(Lack of) Stability

Daniel Karrasch

Consider the example from the lecture, given in polar coordinates (r, θ):

$$\dot{r} = r(1-r), \qquad \dot{\theta} = \sin\frac{\theta}{2}.$$

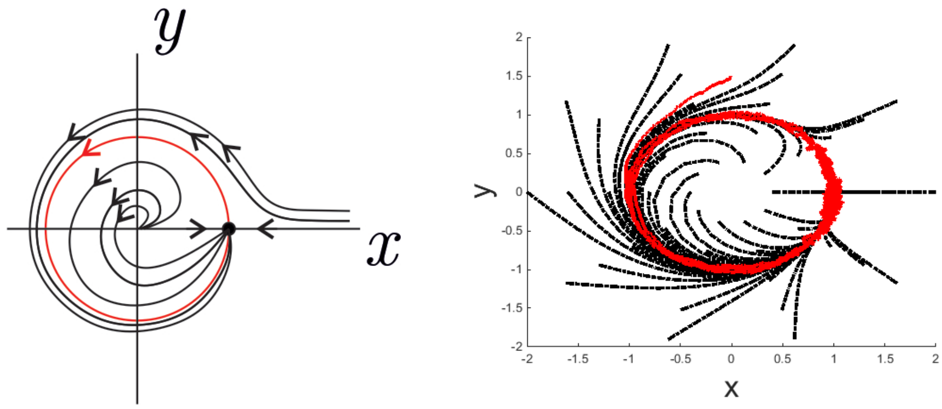
The phase portrait in Euclidean (x, y) coordinates is shown in Fig. 1a. As discussed in the lecture, the point $(x, y) = (1, 0)$ is an attracting equilibrium, which is *not* (Lyapunov) stable and therefore not asymptotically stable.

To better understand the consequence of the lack of stability, consider a (random) perturbation of the above model. Ignoring all mathematical aspects such as well-posedness and regularity of solutions, we perturb the system *computationally* by adding "small noise" to the right hand side, i.e.,

$$\dot{r} = r(1-r) + N(0, 0.3), \qquad \dot{\theta} = \sin\frac{\theta}{2} + N(0, 0.3),$$

where $N(\mu, \sigma)$ refers to a random variable which is distributed according to the normal distribution with mean μ and standard deviation σ. The two random variables are supposed to be independent. The perturbed "phase portrait" is shown in Fig. 1b.

Interestingly, while all trajectories of the deterministic system rotate at most once (recall the invariance of the $\theta = 0$ half-axis), the sample trajectory of the perturbed system rotates 4 times over the time interval $[0, 500]$. Viewed from the equilibrium $(1, 0)$, the perturbed trajectory comes close to it, and is then "kicked" beyond the invariant half-axis, and turns once more. Note also that clearly, the effect of the random perturbation is strongest around the equilibrium position. This is due to the fact that the random perturbation has a fixed "strength"/"amplitude", which is small, and hence dominated by the deterministic vector field, in most regions of the state space, except around the equilibrium position. There, the deterministic right hand side is 0 at $(1, 0)$ and due to continuity arbitrarily small nearby.



(a) Red circle–invariant set, black point–equilibrium.

(b) Black dashed lines–deterministic trajectories, red line–”trajectory” of perturbed system.

Figure 1: (a) Qualitative deterministic phase portrait. (b) Numerical phase portrait for randomly perturbed system.

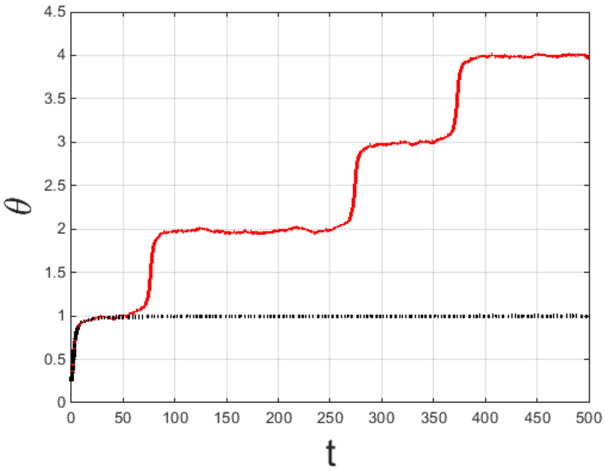


Figure 2: Trajectories’ normalized angular component over time. Black dashed line–deterministic trajectory, red line–trajectory of the perturbed system, both starting at $(r_0, \theta_0) = (1.5, \pi/2)$.