… solutions for robust engineering

Rare events probability estimation

Gaëtan Blondet, Phimeca Engineering SA

‘HPC and Uncertainty Treatment – Examples with Open TURNS and Uranie’

EDF – Phimeca – Airbus Group – IMACS – CEA

PRACE Advanced Training Center – May, 27-29 2019

MAISON DE LA SIMULATION

Outline

- Problem definition
- Brute-force estimation using Monte Carlo sampling
- Importance sampling
- Isoprobabilistic transformation
- Most-probable-failure-point(s)-based methods

Outline

- Problem definition
- Brute-force estimation using Monte Carlo sampling
- Importance sampling
- Isoprobabilistic transformation
- Most-probable-failure-point(s)-based methods

Problem definition

Given

- a *random vector* with known probability distribution:

$$\boldsymbol{X} \sim F_{\boldsymbol{X}}$$

  modelling the uncertainty attached to a component of interest.

- and a *performance model* that characterize its state :

$$g\colon\ \boldsymbol{x} \mapsto g(\boldsymbol{x})$$

  with the convention :

  – if $g(\boldsymbol{x}) \leq 0$, then the system *fails* ;

  – otherwise, it is *safe*.

Objective

Quantify the component safety level
in the form of a *(subjective) failure probability*.

« subjective probability » = A probability that is condionned by assumptions/choices (probabilistic + performance models)

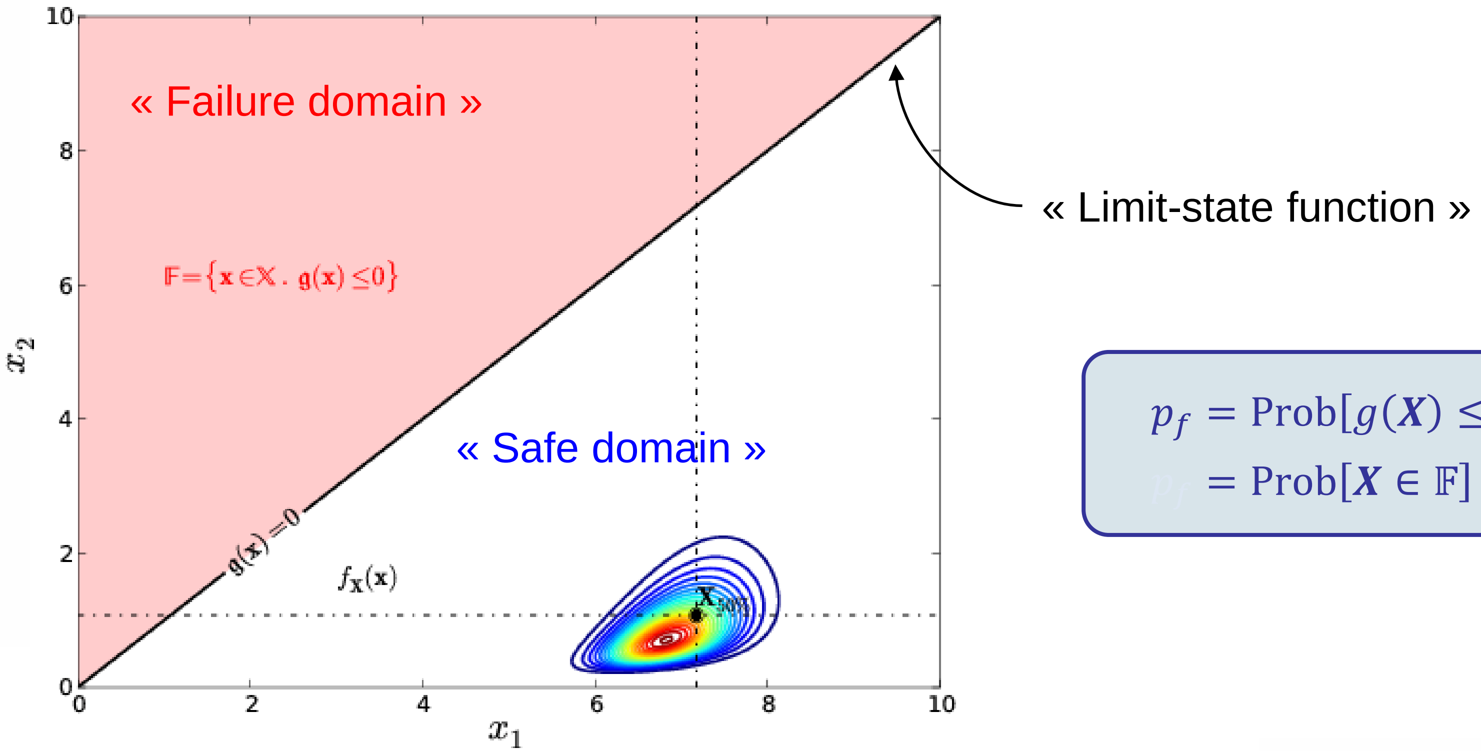
Problem definition

Input

- Ex : Consider a simple *capacity vs demand* example:

$$g(r,s) = r - s$$

with $R \sim \mathcal{LN}(\lambda_R, \zeta_R)$ and $S \sim \mathcal{LN}(\lambda_S, \zeta_S)$ and composed with a Normal copula whose shape matrix only term is $\rho_0 = 0,525$.



$$p_f = \mathrm{Prob}[g(\boldsymbol{X}) \leq 0]$$
$$= \mathrm{Prob}[\boldsymbol{X} \in \mathbb{F}]$$

Problem definition

Definitions for the failure probability

- The failure probability is essentially defined as the *value of the CDF of the safety margin* $G \equiv g(\boldsymbol{X})$ *at point 0*:

$$p_f = F_G(0) = \int_{-\infty}^{0} f_G(t)\mathrm{d}t$$

but G's distribution is rarely *known*.

- It also rewrites as the sum of $\boldsymbol{X}$'s PDF *over the failure domain* $\mathbb{F}$:

$$p_f = \int_{\mathbb{F}=\{\boldsymbol{x}\in\mathbb{X}:g(\boldsymbol{x})\leq 0\}} f_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

- It eventually rewrites as the expectation of the *failure indicator function* $\mathbb{I}_{\mathbb{F}}$ over the support $\mathbb{X}$ of the input probability distribution:

$$p_f = \int_{\mathbb{X}} \mathbb{I}_{\mathbb{F}}(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \mathbb{E}[\mathbb{I}_{\mathbb{F}}(\boldsymbol{X})]$$

Problem definition

Premise

- $G \equiv g(\boldsymbol{X})$'s distribution is *rarely known* (it is in some specific cases: linear combinations of independent random variables, univariate composite distributions)
- *Numerical integration techniques* (*e.g.* quadrature rules) are not suitable for integrating indicator functions (their precision is often less than the probability's order of magnitude).

Dedicated methods

- Brute-force estimation using (intensive) Monte Carlo sampling
- Approximation methods
- Advanced, reduced variance, Monte Carlo sampling methods (not covered in this tutorial)
- Surrogate-model-based methods (not covered in this tutorial)

Outline

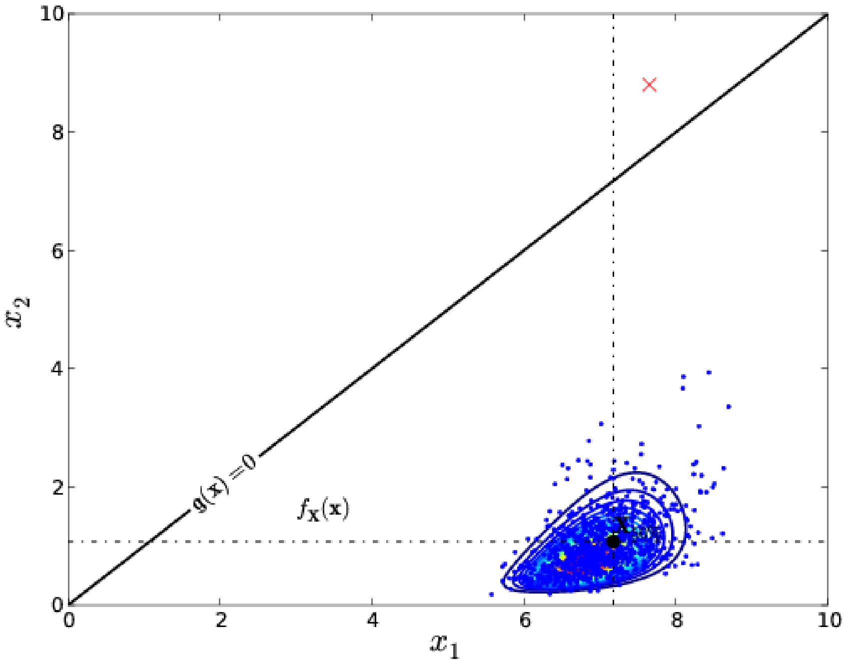
- Problem definition
- Brute-force estimation using Monte Carlo sampling
- Importance sampling
- Isoprobabilistic transformation
- Most-probable-failure-point(s)-based methods

Brute-force Monte Carlo estimation

Principle

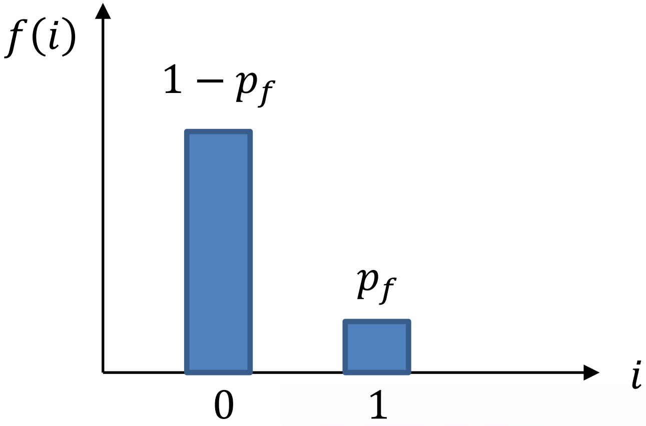
- The crude Monte Carlo estimator of the failure probability is *the empirical average of the Bernoulli failure experiment* :

$$\widehat{P}_{f,\mathrm{MCS}} = \frac{1}{N}\sum_{i=1}^{N} \mathbb{I}_{\mathbb{F}}\left(\boldsymbol{X}^{(i)}\right)$$



where:

$$\mathbb{I}_{\mathbb{F}}(\boldsymbol{X}) \sim \mathcal{B}\mathrm{er}\left(p_f\right)$$



Brute-force Monte Carlo estimation

Convergence

- According to the *central limit theorem* (CLT), this estimator is unbiased and converges as follows :

$$\hat{P}_{f,\mathrm{MCS}} \underset{N\to\infty}{\sim} \mathcal{N}\left(p_f, \sqrt{\frac{p_f(1-p_f)}{N}}\right)$$

- **Rule of thumb:** before applying the CLT, make sure that:

$$\min\{Np_f; N(1-p_f)\} \geq 10$$

- The asymptotic distribution enables the calculation of $(1-\alpha)$-confidence intervals:

$$\hat{p}_{f,\mathrm{MCS}} + \Phi^{-1}\left(\frac{\alpha}{2}\right)\sqrt{\frac{p_f(1-p_f)}{N}} \leq p_f \leq \hat{p}_{f,\mathrm{MCS}} + \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\sqrt{\frac{p_f(1-p_f)}{N}}$$

- <u>Ex</u> : For $1-\alpha = 95\%$, $\Phi^{-1}\left(\frac{\alpha}{2}\right) \approx -1{,}96$ and $\Phi^{-1}\left(1-\frac{\alpha}{2}\right) \approx +1{,}96$.

Brute-force Monte Carlo estimation

Convergence

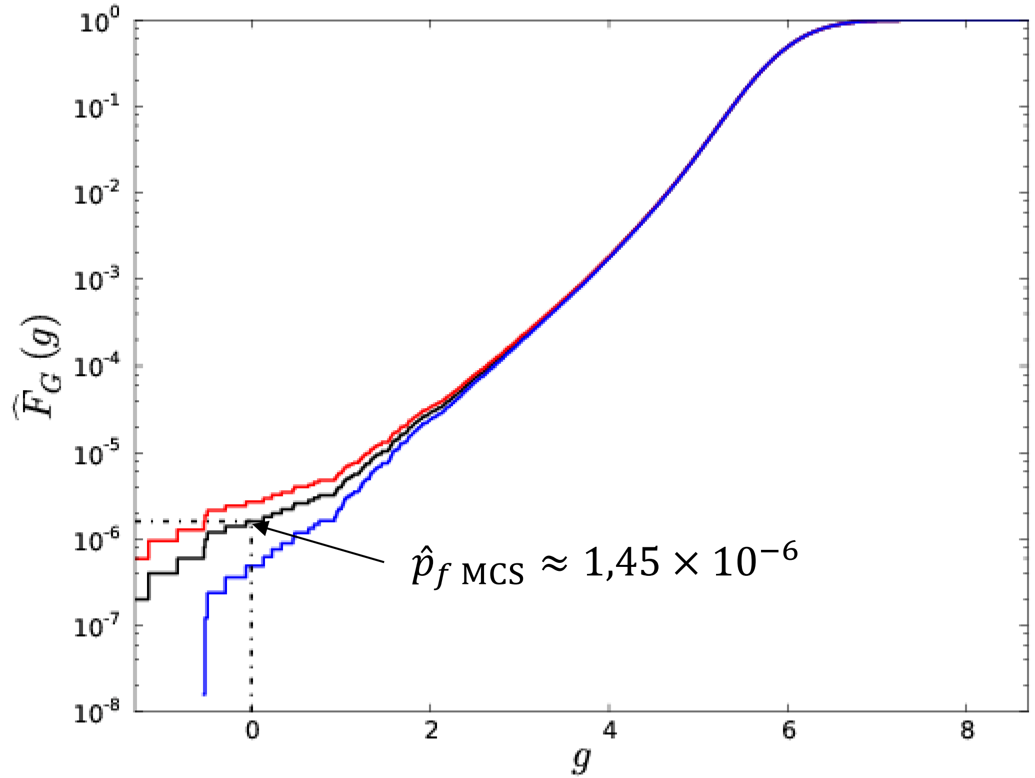
- The *required sample size* drastically increases as the probability gets low:

For a given *10% target coefficient of variation*

$$\delta = \sqrt{\frac{1-p_f}{Np_f}} \approx \frac{1}{\sqrt{Np_f}}$$

$$p_f \approx 10^{-k} \Rightarrow N_{\min} \approx 10^{k+2}$$

p_f	$N_{\min}$
10^{-2}	10 000
10^{-3}	100 000
10^{-4}	1 000 000



Brute-force Monte Carlo estimation

Pros & cons

+ Unbiased, reference, estimator
+ Easy to implement (at least for the statistics part)
+ Rich result (it is then possible to build a good approximation of G's CDF)
+ Highly distributable over high-performance computing means (*e.g.* over a cluster)

– Slow convergence : requires important computing resources

When should it be used?

- When you don't have a choice (when no other more clever method is applicable)!
- When the performance function is fast to evaluate :
  - *Simple closed-form expressions* ;
  - *HPC resources* available.

Outline

- Problem definition
- Brute-force estimation using Monte Carlo sampling
- Importance sampling
- Isoprobabilistic transformation
- Most-probable-failure-point(s)-based methods

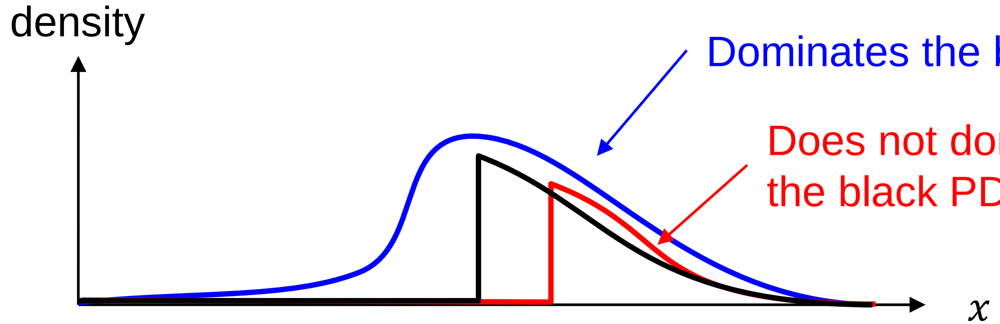
Importance sampling

Principle

- Let H denote some *instrumental probability distribution* with PDF h satisfying the following *dominance condition* :

$$h(\boldsymbol{x}) = 0 \quad \Rightarrow \quad \mathbb{I}_{\mathbb{F}}(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}) = 0$$

that would ideally make the failure event of interest *more frequent*.



- The failure probability rewrites:

$$p_f = \int_{\mathbb{X}} \mathbb{I}_{\mathbb{F}}(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_{\mathbb{X}} \frac{\mathbb{I}_{\mathbb{F}}(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x})}{h(\boldsymbol{x})} h(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

$$\boxed{p_f = \mathbb{E}_{\boldsymbol{Z}}\left[\frac{\mathbb{I}_{\mathbb{F}}(\boldsymbol{Z}) f_{\boldsymbol{X}}(\boldsymbol{Z})}{h(\boldsymbol{Z})}\right]}$$

Importance sampling

Use & properties

- Given an N-sample:

$$\mathcal{Z} = \left\{ \boldsymbol{Z}^{(i)}, \qquad i = 1, \dots, N \right\} \sim h$$

- The importance sampling *estimator* reads:

$$\widehat{P}_{f,h} = \frac{1}{N} \sum_{i=1}^{N} \frac{\mathbb{I}_{\mathbb{F}}\left(\boldsymbol{Z}^{(i)}\right) f_{\boldsymbol{X}}\left(\boldsymbol{Z}^{(i)}\right)}{h\left(\boldsymbol{Z}^{(i)}\right)}$$

- and *converges* according to the central limit theorem:

$$\widehat{P}_{f,h} \underset{N\to\infty}{\sim} \mathcal{N}\left(p_f, \sigma_{p_f}\right)$$

- The estimation *variance obviously depends on h*:

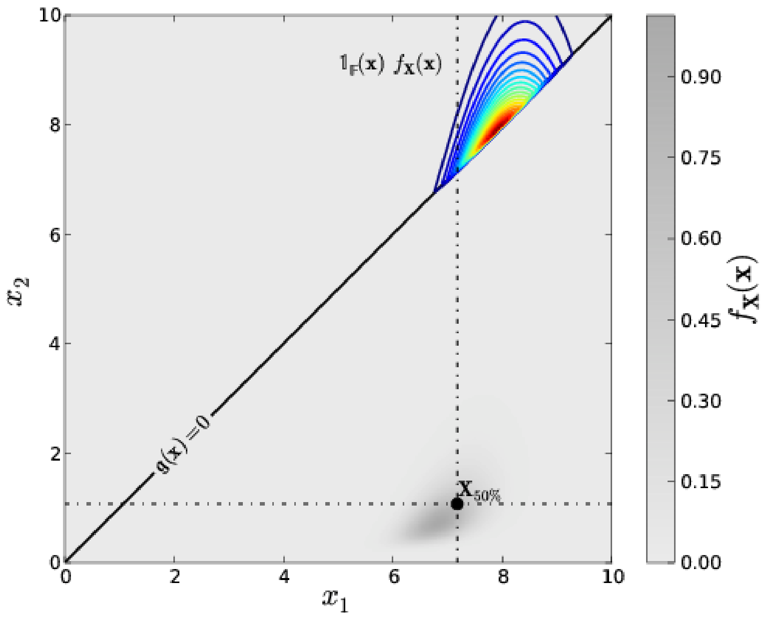
$$\sigma_{p_f}^2 = \frac{1}{N}\left(\mathbb{E}_{\boldsymbol{Z}}\left[\frac{\mathbb{I}_{\mathbb{F}}(\boldsymbol{Z}) f_{\boldsymbol{X}}^2(\boldsymbol{Z})}{h^2(\boldsymbol{Z})} \right] - p_f^2 \right)$$

Importance sampling

Choosing h?

- Any distribution provided the *dominance condition* holds.
- The best instrumental PDF yields a *zero estimation variance* and reads:

$$h^*(x) = \frac{\mathbb{I}_{\mathbb{F}}(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x})}{p_f}$$

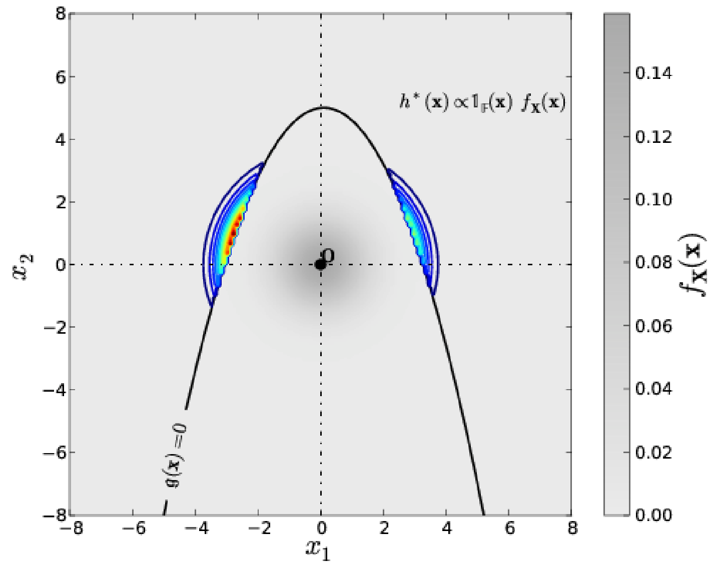


- impractical because its *normalizing constant* is the sought probability p_f!
- confirms *intuition* :
  - it is the probability distribution of the input parameters yielding failure.
  - it barely satisfies the *dominance condition*.

Importance sampling

A fundamental concept in reliability analysis

- The objective is to explore *the tail of the safety margin's probability distribution* (the lower tail in our case: $p_f \equiv \mathrm{Prob}[G \leq 0]$)…
- Using a *biased sampling technique for the input* in order to make failure much more frequent…
- And ideally, by sampling *only and exhaustively failed situations* (*i.e.* without forgetting any (significant) area of the failure domain).



Outline

- Problem definition
- Brute-force estimation using Monte Carlo sampling
- Importance sampling
- Isoprobabilistic transformation
- Most-probable-failure-point(s)-based methods

Isoprobabilistic transformation

Motivation

- *Spherical distributions* are invariant by rotation

$$\boldsymbol{U} \sim \mathbf{R}\boldsymbol{U}, \qquad \forall \mathbf{R} \in \mathcal{SP}_n(\mathbb{R})$$

thus enabling analytical developments (to come).

Available transformations

- Independent copula → Componentwise transformations
- Elliptical copula → Generalized Nataf transformation
- Any other composed distribution → Rosenblatt transformation

The *choice for the most-suitable transformation is automatic* in OpenTURNS.

- Further readings: Lebrun & Dutfoy (2009a,b,c).

Isoprobabilistic transformation

Standard space properties

- Given the *components order* in the input distribution and the Cholesky factor are fixed, the transformation is *unique* and *bijective* (it is *invertible*).

- The *probability measure is preserved*, hence the *failure probability* in the standard space equals the failure probability in the original (physical) space.
ATTENTION : This does not hold for approximations though.

- The transformed performance function is defined by composition:
$$g^{\circ}(\boldsymbol{u}) = g(\boldsymbol{x}) = (g \circ T^{-1})(\boldsymbol{u}), \qquad \boldsymbol{u} \in \mathbb{R}^n$$

- This, in turns, enables the definition of the *failure domain* in the standard space:
$$\mathbb{F}^{\circ} = \{\boldsymbol{u} \in \mathbb{R}^n : g^{\circ}(\boldsymbol{u}) \leq 0\}$$

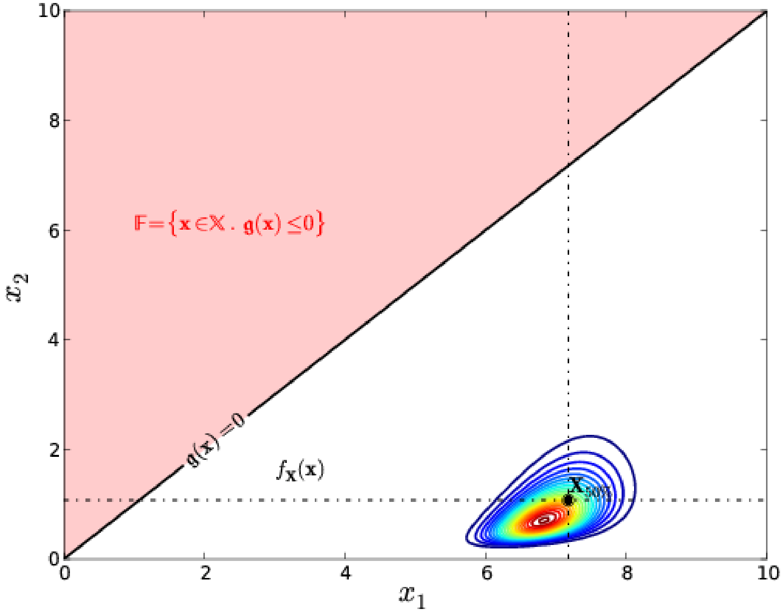
Isoprobabilistic transformation

Standard space properties

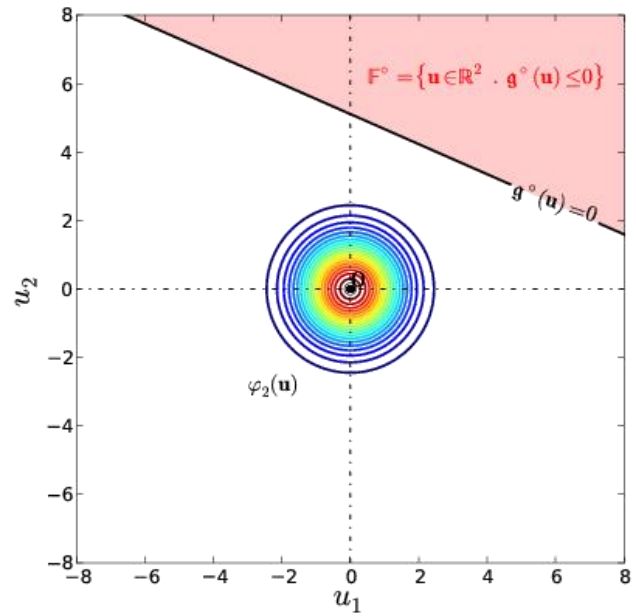
- Ex : Consider the *capacity vs demand* example:

$$g(r, s) = r - s$$

where the variables $R \sim \mathcal{LN}(\lambda_R, \zeta_R)$ and $S \sim \mathcal{LN}(\lambda_S, \zeta_S)$ are composed with a Normal copula with shape parameter $\rho_0 = 0,525$.



Physical space



Standard space

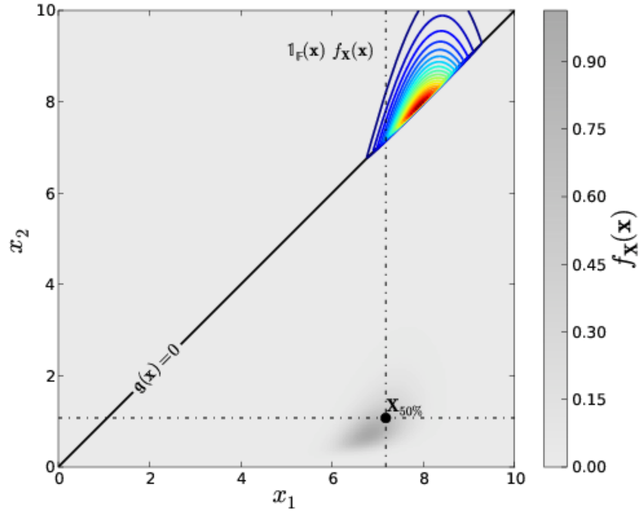
Outline

- Problem definition
- Brute-force estimation using Monte Carlo sampling
- Importance sampling
- Isoprobabilistic transformation
- Most-probable-failure-point(s)-based methods

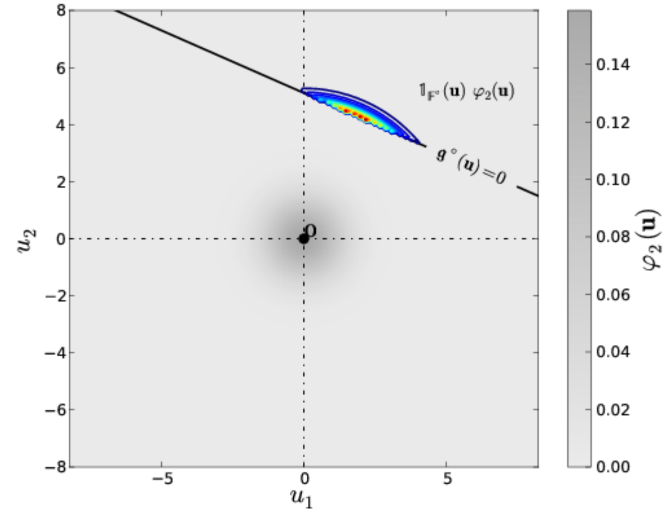
MPFP: FORM, SORM, P*-IS & FORM-Σ

Most probable failure point(s)

- Let's get back to the *optimal importance sampling concept*:



Physical space



Standard space

- We define the most probable failure point(s) as *the mode(s) of the optimal instrumental distribution*:

$$\boldsymbol{u}^* = \arg\max_{\boldsymbol{u}\in\mathbb{R}^n} \mathbb{I}_{\mathbb{F}^\circ}(\boldsymbol{u})\varphi_n(\boldsymbol{u})$$

- The solution for this optimization problem is *not nessarily unique*, although it is often the case in many applications (*e.g.* in structural mechanics).

MPFP: FORM, SORM, P*-IS & FORM-Σ

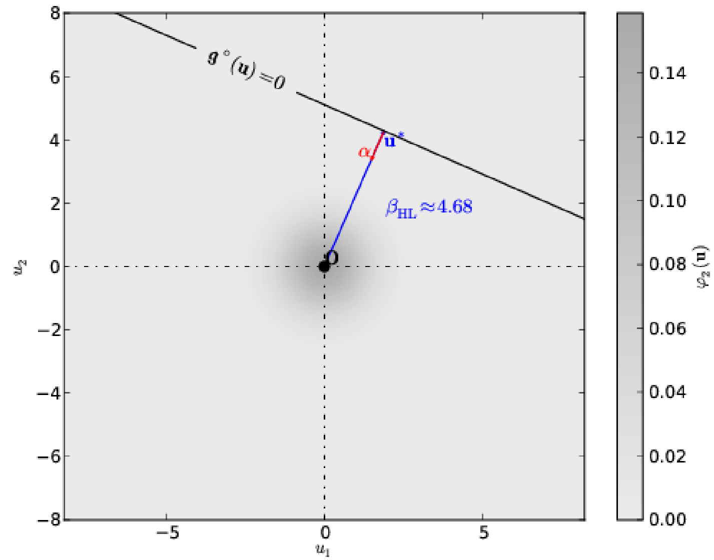
Most probable failure point(s)

- Let's work on the definition:

$$\boldsymbol{u}^* = \arg\max_{\boldsymbol{u}\in\mathbb{R}^n} \mathbb{I}_{\mathbb{F}^\circ}(\boldsymbol{u})\varphi_n(\boldsymbol{u})$$

$$\boldsymbol{u}^* = \arg\max_{\boldsymbol{u}\in\mathbb{R}^n} \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}\boldsymbol{u}^\mathrm{T}\boldsymbol{u}\right):\ g^\circ(\boldsymbol{u}) \le 0$$

$$\boxed{\boldsymbol{u}^* = \arg\min_{\boldsymbol{u}\in\mathbb{R}^n} \boldsymbol{u}^\mathrm{T}\boldsymbol{u}: \quad g^\circ(\boldsymbol{u}) \le 0}$$



- This is then equivalent to searching the *failure point(s) in the standard space that are the closest to the origin*.

MPFP: FORM, SORM, P*-IS & FORM-Σ

Search algorithms (constrained optimization)

- The *Abdo-Rackwitz algorithm* exploits the specificities of the problem at hand:
  - The objective function is quadratic.
  - The constraint is nonlinear, but it is linearized at each step based on the information brought by the gradient.
  - The optimization steps (the moves amplitude) can either be *fixed* (small) or *optimized* (variable) using merit rules such as Goldstein-Armijo's.
  - The algorithm converges when the current point satisfies both:
    - $g(\boldsymbol{u}^*) = 0$ (the point in on the limit-state surface)
    - $\nabla_{\boldsymbol{u}} g^\circ(\boldsymbol{u}^*) \,/\!/\, \boldsymbol{u}^*$ (the gradient of the constraint is colinear to that of the objective function)

- The *COBYLA* (Constrained Optimization BY Linear Approximations) algorithm is an interesting alternative when the partial differences of the performance function are hard to estimate (using finite differences schemes).

MPFP: FORM, SORM, P*-IS & FORM-Σ

First-order reliability method (FORM)

- <u>Assumption</u>: the most probable failure point is *unique*.
- The performance function is linearized at the MPFP:

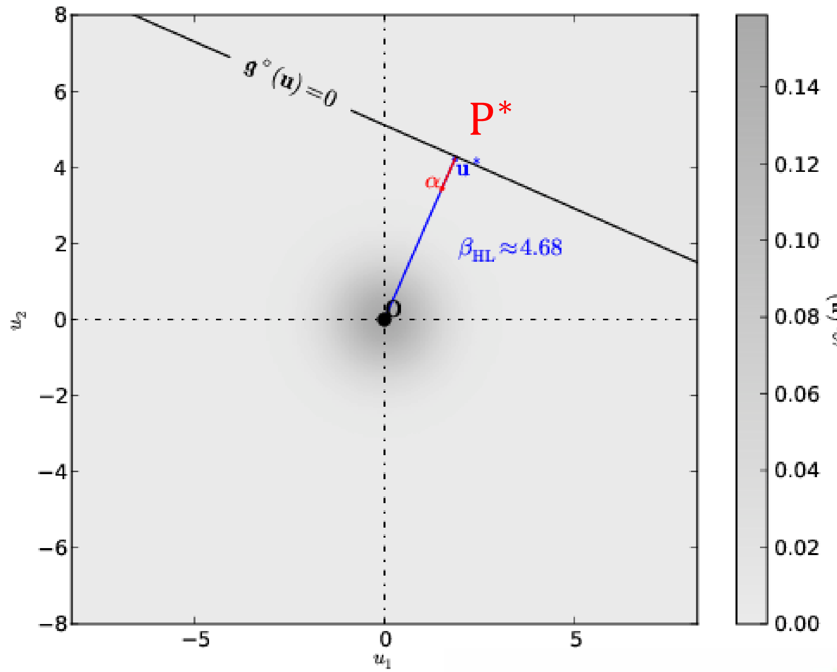
$$g^\circ_{1,\boldsymbol{u}^*}(\boldsymbol{u}) = g^\circ(\boldsymbol{u}^*) + \nabla_{\boldsymbol{u}} g^\circ(\boldsymbol{u}^*)^{\mathrm{T}}(\boldsymbol{u} - \boldsymbol{u}^*) = \nabla_{\boldsymbol{u}} g^\circ(\boldsymbol{u}^*)^{\mathrm{T}}(\boldsymbol{u} - \boldsymbol{u}^*)$$

- We introduce the *unit orientation vector*:

$$\boldsymbol{\alpha} = \frac{\nabla_{\boldsymbol{u}} g^\circ(\boldsymbol{u}^*)}{\|\nabla_{\boldsymbol{u}} g^\circ(\boldsymbol{u}^*)\|_2}$$

- And the *Hasofer-Lind reliability index*:

$$\beta_{\mathrm{HL}} = -\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{u}^* = \overline{\mathrm{OP}^*}$$



MPFP: FORM, SORM, P*-IS & FORM-Σ

First-order reliability method (FORM)

- The *approximate failure domain in the standard space* rewrites:

$$\begin{aligned}\mathbb{F}^{\circ}_{1,\boldsymbol{u}^*} &= \left\{\boldsymbol{u} \in \mathbb{R}^n : g^{\circ}_{1,\boldsymbol{u}^*}(\boldsymbol{u}) \leq 0\right\} \\ &= \{\boldsymbol{u} \in \mathbb{R}^n : \nabla_{\boldsymbol{u}} g^{\circ}(\boldsymbol{u}^*)^{\mathrm{T}}(\boldsymbol{u} - \boldsymbol{u}^*) \leq 0\} \\ &= \{\boldsymbol{u} \in \mathbb{R}^n : \boldsymbol{\alpha}^{\mathrm{T}}(\boldsymbol{u} - \boldsymbol{u}^*) \leq 0\} \\ &= \{\boldsymbol{u} \in \mathbb{R}^n : \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{u} + \beta_{\mathrm{HL}} \leq 0\}\end{aligned}$$

- So that we obtain the following first-order approximation of the failure probability:

$$\begin{aligned}p_{f\ 1,\boldsymbol{u}^*} &= \mathrm{Prob}[\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{U} + \beta_{\mathrm{HL}} \leq 0] \\ &= \mathrm{Prob}[Z \leq -\beta_{\mathrm{HL}}], \text{with } Z = \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{U} \sim \mathcal{N}(0,1)\end{aligned}$$

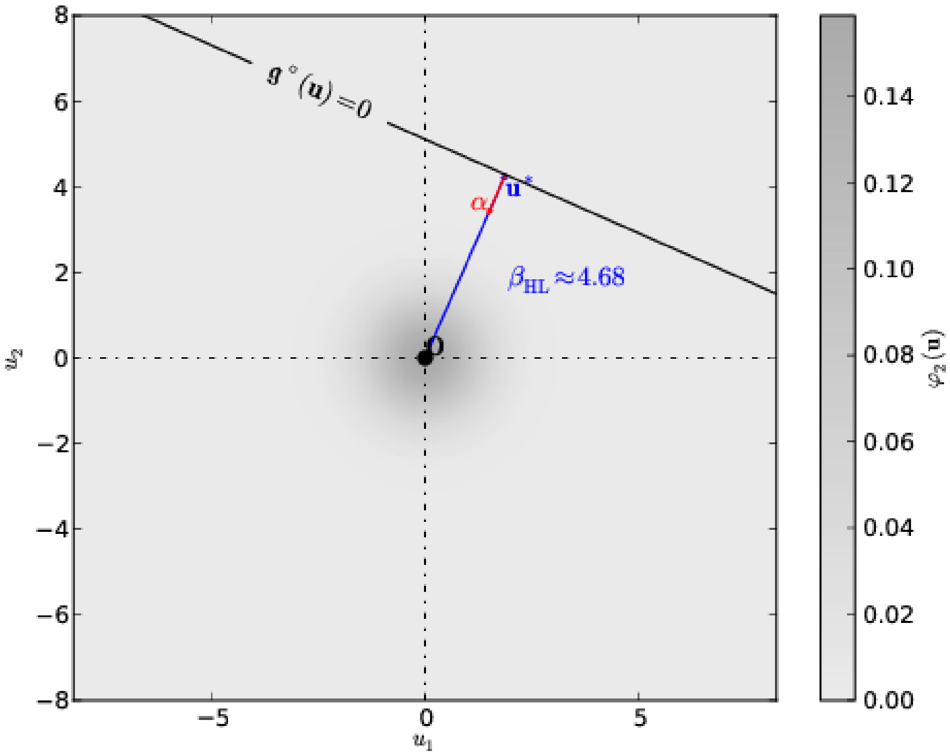
- Hence:

$$\boxed{p_{f\ 1,\boldsymbol{u}^*} = \Phi(-\beta_{\mathrm{HL}})}$$

MPFP: FORM, SORM, P*-IS & FORM-Σ

First-order reliability method (FORM)

- Ex : *(capacity vs demand example)*



$$\beta_{HL} \approx 4{,}68$$

$$p_{f\ 1,\boldsymbol{u}^*} \approx 1{,}44 \times 10^{-6}$$

- The limit-state surface being linear in the standard space, in this particular case, FORM is the reference solution.
- *Generally speaking, this is only an approximation.*

MPFP: FORM, SORM, P*-IS & FORM-Σ

SORM: *accounting for local curvatures*

- Assuming we can compute *the second-order partial derivatives of the performance function* in the standard space.
- Improvement: considering the curvature of g with a 2nd order Taylor development around P^* :

$$g^\circ_{2,\boldsymbol{u}^*} = g^\circ(\boldsymbol{u}^*) + \nabla_{\boldsymbol{u}} g^\circ(\boldsymbol{u}^*)^{\mathrm{T}}(\boldsymbol{u} - \boldsymbol{u}^*) + \frac{1}{2}(\boldsymbol{u} - \boldsymbol{u}^*)^{\mathrm{T}} \boldsymbol{\nabla}_{\boldsymbol{uu}} g^\circ(\boldsymbol{u}^*)(\boldsymbol{u} - \boldsymbol{u}^*)$$

- In case the standard space is spanned by Gaussian variables, *Breitung* has shown the following *asymptotic result* :

$$\boxed{p_{f\ 2,\boldsymbol{u}^*} \underset{\beta_{\mathrm{HL}} \to +\infty}{\to} \Phi(-\beta_{\mathrm{HL}}) \prod_{i=1}^{n} \frac{1}{\sqrt{1 + \beta_{\mathrm{HL}} \kappa_i}}}$$

where κ_i are the *curvatures* calculated from the Hessian matrix. This result is *valid* as soon as $1 + \beta_{\mathrm{HL}} \kappa_i \geq 0, \quad i = 1, \dots, n$.

- Lebrun & Dutfoy (2009a) generalized the approximation to *spherical distributions*.

MPFP: FORM, SORM, P*-IS & FORM-Σ

P*-IS: MPFP(s)-centered importance sampling

- Another correction can be obtained by *importance sampling with an instrumental PDF centered at the identified MPFP(s)*.
- For this, the state-of-the-art consists in using a Gaussian instrumental distribution:

$$\varphi_{n,\boldsymbol{u}^*}(\boldsymbol{u}) = \varphi_n(\boldsymbol{u}-\boldsymbol{u}^*) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{(\boldsymbol{u}-\boldsymbol{u}^*)^{\mathrm{T}}(\boldsymbol{u}-\boldsymbol{u}^*)}{2}\right)$$

- In this case, the failure probability *estimator* simplifies:

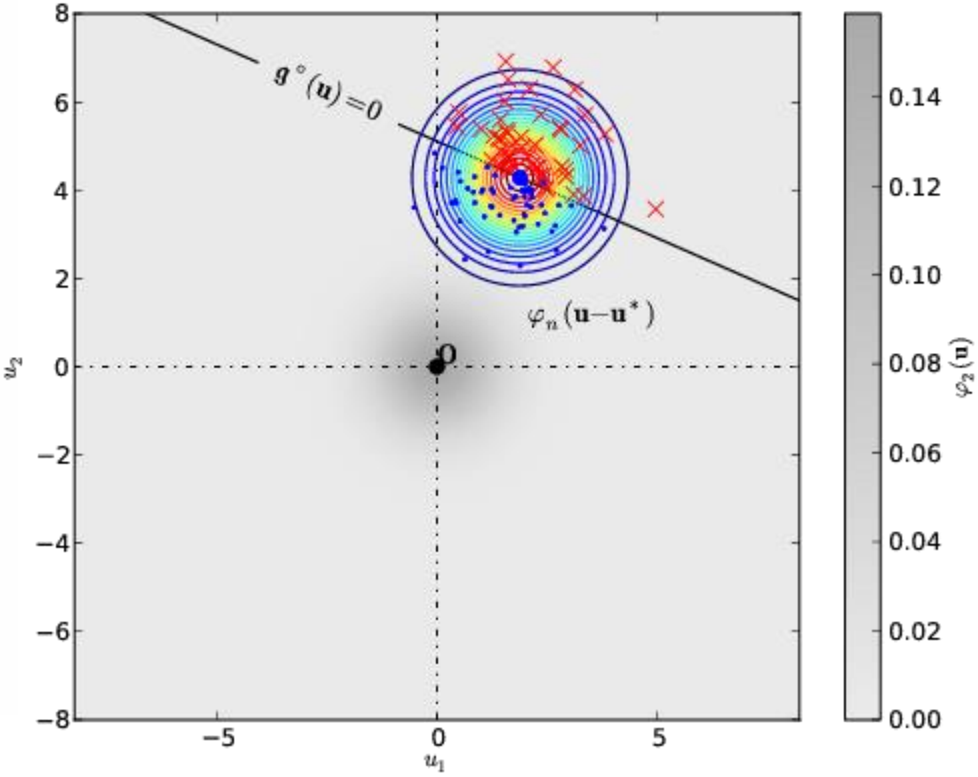
$$\hat{P}_{f,\,\boldsymbol{u}^*\mathrm{IS}} = \frac{\exp\left(-\beta_{\mathrm{HL}}^2/2\right)}{N} \sum_{i=1}^{N} \mathbb{I}_{\mathbb{F}^\circ}\left(\boldsymbol{Z}^{(i)}\right) \exp\left(-\boldsymbol{Z}^{(i)\,\mathrm{T}}\boldsymbol{u}^*\right)$$

- It is unbiased if the *dominance condition* of $\varphi_{n,\boldsymbol{u}^*}$ over $\mathbb{I}_{\mathbb{F}^\circ} \times \varphi_n$ holds (*unicity of the MPFP?*).
- It « *converges* » *much faster*, because the sampled points fails with a probability that is close to 50% (say between 10 and 90%).

MPFP: FORM, SORM, P*-IS & FORM-Σ

P*-IS: MPFP(s)-centered importance sampling

- Ex : *(capacity vs demand example)*



$p_{f,\mathbf{u}^*\mathrm{IS}} \approx 1{,}49 \times 10^{-6}$
up to a 10%
coefficient of variation.

- Convergence is obtained with only *600 additional runs of g*…
- Compared to 10^8 runs for crude Monte Carlo that gave:
  $\hat{p}_{f,\mathrm{MCS}} \approx 1{,}45 \times 10^{-6}$
  up to a *10% coefficient of variation*.

MPFP: FORM, SORM, P*-IS & FORM-Σ

FORM : importance factors

- The unit direction vector indicates how the *reliability index evolves with respect to the MPFP coordinates*:

$$\beta_{\mathrm{HL}} = -\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{u}^{*} = \sum_{i=1}^{n} -\alpha_i u_i^{*} \;\Rightarrow\; \alpha_i = -\frac{\partial \beta_{\mathrm{HL}}}{\partial u_i^{*}}$$

- In case *the distribution has a non-independent copula* though, each standard variable u_i is a function of several original (physical) variable x_i, so that the α_i's *are difficult to read*.

MPFP: FORM, SORM, P*-IS & FORM-Σ

FORM : importance factors

- In case the copula is Normal, Lemaire (2009) defined the following *corrected importance factors*:

$$\gamma_i = \frac{1}{\|\boldsymbol{\gamma}\|_2} \sigma_{X_i} \frac{\partial g}{\partial x_i}\bigg|_{\boldsymbol{x}=\boldsymbol{x}^*}, \qquad i = 1, \dots, n$$

- In the more general case, Lebrun & Dutfoy (2009c) proposed another more general, although unsigned, definition:

$$\gamma_i^2 = \frac{w_i^2}{\|\boldsymbol{w}\|_2^2}, \qquad i = 1, \dots, n$$

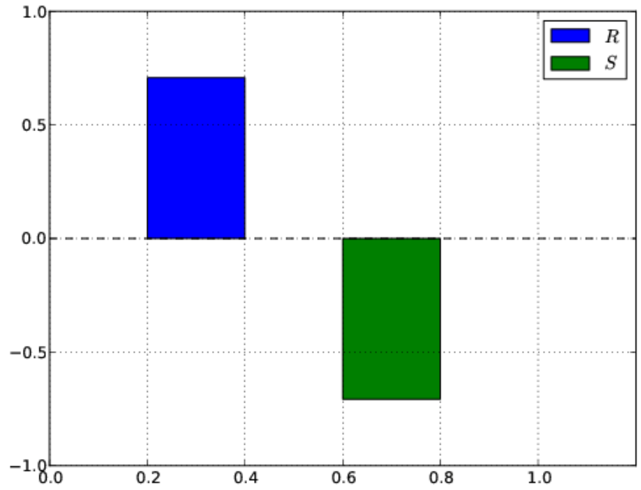
$$\boldsymbol{w} = \begin{pmatrix} E^{-1}\left(F_{X_1}(x_1)\right) \\ \vdots \\ E^{-1}\left(F_{X_n}(x_n)\right) \end{pmatrix}$$

MPFP: FORM, SORM, P*-IS & FORM-Σ

FORM : importance factors

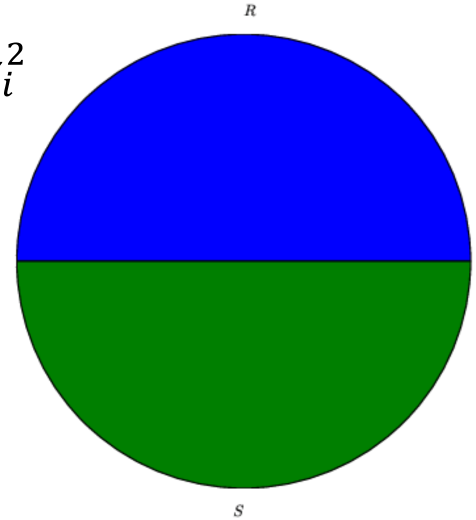
- These results are often presented in either one or both of these two charts:

γ_i



Signed bar chart

γ_i^2



Pie chart

α_i or γ_i *positive* ⇒ X_i is a *capacity variable*
α_i ot γ_i *negative* ⇒ X_i is a *demand variable*

The quadratic sum equals 1.
Qualitative comparison of the importance of variables w.r.t. failure.

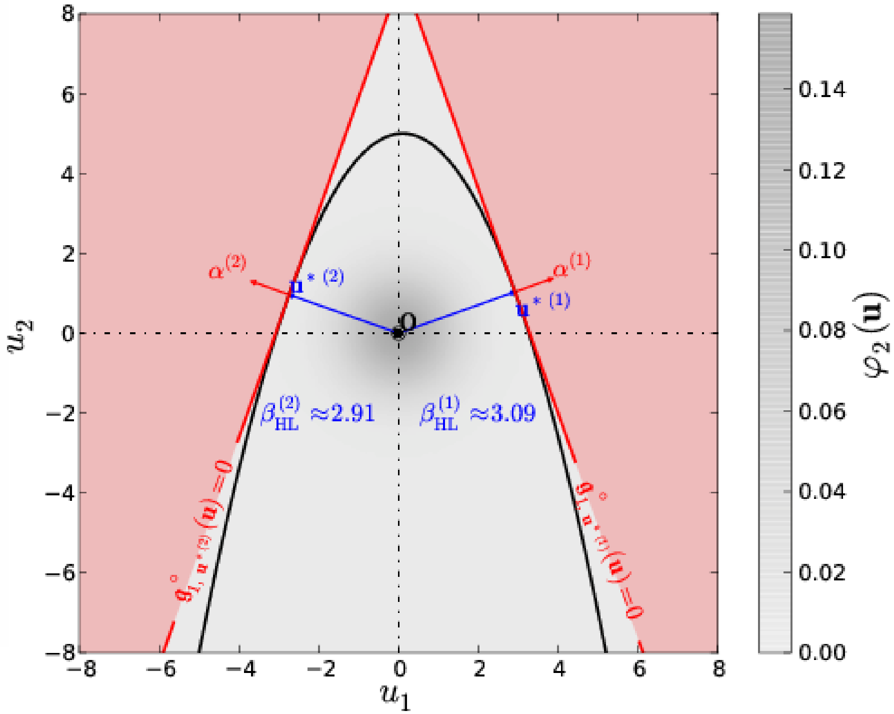
MPFP: FORM, SORM, P*-IS & FORM-Σ

FORM : *Multiple design points*

- Ex : Consider the following limit-state function:

$$g(x_1, x_2) = b - x_2 - \kappa(u_1 - e)^2$$

where $\boldsymbol{X} = \boldsymbol{U} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$, $b = 5$, $\kappa = 0{,}5$ and $e = 0{,}1$.



- PMSoft implements *an algorithm that is devoted to the search of multiple* P^* using:
  - a smart reset strategy;
  - exclusion balls around the already found design points.
  - See Der Kiureghian & Dakessian (1998).
- However the enumeration may lack completeness.

MPFP: FORM, SORM, P*-IS & FORM-Σ

FORM-Σ : *Serial combination of linear limit-states*

- Input: *for the n_{P^*} identified MPFPs* :
  - reliability indices: $\boldsymbol{\beta}_{\mathrm{HL}} = \left(\beta_{\mathrm{HL}}^{(i)},\ i = 1, \ldots, n_{P^*}\right)$
  - importance factors in the standard space: $\mathbf{A} = \left(\boldsymbol{\alpha}^{(i)},\ i = 1, \ldots, n_{P^*}\right)$
- Objective : combine these results into a single probability, the one associated to the *serial system* formed by the contributors.
- Solution:

$$p_{f,1\Sigma} = \mathrm{Prob}\left[\boldsymbol{U} \in \bigcup_{i=1}^{n_{P^*}} \left\{\boldsymbol{u} \in \mathbb{R}^n : \boldsymbol{\alpha}^{(i)\,\mathrm{T}} \boldsymbol{u} + \beta_{\mathrm{HL}}^{(i)} \leq 0\right\}\right] = 1 - \Phi_{n_{P^*}}(\boldsymbol{\beta}_{\mathrm{HL}}; \mathbf{0}, \boldsymbol{\rho})$$

where:

$$\rho_{ij} = \boldsymbol{\alpha}^{(i)\,\mathrm{T}} \boldsymbol{\alpha}^{(j)}, \qquad i, j = 1, \ldots, n_{P^*}$$

are the « *pairwise limit-states' correlation* » $(-1 \leq \rho \leq 1)$.

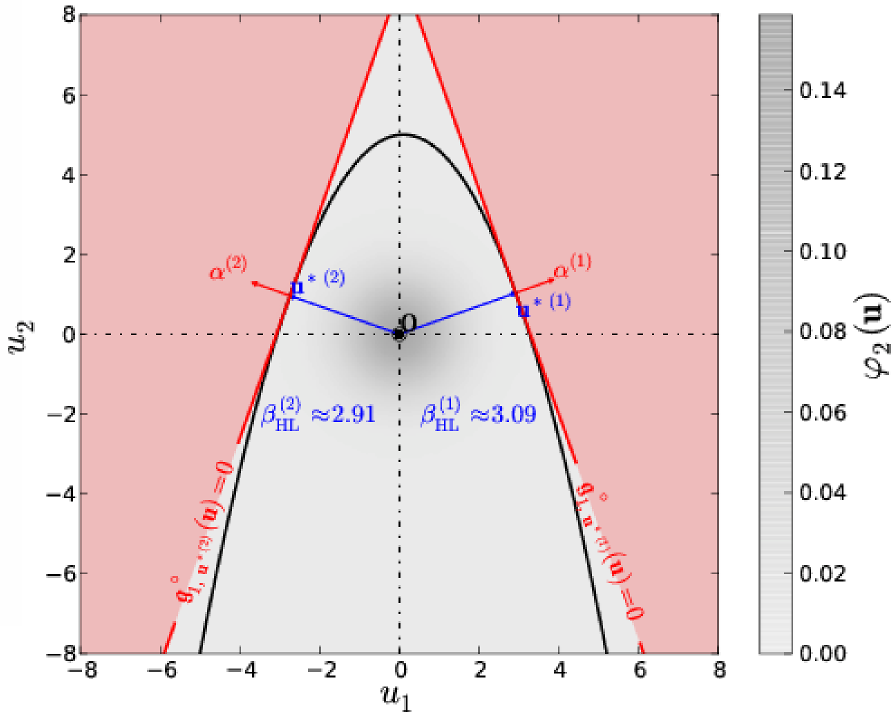
MPFP: FORM, SORM, P*-IS & FORM-Σ

FORM-Σ : *Serial combination of linear limit-states*

- Ex : Consider the following limit-state function:

$$g(x_1, x_2) = b - x_2 - \kappa(u_1 - e)^2$$

where $\boldsymbol{X} = \boldsymbol{U} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$, $b = 5$, $\kappa = 0{,}5$ and $e = 0{,}1$.



The correlation between the two limit-states is:

$$\rho_{12} = \boldsymbol{\alpha}^{(1)\,\mathrm{T}} \boldsymbol{\alpha}^{(2)} \approx -0{,}78$$

Hence the *first-order approximation* of the serial system failure probability is:

$$p_{f1,\Sigma} = 1 - \Phi_2\left(\begin{pmatrix} 3{,}09 \\ 2{,}91 \end{pmatrix}; 0, \begin{bmatrix} 1 & -0{,}78 \\ -0{,}78 & 1 \end{bmatrix}\right)$$

$$\approx 2{,}82 \times 10^{-3}$$

The crude Monte Carlo estimate is:

$$\hat{p}_{f,\mathrm{MCS}} \approx 3{,}12 \times 10^{-3}$$

Up to a *10% coefficient of variation*.

MPFP: FORM, SORM, P*-IS & FORM-Σ

Pros & Cons

+ The most probable failure point concept is interesting because:
    + Coordinates → singular configuration(s) of the system.
    + Importance factors → give clues for improving reliability.
+ Affordable computational cost.

– The most probable failure point concept is dangerous:
    – non-unicity risk (FORM, SORM & basic P*-IS) ;
    – non-completeness risk (FORM-Σ).
– Missing (FORM, SORM, FORM-Σ) or subjective (P*-IS) error metric.

When should it be used?

- As a first *approximation* ;
- *Confirmed by an expert judgement* about the identified failure modes.

Conclusions

- Reliability methods aim at estimating the safety level attached to a component in the form of a ***subjective*** *failure probability* :

$$\boxed{p_f = \mathrm{Prob}[\mathrm{failure} \mid \mathrm{model}]}$$

- The methods reviewed in this presentation are implemented in *OpenTURNS* & *Uranie*.
- *Crude Monte Carlo sampling* enables exploring the model:
  - without requiring any assumption,
  - at a great computational expense though (HPC may help).
- *Most-probable-failure-point(s)-based techniques* enable:
  - a reduction of the computational effort (even if HPC may still help);
  - a deeper investigation of the system thanks to:
    - the most probable failure points coordinates;
    - the importance factors.

Further readings

- Ditlevsen, O. & Madsen, H. (1996).
  Structural reliability methods.
  John Wiley & Sons.
- Lebrun, R. & Dutfoy, A. (2009a).
  A generalization of the Nataf transformation to distributions with elliptical copula.
  Prob. Eng. Mech., 24, 172–178.
- Lebrun, R. & Dutfoy, A. (2009b).
  An innovating analysis of the Nataf transformation from the copula viewpoint.
  Prob. Eng. Mech., 24, 312–320.
- Lebrun, R. & Dutfoy, A. (2009c).
  Do Rosenblatt and Nataf isoprobabilistic transformations really differ?
  Prob. Eng. Mech., 2009, 24, 577–584.
- Lemaire, M. (2009).
  Structural reliability.
  Wiley. 480 pp.