Dynamical Systems MA3081

Exercise sheet 6

Exercise 1 (Transcritical bifurcation)**.** Consider the one-dimensional dynamical system induced by

$$x' = x(p - x)$$

where $p \in \mathbb{R}$ is a parameter. At $p = 0$ a transcritical bifurcation occurs, i.e. the two equilibria $x_1 = 0$ and $x_2 = p$ exchange their stability, compare Example 8.8 in the lecture notes.

1. Draw the bifurcation diagram of the system.
2. Let $\varepsilon > 0$ and perturb the vector field by $\pm\varepsilon$. Analyze the bifurcation behavior of the resulting systems.
3. Show that $x' = x(p - x) + \varepsilon x^3$ exhibits the same bifurcation behavior around the origin.

Exercise 2 (Bifurcations in maps)**.** Consider the following iterated map:

$$x_{k+1} = \alpha x_k e^{-x_k}.$$

It describes a simple population model [Ricker 1954] where $x_k \geq 0$ is the population density in year k, and $\alpha > 0$ is the growth rate. The function on the right-hand side takes into account the negative role of interpopulation competition at high population densities. Determine the fixed points of the map, analyze their stability and the bifurcation behavior.

Exercise 3 (Hénon map)**.** Consider the following iterated map:

$$\begin{aligned} x_{n+1} &= y_n + 1 - a x_n^2, \\ y_{n+1} &= b x_n, \end{aligned}$$

where $(a, b) \in \mathbb{R}^2$ are parameters. It was introduced by the theoretical astronomer Michel Hénon in 1976.

1. Show that the map $F\colon (x, y) \mapsto (y + 1 - ax^2, bx)$ is invertible if $b \neq 0$ and find its inverse. Show that the Hénon map contracts area if $-1 < b < 1$.
2. What are the fixed points of the map? Determine the stability of the fixed points.
3. Show that a bifurcation occurs at $a_l = \frac{3}{4}(1 - b)^2$ and that a period-two orbit exists for $a > a_l$. For which values of a is this period two orbit stable?

4. With your favorite computer program (or should I say with *my* favourite?) analyze the behavior of the map for parameter values $b = 0.3$ and $a = 1.4$ starting in $(0, 0)$. Note, the Hénon map is already implemented in DynamicalSystems.jl!

Exercise 4 (Transcritical bifurcation)**.** Consider the dynamical system induced by

$$x' = r \ln x + x - 1,$$

where $r \in \mathbb{R}$ is a parameter. Show that for a certain value of r a transcritical bifurcation occurs. Find new variables X and R such that the system reduces to the corresponding approximate normal form near the bifurcation.

Numerical excursions: Use the methods demonstrated in this DynamicalSystems.jl tutorial to compute and visualize the Poincaré (return) map for the dynamical system discussed in exercise 2 of sheet 5. Similarly, use the methods demonstrated in the above-mentioned tutorial and in one of the previous solution sheets to produce bifurcation/orbit diagrams for the Ricker problem, exercise 2, and, obviously, for the Hénon map, exercise 3.