Low rank tensor approximation in OpenTURNS

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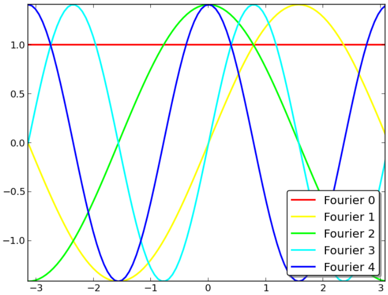
Plan

1. Non-polynomial basis
2. Canonical tensor

Fourier series

Fourier series

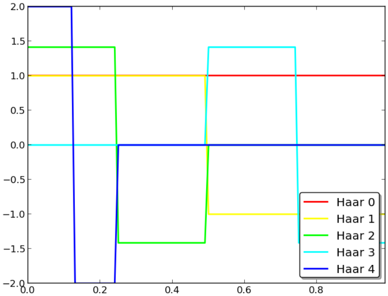
$$
\begin{aligned}
\psi_0(x) &= 1 \\
\psi_{2k+1}(x) &= \sqrt{2}\sin(kx) \\
\psi_{2k+2}(x) &= \sqrt{2}\cos(kx)
\end{aligned}
$$



Haar wavelets

Haar wavelets

$$\psi_0(x) = \mathbb{1}_{[0,1]}(x)$$
$$\psi_n(x) = \frac{1}{2^{j/2}}\left[\mathbb{1}_{[\frac{k}{2^j},\frac{k+1/2}{2^j}]}(x) - \mathbb{1}_{[\frac{k+1/2}{2^j},\frac{k+1}{2^j}]}(x)\right]$$



Usage

In Python...

```
# as a regular function
family = ot.FourierSeriesFactory()
family = ot.HaarWaveletFactory()

for i in range(5):
    f = family.build(i)
    d = f.draw(xmin, xmax, 100)
```

Functional chaos tensorization

Tensorization

Functional chaos decomposition

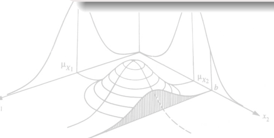
$$Y \equiv h(\underline{X}) = \sum_{j=0}^{\infty} a_j \, \psi_j(\underline{X})$$

upon tensorized basis

$$\psi_{\underline{\alpha}}(\underline{x}) \equiv \pi^{(1)}_{\alpha_1}(x_1) \times \cdots \times \pi^{(d)}_{\alpha_d}(x_d)$$

multi-indices notation

$$\alpha \equiv \{\alpha_1, \ldots, \alpha_d\}$$



Usage

In Python...

```
# polynomial basis
ef = ot.LinearEnumerateFunction(dim)
factC = [LegendreFactory()] * dim
prod = ot.OrthogonalProductPolynomialFactory(factC, ef)

# non-polynomial basis
factC = [ot.FourierSeriesFactory()] * dim
prod = ot.OrthogonalProductFunctionFactory(factC)

algo = ot.FunctionalChaosAlgorithm(...)
```

Rank one tensor

Rank one tensor

$$f(x_1, \dots, x_d) = \prod_{i=1}^{d} v_i(x_i)$$

with

$$v_i = \sum_{j=1}^{n_i} \alpha_j^{(i)} \phi_j(x_i)$$

expanding to

$$\begin{aligned} f(x_1, \dots, x_d) = & (\alpha_1^{(1)} \phi_1(x_1) + \dots + \alpha_{n_1}^{(1)} \phi_{n_1}(x_1)) \\ & \times (\alpha_1^{(2)} \phi_2(x_2) + \dots + \alpha_{n_2}^{(2)} \phi_{n_2}(x_2)) \\ & \times \dots \\ & \times (\alpha_1^{(d)} \phi_1(x_d) + \dots + \alpha_{n_d}^{(d)} \phi_{n_d}(x_d)) \end{aligned}$$

Canonical tensor format

Available representation

$$f(x_1, \ldots, x_d) = \sum_{k=1}^{r} \alpha_k \prod_{i=1}^{d} v_i^{(k)}(x_i)$$

with

$$v_i = \sum_{j=1}^{n_j^{(k)}} \alpha_j^{(i,k)} \phi_j(x_i)$$

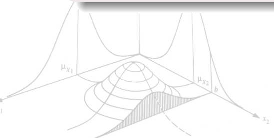
Alternating Least Squares

Alternating Least Squares algorithm

Allows to learn a rank-one tensor.

Algorithm 1 ALS

Initialize $v_i(x_i) = 1$
while v does not converge **do**
  for $i = 1$ to d **do**
    $[\Psi^i(x)]_j = \prod_{u=1 \neq i}^{d} v_u(x_u)\phi_j^i(x_i)$
    Solve $\underset{\beta_i}{\operatorname{argmin}} ||y - \Psi^i(x)^t \beta_i||_2^2$
  end for
end while



Greedy rank-one approximation

Alternating Least Squares algorithm

Allows to learn a rank-r tensor.

Algorithm 2 Greedy rank-one

```
Rank-1 approximation $\prod_{i=1}^{d} v_i^{(1)}(x_i)$
**for** $r = 2$ to $r_{max}$ **do**
    Rank-1 approximation $\prod_{i=1}^{d} v_i^{(r)}(x_i)$
    $y^m = y - \sum_{k=1}^{r} \alpha_k \prod_{i=1}^{d} v_i^{(r)}(x_i)$
    Update $\alpha$ to minimize error (least-squares)
**end for**
```

Usage

In Python...

```
r = 4 # max rank
dim = 5
nk = [10] * dim
factC = [ot.FourierSeriesFactory()] * dim
prod = ot.OrthogonalProductFunctionFactory(factC)

X = dist.getSample(size)
Y = model(X)

algo = ot.TensorApproximationAlgorithm(X, Y, dist, factC, nk, r)

# L1 regularisation
lars = ot.LAR()
loo = ot.CorrectedLeaveOneOut()
aprox = ot.LeastSquaresMetaModelSelectionFactory(lars, loo)
algo.setApproximationAlgorithmFactory(aprox)

algo.run()
```

Conclusion

Conclusions

1. Greedy rank-1
2. Regularized greedy rank-one

Perspectives

- Rank-k