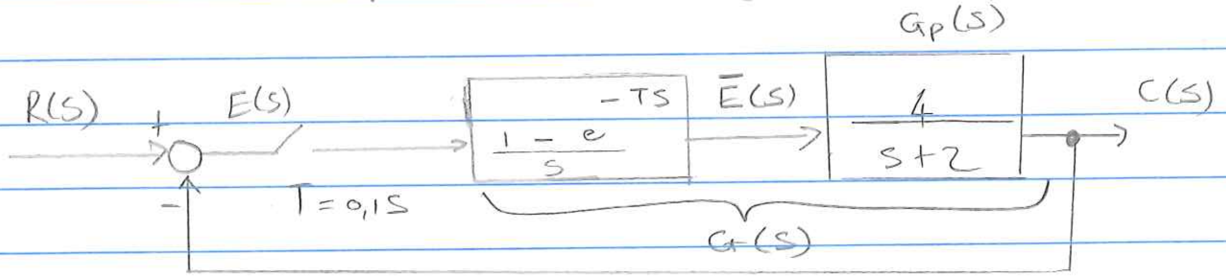
Phillips Chapter 6 — NOTES

§6.1 6.2 SYSTEM TIME-RESPONSE CHARACTERISTICS

We would like to consider the characteristic discrete time response of a system:



Ex 6.1, p 198

The closed-loop transfer function is given by

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)} = T(z)$$

So
$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s}\right] \cdot \left[\frac{4}{s+2}\right]$$
$$= \frac{z-1}{z}\,\mathcal{Z}\left[\frac{4}{s(s+2)}\right]$$

From the table we have

$$\frac{a}{s(s+a)} \Leftrightarrow \frac{z\left(1 - e^{-aT}\right)}{(z-1)\left(z - e^{-aT}\right)}$$

$$G(z) = \left(\frac{z-1}{z}\right) \cdot 2 \cdot \mathcal{Z}\left[\frac{2}{s(s+2)}\right]$$
$$= \left(\frac{z-1}{z}\right) \cdot 2 \cdot \frac{\left(1 - e^{-2T}\right)z}{(z-1)\left(z - e^{-2T}\right)}$$

$$G(z) = \frac{z-1}{z} \, z \cdot \frac{(1-0{,}8187)z}{(z-1)(z-0{,}8187)}$$

$$= \frac{0{,}3625}{z-0{,}8187} \quad \text{where } T = 0{,}1\text{ s}$$

So the closed-loop transfer function $T(z)$ is given by

$$T(z) = \frac{G(z)}{1 + G(z)}$$

$$= \frac{\frac{0{,}3625}{z-0{,}8187}}{1 + \frac{0{,}3625}{z-0{,}8187}}$$

$$= \frac{0{,}3625}{z-0{,}8187+0{,}3625} = \frac{0{,}3625}{z-0{,}4562}$$

So, if the input $R(s)$ is a step input then

$$R(z) = \mathcal{Z}\left[1/s\right] = z/z-1$$

The output is then $C(z) = T(z)R(z)$

$$C(z) = \frac{0{,}3625}{z-0{,}4562} \cdot \frac{z}{z-1}$$

In order to determine the discrete response $c(kT)$ we need to take the inverse z-transform of $C(z)$

so, remember in the case of the partial fraction approach, first devide by z

$$\therefore \quad \frac{C(z)}{z} = \frac{0,3625}{(z-0,4562)(z-1)} = \frac{A}{(z-0,4562)} + \frac{B}{(z-1)}$$

$$A = \left.\frac{0,3625}{(z-1)}\right|_{z=0,4562} = -0,667$$

$$B = \left.\frac{0,3625}{(z-0,4562)}\right|_{z=1} = 0,667$$

$$C(z) = \frac{-0,667\,z}{(z-0,4562)} + \frac{0,667\,z}{(z-1)}$$

from Table we get the sampled output response.

$$c(kT) = -0,667\,(0,4562)^k + 0,667$$

$$= 0,667\left[1-(0,4562)^k\right]$$

The continuous output is derived as follows.

$T_a(s)$ is the analog closed-loop transfer function with the sampler and hold removed.

$$\text{So} \quad T_a(s) = \frac{G_p(s)}{1+G_p(s)} = \frac{\frac{4}{s+2}}{1+\frac{4}{s+2}}$$

The subscript a stands for analogue

$$= \frac{4}{s+2+4}$$

$$= \frac{4}{s+6}$$

$$C_a(s) = T_a(s)\ R(s)$$

$$= \frac{4}{s(s+6)} = \frac{a}{s} + \frac{b}{s+6}$$

$$C_a(s) = \frac{0,667}{s} + \frac{-0,667}{s+6}$$

Taking the inverse Laplace transform gives us

$$c_a(t) = 0,667 - 0,667\,e^{-6t}$$
$$= 0,667\left(1 - e^{-6t}\right)$$

Both the analog and discrete responses can be simulated, see Simulink example.

Clearly the discrete response is the superposition of step responses, hence the form as shown in fig 6-1 (b). Also look at table 6.1

Ex 6.3

For the analogue system, the system response to a unit step is given by

$$C_a(s) = \underbrace{\frac{0,667}{s}}_{\text{Forced response due to step}} + \underbrace{\frac{-0,667}{s+6}}_{\text{natural response / characteristic response of the system}} \Big\} \text{First-order system}$$

and

$$c_a(t) = 0,667 - 0,667e^{-6t}$$

A first-order system has has a transient-response term $Ke^{-t/\tau}$

τ is the time constant

For this system $\frac{1}{\tau} = 6 \Rightarrow \tau = 1/6 = 0,167\,s$

A RULE OF THUMB OFTEN USED FOR SELECTING SAMPLE RATES IS THAT A RATE OF AT LEAST FIVE SAMPLES PER TIME CONSTANT IS A GOOD CHOICE

$\therefore T = 0,167/5 = 0,0334s \approx 0,04\,s$

By choosing the sample period in this way, the sampled-data system is essentially the same as that of the analog system.

The final value of the unit-step response of the sampled-data system can be calculated using the final value theorem of the z-transform.

DISCRETE

$$\lim_{n \to \infty} c(nT) = (z-1)\,C(z)\Big|_{z=1}$$

$$= (z-1)\,\frac{G(z)}{1+G(z)}\,R(z)\Big|_{z=1}$$

$$= \frac{\cancel{(z-1)}\cdot z}{\cancel{(z-1)}}\;\frac{G(z)}{1+G(z)}\Big|_{z=1}$$

$$= \frac{G(z)}{1+G(z)}\Big|_{z=1} = \frac{G(1)}{1+G(1)} = \frac{2}{1+2}$$

$$= 0{,}667$$

CONTINUOUS:

$$C(s) = G_p(s)\,E(s)$$

$$\text{dc gain} = \lim_{s \to 0} G_p(s)$$

$$G_p(s)\Big|_{s=0} = \frac{4}{s+2} = 2 \quad \text{open-loop}$$

$$T(s) = \frac{G_p(s)}{1+G_p(s)} = \frac{4/s+2}{1+\frac{4}{s+2}} \qquad \searrow \quad \frac{2}{1+2} = \frac{2}{3} = 0{,}667$$

$$\Rightarrow \quad \frac{4}{s+6}\Big|_{s=0} = 0{,}667$$

<u>NB</u> FOR A CONSTANT INPUT, THE SAMPLER AND HOLD DOES NOT AFFECT THE GAIN.

Ex 6.4 see Simulink example

This is a second order system

$$T(s) = \frac{G_p(s)}{1 + G_p(s)} = \frac{1/s^2+s}{1 + 1/s^2+s}$$

$$= \frac{1}{s^2 + s + 1} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\therefore \quad 2\zeta\omega_n = 1 \qquad \omega_n^2 = 1$

$\omega_n = \sqrt{1} = 1$ rad/s

$\Rightarrow \quad 2\zeta\,1 = 1$

$\zeta = 1/2 = 0{,}5$

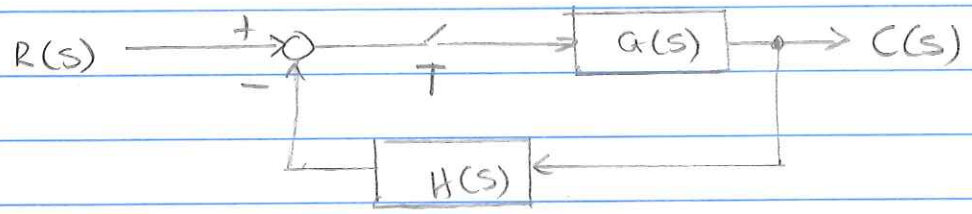
The time constant for a second order system is given by $\tau = 1/\zeta\omega_n$

$= 1/0{,}5 \qquad = 2\,s$

So if $T = 1\,s$ and $\tau = 2$ => sampling rate is too low according to our rule of thumb.

As shown in the response, sampling has a destabilizing effect on the system.

§6.3 System characteristic equation



$$T(z) = \frac{C(z)}{R(z)} = \frac{G(z)}{1 + \overline{GH}(z)}$$

The system characteristic equation is given by

$$1 + \overline{GH}(z) = 0$$

THE ROOTS OF THE CHARACTERISTIC EQUATION ARE THE POLES OF THE CLOSED-LOOP SYSTEM.

§6.4 Mapping the s-plane into the z-plane

Consider a signal $e(t) = e^{-at}$ that is being sampled

So $$E(s) = \frac{1}{s+a}$$

$$E(z) = \frac{z}{z - e^{-aT}}$$

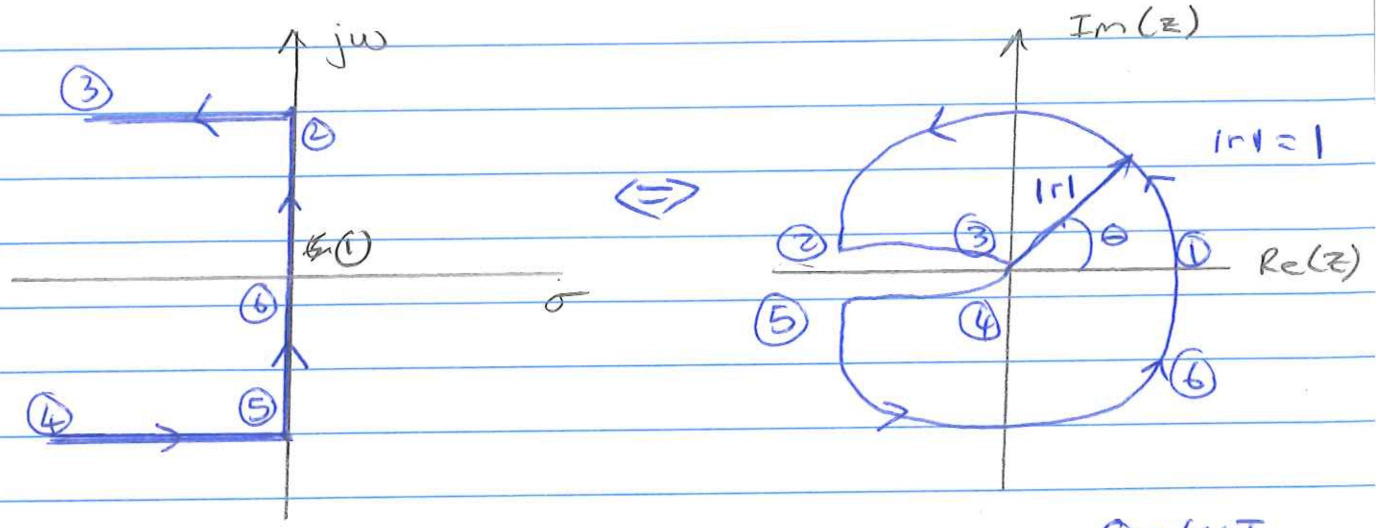
$$E^*(s) = \frac{e^{sT}}{e^{sT} - e^{-aT}} \qquad z = e^{sT}$$

A pole of $E(S)$ at $s = s_1$ results in a z-plane pole of $E(z)$ at $z_1 = e^{s_1 T}$

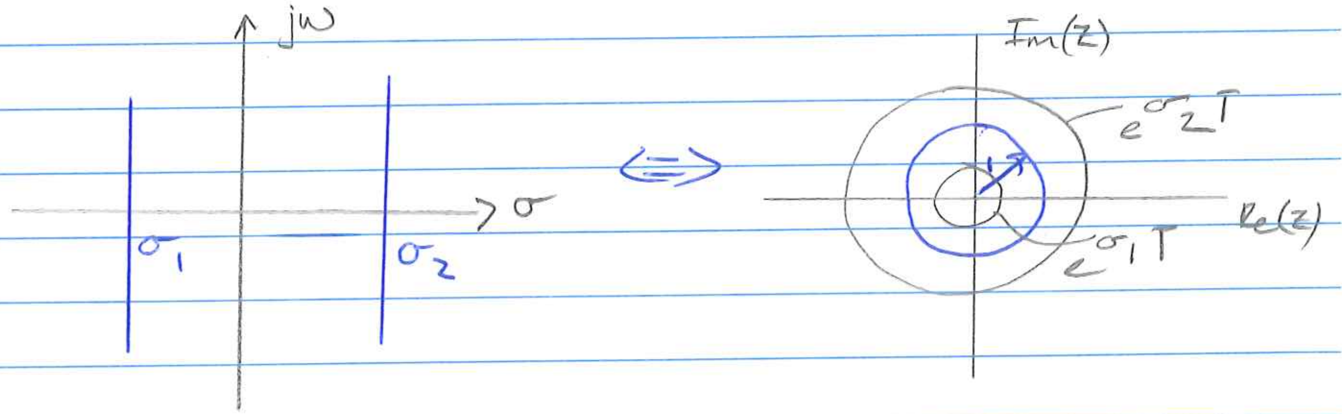
or

A z-plane pole at $z = z_1$ results in a transient response characteristics at the sampling instants of the equivalent s-plane pole s_1 where s_1 and z_1 are related by $z_1 = e^{s_1 T}$

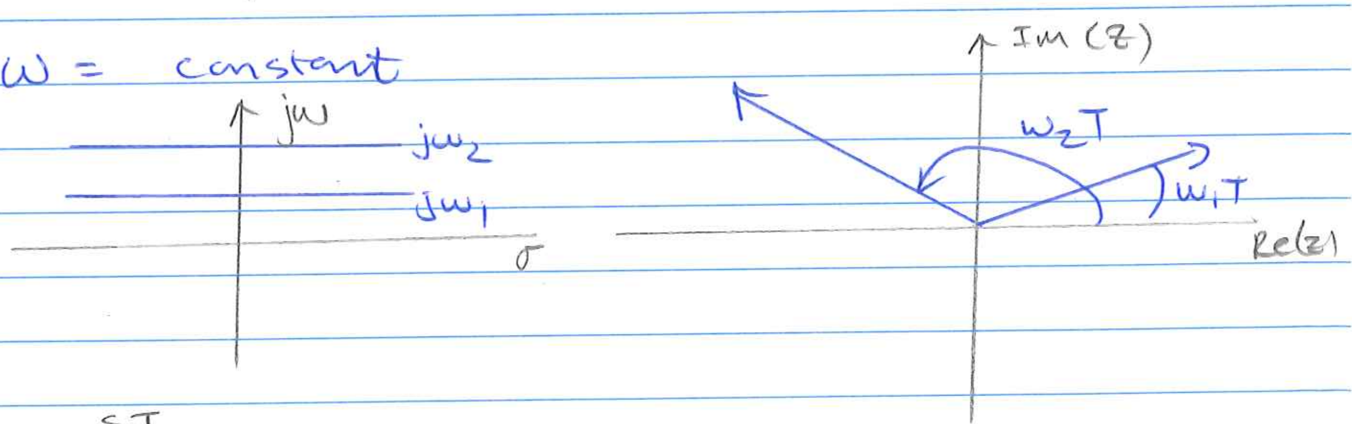
s - plane $\qquad\qquad$ z - plane



σ = constant



ω = constant



$Z = e^{sT}$ $\qquad$ remember $s = \sigma + j\omega$

$= e^{\sigma + j\omega T}$

$= e^{\sigma} e^{j\omega T}$ $\qquad$ and $e^{j\theta} = \cos\theta + j\sin\theta$

$= 1 \angle \omega T$ $\qquad$ $|z| = \sqrt{\cos^2\omega T + \sin^2\omega T}$

$= \sqrt{1} = 1$

POLES LOCATED ON THE UNIT CIRCLE IN THE Z-PLANE ARE EQUIVALENT TO POLE LOCATIONS ON THE IMAGINARY AXIS IN THE S-PLANE

We related the s- and z-plane graphically, now we want to relate it mathematically.

We can express the standard form of the s-plane second-order transfer function as

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

with poles at

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

The equivalent z-plane poles occur at

$$z = e^{sT}\Big|_{s=s_{1,2}}$$

$$= e^{-\zeta\omega_n} \angle \left(\pm\, \omega_n T\sqrt{1-\zeta^2}\right).$$

$$= r \angle \pm\theta$$

So $\quad r = e^{-\zeta\omega_n T}$

so $\quad \ln r = \ln e^{-\zeta\omega_n T}$

$$\ln r = -\zeta\omega_n T$$

$$-\ln r = \zeta\omega_n T$$

Also $$\theta = \omega_n T \sqrt{1-\zeta^2}$$

From the ratio

$$\frac{-\ln r}{\theta} = \frac{\zeta}{\sqrt{1-\zeta^2}}$$

we get $$\boxed{\zeta = \frac{-\ln r}{\sqrt{\theta^2 + (\ln r)^2}}}$$

$$\boxed{\omega_n = \frac{1}{T}\sqrt{(\ln r)^2 + \theta^2}}$$

The time constant is given by

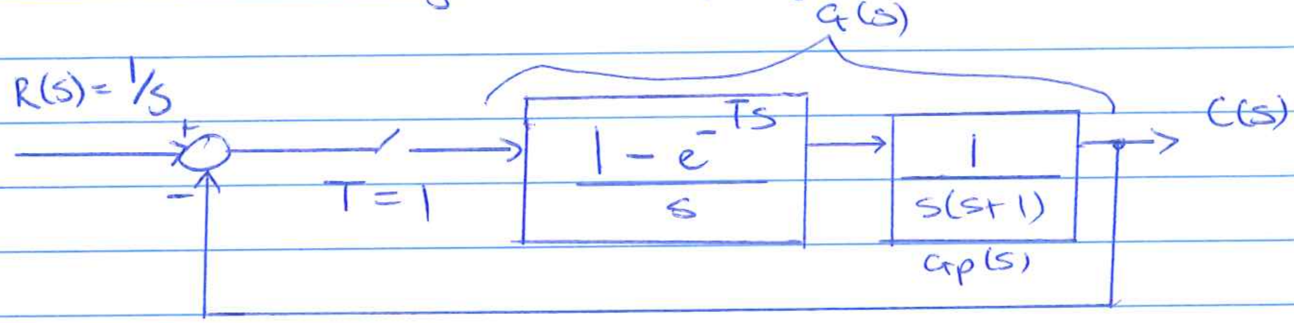
$$\boxed{\tau = 1/\zeta\omega_n = \frac{-T}{\ln r}}$$

or

$$r = e^{-T/\tau}$$

Example 6.6

Consider the following closed-loop system



$$T(z) = \frac{G(z)}{1 + G(z)} = \frac{0{,}368z + 0{,}264}{z^2 - z + 0{,}632} \qquad T = 1s$$

The characteristic equation is given by:

$$z^2 - z + 0{,}632 = 0$$

$$\Rightarrow \quad (z - 0{,}5 - j0{,}618)(z - 0{,}5 + j0{,}618) = 0$$

$$z = 0{,}5 \pm j0{,}618 = 0{,}795 \angle \pm 0{,}890 \text{ rad}$$

$$z = e^{\sigma T} \angle \pm \omega T = r \angle \pm \omega T = 0{,}795 \angle \pm 0{,}890 \text{ rad}$$

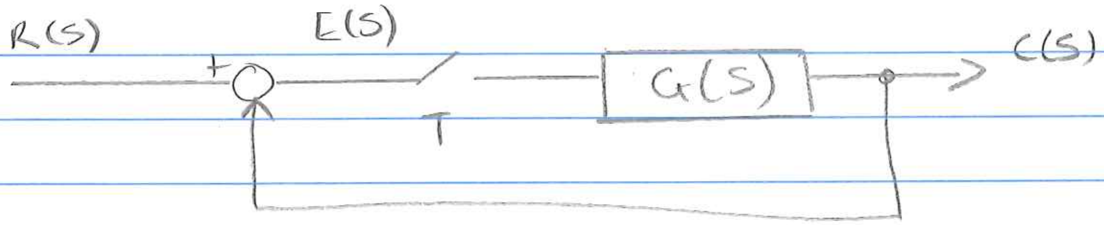
$$\zeta = \frac{-\ln r}{[(\ln 0{,}795)^2 + (0{,}890)^2]^{1/2}} = 0{,}250$$

$$\omega_n = \frac{1}{1}[(\ln 0{,}795)^2 + (0{,}840)^2]^{1/2}$$

$$= 0{,}9191 \text{ rad/sec.}$$

$$\tau = \frac{-1}{\ln 0{,}795} = 4{,}36 \text{ s.}$$

STEADY-STATE ACCURACY



$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)}$$

$$G(z) = \frac{K \prod^{m} (z - z_i)}{(z-1)^N \prod^{p} (z - z_j)} \qquad \begin{matrix} z_i \neq 1 \\ z_j \neq 1 \end{matrix}$$

The value of N is called the system type.

$$K_{dc} = \left. \frac{K \prod^{m} (z - z_i)}{\prod^{p} (z - z_j)} \right|_{z=1}$$

① For a unit-step $R(z) = \frac{z}{z-1}$

$$E(z) = R(z) - C(z)$$

$$= R(z) - \frac{G(z)\, R(z)}{1 + G(z)}$$

$$= \frac{R(z)}{1 + G(z)}$$

$$e_{ss}(kT) = \lim_{z \to 1} (z-1) E(z) = \lim_{z \to 1} \frac{(z-1) R(z)}{1 + G(z)}$$

$$= \frac{1}{1 + \lim_{z \to 1} G(z)}$$

Let $K_p = \lim_{z \to 1} G(z)$

$$e_{ss}(kT) = \frac{1}{1 + K_p}$$

<u>IF $N = 0$</u> $K_p = K_{dc}$

$$e_{ss}(kT) = \frac{1}{1 + K_{dc}}$$

FOR $N \geq 1$ $K_p = \infty$ $\Rightarrow e_{ss} = 0$

② For a unit ramp $R(z) = \frac{zT}{(z-1)^2}$

$$e_{ss} = \lim_{z \to 1} \frac{Tz}{(z-1) + (z-1)G(z)} = \frac{T}{\lim_{z \to 1} (z-1)G(z)}$$

$$K_v = \lim_{z \to 1} \frac{1}{T} (z-1) G(z)$$

<u>IF $N = 0$</u> $K_v = 0$ $e_{ss} = (kT) = \infty$

FOR $N = 1$ $K_v = \frac{K_{dc}}{T}$

$$e_{ss}(kT) = \frac{1}{K_v} = T / K_{dc}$$

For $N \geq 2$ $K_v = \infty$ and $e_{ss}(kT) = 0$