z - 9{,}5] - (10{,}5z - 9{,}5)\frac{d}{dz}[z^2 - 2z + 1] = 0$$

$$\Rightarrow (z^2 - 2z + 1)(10{,}5) - (10{,}5z - 9{,}5)(2z - 2) = 0$$

$$10{,}5z^2 - 21z + 10{,}5 - [21z^2 - 21z - 19z + 19] = 0$$

$$= (10{,}5z^2 - 21z^2) + (-21z + 40z - 8{,}5) = 0$$

$$= -10{,}5z^2 + 19z - 8{,}5 = 0$$

$$\Rightarrow z^2 - 1{,}8095z + 0{,}8095 = 0$$

Breakaway points at

$z = 0{,}8095$ $\qquad z = 1$

$\simeq 0{,}81$

Determine the value of K at the breakaway points

$$1 + K\,\overline{GH}(z) = 0 \qquad H(z) = 1$$

$$1 + KG(z) = 0$$
$$KG(z) = -1$$
$$K = \frac{-1}{G(z)}$$

$$K = \frac{-(z-1)^2}{10{,}5(z - 0{,}9048)}$$

$\therefore$ for $z = 1$, $K = 0$

for $z = 0{,}81$ $\quad K = \dfrac{-(0{,}81-1)^2}{10{,}5(0{,}81 - 0{,}9048)}$

$= 0{,}0363$

characteristic equation :

$$1 + K\,\overline{GH}(z) = 0$$

In this cas $1 + KG(z) = 0$

$$1 + K\frac{(10{,}5z - 9{,}5)}{z^2 - 2z + 1} = 0$$

$$z^2 - 2z + 1 + 10{,}5Kz - 9{,}5K = 0$$

$$Q(z) = z^2 + (10{,}5K - 2)z + (1 - 9{,}5K) = 0$$

Jury	z^0	z^1	z^2
	$(1 - 9{,}5K)$	$(10{,}5K - 2)$	1

① For $Q(1) > 0$ $\quad (-1)^n Q(-1) > 0$

$\Rightarrow (1)^2 + (10{,}5K - 2)1 + 1 - 9{,}5K > 0$

$1 + 10{,}5K - 2 + 1 - 9{,}5K > 0$

$K > 0$

② $(-1)^2 Q(-1) > 0$

$(-1)^2 + (10{,}5K - 2)(-1) + (1 - 9{,}5K) > 0$

$1 + 2 - 10{,}5K + 1 - 9{,}5K > 0$

$-20K > -4$

$K < 0{,}2$

③ $|a_0| < a_2$

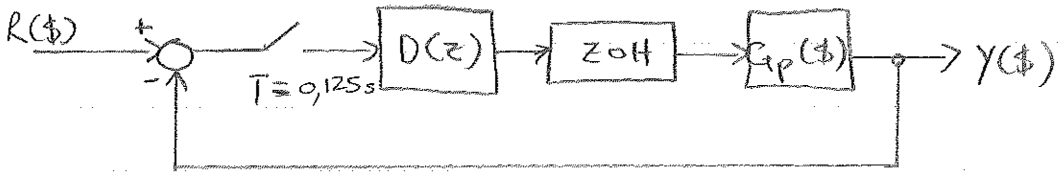
$|1 - 9{,}5K| < 1$ $\qquad 0 < K < 0{,}2$

$-9{,}5K < 0$

$K > 0$

Example Rootlocus

Consider a sampled-data system is given as follows



$$G_p(s) = \frac{10K}{s(s+4)} \quad , \quad D(z) = 1$$

$$KG(s) = \frac{1-e^{-Ts}}{s}\;\frac{10K}{s(s+4)}$$

$$\frac{10}{4}K\left(\frac{4}{s^2(s+4)}\right)$$

$$KG(z) = \frac{z-1}{z}\,\mathcal{Z}\left(\frac{10K}{s^2(s+4)}\right)$$

$$= \frac{z-1}{z} \times \frac{10}{4}K \times \mathcal{Z}\left(\frac{4}{s^2(s+4)}\right)$$

$$= \frac{z-1}{z} \times \frac{10K}{4} \times \left[\frac{(aT-1+e^{-aT})z^2 + (1-e^{-aT}-aTe^{-aT})z}{a(z-1)^2(z-e^{-aT})}\right]$$

$$= \frac{z-1}{z} \cdot \frac{10K}{4} \times \left[\frac{(4\cdot 0{,}125 - 1 + e^{-4\cdot 0{,}125})z^2 + (1-e^{-4\cdot 0{,}125} - 4\cdot 0{,}125\, e^{-4\cdot 0{,}125})z}{4(z-1)^2(z-e^{-4\cdot 0{,}125})}\right] \quad \begin{matrix} a=4 \\ T=0{,}125 \end{matrix}$$

$$= \frac{z-1}{z} \cdot \frac{10K}{4} \times \left[\frac{0{,}1065\,z^2 + 0{,}0902\,z}{4(z-1)^2(z-0{,}607)}\right]$$

$$= \frac{10K}{4\cdot 4} \times \left[\frac{0{,}1065\,z + 0{,}0902}{(z-1)(z-0{,}607)}\right]$$

$$= \frac{10K}{4\cdot 4} \cdot 0{,}1065\,(z+0{,}847)$$
$$= 0{,}06657\,(z+0{,}847)$$

$$= K\left[\frac{0{,}06657\,z + 0{,}05638}{z^2 - 1{,}607\,z + 0{,}6065}\right]$$

$$= \frac{0{,}06657\,K\,(z+0{,}847)}{(z-1)(z-0{,}607)}$$

$n_p = 2$ $\quad z = 1$ and $z = +0,607$

$n_z = 1$ $\quad z = -0,847$

- Number of asymptotes $= n_p - n_z = 2 - 1 = 1$

- The angle of asymptotes

$$\phi_A = \frac{(2k+1)}{n_p - n_z}\pi \qquad k = 0, 1, \ldots, n_p - n_z - 1 = 0$$

$$\therefore \quad \phi_A = \pi = 180^\circ$$

- Asymptotes intersect: $\sigma_A = \dfrac{[(1) + (0,607)] - [0,847]}{2 - 1}$

$$= 0,76$$

- Breakaway point: $P(z) = -\dfrac{1}{G(z)}$

$$= -\frac{(z-1)(z-0,607)}{0,06657(z+0,847)} = K$$

z	0,607	0,78	0,79	0,8025	1
K	0	0,3513	$\underbrace{0,353}$	0,35137	0

$\therefore$ Breakaway point is at $z = 0,79$

Breakaway $\frac{d}{dz} G(z) = 0$

$$G(z) = \frac{0,06657\, z + 0,05638}{[z^2 - 1,607\, z + 0,6065]} = \frac{N(z)}{D(z)}$$

$$\frac{d}{dz} G(z) = D(z) \frac{dN(z)}{dz} - N(z) \frac{dD(z)}{dz} = 0$$

$$\therefore \left(z^2 - 1,607\, z + 0,6065\right)\left(0,06657\right) - \left(0,06657z + 0,05638\right)$$
$$\times \left(2z - 1,607\right) = 0$$

$$\therefore \left(0,06657\, z^2 - 0,107\, z + 0,04\right) - \left(0,133\, z^2 - 0,107\, z\right.$$
$$\left. 0,113\, z - 0,091\right) = 0$$

$$\therefore -0,06643\, z^2 + 0,113\, z + 0,131 = 0$$
$$0,06643\, z^2 - 0,113\, z - 0,131 = 0$$

You can determine the breakaways

* Determine the points of intersection of the root loci with the unit circle by using the Jury stability test

$$Q(z) = 1 + K\, \overline{G}H(z) = 0$$

$$1 + \frac{K\, 0,06657(z + 0,847)}{(z-1)(z-0,607} = 0$$

$$\therefore z^2 - 1,607\, z + 0,6065 + 0,06658\, K z + 0,05638\, K = 0$$
$$z^2 + (0,06658\, K - 1,607)\, z + 0,6065 + 0,05638\, K = 0$$

Jury

① $Q(1) > 0$ $\quad -0{,}0005 + 0{,}12296\,K > 0$

$\Rightarrow K > 0{,}0041$

② $(-1)^2 Q(-1) > 0$ $\quad 3{,}2135 - 0{,}0102\,K > 0 \Rightarrow K < 315$

③ $|a_0| < a_n$ $\quad |\,0{,}6065 + 0{,}05638\,k\,| < 1$

$\Rightarrow K < 6{,}979$

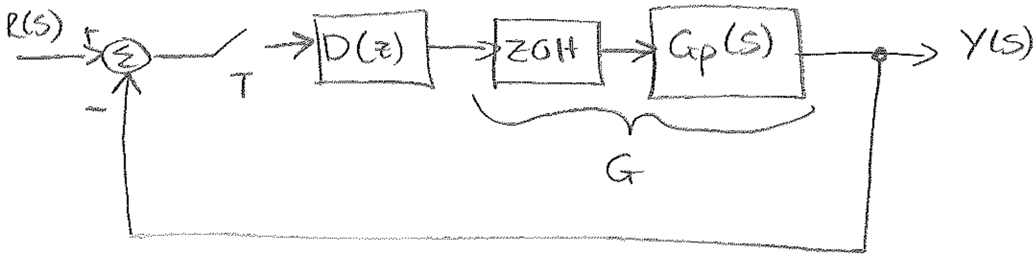
$\therefore$

$K = 6{,}979$

Pole i z by $0{,}563 \pm j\,0{,}826$

Example: Root locus

Given:



- $G_p(s) = \dfrac{10K}{s(s+4)}$

- Diskrete oordragsfunksie van die stelsel:

$$G(z) = \frac{K(0{,}06658z + 0{,}05638)}{\underbrace{z^2 - 1{,}607z + 0{,}6065}_{(z-1)(z-0{,}607)}} \qquad T = 0{,}125\,s$$

$0{,}06658K(z + 0{,}847)$

- $D(z) = 1$

OPLOSSING:

① Wegbreekpunt:

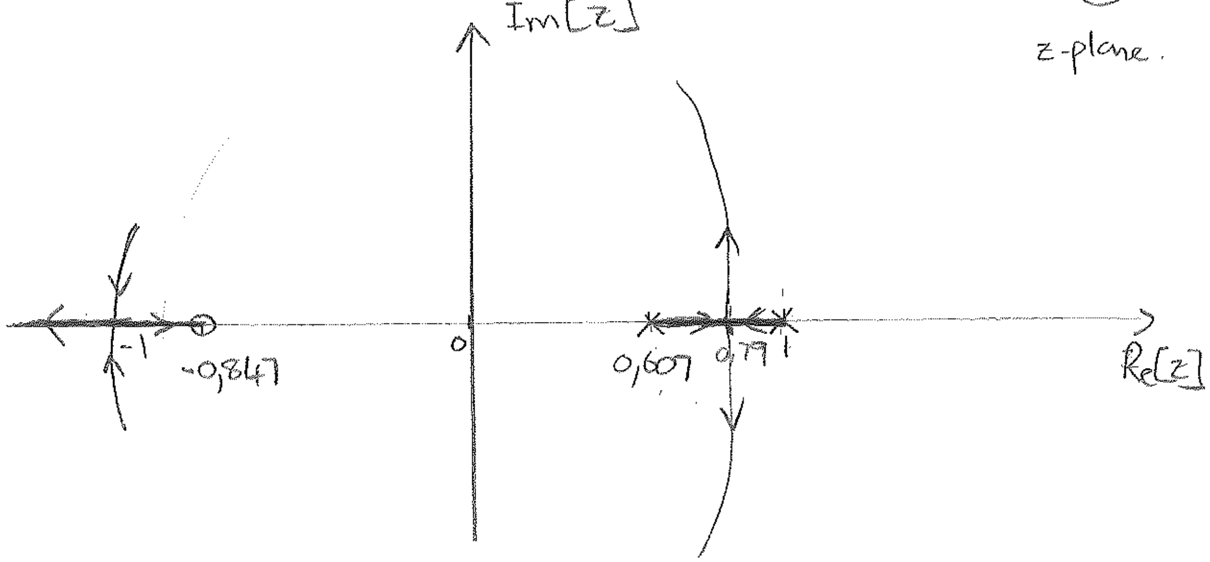
$$1 + KG = 0$$

$$KG = -1$$

$$K = -G^{-1}$$

$$\therefore \quad K = \frac{-(z-1)(z-0{,}607)}{0{,}06658(z+0{,}847)}$$

2



Wegbreekpunt: $k = \dfrac{-(z-1)(z-0,607)}{0,06658\,(z+0,847)}$

z	0,8025	0,79	0,78
k	0,35137	0,353	0,3513

$\therefore$ Wegbreekpunt by $z = 0,79$

KSV $1 + \dfrac{k\,0,0665\,(z+0,847)}{(z-1)(z-0,607)} = 0$

$\therefore\ z^2 - 1,607z + 0,6065 + 0,06658kz + 0,05638k = 0$

$Q = z^2 + (0,06658k - 1,607)z + 0,6065 + 0,05638k = 0$

Gebruik Jury:

① $Q(1) > 0 \quad \therefore -0,0005 + 0,1296k > 0$

$k > 0,0041$

② $(-1)^2 Q(-1) > 0 \quad 3,2135 - 0,0102k > 0 \quad \therefore k < 315$

③ $|a_0| < a_n \quad |0,6065 + 0,05638k| < 1$

$\therefore k < 6,979$

Vir $k = 6,979$

Pole lê by $0,563 \pm j\,0,826$