Notes on decay chain differential equations

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1 Radioactive nucleon production/decay equations and solutions

We are interested in studying the following differential equation, related to nucleon decay/production:

$$\begin{cases} dN_1 &= -\Gamma_1 N_1 dt \\ dN_2 &= \Gamma_1 N_1 dt - \lambda_{23} N_2 dt - \Gamma_{24} N_2 dt \\ dN_3 &= \lambda_{23} N_2 dt - \Gamma_3 N_3 dt \\ dN_4 &= \Gamma_{24} N_2 dt - \lambda_4 N_4 dt \\ dN_5 &= \Gamma_3 N_3 dt + \lambda_4 N_4 dt - \lambda_5 N_5 dt \\ dN_6 &= \lambda_5 N_5 dt \end{cases}$$

Regrouping:

$$\begin{cases} N_1^{'} &= -\Gamma_1 N_1 + 0 \\ N_2^{'} &= -\left(\lambda_{23} + \Gamma_{24}\right) N_2 + \Gamma_1 N_1 \\ N_3^{'} &= -\Gamma_3 N_3 + \lambda_{23} N_2 \\ N_4^{'} &= -\lambda_4 N_4 + \Gamma_{24} N_2 \\ N_5^{'} &= -\lambda_5 N_5 + \left(\Gamma_3 N_3 + \lambda_4 N_4\right) \\ N_6^{'} &= -0 + \lambda_5 N_5 \end{cases}$$

The transition rate due to neutron capture is $\Gamma_i = \sigma_i \cdot F$, F is the neutron flux. λ is the decay constant.

Simple case of constant creation rate of an unstable nucleus:

For a particle being created at a constant rate and decaying into $n = m$ particles:

$$dN = Rdt - \lambda N dt$$

R is the creation rate, λ is the total decay rate for this particle, $\lambda = \sum_{i=1}^{m} \lambda_i$. The above equation is a non-homogenous first order differential equation, which can be solved by the integrating function method:

$$\begin{aligned} \phi N' + \phi\lambda N &= \phi R \\ (\phi N)' &= \phi R \end{aligned}$$

Using an integration function:

$$\begin{aligned} \phi' &= \phi\lambda \\ \phi &= e^{\lambda t} \end{aligned}$$

thus by integrating from $t' = 0$ to $t' = t$:

$$\begin{aligned} e^{\lambda t} N &= \frac{R}{\lambda}\left(e^{\lambda t} - 1\right) \\ N(t) &= \frac{R}{\lambda}\left(1 - e^{-\lambda t}\right) \end{aligned}$$

Case of a time dependent creation rate ($A \to B \to N_f$):

Let's take the example of a time dependent creation rate, for example $A \to B$, where A, B are both unstable. We have that:

$$\begin{aligned} dN_B &= -N_A^{'}(t)dt - \lambda N_B dt \\ dN_B + \lambda N_B dt &= -N_A'(t)dt \end{aligned}$$

This is similar to what was solved previously, but now we have $R = R(t) = -N_A^{'}(t)\,(= \mathcal{A}_A(t))$, we write:

$$N^{'} + \lambda N = \mathcal{A}(t)$$

This can be solved analogously to the constant R case, now we just need to be careful to integrate $\phi\mathcal{A}$ as both are functions of time. We can see that in all decay cases of constant decay rate of the particle at hand (λ = constant) the integrating function will be $\phi = e^{\lambda t}$, meaning that the solution is of the form:

$$N(t) = e^{-\lambda t}\left(\int e^{\lambda t}\mathcal{A}(t)dt + C\right)$$

In the case at hand $\mathcal{A}(t) = \lambda_A N_A(t) = \lambda_A N_A(0) e^{-\lambda_A t}$, giving the solution:

$$N(t) = N_A(0)\frac{\lambda_A}{\lambda_B - \lambda_A} e^{-\lambda_A t} + C e^{-\lambda_B t}$$

We can fix the constant C by imposing the initial conditions, in this case $N(0) = 0$:

$$N(t) = N_A(0)\frac{\lambda_A}{\lambda_B - \lambda_A}\left(e^{-\lambda_A t} - e^{-\lambda_B t}\right)$$

General decay:

Consider a decay $N_i \to B \to N_f$ where all decay rates are constant and in an hierarchy*. We'll denote the decay rate from X into one of the decay products Y as $\lambda_{X\to Y}$.

We define the total decay rate $\lambda_B = \sum_{j=1}^{f} \lambda_{B\to N_f^j}$, and consider that each N_i decays into B with a decay ratio $f_{N_i^j\to B} = \frac{\lambda_{N_i^j\to B}}{\lambda_{N_i^j}}$ (we define $\lambda_{N_i^j}$ similarly to how we defined λ_B). We define the activity into B as $\mathcal{A}_{\to B}(t) = \sum_{j=1}^{i} f_{N_i^j\to B}\mathcal{A}_{N_i^j}(t)$. With this dense notation in mind we get the simple looking differential equation:

$$N_B^{'} + \lambda_B N = \mathcal{A}_{\to B}(t)$$

Which we already solved:

$$\begin{aligned} N_B(t) &= e^{-\lambda_B t}\left(\int e^{\lambda_B t}\mathcal{A}_{\to B}(t)dt + C\right) \\ &= \left[\sum_{j=1}^{i} f_{N_i^j\to B}\int e^{\lambda_B t}\mathcal{A}_{N_i^j}(t)dt + C\right]e^{-\lambda_B t} \end{aligned}$$

All left to do is computing $\int e^{\lambda_B t}\mathcal{A}_{N_i^j}(t)dt$, which is trivial since all $\mathcal{A}_{N_i^j}(t)$ are composed of linear combinations of exponential functions. We also fix C using the initial conditions, usually $N_B(0) = 0$, we expect the final solution to be of the form $N_B(t) = \sum_{i=1}^{\mathcal{H}_B} \alpha_i e^{-\lambda_i t} + Ce^{-\lambda_B t}$, where $\mathcal{H}_B$ denotes every particle above B in the hierarchy and $\alpha_j\left(\{\lambda_{N_j}, \lambda_B, \lambda_k | k \in \mathcal{H}_{N_j}\}\right)$.

This constitutes a algorithm, where we solve the equation layer by layer, the first layers gives that all $N_i \in \{N_k | k \text{ in the first layer}\}$ have solution $N_i(t) = N_i(0)e^{-\lambda_i t}$, we then compute the rest of the solutions layer by layer.

If we apply this to a linear hierarchy we expect to recover the Bateman equations (Krane - Cap. 6.4).

*We define decay in an hierarchy as a chain decay where the products of decays don't decay into eachother, or parent particles of eachother or any previous decay. This is of interest because it can be iterativelly solved: a closed formula for the number of particles in a certain "layer" is easily achieved, as the differential equations depend only on previously solved "layers".

Solutions to the proposed system of equations

Applying the previous method to the case at hand, we assume $\lambda_i \neq \lambda_j$, $i \neq j$:

$$N_1 = N_0 e^{-\lambda_1 t}$$

$$
\begin{aligned}
N_2 &= N_0 \left[\lambda_1 \int e^{\lambda_2 t} e^{-\lambda_1 t}(t)dt + C\right] e^{-\lambda_2 t} \\
&= N_0 \frac{\lambda_1}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)
\end{aligned}
$$

$$
\begin{aligned}
N_3 &= \left[\frac{\lambda_{2,3}}{\lambda_2} \int e^{\lambda_3 t} \mathcal{A}_{N_2}(t)dt + C\right] e^{-\lambda_3 t} \\
&= N_0 \frac{\lambda_1 \lambda_{2,3}}{\lambda_2 - \lambda_1} \left(\int \left(e^{(\lambda_3 - \lambda_1)t} - e^{(\lambda_3 - \lambda_2)t}\right) dt + C\right) e^{-\lambda_3 t} \\
&= N_0 \frac{\lambda_1 \lambda_{2,3}}{\lambda_2 - \lambda_1} \left(\frac{e^{-\lambda_1 t}}{\lambda_3 - \lambda_1} - \frac{e^{-\lambda_2 t}}{\lambda_3 - \lambda_2} + Ce^{-\lambda_3 t}\right) \\
&= N_0 \frac{\lambda_1 \lambda_{2,3}}{\lambda_2 - \lambda_1} \left(\frac{e^{-\lambda_1 t}}{\lambda_3 - \lambda_1} - \frac{e^{-\lambda_2 t}}{\lambda_3 - \lambda_2} - \left(\frac{\lambda_1 - \lambda_2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}\right) e^{-\lambda_3 t}\right)
\end{aligned}
$$

And the same for N_4:

$$
N_4 = N_0 \frac{\lambda_1 \lambda_{2,4}}{\lambda_2 - \lambda_1} \left(\frac{e^{-\lambda_1 t}}{\lambda_4 - \lambda_1} - \frac{e^{-\lambda_2 t}}{\lambda_4 - \lambda_2} - \left(\frac{\lambda_1 - \lambda_2}{(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)}\right) e^{-\lambda_4 t}\right)
$$

Finally N_5:

$$
\begin{aligned}
N_5(t) &= \left[\sum_{j=3}^{4} \lambda_i \int e^{\lambda_5 t} N_i(t)dt + C\right] e^{-\lambda_5 t} \\
&= \sum_{j=3}^{4} N_0 \frac{\lambda_1 \lambda_{2,i} \lambda_i}{\lambda_2 - \lambda_1} \left[\int \frac{e^{(\lambda_5 - \lambda_1)t}}{\lambda_i - \lambda_1} - \frac{e^{(\lambda_5 - \lambda_2)t}}{\lambda_i - \lambda_2} - \left(\frac{\lambda_1 - \lambda_2}{(\lambda_i - \lambda_1)(\lambda_i - \lambda_2)}\right) e^{(\lambda_5 - \lambda_i)t} dt + C\right] e^{-\lambda_5 t} \\
&= \sum_{j=3}^{4} N_0 \frac{\lambda_1 \lambda_{2,i} \lambda_i}{\lambda_2 - \lambda_1} \left[\frac{e^{-\lambda_1 t}}{(\lambda_i - \lambda_1)(\lambda_5 - \lambda_1)} - \frac{e^{-\lambda_2 t}}{(\lambda_i - \lambda_2)(\lambda_5 - \lambda_2)}\right. \\
&\left. - \left(\frac{\lambda_1 - \lambda_2}{(\lambda_i - \lambda_1)(\lambda_i - \lambda_2)(\lambda_5 - \lambda_i)}\right) e^{\lambda_i t} + Ce^{-\lambda_5 t}\right]
\end{aligned}
$$

We find C by imposing $N(0) = 0$ as usual, this becomes tedius at this point, we can simply numerically compute this by evaluating the time dependent terms at $t = 0$.