Prof. Dr. Christian Kühn, Dr. Daniel Karrasch TU München, SS 2018

Dynamical Systems MA3081

Exercise sheet 2

Exercise 1 (Definition of dynamical system)**.** Do the following triplets $(\mathcal{X}, \mathcal{T}, \phi_t)$ define dynamical systems? If so, are they continuous?

1. $\mathcal{X} = \mathbb{R}^2$, $\mathcal{T} = \mathbb{R}$, $\phi_t(x) := \begin{pmatrix} \mathrm{e}^t \sin t & \mathrm{e}^t \cos t \\ \mathrm{e}^t \cos t & -\mathrm{e}^t \sin t \end{pmatrix} x$, for $t \in \mathcal{T}$ and $x =\in \mathcal{X}$;

2. $\mathcal{X} = \mathbb{R}^2$, $\mathcal{T} = \mathbb{R}$, $\phi_t(x) := \begin{pmatrix} \cos(\lambda t) & -\sin(\lambda t) \\ \sin(\lambda t) & \cos(\lambda t) \end{pmatrix} x$, for $t \in \mathcal{T}$, $x \in \mathcal{X}$ and parameter $\lambda \in \mathbb{R}$.

Exercise 2 (ODE-induced dynamical system)**.** Show that the scalar initial value problem

$$x' = x \log x, \qquad\qquad x(0) = x_0 \in \mathbb{R}_{\geq 0},$$

induces the flow $\phi_t(x_0) = (x_0)^{e^t}$ on $\mathcal{T} = \mathbb{R}$. Find the equilibrium points of the dynamical system $(\mathbb{R}, \mathcal{T}, \phi_t)$.

Exercise 3 (Invariant sets)**.**

1. Let $(\mathcal{X}, \mathcal{T}, \phi_t)$ with $\mathcal{T} = \mathbb{R}$ be a dynamical system. Assume that $\mathcal{S} \subset \mathcal{X}$ is a positively invariant set for ϕ_t. Show that the complement $\mathcal{C} = \mathcal{X} \setminus \mathcal{S}$ is negatively invariant for ϕ_t.

2. Consider the linear ODE system

$$x' = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} x.$$

Show that $\mathcal{S} = \operatorname{span}\left\{(-1, 1)^\top\right\}$ is invariant under the induced dynamical system.

3. Consider the (normalized) pendulum equation

$$\ddot{x} + \sin x = 0.$$

Show that the level sets $\{E(x, \dot{x}) = c\}$ of the function

$$E\colon \mathbb{R}^2 \to \mathbb{R}, \qquad\qquad (x, \dot{x}) \mapsto \frac{1}{2}\dot{x}^2 + 1 - \cos x$$

are invariant under the induced dynamical system.

Exercise 4 (Lyapunov function).

1. Consider the following system of differential equations:

$$\dot{x}_1 = 6x_2,$$
$$\dot{x}_2 = -cx_2 - 3ax_1 - 3bx_1^5,$$

with positive coefficients a, b, c. Find all equilibrium points of the time-continuous dynamical system induced by the above ODE system and analyze whether they are stable in the sense of Lyapunov.
(Hint: The function
$$L\colon (x_1, x_2) \mapsto \alpha x_2^2 + \beta x_1^2 + \gamma x_1^6$$
with suitable coefficients might be useful.)

2. Consider the *Lorenz equations*

$$\dot{x} = \sigma(y - x),$$
$$\dot{y} = x(\rho - z) - y,$$
$$\dot{z} = xy - \beta z,$$

with parameters $\sigma, \rho, \beta > 0$. By using the Lyapunov function $L\colon (x, y, z) \mapsto x^2 + \sigma y^2 + \sigma z^2$ show that the origin is globally asymptotically stable if $r < 1$.

Numerical excursions: Make yourself familiar with the julia-packages OrdinaryDiffEq.jl and DynamicalSystems.jl. Find out how to define an ODE or a dynamical system, and plot some trajectories. If you have a local copy of julia, you will first need to include these packages into your package tree by typing `Pkg.add("DifferentialEquations")` and `Pkg.add("DynamicalSystems")`, respectively. After that—or if you're using juliabox—you need to state that you want to use these packages by typing `using OrdinaryDiffEq` or `using DynamicalSystems`.

As for the OrdinaryDiffEq package (DifferentialEquations docs), you could (ab)use it to compare your analytic solutions from this and the previous exercise sheets to the "ground-truth" numerical solution. ;-)

As for the DynamicalSystems package (DynamicalSystems github page), there are a couple of predefined dynamical systems, including the Lorenz system.