# Simple Allocation with Correlated Types*

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**Abstract**

An object is to be allocated among a number of agents. The efficient allocation depends on the agents' information about their peers, but each agent wants the object for themself. Monetary transfers are unavailable. We consider mechanisms where it is a dominant strategy to report truthfully.

On the negative side, deterministic mechanisms do not suffice outside of special cases. Anonymous mechanisms cannot elicit any information.

On the positive side, there are simple mechanisms—*jury mechanisms*—that are optimal when there are three or fewer agents, approximately optimal in symmetric environments with many agents, and the only deterministic mechanisms satisfying a relaxed anonymity notion. In a jury mechanism, each agent is either a juror or a candidate. The jurors decide which of the candidates wins the object; jurors never win.

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# 1 Introduction

We consider a problem where a desirable object must be allocated among a number of agents. The efficient allocation depends on the agents' information about their peers, but monetary transfers cannot be used to elicit this information. A number of applications fit this description:

(1) A planner allocates a good to a community, wishing to target a household with a low income. Each household has some information about their neighbors. Yet, asking a household whether they need the good more than others creates incentives for overstating one's own need.[1] If households are financially constrained, it is infeasible to have them bid for the good.

(2) A group has to select one of its members to serve as president, wishing to select someone with the right skills. Each agent knows the skills of their friends. Yet, if an agent covets (or relishes) the position of president, they may claim that they themself are the most skilled (or least skilled) candidate. Monetary transfers may be excluded on ethical grounds.

(3) A firm splits a budget across subdivisions, wishing to allocate to a subdivision with a high marginal return. Returns are correlated across subdivisions via an underlying state of the world, and hence each subdivision, knowing its own return, can predict the returns of the others. Yet, to maximize one's own allocation, there is an incentive to overstate one's own return. If all parties are risk neutral, the allocated share of the budget can be interpreted as the probability of being allocated the object; additional monetary transfers would be self-defeating.

To understand good allocation rules for such environments, we take a mechanism design approach and consider the *simple allocation problem* with *correlated types*:[2] Each agent wants to win the object and is indifferent to which of the others wins. Allocating to an agent generates a social value. The agents have private information about these values—their *types*. To capture the idea that an agent may have

[1] A literature in development economics studies the idea that community knowledge can improve targeting programs. For example, Hussam et al. (2022), reporting on a field experiment in India, find evidence that community knowledge can help target high-growth microentropreneurs, but that community members also bias their reports towards themselves, friends, or family.

[2] The name "simple allocation problem" is borrowed from Ben-Porath et al. (2019). They study a version of the problem with evidence as an application of their main results.

information about their peers, we allow for arbitrary correlations between types and values across agents.

We study mechanisms for maximizing the expected value of the allocation. In a mechanism, each agent is asked to report their type. To account for the agents' self-interested behavior, we focus on mechanisms where truthfully reporting one's type is a dominant strategy; that is, we focus on dominant-strategy incentive-compatible (DIC) mechanisms. For the assumed preferences of the agents, DIC requires that one's report never influences one's own winning probability.

Let us highlight some of the differences to existing models. There is earlier work on DIC mechanisms where agents nominate a winner for the object (Alon et al., 2011; Holzman and Moulin, 2013). By contrast, we fix a general joint distribution of types and values. Allowing for general types is important as an agent cannot fully express their opinion via a nomination when, say, this opinion is contingent on the opinions of others. Other work considers settings where non-monetary instruments for screening the agents are available, but where each agent has no information about their peers (for example, Ben-Porath et al., 2014, 2019). The present model lacks such instruments, and the focus is on the problem of aggregating agents' information about their peers.

We contribute two results demonstrating the difficulty of designing "good" mechanisms: deterministic mechanisms are not without loss, and anonymity is incompatible with DIC. Further, we contribute three positive results on so-called *jury mechanisms*. These mechanisms solve the problem with three agents, are approximately optimal in symmetric environments with many agents, and the only deterministic DIC mechanisms satisfying a relaxed notion of anonymity. Let us elaborate.

For each agent, there is a trade-off between allocating to the agent and using the agent's information about the others. This trade-off arises since, on the one hand, DIC demands that a change in an agent's type does not affect that agent's own winning probability. On the other hand, since types contain information, a change in the type reveals information about the values from allocating to the others.

Optimally resolving this trade-off may require the use of stochastic mechanisms that cannot be implemented by randomizing over deterministic ones. That is, the set of DIC mechanisms may admit stochastic extreme points, and these can be uniquely optimal. Stochastic extreme points exist if and only if there are at least four agents and the type spaces are not "too small." We further present an example to illuminate why exactly the trade-off leads to randomization as a parameter of the environment is perturbed. These results contribute to the literature on the gap between stochastic and deterministic mechanisms (see, for example, Chen et al., 2019; Pycia and Ünver, 2015). The typical view is that, to reduce complexity of the mechanism, one should use mechanisms that can be implemented by randomizing over deterministic ones. In the present model, doing so is not generally without loss for allocating efficiently.

The second negative result asserts that all anonymous DIC mechanisms must ignore the reports of the agents. Here, anonymity means that all agents can make the same reports and that an agent's winning probability does not change when one permutes the reports of the others. Anonymity is desirable as it reduces the complexity of the mechanism and helps protect agents' privacy. The result implies that is impossible to elicit information in "fully symmetric" environments where anonymity is without loss for allocating efficiently. The result also sheds new light on a characterization due to Holzman and Moulin (2013) and Mackenzie (2015) of a different notion of anonymity.

Our positive results concern the following class of mechanisms. In a *jury mechanism*, each agent is either a *juror* or a *candidate*; the allocation only depends on the reports of the jurors, and the object is always allocated to a candidate. Given that jurors cannot win, all jury mechanisms are DIC.

If there are three (or fewer) agents, then all DIC mechanisms are randomizations over deterministic jury mechanisms. In particular, a deterministic jury mechanism is optimal. This generalizes a known result for deterministic DIC mechanisms (Holzman and Moulin, 2013). Our key insight is that in the three-agent case all DIC mechanisms are actually randomizations over deterministic ones.

Next, deterministic jury mechanisms are approximately optimal when there are many exchangeable agents. Here, exchangeability roughly means that agents are equally good at supplying information about the vector of values. When agents are exchangeable, increasing their number relaxes the aforementioned trade-off. In particular, there is essentially no loss from ignoring the reports of those agents who are sometimes allocated the object—this is the defining property of a jury mechanism.

For the last result, we propose a relaxed notion of anonymity—*partial anonymity*. Whereas the earlier notion of anonymity demands that an agent's winning probability be invariant with respect to *all* permutations of the others, partial anonymity only considers permutations of those agents that in the given mechanism *actually* influence the agent's winning probability. We show that all deterministic partially anonymous DIC mechanism are jury mechanisms. Among jury mechanisms, partial anonymity is also without loss when agents are exchangeable as in the previous paragraph.

We view the positive results on jury mechanisms as our main qualitative insight. The fact that deterministic mechanism do not suffice suggests that exactly optimal DIC mechanisms may be difficult to interpret or implement. This issue is not easily remedied by restricting to deterministic mechanisms, as we explain in Section 7.

To clarify how these results are related, we present them in a different order. We next discuss related work (Section 2) and present the model (Section 3). In Section 4, we introduce jury mechanisms and present the results for the three- and many-agent cases. In Section 5, we characterize when stochastic extreme points exist. In Section 6, we study anonymous mechanisms, presenting the two notions and the associated characterizations side-by-side. We conclude by discussing open questions (Section 7). All omitted proofs are in Appendix A. Supplementary material is collected in Appendices B and C.

## 2 Related literature

Holzman and Moulin (2013) study axioms for peer nomination rules. In such a rule, agents nominate one another to receive a prize. Their central axiom—*impartiality*—is equivalent to DIC when each agent cares only about their own winning probability. As they note, many of their axioms have no obvious counterparts in a model with abstract types. Most relevant for us is their notion and characterization of anonymity, as well the subsequent generalization due to Mackenzie (2015); we discuss the differences in detail in Section 6.4.[3]

Alon et al. (2011) initiated a literature on optimal DIC mechanisms (there called *strategyproof* mechanisms) in a model where each agent nominates a subset of the others, and the aim is to select an agent nominated by many. Mechanisms are ranked according to approximation ratios[4] rather than according to expected values,

[3]Further contributions to the literature following Holzman and Moulin (2013) include Edelman and Por (2021), Mackenzie (2020), Tamura (2016), and Tamura and Ohseto (2014). See also de Clippel et al. (2008).

[4]Given $\alpha \in [0, 1]$, a mechanism has an *approximation ratio* of $\alpha$ if it guarantees a fraction $\alpha$ of some benchmark value. The guarantee is computed across all realizations of the type profile; that is, across all possible approval sets. The benchmark value at a particular realization is the maximal number of approvals across agents.

and this leads to qualitatively different optimal mechanisms. For example, while jury mechanisms can be optimal in our model, the 2-*partition mechanism* of Alon et al. (2011), which is a natural analogue of jury mechanisms, is not optimal in their model.[5,6]

See Olckers and Walsh (2022) for a survey of the literature following Holzman and Moulin (2013) and Alon et al. (2011). Olckers and Walsh also report on some related empirical studies, such as the aforementioned study of Hussam et al. (2022).

Other work on the simple allocation problem studies non-monetary instruments for incentivizing the agents.[7] The typical assumption is that an agent's type is the only source of information about the value of allocating to that agent. In such an environment, the DIC mechanisms that we consider cannot elicit any information from the agents. Most relevant for us are papers that study how a Bayesian incentive-compatible mechanism may exploit correlation across types to screen the agents via their beliefs (Bloch et al., 2022; Kattwinkel, 2019; Kattwinkel and Knoepfle, 2021; Kattwinkel et al., 2022). Although correlation is at the heart of the present model, it is impossible to screen the agents in the DIC mechanisms that we consider. Relatedly, the fundamental insights of Crémer and McLean (1985, 1988) and McAfee and Reny (1992) on mechanisms *with transfers* do not apply here.

The papers of Baumann (2018) and Bloch and Olckers (2021, 2022) study related settings but focus on different questions. For instance, Bloch and Olckers (2022) study whether it is possible to reconstruct the ordinal ranking of agents from their reports when agents prefer a high rank.

Methodologically, we show that the existence of stochastic extreme points can

[5]The 2-partition mechanism randomly splits the agents into two subsets, and then selects an agent from the first subset with the most approvals from agents in the second subset. Alon et al. (2011, Theorem 4.1) show that the 2-partition mechanism has an approximation ratio of $\frac{1}{4}$. Fischer and Klimm (2015) present a mechanism that achieves the strictly higher and optimal ratio of $\frac{1}{2}$.

[6]Further contributions to this literature include Aziz et al. (2016, 2019), Bjelde et al. (2017), Bousquet et al. (2014), Lev et al. (2021), and Mattei et al. (2020). See also Caragiannis et al. (2019, 2021), who consider additive approximations rather than approximation ratios.

[7]Examples of such instruments include promises of future allocations (Guo and Hörner, 2021), delaying the allocation (Condorelli, 2009), costly verification (Ben-Porath et al., 2014, 2019; Epitropou and Vohra, 2019; Erlanson and Kleiner, 2019), costly signaling (Chakravarty and Kaplan, 2013; Condorelli, 2012), allocative externalities (Bhaskar and Sadler, 2019; Goldlücke and Tröger, 2020), or ex-post punishments (Li, 2020; Mylovanov and Zapechelnyuk, 2017). The work of Harris et al. (1982) also roughly fits this description. They study the problem of allocating a budget within in a firm, similar to the example from the introduction. Division managers can exert costly effort to affect their productivity, and this can be used as a screening device.

be understood via a graph-theoretic result due to Chvátal (1975). We elaborate in Appendix B. The main text assumes no familiarity with graph theory.

# 3 Model

A single indivisible object is to be allocated to one of $n$ agents, where $n \geq 2$. For each agent $i$, let $\Omega_i$ be a finite set of reals representing the possible social values from allocating to agent $i$, and let $\Theta_i$ be a finite set representing agent $i$'s possible private types. Let $\Omega = \times_{i=1}^n \Omega_i$ and $\Theta = \times_{i=1}^n \Theta_i$. Values and types are distributed according to a joint distribution $\mu$ over $\Omega \times \Theta$. At all type profiles, agent $i$ strictly prefers winning the object to not winning it; agent $i$ is indifferent to which of the others is allocated the object.

In a *(direct) mechanism*, each agent reports a type, and then the object is allocated to one of the agents according to some lottery. Formally, a mechanism is a function $\varphi\colon \Theta \to [0,1]^n$ satisfying $\sum_{i=1}^n \varphi_i = 1$. Here $\varphi_i\colon \Theta \to [0,1]$ denotes the winning probability of agent $i$. Since the object is allocated to one of the agents, these probabilities sum to 1.[8]

A mechanism $\varphi$ is *dominant-strategy incentive-compatible (DIC)* if truthfully reporting one's type is a dominant strategy. For the assumed preferences of the agents, a mechanism is DIC if and only if one's report never affects one's own winning probability.

To see the previous point in detail, let $u_i(\theta)$ denote the payoff to an agent $i$ when $i$ is allocated the object at a type profile $\theta$. We normalize $i$'s payoff when not allocated the object to 0, and we assume $u_i > 0$. DIC for a mechanism $\varphi$ requires that all $i, \theta_i, \theta_i', \theta_{-i}$, and $\theta_{-i}'$ satisfy $u_i(\theta_i, \theta_{-i})\varphi_i(\theta_i, \theta_{-i}') \geq u_i(\theta_i, \theta_{-i})\varphi_i(\theta_i', \theta_{-i}')$. Since $u_i > 0$ and since $\theta_i$ and $\theta_i'$ are arbitrary, we must have $\varphi_i(\theta_i, \theta_{-i}') = \varphi_i(\theta_i', \theta_{-i}')$. That is, agent $i$'s report never affects $\varphi_i$.[9]

We evaluate a DIC mechanism $\varphi$ via the expected value of the allocation, which is given by $\mathbb{E}_{\omega,\theta}\left[\sum_{i=1}^n \varphi_i(\theta)\omega_i\right]$. The Revelation Principle implies that DIC mechanisms are without loss: if a mechanism can be implemented in some dominant-strategy equilibrium of some game, then it is DIC.

[8]The requirement that the object is always allocated keeps with earlier work (for example, Alon et al., 2011; Holzman and Moulin, 2013). We relax this requirement in Appendix B.

[9]Nothing in this argument changes if $u_i < 0$. Hence we can equally model cases where some agents prefer not to be allocated the object.

Lastly, we define the following: A mechanism is *deterministic* if it maps to a subset of $\{0,1\}^n$. A mechanism is *stochastic* if it is not deterministic.

**Remark 1.** While we framed the model as one where the types are signals about the efficient allocation, this is not the only interpretation. First, the expected value $\mathbb{E}[\omega_i|\theta]$ from allocating to $i$ conditional on a type profile $\theta$ could differ from $i$'s payoff $u_i(\theta)$ from being allocated the object. Second, the value of allocating to $i$ could be given by some deterministic function $\hat{\omega}_i(\theta)$ of the type profile $\theta$; the distribution of types and the functions $\{\hat{\omega}_i\}_{i=1}^n$ then induce some joint distribution of types and values. To give a concrete example, consider the aid-targeting example from the introduction. Suppose the planner wishes to allocate to the household with the lowest per capita income. To capture this, we could let $\theta_i$ encode household $i$'s income as well as $i$'s information about its neighbors. Further, we let $\hat{\omega}_i(\theta) = 1$ if $i$ has the lowest income at $\theta$, and let $\hat{\omega}_i(\theta) = 0$ otherwise.

# 4 Jury mechanisms

In this section, we focus on the following class of mechanisms.

**Definition 1.** A mechanism $\varphi$ is a *jury mechanism* if for all agents $i$ we have the following: if the mechanism is non-constant in agent $i$'s report, then agent $i$ never wins, meaning $\varphi_i = 0$.

Given a jury mechanism, we refer to an agent as a *juror* if the mechanism is non-constant in their report. The set of jurors is called the *jury*, and the remaining agents are called *candidates*. All jury mechanisms are DIC since jurors never win.

Jury mechanisms are restrictive since there could be an agent who makes a good juror and a good candidate. A sophisticated mechanism may allocate to the agent at some type profiles, and at other type profiles let the agent's type inform the allocation. By contrast, a jury mechanism assigns the role of juror or candidate *before* consulting the agents. Nevertheless, we next identify two situations where deterministic jury mechanisms are (approximately) optimal among all DIC mechanisms.

## 4.1 Jury mechanisms in the three-agent case

Our first result asserts that jury mechanisms solve the problem when $n \leq 3$.

**Theorem 4.1.** *Let $n \leq 3$. A mechanism is DIC if and only if it is a convex combination of deterministic jury mechanisms. In particular, there is an optimal DIC mechanism that is a deterministic jury mechanism.*

With three or fewer agents, a jury mechanism admits at most one juror who deliberates between the others. Therefore, all DIC mechanisms with three or fewer agents can be implemented by nominating a juror (according to some distribution over the set of agents), and then asking the juror to pick one of the others as a winner of the object. Optimally, the information of at least two of the agents is ignored.

In the remainder of this section, we discuss the steps in the proof of Theorem 4.1. We begin with a known result (Holzman and Moulin, 2013, Proposition 2.i).

**Lemma 4.2.** *If $n \leq 3$, then all deterministic DIC mechanisms are jury mechanisms.*

Holzman and Moulin (2013) do not refer to these mechanisms as jury mechanisms, but their definition is equivalent to ours for deterministic mechanisms with three or fewer agents.[10] For the sake of completeness, we provide a self-contained proof in Appendix A.1.1.

To the best of our knowledge, Lemma 4.2 has so far been limited to deterministic DIC mechanisms. We now close the gap to stochastic mechanisms.

**Lemma 4.3.** *If $n \leq 3$, then all DIC mechanisms are convex combinations of deterministic DIC mechanisms.*

Lemma 4.3 completes the proof of Theorem 4.1. Indeed, Lemmata 4.2 and 4.3 immediately imply that all DIC mechanisms are convex combinations of deterministic jury mechanisms. Since the expected value is a linear function of the mechanism, at least one deterministic jury mechanism must be optimal.

To prove Lemma 4.3, we study the extreme points of the set of DIC mechanisms. Recall that a point $x$ in a subset $X$ of Euclidean space is an *extreme point* of $X$ if $x$ cannot be written as a convex combination of two other points in $X$. A routine argument shows that the set of DIC mechanisms is convex and compact as a subset of Euclidean space. Hence the Krein-Milman theorem (Aliprantis and Border, 2006, Theorem 7.68) implies that all DIC mechanisms are convex combinations of extreme points of the set of DIC mechanisms.

[10] Holzman and Moulin (2013) note that the result is essentially due to Kato and Ohseto (2002), who study pure exchange economics. For a broader discussion of this relationship, we refer the reader to Section 1.4 of Holzman and Moulin (2013).

It therefore suffices to show that all stochastic DIC mechanisms $\varphi$ fail to be extreme points. We do so by constructing a non-zero function $f$ such that $\varphi + f$ and $\varphi - f$ are two other DIC mechanisms. To understand this construction, recall that a stochastic mechanism is one where, for at least one type profile, at least one agent enjoys an interior winning probability. Since the object is always allocated, some other agent must also enjoy an interior winning probability at the same profile. The function $f$ represents a shift of a small probability mass between these two agents. This shift should be consistent with DIC (since we want $\varphi + f$ and $\varphi - f$ to be DIC), and hence we to shift masses at multiple type profiles. What makes the construction of $f$ difficult is that changing one agent's type may change which of the others enjoys an interior winning probability. Our argument thus intuitively leans on there only being three agents. Indeed, we shall later see that the argument does not go through with four or more agents.

## 4.2 Approximate optimality of jury mechanisms

In this subsection, we identify conditions on the distribution of types and values such that jury mechanisms are approximately optimal if the number of agents is large. The following example conveys the basic idea.

**Example 1.** For each agent $i$, the value $\omega_i$ of allocating to $i$ depends on some common component $s$ and some private component $t_i$. Specifically, for some function $\hat{\omega}_i$ we have $\omega_i = \hat{\omega}_i(s, t_i)$. The agents observe their private components, which are independently and identically distributed across agents and independent of $s$. All agents observe $s$. (So, agent $i$'s type is $\theta_i = (s, t_i)$.) Let $\varphi$ be an arbitrary DIC mechanism for these $n$ agents. Now suppose a new agent $n + 1$, who also observes the common component $s$, joins the group. Agent $n+1$ may observe some additional information, but this will not be relevant. We claim there is a jury mechanism that only uses agent $n + 1$ as juror and that does as well as $\varphi$. To see this, note that, by ignoring the reports of agents 1 to $n$, the information contained in $s$ is not lost. The only information that is potentially lost is the first $n$ agents' knowledge of their private components $t_1, \ldots, t_n$. Each agent $i$'s private component $t_i$ is informative only about $i$'s own value (by independence). However, DIC of the original mechanism $\varphi$ implies that $t_i$ could not have been used to determine $i$'s own allocation. Thus one does not actually lose any information when ignoring the reports of agents 1 to $n$.

The main result of this section generalizes the previous example as follows. Under an assumption on the distribution of types and values, an arbitrary DIC mechanism with $n$ agents can be replicated by a jury mechanism when additional agents are around. If values remain bounded in $n$, an implication is that jury mechanisms become approximately optimal as $n \to \infty$.

We introduce new notation to accomodate the growing number of agents. The agents share a common finite type space ($\Theta_1 = \Theta_i$ for all $i$). The prior distribution of values and types is now a Borel-probability measure $\mu$ on $\times_{i\in\mathbb{N}}(\Omega_i \times \Theta_i)$,[11] where each $\Omega_i$ is a finite set of reals.

The following assumption captures the idea that if $i$, $j$, and $k$ are three distinct agents, then $i$ and $j$ have access to the same sources of information about $\omega_k$.

**Assumption 1.** For all $n \in \mathbb{N}$, all $i \in \{1, \ldots, n\}$, and all $\omega_i \in \Omega_i$, we have the following: Conditional on the value of agent $i$ being equal to $\omega_i$, the distribution of $(\theta_j)_{j\in\{1,\ldots,n\}\setminus\{i\}}$ is invariant with respect to permutations of $\{1, \ldots, n\} \setminus \{i\}$.

We are not assuming that $i$ and $j$ have the same information as $k$ about $\omega_k$. For example, in Example 1, the common component is the only information that $i$ and $j$ have about $\omega_k$, but agent $k$ actually observes $\omega_k$.

When there are $n$ agents (meaning that mechanisms only consult and allocate to the first $n$ agents), let $V_n$ denote the expected value from an optimal DIC mechanism. Let $V_n^J$ denote the expected value from a jury mechanism with $n$ agents that is optimal among jury mechanisms with $n$ agents.

**Theorem 4.4.** *Let Assumption 1 hold. For all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $V_n \leq V_{n+m}^J$. If, additionally, the sequence $\{V_n\}_{n\in\mathbb{N}}$ is bounded, then $\lim_{n\to\infty}(V_n - V_n^J) = 0$.*

In plain words, if $m$ new agents are added to the group, a jury mechanism with $n+m$ agents does as well as an with an arbitrary DIC mechanism with $n$ agents. The proof shows this claim for a jury mechanism that has the new $m$ agents as jurors, and the old $n$ agents as candidates, and where $m = n$. That is, a jury mechanism with the desired properties exists as soon as the number of agents is doubled. Depending on the exact distribution $\mu$, a much smaller number of new agents may be needed; in Example 1, one new agent suffices.

[11] Each of the finite sets $\Omega_i$ and $\Theta_i$ is equipped with the discrete metric. The product $\times_{i\in\mathbb{N}}(\Omega_i \times \Theta_i)$ is equipped with the product metric.

Assumption 1 is stronger than what we really need for this argument. It suffices if, informally, for all groups of agents $\{1, \dots, n\}$ there eventually comes a disjoint group of agents that is at least as well informed as $\{1, \dots, n\}$ about each other. Assumption 2 in Appendix A.1.3 formalizes this idea.

**Remark 2.** Theorem 4.4 does not assert that DIC mechanisms become approximately ex-post optimal conditional on the type profile. In Example 1, the only information that is used in the allocation is the common component. The common component need not pin down the entire profile of values.

**Remark 3.** While the form of Assumption 1 stated above is the one that, in our view, is most intuitive and most convenient for our proofs, the reader may wonder exactly which environments satisfy Assumption 1. To answer this, one can use de Finetti's theorem (Hewitt and Savage, 1955) to characterize, for each $i$ and $\omega_i$, the conditional distribution of types of agents other than $i$. Roughly speaking, this conditional distribution must be Choquet-representable by a Borel-probability measure over distributions where these types are independent and identically distributed.

# 5 Random allocations

Lemma 4.3 from Section 4.1 asserts that it suffices to consider deterministic DIC mechanisms when three or fewer agents are around. In this section, we show that this result does not extend to settings with more agents. This fact not only illustrates the analytical difficulty of the general problem, but will also shed light on the basic economic forces of the model. Further, whether deterministic mechanisms suffice has practical implications for implementation, as we explain below.

## 5.1 Stochastic extreme points

One of way constructing a stochastic DIC mechanism is by taking a convex combination of deterministic ones. In this case, one of the deterministic mechanisms from the combination must generate a weakly higher expected value than the stochastic mechanism (by linearity of the expected value).

We therefore ask whether all stochastic DIC mechanisms can be represented as convex combinations of deterministic ones; that is, whether all extreme points of the set of DIC mechanisms are deterministic. In a nutshell, this is true if and only if $n \leq 3$ and the agents' type spaces are small.

**Theorem 5.1.** *Fix $n$ and $\Theta_1, \ldots, \Theta_n$. All extreme points of the set of DIC mechanisms are deterministic if and only if at least one of the following is true:*

*(1) There are at most three agents; that is, we have $n \leq 3$.*

*(2) All agents have at most two types; that is, for all $i$ we have $|\Theta_i| \leq 2$.*

*(3) At least $(n-2)$-many agents have a degenerate type; that is, we have*

$$|\{i \in \{1, \ldots, n\} \colon |\Theta_i| = 1\}| \geq n - 2.$$

Sufficiency of (1) is the previously seen Lemma 4.3; sufficiency of (2) is related to a generalization of the well-known Birkhoff-von Neumann theorem; sufficiency of (3) is economically and technically uninteresting, but must be included for completeness.[12] As for the other direction: we momentarily give an example of a stochastic extreme point. The general claim that a stochastic extreme point exists when (1) to (3) all fail follows readily by extending this example.

Theorem 5.1 implies that deterministic mechanisms do not suffice for implementation. One can show that for each extreme point there exists at least one distribution of types and values where the extreme point is the unique optimal DIC mechanisms.[13] Hence deterministic mechanisms also do not suffice for optimality.

An implication of the previous paragraph is that jury mechanisms are not generally exactly optimal. Indeed, fixing a jury, it is optimal to allocate to the candidate with the highest expected value conditional on the types of the jurors (breaking ties according to some fixed order). This yields a deterministic jury mechanism. Since deterministic DIC mechanisms are not generally optimal, we conclude that jury mechanisms are not generally optimal.

The fact that optimal mechanisms may require randomization has practical implications. To reduce the complexity and opaqueness of the mechanism, it is desirable to

[12] The reader may wonder whether one can prove sufficiency of (1) to (3) by viewing the set of DIC mechanisms as the set of solutions to a linear system of inequalities, checking for total unimodularity of the constraint matrix, and then invoking the Hoffman-Kruskal theorem (Korte and Vygen, 2018, Theorem 5.21). This approach works for the case where all type spaces are binary; our proof uses a result which can itself be derived from the Hoffman-Kruskal theorem. However, in the difficult case with three agents, the constraint matrix is *not* generally totally unimodular (see Appendix C.5).

[13] This follows essentially by noting that the set of DIC mechanisms is a polytope that does not depend on the distribution of types and values, and that all linear functionals on this polytope can be represented via some distribution of values and types. See Appendix C.1 for the formal details.

resolve all uncertainty as early as possibly by randomizing over deterministic mechanisms. A stochastic extreme point is precisely a mechanism that cannot implemented via such a randomization. Indeed, we show in Appendix C.2 that a stochastic extreme point cannot be implemented via any *indirect* mechanism with a deterministic outcome function. Hence the mechanism designer must commit to honoring the outcome of a random process. See Budish et al. (2013), Chen et al. (2019), and Pycia and Ünver (2015) for further discussion of these points.

We next spell out an example of a stochastic extreme point. Moreover, we construct a parametrized environment where this extreme point is uniquely optimal. This will illustrate the commitment issue concretely and give an intuition for why randomization helps allocating efficiently. Readers wishing to skip the details may jump forward to Section 6 without a serious loss of continuity.

## 5.2 An example of a stochastic extreme point

There are four agents and with the following types.

$$\Theta_1 = \{\ell, r\}, \quad \Theta_2 = \{u, d\}, \quad \Theta_3 = \{f, c, b\}, \quad \Theta_4 = \{0\}. \tag{5.1}$$

Figure 1 shows (among other things that are not yet relevant) the type profiles of agents 1, 2, and 3; the degenerate type of agent 4 is omitted. The types of agents 1, 2, and 3 span a three-dimensional box. (Mnemonically, their types mean "left", "right", "up", "down", "front", "center", and "back".) Each edge of the box represents a set of type profiles along which exactly one agent's type is changing. Hence DIC requires that the winning probability of this agent be constant along the edge. We identify such an edge by a pair $(i, \theta_{-i})$, where $i$ indicates the agent whose type is changing, and $\theta_{-i}$ indicates the fixed types of the others.

Let $\Theta^* = \{\theta^a, \theta^b, \theta^c, \theta^d, \theta^e, \theta^f, \theta^g\}$ be the set of labeled type profiles in Figure 1; these are the profiles

$$\begin{aligned}
&\theta^a = (\ell, d, c, 0), \quad \theta^b = (r, d, c, 0), \quad \theta^c = (r, d, b, 0), \\
&\theta^d = (r, u, b, 0), \quad \theta^e = (r, u, f, 0), \quad \theta^f = (\ell, u, f, 0), \\
&\theta^g = (\ell, u, c, 0).
\end{aligned} \tag{5.2}$$

Let $V^*$ denote the set of bold edges in Figure 1 that connect the profiles in $\Theta^*$; these

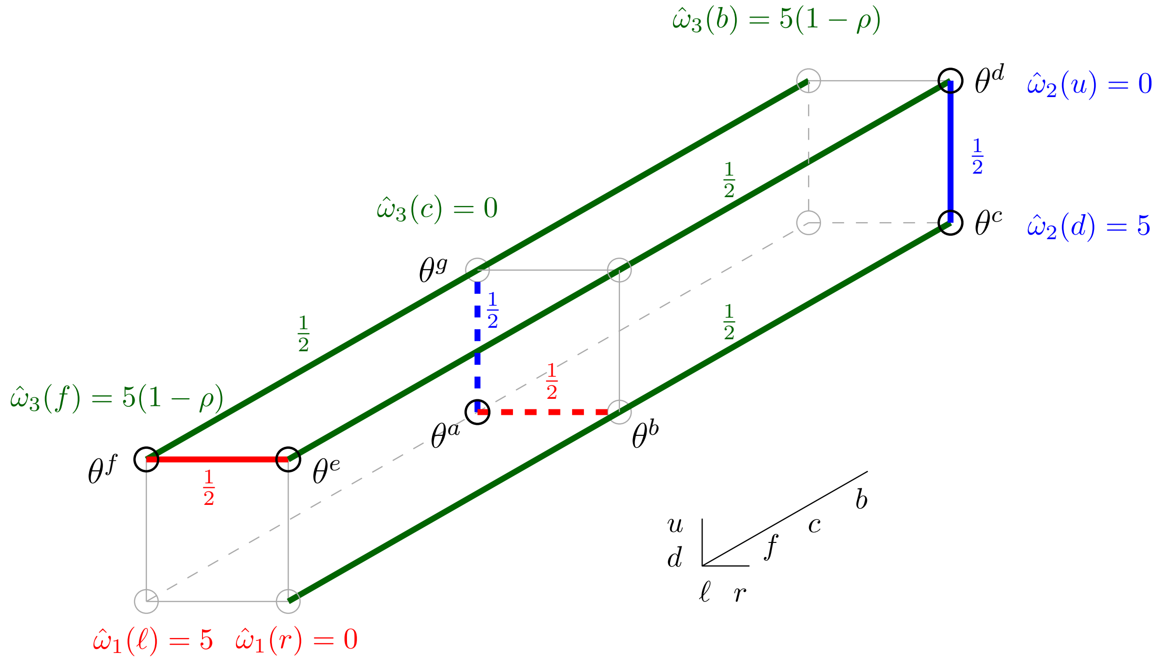



Figure 1: The set of types of agents 1, 2, and 3. The probabilities $\frac{1}{2}$ attached to the edges of the box represent the relevant values of the mechanism $\varphi^*$. The values from the allocation are as defined in (5.4). The distribution $\mu$ assigns probability $\frac{1}{5}$ to the profiles $\{\theta^a, \theta^c, \theta^d, \theta^e, \theta^f\}$. All other profiles have probability 0.

are the edges

$$V^* = \{(1, \theta^a_{-1}), (3, \theta^c_{-3}), (2, \theta^c_{-2}), (3, \theta^e_{-3}), (1, \theta^e_{-1}), (3, \theta^f_{-3}), (2, \theta^a_{-2})\}.$$

Our candidate stochastic extreme point $\varphi^*$ is defined as follows (see Figure 1): For all $i \in \{1, 2, 3\}$ and $\theta \in \Theta$, let

$$\varphi^*_i(\theta) = \begin{cases} \frac{1}{2}, & \text{if } (i, \theta_{-i}) \in V^*, \\ 0, & \text{otherwise.} \end{cases}$$

Further, for all $\theta \in \Theta$ let $\varphi^*_4(\theta) = 1 - \sum_{i\in\{1,2,3\}} \varphi^*_i(\theta)$. In plain words, at all profiles in $\Theta^*$, exactly two bold edges of the box intersect at the profile; the mechanism $\varphi^*$ randomizes evenly between the two agents of these edges. All remaining probability mass is assigned to agent 4. It is easy to verify from Figure 1 that $\varphi^*$ is a well-defined DIC mechanism.

Further below we specify values $\Omega$ and a distribution $\mu$ such that $\varphi^*$ is the unique optimal DIC mechanism. This implies that $\varphi^*$ is an extreme point of the set of DIC mechanisms. Since the proof for uniqueness is somewhat involved, we next present a simple self-contained argument showing that $\varphi^*$ is an extreme point.

Let $\varphi$ be a DIC mechanism that receives non-zero weight in a convex combination that equals $\varphi^*$. We show $\varphi = \varphi^*$. For all profiles $\theta \in \Theta^*$, there are exactly two agents $i$ and $j$ such that $(i, \theta_{-i})$ and $(j, \theta_{-j})$ both belong to $V^*$; these are the two bold edges of the box that intersect at $\theta$. Hence at $\theta$ the mechanism $\varphi^*$ randomizes evenly between $i$ and $j$. Since $\varphi$ is part of a convex combination that equals $\varphi^*$, it follows that at $\theta$ the mechanism $\varphi$ only randomizes between $i$ and $j$, meaning $\varphi_i(\theta) = 1 - \varphi_j(\theta)$. Since $\varphi$ is DIC, repeatedly applying this observation shows:

$$\begin{aligned} \varphi_1(\theta^a) = 1 - \varphi_3(\theta^c) = \varphi_2(\theta^c) &= 1 - \varphi_3(\theta^e) \\ &= \varphi_1(\theta^e) \\ &= 1 - \varphi_3(\theta^f) = \varphi_2(\theta^a) = 1 - \varphi_1(\theta^a). \end{aligned} \tag{5.3}$$

In particular, we have $\varphi_1(\theta^a) = 1 - \varphi_1(\theta^a)$, implying $\varphi_1(\theta^a) = \frac{1}{2}$. Hence all probabilities in (5.3) equal $\frac{1}{2}$. Hence $\varphi$ agrees with $\varphi^*$ at all profiles in $\Theta^*$. By inspecting $\Theta \setminus \Theta^*$, we may easily convince ourselves that $\varphi$ and $\varphi^*$ also agree on $\Theta \setminus \Theta^*$. Thus $\varphi^*$ is an extreme point.

We next construct an environment in which $\varphi^*$ is uniquely optimal. We could do so by invoking a separating hyperplane theorem. However, this would be unsatisfying since we would gain no intuition for why randomization helps or for whether $\varphi^*$ is uniquely optimal in a restricted class of environments. We shall gain both by considering environments in which values are *privately known*, in the following sense: for all agents $i$, the value of allocating to $i$ is pinned down by a function $\hat{\omega}_i$ that depends only on $\theta_i$.[14]

We can describe an environment with privately known values by specifying a distribution $\mu$ over type profiles and, for all agents $i$, a function $\hat{\omega}_i \colon \Theta_i \to \mathbb{R}$ that governs the value of allocating to $i$. Our candidate functions $\hat{\omega}_1, \dots \hat{\omega}_4$ are parametrized by

[14]In the examples from the introduction, this means that household $i$ knows its valuation of the good, agent $i$ knows their ability to serve as president, and subdivision $i$ knows its own marginal return. Privately known values often appear in the literature on the simple allocation problem (for example, Ben-Porath et al., 2014; Mylovanov and Zapechelnyuk, 2017), but not in the peer selection literature following Alon et al. (2011).

$\rho \in [0, \frac{1}{2}]$ and given by (see Figure 1)

$$\begin{aligned} \hat{\omega}_1(r) = \hat{\omega}_2(u) = \hat{\omega}_3(c) &= 0 \\ \hat{\omega}_1(\ell) = \hat{\omega}_2(d) &= 5 \\ \hat{\omega}_3(f) = \hat{\omega}_3(b) &= 5(1-\rho) \\ \hat{\omega}_4 &= 0. \end{aligned} \tag{5.4}$$

Let $\mu$ be defined by

$$\forall_{\theta \in \Theta}, \quad \mu(\theta) = \begin{cases} \frac{1}{5}, & \text{if } \theta \in \{\theta^a, \theta^c, \theta^d, \theta^e, \theta^f\} \\ 0, & \text{else.} \end{cases} \tag{5.5}$$

**Proposition 5.2.** *The mechanism $\varphi^*$ is the unique optimal DIC mechanism if and only if $\rho \in (0, \frac{1}{2})$.*

In the introduction, we intuited that there is a trade-off between allocating to an agent and using that agent's information about others. In the present example, this trade-off involves agent 3 and depends on $\rho$.

To gain an intuition for the trade-off and the result, consider the case $\rho = 0$. Allocating to agent 3 is now ex-post optimal at *all except one* of the five profiles in the support of $\mu$. Indeed, one optimal DIC mechanisms is the constant one that always allocates to agent 3. The mechanism $\varphi^*$ is another optimal mechanism for $\rho = 0$, which is intuitively explained by agent 3's type being informative: if $\theta_3 = c$ realizes, the type profile must be $\theta^a$, where $\theta^a$ is the unique type profile in the support of $\mu$ at which allocating to agents 1 or 2 is better than allocating to agent 3. The mechanism $\varphi^*$ indeed allocates to agents 1 and 2 at $\theta^a$.

Since $\rho$ decreases the value from allocating to agent 3, it is now intuitive that $\varphi^*$ does strictly better than always allocating to agent 3 for small but strictly positive values of $\rho$. In the formal proof, most of our effort goes towards showing that $\varphi^*$ is in fact uniquely optimal for small but strictly positive values of $\rho$. The idea is that, among all DIC mechanisms that are optimal for $\rho = 0$, the mechanism $\varphi^*$ is the unique one minimizing agent 3's overall winning probability.

If we increase $\rho$ further, it is eventually optimal not to allocate to agent 3 at all. In particular, $\varphi^*$ eventually ceases to be optimal. The critical value turns out to be $\rho = \frac{1}{2}$, where $\varphi^*$ is an optimal DIC mechanism, but not the only one. For instance there is another optimal DIC mechanism that allocates to agent 1 at $\theta^a, \theta^b, \theta^e$ and $\theta^f$, and allocates to agent 2 at $\theta^c$ and $\theta^d$ (and otherwise allocates to agent 4).

The above example also helps illustrate the aforementioned commitment issue. At the profile $\theta^e$, a coin flip determines whether agent 1 or 3 wins the object. Yet, at this profile, the value from allocating to agent 3 is strictly higher than the value from allocating to agent 1. In fact, a coin is flipped at all type profiles in the support of the distribution. For $\rho \in (0, \frac{1}{2})$, the mechanism designer is indifferent to the outcome of the coin flip only at one of these profiles.

**Remark 4.** Chen et al. (2019) show that, in certain mechanism design problems, given any stochastic mechanism one there is a deterministic one that induces the same interim-expected allocations. Since the deterministic mechanism is not guaranteed to be DIC, their result does not contradict the suboptimality of deterministic DIC mechanisms in our model.

**Remark 5.** We have given a constructive proof of the existence of a stochastic extreme point. An alternative proof uses a graph-theoretic result due to Chvátal (1975); see Appendix B. To be precise, the results of this appendix concern the relaxed problem where the object must not always be allocated. The associated characterization of extreme points is implied by Theorem 5.1, but not vice versa.

# 6 Anonymous juries

In this section, we study anonymous DIC mechanisms. Anonymity helps protect the agents' privacy and reduces the complexity of the mechanism: it is easier to process an anonymized set of reports than tracking who reported what.

The study of anonymity will shed additional light on the jury mechanisms from Section 4. Recall that a jury mechanism ignores the reports of those agents who have a chance at winning the object (the "candidates"). In this sense, agents are treated asymmetrically as "voters." The first result of this section shows that some asymmetry is unavoidable: all DIC mechanisms that process the agents' reports anonymously must ignore their reports entirely. We then propose a relaxation—*partial anonymity*—and show that all deterministic partially anonymous DIC mechanisms are jury mechanisms.

Throughout this section, we assume that the agents share a common type space, meaning that $\Theta_1 = \Theta_i$ holds for all $i$. In an equally valid interpretation, we can consider indirect mechanisms where all agents have the same message space and are indifferent between sending all messages.

## 6.1 Anonymity notions

Anonymity and partial anonymity are defined next. In plain words, anonymity requires that, for all $k$, the winning probability of agent $k$ does not change if one permutes the reports of the agents other than $k$. An implication is that either all or none of the agents in $\{1, \ldots, n\} \setminus \{k\}$ influence $k$'s winning probability. Partial anonymity relaxes anonymity as follows: When testing whether $k$'s winning probability is affected by permutations, we now only consider permutations of those agents who actually influence agent $k$. In particular, partial anonymity permits the set of agents who influence $k$ to be a proper subset of $\{1, \ldots, n\} \setminus \{k\}$.

**Definition 2.** Let the agents have a common type space. Let $\varphi$ be a mechanism.

(1) Given $i$, $j$, and $k$ that are all distinct, agents $i$ and $j$ are *exchangeable for* $k$ if $\varphi_k$ is invariant with respect to permutations of $i$'s and $j$'s reports; that is, for all profiles $\theta$ and $\theta'$ such that $\theta$ is obtained from $\theta'$ by permuting the types of $i$ and $j$ we have $\varphi_k(\theta) = \varphi_k(\theta')$.

(2) Given distinct $i$ and $k$, agent $i$ *influences* $k$ if $\varphi_k$ is non-constant in $i$'s report; that is, there exist type profiles $\theta$ and $\theta'$ that differ only in $i$'s type and satisfy $\varphi_k(\theta) \neq \varphi_k(\theta')$.

(3) The mechanism is *anonymous* if for all $i$, $j$, and $k$ that are all distinct, agents $i$ and $j$ are exchangeable for $k$.

(4) The mechanism is *partially anonymous* if for all $i$, $j$, and $k$ that are all distinct we have the following: if $i$ and $j$ both influence $k$, then $i$ and $j$ are exchangeable for $k$.

To state the upcoming characterization of partial anonymity, we also define what we mean by an anonymous jury.

**Definition 3.** Let the agents have a common type space. A jury mechanism has an *anonymous jury* if all jurors $i$ and $j$ are exchangeable for all agents $k$.

**Remark 6.** If Assumption 1 holds, then among jury mechanisms it is without loss to use one with an anonymous jury. Indeed, consider the jury mechanism that selects the candidate that is best conditional on the types of the jurors (breaking ties in some fixed order). Under Assumption 1, the identity of the favored candidate does not change when one permutes the jurors' types.

## 6.2 Anonymity is incompatible with DIC

**Theorem 6.1.** *Let the agents have a common type space. All anonymous DIC mechanisms are constant.*

Note well that anonymity does *not* demand that $i$ and $j$ be exchangeable for $i$'s own winning probability. If we did demand this, the theorem would follow rather trivially from DIC. Instead, the theorem is more subtly related to the requirement that the mechanism always allocates the object, and the assumption that all agents have the same type space. (We later discuss what happens when these are relaxed.)

To gain a very rough intuition, fix some type $t$. At the type profile where all agents report $t$, some agent must enjoy a non-zero winning probability. In this sense, type $t$ is a "nomination" for those agents who win with non-zero probability when all report $t$. In an anonymous mechanism, it should not matter who nominates whom. To prevent agents from nominating themselves, the mechanism must ignore all nominations. Example 2 in Appendix C.3 sharpens this intuition by proving the theorem for the special case with three agents and where one of the agents is chosen uniformly at random as a juror to allocate the object.

**Remark 7.** Anonymity is weaker than the following notion of anonymity: Whenever the set of reports is permuted, then the same permutation be applied to the vector of winning probabilities. This stronger notion captures a sense in which agents are treated equally both as "voters" and potential winners. Theorem 6.1 shows that this notion is incompatible with DIC.

**Remark 8.** An interesting implication of Theorem 6.1 is that it is impossible to elicit information in environments where anonymity is without loss. Indeed, if the joint distribution of types and values is invariant with respect to all permutations of the agents, then it is without loss to use a DIC mechanism that satisfies the strong notion of anonymity from Remark 7. In particular, it is without loss to use a constant mechanism.

## 6.3 Partial anonymity

Theorem 6.1 implies that a non-constant DIC mechanism must admit some asymmetry in how it processes the reports of different agents. This brings us to partial anonymity. We offer the following characterization for deterministic mechanisms. (We discuss stochastic ones later.)

**Theorem 6.2.** *Let the agents have a common type space. A mechanism is deterministic, partially anonymous, and DIC if and only if it is a deterministic jury mechanism with an anonymous jury.*

Partial anonymity thus provides an escape route from Theorem 6.1 and characterizes deterministic jury mechanisms.

To better understand the theorem, consider how a partially anonymous jury mechanism could fail to admit an anonymous jury. Given agents $i$ and $j$, partial anonymity is silent on the winning probabilities of those agents $k$ who are influenced by either $i$ or $j$ but not by both. By contrast, anonymity of the jury requires that all candidates are either influenced by all or none of the jurors. Accordingly, most of our effort goes towards proving that, in a deterministic partially anonymous DIC mechanism, if $i$ and $j$ influence *some* third agent $k$, then $i$ and $j$ influence exactly the same set of agents. Equipped with this fact, we show that the agents can be partitioned into equivalence classes with the following property: two agents in the same class do not influence one another, but influence the same (possibly empty) set of agents outside the class. Lastly, there cannot be multiple classes; indeed, else there is a profile where two distinct classes allocate the object to two distinct agents, which is impossible. The unique class defines an anonymous jury.

## 6.4 Discussion of Theorems 6.1 and 6.2

We conclude by discussing limitations of Theorems 6.1 and 6.2.

### 6.4.1 Disposal and randomization

The following definition will be useful: A *mechanism with disposal* is a function $\varphi\colon \Theta \to [0,1]^n$ satisfying $\sum_{i=1}^n \varphi_i \leq 1$. In plain words, this is a mechanism that does not necessarily always allocate the object to the agents. For a mechanism with disposal, DIC and anonymity are defined as above.

The next result shows via an example that Theorem 6.1 does not extend to mechanisms with disposal, and that Theorem 6.2 does not extend to stochastic mechanisms (without disposal).[15]

**Proposition 6.3.** *Let the agents have a common type space $T$ such that $|T| = 7$.*

*(1) If $n = 3$, then the set of DIC mechanisms with disposal admits an extreme point that is stochastic and anonymous.*

*(2) If $n = 4$, then the set of DIC mechanisms (without disposal) admits an extreme point that is stochastic and partially anonymous.*

The extreme point in (1) is non-constant (else it would be a convex combination of deterministic constant mechanisms). The extreme point in (2) is not a jury mechanism (else it would be a convex combination of deterministic jury mechanisms).

The idea of the proof is to "symmetrize" the stochastic extreme point $\varphi^*$ from Section 5.2. Informally, the argument goes as follows. In Section 5.2, there are four agents, the set of type profiles of agents 1 to 3 is a $2 \times 2 \times 3$ set $\hat{\Theta}$, and agent 4 has a singleton type space. Let us view allocating to agent 4 as disposing the object. Let us relabel the types of agents 1 to 3 so that they are all distinct. Across agents 1 to 3 we thus have a set $T$ of seven distinct types. The 3-fold Cartesian product $T^3$ of $T$ with itself contains six permutations of $\hat{\Theta}$ (one for each permutation of $\{1, 2, 3\}$). We can associate to each permutation a permutation of the mechanism $\varphi^*$. The idea is now to extend these permutations to a well-defined DIC mechanism $\psi^*$ on $T^3$. This step uses that the types in $\hat{\Theta}$ are distinct, and hence, by extension, that we have seven types to work with. The mechanism $\psi^*$ is our candidate for part (1) of Proposition 6.3. Part (2) follows by reintroducing agent 4 and viewing $\psi^*$ as a mechanism on $T^4$ that ignores the reports of agent 4.

### 6.4.2 Anonymous ballots

Lastly, we discuss the assumption that all agents can make the same reports. Indeed, a third escape route from Theorem 6.1 (besides partial anonymity and disposal) entails message spaces with some inherent asymmetry across agents. This brings us to the results of Holzman and Moulin (2013) and Mackenzie (2015). They consider DIC

[15] In Appendix C.4, we show that Theorem 6.2 can be extended to stochastic mechanisms by imposing stronger restrictions than partial anonymity. These restrictions are a notion of anonymity inbetween anonymity and partial anonymity, and immunity against certain coalitional manipulations.

mechanisms where agents nominate one another. Let us keep with the terminology of Holzman and Moulin by referring to these mechanisms as *impartial nomination rules*. This is the same mathematical object as a DIC mechanism when each agent $i$'s type space is $\{1, \ldots, n\} \setminus \{i\}$. Their notion of anonymity—*anonymous ballots*—requires that the winning probabilities depend only on the number of nominations received by each agent.[16] Importantly, in a nomination rule agents cannot nominate themselves, and hence they all have distinct message spaces. By contrast, we have assumed that the agents have the same type space. Hence our notion of anonymity neither nests nor is nested by anonymous ballots.

The different notions lead to different results. Contrasting Theorem 6.1, there are *non-constant* impartial nomination rules with anonymous ballots. For one example, suppose one of the agents is selected uniformly at random as a juror, following which the juror's nomination determines a winner. The aforementioned Example 2 (Appendix C.3) clarifies that this construction does not work when all agents can make the same reports. See Mackenzie (2015, Theorem 1) for a full characterization of anonymous ballots. Mackenzie's result generalizes Theorem 3 of Holzman and Moulin (2013), who had previously shown that all *deterministic* impartial nomination rules with anonymous ballots are constant.

# 7 Conclusion

We saw that jury mechanisms are optimal with three agents, and approximately-optimal when there are many exchangeable agents in the sense of Assumption 1. While DIC mechanisms cannot process all reports anonymously, jury mechanisms are the only deterministic partially anonymous DIC mechanisms. We also characterized when the set of DIC mechanisms admits stochastic extreme points.

A full characterization of optimal DIC mechanisms is a daunting open problem. Without assumptions on the distribution of types and values, this problem amounts to characterizing all extreme points of the set of DIC mechanisms. We already discussed concerns with stochastic extreme points in Section 5.1. What if one restricts to deterministic ones? All deterministic DIC mechanisms are extreme points, and

[16]Equivalently, the allocation is unchanged if one permutes the profile in a way that does not yield self-nominations (Mackenzie, 2015, Lemma 1.1). Mackenzie uses the name *voter anonymity* instead of anonymous ballots.

they should not be expected to admit an interpretable or tractable characterization. To gain a sense for the complexity of these mechanisms, consider the simplest case where all agents' type spaces are binary. In this case, there is a one-to-one mapping from deterministic DIC mechanisms with $n$ agents to so-called *$n$-hypercube perfect matchings* (see the proof of Lemma A.6 in Appendix A.2). While it is known, for instance, that there are least $2^{2^{n-2}}$ distinct perfect matchings, their exact number has yet to be discovered for general $n$. Indeed, the problem of counting the number of perfect matchings is NP-hard.

A more fruitful way of making progress may thus entail additional assumptions on the distribution of types and values. On the positive side, Theorem 4.4 shows that such assumptions can quickly lead to approximately optimal mechanisms that are interpretable. On the negative side, Theorem 6.1 shows that there is no hope for eliciting information in "fully symmetric" environments (Remark 8), unless one allows for disposal of the object. If one does allow for disposal, one has to contend with stochastic extreme points (Proposition 6.3).

It is naturally interesting to extend the analysis to settings with multiple objects, allocated simultaneously or over many periods.[17] If the mechanism designer can commit to future allocations, this should lead to stronger foundations for jury mechanisms. Agents serving as jurors today can be promised a future spot as candidates, which may help justify excluding jurors as potential winners in the present. Alternatively, past winners may be expected to volunteer as jurors in the future.

The problem of finding an optimal composition of the jury is an interesting problem in itself. We expect interesting comparative statics when agents who are likely to have good information are also likely to yield a high value. In the example from the introduction where a group selects a president, say, an agent who is popular with others may be a suitable candidate (being well-liked for their pleasant qualities) but also have good information about others (being well-acquainted with everyone).

An important line of future research concerns optimal DIC mechanisms when agents care about the allocation to their peers. While DIC has different implications in such a model, our results provide insight in at least two cases. Firstly, in situations where agents evaluate their peers, it is seems inherently interesting to

[17] See Guo and Hörner (2021) for recent work in this direction with a single agent. The literature following Alon et al. (2011) has also studied settings with multiple objects. Lipnowski and Ramos (2020) and de Clippel et al. (2021) study settings with limited or no commitment.

use a mechanism where agents cannot influence their individual chances of winning. Secondly, suppose agents have the following lexicographic preferences: each agent $i$ strictly prefers one allocation to another if the former has $i$ winning with strictly higher probability; if two allocations have the same winning probability for $i$, agent $i$ ranks them according to some type-dependent preference. In some applications, this preference could reasonably capture $i$'s opinion about who is the most deserving winner if it cannot be $i$ themself. In particular, it could coincide with the preference of the mechanism designer. In this case, optimal jury mechanism are DIC. However, an agent's preferences may also differ from those of the designer. This is plausibly the case when agents are biased in favor of friends or family, biased against minorities within the group, or simply have a different notion of who deserves to win.[18] Fixing a jury of agents, the designer therefore also has to come up with a voting rule for eliciting the jurors' information.

# Appendices

In Appendices A.1 to A.3, respectively, we present the omitted proofs for Sections 4 to 6, respectively. Appendix B studies the model where the object does not have to be allocated. Appendix C contains results that were previously mentioned in passing.

## Appendix A Omitted proofs

### A.1 Jury mechanisms

#### A.1.1 Proof of Lemma 4.2

*Proof of Lemma 4.2.* Let $\varphi$ be deterministic and DIC. First, let $n = 2$. We show that $\varphi$ is constant (and hence trivially a jury mechanism). All type profiles $(\theta_1, \theta_2)$ satisfy $\varphi_1(\theta_1, \theta_2) = 1 - \varphi_2(\theta_1, \theta_2)$. DIC implies that the left and right sides of this

[18] For example, Alatas et al. (2012), reporting on a field experiment on targeting the poor, find evidence of nepotism, though the welfare impact may be small relative to other upsides from involving the agents in the decision. They also find evidence that households have a poverty notion that differs from poverty as defined by per capita income. In this sense, if the central government (in the role of the mechanism designer) wishes to target based only on per capita income, agents indeed hold a different notion of who deserves to win.

equation, respectively, are constant in $\theta_1$ and $\theta_2$, respectively. Thus both sides must be constant in both $\theta_1$ and $\theta_2$, implying that $\varphi$ is constant.

Now let $n = 3$. There is nothing to show if $\varphi$ is constant. Since $\varphi$ is determinstic, there is exactly one winner at each type profile. By possibly relabelling the agents, it follows that there are profiles $\theta$ and $\theta'$ that differ in exactly one entry and are such that agent 1 wins at $\theta$ and agent 2 wins at $\theta'$. DIC implies that $\theta$ and $\theta'$ differ exactly in agent 3's type. We thus denote $\theta = (\theta_1, \theta_2, \theta_3)$ and $\theta' = (\theta_1, \theta_2, \theta_3')$. To prove that $\varphi$ is a jury mechanism, we show that agent 3 loses at all profiles, and that $\varphi$ is constant in the reports of agents 1 and 2.

Let $\theta_2^* \in \Theta_2$. By DIC, agent 2 loses at $(\theta_1, \theta_2^*, \theta_3)$, but wins at $(\theta_1, \theta_2^*, \theta_3')$. Hence agent 3 loses at $(\theta_1, \theta_2^*, \theta_3')$, and hence also loses at $(\theta_1, \theta_2^*, \theta_3)$. Since agents 2 and 3 both lose at $(\theta_1, \theta_2^*, \theta_3)$, agent 1 wins at this profile. By DIC, agent 1 wins whenever agents 2 and 3, respectively, report $\theta_2^*$ and $\theta_3$, respectively. Since $\theta_2^*$ was arbitrary, agent 1 wins if agent 3 reports $\theta_3$. In particular, agent 3 loses when reporting $\theta_3$. By DIC, agent 3 loses at all profiles, meaning $\varphi_3 = 0$. Hence $\varphi_1 = 1 - \varphi_2$. As in the two-agent case, this equation implies that $\varphi_1$ and $\varphi_2$ are constant in the reports of agents 1 and 2. $\square$

### A.1.2 Proof of Lemma 4.3

*Proof of Lemma 4.3.* If $n = 1$ or $n = 2$, it is easy to verify that all DIC mechanisms are constant. All constant mechanisms are convex combination of deterministic constant mechanisms, proving the claim. In what follows, let $n = 3$. Given an arbitrary stochastic DIC mechanism $\varphi$, we will find a non-zero function $f$ such that $\varphi + f$ and $\varphi - f$ are two other DIC mechanisms. This shows that all extreme points of the set of DIC mechanisms are deterministic. Since this set is non-empty, convex and compact as a subset of Euclidean space, the claim follows from the Krein-Milman theorem.

In what follows, we fix a stochastic DIC mechanisms $\varphi$. Let us agree to the following terminology. In view of DIC, we drop $i$'s type from $\varphi_i$. Given a profile $\theta$, we refer to the equation $\sum_{i \in \{1,2,3\}} \varphi_i(\theta_{-i}) = 1$ as the *feasibility constraint* at profile $\theta$. We refer to $(i, \theta_{-i})$ as the *node of agent $i$ with coordinates $\theta_{-i}$*. Lastly, when we say $\varphi_i(\theta_{-i})$ is interior we naturally mean $\varphi_i(\theta_{-i}) \in (0, 1)$.

Most of the work will go towards proving the following auxiliary claim.

**Claim A.1.** *There are non-empty disjoint subsets $R$ and $B$ ("red" and "blue") of*

$\cup_{i\in\{1,2,3\}}(\{i\} \times \Theta_{-i})$ *such that all of the following are true:*

*(1) If* $(i, \theta_{-i}) \in R \cup B$*, then* $\varphi_i(\theta_{-i})$ *is interior.*

*(2) For all* $\theta \in \Theta$*, exactly one of the following is true:*

- *(a) There does not exist* $i \in \{1, 2, 3\}$ *such that* $(i, \theta_{-i}) \in R \cup B$*.*
- *(b) There exists exactly one* $i \in \{1, 2, 3\}$ *such that* $(i, \theta_{-i}) \in R$*, exactly one* $j \in \{1, 2, 3\}$ *such that* $(j, \theta_{-j}) \in B$*, and exactly one* $k \in \{1, 2, 3\}$ *such that* $(k, \theta_{-k}) \notin R \cup B$*.*

Before proving Claim A.1, let us use it to complete the proof of Lemma 4.3. For a number $\varepsilon$ to be chosen in a moment, let $f : \Theta \to \{-\varepsilon, 0, \varepsilon\}^3$ be defined as follows:

$$\forall_{\theta\in\Theta}, \quad f_i(\theta) = \begin{cases} -\varepsilon, & \text{if } (i, \theta_{-i}) \in R, \\ \varepsilon, & \text{if } (i, \theta_{-i}) \in B, \\ 0, & \text{if } (i, \theta_{-i}) \notin R \cup B. \end{cases}$$

By finiteness of $\Theta$ and Claim A.1, if we choose $\varepsilon > 0$ sufficiently close to 0, then $\varphi + f$ and $\varphi - f$ are two DIC mechanisms. Since $f$ is non-zero, it follows that $\varphi$ is not an extreme point. It remains to prove Claim A.1.

*Proof of Claim A.1.* Given candidate sets $R$ and $B$, let us say a profile $\theta$ is *uncolored* if it falls into case (2.a) of Claim A.1. A profile *two-colored* if it falls into case (2.a) of Claim A.1. In this terminology, our goal is to construct sets $R$ and $B$ such that all $(i, \theta_{-i}) \in R \cup B$ satisfy $\varphi_i(\theta_{-i}) \in (0, 1)$, and such that all type profiles are either uncolored or two-colored.

Since $\varphi$ is stochastic, we may assume (after possibly relabelling the agents and types) that there exists a profile $\theta^0$ such that $\varphi_1(\theta^0_2, \theta^0_3)$ and $\varphi_2(\theta^0_1, \theta^0_3)$ are interior.

Let $\Theta^\circ_2$ denote the set of types $\theta_2$ for which $\varphi_1(\theta_2, \theta^0_3)$ is interior. Let $\Theta^\partial_2 = \Theta_2 \setminus \Theta^\circ_2$. Similarly, let $\Theta^\circ_1$ denote the set of types $\theta_1$ such that $\varphi_2(\theta_1, \theta^0_3)$ is interior, and let $\Theta^\partial_1 = \Theta_1 \setminus \Theta^\circ_1$. Notice that $\Theta^\circ_1$ and $\Theta^\circ_2$ are non-empty as, by assumption, agents 1 and 2 are enjoying interior winning probabilities at $\theta^0$.

We consider two cases.

**Case 1.** Let $\Theta^\partial_1 \neq \emptyset$ and $\Theta^\partial_2 \neq \emptyset$.

We establish two auxiliary claims.

**Claim A.2.** *If* $\theta_1 \in \Theta^\partial_1$*, then* $\varphi_2(\theta_1, \theta^0_3) = 0$*. Similarly, if* $\theta_2 \in \Theta^\partial_2$*, then* $\varphi_1(\theta_2, \theta^0_3) = 0$*. If* $(\theta_1, \theta_2) \in (\Theta^\circ_1 \times \Theta^\partial_2) \cup (\Theta^\partial_1 \times \Theta^\circ_1)$*, then* $\varphi_3(\theta_1, \theta_2)$ *is interior.*

*Proof of Claim A.2.* Consider the first part of the claim. Let $\theta_1 \in \Theta_1^\partial$. Let us find a type $\theta_2$ in $\Theta_1^\circ$; by assumption of Case 1, such a type exists. By definition, $\varphi_1(\theta_2, \theta_3^0)$ is interior. By definition of $\Theta_1^\partial$, we also know that $\varphi_2(\theta_1, \theta_3^0)$ must either equal 0 or 1. But it cannot equal 1 since $\varphi_2(\theta_1, \theta_3^0)$ and $\varphi_1(\theta_2, \theta_3^0)$ both appear in the feasibility constraint at the profile $(\theta_1, \theta_2, \theta_3^0)$, and since $\varphi_1(\theta_2, \theta_3^0)$ is interior. Thus $\varphi_2(\theta_1, \theta_3^0) = 0$, as desired.

A similar argument establishes the second claim.

As for the third claim, let $(\theta_1, \theta_2) \in \Theta_1^\circ \times \Theta_2^\partial$. The previous two paragraphs imply that at the profile $(\theta_1, \theta_2, \theta_3^0)$ the winning probability of agent 1 is 0. Moreover, by definition of $\Theta_1^\circ$, the winning probabiltiy of agent 2 is interior. Thus agent 3's winning probability at this profile must be interior, meaning $\varphi_3(\theta_1, \theta_2)$ is interior. A similar argument shows that $\varphi_3(\theta_1, \theta_2)$ is interior whenever $(\theta_1, \theta_2)$ is in $\Theta_1^\partial \times \Theta_1^\circ$. □

The second auxiliary result is:

**Claim A.3.** *Let $\theta_3 \in \Theta_3$. If $\theta_2 \in \Theta_2^\circ$, then $\varphi_1(\theta_2, \theta_3)$ is interior. Similarly, if $\theta_1 \in \Theta_1^\circ$, then $\varphi_2(\theta_1, \theta_3)$ is interior.*

*Proof of Claim A.3.* We will prove the first part of the claim, the second being similar. Thus let $\theta_2 \in \Theta_2^\circ$. By assumption of Case 1, we may find $\theta_1^\partial \in \Theta_1^\partial$ and $\theta_2^\partial \in \Theta_2^\partial$. We make two auxiliary observations.

First, consider the profile $(\theta_1^\partial, \theta_2^\partial, \theta_3^0)$. According to Claim A.2, both agent 1's and agent 2's winning probabilities at this profile equal 0. Thus $\varphi_3(\theta_1^\partial, \theta_2^\partial) = 1$. But $\varphi_3(\theta_1^\partial, \theta_2^\partial)$ and $\varphi_2(\theta_1^\partial, \theta_3)$ both appear in the feasibility constraint at the profile $(\theta_1^\partial, \theta_2^\partial, \theta_3)$. Hence $\varphi_2(\theta_1^\partial, \theta_3) = 0$.

Second, since $\theta_1^\partial \in \Theta_1^\partial$ and $\theta_2 \in \Theta_2^\circ$, we infer from Claim A.2 that $\varphi_3(\theta_1^\partial, \theta_2)$ is interior.

The previous two observations imply that at the profile $(\theta_1^\partial, \theta_2, \theta_3)$ agent 2's winning probability is 0 and that agent 3's winning probability is interior. Hence $\varphi_1(\theta_2, \theta_3)$ is interior, as promised. □

We are ready to define the sets $R$ and $B$. We assign the following colors (recall the terminology introduced in the paragraph before Claim A.1):

- red to all nodes of agent 1 with coordinates in $\Theta_2^\circ \times \Theta_3$,
- blue to all nodes of agent 3 with coordinates in $\Theta_1^\partial \times \Theta_2^\circ$,
- blue to all nodes of agent 2 with coordinates in $\Theta_1^\circ \times \Theta_3$,

- red to all nodes of agent 3 with coordinates in $\Theta_1^\circ \times \Theta_2^\partial$.

According to Claims A.2 and A.3, all of these nodes are interior. Moreover, all profiles are now either two-colored or uncolored: The profiles in $\Theta_1^\partial \times \Theta_2^\circ \times \Theta_3$ are two-colored via red nodes of agent 1 and blue nodes of agent 3; the profiles in $\Theta_1^\circ \times \Theta_2^\circ \times \Theta_3$ are two-colored via red nodes of agent 1 and blue nodes of agent 2; the profiles in $\Theta_1^\circ \times \Theta_2^\partial \times \Theta_3$ are two-colored via blue nodes of agent 2 and red nodes of 3; and the profiles in $\Theta_1^\partial \times \Theta_2^\partial \times \Theta_3$ are uncolored. ▲

**Case 2.** Suppose at least one of the sets $\Theta_1^\partial$ and $\Theta_2^\partial$ is empty. In what follows, we assume that $\Theta_2^\partial$ is empty, the other case being analogous (switch the roles of agents 1 and 2).

The assumption that $\Theta_2^\partial$ is empty means that $\varphi_1(\theta_2, \theta_3^0)$ is interior for all $\theta_2$. Let $\Theta_1^*$ be the set of types $\theta_1$ such that for all $\theta_2 \in \Theta_2$ the probability $\varphi_3(\theta_1, \theta_2)$ is interior. Notice that at this point $\Theta_1^*$ may or may not be empty; we will make a case distinction further below.

We first claim that if $\theta_1 \in (\Theta_1 \setminus \Theta_1^*)$, then $\varphi_2(\theta_1, \theta_3^0)$ is interior. Towards a contradiction, suppose this were false for some $\theta_1 \in (\Theta_1 \setminus \Theta_1^*)$. This means that we can find a type $\theta_2 \in \Theta_2$ such that $\varphi_2(\theta_1, \theta_3^0)$ and $\varphi_3(\theta_1, \theta_2)$ both fail to be interior. Recall from the previous paragraph that $\varphi_1(\theta_2, \theta_3^0)$ is interior for all $\theta_2$. Hence at the profile $(\theta_1, \theta_2, \theta_3^0)$ only agent 1 is enjoying an interior winning probability; this is impossible.

Before proceeding further, let us assign the following colors:

- red to all nodes of agent 1 with coordinates in $\Theta_2 \times \{\theta_3^0\}$. These nodes are all interior since $\Theta_2^\partial$ is empty.
- blue to all nodes of agent 2 with coordinates in $(\Theta_1 \setminus \Theta_1^*) \times \{\theta_3^0\}$. The previous paragraph implies that these nodes are all interior.
- blue to all nodes of agent 3 with coordinates in $\Theta_1^* \times \Theta_2$. These nodes are all interior by definition of $\Theta_1^*$.

Observe that all profiles in $\Theta_1 \times \Theta_2 \times \{\theta_3^0\}$ are now either two-colored or uncolored.

If $\Theta_1^*$ is empty, then the colors assigned above already define sets $R$ and $B$ with the desired properties, completing the proof. Thus suppose $\Theta_1^*$ is non-empty.

Let $\theta_3 \in \Theta_3 \setminus \{\theta_3^0\}$ be arbitrary. The fact that we have already assigned blue to the nodes of agent 3 with coordinates $\Theta_1^* \times \Theta_2$ requires us to assign some colors to the nodes of agents 1 or 2 whose 3'rd coordinate is $\theta_3$. In this step, we will not color any further nodes of agent 3. We make a case distinction.

(1) Suppose that for all $\theta_1$ in $\Theta_1^*$ the probability $\varphi_2(\theta_1, \theta_3)$ is interior. We assign red to all nodes of agent 2 with coordinates in $\Theta_1^* \times \{\theta_3\}$. This yields a coloring of the profiles in $\Theta_1 \times \Theta_2 \times \{\theta_3^0\}$ with the desired properties: The profiles in $\Theta_1^* \times \Theta_2 \times \{\theta_3\}$ are two-colored via red nodes of agent 2 and blue nodes of 3; the profiles in $(\Theta_1 \setminus \Theta_1^*) \times \Theta_2 \times \{\theta_3\}$ are uncolored.

(2) Suppose there exists $\tilde{\theta}_1 \in \Theta_1^*$ such that $\varphi_2(\theta_1, \theta_3)$ is interior. Given that $\varphi_3(\tilde{\theta}_1, \theta_2)$ is interior for all $\theta_2 \in \Theta_2$ (recall the definition of $\Theta_1^*$), it must be the case that, for all $\theta_2 \in \Theta_2$, the probability $\varphi_1(\theta_2, \theta_3)$ is interior.

We next claim that $\varphi_2(\theta_1, \theta_3)$ is interior for all $\theta_1 \in (\Theta_1 \setminus \Theta_1^*)$. Suppose this were false for some $\theta_1 \in (\Theta_1 \setminus \Theta_1^*)$. The previous paragraph tells us that $\varphi_1(\theta_2, \theta_3)$ is interior for all $\theta_2$. Thus, if $\varphi_2(\theta_1, \theta_3)$ fails to be interior, then $\varphi_3(\theta_1, \theta_2)$ would have to be interior for all $\theta_2 \in \Theta_2$; this is a contradiction since $\theta_1 \in (\Theta_1 \setminus \Theta_1^*)$.

We now assign red to all nodes of agent 1 with coordinates in $\Theta_2 \times \{\theta_3\}$, and assign blue to all nodes of agent 2 with coordinates in $(\Theta_1 \setminus \Theta_1^*) \times \{\theta_3\}$. The previous two paragraphs imply that all of these nodes are interior. Moreover the profiles in $\Theta_1^* \times \Theta_2 \times \{\theta_3\}$ are two-colored via red nodes of agent 1 and blue nodes of agent 3, and the profiles in $(\Theta_1 \setminus \Theta_1^*) \times \Theta_2 \times \{\theta_3\}$ are two-colored via red nodes of agent 1 and blue nodes of agent 2.

If we apply this case distinction separately to all $\theta_3$ in $\Theta_3 \setminus \{\theta_3^0\}$, this completes the construction of $R$ and $B$ in Case 2. ▲

Cases 1 and 2 together complete the proof of Claim A.1. □

□

### A.1.3 Approximate optimality of jury mechanisms

In this part of the appendix, we prove Theorem 4.4. To distinguish a random variable from its realization, we denote the former using a tilde $\sim$. Given a set $N$ of agents, we denote the profile of their types by $\theta_N$, and the set of these profiles by $\Theta_N$. For example, given $i \in N$, $\omega_i \in \Omega_i$, and $\theta_{N\setminus\{i\}} \in \Theta_{N\setminus\{i\}}$, we write $\mu\left(\tilde{\omega}_i = \omega_i, \tilde{\theta}_{N\setminus\{i\}} = \theta_{N\setminus\{i\}}\right)$ to mean the probability of the event that $i$'s value is $\omega_i$ and the types of the other agents in $N$ are $\theta_{N\setminus\{i\}}$.

**Assumption 2.** For all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ with the following property: Denoting $N = \{1, \ldots, n\}$ and $N' = \{n+1, \ldots, n+m\}$, there is a function $g \colon \Theta_{N'} \times \Theta_N \to \mathbb{R}_+$ with the following two properties:

(1) For all $i \in N$, all $\omega_i \in \Omega_i$ and $\theta_{N\setminus\{i\}} \in \Theta_{N\setminus\{i\}}$ we have

$$\begin{aligned} &\mu\left(\tilde{\omega}_i = \omega_i, \tilde{\theta}_{N\setminus\{i\}} = \theta_{N\setminus\{i\}}\right) \\ &= \sum_{\theta_{N'} \in \Theta_{N'}} \sum_{\theta_i \in \Theta_i} g(\theta_{N'}, \theta_{N\setminus\{i\}}, \theta_i) \mu\left(\tilde{\omega}_i = \omega_i, \tilde{\theta}_{N'} = \theta_{N'}\right). \end{aligned} \tag{A.1}$$

(2) For all $\theta_{N'} \in \Theta_{N'}$ we have

$$\sum_{\theta_N \in \Theta_N} g(\theta_{N'}, \theta_N) = 1. \tag{A.2}$$

**Lemma A.4.** *Assumption 1 implies Assumption 2.*

*Proof of Lemma A.4.* Let $m = n$. Let $N = \{1, \ldots, n\}$ and $N' = \{n+1, \ldots, 2n\}$, and let $\xi \colon N \to N'$ be a bijection. It is straightforward to verify that the function $g$ defined as follows has the desired properties: For all $(\theta_N, \theta_{N'})$, let $g(\theta_N, \theta_{N'}) = 1$ if for all $i \in N$ the types of $i$ and $\xi(i)$ agree; else, let $g(\theta_N, \theta_{N'}) = 0$. □

*Proof of Theorem 4.4.* The second part of the claim is immediate from the first. For the first part, let $\varphi$ be an arbitrary DIC mechanism with $n$ agents. Let $N = \{1, \ldots, n\}$. For this choice of $N$, we invoke Lemma A.4 to find $m$ and $g$ as in Assumption 2. Let $N' = \{n+1, \ldots, n+m\}$. We define our candidate jury mechanism as follows: For all $i \in N$, let $\psi_i \colon \Theta_{N'} \to \mathbb{R}^n$ be defined by

$$\forall_{\theta_{N'} \in \Theta_{N'}}, \quad \psi_i(\theta_{N'}) = \sum_{\theta_N \in \Theta_N} g(\theta_{N'}, \theta_N) \varphi_i^*(\theta_{N\setminus\{i\}}).$$

For all $i \in N'$, let $\psi_i = 0$. Let $\psi = (\psi_1, \ldots, \psi_m)$.

Notice that $\psi$ only depends on the reports of agents in $N'$. Since $N'$ is disjoint from $N$, we can show that $\psi$ is a jury mechanism in the setting with $n + m$ agents by showing that $\psi$ maps to probability distributions over $N$. It is clear that $\varphi$ is non-negative (as $g$ and $\psi^*$ are non-negative). To verify that $\psi$ almost surely allocates to an agent in $N$, we observe that for all profiles $\theta_{N'}$ we have the following (the first equality is by definition of $\psi$; the second is from the fact that $\varphi^*$ is a well-defined mechanism when the set of agents is $N$; the third is from (A.2)):

$$\sum_{i \in N} \psi_i(\theta_{N'}) = \sum_{i \in N} \sum_{\theta_N \in \Theta_N} g(\theta_{N'}, \theta_N) \varphi_i^*(\theta_{N\setminus\{i\}}) = \sum_{\theta_N \in \Theta_N} g(\theta_{N'}, \theta_N) = 1,$$

as desired. We complete the proof by verifying that $\varphi$ and $\psi$ lead to the same expected value. We write the expected value from $\varphi$ as follows (the first equality follows from (A.1); the remaining equalities obtain by rearranging):

$$\begin{aligned}
&\sum_{i\in N}\sum_{\theta_{N\setminus\{i\}}}\sum_{\omega_i}\omega_i\mu\left(\tilde{\omega}_i=\omega_i,\tilde{\theta}_{N-i}=\theta_{N\setminus\{i\}}\right)\varphi_i^*(\theta_{N\setminus\{i\}})\\
=&\sum_{i\in N}\sum_{\theta_{N\setminus\{i\}}}\sum_{\omega_i}\omega_i\sum_{\theta_{N'}}\sum_{\theta_i}g(\theta_{N'},\theta_{N\setminus\{i\}},\theta_i)\mu\left(\tilde{\omega}_i=\omega_i,\tilde{\theta}_{N'}=\theta_{N'}\right)\varphi_i^*(\theta_{N\setminus\{i\}})\\
=&\sum_{i\in N}\sum_{\omega_i}\sum_{\theta_{N'}}\omega_i\mu\left(\tilde{\omega}_i=\omega_i,\tilde{\theta}_{N'}=\theta_{N'}\right)\sum_{\theta_{N\setminus\{i\}}}\sum_{\theta_i}g(\theta_{N'},\theta_{N\setminus\{i\}},\theta_i)\varphi_i^*(\theta_{N\setminus\{i\}})\\
=&\sum_{i\in N}\sum_{\omega_i}\sum_{\theta_{N'}}\omega_i\mu\left(\tilde{\omega}_i=\omega_i,\tilde{\theta}_{N'}=\theta_{N'}\right)\psi_i(\theta_{N'}).
\end{aligned}$$

This last expression is precisely the expected value from $\psi$. □

## A.2 Random allocations

### A.2.1 Proof of Proposition 5.2

*Proof of Proposition 5.2.* To keep calculations readable, it will be convient to adopt the following notation: When a DIC mechanism $\varphi$ is given, we denote

$$\begin{aligned}
&\varphi_1(\theta^a)=p^{a|b},\quad \varphi_3(\theta^c)=p^{b|c},\quad \varphi_2(\theta^c)=p^{c|d},\quad \varphi_3(\theta^e)=p^{d|e},\\
&\varphi_1(\theta^e)=p^{e|f},\quad \varphi_3(\theta^f)=p^{f|g},\quad \varphi_2(\theta^a)=p^{g|a}.
\end{aligned}$$

The probabilities in the previous display do not fully describe the mechanism, but these are the only ones needed to evaluate the mechanism. For a given value of $\rho$, we denote the expected value from $\varphi$ by $V_\rho(\varphi)$. Direct computation shows

$$V_\rho(\varphi)=p^{a|b}+p^{b|c}+p^{c|d}+2p^{d|e}+p^{e|f}+p^{f|g}+p^{g|a}-\rho\left(p^{b|c}+2p^{d|e}+p^{f|g}\right). \tag{A.3}$$

In particular, $V_\rho(\varphi^*)=4-2\rho$.

Following the paragraphs after Proposition 5.2, one can show by direct computation that $\varphi^*$ fails to be uniquely optimal if $\rho\notin(0,\frac{1}{2})$.

It remains to show that $\varphi^*$ is uniquely optimal if $\rho\in(0,\frac{1}{2})$. We establish the following auxiliary claim.

**Claim A.5.** *Let $\varphi$ be a DIC mechanism distinct from $\varphi^*$. We have $V_{\frac{1}{2}}(\varphi) \leq V_{\frac{1}{2}}(\varphi^*)$. Further, there exists $\rho_\varphi \in (0, \frac{1}{2})$ such that $\rho \in (0, \rho_\varphi)$ implies $V_\rho(\varphi) < V_\rho(\varphi^*)$.*

*Proof of Claim A.5.* Inspection of Figure 1 shows that $\varphi$ must satisfy the following system of inequalities:

$$\begin{aligned} &p^{a|b} + p^{g|a} \leq 1, \quad p^{a|b} + p^{b|c} \leq 1, \quad p^{c|d} + p^{b|c} \leq 1, \quad p^{c|d} + p^{d|e} \leq 1, \\ &p^{e|f} + p^{d|e} \leq 1, \quad p^{e|f} + p^{f|g} \leq 1, \quad p^{g|a} + p^{f|g} \leq 1. \end{aligned} \tag{A.4}$$

Turning to the first part of the claim, we have to show $V_{\frac{1}{2}}(\varphi) \leq V_{\frac{1}{2}}(\varphi^*)$. Direct computation shows $V_{\frac{1}{2}}(\varphi^*) = 3$. Using (A.4), we can bound $V_{\frac{1}{2}}(\varphi)$ as follows.

$$\begin{aligned} V_{\frac{1}{2}}(\varphi) =& p^{a|b} + p^{b|c} + p^{c|d} + 2p^{d|e} + p^{e|f} + p^{f|g} + p^{g|a} - \frac{1}{2}\left(p^{b|c} + 2p^{d|e} + p^{f|g}\right) \\ =& p^{a|b} + \frac{p^{b|c}}{2} + p^{c|d} + p^{d|e} + p^{e|f} + \frac{p^{f|g}}{2} + p^{g|a} \\ =& \underbrace{p^{a|b} + p^{g|a}}_{\leq 1} + \underbrace{\frac{p^{b|c} + p^{c|d}}{2}}_{\leq \frac{1}{2}} + \underbrace{\frac{p^{c|d} + p^{d|e}}{2}}_{\leq \frac{1}{2}} + \underbrace{\frac{p^{d|e} + p^{e|f}}{2}}_{\leq \frac{1}{2}} + \underbrace{\frac{p^{e|f} + p^{f|g}}{2}}_{\leq \frac{1}{2}} \\ \leq& 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \\ =& 3. \end{aligned}$$

Hence $V_{\frac{1}{2}}(\varphi) \leq V_{\frac{1}{2}}(\varphi^*)$, as promised.

Now consider the second part of the claim. We show the contrapositive: If there exists a sequence $\{\rho_k\}_{k\in\mathbb{N}}$ in $(0, \frac{1}{2})$ that converges to 0 and such that $V_{\rho_k}(\varphi) \geq V_{\rho_k}(\varphi^*)$ holds for all $k$, then $\varphi = \varphi^*$. Let $\{\rho_k\}_{k\in\mathbb{N}}$ be such a sequence. For all $\rho_k$, the system (A.4) implies the following upper bound on $V_{\rho_k}(\varphi)$:

$$\begin{aligned} V_{\rho_k}(\varphi) =& \underbrace{p^{a|b} + p^{b|c}}_{\leq 1} + \underbrace{p^{c|d} + p^{d|e}}_{\leq 1} + \underbrace{p^{d|e} + p^{e|f}}_{\leq 1} + \underbrace{p^{f|g} + p^{g|a}}_{\leq 1} \\ &- \rho_k\left(p^{b|c} + 2p^{d|e} + p^{f|g}\right) \\ \leq& 4 - \rho_k\left(p^{b|c} + 2p^{d|e} + p^{f|g}\right). \end{aligned} \tag{A.5}$$

Since $V_{\rho_k}(\varphi) \geq V_{\rho_k}(\varphi^*) = 4 - 2\rho_k$ and $\rho_k > 0$, we find

$$p^{b|c} + 2p^{d|e} + p^{f|g} \leq 2. \tag{A.6}$$

Further, since $V_{\rho_k}(\varphi) \geq 4 - 2\rho_k$ holds for all $k$, taking limits implies $V_0(\varphi) \geq 4$. Together with the bound in (A.5) we get $V_0(\varphi) = 4$; that is,

$$V_0(\varphi) = p^{a|b} + p^{b|c} + p^{c|d} + p^{d|e} + p^{d|e} + p^{e|f} + p^{f|g} + p^{g|a} = 4 \tag{A.7}$$

Hence (A.4) and (A.7) imply

$$p^{a|b} + p^{b|c} = p^{c|d} + p^{d|e} = p^{d|e} + p^{e|f} = p^{f|g} + p^{g|a} = 1. \tag{A.8}$$

We now bound $V_0(\varphi)$ a second time (the equality is by direct computation; the inequality follows from (A.4)):

$$V_0(\varphi) = p^{a|b} + p^{g|a} + p^{b|c} + p^{c|d} + 2p^{d|e} + p^{e|f} + p^{f|g} \leq 3 + 2p^{d|e}. \tag{A.9}$$

Hence $V_0(\varphi) = 4$ implies $p^{d|e} \geq \frac{1}{2}$. We next claim $p^{d|e} = \frac{1}{2}$. Towards a contradiction, suppose not, meaning $p^{d|e} > \frac{1}{2}$. Hence (A.8) implies $p^{c|d} = p^{e|f} < \frac{1}{2}$. Now, we also know from (A.6) and (A.7) that

$$p^{a|b} + p^{c|d} + p^{e|f} + p^{g|a} \geq 2$$

holds. However, in light of (A.4) we have $p^{a|b} + p^{g|a} \leq 1$, and hence the previous display requires $p^{c|d} + p^{e|f} \geq 1$. This contradicts $p^{c|d} = p^{e|f} < \frac{1}{2}$. Thus $p^{d|e} = \frac{1}{2}$.

Let us now return to the bound derived in (A.9). In view of $p^{d|e} = \frac{1}{2}$ and (A.4), we can infer from (A.9) that $p^{a|b} + p^{g|a} = p^{b|c} + p^{c|d} = p^{e|f} + p^{f|g} = 2p^{d|e} = 1$ holds. Together with (A.8), we find

$$p^{a|b} = 1 - p^{b|c} = p^{c|d} = 1 - p^{d|e} = p^{e|f} = 1 - p^{f|g} = p^{g|a}. \tag{A.10}$$

We already know that $p^{d|e} = \frac{1}{2}$ holds. Hence all probabilities (A.10) must equal $\frac{1}{2}$. This shows that $\varphi$ agrees with $\varphi^*$ at all profiles in $\Theta^* = \{\theta^a, \theta^b, \theta^c, \theta^d, \theta^e, \theta^f, \theta^g\}$. By inspecting $\Theta \setminus \Theta^*$, it is now easy to verify that $\varphi$ and $\varphi^*$ also agree on $\Theta \setminus \Theta^*$. □

We now use Claim A.5 to show that all $\rho \in (0, \frac{1}{2})$ and all DIC mechanisms $\varphi$ different from $\varphi^*$ satisfy $V_\rho(\varphi) < V_\rho(\varphi^*)$. Fixing $\varphi$, inspection of (A.3) shows that the difference $V_\rho(\varphi) - V_\rho(\varphi^*)$ is an affine function of $\rho$; that is, there exist reals $a_\varphi$ and $b_\varphi$ such that $V_\rho(\varphi) - V_\rho(\varphi^*) = a_\varphi + b_\varphi \rho$ holds for all $\rho \in [0, \frac{1}{2}]$. Let $\rho_\varphi \in (0, \frac{1}{2})$

be as in the conclusion of Claim A.5.

If $\rho \in (0, \rho_\varphi)$, the choice of $\rho_\varphi$ implies $V_\rho(\varphi) < V_\rho(\varphi^*)$, and so we are done. Hence in what follows we assume $\rho \in [\rho_\varphi, \frac{1}{2})$. We distinguish two cases.

If $b_\varphi \le 0$, then

$$V_\rho(\varphi) - V_\rho(\varphi^*) = a_\varphi + b_\varphi \rho \le a_\varphi + b_\varphi \frac{\rho_\varphi}{2} = V_{\frac{\rho_\varphi}{2}}(\varphi) - V_{\frac{\rho_\varphi}{2}}(\varphi^*).$$

Now $\frac{\rho_\varphi}{2} \in (0, \rho_\varphi)$ and the choice of $\rho_\varphi$ imply $V_{\frac{\rho_\varphi}{2}}(\varphi) - V_{\frac{\rho_\varphi}{2}}(\varphi^*) < 0$, and we are done.

If $b_\varphi > 0$, then

$$V_\rho(\varphi) - V_\rho(\varphi^*) = a_\varphi + b_\varphi \rho < a_\varphi + b_\varphi \frac{1}{2} = V_{\frac{1}{2}}(\varphi) - V_{\frac{1}{2}}(\varphi^*).$$

Now Claim A.5 implies $V_{\frac{1}{2}}(\varphi) - V_{\frac{1}{2}}(\varphi^*) \le 0$, and we are done. □

### A.2.2 Proof of Theorem 5.1

**Lemma A.6.** *If for all agents $i$ we have $|\Theta_i| \le 2$, then all extreme points of the set of DIC mechanisms are deterministic.*

For the proof, recall the following definitions for a given (simple undirected) graph $G$ with node set $V$ and edge set $E$. Given a node $v$, the set of edges which are incident to $v$ is denoted $E(v)$. A *perfect matching* is a function $\psi\colon E \to \{0,1\}$ such that all $v \in V$ satisfy $\sum_{e \in E(v)} \psi(e) = 1$. The *perfect matching polytope* is the set $\{\psi\colon E \to [0,1] \colon \forall_{v \in V}, \sum_{e \in E(v)} \psi(e) = 1\}$.

*Proof of Lemma A.6.* Let us relabel types such that we have $\Theta_i \subseteq \{0,1\}$ for all $i$. First, suppose we have $\Theta_i = \{0,1\}$ for all $i$.

For all DIC mechanisms $\varphi$, all agents $i$ and all profiles $\theta$, we may drop $i$'s report from $i$'s winning probability, writing $\varphi_i(\theta_{-i})$ instead of $\varphi_i(\theta)$. Under this convention, we claim that the set of DIC mechanisms is the perfect matching polytope of the graph $G$ that has node set $\{0,1\}^n$ and where two nodes are adjacent if and only if they differ in exactly one coordinate. (This graph is known as the *$n$-hypercube*.) Indeed, each node of the graph is a type profile $\theta$, and each edge may be identified with a pair of the form $(i, \theta_{-i})$. The set of edges incident to $\theta$ is the set $\{(i, \theta_{-i})\}_{i=1}^n$. Hence the constraint $\sum_{e \in E(v)} \psi(e) = 1$ is exactly the constraint that the object be allocated to one of the agents.

Now, the graph $G$ described in the previous paragraph is bi-partite (partition the type profiles (that is, the nodes of $G$) according to whether the profile has an odd or even number of entries equal to 0). It follows from Theorem 11.4 of Korte and Vygen (2018) that all extreme points of the perfect matching polytope are perfect matchings. All perfect matchings represent deterministic DIC mechanisms. Hence all extreme points of the set of DIC mechanisms are deterministic.

The claim for the general case, where we have $\Theta_i \subseteq \{0, 1\}$ for all $i$, follows from the previous paragraph by viewing a DIC mechanism on $\Theta$ as a mechanism on $\{0, 1\}^n$ that ignores the reports of those whose type spaces are singletons. $\square$

**Lemma A.7.** *If $|\{i \in \{1, \dots, n\} \colon |\Theta_i| \geq 2\}| \leq 2$, then all extreme points of the set of DIC mechanisms are deterministic.*

*Proof of Lemma A.7.* We may assume $n \geq 3$, as otherwise the claim follows from Lemma A.6. We will prove the claim for the case where $|\{i \in \{1, \dots, n\} \colon |\Theta_i| \geq 2\}| = 2$, the other cases being simpler. After possibly relabelling the agents, suppose we have $|\Theta_1| \geq 2$ and $|\Theta_2| \geq 2$. Let $\varphi$ be a stochastic DIC mechanism. Notice that at all profiles $\theta$ where either agent 1 or agent 2 but not both is enjoying an interior winning probability, there must be an agent in $\{3, \dots, n\}$ who is also enjoying an interior winning probability; let $i_\theta$ denote one such agent. For a number $\varepsilon > 0$ to be chosen later, consider $f \colon \Theta \to \{-\varepsilon, 0, \varepsilon\}^n$ defined for all $\theta$ as follows:

(1) If $\varphi_1(\theta) \in (0, 1)$ and $\varphi_2(\theta) \in (0, 1)$, let $f_1(\theta) = \varepsilon$, let $f_2(\theta) = -\varepsilon$, and let $f_i(\theta) = 0$ for all $i \notin \{1, 2\}$.
(2) If $\varphi_1(\theta) \in (0, 1)$ and $\varphi_2(\theta) \notin (0, 1)$, let $f_1(\theta) = \varepsilon$, let $f_{i_\theta}(\theta) = -\varepsilon$, and let $f_i(\theta) = 0$ for all $i \notin \{1, i_\theta\}$.
(3) If $\varphi_1(\theta) \notin (0, 1)$ and $\varphi_2(\theta) \in (0, 1)$, let $f_2(\theta) = -\varepsilon$, let $f_{i_\theta}(\theta) = \varepsilon$, and let $f_i(\theta) = 0$ for all $i \notin \{2, i_\theta\}$.

Using that, for all $\theta$, agent $i_\theta$ has a singleton type space, it is easy to see that $\varphi + f$ and $\varphi - f$ are two DIC mechanisms distinct from $\varphi$ whenever $\varepsilon$ is sufficiently small. Thus $\varphi$ is not an extreme point. $\square$

*Proof of Theorem 5.1.* Lemmata 4.3, A.6 and A.7 imply that all extreme points are deterministic if one of the conditions (1) to (3) holds. Now let conditions (1) to (3) all fail. We know from Section 5.2 that a stochastic extreme point exists in the hypothetical situation where $n = 4$ and the set of type profiles is $\hat{\Theta} = \{\ell, r\} \times \{u, d\} \times$

$\{f, c, b\} \times \{0\}$. Since (1) to (3) all fail, we can relabel the agents and types such that agents 1 to 4 have these sets as subsets of their respective sets of types. Let $\varphi^*$ denote the stochastic extreme point Section 5.2. Using $\varphi^*$, it is straightforward to define a stochastic extreme point for the actual set of type profiles with $n$ agents. To see this in detail, let us agree to the following notation: when $i \in \{1, 2, 3\}$, then $\hat{\Theta}_{-i}$ means the sets of type profiles of agents $\{1, 2, 3, 4\} \setminus \{i\}$ that belong to $\hat{\Theta}$. Now consider $\psi^* \colon \Theta \to \mathbb{R}^n$ defined as follows: For all $i \in \{1, \dots, n\} \setminus \{1, 2, 3, 4\}$, let $\psi_i^* = 0$; for all $i \in \{1, 2, 3\}$ and all $\theta \in \Theta$, let $\psi_i^*(\theta) = \varphi_i^*(\theta_1, \theta_2, \theta_3, \theta_4)$ if $(\theta_j)_{j \in \{1,2,3,4\} \setminus \{i\}} \in \hat{\Theta}_{-i}$, and let $\psi_i^*(\theta) = 0$ if $(\theta_j)_{j \in \{1,2,3,4\} \setminus \{i\}} \notin \hat{\Theta}_{-i}$; let $\psi_4^* = 1 - \sum_{i=1}^{3} \psi_i^*$. A moment's thought reveals that $\psi^*$ is a well-defined DIC mechanism. To see that it is a stochastic extreme point, consider an arbitrary DIC mechanism $\psi$ that appears in a convex combination that equals $\psi^*$. We know from Section 5.2 that $\psi$ must agree with $\psi^*$ whenever the types of agents 1 to 4 are in $\hat{\Theta}$. From here it is easy to see that $\psi$ must agree with $\psi^*$ at all other profiles, too. □

## A.3 Anonymous juries

### A.3.1 Proof of Theorem 6.1

*Proof of Theorem 6.1.* Let $\varphi$ be DIC and anonymous.

The following notation is useful. Let $T$ denote the common type space. Let $T^{n-1}$ with generic element $\theta^{n-1}$ denote the $(n-1)$-fold Cartesian product of $T$. We will frequently consider profiles obtained from a profile $\theta^{n-1}$ in $T^{n-1}$ by replacing one entry of $\theta^{n-1}$. For instance, we write $(t, \theta^{n-1}_{-j})$ to denote the profile obtained by replacing the $j$'th entry of $\theta^{n-1}$ by $t$.

By DIC, for all $i$, we may drop $i$'s type from $i$'s winning probability. Thus we write $\varphi_i(\theta^{n-1})$ for $i$'s winning probability when the types of the others are $\theta^{n-1} \in T^{n-1}$. Anonymity implies that $\varphi_i(\theta^{n-1})$ is invariant to permutations of $\theta^{n-1}$.

We use the following auxiliary claim.

**Claim A.8.** *Let $i \in \{1, \dots, n\}$, $t \in T$, $t' \in T$, and $\theta^{n-1} \in T^{n-1}$. Then*

$$\sum_{j=1}^{n-1} \left( \varphi_i(t, \theta^{n-1}_{-j}) - \varphi_i(t', \theta^{n-1}_{-j}) \right) = 0. \tag{A.11}$$

*Proof of Claim A.8.* Let us arbitrarily label $\theta^{n-1}$ as $(\theta_j)_{j \in N \setminus \{i\}}$. Let us also fix an arbitrary type $\theta_i \in T$.

In an intermediate step, let $j$ be distinct from $i$. For clarity, we spell out winning probabilities as follows: $\varphi_i(r_i = t, r_j = t', r_{-ij} = \theta_{-ij})$ means $i$'s winning probability when $i$ reports $t$, $j$ reports $t'$, and all remaining agents report $\theta_{-ij}$. A permutation of $i$'s and $j$'s reports does not change the winning probabilities of the agents other than $i$ and $j$. Since the object is allocated with probability one, we have

$$\begin{aligned}&\varphi_i(r_i = t, r_j = t', r_{-ij} = \theta_{-ij}) + \varphi_j(r_i = t, r_j = t', r_{-ij} = \theta_{-ij})\\ =&\varphi_i(r_i = t', r_j = t, r_{-ij} = \theta_{-ij}) + \varphi_j(r_i = t', r_j = t, r_{-ij} = \theta_{-ij}).\end{aligned}$$

By rearranging the previous display, and by DIC, we obtain

$$\begin{aligned}&\varphi_i(r_i = t, r_j = t', r_{-ij} = \theta_{-ij}) - \varphi_i(r_i = t', r_j = t, r_{-ij} = \theta_{-ij})\\ =&\varphi_j(r_i = t', r_j = \theta_j, r_{-ij} = \theta_{-ij}) - \varphi_j(r_i = t, r_j = \theta_j, r_{-ij} = \theta_{-ij}).\end{aligned} \tag{A.12}$$

Now consider summing (A.12) over all $j \in \{1, \ldots, n\} \setminus \{i\}$. This summation yields

$$\sum_{j:\, j\neq i} \left(\varphi_i(r_i = t, r_j = t', r_{-ij} = \theta_{-ij}) - \varphi_i(r_i = t', r_j = t, r_{-ij} = \theta_{-ij})\right) \tag{A.13}$$

$$= \sum_{j:\, j\neq i} \left(\varphi_j(r_i = t', r_j = \theta_j, r_{-ij} = \theta_{-ij}) - \varphi_j(r_i = t, r_j = \theta_j, r_{-ij} = \theta_{-ij})\right). \tag{A.14}$$

In (A.14), the profiles considered are all of the form $(r_i = t', r_{-i} = \theta_{-i})$ and $(r_i = t, r_{-i} = \theta_{-i})$, respectively. Note that by DIC we have $\varphi_i(r_i = t', r_{-i} = \theta_{-i}) - \varphi_i(r_i = t, r_{-i} = \theta_{-i}) = 0$. Hence (A.14) equals

$$\sum_{j=1}^{n} \left(\varphi_j(r_i = t', r_{-i} = \theta_{-i}) - \varphi_j(r_i = t, r_{-i} = \theta_{-i})\right).$$

Since the object is always allocated, the term in the previous display equals 0. Hence the sum in (A.13) equals

$$\sum_{j:\, j\neq i} \left(\varphi_i(r_i = \theta_i, r_j = t', r_{-ij} = \theta_{-ij}) - \varphi_i(r_i = \theta_i, r_j = t, r_{-ij} = \theta_{-ij})\right) = 0.$$

We now revert to our usual notation. By DIC, we may drop $i$'s report from $\varphi_i$. Since

$\varphi_i$ is permutation-invariant with respect to $N \setminus \{i\}$, we may also write

$$\varphi_i(r_i = \theta_i, r_j = t', r_{-ij} = \theta_{-ij}) = \varphi_i(t', \theta^{n-1}_{-j}) \qquad \text{and}$$
$$\varphi_i(r_i = \theta_i, r_j = t, r_{-ij} = \theta_{-ij}) = \varphi_i(t, \theta^{n-1}_{-j}).$$

Thus we obtain the desired equality $\sum_{j=1}^{n-1} \left(\varphi_i(t', \theta^{n-1}_{-j}) - \varphi_i(t, \theta^{n-1}_{-j})\right) = 0.$ □

In what follows, let $i$ be an arbitrary agent. We show $i$'s winning probability is constant in the reports of others. To that end, let us fix an arbitrary type $t^* \in T$. For all $k \in \{0, \ldots, n-1\}$, let $T^{n-1}_k$ denote the subset of profiles in $T^{n-1}$ where exactly $k$-many entries are distinct from $t^*$. Let $p_i$ denote $i$'s winning probability when all other agents report $t^*$. We will show via induction over $k$ that $i$'s winning probability is equal to $p_i$ whenever the others report a profile in $T^{n-1}_k$. This completes the proof since $T^{n-1} = \cup_{k=0}^{n-1} T^{n-1}_k$ holds.

*Base case* $k = 0$. Immediate from the definitions of $p_i$ and $T_0$.

*Induction step.* Let $k \geq 1$. Let all $\hat{\theta}^{n-1} \in \cup_{\ell=0}^{k-1} T^{n-1}_\ell$ satisfy $\varphi_i(\hat{\theta}^{n-1}) = p_i$. Letting $\theta^{n-1} \in T^{n-1}_k$ be arbitrary, we show $\varphi_i(\theta^{n-1}) = p_i$.

By anonymity, we may assume that exactly the first $k$ entries of $\theta^{n-1}$ are distinct from $t^*$. That is, there exist types $t_1, \ldots, t_k$ all distinct from $t^*$ such that $\theta^{n-1} = (t_1, \ldots, t_k, t^*, \ldots, t^*)$.

Let $\tilde{\theta}^{n-1} = (t_1, \ldots, t_{k-1}, t^*, \ldots, t^*)$. This profile is obtained from $\theta^{n-1}$ by replacing $t_k$ by $t^*$. We now invoke Claim A.8 to infer

$$\sum_{j=1}^{n-1} \varphi_i(t_k, \tilde{\theta}^{n-1}_{-j}) = \sum_{j=1}^{n-1} \varphi_i(t^*, \tilde{\theta}^{n-1}_{-j}). \tag{A.15}$$

Consider the profiles appearing in the sum on the left of (A.15) as $j$ varies from 1 to $n-1$.

(1) Let $j \leq k-1$. Since exactly the first $k-1$ entries of $\tilde{\theta}$ are distinct from $t^*$, it follows that $(t_k, \tilde{\theta}^{n-1}_{-j})$ is another profile where exactly $k-1$ entries differ from $t^*$. Hence the induction hypothesis implies $\varphi_i(t_k, \tilde{\theta}^{n-1}_{-j}) = p_i$.

(2) Let $j > k-1$. In the profile $(t_k, \tilde{\theta}^{n-1}_{-j})$, the first $k-1$ entries are $t_1, \ldots, t_{k-1}$, the $j$'th entry is $t_k$, and all remaining entries are $t^*$. Hence $(t_k, \tilde{\theta}^{n-1}_{-j})$ is a permutation of $\theta^{n-1}$. Anonymity implies $\varphi_i(t_k, \tilde{\theta}^{n-1}_{-j}) = \varphi_i(\theta^{n-1})$.

Hence the sum on the left of (A.15) equals $\sum_{j=1}^{n-1} \varphi_i(t, \tilde{\theta}^{n-1}_{-j}) = (k-1)p_i + (n-k)\varphi_i(\theta^{n-1})$

Now consider the sum on the right of (A.15). For all $j$, a moment's thought reveals that the profile $(t^*, \tilde{\theta}^{n-1}_{-j})$ contains at most $(k-1)$-many entries different from $t^*$. By the induction hypothesis, therefore, the sum on the right of (A.15) equals $(n-1)p_i$.

The previous two paragraphs and (A.15) imply $(k-1)p_i + (n-k)\varphi_i(\theta^{n-1}) = (n-1)p_i$. Equivalently, $(n-k)(\varphi_i(\theta^{n-1}) - p_i) = 0$. Since $k \leq n-1$, we find $\varphi_i(\theta^{n-1}) = p_i$, as promised. $\square$

### A.3.2 Proof of Theorem 6.2

*Proof of Theorem 6.2.* We omit the straightforward verification that a jury mechanism with an anonymous jury is partially anonymous.

For the converse, let $\varphi$ be deterministic, partially anonymous, and DIC. Let $N$ denote the set of agents, and let $T$ denote the common type space. For this proof, we write $\varphi(\theta)$ to mean the agent who wins at profile $\theta$; this makes sense since $\varphi$ is deterministic.

Let $I_i$ denote the set of agents that influence agent $i$'s winning probability. For all $j \in N$, let $A_j = \{i \in N \colon j \in I_i\}$ be the set of agents that are influenced by $j$. Let $I = \{i \in N \colon A_i \neq \emptyset\}$. We may assume that $\varphi$ is non-constant, meaning $I \neq \emptyset$, as otherwise the proof is trivial.

Given two agents $i$ and $j$, let $D_{i-j} = A_i \setminus A_j$, and $D_{j-i} = A_j \setminus A_i$, and $C_{ij} = A_j \cap A_i$, and $N_{ij} = N \setminus (A_i \cup A_j)$. Note that, by DIC, the set $C_{ij}$ contains neither $i$ nor $j$. Hence partial anonymity implies that for all $k \in C_{ij}$ the winning probability of $k$ is invariant with respect to permutations of $i$ and $j$.

When $i$, $j$, and $k$ are given, we write $(t, t', t'', \theta_{-ijk})$ to mean the profile where $i$, $j$, and $k$, respectively, report $t$, $t'$, and $t''$, respectively, and all others report $\theta_{-ijk}$.

**Claim A.9.** *Let $i$ and $j$ be distinct. Let $\theta_{-ij} \in \Theta_{-ij}$. If there exists $\theta_i, \theta_j \in T$ such that $\varphi(\theta_i, \theta_j, \theta_{-ij}) \in D_{i-j}$, then all $\theta'_i, \theta'_j \in T$ satisfy $\varphi(\theta'_i, \theta'_j, \theta_{-ij}) \in D_{i-j}$.*

*Proof of Claim A.9.* We drop the fixed type profile $\theta_{-ij}$ of the others from the notation. To show $\varphi(\theta'_i, \theta'_j) \in D_{i-j}$, it suffices to show $\varphi(\theta'_i, \theta_j) \in D_{i-j}$ since if the latter is true then definition of $D_{i-j}$ implies $\varphi(\theta'_i, \theta'_j) = \varphi(\theta'_i, \theta_j)$.

We first claim $\varphi(\theta_j, \theta_i) \in D_{i-j}$. If $\varphi(\theta_j, \theta_i) \in N_{ij}$, then $\varphi(\theta_j, \theta_i) = \varphi(\theta_i, \theta_j)$, and we have a contradiction to $\varphi(\theta_i, \theta_j) \in D_{i-j}$. If $\varphi(\theta_j, \theta_i) \in C_{ij}$, then partial anonymity implies $\varphi(\theta_i, \theta_j) \in C_{ij}$, and we have another contradiction to $\varphi(\theta_i, \theta_j) \in D_{i-j}$. If $\varphi(\theta_j, \theta_i) \in D_{j-i}$, then $\varphi(\theta_j, \theta_i) = \varphi(\theta_i, \theta_i) \in D_{j-i}$. However, from $\varphi(\theta_i, \theta_j) \in D_{i-j}$ we know $\varphi(\theta_i, \theta_j) = \varphi(\theta_i, \theta_i) \in D_{i-j}$; contradiction. Thus $\varphi(\theta_j, \theta_i) \in D_{i-j}$.

We next claim $\varphi(\theta'_i, \theta_j) \in (D_{i-j} \cup C_{ij})$. Towards a contradiction, suppose not. Then $\varphi(\theta'_i, \theta_j) \in (D_{j-i} \cup N_{ij})$, and hence $\varphi(\theta'_i, \theta_j) = \varphi(\theta_i, \theta_j) \notin D_{i-j}$. This contradicts the assumption $\varphi(\theta_i, \theta_j) \in D_{i-j}$.

In view of the previous paragraph, we can complete the proof by showing $\varphi(\theta'_i, \theta_j) \notin C_{ij}$. Towards a contradiction, let $\varphi(\theta'_i, \theta_j) \in C_{ij}$. Partial anonymity implies $\varphi(\theta_j, \theta'_i) \in C_{ij}$. We have shown earlier that $\varphi(\theta_j, \theta_i) \in D_{i-j}$ holds. Hence $\varphi(\theta_j, \theta'_i) \in D_{i-j}$, and this contradicts $\varphi(\theta_j, \theta'_i) \in C_{ij}$. Thus $\varphi(\theta'_i, \theta_j) \notin C_{ij}$ and the proof is complete. $\square$

**Claim A.10.** *Let $i, j, k$ be distinct. Let $\theta_k \in T$ and $\theta_{-ijk} \in \Theta_{-ijk}$ be such that all $\theta'_i, \theta'_j \in T$ satisfy $\varphi(\theta'_i, \theta'_j, \theta_k, \theta_{-ijk}) \in (C_{ij} \cup N_{ij})$. Then, all $\theta'_i, \theta'_j, \theta'_k \in T$ satisfy $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in (C_{ij} \cup N_{ij})$.*

*Proof of Claim A.10.* Towards a contradiction, suppose $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in (D_{i-j} \cup D_{j-i})$. Suppose $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in D_{i-j}$, the other case being similar. The inclusions $\varphi(\theta'_i, \theta'_j, \theta_k, \theta_{-ijk}) \in (C_{ij} \cup N_{ij})$ and $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in D_{i-j}$ together imply $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in A_k$. Hence $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in D_{k-j}$. We now invoke Claim A.9 to infer $\varphi(\theta'_i, \theta'_j, \theta_k, \theta_{-ijk}) \in D_{k-j}$. Since we also have $\varphi(\theta'_i, \theta'_j, \theta_k, \theta_{-ijk}) \in (C_{ij} \cup N_{ij})$, we infer $\varphi(\theta'_i, \theta'_j, \theta_k, \theta_{-ijk}) \in N_{ij}$. In particular, we have $\varphi(\theta'_i, \theta'_j, \theta_k, \theta_{-ijk}) \notin A_i$. Hence $\varphi(\theta'_i, \theta'_j, \theta_k, \theta_{-ijk}) \in D_{k-i}$. We now invoke Claim A.9 to infer $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in D_{k-i}$. In particular, we have $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \notin A_i$. This contradicts the assumption $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in D_{i-j}$. $\square$

**Claim A.11.** *If $C_{ij} \neq \emptyset$, then $D_{i-j} \cup D_{j-i} = \emptyset$.*

*Proof of Claim A.11.* Let $k \in C_{ij}$. We may find a profile $\theta$ such that $\varphi(\theta) = k$ as else $k$'s winning probability is constantly 0 (which would contradict $k \in C_{ij}$). Denoting by $\theta_{-ij}$ the types of agents other than $i$ and $j$ at $\theta$, we appeal to Claim A.9 to infer that all $\theta'_i, \theta'_j \in T$ satisfy $\varphi(\theta'_i, \theta'_j, \theta_{-ij}) \in (C_{ij} \cup N_{ij})$. Repeatedly applying Claim A.10 implies that all profiles $\theta'$ satisfy $\varphi(\theta') \in (C_{ij} \cup N_{ij})$. It follows that all agents in $D_{i-j} \cup D_{j-i}$ enjoy a winning probability that is constantly equal to 0. Recalling the definitions $D_{i-j} = A_i \setminus A_j$, and $D_{j-i} = A_j \setminus A_i$, it follows that $D_{i-j} \cup D_{j-i}$ is empty. $\square$

Recall the definition $I = \{i \in N \colon A_i \neq \emptyset\}$. Consider the binary relation $\sim$ on $I$ defined as follows: Given $i$ and $j$ in $I$, we let $i \sim j$ if and only if $C_{ij} \neq \emptyset$.

**Claim A.12.** *The relation $\sim$ is an equivalence relation. For all $i, j \in I$, if $i \sim j$, then $i \notin A_j$ and $A_i = A_j$.*

*Proof of Claim A.12.* It is clear that $\sim$ is symmetric. As for reflexivity, note that $i \in I$ implies $A_i = C_{ii} \neq \emptyset$. Turning to transitivity, suppose $i \sim j$ and $j \sim k$. Hence $C_{ij} \neq \emptyset$ and $C_{jk} \neq \emptyset$. Let $\ell \in C_{jk}$. Claim A.11 and $C_{ij} \neq \emptyset$ together imply $D_{j-i} = \emptyset$. Hence $\ell \in C_{jk}$ implies $\ell \in C_{ij}$. Hence $\ell \in C_{jk} \cap C_{ij}$, implying $\ell \in C_{ik}$. Hence $i \sim k$.

As for the second part of the claim, let $i \sim j$. Thus $C_{ij} \neq \emptyset$. Claim A.11 implies $D_{j-i} = D_{i-j} = \emptyset$. This immediately implies $A_i = A_j$. Together with DIC, we also infer $i \notin A_j$. $\square$

Claim A.12 implies that we may partition $I$ into finitely-many non-empty $\sim$-equivalence classes. (Recall that $I$ is non-empty.) We now claim that there is exactly one $\sim$-equivalence class. Towards a contradiction, suppose not. In view of Claim A.12, this means that there are distinct $i$ and $j$ such that $A_i \cap A_j = \emptyset$ and $A_i \neq \emptyset \neq A_j$. Let $J_i$ and $J_j$, respectively, denote the equivalence classes containing $i$ and $j$, respectively. Let $k \in A_i$ and $\ell \in A_j$. Claim A.12 implies $k \notin J_i$ and $\ell \notin J_j$ and $k \neq \ell$. Since $k \in A_i$ and $\varphi$ is deterministic, there is a type profile $\theta$ such that $\varphi(\theta) = k$; there must be another type profile $\theta'$ such that $\varphi(\theta') = \ell$. However, the definition of equivalence classes implies that $k$'s winning probability depends only on the types of agents in $J_i$, and that $\ell$'s winning probability depends only on the types of agents in $J_j$. Hence there is a type profile where both $k$ and $\ell$ are winning with probability 1 (such a type profile is obtained by changing at the profile $\theta$ the types of agents in $J_j$ to their respective types at $\theta'$, and keeping all other types fixed). Contradiction.

Now, Claim A.12 implies that the members of the unique $\sim$-equivalence class do not influence one another, and that they influence the same set of others. By partial anonymity, it follows $\varphi$ that is a deterministic jury mechanism with an anonymous jury. $\square$

### A.3.3 Proof of Proposition 6.3

*Proof of Proposition 6.3.* We first prove part (2) of the claim, assuming for a moment that part (1) is true. Thus, assume that if $n = 3$, then there is an anonymous stochastic DIC mechanism with disposal that is an extreme point of the set of all DIC mechanisms with disposal. Denote this mechanism with disposal by $\varphi^*$. Now let $n = 4$. When there are 4 agents, we may view $\varphi^*$ as a mechanism (without disposal) $\psi^*$ that ignores the reports of agent 4 but otherwise coincides with $\varphi^*$.[19] A moment's thought reveals that, since $\varphi^*$ is stochastic, DIC, and anonymous, the mechanism $\psi^*$ is stochastic, DIC, and partially anonymous. To see that $\psi^*$ is an extreme point of the set of all DIC mechanisms (without disposal), consider an arbitrary convex combination of DIC mechanisms that agrees with $\psi^*$. Let $\psi$ be a mechanism in this convex combination. Fixing an arbitrary type $\theta_4$ of agent 4, the restriction of $\psi$ to $T^3 \times \{\theta_4\}$ defines a mechanism with disposal for three agents in the obvious way. Since $\varphi^*$ is an extreme point of DIC mechanisms with disposal for three agents, it follows that this restriction agrees with $\varphi^*$, and hence with the restriction of $\psi^*$ to $T^3 \times \{\theta_4\}$. Since $\theta_4$ was arbitrary, it follows that $\psi$ agrees with $\psi^*$ everywhere.

It remains to prove part (1) of the claim. That is, we show that if $n = 3$, then there is an anonymous stochastic DIC mechanism with disposal that is an extreme point of the set of all DIC mechanisms with disposal.

Let us relabel the common type space as $T = \{1, 2, 3, 4, 5, 6, 7\}$. Let $T^3 = \times_{i=1}^3 T$ denote the 3-fold Cartesian product of $T$. Let $T_1 = \{1, 2\}$, $T_2 = \{3, 4\}$ and $T_3 = \{5, 6, 7\}$ and $\hat{\Theta} = T_1 \times T_2 \times T_3$. In Section 5.2, we constructed a stochastic DIC mechanism $\varphi^*$ without disposal in a setting with 4 agents, where the types of agents 1, 2, and 3, respectively, are $\{\ell, r\}$, $\{u, d\}$, $\{f, c, b\}$, respectively, and where agent 4's type is degenerate. By relabelling types, we can view $\varphi^*$ as being a mechanism with disposal with 3 agents on the set of type profiles $\hat{\Theta}$, and where allocating to agent 4 is identified with disposing the object. The arguments from Section 5.2 show that, if $n = 3$ and the set of type profiles is $\hat{\Theta}$, then $\varphi^*$ is an extreme point of the set of DIC mechanisms with disposal.

For later reference, we note that, at all type profiles $\theta \in \hat{\Theta}$ and all $i \in \{1, 2, 3\}$, agent $i$'s winning probability at $\theta$ under $\varphi^*$ is either 0 or 1/2. We now use $\varphi^*$ to

[19] Formally, the mechanism $\psi^* \colon T^4 \to [0, 1]^n$ is defined as follows: For all $(\theta_1, \theta_2, \theta_3, \theta_4) \in T^4$ and all $i \in \{1, 2, 3, 4\}$, let $\psi_i^*(\theta_1, \theta_2, \theta_3, \theta_4) = \varphi_i^*(\theta_1, \theta_2, \theta_3)$ if $i \neq 4$, and let $\psi_4^*(\theta_1, \theta_2, \theta_3, \theta_4) = 1 - \sum_{i=1}^3 \varphi_i^*(\theta_1, \theta_2, \theta_3)$.

define a mechanism as in the claim.

Our candidate mechanism will be denoted $\psi^*$. Let $\Xi$ denote the set of permutations of $\{1,2,3\}$. Let $\Theta^* = \{\xi(\theta)\colon \theta \in \hat{\Theta}, \xi \in \Xi\}$ denote the set of type profiles obtained by permuting a type profile in $\hat{\Theta}$; see Figure 2. We return to this figure later. Note that, fixing an arbitrary type profile in $\hat{\Theta}$, the types of the agents at this type profile are all distinct. Consequently, for all $\theta^*$ in $\Theta^*$ there is a unique profile $\theta$ in $\hat{\Theta}$ and $\xi$ in $\Xi$ such that $\theta^* = \xi(\theta)$.

For later reference, we also note the following: At an arbitrary type profile in $\Theta^*$, the types of distinct agents must belong to distinct elements of the partition $\{T_1, T_2, T_3\}$.

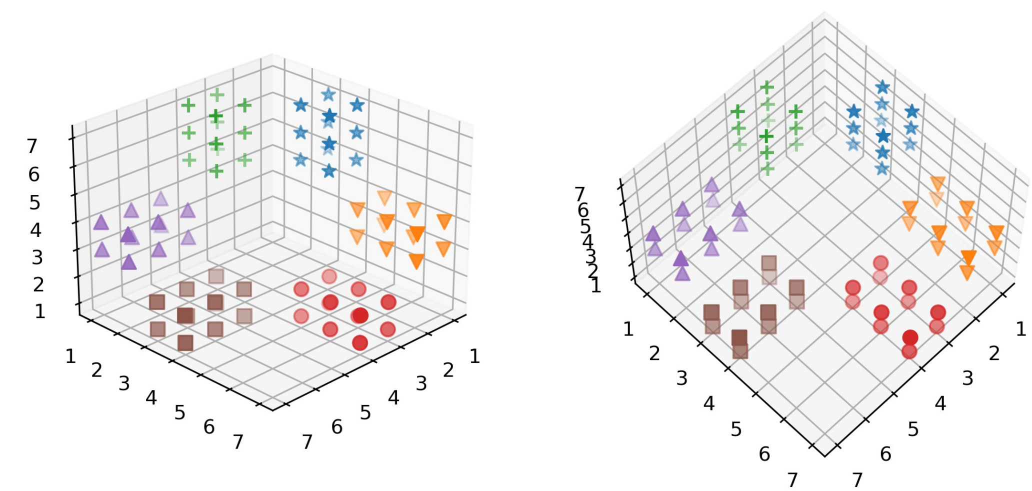


Figure 2: The set $\Theta^*$ viewed from two different angles. Each agent is associated with a distinct axis. Each symbol (square, circle, upward-pointing triangle, etc.) identifies a particular permutation of $\{1,2,3\}$. For instance, the upward-pointing triangles are obtained from the downward-pointing triangles by permuting the two agents on the horizontal axes.

We now define $\psi^*$ as follows: For all $\theta^*$ in $\Theta^*$, we find the unique $(\theta, \xi) \in T \times \Xi$

such that $\theta^* = \xi(\theta)$, and then let

$$(\psi_i^*(\theta^*))_{i=1}^n = \left(\varphi_{\xi(i)}^*(\xi(\theta))\right)_{i=1}^n. \tag{A.16}$$

In words, if $\theta^*$ is obtained from, say, $\theta$ by permuting the entries of agents 1 and 2, then $\psi^*(\theta^*)$ swaps the winning probabilities of agents 1 and 2 while leaving agent 3's winning probability unchanged. For the remaining profiles, we proceed as follows: For all agents $i$ and profiles $\theta$, if $\theta$ differs from at least one profile $\theta^*$ in $\Theta^*$ in agent $i$'s type and no other agent's type, then $i$'s winning probability at $\theta$ is set equal to $i$'s winning probability at $\theta^*$ (which makes sense since the latter probability has already been defined in (A.16)); else, if no such profile $\theta^*$ in $\Theta^*$ exists, then agent $i$'s winning probability is set equal to 0.

To complete the argument, we have to show that $\psi^*$ is a (1) well-defined mechanism, and (2) that it is DIC, stochastic, anonymous, and an extreme point of the set of DIC mechanisms with disposal. Assuming for a moment that (1) is true, it is clear that the mechanism is stochastic, and one can easily verify from the definition that it is DIC and anonymous. Moreover, to show that it is an extreme point of the set of DIC mechansims, we can proceed essentially via the arguments from Section 5.2. Indeed, we know from Section 5.2 that all DIC mechanisms $\psi$ with disposal that appear in a candidate convex combination must agree with $\psi^*$ on $\hat{\Theta}$, and hence on $\Theta^*$; it is then straightforward to verify that such a mechanism $\psi$ must also agree with $\psi^*$ on $\Theta \setminus \Theta^*$.

We now turn to (1). To show that $\psi^*$ is a well-defined mechanism, we have to show that the winning probabilities of the agents do not sum to a number strictly above 1. Before delving into the details, consider Figure 2. The different symbols (squares, circles, upward-pointing triangles, etc.) partition $\Theta^*$ into six subsets (one for each permutation of $\{1, 2, 3\}$). For each of these subsets, imagine rays emanating from the subset and travelling parallel to the axes. These rays identify sets of type profiles along which exactly one agent's type is changing. Now consider the intersection of rays originating from subsets with different symbols. The geometry of $\Theta^*$ implies that *at most* two such rays intersect simultaneously. This is one critical observation that we will use to argue that $\psi^*$ is well-defined.

The second critical observation is that at all type profiles $\theta \in \Theta^*$ and all $i \in \{1, 2, 3\}$ agent $i$'s winning probability under $\varphi^*$ at $\theta$ is either 0 or 1/2. The mechanism

$\psi^*$ inherits this property.

Towards a contradiction, suppose there is a profile $\theta = (\theta_1, \theta_2, \theta_3)$ in $\Theta$ where the winning probabilities under $\psi^*$ sum to a number strictly above 1. By the previous paragraph, all three agents are therefore enjoying non-zero winning probabilities. By definition of $\psi^*$, we can infer the following: Since agent 1's winning probability is non-zero, there exists $t_1$ such that $(t_1, \theta_2, \theta_3) \in \Theta^*$. Similarly, there are $t_2$ and $t_3$ such that $(\theta_1, t_2, \theta_3) \in \Theta^*$ and $(\theta_1, \theta_2, t_3) \in \Theta^*$. Recall that $\{T_1, T_2, T_3\}$ is a partition of the common type space. Hence, for all agents $i$, there is a unique interger $\xi(i)$ in $\{1, 2, 3\}$ such that $\theta_i \in T_{\xi(i)}$. Let us now recall the follwing from the definition of $\Theta^*$: If a profile is in $\Theta^*$, then the types of distinct agents must belong two distinct elements of the partition $\{T_1, T_2, T_3\}$. Hence, we can infer from $(t_1, \theta_2, \theta_3) \in \Theta^*$ that $\xi(2) \neq \xi(3)$ holds. Similarly, from $(\theta_1, t_2, \theta_3) \in \Theta^*$ and $(\theta_1, \theta_2, t_3) \in \Theta^*$ we infer $\xi(1) \neq \xi(2)$ and $\xi(1) \neq \xi(3)$. Taken together, we infer that $\theta$ must itself be in $\Theta^*$. Hence the vector of winning probabilities at $\theta$ is a permutation of the vector of winning probabilities at a profile $\theta'$ in $\hat{\Theta}$. At the profile $\theta'$, the winning probabilites under $\psi^*$ agree with $\varphi^*$. Thus there is a profile where the winning probabilities under $\varphi^*$ sum to a number strictly greater than 1; this is a contradiction since $\varphi^*$ is a well-defined mechanism. □

# Appendix B Supplementary material: Disposal

In this part of the appendix, we relax the requirement that the object always be allocated. An intepretation is that the mechanism designer can dispose or privately consume the object. Accordingly, we refer to such mechanisms as mechanisms with disposal. We discuss how this affects our results from the main text (Appendix B.1). Further, we show how the existence of stochastic extreme points of the set of DIC mechanisms with disposal can be related to a certain graph (Appendix B.2).

Beginning with the definitions, a *mechanism with disposal* is a function $\varphi\colon \Theta \to [0,1]^n$ satisfying

$$\forall_{\theta \in \Theta}, \quad \sum_{i=1}^{n} \varphi_i(\theta) \leq 1.$$

A mechanism from the main text will be referred to as a mechanism with no disposal. If there is no risk of confusion, we will drop the qualifiers "with disposal" or "with no disposal".

A mechanism with disposal is DIC if and only if for arbitrary $i$ the winning probability $\varphi_i$ is constant in $i$'s report. We will sometimes drop $i$'s report $\theta_i$ from $\varphi_i(\theta_i, \theta_{-i})$.

A jury mechanism with disposal is defined as in the basic model: For all $i$, if agent $i$ influences the allocation, then $i$ never wins the object.

We normalize the value from not allocating the object to 0.

A mechanism with $n$ agents and disposal can be viewed as a mechanism with no disposal and with $n+1$ agents where agent $n+1$ has a singleton type space; the value from allocating to $n+1$ is always 0. Likewise, if there are other agents with singleton type spaces, we can always renormalize values and view allocating to one of these agents as disposing the object. In what follows, whenever considering mechanisms with disposal, let us thus simplify by assuming that no agent has a singleton type space; that is, for all agents $i$ we have $|\Theta_i| \geq 2$.

## B.1 Results from the main text

Here we discuss how our results change when the mechanism can dispose the object.

To begin with, we have the following analogue of Theorem 5.1.

**Theorem B.1.** *Fix $n$ and $\Theta_1, \ldots, \Theta_n$. For all agents $i$, let $|\Theta_i| \geq 2$. All extreme points of the set of DIC mechanisms with disposal are deterministic if and only if at least one of the following is true:*

*(1) We have $n \leq 2$.*

*(2) For all agents $i$ we have $|\Theta_i| = 2$.*

*Proof of Theorem B.1.* As discussed above, a DIC mechanism with $n$ agents and disposal is a DIC mechanism with $n+1$ agents and no disposal. The claim follows from Theorem 5.1. □

Further below, we provide an alternative proof of Theorem B.1 that does not invoke Theorem 5.1 but relies on graph-theoretic results. We emphasize that Theorem B.1 does not imply Theorem 5.1. Namely, we cannot conclude from Theorem B.1 that if $n = 3$ all extreme points of the set of DIC mechanisms with no disposal are deterministic.

It follows from Theorem B.1 that Theorem 4.1 (jury mechanisms with 3 agents) carries over to mechanisms with disposal in the sense that all mechanisms with disposal and 2 agents are convex combinations of deterministic jury mechanisms with disposal. Note that, according to Theorem B.1, this result does not extend to $n = 3$. With $n = 2$, a jury mechanism with disposal admits a single juror whose report determines whether or not the object is disposed or allocated to the other agent.

Proposition 5.2 (on the suboptimality of deterministic DIC mechanisms) analogizes straightforwardly to mechanisms with disposal. Indeed, note that in our proof of Proposition 5.2 agent 4 was simply a dummy agent with value normalized to 0.

Theorem 4.4 (approximate optimality of jury mechanisms under Assumption 1 and large $n$) extends to mechanisms with disposal in a straightforward way, with no changes to the proof.

We already showed via Proposition 6.3 that Theorem 6.1 does not extend to mechanisms with disposal. In fact, the non-constant mechanism constructed in the proof of Proposition 6.3 actually satisfies an even stronger notion of anonymity. Namely, whenever one permutes the type profiles, the vector of winning probabilities is permuted in the same manner.

We next turn to partial anonymity for mechanisms with disposal. In particular, we show that Theorem 6.2 extends under a slight strengthening of partial anonymity. Given a mechanism $\varphi$, let $\varphi_0 = 1 - \sum_{i=1}^{n} \varphi_i$ denote the probability that the object is not allocated.

**Definition 4.** Let $\varphi$ be a mechanism with disposal. Let $N = \{1, \ldots, n\}$ and $N_0 = N \cup \{0\}$.

(1) Given distinct $i \in N$ and $k \in N_0$, agent $i$ *influences* $k$ if $\varphi_k$ is non-constant in $i$'s report.

(2) The mechanism is *partially $*$-anonymous* if for all $i \in N$, $j \in N$, and $k \in N_0$ that are all distinct and are such that $i$ and $j$ influence $k$, agents $i$ and $j$ are exchangeable for $k$.

In words, partial anonymity is strengthened by demanding that the disposal probability $\varphi_0$ is permutation-invariant with respect to those agents who influence $\varphi_0$.

It follows from Theorem 6.2 that a deterministic partially $*$-anonymous DIC mechanism with disposal is a deterministic jury mechanism with an anonymous jury. To see this, let us view disposing the object as allocating to agent 0. Now, agent 0 does not have the same type space as the other agents. Since this was a maintained assumption of Section 6, we cannot yet appeal to Theorem 6.2. But, we can simply view the mechanism as a mechanism where agent 0's type space is same as the type spaces of the others, and where agent 0's report is always ignored. By now appealing to Theorem 6.2, the claim follows.

## B.2 Stochastic extreme points and perfect graphs

In this section, we relate the existence of stochastic extreme points with disposal to a graph-theoretic property called perfection.

### B.2.1 Preliminaries

We first introduce several definitions for a general graph $G$ with nodes $V$ and edges $E$. All graphs are understood to be simple and undirected.

An *induced cycle of length* $k$ is a subset $\{v_1, \ldots, v_k\}$ of $V$ such that, denoting $v_{k+1} = v_1$, two nodes $v_\ell$ and $v_{\ell'}$ in the subset are adjacent if and only if $|\ell - \ell'| = 1$.

The *line graph* of $G$ is the graph that has as node set the edge set of $G$; two nodes of the line graph are adjacent if and only if the two associated edges of $G$ share a node in $G$.

A *clique* of $G$ is a set of nodes such that every pair in the set are adjacent. A clique is *maximal* if it is not a strict subset of another clique. A *stable set* of $G$ is a subset of nodes of which no two are adjacent. The *incidence vector* of a subset of nodes $\hat{V}$ is the function $x \colon V \to \{0, 1\}$ that equals one on $\hat{V}$ and equals zero otherwise. Let $S(G)$ denote the set of incidence vectors belonging to some stable set of $G$.

The upcoming result uses another definition called *perfection*. For our purposes, it will be enough to know the following.

**Lemma B.2.** *All bi-partite graphs and line graphs of bi-partite graphs are perfect. If a graph admits an induced cycle of odd length greater than five, then it is not perfect.*

These facts may be found in Korte and Vygen (2018).

Our interest in perfect graphs is due to the following theorem due to Chvátal (1975, Theorem 3.1); one may also find it in Korte and Vygen (2018, Theorem 16.21).

**Theorem B.3.** *A graph $G$ with node set $V$ and edge set $E$ is perfect if and only if the convex hull* $\operatorname{co} S(G)$ *is equal to the set*

$$\left\{ x\colon V \to [0,1]\colon \textit{all maximal cliques } X \textit{ of } G \textit{ satisfy} \sum_{v \in X} x(v) \leq 1 \right\}. \tag{B.1}$$

The set $\operatorname{co} S(G)$ is the *stable set polytope* of $G$. The set in (B.1) is the *clique-constrained stable set polytope* of $G$.

We next define a graph $G$ such that the set of deterministic DIC mechanisms with disposal corresponds to $S(G)$, and such that the set of all DIC mechanisms with disposal coincides with the clique-constrained stable set polytope of $G$. In view of Theorem B.3, the question of whether all extreme points are deterministic thus reduces to checking whether $G$ is a perfect graph.

### B.2.2 The feasibility graph

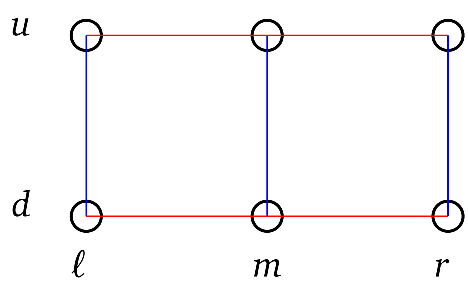
Consider the following graph $G$ with node set $V$ and edge set $E$. Let $V = \cup_{i=1}^{n} (\{i\} \times \Theta_{-i})$, and let two nodes $(i, \theta_{-i})$ and $(j, \theta'_{-j})$ be adjacent if and only if $i \neq j$ and there is a type profile $\hat{\theta}$ satisfying $\hat{\theta}_{-i} = \theta_{-i}$ and $\hat{\theta}_{-j} = \theta'_{-j}$. We refer to $G$ as the *feasibility graph*.

Informally, a node $(i, \theta_{-i})$ is the index for agent $i$'s winning probability when the type profile of the others is $\theta_{-i}$. Two nodes are adjacent if and only if there is a profile $\hat{\theta}$ such that the associated winning probabilities simultaneously appear in the feasibility constraint

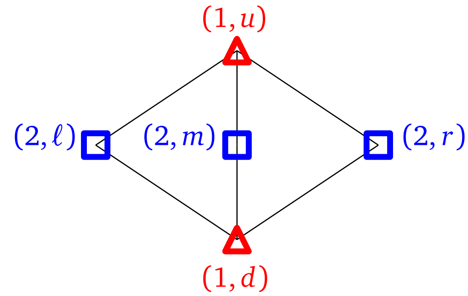
$$\sum_{i=1}^{n} \varphi_i(\hat{\theta}_{-i}) \leq 1. \tag{B.2}$$

Figure 3 shows the feasibility graph in an example with two agents; Figure 4 shows it in an example with three agents

Given a node $v = (i, \theta_{-i})$ of $G$, let us write $\varphi(v) = \varphi_i(\theta_{-i})$. Note that a clique in the feasibility graph is a subset of nodes of $V$ such that the winning probabilities associated with these nodes all appear in the same feasibility constraint (B.2). It follows that there is a one-to-one mapping between maximal cliques of $G$ and type profiles. For a DIC mechanism with disposal, the feasibility constraint (B.2) may thus be equivalently stated as follows: For all maximal cliques of $X$ of $G$, we have




(a) The set of type profiles $\Theta$. Circles represent type profiles.




(b) The graph $G$. Red triangles represent nodes of $G$ that are associated with agent 1. Blue squares represent nodes associated with agent 2.

Figure 3: There are two agents with types $\Theta_1 = \{\ell, m, r\}$ and $\Theta_2 = \{u, d\}$.

$\sum_{v \in X} \varphi(v) \leq 1$. Thus the set of DIC mechanisms with disposal coincides with the set (B.1). One may similarly verify that the set of deterministic DIC mechanisms with disposal coincides with $S(G)$. In view of Theorem B.3, we deduce:

**Lemma B.4.** *All extreme points of the set of DIC mechanisms with disposal are deterministic if and only if $G$ is perfect.*

This leads us to the following alternative proof of Theorem B.1.

*Alternative proof of Theorem B.1.* Let $n = 2$. Observe that the node set of $G$ may be partitioned into the sets $\{1\} \times \Theta_2$ and $\{2\} \times \Theta_1$. By definition, two nodes $(i, \theta_{-i})$ and $(j, \theta_{-j})$ are adjacent only if $i \neq j$. Thus $G$ is bi-partite. Since every bi-partite graph is perfect (Lemma B.2), the claim follows from Theorem B.3.

Suppose $|\Theta_i| = 2$ holds for all $i$. We may relabel the types so that $\Theta_i = \{0, 1\}$ holds for all $i$. In this case $G$ is the line graph of a bi-partite graph; namely the bi-partite graph with node set $\{0, 1\}^n$ and where two nodes are adjacent if and only if they differ in exactly one entry. The line graph of a bi-partite graph is perfect (Lemma B.2), and so the claim again follows from Theorem B.3.

Lastly, suppose $n \geq 3$ and $|\Theta_i| > 2$ for at least one $i$. We will show that $G$ admits an odd induced cycle of length seven. In view of Lemma B.2 and Theorem B.3, this proves that there exists a stochastic extreme point. Let us relabel the agents and types such that the type spaces contain the following subsets of types:

$$\tilde{\Theta}_1 = \{\ell, r\} \quad \text{and} \quad \tilde{\Theta}_2 = \{u, d\} \quad \text{and} \quad \tilde{\Theta}_3 = \{f, c, b\}$$

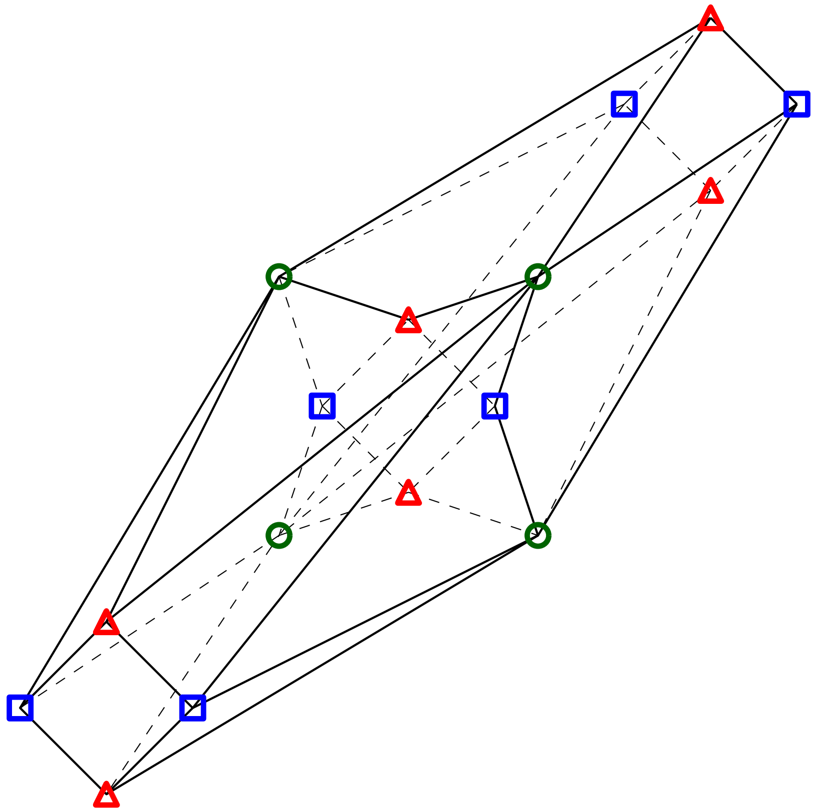


Figure 4: The feasibility graph $G$ in an example with three agents. Agents 1 and 2 each have two possible types. The nodes of $G$ associated with agents 1 and 2, respectively, are depicted by red triangles and blue squares, respectively. Agent 3 has three possible types; the associated nodes are depicted by green circles. One may view this as the graph $G$ associated with the four-agent environment of Section 5.2, except that all nodes of the dummy agent 4 are omitted.

all hold. Let $\theta_{-123}$ be an arbitrary type profile of agents other than 1, 2 and 3 (assuming such agents exist). One may verify that the following is an induced cycle of length seven:

$$\begin{aligned}(2,(\ell,c,\theta_{-123})) &\leftrightarrow (1,(d,c,\theta_{-123}))\\ &\leftrightarrow (3,(r,d,\theta_{-123}))\\ &\leftrightarrow (2,(r,b,\theta_{-123}))\\ &\leftrightarrow (3,(r,u,\theta_{-123}))\\ &\leftrightarrow (1,(u,f,\theta_{-123}))\\ &\leftrightarrow (3,(\ell,u,\theta_{-123}))\\ \leftrightarrow (2,(\ell,c,\theta_{-123}))\,. \end{aligned}$$

□

The proof in the main text for the existence of a stochastic extreme point is slightly more elaborate than the one given above since in the former we explicitly spell out the extreme point. (The proof in the main text uses one of the agents as a dummy, and therefore also works for mechanisms with disposal.) In our view, the advantage of the more elaborate argument is that it facilitates the construction of environments where all deterministic DIC mechanisms fail to be optimal. This lets us give an interpretation as to why it may be optimal to use a lottery. That said, it is clear how the induced cycle defined in the proof of Theorem B.1 relates to the construction from the main text.

# Appendix C Supplementary material: Additional results

## C.1 All extreme points are candidates for optimality

**Lemma C.1.** *Let $n \in \mathbb{N}$. Let $\Theta_1, \dots, \Theta_n$ be finite sets, and let $\Theta = \times_{i=1}^n \Theta_i$. If $\varphi$ is an extreme point of the set of DIC mechanisms when there are $n$ agents and the set of type profiles is $\Theta$, then there exists a set $\Omega$ of value profiles and a distribution $\mu$ over $\Omega \times \Theta$ such that in the environment $(n, \Omega, \Theta, \mu)$ the mechanism $\varphi$ is the unique optimal DIC mechanism.*

Note that the set of DIC mechanisms depends only on the number of agents and (the cardinalities of) their type spaces.

*Proof of Lemma C.1.* The set of DIC mechanisms is a polytope in Euclidean space (being the set of solutions to a finite system of linear inequalities). Hence all its extreme points are exposed (Aliprantis and Border, 2006, Corollary 7.90). In particular, since $\varphi$ is an extreme point, there is a function $p\colon \{1,\ldots,n\}\times\Theta \to \mathbb{R}$ such that for all DIC mechanisms $\varphi'$ different from $\varphi$ we have $\sum_{i,\theta} p_i(\theta)(\varphi_i(\theta) - \varphi'_i(\theta)) > 0$. By suitably choosing $\Omega$ and $\mu$, the function $p$ represents the objective function of our model. For example, one possible choice of $\Omega$ and $\mu$ is as follows: Let the marginal of $\mu$ on $\Theta$ be uniform; for all agents $i$, let $\Omega_i$ be the image of $p_i$; for all $\theta$, conditional on the type profile realizing as $\theta$, let the value of allocating to agent $i$ be $|\Theta|p_i(\theta)$. $\square$

## C.2 Implementation with deterministic outcome functions

An indirect mechanism specifies a tuple $M = (M_1, \ldots, M_n)$ of finite message sets, and an outcome function $g\colon \times_i M_i \to \Delta\{0,\ldots,n\}$. (Given a finite set $X$, we denote by $\Delta X$ the set of distributions over $X$.) The outcome function is *deterministic* if for all $m$ the distribution $g(m)$ is degenerate. A *strategy* of agent $i$ is a function $\sigma_i\colon \Theta \to \Delta M_i$. A DIC mechanism $\varphi$ is *implementable* via $(M, g)$ if there is a dominant-strategy equilibrium $(\sigma_1, \ldots, \sigma_n)$ of $(M, g)$ such that all profiles $\theta$ satisfy $\varphi(\theta) = \sum_m g(m)\prod_i \sigma_i(m_i|\theta_i)$.

**Lemma C.2.** *If a stochastic DIC mechanism $\varphi$ is implementable via an indirect mechanism with a deterministic outcome function, then $\varphi$ is not an extreme point of the set of DIC mechanisms.*

*Proof of Lemma C.2.* Let $(M, g, \sigma)$ implement $\varphi$. We may assume $g_i$ does not depend on $i$'s message; the reason is that for $\sigma$ to be a dominant-strategy equilibrium, agent $i$'s strategy must be supported on messages that give $i$ the same winning probability. Let us abbreviate $\sigma(m|\theta) = \prod_i \sigma_i(m_i|\theta_i)$. Let $\Sigma$ denote the set of functions from $\Theta$ to $\Delta(\times_i M_i)$, and notice that $\Sigma$ contains $\sigma$. The extreme points of $\Sigma$ are deterministic; that is, they are mappings from $\Theta$ to $\times_i M_i$. Since $\Sigma$ is also compact, convex and nonempty, the Krein-Milman theorem implies that $\sigma$ is a convex combination of functions from $\Theta$ to $\times_i M_i$. Denote these functions by $\{\hat{\sigma}_k\}_k$, and the weights in the combination by $\{\alpha_k\}_k$. For all $k$, define the mechanism $\varphi_k$ for all $\theta$ by $\varphi_k(\theta) = \sum_m \hat{\sigma}_k(m|\theta)g(m)$. Since $g$ and $\hat{\sigma}_k$ are deterministic, it follows that $\varphi_k$ is deterministic. Moreover $\varphi_k$ is

DIC since for all $i$ we have that $g_i$ is constant in $i$'s message. Since for all $\theta$ we have $\varphi(\theta) = \sum_m \sigma(m|\theta)g(m) = \sum_m \sum_k \alpha_k \hat{\sigma}_k(m|\theta)g(m) = \sum_k \alpha_k \varphi_k(\theta)$, we conclude that $\varphi$ is a convex combination of deterministic DIC mechanisms. Since $\varphi$ is stochastic, it follows that $\varphi$ is not an extreme point of the set of DIC mechanisms. $\square$

**Remark 9.** The variant of the Revelation Principle of Jarman and Meisner (2017) implies that, in the model of the present paper, deterministic DIC direct mechanisms are without loss for optimality among DIC direct mechansims that can be implemented via indirect mechanisms with a deterministic outcome function. Lemma C.2 slightly strengthens this point.

## C.3 Example: Theorem 6.1 in a special case

**Example 2.** Let $n = 3$. Let $\varphi$ be a mechanism where one of the agents is randomly selected as a juror to allocate the object deterministically. When $j$ is the juror, let $f_i^j(\theta_j) \in \{0, 1\}$ denote the probability that $j$ allocates to candidate $i$ at type $\theta_j$. An agent $i$'s winning probability at a profile $\theta$ is thus $\varphi_i(\theta) = \frac{1}{3} \sum_{j:\, j \neq i} f_i^j(\theta_j)$.

Towards a contradiction, suppose the probabilities $\{f_i^j\}_{i,j:\, j\neq i}$ are such that $\varphi$ is anonymous and non-constant. Hence there are distinct agents $i$ and $j$ and distinct types $t'$ and $t$ such that $f_i^j(t') - f_i^j(t) = 1$. Fixing some arbitrary type $t''$, consider the profile where the types of $i$, $j$, and $k$, respectively, are $t$, $t'$, and $t''$. Now, by anonymity, a permutation of $i$'s and $j$'s report does not affect the winning probability of the third agent $k$. Since the object is always allocated, a permutation also does not affect the probability that $i$ or $j$ win. Hence

$$\frac{1}{3}\left(f_i^j(t') + f_i^k(t'') + f_j^i(t) + f_j^k(t'')\right) = \frac{1}{3}\left(f_i^j(t) + f_i^k(t'') + f_j^i(t') + f_j^k(t'')\right).$$

This equation further implies

$$1 = f_i^j(t') - f_i^j(t) = f_j^i(t') - f_j^i(t). \tag{C.1}$$

A similar argument shows

$$\begin{aligned} f_k^j(t') - f_k^j(t) &= f_j^k(t') - f_j^k(t), \qquad \text{and} \\ f_i^k(t') - f_i^k(t) &= f_k^i(t') - f_k^i(t). \end{aligned} \tag{C.2}$$

Now, the equality (C.1) implies $f_i^j(t') = 1$ and $f_i^j(t) = 0$. Hence $f_k^j(t') = 0$. Since $f^j$ must allocate to either $i$ or $k$, we also have $f_k^j(t) = 1$. Via a similar argument, we infer from (C.1) that $f_k^i(t') = 0$ and $f_k^i(t) = 1$ hold. In particular, we infer $-1 = f_k^j(t') - f_k^j(t) = f_k^i(t') - f_k^i(t)$. Hence (C.2) implies $-1 = f_j^k(t') - f_j^k(t) = f_i^k(t') - f_i^k(t)$. Hence $f_j^k(t) = f_i^k(t) = 1$, meaning that $i$ and $j$ both win when $k$ is the juror and reports $t$; contradiction.

## C.4 Balanced coalitions and anonymous juries

In this part of the appendix, we extend Theorem 6.2 to stochastic mechanisms under an additional axiom that requires immunity against certain coalitional manipulations, and a strengthening of partial anonymity.

### C.4.1 Balanced coalitions

As in the main text, given a mechanism $\varphi$, we say agent $i$ *influences* agent $\ell$ if $\varphi_\ell$ is non-constant in $i$'s report

**Definition 5.** Let $\varphi$ be a mechanism.

A subset $J$ of agents is *balanced* if it satisfies the following: An agent is in $J$ if and only if the agent influences all other agents in $J$.

The mechanism is *immune against balanced coalitions* if for all non-empty balanced subsets $J$ of agents the probability $\sum_{i \in J} \varphi_i$ that the object is allocated to an agent in $J$ is constant in the reports of agents in $J$.

All jury mechanisms are immune against balanced coalitions. Indeed, in a jury mechanism all balanced subsets contain at most one agent, and hence immunity follows from DIC.

Balancedness captures the idea that some agents may find it easier to cooperate than others. For instance, suppose that, in some given mechanism, agent $j$ cannot influence agent $i$. Arguably, agent $i$ now has little to fear when *not* entering a coalition with agent $j$. After all, if the others report truthfully, agent $j$ has no way of punishing agent $i$ within the mechanism. Thus there is a sense in which coalitions where some members cannot influence other members are less plausible than balanced coalitions. Demanding immunity with respect to a larger set of coalitions would lead to a more restrictive notion of coalition-proofness. Indeed, the only mechanisms immune against *all* coalitions are constant ones.[20]

### C.4.2 Strong partial anonymity

We now turn to a strengthening of partial anonymity. Recall the following definitions from the main text for a given mechanism $\varphi$ and distinct agents $i$, $j$ and $k$: Agents $i$ and $j$ are *exchangeable for $k$* if $\varphi_k$ is invariant with respect to all permutations of $i$'s and $j$'s reports. Agent $i$ *influences agent $k$* if $\varphi_k$ is non-constant in $i$'s report.

**Definition 6.** Let $\varphi$ be a mechanism.

(1) Let $i$ and $j$ be distinct. Agents $i$ and $j$ *influence a common agent* if there exists $\ell$ such that $i$ and $j$ both influence $\ell$.

(2) The mechanism $\varphi$ is *strongly partially anonymous* if for all $i$, $j$ and $k$ that are all distinct, if $i$ and $j$ influence a common agent, then $i$ and $j$ are exchangeable for $k$.

This notion strengthens partial anonymity from the main text (Definition 2). Recall that partial anonymity demands that $i$ and $j$ be exchangeable for $k$ if $i$ and $j$ both influence $k$. Strong partial anonymity also demands that $i$ and $j$ be exchangeable for $k$ as soon as $i$ and $j$ both influence *some* agent $\ell$, where $\ell$ may differ from $k$. We observe, however, that strong partial anonymity does not imply that all agents influence all others. It may be the case that agents $i$ and $j$ both influence $\ell$ and both fail to influence agent $k$. The content of the axiom is that if $i$ and $j$ influence a common agent, then one of $i$ and $j$ influences $k$ if and only if both of them influence $k$ (and, if they influence $k$, then they are also exchangeable for $k$). We formalize this fact in the proof of the upcoming result (see Claim C.4).

An implication of Theorem 6.2 is that partial anonymity and strong partial anonymity are equivalent for deterministic DIC mechanisms. They are not equivalent for stochastic DIC mechanisms, as the following example shows.

**Example 3.** There are fives agents with common type space $\{0, 1\}$. Let $\varphi_1$ be the jury mechanism that allocates to agent 3 if $\theta_1 = 0$, and else allocates to agent 5. Let

[20] To briefly verify this, fix a mechanism $\varphi$ that is immune against all coalitions. Consider two arbitrary agents $i$ and $j$. A change in $i$'s report affects neither $\varphi_i + \varphi_j$ (by immunity against the coalition $\{i, j\}$) nor $\varphi_i$ (by immunity against $\{i\}$). Hence $\varphi_j$ is also constant in $i$'s report. Hence $\varphi$ is constant.

$\varphi_2$ be the mechanism that allocates to agent 4 if $\theta_2 = 0$, and else allocates to agent 5. The convex combination $\frac{\varphi_1+\varphi_2}{2}$ is DIC and partially anonymous. However, it is not strongly partially anonymous: Agents 1 and 2 influence a common agent (agent 5), but they are not exchangeable for agent 3 nor agent 4.

### C.4.3 Characterization

Let $\Phi^*$ denote the set of mechanisms that are DIC, strongly partially anonymous, and immune against balanced coalitions. The next result characterizes the extreme points of $\Phi^*$. One may verify that $\Phi^*$ is compact (as a subset of Euclidean space), and hence $\Phi^*$ is contained in the convex hull of its extreme points.[21]

**Theorem C.3.** *A mechanism is an extreme point of $\Phi^*$ if and only if it is a deterministic jury mechanism with an anonymous jury.*

In words, all mechanisms in $\Phi^*$ (or randomizations thereof) can be implemented by randomizing over deterministic jury mechanisms with anonymous juries. Hence the extreme points of $\Phi^*$ are exactly the mechanisms characterized by Theorem 6.2.

Let us give a brief sketch of the proof. Strong partial anonymity implies a partition of the agents into equivalence class, where two agents are equivalent if and only if they influence the same set of others. These equivalence classes are the candidates for the juries of the jury mechanisms that we eventually define. The main step for the remainder of the argument is then to show that agents within one equivalence class cannot influence one another (for else these would not be well-defined juries). This is the only step where we appeal to immunity against balanced coalitions, and where the impossibility result Theorem 6.1 turns out to be useful.

**Remark 10.** We showed in Proposition 6.3 that there are partially anonymous DIC mechanisms that fail to be convex combinations of jury mechanisms. A slight modification of the mechanism constructed in this proposition shows that in Theorem C.3 one cannot relax strong partially anonymity to partial anonymity (while keeping immunity against balanced coalitions), and that one cannot drop immunity (while keeping strong partial anonymity).

[21] The set $\Phi^*$ is not generally convex. Example 3 actually shows this. The mechanisms $\varphi_1$ and $\varphi_2$ are both in $\Phi^*$, but the convex combination $\frac{\varphi_1+\varphi_2}{2}$ fails to be strongly partially anonymous.

### C.4.4 Proof of Theorem C.3

*Proof of Theorem C.3.* Let $\varphi$ be a deterministic jury mechanism with an anonymous jury. All jury mechanisms are immune against balanced coalitions. Thus $\varphi$ is in $\Phi^*$. Since $\varphi$ is determinstic, it is an extreme point of $\Phi^*$.

Now let $\varphi \in \Phi^*$. We complete the proof by finding a convex combination of deterministic jury mechanisms with anonymous juries that equals $\varphi$.

For all agents $i$, let $I_i$ denote the set of agents that influence $i$. Let $A_i$ denote the set of agents that are influence by $i$. Let $I = \cup_{i=1}^n I_i$ denote the set of agents with respect to whose reports $\varphi$ is non-constant. We may assume that $\varphi$ is non-constant, meaning $I \neq \emptyset$, as otherwise the proof is trivial.

We proceed along a series of claims. The first of these is essentially a restatement of strong partial anonymity.

**Claim C.4.** *All agents $i$ and $j$ satisfy the following.*

*(1) The allocation $\varphi_i$ is invariant with respect to all permutations of $I_i$.*
*(2) If $(A_i \setminus \{j\}) \cap (A_j \setminus \{i\}) \neq \emptyset$, then $A_i \setminus \{j\} = A_j \setminus \{i\}$.*

*Proof of Claim C.4.* Consider (1). The agents in $I_i$ influence a common agents, namely $i$. Hence $\varphi_i$ is invariant with respect to pairwise permutations of $I_i$. Hence $\varphi_i$ is invariant with respect to all permutations of $I_i$.

Consider (2). If $(A_i \setminus \{j\}) \cap (A_j \setminus \{i\}) \neq \emptyset$, then $i$ and $j$ influence a common agent. Hence, if $k$ is distinct from $i$ and $j$, then $\varphi_k$ is invariant with respect to permutations of $i$ and $j$. Thus $i$ influences $k$ if and only if $j$ influences $k$. □

Consider the binary relation $\sim$ on $I$ defined as follows: Given $i$ and $j$ in $I$, we have $i \sim j$ if and only if $(A_i \setminus \{j\}) \cap (A_j \setminus \{i\}) \neq \emptyset$. Claim C.4 implies that $i \sim j$ is equivalent to $\emptyset \neq A_i \setminus \{j\} = A_j \setminus \{i\}$.

**Claim C.5.** *Let $i$ and $j$ in $I$ be such that $i \sim j$. If $i \in A_j$, then $j \in A_i$.*

*Proof of Claim C.5.* We will show the contrapositive. Let $j \notin A_i$. We have to show $i \notin A_j$. For this proof, let us write $\varphi_i(t, t', \theta_{-ij})$ and $\varphi_j(t, t', \theta_{-ij})$, respectively, for $i$'s and $j$'s winning probabilites, respectively, when $i$ reports some type $t$, $j$ reports some type $t'$, and the others report some profile $\theta_{-ij}$.

Let $\theta$ be an arbitrary profile. Consider the permutation where exactly the types of $i$ and $j$ are swapped. Claim C.4, DIC, and the definition of $\sim$ imply $A_i \setminus \{i, j\} =$

$A_i \setminus \{j\} = A_j \setminus \{i\} = A_j \setminus \{i,j\}$. Hence $A_i^c \setminus \{i,j\} = A_j^c \setminus \{i,j\}$. By definition of $A_i^c$ and $A_j^c$, the permutation does not affect the allocation of agents in $A_i^c \setminus \{i,j\} = A_j^c \setminus \{i,j\}$. Anonymity implies that the permutation does not affect the allocation of agents in $A_i \setminus \{i,j\} = A_j \setminus \{i,j\}$. Since the object is always allocated, we find

$$\varphi_i(\theta_i, \theta_j, \theta_{-ij}) + \varphi_j(\theta_i, \theta_j, \theta_{-ij}) = \varphi_i(\theta_j, \theta_i, \theta_{-ij}) + \varphi_j(\theta_j, \theta_i, \theta_{-ij}).$$

By DIC and since $j \notin A_i$, we have that $\varphi_j$ depends neither on $i$'s nor $j$'s report. Hence the previous equation is equivalent to

$$\varphi_i(\theta_i, \theta_j, \theta_{-ij}) = \varphi_i(\theta_j, \theta_i, \theta_{-ij}).$$

DIC implies that $\varphi_i(\theta_j, \theta_i, \theta_{-ij})$ is constant in $\theta_j$. Hence the previous display implies that $\varphi_i(\theta_i, \theta_j, \theta_{-ij})$ must be constant in $\theta_j$, too. This shows that $\varphi_i$ is constant in $j$'s report, meaning $i \notin A_j$. □

**Claim C.6.** *The relation $\sim$ is an equivalence relation.*

*Proof of Claim C.6.* It is clear that $\sim$ is symmetric. To see that $\sim$ is reflexive, note that $i \in I$ is equivalent to $A_i \neq \emptyset$. Hence $A_i \cap A_i \neq \emptyset$, implying $i \sim i$.

Turning to transitivity, let $i$, $j$ and $k$ be agents in $I$ such that $(A_i \setminus \{j\}) \cap (A_j \setminus \{i\}) \neq \emptyset$ and $(A_j \setminus \{k\}) \cap (A_k \setminus \{j\}) \neq \emptyset$. Claim C.4 implies $A_i \setminus \{j\} = A_j \setminus \{i\}$ and $A_j \setminus \{k\} = A_k \setminus \{j\}$. We distinguish two cases.

First, suppose there exists $\ell \in A_j \setminus \{i,k\}$. Then $j \notin A_j$ and $A_i \setminus \{j\} = A_j \setminus \{i\}$ imply $\ell \in A_i \setminus \{k\}$. Similarly, we have $\ell \in A_k \setminus \{i\}$. Thus $i \sim k$.

Second, suppose $A_j \setminus \{i,k\} = \emptyset$. Since $A_j \setminus \{i\}$ and $A_j \setminus \{k\}$ are both non-empty, we have $A_j = \{i,k\}$. Claim C.5 implies $j \in A_i$ and $j \in A_k$. In view of DIC, this implies $i \neq j \neq k$. In particular, we have $j \in A_i \setminus \{k\} \cap A_k \setminus \{i\}$. Thus $i \sim k$. □

Claim C.6 implies that we may partition $I$ into finitely-many non-empty $\sim$-equivalence classes. (Recall that $I$ is non-empty.) Let $\mathcal{J}$ denote the collection of $\sim$-equivalence classes.

**Claim C.7.** *Let $J$ and $J'$ be distinct sets in $\mathcal{J}$. Let $i$ and $j$ be in $J$. If $j \in A_i$, then both of the following are true:*

*(1) For all distinct $\ell$ and $k$ in $J$ we have $\ell \in A_k$; that is, all agents in $J$ influence all others in $J$.*

(2) *If $k$ is in $J'$, then $J \cap A_k = \emptyset$; that is, no agent outside of $J$ influences an agent in $J$.*

*Proof of Claim C.7.* Consider (1). If $\ell = j$, then the claim follows from $j \in A_i$ and $i \sim k$. If $\ell \neq j$, then $\ell \sim i$ and $j \in A_i$ imply $j \in A_\ell$. Hence, by Claim C.5, we have $\ell \in A_j$. Now $j \sim k$ and $\ell \neq k$ imply $\ell \in A_k$.

Consider (2). Towards a contradiction, suppose there exists $k$ in $J'$ such that $\ell \in J \cap A_k$. We will show that $k \in J$; this contradicts the fact that $J$ and $J'$ are disjoint.

Note that $j \in A_i$ and $\ell \in A_k$ imply $i \neq j$ and $k \neq \ell$ (else DIC is contradicted). Suppose for a moment $\ell = j$. This implies $j \in (A_k \setminus \{i\}) \cap (A_i \setminus \{k\})$, and hence $i \sim k$, and hence $k \in J$. In what follows, we may thus assume $\ell \neq j$.

As an intermediate step, we claim $j \in A_\ell$. Since $i \sim \ell$, we have $A_i \setminus \{\ell\} = A_\ell \setminus \{i\}$. Using $j \in A_i$ and $\ell \neq j$, we infer $j \in A_i \setminus \{\ell\}$, and hence $j \in A_\ell$.

Claim C.5 and $j \in A_\ell$ together imply $\ell \in A_j$. Altogether, we now know that $\ell \in A_j \cap A_k$. We also have $j \neq \ell$ (by assumption) and $k \neq \ell$ (by $\ell \in A_k$ and DIC). Thus $\ell \in A_j \setminus \{k\}$ and $\ell \in A_k \setminus \{j\}$ hold. In particular, we find $j \sim k$, and hence $k \in J$. □

**Claim C.8.** *Let $J \in \mathcal{J}$. The allocation $(\varphi_i)_{i \in J}$ is constant in the reports of agents in $J$.*

*Proof of Claim C.8.* Towards a contradiction, suppose $(\varphi_i)_{i \in J}$ is non-constant in the reports of agents in $J$. Part (1) of Claim C.7 implies that all agents in $J$ can influence all others agents in $J$. Part (2) of Claim C.7 implies that no agent outside of $J$ influences an agent in $J$. Thus $J$ is balanced. Since $\varphi$ is immune against balanced coalitions, therefore, the sum $\sum_{i \in J} \varphi_i$ is constant in the reports of all agents. Let $p$ denote this constant probability. We have $p > 0$ since else the agents in $J$ enjoy a constant winning probability, contradicting the assumption that $(\varphi_i)_{i \in J}$ is non-constant in the reports of agents in $J$. Consider the functions $(\varphi_i/p)_{i \in J}$ obtained by scaling the winning probabilities of agents in $J$ by $1/p$. These functions define a DIC mechanism in a setting where the set of agents is $J$. This mechanism satifies the hypotheses of Theorem 6.1 since $\varphi$ is DIC and strongly partially anonymous, and since, as observed above, all agents in $J$ influence all other agents in $J$. Thus $(\varphi_i/p)_{i \in J}$ is constant; contradiction. □

Given $J$ in $\mathcal{J}$ and $i$ in $J$, Claims C.4 and C.8 imply that the set $A_i$ is disjoint from $J$ and the same across all $i \in J$. Let us denote this common set by $A_J$. Let $A_\emptyset$ denote the (possibly empty) set of agents $i$ such that $\varphi_i$ is constant in the reports of all agents. The following fact is immediate

**Claim C.9.** *The collection $\{A_J\}_{J \in \mathcal{J} \cup \{\emptyset\}}$ partitions the set of agents.*

Let $J$ be in $\mathcal{J}$. If an agent $i$ is in $A_J$, then Claim C.9 implies that $i$'s winning probability only depends on the reports of agents in $J$. Thus in what follows we write $\varphi_i(\theta_J)$ for $i$'s winning probability. By definition of $A_J$, the sum $1 - \sum_{i \in A_J} \varphi_i = \sum_{i \notin A_J} \varphi_i$ is constant in the reports of $J$. Thus $\sum_{i \in A_J} \varphi_i$ is constant in the reports of $J$, too. Claim C.9 implies $A_J \cap A_{J'} = \emptyset$ whenever $J'$ is distinct from $J$. Hence $\sum_{i \in A_J} \varphi_i$ is constant in the reports of agents outside of $J$. Thus $\sum_{i \in A_J} \varphi_i$ must be constant in all reports. We denote the constant probability by $\alpha_J$.

The probability $\sum_{i \in A_\emptyset} \varphi_i$ is constant by the definition of $A_\emptyset$. We denote the constant value by $\alpha_\emptyset$. Since $\{A_J\}_{J \in \mathcal{J} \cup \{\emptyset\}}$ partitions the set of agents (Claim C.9), we have $\sum_{J \in \mathcal{J} \cup \{\emptyset\}} \alpha_J = 1$.

We next define an auxiliary collection of jury mechanisms.

For all $J$ in $\mathcal{J} \cup \{\emptyset\}$ such that $\alpha_J > 0$, let $\psi_J$ denote the following mechanism: For all $i$ in $A_J$, agent $i$ is allocated the object with probability $\varphi_i(\theta_J)/\alpha_J$. For all $i$ not in $A_J$, agent $i$ is allocated the object with probability 0. For all $J$ in $\mathcal{J} \cup \{\emptyset\}$ such that $\alpha_J = 0$, let $\psi_J$ be an arbitrary constant mechanism.

**Claim C.10.** *The collection $\{\psi_J\}_{J \in \mathcal{J} \cup \{\emptyset\}}$ is a collection of jury mechanisms with anonymous juries satisfying $\sum_{J \in \mathcal{J} \cup \{\emptyset\}} \alpha_J \psi_J = \varphi$.*

Note that the jury mechanisms $\{\psi_J\}_{J \in \mathcal{J} \cup \{\emptyset\}}$ have not been proven to be deterministic.

*Proof of Claim C.10.* For all $J$, the mechanism $\psi_J$ is a well-defined mechanism since $\alpha_J$ is the constant probability that the object is allocated to an agent in $J$. Anonymity of the jury is inherited from strong partial anonymity of $\varphi$. If $i \in A_J$, then $i \notin J$ (recall the discussion preceding Claim C.9) and $i$'s' winning probability depends only on the reports of agents in $J$. This argument shows that $\psi_J$ is a jury mechanism and that $\sum_{J \in \mathcal{J} \cup \{\emptyset\}} \alpha_J \psi_J = \varphi$ holds. $\square$

The following claim is easily verified using the Krein-Milman theorem.

**Claim C.11.** *For all $J \in \mathcal{J} \cup \{\emptyset\}$, the mechanism $\psi_J$ is a convex combination of deterministic jury mechanisms with anonymous juries.*

*Proof of Claim C.11.* Let $J \in \mathcal{J} \cup \{\emptyset\}$. Let $\Psi$ be the set of jury mechanisms having all following properties: For all $i$, the mechanism is non-constant in the report of agent $i$ only if $i$ is in $J$; the mechanism is invariant with respect to permutations of agents in $J$. The set $\Psi$ is compact and convex, and it contains $\psi_J$. Hence the claim follows from the Krein-Milman theorem if we can show that all extreme points are deterministic. To that end, consider a stochastic mechanism in $\Psi$. At a profile where the mechanism randomizes, shift a small mass between two agents with interior winning probabilities; do the same at all profile obtained by permuting the reports of agents in $J$. Using that the mechanism is in $\Psi$, it easy to see that this yields two other mechanisms in $\Psi$, the convex hull of which contains the given stochastic one. $\square$

Claims C.10 and C.11, together with the equation $\sum_{J \in \mathcal{J} \cup \{\emptyset\}} \alpha_J = 1$ imply that $\varphi$ is a convex combination of deterministic jury mechanisms with anonymous juries. $\square$

## C.5 Total unimodularity

This section of the appendix discusses another potential approach for showing that all extreme points are deterministic. Our aim is to give a brief explanation for why this approach, that is based on total unimodularity, does not actually help us for the proof of Theorem 5.1 in the difficult case with three agents.

For a function $\varphi\colon \Theta \to [0,1]^n$ to be a DIC mechanism, it should satisfy the following:

$$\begin{aligned} \forall_{i,\theta}, \quad & 1 \geq \varphi_i(\theta) \\ \forall_{i,\theta_i,\theta_i',\theta_{-i}}, \quad & 0 \geq \varphi_i(\theta_i,\theta_{-i}) - \varphi_i(\theta_i',\theta_{-i}) \geq 0 \\ \forall_{\theta}, \quad & 1 \geq \sum_i \varphi_i(\theta) \geq 1 \end{aligned} \tag{C.3}$$

For a suitable matrix $A$ and vector $b$, the set of DIC mechanisms is then the polytope $\{\varphi\colon A\varphi \geq b, \varphi \geq 0\}$. Here, the matrix $A$ has one row for every constraint in (C.3) (after splitting the constraints into one-sided inequalities). Each column of $A$ identifies a pair of the form $(i,\theta)$.

A matrix or a vector is *integral* if its entries are all in $\mathbb{Z}$. A polytope is *integral* if all its extreme points are integral. In this language, all extreme points of the set of DIC mechanisms are deterministic if and only if the polytope $\{\varphi\colon A\varphi \geq b, \varphi \geq 0\}$ is integral.

Recall that a matrix is *totally unimodular* if all its square submatrices have a determinant equal to $-1$, 0, or 1. For later reference, notice that a submatrix of a totally unimodular matrix is itself totally unimodular.

Our interest in total unimodularity is due the Hoffman-Kruskal theorem; see Korte and Vygen (2018, Theorem 5.21).

**Theorem C.12.** *An integral matrix $A$ is totally unimodular if and only if for all integral vectors $b$ all extreme points of the set $\{\varphi\colon A\varphi \geq b, \varphi \geq 0\}$ are integral.*

Thus a sufficient condition for all extreme points of the set of DIC mechanisms to be deterministic is that the constraint matrix $A$ be totally unimodular. Unfortunately:

**Lemma C.13.** *For all agents $i$, let $|\Theta_i| \geq 2$. Let $n = 3$. If there exists $i$ such that $|\Theta_i| \geq 3$, then $A$ is not totally unimodular.*

This explains why our approach to integrality in the case with three agents is not based on total unimodularity.[22]

*Proof of Lemma C.13.* Towards a contradiction, suppose $A$ is totally unimodular. Let us consider the constraint matrix $\tilde{A}$ and vector $\tilde{b}$ that define the set of DIC mechanisms with disposal (as defined in Appendix B). That is, $\varphi$ is a DIC mechanism with disposal if and only if $\tilde{A}\varphi \geq \tilde{b}$ and $\varphi \geq 0$. Notice that $\tilde{A}$ is obtained from $A$ by dropping all rows corresponding to constraints of the form $\sum_i \varphi_i(\theta) \geq 1$; the vector $\tilde{b}$ is obtained from $b$ by dropping the corresponding entries. In particular, the matrix $\tilde{A}$ is a submatrix of $A$. Hence, since $A$ is totally unimodular, we conclude that $\tilde{A}$ is totally unimodular. We therefore infer from Theorem C.12 that all extreme points of the set $\{\varphi\colon \tilde{A}\varphi \geq \tilde{b}, \varphi \geq 0\}$ are integral. That is, all extreme points of the set of DIC mechanism with disposal are deterministic. Since $n = 3$, all agents have at least

[22]Note that total unimodularity of $A$ is sufficient, but not necessary, for the polytope to be integral when $b$ is held fixed. Therefore, the fact that $A$ is not always totally unimodular when $n = 3$ does not imply a contradiction to the fact that, according to Theorem 5.1, all extreme points are deterministic when $n = 3$.

binary types, and at least one agent has non-binary types, we have a contradiction to Theorem B.1. $\square$

We can also give an alternative proof of Lemma C.13 that does not require Theorem B.1. Consider the following characterization of total unimodularity due to Ghouila-Houri (1962); see Korte and Vygen (2018, Theorem 5.25).

**Theorem C.14.** *A matrix $A$ with entries in $\{-1,0,1\}$ is totally unimodular if and only if all subsets $C$ of columns of $A$ satisfy the following: There exists a partition of $C$ into subsets $C^+$ and $C^-$ such that for all rows $r$ of $A$ we have*

$$\left(\sum_{c\in C^+} A(r,c) - \sum_{c\in C^-} A(r,c)\right) \in \{-1,0,1\}. \tag{C.4}$$

*Alternative proof of Lemma C.13.* Let us, once again, relabel the agents and types such that the type spaces contain the following subsets:

$$\tilde{\Theta}_1 = \{\ell, r\} \quad \text{and} \quad \tilde{\Theta}_2 = \{u, d\} \quad \text{and} \quad \tilde{\Theta}_3 = \{f, c, b\}$$

Fixing an arbitrary type profile $\theta_{-123}$ of agents other than 1, 2, and 3, let us define the type profiles $\{\theta^a, \theta^b, \theta^c, \theta^e, \theta^f, \theta^g\}$ as in (5.2) in Section 5.2. That is, let

$$\begin{aligned}
\theta^a = (\ell, d, c, \theta_{-123}), \quad \theta^b &= (r, d, c, \theta_{-123}), \quad \theta^c = (r, d, b, \theta_{-123}),\\
\theta^d &= (r, u, b, \theta_{-123}), \quad \theta^e = (r, u, f, \theta_{-123}),\\
\theta^f &= (\ell, u, f, \theta_{-123}), \quad \theta^g = (\ell, u, c, \theta_{-123}).
\end{aligned}$$

Recall that each column of $A$ corresponds to an entry of the form $(i,\theta)$. We will argue that the following set $C$ of columns does not admit a partition in the sense of Theorem C.14.

$$\begin{aligned}
C = \{&(1,\theta^a), (1,\theta^b), (3,\theta^b), (3,\theta^c),\\
&(2,\theta^c), (2,\theta^d), (3,\theta^d), (3,\theta^e),\\
&(1,\theta^e), (1,\theta^f), (3,\theta^f), (3,\theta^g),\\
&(2,\theta^g), (2,\theta^a)\}
\end{aligned}$$

Towards a contradiction, suppose $C$ does admit a partition into sets $C^+$ and $C^-$

in the sense of Theorem C.14. Let us assume $(1, \theta^a) \in C^+$, the other case being similar. Note that $\theta^a$ and $\theta^b$ differ only in the type of agent 1. Consider the row of $A$ corresponding to the DIC constraint for agent 1 at these type profiles. By referring to (C.4) for this row, we deduce $(1, \theta^b) \in C^+$. Next, via a similar argument, the constraint that the object is allocated at $\theta^b$ requires $(3, \theta^b) \in C^-$. Continuing in this manner, it is easy to see that $(1, \theta^a)$ must be in $C^-$. Since $(1, \theta^a)$ is assumed to be in $C^+$, we have a contradiction to the assumption that $C^+$ and $C^-$ are a partition of $C$. □

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