Rice distribution

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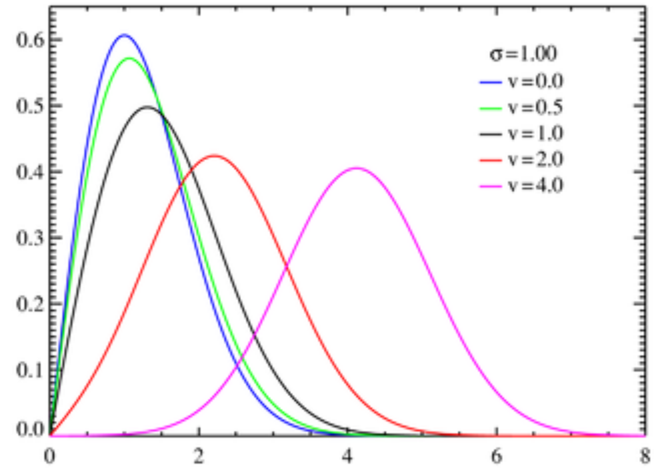
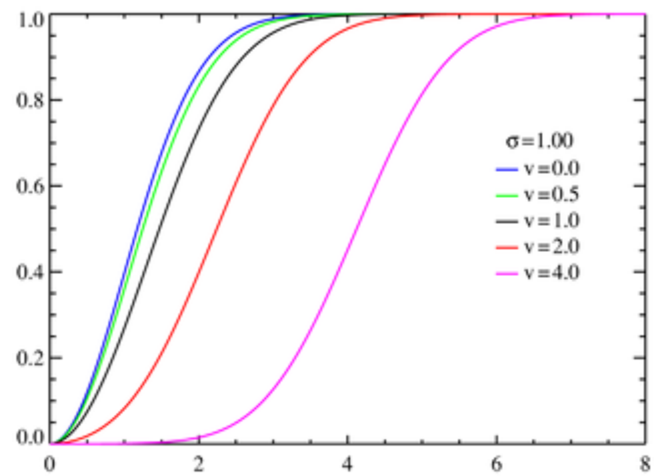
In probability theory, the **Rice distribution** or **Rician distribution** is the probability distribution of an absolute value of a circular bivariate normal random variable with potentially non-zero mean. It was named after Stephen O. Rice.

Probability density function



Cumulative distribution function



parameters:	$\nu \geq 0$ — distance between the reference point and the center of the bivariate distribution, $\sigma \geq 0$ — scale
support:	$x \in [0, +\infty)$
pdf:	$\frac{x}{\sigma^2}\exp\left(\frac{-(x^2+\nu^2)}{2\sigma^2}\right) I_0\left(\frac{x\nu}{\sigma^2}\right)$
cdf:	$1 - Q_1\left(\frac{\nu}{\sigma}, \frac{x}{\sigma}\right)$ where Q_1 is the Marcum

Contents

Characterization

The probability density function is

	Q-function
mean:	$\sigma\sqrt{\pi/2}\ L_{1/2}(-\nu^2/2\sigma^2)$
median:	
mode:	
variance:	$2\sigma^2+\nu^2-\frac{\pi\sigma^2}{2}L_{1/2}^2\left(\frac{-\nu^2}{2\sigma^2}\right)$
skewness:	(complicated)
ex.kurtosis:	(complicated)
entropy:	
mgf:	
cf:	

$$f(x|\nu,\sigma)=\frac{x}{\sigma^2}\exp\left(\frac{-(x^2+\nu^2)}{2\sigma^2}\right)I_0\left(\frac{x\nu}{\sigma^2}\right),$$

where $I_0(z)$ is the modified Bessel function of the first kind with order zero. When $\nu = 0$, the distribution reduces to a Rayleigh distribution.

The characteristic function is:[1][2]

$$\chi_X(t|\nu,\sigma)=\exp\left(-\frac{\nu^2}{2\sigma^2}\right)\left[\Psi_2\left(1;1,\frac{1}{2};\frac{\nu^2}{2\sigma^2},-\frac{1}{2}\sigma^2t^2\right)\right.$$
$$\left.+i\sqrt{2}\sigma t\Psi_2\left(\frac{3}{2};1,\frac{3}{2};\frac{\nu^2}{2\sigma^2},-\frac{1}{2}\sigma^2t^2\right)\right],$$

where $\Psi_2\left(\alpha;\gamma,\gamma';x,y\right)$ is one of Horn's confluent hypergeometric functions with two variables and convergent for all finite values of x and y. It is given by:[3][4]

$$\Psi_2\left(\alpha;\gamma,\gamma';x,y\right)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha)_{m+n}}{(\gamma)_m(\gamma')_n}\frac{x^my^n}{m!n!},$$

where

$$(x)_n=x(x+1)\cdots(x+n-1)=\frac{\Gamma(x+n)}{\Gamma(x)}$$

is the rising factorial.

Properties

Moments

The first few raw moments are:

$$\begin{aligned}
&\mu_1' = \sigma\sqrt{\pi/2}\;L_{1/2}(-\nu^2/2\sigma^2)\\
&\mu_2' = 2\sigma^2 + \nu^2\\
&\mu_3' = 3\sigma^3\sqrt{\pi/2}\;L_{3/2}(-\nu^2/2\sigma^2)\\
&\mu_4' = 8\sigma^4 + 8\sigma^2\nu^2 + \nu^4\\
&\mu_5' = 15\sigma^5\sqrt{\pi/2}\;L_{5/2}(-\nu^2/2\sigma^2)\\
&\mu_6' = 48\sigma^6 + 72\sigma^4\nu^2 + 18\sigma^2\nu^4 + \nu^6\\
&L_\nu(x) = L_\nu^0(x) = M(-\nu,1,x) = \;{}_1F_1(-\nu;1;x)
\end{aligned}$$

where, $L_\nu(x)$ denotes a Laguerre polynomial.

For the case $\nu = 1/2$:

$$\begin{aligned}
&L_{1/2}(x) = \;{}_1F_1\left(-\frac{1}{2};1;x\right)\\
&= e^{x/2}\left[(1-x)\,I_0\left(\frac{-x}{2}\right) - xI_1\left(\frac{-x}{2}\right)\right].
\end{aligned}$$

Generally the moments are given by

$$\mu_k' = s^k 2^{k/2}\,\Gamma(1+k/2)\,L_{k/2}(-\nu^2/2\sigma^2),$$

where $s = \sigma^{1/2}$.

When k is even, the moments become actual polynomials in σ and ν.

The second central moment, equals the variance equation below (which is listed to the right):

$$\mu_2 = 2\sigma^2 + \nu^2 - (\pi\sigma^2/2)\,L_{1/2}^2(-\nu^2/2\sigma^2)$$

When the Rice distribution parameter $\nu = 0$, the distribution becomes the Rayleigh distribution.

$$\begin{aligned}
&\mu_2 = 2\sigma^2 + 0^2 - (\pi\sigma^2/2)\,L_{1/2}^2(0)\\
&L_{1/2}^2(0) = (e^0[(1-0)I_0(0) - 0I_1(0)])^2
\end{aligned}$$

$$L_{1/2}^2(0) = (1 * [I_0(0)])^2$$
$$L_{1/2}^2(0) = (1 * 1])^2 = 1$$
$$\mu_2 = 2\sigma^2 - (\pi\sigma^2/2)$$
$$\mu_2 = (4\sigma^2 - \pi\sigma^2)/2$$
$$\mu_2 = \frac{4-\pi}{2}\sigma^2$$

which is the variance of the Rayleigh distribution.

Related distributions

- $R \sim \mathrm{Rice}\,(\nu, \sigma)$ has a Rice distribution if $R = \sqrt{X^2 + Y^2}$ where $X \sim N\left(\nu\cos\theta, \sigma^2\right)$ and $Y \sim N\left(\nu\sin\theta, \sigma^2\right)$ are statistically independent normal random variables and θ is any real number.
- Another case where $R \sim \mathrm{Rice}\,(\nu, \sigma)$ comes from the following steps:

  1. Generate P having a Poisson distribution with parameter (also mean, for a Poisson) $\lambda = \frac{\nu^2}{2\sigma^2}$.

  2. Generate X having a chi-squared distribution with $2P + 2$ degrees of freedom.

  3. Set $R = \sigma\sqrt{X}$.
- If $R \sim \mathrm{Rice}\,(\nu, 1)$ then R^2 has a noncentral chi-square distribution with two degrees of freedom and noncentrality parameter ν^2.
- If $R \sim \mathrm{Rice}\,(\nu, 0)$ then R^2 has an exponential distribution.[5]

Limiting cases

For large values of the argument, the Laguerre polynomial becomes (see Abramowitz and Stegun §13.5.1 (http://www.math.sfu.ca/~cbm/aands/page_508.htm))

$$\lim_{x\rightarrow-\infty} L_\nu(x) = \frac{|x|^\nu}{\Gamma(1+\nu)}.$$

It is seen that as ν becomes large or σ becomes small the mean becomes ν and the variance becomes σ^2.

Parameter estimation (the Koay inversion technique)

There are three different methods for estimating the Rice parameters, (1) method of moments, (2) method of maximum likelihood, and (3) method of least squares. The first two methods have been investigated by Talukdar et al.[6] and Bonny et al.[7] and Sijbers et al.[8]

Here the interest is in estimating the parameters of the distribution, ν and σ, from a sample of data. This can be done using the method of moments, e.g., the sample mean and the sample standard deviation. The sample mean is an estimate of $\mu_1{}'$ and the sample standard deviation is an estimate of $\mu_2{}^{1/2}$.

The following is an efficient method, known as the "Koay inversion technique", published by Koay et al.[9] for solving the estimating equations, based on the sample mean and the sample standard deviation, simultaneously . This inversion technique is also known as the fixed point formula of SNR. Earlier works [10][11] on the method of moments usually use a root-finding method to solve the problem, which is not efficient.

First, the ratio of the sample mean to the sample standard deviation is defined as *r*, i.e., $r = \mu_1'/\mu_2^{1/2}$. The fixed point formula of SNR is expressed as

$$g(\theta) = \sqrt{\xi(\theta)\left[1 + r^2\right] - 2},$$

where θ is the ratio of the parameters, i.e., $\theta = \frac{\nu}{\sigma}$, and $\xi(\theta)$ is given by:

$$\xi(\theta) = 2 + \theta^2 - \frac{\pi}{8}\exp\left(-\theta^2/2\right)\left[(2+\theta^2)I_0(\theta^2/4) + \theta^2 I_1(\theta^2/4)\right]^2,$$

Note that $\xi(\theta)$ is a scaling factor of σ and is related to μ_2 by:

$$\mu_2 = \xi(\theta)\sigma^2.$$

To find the fixed point, θ^*, of *g*, an initial solution is selected, θ_0, that is greater than the lower bound, which is equal to $\sqrt{\pi/(4-\pi)}$. This provides a starting point for the iteration, which uses functional composition, and this continues until $\left|g^i(\theta_0) - \theta_{i-1}\right|$ is less than some small positive value. Here, g^i denotes the composition of the same function, *g*, *i*-th times. In practice, we associate the final θ_n for some integer *n* as the fixed point, θ^*, i.e., $\theta^* = g(\theta^*)$.

Once the fixed point is found, the estimates ν and σ are found through the scaling function, $\xi(\theta)$, as follows:

$$\sigma = \frac{\mu_2^{1/2}}{\sqrt{\xi\left(\theta^*\right)}},$$

and

$$\nu = \sqrt{\left(\mu_1'^{\,2} + \left(\xi\left(\theta^*\right) - 2\right)\sigma^2\right)}.$$

To speed up the iteration even more, one can use the Newton's method of root-finding as presented by Koay et al.[12] This particular approach is highly efficient.

The author has also provided an online calculator (http://sites.google.com/site/hispeedpackets/Home/launchSNRAnalysisI.jnlp?attredirects=0) for computing the fixed point, which is also known as the underlying SNR from $r = \mu_1'/\mu_2^{1/2}$, the magnitude SNR. See the link here under the subtitle called HI-SPEED SNR Analysis I (http://sites.google.com/site/hispeedpackets/) . Note that the number of combined channel is 1 for the Rician distribution.

See also

- Rayleigh distribution
- Rician fading
- Stephen O. Rice (1907–1986)

Notes

1. ^ Liu 2007 (in one of Horn's confluent hypergeometric functions with two variables).
2. ^ Annamalai 2000 (in a sum of infinite series).
3. ^ Erdelyi 1953.
4. ^ Srivastava 1985.
5. ^ Richards, M.A., Rice Distribution for RCS (http://users.ece.gatech.edu/mrichard/Rice%20power%20pdf.pdf) , Georgia Institute of Technology (Sep 2006)
6. ^ Talukdar 1991
7. ^ Bonny 1996
8. ^ Sijbers 1998
9. ^ Koay 2006 (known as the SNR fixed point formula).
10. ^ Talukdar 1991
11. ^ Abdi 2001

12. ^ Koay 2006 (in Appendix A).

References

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External links

- MATLAB code for Rice/Rician distribution (PDF, mean and variance, and generating random samples)

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