2. a) The linearized flow near the origin satisfies

$$\dot{\underline{\xi}} = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho+1 & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix} \underline{\xi}$$

The eigenvalues of the Jacobian matrix are

$$\lambda_1 = \frac{-(1+\sigma) + \sqrt{(1+\sigma)^2 + 4\sigma\rho}}{2}$$

$$\lambda_2 = \frac{-(1+\sigma) - \sqrt{(1+\sigma)^2 + 4\sigma\rho}}{2}$$

$$\lambda_3 = -\beta$$

$\underline{-1 \leq \rho < 0}$: $\mathrm{Re}(\lambda_i) < 0$ for $i = 1,2,3 \Rightarrow$ Origin is a stable fixed point

$\underline{\rho > 0}$: $\mathrm{Re}(\lambda_1) > 0 \Rightarrow$ origin is unstable

$\underline{\rho = 0}$: $\lambda_1 = 0$, $\lambda_2 = -(1+\sigma)$, $\lambda_3 = -\beta$

$\hookrightarrow$ By the center manifold theorem the system has a one-dimensional center manifold passing through the origin.

b) Consider the extended system

$$\dot{\rho} = 0$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \underbrace{\begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ x(\rho - z) \\ xy \end{pmatrix}$$

The matrix A has the eigenvalues and eigenvectors:

$$\lambda_1 = 0, \quad \lambda_2 = -(\sigma+1), \quad \lambda_3 = -\beta$$

$$e_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} \sigma \\ -1 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Perform the coordinate transformation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = T \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ where

$$T = (e_1 | e_2 | e_3) = \begin{pmatrix} 1 & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x = u + \sigma v \\ y = u - v \\ z = w \end{cases}$$

therefore, $\begin{pmatrix} u \\ v \\ w \end{pmatrix} = T^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{pmatrix} \dot u \\ \dot v \\ \dot w \end{pmatrix} = T^{-1}\begin{pmatrix} \dot x \\ \dot y \\ \dot z \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} \dot u \\ \dot v \\ \dot w \end{pmatrix} = T^{-1}AT\begin{pmatrix} u \\ v \\ w \end{pmatrix} + T^{-1}\begin{pmatrix} 0 \\ \rho x - xz \\ xy \end{pmatrix}$$

Substituting (x,y,z) in terms of (u,v,w) and performing the matrix multiplications, we get

$$(1)\begin{cases} \dot u = \dfrac{\sigma}{1+\sigma}(u+\sigma v)(\rho - w) = \dfrac{\sigma}{1+\sigma}(\rho u - uw + \sigma\rho v - \sigma v w) \\ \dot v = -(1+\sigma)v - \dfrac{1}{1+\sigma}(u+\sigma v)(\rho - w) = -(1+\sigma)v - \dfrac{1}{1+\sigma}(\rho u - uw + \sigma\rho v - \sigma v w) \\ \dot w = -\beta w + (u+\sigma v)(u-v) = -\beta w + (u^2 - uv + \sigma u v - \sigma v^2) \end{cases}$$

The center manifold is given by $v = h_1(u,\rho)$ and $w = h_2(u,\rho)$ with the quadratic approximations

$$\left.\begin{aligned} h_1(u,\rho) &= a_1 u^2 + a_2 u\rho + a_3\rho^2 + O(3) \\ h_2(u,\rho) &= b_1 u^2 + b_2 u\rho + b_3\rho^2 + O(3) \end{aligned}\right\} (2)$$

By invariance of the center manifold we have:

$$\left.\begin{aligned} \dot v &= 2a_1 u\dot u + a_2\rho\dot u + O(3) \\ \dot w &= 2b_1 u\dot u + b_2\rho\dot u + O(3) \end{aligned}\right\} (3)$$

Substituting from (1) and (2) into (3) we get

$$\begin{cases} -(1+\sigma)\left[a_1u^2 + a_2\rho u + a_3\rho^2\right] - \dfrac{1}{1+\sigma}\rho u + O(3) = O(3) \\ -\beta\left[b_1 u^2 + b_2\rho u + b_3\rho^2\right] + u^2 + O(3) = O(3) \end{cases}$$

Matching exponents we obtain: $\begin{cases} a_1 = 0, \ a_2 = \dfrac{-1}{(1+\sigma)^2}, \ a_3 = 0 \\ b_1 = 1/\beta, \ b_2 = b_3 = 0 \end{cases}$

Therefore, the graph of the center manifold is given by

$$(4) \qquad v = \frac{-1}{(1+\sigma)^2}\rho u + O(3), \quad w = \frac{1}{\beta}u^2 + O(3) \quad \text{for } 0 \le |\rho| \ll 1$$

c) Substituting approximations (4) in the $\dot{u} = \cdots$ of eq. (1) we get the reduced equations on the center manifold.

$$\dot{u} = \frac{\sigma}{1+\sigma}\left(1 - \frac{\sigma\rho}{(1+\sigma)^2}\right) u \left(\rho - \frac{u^2}{\beta}\right)$$

For $\rho < 0$, there is only one fixed point $u = 0$

For $\rho > 0$, there are three fixed points $u = 0$, $u = \pm\sqrt{\rho\beta}$

The stability diagram looks like:

Note that $u = 0$ changes from stable to unstable as ρ changes from negative to positive.