CSE 441T/541T Advanced Algorithms **Homework 0**

Practice Your Proofs!

Assigned: August 25, 2014 *Due Date: None*

An important prerequisite for this course is some ability to create and write correct proofs of simple propositions and to realize when you have failed to properly complete a proof. The proof techniques that are used most in this course are proof by contradiction (or indirect proof), induction (both weak and strong), and proof by cases.

To help each of you evaluate your readiness to use these proof techniques, I am providing this *optional* homework. This homework will in no way be part of your grade, so it would be pointless to look up solutions in notes or books that you have. Also, remember that it is much easier to understand a proof than create one; hence, this homework will be of little value if a friend tells you how he/she did the proof – even if you understand it completely once it is explained to you. The goal here is to see if you can solve these problems and provide convincing proofs *on your own*.

Some of these proofs are non-trivial and require some thought, but so do the proofs you will be doing throughout this course. If you need some guidance, that is fine – just come by office hours or make an appointment. If you aren't sure about whether or not your proofs for these problems are valid proofs, then I recommend that you submit your solutions to me. I will read them and return them with comments. Our recitation exercises will also help you brush up your proof skills.

1. Let $P(n)$ be the proposition that any n lines, where no two are parallel and no three pass through the same point, divide the plane into n^2+1 regions. What is wrong with the following inductive proof? It is not sufficient to give a counterexample to the theorem. Rather, you must find and describe the flaw in the proof.

   Theorem: $\forall n \geq 1,\ P(n)$

   Proof: By induction on n.
   Basis Step: 1 line divides the plane into 2 regions and $1^2 + 1 = 2$. Hence $P(1)$ is true.

   Inductive Step: We must show that $\forall n \geq 1\ P(n) \rightarrow P(n+1)$. By the inductive hypothesis there are $n^2 + 1$ regions formed with n lines. Note that $(n+1)^2 + 1 = n^2 + 1 + 2n + 1$. So adding the $(n+1)$st line creates $2n+1$ new regions. Hence the number of regions with $n+1$ lines is $n^2 + 1 + 2n + 1 = (n+1)^2 + 1$

   Since $P(1)$ is true and $\forall n \geq 1,\ (P(n) \rightarrow P(n+1))$, by the principle of mathematical induction we have that $\forall n \geq 1,\ P(n)$. ■

2. Prove that n lines separate the plane into $(n^2 + n + 2)/2$ regions if no two of these lines are parallel and no three pass through a common point.

3. Consider an ice cream parlor that offers cones with two scoops. There are n flavors available, and it is possible to get two scoops of the same flavor. Prove that there are n^2 possible ways to make a two-scoop ice cream cone.

4. A *perfect number* is an integer which is equal to the sum of all its divisors except the number itself. Thus 6 is a perfect number, since $6 = 1+2+3$, and so is 28. By definition, 1 is not considered to be a prime number.

   Prove that no perfect number is prime.

5. Prove that there is no largest prime number. That is, prove that for any prime number p, there is a prime number p' such that $p' > p$. (*Note*: it may help to remember that every integer can be decomposed as a product of prime factors.)

6. Suppose you have a set of homes that you want to connect via a communications network. Assume that you can directly place a connection between any two homes with a cost that is proportional to the distance between the two homes. Your goal is to choose which set of connections to install so that you incur the minimum possible cost under the constraint that you must ensure that all homes are connected (i.e. between any two homes you can follow a sequence of connections from one to the other).

   Prove that there exists a solution to this problem that is optimal (i.e. it connects all the homes and has the minimum possible cost) and that includes a direct connection between the two closest homes.

7. Prove that for all configurations of four points in the plane, it is possible to color each point either red or blue, so that *no* line can be drawn so as to place all red points on one side of the line and all blue points on the other.

   Hint: Use a proof by cases.