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Dynamical Systems MA3081

Exercise sheet 8

Exercise 1 (Center manifold (reduction))**.** Consider the two-dimensional dynamical system induced by

$$\begin{aligned}\dot{x} &= xy,\\ \dot{y} &= -y + \alpha x^2,\end{aligned}$$

where $\alpha \in \mathbb{R}$ is a parameter. For any α, the origin is clearly nonhyperbolic.

1. Determine its stability by a center manifold reduction up to (and including) fourth-order, depending on α. For all cases, determine the stable set $W^{\mathrm{s}}(0)$.
2. Sketch phase portraits for the distinct cases.

Exercise 2 (1-DOF mechanical systems)**.** Consider a mechanical system of one degree of freedom

$$\ddot{x} + V'(x) = 0,$$

where V is a smooth potential with $|V(x)| \to \infty$ for $x \to \infty$. Draw your favourite potential and sketch the corresponding phase portrait. (If you don't have favourite potentials, start out with $V\colon x \mapsto x^2$ and increase complexity gradually.) The aim of the exercise is to be able to sketch the phase portrait for any given potential.

Next, let me pose a problem which you can think about in the beergarden, enjoying sun, life and alcohol (Drink responsibly!) instead of sitting in some dark room, getting ever more bored and frustrated about stupid calculation problems. The goal is to appreciate the actual power of the Center Manifold Theorem!

First, recall the Center Manifold Theorem! One of the assumptions is that we have a nonhyperbolic equilibrium, with (un)stable and center subspaces. In parametric ODE problems with some vector field $x \mapsto f(x, \mu)$, we would need to fix the parameter, find such a nonhyperbolic equilibrium and then we can apply center manifold reduction and determine stability. But what happens under parameter variation? While hyperbolicity is a persistent feature (no eigenvalues on the imaginary axis is an "open" condition), nonhyperbolicity is not robust (some eigenvalues on the imaginary axis can be generically perturbed away!). So the question is: is the center manifold reduction useful only in degenerate cases?

The answer is clearly no! To see this, write out the ODE problem in all "variables":

$$\begin{aligned}\dot{x} &= f(x, \mu),\\ \dot{\mu} &= 0.\end{aligned}$$

At some nonhyperbolic (for some scalar parameter value μ^*) equilibrium x^* with stable, center and unstable dimensions n^+, n^0, and n^-, resp., this extended ODE system has a nonhyperbolic equilibrium (x^*, μ^*) with dimensions n^+, $n^0 + 1$, and n^-, resp. Now, we can apply the Center Manifold Theorem, and get a low-order approximation of $W^c(x^*, \mu^*)$ in the n^0 center coordinates in x and the single parameter coordinate μ. Make a plot of the described situation!

The remarkable thing here is that all we need is that single equilibrium (x^*, μ^*), nothing is said about equilibria for μ close to but different from μ^*! Let's consider a simple (stupid) example:

$$\begin{aligned}\dot{x} &= \mu + x^2,\\ \dot{y} &= -y.\end{aligned}$$

In the first equation, you recognize the normal form of the fold bifurcation, and the second equation corresponds simply to exponential decay. Clearly, for $\mu = 0$, the origin of this system is a nonhyperbolic equilibrium, with center subspace spanned by e_x and stable subspace spanned by e_y. Extending this system by $\dot{\mu} = 0$ we obtain a center subspace spanned e_x and e_μ. Because of the decoupling, the center manifold is the (x, μ)-plane, and the phase portrait of the reduced dynamics is nothing but the bifurcation diagram for the fold bifurcation.

Similarly, we can consider a system like

$$\begin{aligned}\dot{x} &= \text{normal form of Hopf bifurcation,}\\ \dot{y} &= \text{or any other topologically equivalent system, with higher-order terms etc!}\\ \dot{z} &= -z.\end{aligned}$$

Again, because all characteristic directions are fully aligned with our coordinate system, the center manifold is given by the (x, y, μ)-space, and the phase portrait is nothing but the bifurcation diagram of the Hopf bifurcation.

Now, why is all this of interest? Simply because in practice the stable, unstable and center directions will not be aligned with the coordinate axes. Moreover, stable, unstable and center manifolds will not be flat, so we will be concerned with objects that are positioned diagonally in space and curved. Second, in the fold bifurcation we have two equilibria on one side and none on the other parameter side. In both cases, there are no nonhyperbolic equilbria to which we can apply center manifold theory. So, the actual power of the center manifold theory in parametric dynamical systems is that it helps determining bifurcations in higher-dimensional systems.

To demonstrate what kind of problems I could have posed here (but don't do because (i) I'm a nice guy and (ii) I'm too lazy), imagine we were observing the above simple examples in a different coordinate system. If I had an explicit example for a 3D-diffeo Φ, I could have represented the examples in coordinates $(\tilde{x}, \tilde{y}, \tilde{z}) = \Phi(x, y, z)$, where Φ could shift the equilibrium, rotate the coordinate system and bend the coordinate subspaces. If I gave you the task of doing a stability analysis, you would have computed the invariant subspaces, graph representations of the center manifold, low-order approximations of the reduced dynamics, just to recover the bifurcation diagrams of the two bifurcations.

Now, go out, have a beer (or Huge Spritz), think about all this and make sure you understand what I was talking about. Cheers!