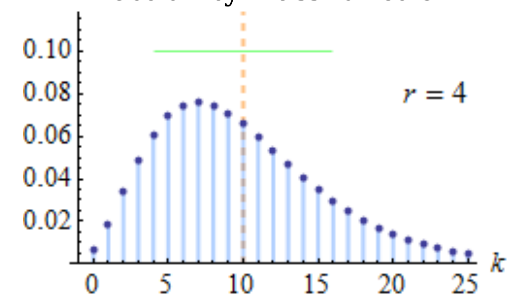
Negative binomial distribution

From Wikipedia, the free encyclopedia

In probability theory and statistics, the **negative binomial distribution** is a discrete probability distribution of the number of successes in a sequence of Bernoulli trials before a specified (non-random) number *r* of failures occurs. For example, if one throws a die repeatedly until the third time "1" appears, then the probability distribution of the number of non-"1"s that had appeared will be negative binomial.

The **Pascal distribution** (after Blaise Pascal) and **Polya distribution** (for George Pólya) are special cases of the negative binomial. There is a convention among engineers, climatologists, and others to reserve "negative binomial" in a strict sense or "Pascal" for the case of an integer-valued stopping-time parameter *r*, and use "Polya" for the real-valued case. The Polya distribution more accurately models occurrences of "contagious" discrete events, like tornado outbreaks, than the Poisson distribution.

Probability mass function



The orange line represents the mean, which is equal to 10 in each of these plots; the green line shows the standard deviation.

notation:	$\mathrm{NB}(r,\,p)$
parameters:	$r > 0$ — number of failures until the experiment is stopped (integer, but the definition can also be extended to reals) $p \in (0,1)$ — success probability in each experiment (real)
support:	$k \in \{\,0,\,1,\,2,\,3,\,\ldots\,\}$
pmf:	$\binom{k+r-1}{k}\cdot(1-p)^r p^k,$ involving a binomial coefficient
cdf:	$1-I_p(k+1,\,r)$, the regularized incomplete beta function
mean:	$\frac{pr}{1-p}$
median:	
mode:	$\begin{cases}\big\lfloor\frac{p(r-1)}{1-p}\big\rfloor & \text{if}\ r>1\\ 0 & \text{if}\ r\leq 1\end{cases}$
variance:	$\frac{pr}{(1-p)^2}$

Contents

skewness:	$\frac{1+p}{\sqrt{pr}}$
ex.kurtosis:	$\frac{6}{r} + \frac{(1-p)^2}{pr}$
entropy:	
mgf:	$\left(\frac{1-p}{1-pe^t}\right)^r$ for $t < -\log p$
cf:	$\left(\frac{1-p}{1-pe^{it}}\right)^r$ with $t \in \mathbb{R}$
pgf:	$\left(\frac{1-p}{1-pz}\right)^r$ for $\lvert z\rvert < \frac{1}{p}$

Definition

Suppose there is a sequence of independent Bernoulli trials, each trial having two potential outcomes called "success" and "failure". In each trial the probability of success is p and of failure is $(1 - p)$. We are observing this sequence until a predefined number r of failures has occurred. Then the random number of successes we have seen, X, will have the **negative binomial** (or **Pascal**) distribution:

$$X \ \sim\ \mathrm{NB}(r,\, p)$$

When applied to real-world situations, the words *success* and *failure* need not necessarily be associated with outcomes which we see as good or bad. Say in one case we may use negative binomial distribution to model the number of days a certain machine works before it breaks down. In such a case the "success" would mean the machine was working properly, whereas "failure" would mean it broke down. In another case we can use negative binomial distribution to model the number of attempts needed for a sportsman to score a goal. Then the "failure" will be his/her scoring the goal, whereas "successes" are misses.

The probability mass function of the negative binomial distribution is

$$f(k) \equiv \Pr(X = k) = \binom{k+r-1}{k} (1-p)^r p^k \quad \text{for } k = 0, 1, 2, \dots$$

Here the quantity in parentheses is called the binomial coefficient, and is equal to

$$\binom{k+r-1}{k} = \frac{(k+r-1)!}{k!\,(r-1)!} = \frac{(k+r-1)(k+r-2)\cdots(r)}{k!}.$$

This quantity can alternatively be written in the following manner, explaining the name “negative binomial”:

$$\frac{(k+r-1)\cdots(r)}{k!} = (-1)^k \frac{(-r)(-r-1)(-r-2)\cdots(-r-k+1)}{k!} = (-1)^k \binom{-r}{k}$$

To understand the above definition of the probability mass function, note that the probability for every specific sequence of k successes and r failures is $(1 - p)^r p^k$, because the outcomes of the $k + r$ trials are supposed to happen independently. Since the r^{th} failure comes last, it remains to choose the k trials with successes out of the remaining $k + r - 1$ trials. The above binomial coefficient, due to its combinatorial interpretation, gives precisely the number of all these sequences of length $k + r - 1$.

Extension to real-valued *r*

It is possible to extend the definition of the negative binomial distribution to the case of a positive real parameter r. Although it is impossible to visualize a non-integer number of “failures”, we can still formally define the distribution through its probability mass function.

As before, we say that X has a **negative binomial** (or **Pólya**) distribution if it has a probability mass function:

$$f(k) \equiv \Pr(X = k) = \binom{k+r-1}{k}(1-p)^r p^k \quad \text{for } k = 0, 1, 2, \dots$$

Here r is a real, positive number. The binomial coefficient is then defined by the multiplicative formula and can also be rewritten using the gamma function:

$$\binom{k+r-1}{k} = \frac{(k+r-1)(k+r-2)\cdots(r)}{k!} = \frac{\Gamma(k+r)}{k!\,\Gamma(r)}.$$

Note that by the binomial series and (*) above, for every $0 \le p < 1$,

$$(1-p)^{-r} = \sum_{k=0}^{\infty} \binom{-r}{k}(-p)^k = \sum_{k=0}^{\infty} \binom{k+r-1}{k} p^k,$$

hence the terms of the probability mass function indeed add up to one.

Alternative formulations

Some textbooks may define the negative binomial distribution slightly differently than it is done here. The most common variations are:

- The definition where X is the total number of **trials** needed to get r failures, not simply the number of successes. Since the total number of trials is equal to the number of successes plus the number of failures, this definition differs from ours by adding constant r. In order to convert formulas written with this definition into the one used in the article, replace everywhere "k" with "$k + r$", and also subtract r from the mean, the median, and the mode. In order to convert formulas of this article into this alternative definition, replace "k" with "$k - r$" and add r to the mean, the median and the mode. Effectively, this implies using the probability mass function
$$f(k) \equiv \Pr(X = k) = \binom{k-1}{k-r}(1-p)^r p^{k-r} \quad \text{for } k = r, r+1, r+2, \dots,$$
which perhaps resembles the binomial distribution more closely than the version above. Note that the arguments of the binomial coefficient are decremented due to *order*: the last "failure" must occur *last*, and so the other events have one fewer positions available when counting possible orderings. Note that this definition of the negative binomial distribution does not easily generalize to a positive, real parameter r.
- The definition where p denotes the probability of a **failure**, not of a success. This may also be formulated as "X is the number of failures before r successes", in which case p will be the probability of a success, but the words "failure" and "success" have been swapped around. In order to convert formulas between this definition and the one used in the article, replace "p" with "$1 - p$" everywhere.
- The two alterations above may be applied simultaneously.

Occurrence

Waiting time in a Bernoulli process

For the special case where r is an integer, the negative binomial distribution is known as the **Pascal distribution**. It is the probability distribution of a certain number of failures and successes in a series of independent and identically distributed Bernoulli trials. For $k + r$ Bernoulli trials with success probability p, the negative binomial gives the probability of k successes and r failures, with a failure on the last trial. In other words, the negative binomial distribution is the probability distribution of the number of successes before the rth failure in a Bernoulli process, with probability p of successes on each trial. A Bernoulli process is a discrete time process, and so the number of trials, failures, and successes are integers.

Consider the following example. Suppose we repeatedly throw a die, and consider a “1” to be a “failure”. The probability of failure on each trial is 1/6. The number of successes before the third failure belongs to the infinite set { 0, 1, 2, 3, ... }. That number of successes is a negative-binomially distributed random variable.

When $r = 1$ we get the probability distribution of number of successes before the first failure (i.e. the probability of the first failure occurring on the $(k + 1)^{\text{st}}$ trial), which is a geometric distribution:

$$f(k) = (1-p) \cdot p^k$$

Overdispersed Poisson

The negative binomial distribution, especially in its alternative parameterization described above, can be used as an alternative to the Poisson distribution. It is especially useful for discrete data over an unbounded positive range whose sample variance exceeds the sample mean. In such cases, the observations are overdispersed with respect to a Poisson distribution, for which the mean is equal to the variance. Hence a Poisson distribution is not an appropriate model. Since the negative binomial distribution has one more parameter than the Poisson, the second parameter can be used to adjust the variance independently of the mean. See Cumulants of some discrete probability distributions. An application of this is to annual counts of tropical cyclones in the North Atlantic or to monthly to 6-monthly counts of wintertime extratropical cyclones over Europe, for which the variance is greater than the mean.[1][2][3] In the case of modest overdispersion, this may produce substantially similar results to an overdispersed Poisson distribution.[4][5]

Related distributions

- The geometric distribution is a special case of the negative binomial distribution, with

$$\mathrm{Geom}(1-p) = \mathrm{NB}(1, p).$$

- The negative binomial distribution is a special case of the discrete phase-type distribution.

Poisson distribution

Consider a sequence of negative binomial distributions where the stopping parameter r goes to infinity, whereas the probability of success in each trial, p, goes to zero in such a way as to keep the mean of the distribution constant. Denoting this mean λ, the parameter p will have to be

$$\lambda = r\,\frac{p}{1-p} \quad \Rightarrow \quad p = \frac{\lambda}{r+\lambda}.$$

Under this parametrization the probability mass function will be

$$f(k) = \frac{\Gamma(k+r)}{k!\cdot\Gamma(r)}(1-p)^r p^k = \frac{\lambda^k}{k!}\cdot\frac{\Gamma(r+k)}{\Gamma(r)\;(r+\lambda)^k}\cdot\frac{1}{\left(1+\frac{\lambda}{r}\right)^r}$$

Now if we consider the limit as $r \to \infty$, the second factor will converge to one, and the third to the exponent function:

$$\lim_{r\to\infty} f(k) = \frac{\lambda^k}{k!}\cdot 1\cdot\frac{1}{e^\lambda},$$

which is the mass function of a Poisson-distributed random variable with expected value λ.

In other words, the alternatively parameterized negative binomial distribution converges to the Poisson distribution and r controls the deviation from the Poisson. This makes the negative binomial distribution suitable as a robust alternative to the Poisson, which approaches the Poisson for large r, but which has larger variance than the Poisson for small r.

$$\text{Poisson}(\lambda) = \lim_{r\to\infty} \text{NB}\Big(r,\ \frac{\lambda}{\lambda+r}\Big).$$

Gamma–Poisson mixture

The negative binomial distribution also arises as a continuous mixture of Poisson distributions where the mixing distribution of the Poisson rate is a gamma distribution. That is, we can view the negative binomial as a Poisson(λ) distribution, where λ is itself a random variable, distributed according to *Gamma*(r, $p/(1-p)$).

Formally, this means that the mass function of the negative binomial distribution can be written as

$$\begin{aligned}
f(k) &= \int_0^\infty f_{\mathrm{Poisson}(\lambda)}(k) \cdot f_{\mathrm{Gamma}\left(r,\, \frac{p}{1-p}\right)}(\lambda)\;\mathrm{d}\lambda \\
&= \int_0^\infty \frac{\lambda^k}{k!} e^{-\lambda} \cdot \lambda^{r-1} \frac{e^{-\lambda(1-p)/p}}{\left(\frac{p}{1-p}\right)^r \Gamma(r)}\;\mathrm{d}\lambda \\
&= \frac{(1-p)^r p^{-r}}{k!\,\Gamma(r)} \int_0^\infty \lambda^{r+k-1} e^{-\lambda/p}\;\mathrm{d}\lambda \\
&= \frac{(1-p)^r p^{-r}}{k!\,\Gamma(r)}\; p^{r+k}\,\Gamma(r+k) \\
&= \frac{\Gamma(r+k)}{k!\;\Gamma(r)}\;(1-p)^r p^k.
\end{aligned}$$

Because of this, the negative binomial distribution is also known as the **gamma–Poisson (mixture) distribution**.

Sum of geometric distributions

If Y_r is a random variable following the negative binomial distribution with parameters r and p, and support {0, 1, 2, ...}, then Y_r is a sum of r independent variables following the geometric distribution with parameter p. As a result of the central limit theorem, Y_r (properly scaled and shifted) is therefore approximately normal for sufficiently large r.

Furthermore, if B_{s+r} is a random variable following the binomial distribution with parameters $s + r$ and $1 - p$, then

$$\begin{aligned}
\Pr(Y_r \leq s) &= 1 - I_p(s+1, r) \\
&= 1 - I_p((s+r)-(r-1), (r-1)+1) \\
&= 1 - \Pr(B_{s+r} \leq r-1) \\
&= \Pr(B_{s+r} \geq r) \\
&= \Pr(\text{after } s+r \text{ trials, there are at least } r \text{ successes}).
\end{aligned}$$

In this sense, the negative binomial distribution is the "inverse" of the binomial distribution.

The sum of independent negative-binomially distributed random variables with the same value of the parameter p but the "r-values" r_1 and r_2 is negative-binomially distributed with the same p but with "r-value" $r_1 + r_2$.

The negative binomial distribution is infinitely divisible, i.e., if Y has a negative binomial distribution, then for any positive integer n, there exist independent identically distributed random variables $Y_1, \ldots, Y_n$ whose sum has the same distribution that Y has. These will not be negative-binomially distributed in the sense defined above unless n is a divisor of r (more on this below).

Representation as compound Poisson distribution

The negative binomial distribution NB(r,p) can be represented as a compound Poisson distribution: Let $\{Y_n, n \in \mathbb{N}_0\}$ denote a sequence of independent and identically distributed random variables, each one having the logarithmic distribution Log(p). Let N be a random variable, independent of the sequence, and suppose that N has a Poisson distribution with parameter $\lambda = -r \ln(1 - p)$. Then the random sum

$$X = \sum_{n=1}^{N} Y_n$$

is NB(r,p)-distributed. To prove this, we calculate the probability generating function G_X of X, which is the composition of the probability generating functions G_N and G_{Y_1}. Using

$$G_N(z) = \exp(\lambda(z-1)), \qquad z \in \mathbb{R},$$

and

$$G_{Y_1}(z) = \frac{\ln(1-pz)}{\ln(1-p)}, \qquad |z| < \frac{1}{p},$$

we obtain

$$\begin{aligned} G_X(z) &= G_N(G_{Y_1}(z)) \\ &= \exp\left(\lambda\left(\frac{\ln(1-pz)}{\ln(1-p)} - 1\right)\right) \\ &= \exp\big(-r(\ln(1-pz) - \ln(1-p))\big) \\ &= \left(\frac{1-p}{1-pz}\right)^r, \qquad |z| < \frac{1}{p}, \end{aligned}$$

which is the probability generating function of the NB(r,p) distribution.

Properties

Cumulative distribution function

The cumulative distribution function can be expressed in terms of the regularized incomplete beta function:

$$F(k) \equiv \Pr(X \leq k) = 1 - I_p(k+1, r).$$

Sampling and point estimation of *p*

Suppose p is unknown and an experiment is conducted where it is decided ahead of time that sampling will continue until r successes are found. A sufficient statistic for the experiment is k, the number of failures.

In estimating p, the minimum variance unbiased estimator is

$$\hat{p} = \frac{r-1}{r+k-1}.$$

The maximum likelihood estimate of p is

$$\tilde{p} = \frac{r}{r+k},$$

but this is a biased estimate. Its inverse $(r + k)/r$, is an unbiased estimate of $1/p$, however.[6]

Relation to the binomial theorem

Suppose K is a random variable with a negative binomial distribution with parameters r and p. The statement that the sum from $k = 0$ to infinity, of the probability $\Pr[K = k]$, is equal to 1, can be shown algebraically to be equivalent to the statement that $(1 - p)^{-r}$ is what Newton's binomial theorem says it should be.

Suppose Y is a random variable with a binomial distribution with parameters n and p. The statement that the sum from $y = 0$ to n, of the probability $\Pr[Y = y]$, is equal to 1, says that $1 = (p + (1 - p))^n$ is what the strictly finitary binomial theorem of rudimentary algebra says it should be.

Thus the negative binomial distribution bears the same relationship to the negative-integer-exponent case of the binomial theorem that the binomial distribution bears to the positive-integer-exponent case.

Assume $p + q = 1$. Then the binomial theorem of elementary algebra implies that

$$1 = 1^n = (p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}.$$

This can be written in a way that may at first appear to some to be incorrect, and perhaps perverse even if correct:

$$(p+q)^n = \sum_{k=0}^{\infty} \binom{n}{k} p^k q^{n-k},$$

in which the upper bound of summation is infinite. The binomial coefficient

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

is defined even when n is negative or is not an integer. But in our case of the binomial distribution it is zero when $k > n$. So *why* would we write the result in that form, with a seemingly needless sum of infinitely many zeros? The answer comes when we generalize the binomial theorem of elementary algebra to Newton's binomial theorem. Then we can say, for example

$$(p+q)^{8.3} = \sum_{k=0}^{\infty} \binom{8.3}{k} p^k q^{8.3-k}.$$

Now suppose $r > 0$ and we use a negative exponent:

$$1 = p^r \cdot p^{-r} = p^r (1-q)^{-r} = p^r \sum_{k=0}^{\infty} \binom{-r}{k} (-q)^k.$$

Then all of the terms are positive, and the term

$$p^r \binom{-r}{k} (-q)^k$$

is just the probability that the number of failures before the rth success is equal to k, provided r is an integer. (If r is a negative non-integer, so that the exponent is a positive non-integer, then some of the terms in the sum above are negative, so we do not have a probability distribution on the set of all nonnegative integers.)

Now we also allow non-integer values of r. Then we have a proper negative binomial distribution, which is a generalization of the Pascal distribution, which coincides with the Pascal distribution when r happens to be a positive integer.

Recall from above that

> The sum of independent negative-binomially distributed random variables with the same value of the parameter p but the "r-values" r_1 and r_2 is negative-binomially distributed with the same p but with "r-value" $r_1 + r_2$.

This property persists when the definition is thus generalized, and affords a quick way to see that the negative binomial distribution is infinitely divisible.

Parameter estimation

Maximum likelihood estimation

The likelihood function for N iid observations (k_1, ..., k_N) is

$$L(r,p) = \prod_{i=1}^{N} f(k_i; r, p)$$

from which we calculate the log-likelihood function

$$\ell(r,p) = \sum_{i=1}^{N} \ln\left(\Gamma(k_i + r)\right) - \sum_{i=1}^{N} \ln(k_i!) - N \ln\left(\Gamma(r)\right) + Nr \ln\left(1-p\right) + \sum_{i=1}^{N} k_i \ln(p).$$

To find the maximum we take the partial derivatives with respect to r and p and set them equal to zero:

$$\frac{\partial \ell(r,p)}{\partial p} = -Nr\frac{1}{1-p} + \sum_{i=1}^{N} k_i \frac{1}{p} = 0$$ and

$$\frac{\partial \ell(r,p)}{\partial r} = \sum_{i=1}^{N} \psi(k_i + r) - N\psi(r) + N\ln\left(1-p\right) = 0$$

where

$$\psi(k) = \frac{\Gamma'(k)}{\Gamma(k)}$$ is the digamma function.

Solving the first equation for p gives:

$$p = \frac{\sum_{i=1}^{N} k_i / N}{r + \sum_{i=1}^{N} k_i / N}$$

Substituting this in the second equation gives:

$$\frac{\partial \ell(r,p)}{\partial r} = \sum_{i=1}^{N} \psi(k_i + r) - N\psi(r) + N\ln\left(\frac{r}{r + \sum_{i=1}^{N} k_i / N}\right) = 0$$

This equation cannot be solved directly. If a numerical solution is desired, an iterative technique such as Newton's Method can be used.

Examples

Pat is required to sell candy bars to raise money for the 6th grade field trip. There are thirty houses in the neighborhood, and Pat is not supposed to return home until five candy bars have been sold. So the child goes door to door, selling candy bars. At each house, there is a 0.4 probability of selling one candy bar and a 0.6 probability of selling nothing.

What's the probability mass function for selling the last candy bar at the n^{th} house?

Recall that the NegBin(r, p) distribution describes the probability of k failures and r successes in $k + r$ Bernoulli(p) trials with success on the last trial. Selling five candy bars means getting five successes. The number of trials (i.e. houses) this takes is therefore $k + 5 = n$. The random variable we are interested in is the number of houses, so we substitute $k = n - 5$ into a NegBin(5, 0.4) mass function and obtain the following mass function of the distribution of houses (for $n \geq 5$):

$$f(n) = \binom{(n-5)+5-1}{n-5}\, 0.4^5\, 0.6^{n-5} = \binom{n-1}{n-5}\, 2^5\, \frac{3^{n-5}}{5^n}.$$

What's the probability that Pat finishes on the tenth house?

$$f(10) = 0.1003290624.$$

What's the probability that Pat finishes on or before reaching the eighth house?

To finish on or before the eighth house, Pat must finish at the fifth, sixth, seventh, or eighth house. Sum those probabilities:

$$f(5) = 0.01024$$
$$f(6) = 0.03072$$
$$f(7) = 0.055296$$
$$f(8) = 0.0774144$$
$$\sum_{j=5}^{8} f(j) = 0.17367.$$

What's the probability that Pat exhausts all 30 houses in the neighborhood?

This can be expressed as the probability that Pat does not finish on the fifth through the thirtieth house:

$$1 - \sum_{j=5}^{30} f(j) = 1 - I_{0.4}(5, 30-5+1) \approx 1 - 0.99849 = 0.00151.$$

Data on polygyny among a wide range of traditional African societies suggest that the distribution of wives follow a range of binomial profiles. The majority of these are negative binomial indicating the degree of competition for wives. However some tend towards a Poisson Distribution and even beyond towards a true binomial, indicating a degree of conformity in the allocation of wives. Further analysis of these profiles indicates shifts along this continuum between more competitiveness or more conformity according to the age of the husband and also according to the status of particular sectors within a society. In this way, these binomial distributions provide a tool for comparison, between societies, between sectors of societies, and over time. [7]

See also

- Coupon collector's problem
- Extended negative binomial distribution
- Negative multinomial distribution

References

1. ^ Villarini, G.; Vecchi, G.A. and Smith, J.A. (2010). "Modeling of the dependence of tropical storm counts in the North Atlantic Basin on climate indices". *Monthly Weather Review* **138** (7): 2681–2705. doi:10.1175/2010MWR3315.1 (http://dx.doi.org/10.1175%2F2010MWR3315.1) .
2. ^ Mailier, P.J.; Stephenson, D.B.; Ferro, C.A.T.; Hodges, K.I. (2006). "Serial Clustering of Extratropical Cyclones". *Monthly Weather Review* **134** (8): 2224–2240. doi:10.1175/MWR3160.1 (http://dx.doi.org/10.1175%2FMWR3160.1) .
3. ^ Vitolo, R.; Stephenson, D.B.; Cook, Ian M.; Mitchell-Wallace, K. (2009). "Serial clustering of intense European storms". *Meteorologische Zeitschrift* **18** (4): 411–424. doi:10.1127/0941-2948/2009/0393 (http://dx.doi.org/10.1127%2F0941-2948%2F2009%2F0393) .
4. ^ McCullagh, Peter; Nelder, John (1989). *Generalized Linear Models, Second Edition*. Boca Raton: Chapman and Hall/CRC. ISBN 0-412-31760-5.
5. ^ Cameron, Adrian C.; Trivedi, Pravin K. (1998). *Regression analysis of count data*. Cambridge University Press. ISBN 0-521-63567-5.
6. ^ J. B. S. Haldane, "On a Method of Estimating Frequencies", *Biometrika*, Vol. 33, No. 3 (Nov., 1945), pp. 222–225. JSTOR 2332299 (http://www.jstor.org/stable/2332299)
7. ^ Spencer, Paul, 1998, *The Pastoral Continuum: the Marginalization of Tradition in East Africa*, Clarendon Press, Oxford (pp. 51-92).

Further reading

- Hilbe, Joseph M., *Negative Binomial Regression*, Cambridge, UK: Cambridge University Press (2007) Negative Binomial Regression – Cambridge University Press (http://www.cambridge.org/uk/catalogue/catalogue.asp?isbn=9780521857727)

External links

- Online calculator of Negative Binomial Distribution (http://www.stud.feec.vutbr.cz/~xvapen02/vypocty/nb.php?language=english)

Retrieved from "http://en.wikipedia.org/wiki/Negative_binomial_distribution"
Categories: Discrete distributions | Factorial and binomial topics

- This page was last modified on 8 March 2011 at 15:13.