ASREC Graduate Workshop

Note: The Replicator Dynamic

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Evolutionary Game Theory

- Game theory was initially developed by mathematicians and economists.
- Evolutionary biologists adapted these techniques/concepts in developing evolutionary game theory—see for e.g. the pioneering work of British biologist John Maynard Smith. EGT was later imported back into economics.
- Owing to this intellectual history, we shall derive the **replicator dynamic** from a basic biological approach to evolution in which payoffs represent reproduction rates.
- Note that the replicator dynamic can also be generated at the macro level by imitation of more successful agents.

The Biological Approach

Ingredients:

1. **Inheritance**:
   - Players are *programmed* with a strategy. (Players are essentially strategies.)
2. **Selection**:
   - Strategies that do better, given what everyone else is doing, proliferate.
   - In particular, payoffs are interpreted as *reproduction rates* of strategies.
   - Extends Darwin's notion of survival of the fittest from an exogenous environment to an interactive setting.
3. **Mutation**:
   - Equilibrium states can be perturbed by random shocks.
   - To be *stable*, an equilibrium must be resistant to invasion by "mutant strategies".

The Strategic Context

Players

- The population consists of a continuum of agents.

Strategies

- The set of (pure) strategies is $S = \{1, \dots, n\}$, with typical members i, j and s.
- The mass of agents programmed with strategy i is m_i, where $\sum_{i=1}^{n} m_i = m$.
- Let $x_i = \frac{m_i}{m}$ denote the proportion of players choosing strategy $i \in S$.

The Strategic Context

Population States

- The set of population states (or strategy distributions) is $X = \{x \in \mathbb{R}^n_+ : \sum_{i \in S} x_i = 1\}$.
- X is the unit simplex in $\mathbb{R}^n$.
- The set of vertices of X are the pure population states—those in which all agents choose the same strategy.
- These are the standard basis vectors in $\mathbb{R}^n$:

$$e_1 = (1, 0, 0, \ldots), e_2 = (0, 1, 0, \ldots), e_3 = (0, 0, 1, \ldots), \quad \ldots$$

Payoffs

- A *continuous* payoff function $F : X \to \mathbb{R}^n$ assigns to each population state a vector of payoffs, consisting of a real number for each strategy.
- $F_i : X \to \mathbb{R}$ denotes the payoff function for strategy i.

Some Useful Facts

Consider the expected payoff to strategy i if i is matched with another strategy drawn uniformly at random from the population to play the following *two-player* game:

	1	2	...	n
i	$u(i,1)$	$u(i,2)$	...	$u(i,n)$

The expected payoff to strategy i in state x is:

$$
\begin{aligned}
F_i(x) &= x_1 u(i,1) + x_2 u(i,2) \ldots + x_n u(i,n) \\
&= \sum_{j=1}^{n} x_j u(i,j) \\
&= \sum_{j=1}^{n} x_j F_i(e_j).
\end{aligned}
$$

Some Useful Facts

The **average payoff** in the population is:

$$\begin{aligned}\overline{F}(x) &= x_1 F_1(x) + x_2 F_2(x) \ldots + x_n F_n(x) \\ &= \sum_{i=1}^{n} x_i F_i(x).\end{aligned}$$

Note: this is the same as the payoff from playing the mixed strategy x against itself.

The Replicator Dynamic

- Suppose that payoffs represent *fitness* (rates of reproduction) and reproduction takes place in continuous time.
- This yields a continuous-time evolutionary dynamic called the **replicator dynamic** (Taylor and Jonker 1978).
- The replicators here are pure strategies that are copied without error from parent to child.
  - As the population state x changes, so do the payoffs and thereby the fitness of each strategy.
- The replicator dynamic is formalized as a (deterministic) system of ordinary differential equations without mutation.
- We will (at first) analyze the role of mutations indirectly by examining the local stability of various limit states of the replicator dynamic.

The Replicator Dynamic

- Let the rate of growth of strategy i be:

$$\frac{\dot{m}_i}{m_i} = [\beta - \delta + F_i(x)],$$

  where β and δ are “background” birth and death rates (which are independent of payoffs).

- This is the interpretation of payoffs as fitness (reproduction rates) in biological models of evolution.

Derivation

What is the rate of growth of x_i, the *population share* of strategy i?

By definition:

$$\begin{aligned} x_i &= \frac{m_i}{m} \\ \ln(x_i) &= \ln(m_i) - \ln(m) \\ \frac{\dot{x}_i}{x_i} &= \frac{\dot{m}_i}{m_i} - \frac{\dot{m}}{m} \\ &= [\beta - \delta + F_i(x)] - \sum_{j=1}^{n} x_j[\beta - \delta + F_j(x)] \\ &= F_i(x) - \overline{F}(x). \end{aligned}$$

That is, the growth rate of a strategy equals the excess of its payoff over the average payoff.

Some Properties of the Replicator Dynamic

The following results are immediate:

- Those subpopulations that are associated with better than average payoffs grow and *vice versa*.
- The subpopulations associated with pure best replies to the current population state $x \in X$ have the highest growth rate.
- $\dot{x}_i = x_i[F_i(x) - \overline{F}(x)]$, so that if $m_i = 0$ at T, then $m_i = 0$ for all $t > T$.

Example: Pure Coordination

Pure Coordination

	1	2
1	1, 1	0, 0
2	0, 0	2, 2

$$\frac{\dot{x}_1}{x_1} = (1 - x_1)(3x_1 - 2).$$

Therefore, $\frac{\dot{x}_1}{x_1} > 0$ iff $3x_1 > 2$ or $x_1 > \frac{2}{3}$.

Example: Impure Coordination

Stag Hunt

	1	2
1	2, 2	2, 0
2	0, 2	3, 3

$$\frac{\dot{x}_1}{x_1} = (1 - x_1)(3x_1 - 1).$$

Therefore, $\frac{\dot{x}_1}{x_1} > 0$ iff $3x_1 > 1$ or $x_1 > \frac{1}{3}$.

Example: Anti-Coordination

Hawk Dove

	1	2
1	-2, -2	4, 0
2	0, 4	0, 0

$$\frac{\dot{x}_1}{x_1} = (1 - x_1)(4 - 6x_1).$$

Therefore, $\frac{\dot{x}_1}{x_1} > 0$ iff $6x_1 < 4$ or $x_1 < \frac{2}{3}$.