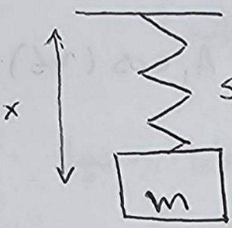
# Harmonic oscillator



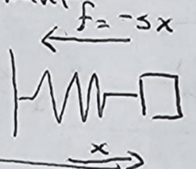


Recall $f=ma$, in terms of $x$ (displacement)

$$\boxed{f = m\frac{d^2x}{dt^2}}$$




spring stiffness (s) $\rightarrow$ restoring force (working against ~~gravity~~ displacement): $\boxed{f=-sx}$

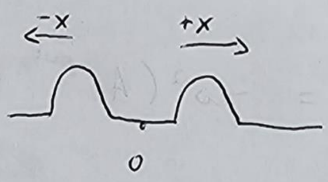



substitution $f = m\frac{d^2x}{dt^2} \Rightarrow \boxed{-sx = \frac{md^2x}{dt^2}}$ now lets define: $\boxed{\omega^2=\frac{s}{m}}$ angular frequen

$$\frac{1}{m}-sx = \frac{md^2x}{dt^2}\frac{1}{m} \Rightarrow -\omega^2 x = \frac{\cancel{m}d^2x}{dt^2} \Rightarrow 0=\frac{d^2x}{dt^2}+\omega^2 x$$

replace $\frac{d^2x}{dt^2}$ with $\ddot{x}$ $\Rightarrow \boxed{\ddot{x} + \omega^2 x = 0}$ this is a different equation! Thankfully, to solve we guess & check

Generally some function moving left and another moving right




$f(t+\frac{x}{c})$ $\quad f(t-\frac{x}{c})$

for now assume

$$x = A\cos \eta t$$

Lets find first & second derivitives

$x = A_1 \cos \gamma t \longrightarrow A_1 \cos(\gamma t)$

$\dot{x} = -A_1 \gamma \sin \gamma t \longrightarrow -A_1 \gamma \sin(\gamma t)$

$\ddot{x} = -A_1 \gamma^2 \cos \gamma t \rightarrow -A_1 \gamma^2 \cos(\gamma t)$

use substitution

$$\ddot{x} + \omega^2 x = 0 \Rightarrow \ddot{x} = -\omega^2 x \Rightarrow -A_1 \underline{\gamma}^2 \cos(\gamma t) = \underline{-\omega^2} A_1 \cos(\gamma t)$$

$\therefore$ if $\gamma^2 = \omega^2$ or $\gamma = \omega$ this solution works!

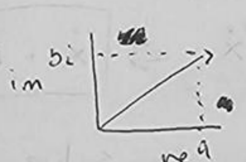
$\boxed{\text{Hw: Show that } A_2 \sin(\omega t) \text{ is also a solution}}$

Complex notation - makes differentiation & integration easier widely used in signal processing $\quad j = \sqrt{-1}$

$\frac{d}{dx} e^x = e^x$, $\boxed{\frac{d}{dx} e^{nx} = n e^{nx}}$ also we know Euler's formula $\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$

$\text{Real}(e^{j\theta}) = \cos\theta$, $\text{Imag}(e^{j\theta}) = j\sin\theta$

Magnitude & phase $\quad z = a + bi$




magnitude $|z| = \sqrt{a^2 + b^2}$

phase $= \tan^{-1}\left(\frac{b}{a}\right)$

New assumed solution to $sx = m\ddot{x}$

$x = A_1 e^{j\omega t} + A_2 e^{-j\omega t}$

$\dot{x} = j\omega A_1 e^{j\omega t} - j\omega A_2 e^{-j\omega t}$

$\ddot{x} = j^2 \omega^2 A_1 e^{j\omega t} - (-j^2)\omega^2 A_2 e^{j\omega t} = -\omega^2 (A_1 e^{j\omega t} + A_2 e^{-j\omega t})$

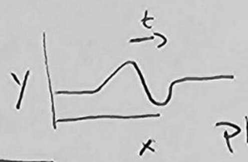
$s(A_1 e^{j\omega t} + A_2 e^{-j\omega t}) = -m\omega^2 (A_1 e^{j\omega t} + A_2 e^{-j\omega t})$

$s = -m\omega^2 \quad \therefore$ if $\omega^2 = \frac{s}{m}$ its a solution!

1-dimensional wave equation 

string tension $= T$, linear density $= \rho_L$, three variables

$x, y, \ \& \ t$




$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}}$$

phase speed

$$\boxed{c^2 = \frac{T}{\rho_L}}$$

$$y(x,t) = \overbrace{y_1(ct - x)}^{\text{some function}} + y_2(ct$$

wave number

$$\boxed{k^2 = \frac{\omega^2}{c^2}}$$

Again, lets assume a solution $\boxed{y = Ae^{j(\omega t - kx)}}$

recall the <u>chain rule</u>: $\frac{d}{dx} e^{fx} = e^{fx} \frac{d\, f(x)}{dx}$

for partial dirivitives, treat one variable as constant

$\frac{\partial^2 y}{\partial x^2} Ae^{j(\omega t - kx)}$ -> treat $t$ as constant and differentiate w.r.t $x$

$$\frac{d}{dx} Ae^{j(\omega t - kx)} \Rightarrow \text{chain rule} \Rightarrow Ae^{j(\omega t - kx)} \frac{d}{dx} j(\omega t - kx) = Ae^{j(\omega t - kx)} \cdot -jk$$

$$\frac{d}{dx} Ae^{j(\omega t - kx)} = -jk\, Ae^{j(\omega t - kx)}$$

$$\frac{d^2}{dx^2} Ae^{j(\omega t - kx)} = \frac{d}{dx} -jk\, Ae^{j(\omega t - kx)} \Rightarrow \text{chain rule} \Rightarrow -jk\, Ae^{j(\omega t - kx)} (-jk) =$$

$$\frac{\partial^2 y}{\partial x^2} Ae^{j(\omega t - kx)} = -j^2 k^2 Ae^{j(\omega t - kx)} = k^2 Ae^{j(\omega t - kx)}$$

do the same for $\frac{\partial^2 y}{\partial t^2} = \omega^2 Ae^{j(\omega t - kx)}$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

$$k^2 Ae^{j(\omega t - kx)} = \frac{1}{c^2}\omega^2 Ae^{j(\omega t - kx)} \quad \therefore \text{ if } k = \frac{\omega}{c} \text{ solution}$$