Inverse Gaussian distribution

From Wikipedia, the free encyclopedia

In probability theory, the **inverse Gaussian distribution** (also known as the **Wald distribution**) is a two-parameter family of continuous probability distributions with support on (0,∞).

Its probability density function is given by

$$f(x;\mu,\lambda) = \left[\frac{\lambda}{2\pi x^3}\right]^{1/2} \exp\frac{-\lambda(x-\mu)^2}{2\mu^2 x}$$

for x > 0, where μ > 0 is the mean and λ > 0 is the shape parameter.

As λ tends to infinity, the inverse Gaussian distribution becomes more like a normal (Gaussian) distribution. The inverse Gaussian distribution has several properties analogous to a Gaussian distribution. The name can be misleading: it is an "inverse" only in that, while the Gaussian describes a Brownian Motion's level at a fixed time, the inverse Gaussian describes the distribution of the time a Brownian Motion with positive drift takes to reach a fixed positive level.

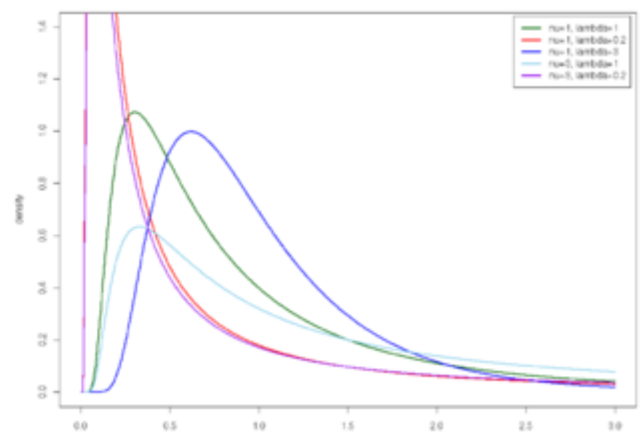
Its cumulant generating function (logarithm of the characteristic function) is the inverse of the cumulant generating function of a Gaussian random variable.

To indicate that a random variable X is inverse Gaussian-distributed with mean μ and shape parameter λ we write

$$X \sim IG(\mu,\lambda).$$

Inverse Gaussian

Probability density function



parameters:	λ > 0 μ > 0
support:	$x \in (0,\infty)$
pdf:	$\left[\frac{\lambda}{2\pi x^3}\right]^{1/2} \exp\frac{-\lambda(x-\mu)^2}{2\mu^2 x}$
cdf:	$\Phi\left(\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu}-1\right)\right)$ $+\exp\left(\frac{2\lambda}{\mu}\right)\Phi\left(-\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu}+1\right)\right)$ where $\Phi\left(\right)$ is the standard normal (standard Gaussian) distribution c.d.f.

Contents

- 1 Properties
  - 1.1 Summation
  - 1.2 Scaling
  - 1.3 Exponential family
- 2 Relationship with Brownian motion
  - 2.1 When drift is zero
- 3 Maximum likelihood
- 4 Generating random variates from an inverse-Gaussian distribution
- 5 See also
- 6 Notes
- 7 References
- 8 External links

Properties

Summation

If X_i has a IG($\mu_0 w_i$, $\lambda_0 w_i^2$) distribution for i = 1, 2, ..., n and all X_i are independent, then

$$S=\sum_{i=1}^n X_i \sim IG\left(\mu_0 \sum w_i, \lambda_0 \left(\sum w_i\right)^2\right).$$

Note that

$$\frac{\mathrm{Var}(X_i)}{\mathrm{E}(X_i)}=\frac{\mu_0^2 w_i^2}{\lambda_0 w_i^2}=\frac{\mu_0^2}{\lambda_0}$$

is constant for all i. This is a necessary condition for the summation. Otherwise S would not be inverse Gaussian.

Scaling

For any $t > 0$ it holds that

$$X \sim IG(\mu,\lambda) \quad \Rightarrow \quad tX \sim IG(t\mu, t\lambda).$$

Exponential family

The inverse Gaussian distribution is a two-parameter exponential family with natural parameters $-\lambda/(2\mu^2)$ and $-\lambda/2$, and natural statistics *X* and *1/X*.

Relationship with Brownian motion

The stochastic process X_t given by

$$X_0 = 0$$
$$X_t = \nu t + \sigma W_t$$

(where W_t is a standard Brownian motion and $\nu > 0$) is a Brownian motion with drift ν.

Then, the first passage time for a fixed level $\alpha > 0$ by X_t is distributed according to an inverse-gaussian:

$$T_\alpha = \inf\{0 < t \mid X_t = \alpha\} \sim IG(\tfrac{\alpha}{\nu}, \tfrac{\alpha^2}{\sigma^2}).$$

When drift is zero

A common special case of the above arises when the Brownian motion has no drift. In that case, parameter μ tends to infinity, and the first passage time for fixed level α has probability density function

$$f\left(x; \infty, \left(\frac{\alpha}{\sigma}\right)^2\right) = \frac{\alpha}{\sigma\sqrt{2\pi x^3}} \exp\left(-\frac{\alpha^2}{2x\sigma^2}\right).$$

This is a Lévy distribution with parameter $c = \frac{\alpha^2}{\sigma^2}$.

Maximum likelihood

The model where

$$X_i \sim IG(\mu, \lambda w_i), \quad i = 1, 2, \ldots, n$$

with all w_i known, (μ, λ) unknown and all X_i independent has the following likelihood function

$$L(\mu,\lambda)=\left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}}\left(\prod_{i=1}^{n}\frac{w_i}{X_i^3}\right)^{\frac{1}{2}}\exp\left(\frac{\lambda}{\mu}-\frac{\lambda}{2\mu^2}\sum_{i=1}^{n}w_iX_i-\frac{\lambda}{2}\sum_{i=1}^{n}w_i\frac{1}{X_i}\right).$$

Solving the likelihood equation yields the following maximum likelihood estimates

$$\hat{\mu}=\frac{\sum_{i=1}^{n}w_iX_i}{\sum_{i=1}^{n}w_i},\qquad \frac{1}{\hat{\lambda}}=\frac{1}{n}\sum_{i=1}^{n}w_i\left(\frac{1}{X_i}-\frac{1}{\hat{\mu}}\right).$$

$\hat{\mu}$ and $\hat{\lambda}$ are independent and

$$\hat{\mu}\sim IG\left(\mu,\lambda\sum_{i=1}^{n}w_i\right)\qquad \frac{n}{\hat{\lambda}}\sim\frac{1}{\lambda}\chi^2_{n-1}.$$

Generating random variates from an inverse-Gaussian distribution

The following algorithm may be used.[1]

Generate a random variate from a normal distribution with a mean of 0 and 1 standard deviation

$$\nu=N(0,1).$$

Square the value

$$y=\nu^2$$

and use this relation

$$x=\mu+\frac{\mu^2y}{2\lambda}-\frac{\mu}{2\lambda}\sqrt{4\mu\lambda y+\mu^2y^2}.$$

Generate another random variate, this time sampled from a uniformed distribution between 0 and 1

$$z=U(0,1).$$

If

$$z\le\frac{\mu}{\mu+x}$$

then return

$$x$$

else return

$$\frac{\mu^2}{x}.$$

Sample code in Java language:

```
public double inverseGaussian(double mu, double lambda) {
       Random rand = new Random();
       double v = rand.nextGaussian();   // sample from a normal dis
       double y = v*v;
       double x = mu + (mu*mu*y)/(2*lambda) - (mu/(2*lambda)) * Math
       double test = rand.nextDouble();  // sample from a uniform di
       if (test <= (mu)/(mu + x))
              return x;
       else
              return (mu*mu)/x;
}
```

See also

- Generalized inverse Gaussian distribution
- Tweedie distributions

Notes

1. ^ *Generating Random Variates Using Transformations with Multiple Roots* by John R. Michael, William R. Schucany and Roy W. Haas, *American Statistician*, Vol. 30, No. 2 (May, 1976), pp. 88-90

References

- *The inverse gaussian distribution: theory, methodology, and applications* by Raj Chhikara and Leroy Folks, 1989 ISBN 0-8247-7997-5
- *System Reliability Theory* by Marvin Rausand and Arnljot Høyland
- *The Inverse Gaussian Distribution* by Dr. V. Seshadri, Oxford Univ Press, 1993

External links

- Inverse Gaussian Distribution (http://mathworld.wolfram.com/InverseGaussianDistribution.html) in Wolfram website.