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Dynamical Systems MA3081

Exercise sheet 9

Exercise 1 (1-DOF mechanical systems)**.** Consider a mechanical system of one degree of freedom

$$\ddot{x} + V'(x) = 0,$$

where V is a smooth potential with $|V(x)| \to \infty$ for $x \to \infty$. Draw your favourite potential and sketch the corresponding phase portrait. (If you don't have favourite potentials, start out with $V\colon x \mapsto x^2$ and increase complexity gradually.) The aim of the exercise is to be able to sketch the phase portrait for any given smooth potential.

Exercise 2 (Homoclinic bifurcation)**.** Consider the following system:

$$\begin{aligned} x' &= -x + 2y + x^2, \\ y' &= (2-\alpha)x - y - 3x^2 + \frac{3}{2}xy, \end{aligned}$$

where α is a parameter.

1. Obviously, $(x, y) = (0, 0)$ is an equilibrium of the system for all parameter values. Determine its stability for $|\alpha| < 3/2$.
2. Show that for $\alpha = 0$, $x^2(1-x) - y^2 = 0$ describes a homoclinic orbit of the corresponding system.
3. Let $\alpha \mapsto \xi(\alpha)$ be a corresponding split function. Assume $\xi'(0) \neq 0$ and show that under small variations of α a unique and stable limit cycle bifurcates from the homoclinic orbit found in (2). Sketch the phase portraits for $\xi < 0$, $\xi = 0$ and $\xi > 0$.

Exercise 3 (Center manifold)**.** We consider the system

$$\begin{aligned} \dot{x} &= Ax + F_2(y) + G_1(x, y) \cdot x + H_3(x, y), \\ \dot{y} &= K_2(x, y), \end{aligned}$$

of ODEs in $\mathbb{R}^{m+d}$, where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^d$. Here $A \in \mathbb{R}^{m\times m}$ is a matrix whose eigenvalues have nonvanishing real parts. Furthermore, F_2, G_1, H_3, K_2 are smooth functions, where the subindices indicate the order of convergence to zero, e.g., $F_2 = O(\|(x, y)^T\|^2)$ for $x, y \to 0$.

1. Show that any center manifold associated to $(0, 0)^\top \in \mathbb{R}^m \times \mathbb{R}^d$ is given as the graph of a map h^c of the form

$$h^c(y) = -A^{-1}F_2(y) + O(\|y\|^3).$$

2. Now consider the ODE given by

$$\dot{x} = -2x + y^2 + 3xy + 2x^2 + 2x\sin^2 y,$$
$$\dot{y} = -y^3 + x^2,$$

for $x^2 + y^2 < 9$. Use (1) to sketch the corresponding phase portrait in a neighbourhood of the equilibrium $(0,0)^\top$.

Exercise 4 (Center manifold)**.** Show that the ODE given by

$$\dot{x} = -x^3,$$
$$\dot{y} = -y + x^2,$$

has no analytic but infinitely many locally C^1 center manifolds to the equilibrium $x_e = (0,0)^\top$.