Dynamical Systems MA3081

Exercise sheet 11

Exercise 1 (Sensitive dependence on initial conditions)**.** Let $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$. Consider $\mathbb{R}^n$ equipped with the Euclidean metric d and let $(\mathbb{R}^n, \Phi)$ be a C^0-dynamical system with flow map $\Phi\colon \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$. Show:

1. Let $x_0 \in X$ and $T > 0$. Then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $t \in [-T, T]$

$$d(x_0, y_0) < \delta \quad \Rightarrow \quad d(\Phi(t, x_0), \Phi(t, y_0)) < \varepsilon.$$

2. The following statements are equivalent:

   (a) The dynamical system $(\mathbb{R}^n, \Phi)$ depends sensitively on initial conditions, i.e., there exists $\Lambda > 0$ such that for each $x \in \mathbb{R}^n$ and $\delta > 0$ there is a $y \in \mathbb{R}^n$ and a $t \geq 0$ with $d(x, y) < \delta$ and $d(\Phi(t, x), \Phi(t, y)) \geq \Lambda$.

   (b) The dynamical system $(\mathbb{R}^n, \Phi)$ depends sensitively on initial conditions *and* the minimal separation distance Λ occurs arbitrarily late, i.e., there is a $\Lambda > 0$ such that for each $x \in \mathbb{R}^n$, $\delta > 0$ and $T \geq 0$ there are $y \in \mathbb{R}^n$ and $t \geq T$ with $d(x, y) < \delta$ and $d(\Phi(t, x), \Phi(t, y)) \geq \Lambda$.

   (c) There is a $\Lambda' > 0$ such that for each open, non-empty $U \subseteq \mathbb{R}^n$ and for each $T \geq 0$ there exits a $t \geq T$ with

$$\operatorname{diam}(\Phi(t, U)) \geq \Lambda'.$$

   Here, $\operatorname{diam}(M) := \sup_{x,y \in M} d(x, y) \in [0, \infty]$ denotes the diameter of $M \subseteq \mathbb{R}^n$.

Hint: Show (a)⇒(c)⇒(b)⇒(a), utilizing the result in (1) for the first implication.

Exercise 2 (Parameter-dependent center manifolds)**.** Consider the Lorenz system

$$\begin{aligned}
\dot{x} &= \sigma(y - x), \\
\dot{y} &= (\rho + 1)x - y - xz, \\
\dot{z} &= xy - \beta z,
\end{aligned}$$

where $\sigma, \beta > 0$ and $\rho \geq -1$ are parameters.

1. Using linearization show that the origin is a stable equilibrium for $-1 \leq \rho < 0$ and an unstable equilibrium for $\rho > 0$. For $\rho = 0$, show the existence of a center manifold near the origin.

2. For small ρ around 0, construct a ρ–dependent, quadratic-order local continuation of the center manifold. To do so, carry out the following steps.

   - Using an appropriate linear transformation $T\colon (x, y, z) \mapsto (u, v, w)$, write the Lorenz system as

   $$\dot{u} = f_1(u, v, w, \rho),$$
   $$\begin{pmatrix} \dot{v} \\ \dot{w} \end{pmatrix} = B \begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} f_2(u, v, w, \rho) \\ f_3(u, v, w, \rho) \end{pmatrix},$$

     where $B \in \mathbb{R}^{2\times 2}$ is a diagonal matrix independent of ρ, and the f_i's are smooth functions.
   - The center manifold then satisfies $v = h_1(u, \rho)$ and $w = h_2(u, \rho)$ for appropriate functions h_1 and h_2. Use the invariance of the center manifold to find quadratic-order approximations for h_1 and h_2.

3. Construct a bifurcation diagram for the reduced system on the center manifold using ρ as a bifurcation parameter.

Exercise 3 (Melnikov function). 1. Recall the setting of Melnikov's method and the definition of the Melnikov function. Show that if g is not explicitly time dependent we have:

$$M(t_0) = \int_{\mathrm{int}\Gamma_0} \mathrm{trace} Dg(x) dx,$$

where $\Gamma_0 = \{\gamma(t) | t \in \mathbb{R}\} \cup \{p\}$ and γ is the homoclinic orbit to the equilibrium p.

2. In the proof of Theorem 13.5 (Melnikov Zeros for Homoclinic Orbits) we used that

$$Df(\gamma)f(\gamma)\wedge\gamma_1^s + f(\gamma)\wedge(Df(\gamma)\gamma_1^s + g(\gamma, t)) = \mathrm{trace}(Df(\gamma))\delta^s + f(\gamma)\wedge g(\gamma, t).$$

   Prove this equality.

Exercise 4 (Linear twist map). Let $T\colon \mathbb{T}^2 \to \mathbb{T}^2$ be the linear twist map given by

$$T(x, y) = (x + y \mod 1, y).$$

Consider $T\colon X \to X$ as a dynamical system where $X = \mathbb{T}^2$ is a metric space with the distance d induced by the Euclidean distance.

1. Show that T is not expansive.
2. Show that T has sensitive dependence on initial conditions.
3. Show that T is not topologically transitive.