Dynamical Systems MA3081

Exercise sheet 7

Exercise 1 (Normal forms)**.** Show that a system of the form

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \text{higher order terms} \tag{1}$$

can be locally transformed into the following Poincaré normal form:

$$y' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y + \begin{pmatrix} 0 \\ p_1 y_1^2 + p_2 y_1 y_2 \end{pmatrix} + \mathcal{O}(|y|^3).$$

Hint: cf. Theorem 9.4.

Exercise 2 (Center manifold (reduction))**.** Consider a dynamical system in $\mathbb{R}^2$, generated by

$$\begin{aligned} \dot{x} &= 16y^2 - x, \\ \dot{y} &= y^2 - x^3 \end{aligned}$$

for $x^2 + y^2 < 1$. Determine the equilibria for $x^2 + y^2 < 1$ and their stability.

Exercise 3 (Center manifold (reduction))**.** Let $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism, such that in a neighbourhood of the origin

$$\Psi(x, y) = \begin{pmatrix} x + xy \\ \frac{1}{2}y - x^2 - xy \end{pmatrix}.$$

Sketch a phase portrait close to the equilibrium $(0, 0)$.

Exercise 4 (Transcritical bifurcation)**.** Show that in order for a vector field $x' = f(x, \mu)$, $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ and $f \in C^k$ for $k \geq 2$ to undergo a transcritical bifurcation at $(0, 0)$ it must hold:

- $f(0, 0) = 0$,
- $\frac{\partial f}{\partial x}(0, 0) = 0$
- $\frac{\partial f}{\partial \mu}(0, 0) = 0$,
- $\frac{\partial^2 f}{\partial x \partial \mu}(0, 0) \neq 0$,
- $\frac{\partial^2 f}{\partial x^2}(0, 0) \neq 0$.

That means, we want (or assume) that for any μ the origin is an equilibrium, and that we have another curve of equilibria $\mu \mapsto x^*(\mu)$ (different from 0 for $\mu \neq 0$) that crosses 0 for $\mu = 0$, at which point the equilibria interchange the stability type.