Solutions Exercise Sheet 11

Exercise 11.1 (Sensitive dependence on initial conditions)

a) Let $x_0 \in X$ and $T > 0$. Take $\eta > 0$. Then $C := [-T, T] \times \overline{B}_\eta(x_0)$ is a compact subset of $\mathbb{T} \times \mathbb{R}^n$. Since Φ is continuous on $\mathbb{T} \times \mathbb{R}^n$, it is uniformly continuous on the compact subset C. Thus we know that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $t_1, t_2 \in [-T, T]$ and $x, y \in \overline{B}_\eta(x_0)$:

$$|t_1 - t_2| < \delta, \quad d(x, y) < \delta \quad \Rightarrow \quad d(\Phi(t_1, x), \Phi(t_2, y)) < \varepsilon.$$

Setting $t_1 = t_2$ and $x = x_0$, $y = y_0$ the claim follows.

b) We now prove the equivalences by proving the following implications (1)⇒(3)⇒(2)⇒(1).

(1)⇒(3): Let $\emptyset \neq U \subset X$, U open, $x_0 \in U$. Choose δ_1 sufficiently small such that $B_{\delta_1}(x_0) \subset U$. By part a) we know that we can find a sufficiently small δ_2 for which

$$d(\Phi(t, x_0), \Phi(t, y_0)) < \Lambda \ (= \varepsilon) \tag{1}$$

for $t \in [0, T]$ and $y_0 \in B_{\delta_2}(x_0)$. We now choose $\delta = \min\{\delta_1, \delta_2\}$. Since we assumed that (1) holds, there exists $\widehat{y} \in B_\delta(x_0)$ and $\widehat{t} \geq 0$ such that

$$d(\Phi(\widehat{t}, \widehat{y}), \Phi(\widehat{t}, x_0)) \geq \Lambda.$$

Because of (1) we know that $\widehat{t} > T$. Therefore

$$\operatorname{diam}(\Phi(\widehat{t}, U)) \geq d(\Phi(\widehat{t}, \widehat{y}), \Phi(\widehat{t}, x_0)) \geq \Lambda =: \Lambda'$$

which proves the first implication.

(3)⇒(2): Let $x \in X$, $\delta > 0$ and $T \geq 0$. We set $U := B_\delta(x)$. By the assumption that (3) holds we know that there is a $t \geq T$ such that

$$\operatorname{diam}(\Phi(t, U)) \geq \Lambda'.$$

Hence there are $y, z \in B_\delta(x)$ such that

$$d(\Phi(t, y), \Phi(t, z)) > \frac{\Lambda'}{2}. \tag{2}$$

Putting $\Lambda := \frac{\Lambda'}{4}$ yields that

$$d(\Phi(t, y), \Phi(t, x)) \geq \Lambda \quad \text{or} \quad d(\Phi(t, z), \Phi(t, x)) \geq \Lambda$$

(since otherwise $d(\Phi(t, y), \Phi(t, z)) \leq d(\Phi(t, y), \Phi(t, x)) + d(\Phi(t, z), \Phi(t, x)) \leq \Lambda + \Lambda = \frac{\Lambda'}{2}$, contradicting (2)). This however already implies that (2) holds.

(2)⇒(1): This implication is easy to prove, since (2) is seemingly stronger than (1) in the sense that putting $T = 0$ yields statement (1).

□

Exercise 11.2 (Lorenz system)

All steps are straightforward following the hints in the exercise. After you did the calculations, take a moment to appreciate the fact that the bifurcation parameter ρ is a coordinate on the center manifold, and that the phase portrait of the reduced dynamics in fact corresponds to the bifurcation diagram of the pitchfork bifurcation.

Exercise 11.3 (Melnikov function)

(a) We calculate for g:

$$
\begin{aligned}
M(t_0) &= \int_{-\infty}^{\infty} f(\gamma(t-t_0)) \wedge g(\gamma(t-t_0))dt \\
&= \int_{\Gamma_0} f(\gamma(x)) \wedge g(\gamma(x) \frac{1}{\|f(x)\|} dx \\
&= \int_{\Gamma_0} g(\gamma(x) \cdot \frac{f^\perp}{\|f(x)\|} dx \\
&= \pm \int_{\Gamma_0} g(\gamma(x) \cdot n(x) dx \\
&= \pm \int_{\Gamma_0} g(\gamma(x) \cdot n(x) dx \\
&= \pm \int_{\mathrm{int}\Gamma_0} \mathrm{div} g(x) dx \\
&= \pm \int_{\mathrm{int}\Gamma_0} \mathrm{trace} Dg(x) dx,
\end{aligned}
$$

where $f^\perp$ is f rotated by $\pi/2$. We omitted a discussion of orientation, so the result is only valid up to sign.

(b) We calculate:

$$
Df(\gamma)f(\gamma) = \begin{pmatrix} \partial_x f_1(\gamma) & \partial_y f_1(\gamma) \\ \partial_x f_2(\gamma) & \partial_y f_2(\gamma) \end{pmatrix} \begin{pmatrix} f_1(\gamma) \\ f_2(\gamma) \end{pmatrix} = \begin{pmatrix} \partial_x f_1(\gamma) f_1(\gamma) + \partial_y f_1(\gamma) f_2(\gamma) \\ \partial_x f_2(\gamma) f_1(\gamma) + \partial_y f_2(\gamma) f_2(\gamma) \end{pmatrix}
$$

and likewise

$$
Df(\gamma)\gamma_1^s = \begin{pmatrix} \partial_x f_1(\gamma)(\gamma_1^s)_1 + \partial_y f_1(\gamma)(\gamma_1^s)_2 \\ \partial_x f_2(\gamma)(\gamma_1^s)_1 + \partial_y f_2(\gamma)(\gamma_1^s)_2 \end{pmatrix}
$$

hence,

$$
\begin{aligned}
& Df(\gamma)f(\gamma) \wedge \gamma_1^s + f(\gamma) \wedge Df(\gamma)\gamma_1^s \\
=& (Df(\gamma)f(\gamma))_1(\gamma_1^s)_1 - (Df(\gamma)f(\gamma))_2(\gamma_1^s)_1 + f_1(\gamma)(Df(\gamma)\gamma_1^s)_2 - f_2(\gamma)(Df(\gamma)\gamma_1^s)_1 \\
=& (\partial_x f_1(\gamma) f_1(\gamma) + \partial_y f_1(\gamma) f_2(\gamma))(\gamma_1^s)_2 - (\partial_x f_2(\gamma) f_1(\gamma) + \partial_y f_2(\gamma) f_2(\gamma))(\gamma_1^s)_1 \\
& + f_1(\gamma)(\partial_x f_1(\gamma)(\gamma_1^s)_1 + \partial_y f_1(\gamma)(\gamma_1^s)_2) - f_2(\gamma)(\partial_x f_2(\gamma)(\gamma_1^s)_1 + \partial_y f_2(\gamma)(\gamma_1^s)_2) \\
=& \partial_x f_1(\gamma) f_1(\gamma)(\gamma_1^s)_2 - \partial_y f_2(\gamma) f_2(\gamma)(\gamma_1^s)_1 + \partial_y f_2(\gamma) f_1(\gamma)(\gamma_1^s)_2 - \partial_x f_1(\gamma) f_2(\gamma)(\gamma_1^s)_1 \\
=& (\partial_x f_1(\gamma) + \partial_y f_2(\gamma))(f_1(\gamma)(\gamma_1^s)_2 - f_2(\gamma)(\gamma_1^s)_1) \\
=& \mathrm{trace}(Df(\gamma))(f(\gamma) \wedge \gamma_1^s) \\
=& \mathrm{trace}(Df(\gamma))\delta^s
\end{aligned}
$$

and it follows:

$$
Df(\gamma)f(\gamma) \wedge \gamma_1^s + f(\gamma) \wedge (Df(\gamma)\gamma_1^s + g(\gamma,t)) = \mathrm{trace}(Df(\gamma))\delta^s + f(\gamma) \wedge g(\gamma,t).
$$

Exercise 11.4 (Linear twist map)

(a) We will show that the linear twist map is not expansive. Fix $\nu > 0$. Consider points (x_1, y) and (x_2, y) such that $|x_1 - x_2| < \nu$. Then since $T^n(x_1, y) = (x_1 + ny \mod 1, y)$ and $T^n(x_2, y) = (x_2 + ny \mod 1, y)$ we have

$$\begin{aligned} d(T^n(x_1,y), T^n(x_2,y)) &= \sqrt{((x_1 + ny \mod 1) - (x_2 + ny \mod 1))^2 + (y-y)^2} \\ &\leq \sqrt{((x_1 + ny) - (x_2 + ny))^2} \\ &= |x_1 - x_2| \\ &< \nu \end{aligned}$$

for any $n \in \mathbb{N}$. Hence, ν is not an expansivity constant and T is not expansive.

(b) We want to show that the linear twist map has sensitive dependence on initial conditions with sensitivity constant $\triangle = 1/2$.
Let $(x_1, y_1) \in \mathbb{T}^2$ and $\varepsilon > 0$. Now choose $y_2 \neq y_1$ such that $y_1 - y_2 = \frac{1}{2N} < \varepsilon$. Thus we have: $d((x_1, y_1), (x_1, y_2)) = \sqrt{(x_1 - x_1)^2 + (y_1 - y_2)^2} = |y_1 - y_2| < \varepsilon$. Furthermore:

$$T^n(x_1, y_1) = (\underbrace{x_1 + ny_1 \mod 1}_{=:R^n_{y_1}(x_1)}, y_1)$$

and

$$T^n(x_1, y_2) = (\underbrace{x_1 + ny_2 \mod 1}_{=:R^n_{y_2}(x_1)}, y_2).$$

Now, note that $x_1 + ny_1 = x_1 + ny_2 + n(y_1 - y_2) = x_1 + ny_2 + n\frac{1}{2N}$ so that for $n = N$ we have

$$\begin{aligned} R^n_{y_1}(x_1) &= x_1 + ny_1 \mod 1 \\ &= x_1 + ny_2 + \frac{1}{2} \mod 1 \\ &= R^n_{y_2}(x_1) + \frac{1}{2} \mod 1 \\ &= \begin{cases} R^n_{y_2}(x_1) + \frac{1}{2} & R^n_{y_2}(x_1) \in [0, 1/2) \\ R^n_{y_2}(x_1) - \frac{1}{2} & R^n_{y_2}(x_1) \in [1/2, 1) \end{cases} \end{aligned}$$

In both cases we showed that

$$|R^n_{y_1}(x_1) - R^n_{y_2}(x_1)| = \frac{1}{2}$$

and thus since the Euclidean distance is at least the distance between the horizontal components, we have:

$$d(T^n(x_1, y_1), T^n(x_1, y_2)) \geq |R^n_{y_1}(x_1) - R^n_{y_2}(x_1)| = \frac{1}{2}$$

i.e. T has sensitive dependence on initial conditions.

(c) We want to show that T is not topologically transitive. Given any $(x, y) \in \mathbb{T}^2$, its orbit $\phi_t((x, y))$ will always be contained in $[0, 1) \times \{y\}$, since the y coordinate is not changed by the map T. Now, take $(x, v) \in \mathbb{T}^2$ with $|y - v| > \varepsilon$. The orbit $\phi_t((x, y))$ for any $t \in \mathbb{Z}$ will never enter the nonempty open ball centered at (x, v) with radius ε. Thus no orbit can be dense in $\mathbb{T}$ and T is not topologically transitive.