Dynamical Systems MA3081

Exercise sheet 5

Finish the last exercise of the last sheet: compute graph representations of the invariant manifolds at the hyperbolic equilibria (as good as you can), without relying on energy level-set information.

Exercise 1 (Topological equivalence)**.** Consider the two linear dynamical systems induced by the following ODE systems:

$$\begin{cases} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= -x_2 \end{cases} \quad (1) \qquad \text{and} \qquad \begin{cases} \dot{x}_1 &= -x_1 - x_2 \\ \dot{x}_2 &= x_1 - x_2 \end{cases} \quad (2)$$

1. Rewrite the systems in polar coordinates and solve them for initial conditions $(r(0), \theta(0)) = (r_0, \theta_0)$.
2. Clearly, the origin is an equilibrium for both system. Classify this equilibrium for both systems and draw the phase portraits of the flows around the origin.
3. Show that the two systems are topologically equivalent in the closed unit disc $U = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\} = \{(r, \theta) : r \leq 1\}$.

Exercise 2 (Poincaré map)**.** Consider the ODE system

$$\begin{aligned} \dot{x} &= \mu x - y - x(x^2 + y^2), \\ \dot{y} &= x + \mu y - y(x^2 + y^2), \end{aligned}$$

where $(x, y) \in \mathbb{R}^2$ and $\mu > 0$ is a parameter.

1. Find the periodic orbit of the induced dynamical system. *Hint:* Use polar coordinates.
2. Analyze the stability of the periodic orbit with the help of a Poincaré map.

Exercise 3 (Structural stability)**.** Show that the vector field

$$f(x, y) = \begin{pmatrix} -y \\ x \end{pmatrix}$$

is not structurally stable on any open set $K \subset \mathbb{R}^2$ containing the origin.

Exercise 4 (Topological conjugacy)**.** Consider the two-dimensional systems $\dot{x} = Ax$ and $\dot{y} = By$ with matrices:

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}.$$

Show that the matrices are not similar, but that the corresponding flows of the two systems are topologically conjugate.

Numerical excursions: Along the lines of the solution notebook for sheet 3, produce phase portraits for the systems discussed on this exercise sheet.