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Dynamical Systems MA3081

Exercise sheet 10

Exercise 1 (Normal forms for maps). We consider a diffeomorphism $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ such that for $(x, y)^T \in \mathbb{R}^2$ in a neighborhood of $(0, 0)^T$ one has

$$\Psi(x, y) = \begin{pmatrix} x + 2y + xy \\ 3y - x^2 + x^2 y \end{pmatrix}.$$

Determine its normal form up to and including second order in a neighborhood of $(0, 0)^\top$.

Exercise 2 (Symbolic dynamics). Let $N \in \mathbb{N} \setminus \{1\}$ and define the space on N symbols as $\Sigma_N = \{0, 1, ..., N - 1\}^{\mathbb{N}_0}$. Together with

$$d(x, y) = \sum_{n \in \mathbb{N}_0} \frac{|x_n - y_n|}{N^n}$$

(Σ_N, d) is a metric space.

1. Show that $d(x, y) \leq N^{-n}$ if $x_j = y_j$ for all $j \leq n$, and that $d(x, y) \geq N^{-n}$ if $x_j \neq y_j$ for at least one $j \leq n$. In other words, $x, y \in \Sigma_N$ are close if and only if their first n values coincide.
2. Show that Σ_N is a Cantor set, i.e. it is compact, perfect and totally disconnected.
3. On Σ_N we define the shift map as

   $$\sigma \colon \Sigma_N \to \Sigma_N, \qquad x_n \mapsto x_{n+1}.$$

   Show that the shift map has a dense orbit, i.e., there exists $s \in \Sigma_N$ such that for all $s' \in \Sigma_N$ and $\varepsilon > 0$ there exists k such that $d(\sigma^k(s), s') < \varepsilon$.

Exercise 3 (Mass on a hoop). Consider a ball of mass m that slides on a rotating hoop (see Fig. 1). The angular velocity of the hoop is Ω, the viscous friction coefficient between the hoop and the ball is b, and the constant of gravity is g. The equation of motion for the sliding ball is given by

$$mR^2\ddot{\alpha} + bR^2\dot{\alpha} + mR^2 \left(g/R - \Omega^2 \cos\alpha\right) \sin\alpha = 0.$$

1. Plot the location of equilibria of the ball as a function of the non-dimensionalized rotation parameter $\nu = R\Omega^2/g$.

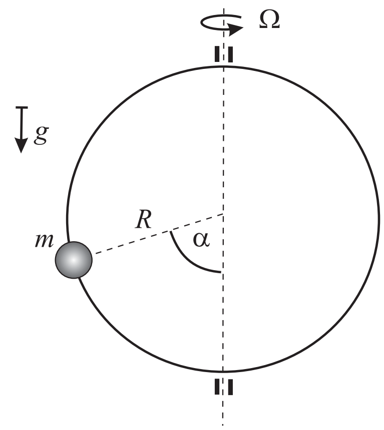

Figure 1: Illustration of the sliding mass on a rotating hoop.

2. Determine the stability type of the different equilibrium branches on the plot. Identify the critical angular velocity at which a bifurcation of equilibria occurs.

In the following, assume that there is no friction, i.e., $b = 0$, and that the parameter values are such that the lower equilibrium position of the ball is stable in linear approximation.

3. Show that in this case, the equilibrium is also nonlinearly stable.

4. Prove that the equilibrium *cannot* be asymptotically stable for the nonlinear system.

Exercise 4 (Sensitive dependence on initial conditions)**.** Let $\mathbb{T} \in \{\mathbb{R}, \mathbb{Z}\}$. Consider $\mathbb{R}^n$ equipped with the Euclidean metric d and let $(\mathbb{R}^n, \Phi)$ be a C^0-dynamical system with flow map $\Phi\colon \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$. Show:

1. Let $x_0 \in X$ and $T > 0$. Then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $t \in [-T, T]$
$$d(x_0, y_0) < \delta \quad \Rightarrow \quad d(\Phi(t, x_0), \Phi(t, y_0)) < \varepsilon.$$

2. The following statements are equivalent:

   (a) The dynamical system $(\mathbb{R}^n, \Phi)$ depends sensitively on initial conditions, i.e., there exists $\Lambda > 0$ such that for each $x \in \mathbb{R}^n$ and $\delta > 0$ there is a $y \in \mathbb{R}^n$ and a $t \geq 0$ with $d(x, y) < \delta$ and $d(\Phi(t, x), \Phi(t, y)) \geq \Lambda$.

   (b) The dynamical system $(\mathbb{R}^n, \Phi)$ depends sensitively on initial conditions *and* the minimal separation distance Λ occurs arbitrarily late, i.e., there is a $\Lambda > 0$ such that for each $x \in \mathbb{R}^n$, $\delta > 0$ and $T \geq 0$ there are $y \in \mathbb{R}^n$ and $t \geq T$ with $d(x, y) < \delta$ and $d(\Phi(t, x), \Phi(t, y)) \geq \Lambda$.

(c) There is a $\Lambda' > 0$ such that for each open, non-empty $U \subseteq \mathbb{R}^n$ and for each $T \geq 0$ there exits a $t \geq T$ with

$$\operatorname{diam}\left(\Phi(t, U)\right) \geq \Lambda'.$$

Here, $\operatorname{diam}(M) := \sup_{x,y \in M} d(x, y) \in [0, \infty]$ denotes the diameter of $M \subseteq \mathbb{R}^n$.

Hint: Show (a)$\Rightarrow$(c)$\Rightarrow$(b)$\Rightarrow$(a), utilizing the result in (1) for the first implication.