Simple Allocation with Correlated Types*

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Abstract

A principal allocates a single indivisible object to one of n agents who all want it. The agents have private information that is valuable to the principal. In particular, each agent may have information about the principal's payoff from allocating to the others. Without using monetary transfers, the principal designs a dominant-strategy IC mechanism. Our main results make a case for *jury mechanisms*. Such a mechanism splits the agents into a set of jurors and a set of candidates. The jury decides which of the candidates wins the object. Jury mechanisms are optimal within a restricted set of mechanisms, and approximately optimal in symmetric environments with many agents. In general, exactly-optimal DIC mechanisms may require the principal to commit to random allocations.

Keywords: Allocation, Correlation, Mechanism design without transfers, Peer selection

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1. Introduction

1.1. Motivation

The problem of allocating a scarce but desirable resource is a basic problem in economics. In this paper, we study the following instance of the problem—*the simple allocation problem*. There is an object that each member of a group of agents would like to have. A principal chooses who gets it, and the agents have private information that is relevant for the principal's payoff from this choice. To elicit this information, the principal designs a mechanism that allocates the object as a function of the agents' reported information. The allocation is the only instrument at the principal's disposal—monetary transfers are unavailable.

The simple allocation problem is the subject of a recent growing literature in mechanism design, reviewed later.[1] If the principal could set prices, the principal could design an auction to raise revenue or allocate the object efficiently. Yet, there are many interesting and important applications where money is not available. The absence of money is what distinguishes the simple allocation problem from auctions.

To have some applications in mind, imagine the principal is a benevolent planner who allocates a valuable good as part of an aid program. The planner would like to allocate the good to the household who needs it most. However, the potential recipients are severely financially constrained—the planner cannot have them bid for the good. Or imagine the assignment of a prestigious task, such as serving as the president of a club. Everyone wants to be president. For the club as a whole it would be best to select the most qualified member, and we capture this preference via a metaphorical principal. In this setting, monetary transfers may be excluded on ethical grounds. Or imagine a manager of a firm who splits a budget across a group of risk-neutral divisions. All subdivisions want to be allocated the full budget, but the manager cares about the marginal returns that the divisions can produce. Here, the allocation itself is a transfers of money; the probability of getting the object is interpreted as the allocated share of the budget.

In all of these applications, the principal chooses the allocation, and nothing else. It is also natural that the agents in these applications have some information about one another. Neighbouring households may know more about one another's incomes

[1] The name of the problem is borrowed from Ben-Porath, Dekel, and Lipman (2019), who among other things study a version of the problem with hard evidence.

than the external planner. When selecting the president, each club member has an informed opinion about the qualities of their peers. In the budget example, the revenue generated by the divisions may be correlated through an underlying state of the world, and hence each division may be able to predict the performance of others. Our contribution is to study how to optimally elicit the information that agents have about their peers.

In our model, the principal's payoff from allocating to an agent i is given by the realization ω_i of an unobservable state. Each agent has a private type θ_i that is informative about these payoffs. For the most part, our model is quite general in that we impose no structure on the joint distribution of $(\omega_1, \theta_1, \ldots, \omega_n, \theta_n)$. This allows for the payoffs to be correlated across agents, and for each agent's type to be informative about the types and payoffs of others. The principal designs a mechanism for which it is a dominant strategy to truthfully report one's type, i.e. a mechanism which is dominant-strategy incentive-compatible (DIC). Monetary transfers are unavailable; the mechanism specifies the allocation, and nothing else. Since all agents want the object, DIC means that agents must be unable to influence their individual allocation with their reports. Nevertheless, agents are willing to share what they know about one another as long as it does not affect their individual chances of winning.

Aside from understanding applications in which correlation appears natural, allowing for agents to be informed about one another is of inherent theoretical importance. If agents have no information about one another's payoffs, then the principal would not be able to benefit from consulting the agent (we make this point precise later). By contrast, when agents' private information is correlated, we find sophisticated DIC mechanisms that elicit information. Thus it would be amiss to conclude from the analysis with independent private information that meaningful mechanism design without money is infeasible. This loosely mirrors the classical insight from mechanism design theory *with* money that the principal can be much better off when agents' private information is correlated rather than independent (Crémer and McLean, 1985, 1988; McAfee and Reny, 1992). That said, the channel through which correlation benefits the principal in our model is quite different from the one the principal exploits when monetary transfers are available. With money, the principal can construct side bets that screen the agents along their beliefs about the types of the others. This construction is infeasible in our setting. Our principal benefits from correlation directly via the fact that an agent can share information about the principal's payoffs from allocating to one of other agents.

1.2. Results

What do optimal DIC mechanisms look like? We give two partial answers to this question. First, we show that randomization is a pervasive feature of optimal DIC mechanisms. Second, we identify a simple and interpretable class of DIC mechanisms—*jury mechanisms*—that perform well in important economic settings. Let us elaborate.

Our first main result fully characterizes when it suffices to consider deterministic DIC mechanisms. We show that, outside of special cases, there are stochastic DIC mechanisms that cannot be implemented by randomizing over deterministic ones. Moreover, stochastic DIC mechanisms cannot be ignored as candidates for optimality.

The analysis that establishes these results reveals much about the basic economic forces of the model. In the main text, we illustrate these in detail in a simple example with 4 agents where the optimal DIC mechanism, which we spell out explicitly, is unique and stochastic. Let us give a brief intuition. The heart of the principal's problem is that each agent is simultaneously a voter and a candidate for winnining the object. When the principal allocates to an agent i at some type profile, then DIC requires that the principal also allocate to i at all type profiles obtained by a unilateral change of i's type. If types are correlated, agent i's type contains information about the principal's payoffs from allocating to one of the others. In a deterministic mechanism, however, allocating to agent i at the original profile prevents the principal from using this information. Thus there is a tension between allocating to agent i (to enjoy the direct payoff of agent i) and using i's information (to allocate efficiently to the others). By contrast, suppose the principal commits to a random allocation that only allocates to agent i with, say, probability $1/2$ at the two profiles. The principal can then let agent i's type inform how to split the remaining probability mass of $1/2$ among the other agents.

The need to randomize the allocation is problematic as it leads to mechanisms that are difficult to implement and interpret. One concern is that stochastic mechanisms require strong commitment assumptions on the part of the principal, as other authors have discussed in other contexts (Budish, Che, Kojima, and Milgrom, 2013; Chen, He, Li, and Sun, 2019; Pycia and Ünver, 2015). We illustrate these commitment issues in the aforementioned example where the unique optimal DIC mechanism is stochastic. The commitment issues can be side-stepped if we restrict attention to deterministic mechanisms. Unfortunately, deterministic mechanisms turn out to be no less complex to characterize or interpret than stochastic ones.

The previous findings motivate us to search for a simpler class of DIC mechanisms that, although not generally optimal, perform well in important settings. Our second contribution is to propose *jury mechanisms* as such a class of mechanisms. A jury mechanism partitions the agents into a set of *jurors* and a set of *candidates*. The allocation only depends on the reports of jurors, and the object is always allocated to one of the candidates. Jury mechanisms represent a particularly simple way of resolving the conflict between allocating to an agent and using that agent's information. Loosely speaking, instead of resolving the conflict for each type profile individually, the principal makes a choice *before* consulting the agents. Jury mechanisms also capture a strong sense of impartiality—the agents who influence the allocation with their reports have no stake in it.

Jury mechanisms are reminiscent of other mechanisms that have appeared in the literature, such as the 2-partition mechanisms of Alon, Fischer, Procaccia, and Tennenholtz (2011), or the partition methods of Holzman and Moulin (2013). The contribution of our remaining results is to make a normative and positive case for jury mechanisms in our model.

Firstly, when there are three agents, it turns out that jury mechanisms are exactly the extreme points of the set of DIC mechanisms—and therefore optimal.[2] With three agents, jury mechanisms are particularly simple to implement: The principal nominates a single juror, and the juror points out which of the other two should optimally be allocated the object conditional on the juror's information.

Secondly, we consider constraints on the set of allowed DIC mechanisms. Two natural properties are that the mechanism handle the reports of the agents anonymously, and that coalitions of agents be unable to manipulate the allocation in their favor. We show that jury mechanisms fully characterize and are optimal within this constrained set of mechanisms for two appropriate notions of anonymity and coalition-proofness. Along the way, we uncover an impossibility result that is of independent interest and that obtains when the two constraints are interpreted too stringently.

Thirdly, we identify an assumption on the distribution of types and payoffs such that jury mechanisms are approximately optimal (among all DIC mechanisms) when

[2]The case with three agents is one of the aforementioned special cases where deterministic DIC mechanisms do suffice.

the number of agents is large. The content of the assumption is that, for all agents i, all agents other than i are exchangeable in terms of supplying information about the payoff ω_i from allocating to i. (But agent i may still be better informed than others about ω_i.) Intuitively, when there are many exchangeable agents, the aforementioned tension between allocating to an agent and using that agent's information is diminished. The principal loses little when not consulting the agents who are sometimes allocated the object—this is the defining feature of a jury mechanism.

1.3. Related Literature

We see several strands to which our paper relates: The literature on the simple allocation problem, on axiomatic peer selection, on approximately-optimal mechanisms for peer selection, and on stochastic vs. deterministic mechanisms.

The simple allocation problem. Closest to us are the papers of Bloch, Dutta, and Dziubiński (2022), Kattwinkel (2019), Kattwinkel and Knoepfle (2021), and Kattwinkel, Niemeyer, Preusser, and Winter (2022).[3] The focus of these papers is on how the principal benefits from correlated information. Informally speaking, correlation may help through two channels: Agents can provide information about the principal's payoffs from allocating to their peers, and, if a Bayesian IC mechanism is used, the principal can try to screen the agents via their beliefs.

Bloch et al. (2022), Kattwinkel (2019), and Kattwinkel and Knoepfle (2021) consider settings where the principal has a private signal about the types of the agent(s). Bloch et al. (2022) and Kattwinkel (2019) show that the principal can use this signal to screen the agent(s) via their beliefs. For instance, Kattwinkel (2019) shows that the principal may to this end commit to highly inefficient decisions. Since we require dominant-strategy IC, our principal cannot screen the agents via their beliefs. In the single-agent setting of Kattwinkel and Knoepfle (2021), the principal can additionally verify the agent's type at a cost. They find an optimal mechanism that is

[3]Most existing work on the simple allocation problem assumes independent private information and studies how particular non-monetary screening devices or features of the environment help provide incentives. There are no such devices in our model. Examples include promises of future allocations (Guo and Hörner, 2021), delaying the allocation (Condorelli, 2009), costly verification (Ben-Porath, Dekel, and Lipman, 2014; Epitropou and Vohra, 2019; Erlanson and Kleiner, 2019), costly signaling (Chakravarty and Kaplan, 2013; Condorelli, 2012), allocative externalities (Bhaskar and Sadler, 2019; Goldlücke and Tröger, 2020), or ex-post punishments (Li, 2020; Mylovanov and Zapechelnyuk, 2017).

transparent—the principal is not made worse off by revealing the signal to the agent.

Kattwinkel et al. (2022) consider a setting with two agents, correlated types, and an uninformed principal. They find that in this setting Bayesian IC mechanisms cannot screen the agents via their beliefs. Optimal mechanisms instead exploit the fact that the principal's payoffs may be non-additive in the types of the agents. Non-additivity is interpreted as informational spillovers—agent i's type affects the payoff of allocating to another agent j. This is the kind of information our principal extracts, too, if the agents' types are independent but nevertheless informative about the principal's payoffs.[4]

Given that the allocation is the principal's only instrument in the simple allocation problem, characterizing optimal ways of screening the agents via their beliefs is extremly difficult; no general characterization is known. By focusing on DIC mechanisms, we focus on the direct channel through which correlaton helps the principal, and we make progress on the simple allocation problem in cases hitherto unexplored. A further appeal of DIC mechanisms is that they require no assumptions on agents' beliefs and are simple for the agents to play.

Axiomatic peer selection. Holzman and Moulin (2013) study mechanisms where each agent submits the name of one of the others, interpreted as a nomination. The problem is to find reasonable rules for selecting a winner as a function of these nominations. Their central axiom—*impartiality*—demands that agents be unable to influence their individual winning probability. Hence impartial nomination rules correspond to indirect mechanisms with particular messages sets and in which all strategies of all agents are dominant. Our results on anonymous and coalition-proof DIC mechanisms contribute to the literature following Holzman and Moulin (2013),[5] the novelty being that our results apply to mechanisms where the agents have a common abstract space of messages. The analysis of abstract message spaces is important since the space of nominations is not sufficiently rich for agents to convey all private information. Passing from nominations to abstract message spaces also leads to new hurdles and insights, which we discuss in detail in Section 5.6.

[4] He, Sandomirskiy, and Tamuz (2021) consider the problem of designing information structures where the agents' private types are uninformative about oneanother but, simultaneously, informative about some underlying state.

[5] For further contributions to this literature, see Edelman and Por (2021), Mackenzie (2015, 2020), and Tamura and Ohseto (2014). See also De Clippel, Moulin, and Tideman (2008) for earlier work.

Approximately-optimal peer selection. Alon et al. (2011) initiated a literature on approximately-optimal DIC mechanisms for the peer selection problem (there called strategy-proof mechanisms).[6] That is, each agent approves of a subset of the others, and the principal wishes to select the agent with the most approvals. While our setup is general enough to nest this environment, the papers in this literature rank mechanisms according to approximation ratios rather than expected utility.[7,8] This leads to rather different result, and thus we view our results as complementary to this literature. For instance, Alon et al. (2011, Theorem 3.1) show that deterministic mechanisms perform poorly no matter the number of agents; we find that deterministic mechanisms are exactly-optimal when there are three agents (in a more general model).

We note that the complexity of exactly-optimal mechanisms, broadly defined, has been used in other design problems to motivate the study of approximately-optimal ones; for further discussion, see the survey article of Roughgarden and Talgam-Cohen (2019) on approximately-optimal mechanisms. As far as we are aware, we are the first to study exactly-optimal DIC mechanisms for the simple allocation problem with a fully-specified prior and at our level of generality. Our results indeed suggest that such mechanisms are unlikely to be of direct practical use or are too complex to yield a tractable theoretical benchmark. Together with our analysis of jury mechansims, we thus make a case for approximately-optimal mechanisms.

Stochastic vs. deterministic mechanisms. It is well-known that deterministic mechanisms are not generally without loss.[9] Among the papers that shed light on

[6]Further contributions to this literature include Aziz, Lev, Mattei, Rosenschein, and Walsh (2016, 2019), Bjelde, Fischer, and Klimm (2017), Bousquet, Norin, and Vetta (2014), Caragiannis, Christodoulou, and Protopapas (2019), Fischer and Klimm (2015), Lev, Mattei, Turrini, and Zhydkov (2021), and Mattei, Turrini, and Zhydkov (2020). Many of these papers consider the more difficult problem of allocating multiple homogenous objects.

[7]A mechanism has an approximation ratio of α if it is guaranteed to generate a fraction of at least α of some benchmark value. The guarantee is computed over all possible approval sets of the agents, and the benchmark value is the maximal number of nominations received by some agent.

[8]An exception is the paper of Caragiannis, Christodoulou, and Protopapas (2021). They study a particular DIC mechanism, focusing on its performance as the number of agents grows. This mechanism does not have an immediate analogue in our setup with abstract types.

[9]For instance, Strausz (2003) shows that the Revelation Principle may fail if one restricts to deterministic mechanisms. Budish et al. (2013) and Pycia and Ünver (2015) show that deterministic mechanisms are not without loss in certain allocation problems. Optimal mechanisms for selling multiple objects to a single buyer may be random; see Hart and Reny (2015) for simple examples and further references.

the gap between stochastic and deterministic mechanisms, the closest one is that of Chen et al. (2019). They prove an equivalence result for stochastic and deterministic mechanism in a large class of social choice settings. In Section 4.3.1, we comment on how to reconcile their results with ours.

Further related work. Finally, we mention the more distantly related papers of Baumann (2018) and Bloch and Olckers (2020, 2021). They study the problem of allocating to agents who are arranged on a network and have information about their neighbors. The focus of these papers is on questions quite different from ours. For instance, Bloch and Olckers (2020) ask whether the principal can reconstruct the ordinal ranking of agents from their reports in an ex-post IC manner when agents prefer that the principal assign them a high rank.

2. Model

A principal allocates a single indivisible object to one of n agents, where $n \geq 3$. Each agent i has a private type θ_i from a finite set Θ_i. The set of type profiles θ is $\Theta = \times_{i=1}^{n}\Theta$. Agent i gets a payoff of $u_i(\theta)$ if allocated the object at type profile θ, where $u_i \colon \Theta \to \mathbb{R}_{++}$ is some function. Agent i's payoff if not allocated the object is normalized to 0. We assume $u_i(\theta) > 0$, meaning that winning the object is strictly preferred to not winning it.

The principal also enjoys payoffs from the allocation. Specifically, the principal's payoff from allocating to agent i is the realization ω_i of an unobservable state from a finite set Ω_i of reals. Let $\Omega = \times_{i=1}^{n}\Omega_i$. The principal's prior about the state and the agents' types is given by a distribution μ over $\Omega \times \Theta$. (The agents may or may not share this prior.) The marginals of μ on each of the type spaces have full support.

As we allow for arbitrary distributions μ, the type of agent i may be informative about all payoffs and all other types. It will occasionally be useful to consider environments where agent i's type pins down the payoff ω_i from allocating to i. Formally, the principal's payoffs are *privately-known (to the agents)* if for all i there is a function $\hat{\omega}_i \colon \Theta_i \to \mathbb{R}$ such that $\omega_i = \hat{\omega}_i(\theta_i)$ holds with μ-probability 1.[10]

[10] A number of papers in the literature assume privately-known payoffs; see e.g. Ben-Porath et al. (2014) and Mylovanov and Zapechelnyuk (2017). If we interpret the principal as a benevolent planner, the payoff ω_i would represent i's valuation for the object. Privately-known payoffs then correspond to private values from auction theory.

The principal designs a mechanism in which the agents send cheap-talk messages to the principal, possibly over multiple rounds of communication. The principal allocates the object to one of the agents as function of these messages, and this allocation may be randomized. The principal cannot keep the object or dispose of it. Throughout this paper, we focus on social choice functions that are implementable in a dominant-strategy equilibrium. By the Revelation Principle, it therefore suffices to consider direct mechanisms that are dominant-strategy incentive-compatible. We henceforth refer to direct mechanisms simply as mechanisms.

Formally, a *mechanism* φ is a collection $\varphi_1 \colon \Theta \to [0,1], \ldots, \varphi_n \colon \Theta \to [0,1]$ of functions such that all θ in Θ satisfy $\sum_{i=1}^n \varphi_i(\theta) = 1$. Here we interpret $\varphi_i(\theta)$ as the probability that agent i is allocated the object when the reported type profile is θ. Since the object is allocated to one of the agents, these probabilites sum to 1. A mechanism φ is *dominant-strategy incentive-compatible (DIC)* if all agents i, all θ_i and θ_i' in Θ_i, and all θ_{-i} and θ_{-i}' in Θ_{-i} satisfy

$$u_i(\theta_i, \theta_{-i})\varphi_i(\theta_i, \theta_{-i}') \geq u_i(\theta_i, \theta_{-i})\varphi_i(\theta_i', \theta_{-i}').$$

The principal's expected utility from a DIC mechanism φ (when agents tell the truth) is given by $\mathbb{E}_{\omega,\theta}\left[\sum_{i=1}^n \varphi_i(\theta)\omega_i\right]$. The principal's problem consists of finding a DIC mechanism to maximize expected utility.

To wrap up, we comment on the requirement that the principal always allocate the object. This requirement keeps with the peer selection literature. One motivation is that not allocating is Pareto inefficient in the applications we have in mind. In the example from the introduction where a planner distributes a valuable good to financially-constrained households, the planner, by not allocating, would consume it privately or destroy it; in the peer selection example, the club goes on without leadership; in the budget-allocation example, the manager burns money. That said, in Appendix D.1 of the supplementary material we discuss how our results change if the principal can dispose the object. Some (but not all) results are completely unchanged. The reason is that the present setup allows for there to be a "dummy" agent i who has a singleton type space, $|\Theta_i| = 1$. A mechanism with n agents and disposal is really a mechanism with $n+1$ agents where agent $n+1$ is a dummy and the mechanism always allocates.

3. Preliminaries

To begin with, we note that truthtelling incentives take a particularly simple form. Namely, in a DIC mechanism, all agents must be unable to influence their individual winning probabilities with their reports. To see this, recall that DIC for agent i requires $u_i(\theta_i, \theta_{-i})\varphi_i(\theta_i, \theta'_{-i}) \geq u_i(\theta_i, \theta_{-i})\varphi_i(\theta'_i, \theta'_{-i})$. The agent's payoff from winning the object, u_i, is strictly positive at all type profiles. Hence DIC is equivalent to $\varphi_i(\theta_i, \theta'_{-i}) \geq \varphi_i(\theta'_i, \theta'_{-i})$. This inequality must hold for all θ_i and θ'_i. Therefore, by switching the roles of θ_i and θ'_i, we obtain $\varphi_i(\theta_i, \theta'_{-i}) = \varphi_i(\theta'_i, \theta'_{-i})$. For emphasis, we record this simple but important fact as a lemma.

Lemma 3.1. *A mechanism is DIC if and only if for all i the winning probability φ_i of agent i is constant in agent i's report θ_i.*[11,12]

We may therefore drop agent i's report θ_i from i's winning probability, writing $\varphi_i(\theta_{-i})$ instead of $\varphi_i(\theta_i, \theta_{-i})$. At various points in the paper, it will be more convenient include i's report, but no confusion should arise.

What does the fact that agents cannot influence their own winning probability imply for the principal's problem? One implication is that, in very special environments, it is optimal to ignore the reports of the agents; that is, a constant mechanism is optimal. Consider the following.

Proposition 3.2. *If types are independent and payoffs are privately-known, then there is an optimal DIC mechanism that constantly allocates to a single agent.*

Intuitively, for all agents i, in this environment the only source of information about the payoff from allocating to agent i is i's own type. If the principal attempts to use this information to determine i's allocation, agent i would misreport it.

Proof of Proposition 3.2. Let $\hat{\omega}_1, \ldots, \hat{\omega}_n$ denote the functions giving the principal's payoffs. Let φ be an arbitrary DIC mechanism. We can write the principal's utility from φ as follows (the first equality follows from DIC and privately-known payoffs;

[11] The fact that agents are indifferent to between all reports implies that truth-telling is never a unique equilibrium. Hence there is no hope for implementing a given social choice function in all dominant-strategy equilibria unless that social choice function is constant.

[12] Throughout the paper, when we say that a function of multiple arguments is constant in one argument, we mean that it is constant in that argument no matter the value of the other arguments.

the second equality follows from independent types):

$$\begin{aligned}\mathbb{E}_{\omega,\theta}\left[\sum_{i=1}^{n}\varphi_i(\theta)\omega_i\right] =&\mathbb{E}_{\theta}\left[\sum_{i=1}^{n}\varphi_i(\theta_{-i})\hat{\omega}_i(\theta_i)\right]\\ =&\sum_{i=1}^{n}\mathbb{E}_{\theta_{-i}}\left[\varphi_i(\theta_{-i})\right]\mathbb{E}_{\theta_i}\left[\hat{\omega}_i(\theta_i)\right].\end{aligned}$$

We also have $\sum_{i=1}^{n}\mathbb{E}_{\theta_{-i}}[\varphi_i(\theta_{-i})] = \mathbb{E}_{\theta}[\sum_{i=1}^{n}\varphi_i(\theta_{-i})] = 1$ from the fact that the object is always allocated. Hence the principal's utility is no greater than $\max_{i=1}^{n}\mathbb{E}_{\theta_i}[\hat{\omega}_i(\theta_i)]$. The principal can obtain this upper bound in a constant mechanism. $\square$

In the environments of interest, therefore, agents have information about one another. We shall see that constant DIC mechansims will cease being optimal—the principal can elicit information.

4. The set of DIC mechanisms

A natural starting point for the analysis are the extreme points of the set of DIC mechanisms. To see why, note that the set of DIC mechanisms is convex and compact as a subset of Euclidean space. Thus it coincides with the convex hull of its extreme points.[13] Since the principal's utility is linear in the mechanism, there always exists an optimal DIC mechanism that is also an extreme point.

Observe that the set of DIC mechanisms does not depend on the payoffs Ω or the distribution μ of types and payoffs—after all, the agents do not care about the payoffs or beliefs of the principal. We use this observation to clarify our interest in extreme points: All extreme points are candidates for optimal mechanisms.

Lemma 4.1. *Let $n \in \mathbb{N}$. Let $\Theta_1, \dots, \Theta_n$ be finite sets, and let $\Theta = \times_{i=1}^{n}\Theta_i$. If φ is an extreme point of the set of DIC mechanisms when there are n agents and the set of type profiles is Θ, then there exists a set Ω of payoff profiles and a distribution μ over $\Omega \times \Delta$ such that in the environment (n, Ω, Θ, μ) the mechanism φ is the unique optimal DIC mechanism.*

[13] Recall that a point x in a subset X of Euclidean space is an extreme point of X if x cannot be written as a convex combination of two other points in X. According to the Krein-Milman theorem (Aliprantis and Border, 2006, Theorem 7.68), a non-empty, convex, compact subset of Euclidean space coincides with the convex hull of its extreme points.

Proof of Lemma 4.1. Since the set of DIC mechanisms is compact, convex, and finite dimensional, a separating hyperplane theorem (e.g. Theorem 7.31 of Aliprantis and Border (2006)) implies that there is a function $p\colon \{1,\dots,n\}\times\Theta \to \mathbb{R}$ with the following property: For all DIC mechanisms φ' different from φ we have $\sum_{i,\theta} p_i(\theta)(\varphi_i(\theta)-\varphi'_i(\theta)) > 0$. By suitably choosing Ω and μ, the function p represents the principal's objective function. For example, one possible choice of Ω and μ is as follows: Let the marginal of μ on Θ be uniform; for all agents i, let Ω_i be the image of p_i; conditional on an arbitrary profile, let the payoff of allocating to agent i equal $|\Theta|p_i(\theta)$ with probability 1. □

The remainder of this section is divided into three parts. We first show that outside of special cases the set of DIC mechanisms admits stochastic extreme points. We then go on to discuss the implications of these results for the principal's problem. Lastly, we use these findings to motivate a simpler class of DIC mechanisms.

4.1. Stochastic extreme points

A mechanism is *deterministic* if it only assigns probabilities of 0 or 1, i.e. if it maps to a subset of $\{0,1\}^n$. A mechanism is *stochastic* if it is not deterministic. In a nutshell, we find that stochastic extreme points exist outside of special cases.

Theorem 4.2. *All extreme points of the set of DIC mechanisms are deterministic if and only if at least one of the following is true:*

(1) There are at most three agents; that is $n \leq 3$.
(2) All agents have at most two types; that is, for all i we have $|\Theta_i| \leq 2$.
(3) At least $(n-2)$-many agents have a degenerate type; that is, we have

$$|\{i \in \{1,\dots,n\}\colon |\Theta_i| = 1\}| \geq n-2.$$

Theorem 4.2 implies that deterministic DIC mechanisms do not generally suffice for characterizing the full set of DIC mechanisms. Whenever a stochastic extreme point exists, we know from Lemma 4.1 that the rest of the environment—the payoffs Ω and the distribution μ—may be such that all deterministic DIC mechanisms fail to be optimal. Hence deterministic DIC mechanisms do not generally suffice for optimality.

We can say a more about environments in which randomization strictly benefits the principal. Our proof for the existence of a stochastic extreme point in Theorem 4.2

is constructive. For the main step of this construction, which we sketch momentarily, we consider an example with 4 agents. We show that there is an environment with *privately-known payoffs* in which the unique optimal DIC mechanism, which we spell out explictly, is stochastic.

The main insight from this exercise is into the basic economics of the principal's problem.[14] In the introduction, we intuited that the principal has a trade-off between allocating to an agent i and using the information contained in i's type. To support the interpretation that this trade-off drives the need to randomize, recall Proposition 3.2: When payoffs are privately-known and types are independent—and hence uninformative about one another—, then there is a constant optimal mechanism that allocates deterministically.[15]

We explore the above ideas more carefully in the next subsection. In that subsection, we also sketch the proofs for the cases where all extreme points are deterministic.

4.2. Ideas for the proof

4.2.1 The feasibility graph. Our approach to studying the extreme points rests on an auxiliary graph—the *feasibility graph*. The idea is as follows: When the principal promises to allocate to agent i at some profile, DIC requires that the principal also promise to allocate to i at all profiles obtained by a unilateral change of i's type. As there is only one good to allocate, the principal must reconcile these promises across agents and type profiles. Recall that the feasibility constraint that the single object be allocated reads

$$\forall_{\theta \in \Theta}, \quad \sum_{i=1}^{n} \varphi_i(\theta_{-i}) = 1. \tag{1}$$

[14] Analytically, focusing on an environment with privately-known payoffs is non-trivial since the separating-hyperplane argument from Lemma 4.1 does not go through. The reason is that not all functions p can be represented by environments with privately-known payoffs.

[15] In the literature on optimal mechanisms for selling multiple objects to a single buyer, it is well-known that randomization may be necessary for maximizing revenue (whereas deterministic mechanisms are optimal for selling a single object). See Hart and Reny (2015) for simple examples and further references. We loosely share with this literature the idea that multi-dimensionality of the agent's type is related to the need to randomize the allocation. In our model, an agent's type is multi-dimensional in the sense that it influences the principal's payoff from allocating to the agent, and, when types are correlated, also contains information about the types of the others. When types are independent, the second "dimension" is shut down.

Each node of the feasibility graph identifies a probability of the form $\varphi_i(\theta_{-i})$. Two nodes are adjacent if and only if there is a type profile such that the two probabilities both appear in the feasibility constraint (1) at that profile.

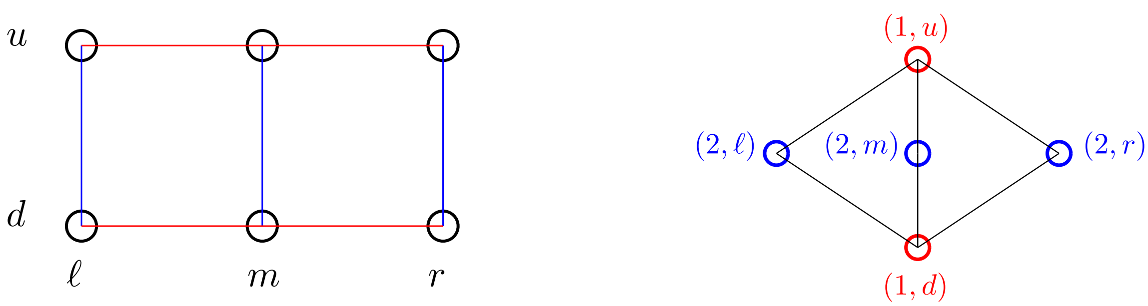
Formally, let the (simple) graph G with nodes V and edges E be defined as follows: Let

$$V = \bigcup_{i=1}^{n} (\{i\} \times \Theta_{-i}),$$

and let two nodes (i, θ_{-i}) and (j, θ'_{-j}) be adjacent if and only if $i \neq j$ and there is a type profile $\hat{\theta}$ satisfying $\hat{\theta}_{-i} = \theta_{-i}$ and $\hat{\theta}_{-j} = \theta'_{-j}$.

Let us say that the profile $\hat{\theta}$ *contains* the nodes $\{(i, \hat{\theta}_{-i})\}_{i=1}^{n}$. Thus two nodes are adjacent if and only if they correspond to two distinct agents and there is a profile that contains both of them.

Figure 1 shows the feasibility graph in an example with two agents; Figure 4 in Appendix A shows it in an example with three agents. Notice that the nodes of G are not found by "replacing" the lines between neighboring type profiles. For example, the line between the profiles (ℓ, d) and (m, d) in Figure 1a is not a node of G.



(a) The set of type profiles Θ. Circles represent type profiles.

(b) The graph G. Circles represent nodes of G.

Figure 1: There are two agents with types $\Theta_1 = \{\ell, m, r\}$ and $\Theta_2 = \{u, d\}$.

Having introduced the feasibility graph, this is a good moment to pause and comment on a graph-theoretic result that provided us with the right conjecture for (but does not imply) Theorem 4.2.

Remark 1 (Disposal and the feasibility graph)**.** When the principal is not required to allocate the object (say, because the principal can dispose of it or keep it), we obtain a result that is analogous to Theorem 4.2, except that the threshold for the number of agents drops from 3 to 2; see Theorem D.1 in Appendix D.1 and the proof in Appendix D.1.3. This proof is based on a graph-theoretic result due to Chvátal (1975). Chvátal's result implies that a stochastic extreme point with disposal exists if and only if the graph G is not perfect. In our proof, we check whether G is perfect. We also explain in said appendix how Theorem 4.2 can be deduced from Theorem D.1, but not vice versa.

4.2.2 Stochastic extreme points with four agents. In this section, we consider environments in which stochastic extreme point exist. To prove Theorem 4.2, we have to show that a stochastic extreme point exists whenever there are at least 4 agents, at least 3 of these have more than 2 types, and 1 of these has more than 3 types. The full argument turns out to be no more complicated than the argument in an example with 4 agents, and where 2 agents have 2 types, one agent has 3 types, and one agent has a singleton type space. Specifically, suppose $n = 4$ and suppose that type spaces are as follows.

$$\Theta_1 = \{\ell, r\}, \quad \Theta_2 = \{u, d\}, \quad \Theta_3 = \{L, M, R\}, \quad \Theta_4 = \{0\}. \tag{2}$$

Below, we define a stochastic DIC mechanism, payoffs, and a distribution such that this mechanism is the unique optimal DIC mechanism.

Figure 2 shows the type profiles of agents 1 to 3; the degenerate type of agent 4 is omitted. The types of agents 1 to 3 span a 3-dimensional box. Each physical edge of the box (not to be confused with an edge of G) shown in the figure represents a set of type profiles along which exactly one agent's type is changing. That is, each edge of the box represents a node of G. Hence DIC requires that the winning probability of this agent be constant along the edge. For instance, agent 1's winning probability is constant along the profiles θ^a and θ^b; agent 3's winning probability is constant along the profiles θ^f, θ^g and the profile (ℓ, u, R).

Let Θ^* be the set $\{\theta^a, \theta^b, \theta^c, \theta^d, \theta^e, \theta^f, \theta^g\}$ of type profiles shown in Figure 2. Formally,

$$\begin{aligned} \theta^a = (\ell, d, M, 0), \quad \theta^b &= (r, d, M, 0), \quad \theta^c = (r, d, R, 0), \\ \theta^d &= (r, u, R, 0), \quad \theta^e = (r, u, L, 0), \\ \theta^f &= (\ell, u, L, 0), \quad \theta^g = (\ell, u, M, 0). \end{aligned} \tag{3}$$

By inspecting Figure 2, we can see that the following are adjacencies in G:

$$\begin{aligned}(2,\theta^g_{-2}) \leftrightarrow (1,\theta^a_{-1}) \leftrightarrow (3,\theta^b_{-3}) \leftrightarrow (2,\theta^c_{-2}) \leftrightarrow (3,\theta^d_{-3}) \leftrightarrow{}& (1,\theta^e_{-1}) \\ \leftrightarrow{}& (3,\theta^f_{-3}) \\ \leftrightarrow{}& (2,\theta^g_{-2}).\end{aligned} \tag{4}$$

Let V^* denote the nodes of G shown in (4). These nodes correspond precisely to the colored physical edges in Figure 2.

We now define our candidate stochastic extreme point φ^*. For all i in $\{1,2,3\}$ and θ_{-i}, let

$$\varphi^*_i(\theta_{-i}) = \begin{cases} 1/2, & \text{if } (i,\theta_{-i}) \in V^*, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Let φ^*_4 be defined at all profiles θ by

$$\varphi^*_4(\theta) = 1 - \sum_{i\in\{1,2,3\}} \varphi^*_i(\theta_{-i}). \tag{6}$$

The winning probabilities of agents 1, 2 and 3 are depicted in Figure 2. It is easy to verify from the figure that φ^* is indeed a DIC mechanism.

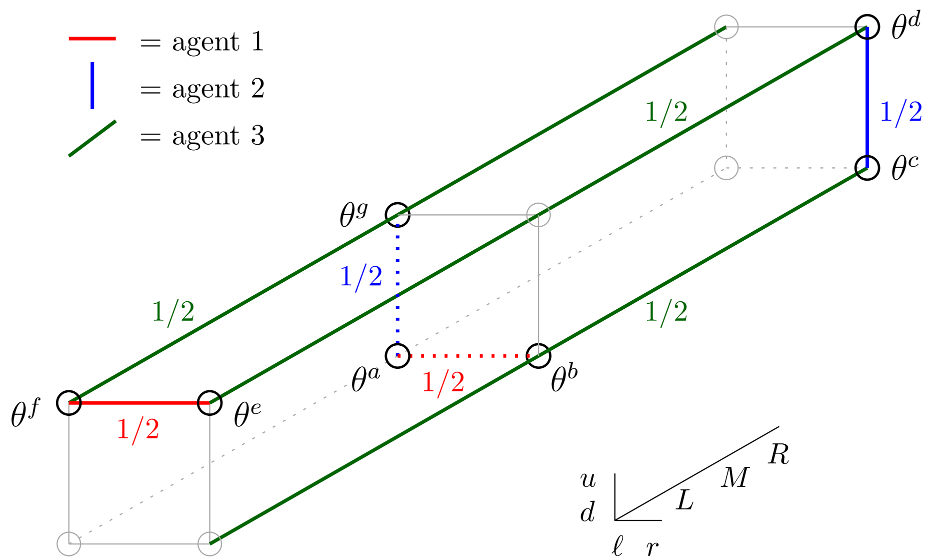


Figure 2: The set of types of agents 1 to 3. The probabilities 1/2 attached to the edges of the box represent the relevant values of the mechanism φ^*.

Further below we specify payoffs Ω and a distribution μ such that φ^* is the unique optimal DIC mechanism. This implies that φ^* is an extreme point of the set of DIC mechanisms. Since the proof for uniqueness is somewhat involved, we present here a straightforward argument showing that φ^* is an extreme point.

Proposition 4.3. *Let $n = 4$. For the type spaces given in (2), the stochastic DIC mechanism defined in (5) and (6) is an extreme point of the set of DIC mechanisms.*

Proof of Proposition 4.3. Let ψ be a DIC mechanism that receives non-zero weight in a convex combination that equals φ^*. We will show that ψ equals φ^*. For all i, let ψ_i denote agent i's winning probability under ψ. For all profiles θ in Θ^*, there are exactly two nodes (i, θ_{-i}) and (j, θ_{-j}) in V^* that are contained in θ, and hence φ^* randomizes evenly between i and j. Therefore, in the same situation, the mechanism ψ must also randomize between i and j (though at this point we do not know using which probabilities). Repeatedly applying this observation shows:

$$\begin{aligned}\psi_1(\theta_1^a) = 1 - \psi_3(\theta_{-3}^c) = \psi_2(\theta_{-2}^c) &= 1 - \psi_3(\theta_{-3}^e) \\ &= \psi_1(\theta_{-1}^e) \\ &= 1 - \psi_3(\theta_{-3}^f) = \psi_2(\theta_{-2}^a) = 1 - \psi_1(\theta_1^a).\end{aligned} \tag{7}$$

In particular, we have $\psi_1(\theta_1^a) = 1 - \psi_1(\theta_1^a)$, and therefore $\psi_1(\theta_1^a) = 1/2$. Hence all probabilities in (7) must equal $1/2$. This shows that ψ agrees with φ^* at all profiles in $\Theta^* = \{\theta^a, \theta^b, \theta^c, \theta^d, \theta^e, \theta^f, \theta^g\}$. By inspecting $\Theta \setminus \Theta^*$, it is now easy to verify that ψ and φ^* also agree on $\Theta \setminus \Theta^*$. □

With Proposition 4.3 in hand, it is easy to prove the part of Theorem 4.2 concerned with situations where a stochastic extreme point exists. The idea is simply to view the current example as embedded into a larger environment with additional agents and additional types. By appropriately extending φ^*, we readily obtain a stochastic extreme point of this larger environment. The details are in Appendix A.3.

We now turn to the more difficult problem of finding an environment with privately-known payoffs in which φ^* is the unique optimal DIC mechanism. Our candidate payoffs are parametrized by a number $\delta \in [0, 1/2]$. Consider $\hat{\omega}_1 \colon \Theta_1 \to \mathbb{R}, \ldots, \hat{\omega}_4 \colon \Theta_4 \to$

$\mathbb{R}$ and μ defined as follows (see Figure 3):

$$
\begin{aligned}
\hat{\omega}_1(r) = \hat{\omega}_2(u) = \hat{\omega}_3(M) &= 0 \\
\hat{\omega}_1(\ell) = \hat{\omega}_2(d) &= 5 \\
\hat{\omega}_3(L) = \hat{\omega}_3(R) &= 5(1-\delta) \\
\hat{\omega}_4 &= 0.
\end{aligned} \tag{8}
$$

Moreover, let μ be defined by

$$
\forall_{\theta \in \Theta}, \quad \mu(\theta) = \begin{cases} 1/5, & \text{if } \theta \in \{\theta^a, \theta^c, \theta^d, \theta^e, \theta^f\} \\ 0, & \text{else.} \end{cases} \tag{9}
$$

The functions $\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3, \hat{\omega}_4$ together with μ define an environment with privately-known payoffs.

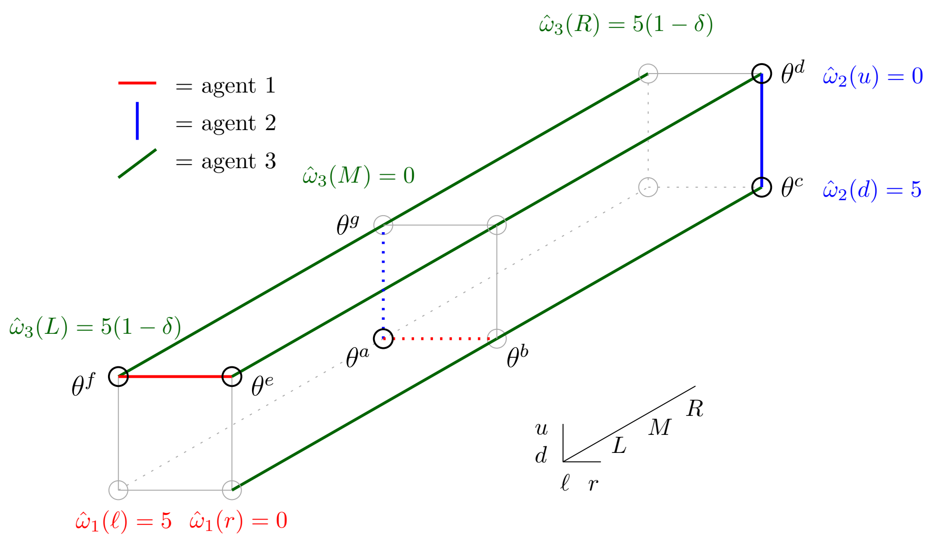


Figure 3: The payoffs defined in (8). The distribution μ assigns probability 1/5 to the profiles $\{\theta^a, \theta^c, \theta^d, \theta^e, \theta^f\}$. All other profiles have probability 0.

Proposition 4.4. *Let $n = 4$, let the type spaces be given by (2), and let φ^* be defined by (5) and (6). Let the payoffs and the distribution be parametrized by $\delta \in [0, 1/2]$ and given by (8) and (9). If $\delta \in (0, 1/2)$, then φ^* is the unique DIC mechanism that maximizes the principal's expected utility.*

To guide our intuition, recall the aforementioned idea that the principal has to trade off between allocating to an agent and using that agent's information. When δ

is small, the payoff from allocating to agent 3 is so high that it is optimal to ignore the information in 3's type; when δ is large, it is not enticing to allocate to agent 3. However for intermediate values of δ, it is strictly optimal to both allocate to agent 3 and to use the information in 3's type. Let us be more precise.

Suppose $\delta = 0$. Now, allocating to agent 3 is ex-post optimal at *all except one* of the 5 profiles in the support of μ. The constant mechanism that always allocates to agent 3 is in fact an optimal DIC mechanism when $\delta = 0$. To explain why φ^* is another optimal mechanism, note that agent 3's type contains information. When $\theta_3 = M$, the type profile must be θ^a, which is the unique type profile in the support of μ where allocating to agent 3 is not optimal. The mechanism φ^* uses this information by randomizing between agents 1 and 2 at θ^a. If we now increase δ from 0, the expected utility from always allocating to agent 3 decreases more rapidly than the expected utility from φ^*. A general version of this argument shows that φ^* is indeed uniquely optimal for non-zero but small values of δ.

If we increase δ further, then φ^* ceases to be optimal since the principal optimally avoids allocating to agent 3. The critical value turns out to be $\delta = 1/2$, where φ^* is an optimal DIC mechanism, but not the only one. For instance there is another optimal DIC mechanism that allocates to agent 1 at $\theta^a, \theta^b, \theta^e$ and θ^f, and allocates to agent 2 at θ^c and θ^d (and otherwise allocates to agent 4).

To summarize, the mechanism φ^* is optimal for $\delta = 1/2$, and uniquely optimal for δ close to 0. In a final step, we note that the principal's utility is an affine function of δ, and use this to conclude that φ^* is uniquely optimal for all δ in $(0, 1)$. The details are in Appendix A.1.

4.2.3 Deterministic extreme points. Let us now consider the cases in Theorem 4.2 where all extreme points are claimed to be deterministic.

First, suppose all agents have at most two types. We verify in the appendix that here the set of DIC mechanisms equals the perfect matching polytope of a bipartite graph. A generalization of the Birkhoff-von Neumann theorem implies that the extreme points of this polytope are deterministic (see Korte and Vygen (2018) for a definition of these terms; the generalization in question is Theorem 11.4 in Korte and Vygen (2018)).

The proof that all extreme points are deterministic if there are three or fewer agents follows a simple idea but is the most difficult part of the theorem to prove.

Given an arbitrary stochastic DIC mechanism φ, we construct a non-zero perturbation f such that $\varphi + f$ and $\varphi - f$ are two other DIC mechanisms. Since φ is otherwise arbitrary, this shows that all extreme points are deterministic. To understand the construction of f, consider the following: If at some type profile one agent is enjoying an interior winning probability, then, since the object is always allocated, there must be at least one other agent who also enjoys an interior winning probability at that type profile. The function f represents a shift of a small probability mass between these two agents. Note that an agent's winning probability appears in the feasibility constraint of many different type profile, and hence we have to shift masses in a manner that is consistent across profiles. What makes this difficult is that the identities of the agents who enjoy an interior winning probability may change from one profile to the next. We reformulate the construction of f as finding a particular partition of the node set of the feasibility graph. Our proof leans heavily on the combinatorial structure of the feasibility graph G for three agents.

Lastly, if at most two agents have two or more types, we construct a function f following the same idea as in the previous paragraph. However, the assumption on the type spaces dramatically simplifies the construction of f.

Remark 2 (Total unimodularity)**.** The reader may wonder whether one can also approach the problem by viewing the set of DIC mechanisms as the set of solutions to a linear system of inequalities, and checking for total unimodularity of the constraint matrix, and then invoking the Hoffman-Kruskal theorem (Korte and Vygen, 2018, Theorem 5.21). For example, in the mechanism design literature, Pycia and Ünver (2015) pursue this approach. This approach works for the case where all type spaces are binary (the aforementioned generalization of the Birkhoff-von Neumann theorem can itself be derived from the Hoffman-Kruskal theorem). However, for the difficult case with three agents, we show in Appendix D.2 of the supplementary that the constraint matrix is *not* generally totally unimodular.

4.3. Discussion

In this subsection, we relate our findings to those of Chen et al. (2019). We then discuss the implications of our results for the principal's problem.

4.3.1 Stochastic vs. deterministic mechanisms. Chen et al. (2019) establish an equivalence result between stochastic and deterministic mechanisms (with transfers) in certain mechanism design problems. Their Theorem 1 and Remark 2 imply the following (when applied to our setup): *L*et the type profile follow an atomless distribution in Euclidean space. For all mechanisms φ, there is a deterministic mechanism φ' that induces the same interim-expected allocation probabilities as φ; that is, all i and θ_i satisfy $\mathbb{E}_{\theta_{-i}}[\varphi_i(\theta_{-i})|\theta_i] = \mathbb{E}_{\theta_{-i}}[\varphi_i'(\theta_{-i})|\theta_i]$. In light of this result, one may not have expected that in our setup there are environments with privately known-payoffs in which all deterministic DIC mechanisms are suboptimal. After all, if payoffs are privately-known, the contribution of allocating to agent i at θ_i, namely $\hat{\omega}_i(\theta_i)\mathbb{E}_{\theta_{-i}}[\varphi_i(\theta_{-i})|\theta_i]$, depends on φ only through the interim-expected allocation probability $\mathbb{E}_{\theta_{-i}}[\varphi_i(\theta_{-i})|\theta_i]$. The results are reconciled by noting that the mechanism φ' is not guaranteed to be DIC even if φ is DIC. (Unlike us, Chen et al. (2019) also assume an atomless type distribution, and they show that this assumption cannot be dropped as an hypothesis in the above result.)

4.3.2 Implications for the principal's problem. The preceding analysis produced the qualitative insight that the principal may strictly benefit from committing to random allocations. Beyond this insight, what can we say about optimal mechanisms? Is there a class of optimal mechanisms that are of practical use or that are interpretable as a real-world institutions? In this section we explain why we do not believe it likely that such a class exists.

We have seen that it is not generally without loss to restrict attention to deterministic DIC mechanisms. This points to an important conceptual issue. Namely, stochastic mechanisms are less transparent and require a stronger form of commitment than deterministic ones—the principal must commit to honoring the outcome of a random process. This commitment requirement diminishes the practical usefulness of stochastic mechanisms. We refer the reader to the existing literature for further discussion; e.g. Budish et al. (2013), Chen et al. (2019), Laffont and Martimort (2009), and Pycia and Ünver (2015).

To illustrate further, consider again the example from Proposition 4.4. The principal commits to randomizing evenly between agents 1 and 3 at the profile θ^e. Yet, at this profile, the payoff of allocating to 3 is strictly higher than the payoff of allocating to agent 1. Thus the principal has to commit to honoring the coin flip that determines which of the two agents wins. In fact, we note that in this example the principal commits to flipping a coin at *all* type profiles in the support of the distribution.[16] That is, the commitment issue does not only arise in a low probability event.

The previous point leads to the question of whether the principal's problem has a satisfying solution when we restrict attention to deterministic mechanisms. We next argue that this is unlikely to be the case. Suppose all agents have binary types, for we know that then deterministic mechanisms are in fact without loss. For binary types, the set of deterministic DIC mechanisms with n agents coincides with the set of perfect matchings on the n-dimensional hypercube graph (see Korte and Vygen (2018) for a definition of these terms). The characterization of the set of perfect matchings of this graph is an open problem; in fact, it is even unknown how many perfect matchings there are.[17] A priori, we must consider all deterministic DIC mechanisms as candidates for optimal mechanisms. The reason is that all deterministic DIC mechanisms are extreme points.[18] Hence Lemma 4.1 implies that for all deterministic DIC mechanisms there is at least one environment in which it is the unique optimal DIC mechanism. In summary, even in the simplest case of binary types, the sheer combinatorial complexity of the set of DIC mechanisms stands in the way of finding a tractable and interpretable solution.

Although the economics literature is typically interested in qualitative rather than computational solutions, it is worth pointing out the computational complexity of the principal's problem. When stochastic DIC mechanisms are considered, the principal's problem is a linear program. However, the input size of this program grows exponentially with the number of agents.[19] When only deterministic mechanisms are considered, the problem is even more challenging. The problem of finding an optimal deterministic DIC mechanisms is an instance of the so-called *maximum-weight stable set problem* on the feasibility graph G. The general version of this combinatorial

[16]Granted, the support of the distribution in this example does not include all types profiles. The fact that some type profiles are outside the support is in line with the point made earlier that the benefit from randomization stems from the information that agents have about one another.

[17]Östergård and Pettersson (2013) have discovered the *number* of perfect matchings on the 7-dimensional hypercube; it exceeds 10^{27}. As far as we are aware, the number of perfect matchings in 8 dimensions and beyond is unknown.

[18]To see this, fix a deterministic DIC mechanism φ and a convex combination that equals φ. Consider a type profile where φ allocates to, say, agent i with probability 1. Since i's winning probability is always less than 1, all mechanisms in the convex combination must also allocate to i with probability 1 at this profile. Since at all type profiles there is exactly one agent to whom φ, being deterministic, allocates with probability 1, we conclude that all mechanisms in the convex combination must coincide with φ. Thus φ is an extreme point.

[19]If, say, all agents have m possible types, the number of variables in the program is nm^{n-1}.

optimization problem is NP-hard (Korte and Vygen, 2018, p. 441).[20]

We take this discussion as motivation for looking for a simpler and more interpretable class of DIC mechanisms that, although perhaps not exactly-optimal, perform well in relevant settings. The class we propose is the class of jury mechanisms, defined next.

4.4. Jury mechanisms

Definition 1. *A mechanism φ is a **jury mechanism** if there is a subset J of agents with the following two properties:*

(1) For all i not in J, the mechanism φ is constant with respect to agent i's report.

(2) For all i in J, the allocation φ_i to agent i is constantly equal to zero.

*An agent i is a **juror** if i is in J, and a **candidate** if i is not in J.*

In words, the principal nominates a jury of experts prior to consulting the agents. The collective opinion of the jury determines the allocation to the remaining agents, and jury members cannot abuse their power by allocating the object to themselves. All jury mechanisms are therefore DIC.

One appealing feature of jury mechanisms is their relative simplicity of their indirect implementation. We do not literally envision that the principal, who may simply be a metaphor, collects reports of agents' abstract types. In an indirect mechanism, each agent has some set of actions and is aware of how action profiles determine a winner. To implement a given direct mechanisms, agents have to be convinced that their actions will not affect their individual winning probabilities. Given that they are indifferent between all actions, they also have to be instructed on how to map their private information into actions. Compare this to the implementation of a jury mechanism: The jurors are asked to share their information with one another so as to agree on which of the candidates should win (according to the principal's payoffs).

To motivate jury mechanisms out of our earlier results, consider the case where three or fewer agents are around, meaning $n \leq 3$. As it turns out, jury mechanism fully characterize the set of DIC mechanisms with three or fewer agents.

Corollary 4.5. *Let $n \leq 3$. A mechanism is DIC if and only if it is a convex combination of jury mechanisms. In particular, there is a deterministic jury mechanism that maximizes the principal's utility over the set of DIC mechanisms.*

[20] The maximum-weight stable set problem is solvable in polynomial time if the graph in question is perfect. Unfortunately, our graph G is often imperfect, as we show in Appendix D.1.

With three or fewer agents, a jury mechanism admits a single juror who deliberates between the other two agents. Hence the content of the result is that all DIC mechanisms with three or fewer agents can be implemented by nominating a juror (according to some distribution over the set of agents), and then asking the juror to pick one of the others as a winner of the object.

Proof of Corollary 4.5. According to Theorem 4.2 and the Krein-Milman theorem, a mechanism is DIC if and only if it is a convex combination of deterministic DIC mechanisms. Proposition 2.i of Holzman and Moulin (2013), adapted to our terminology, implies that for an arbitrary deterministic DIC mechanism there is at most one agent whose report changes the allocation. That is, all deterministic DIC mechanisms with three or fewer agents are jury mechanisms. □

To summarize, the simple class of jury mechanisms emerges as optimal when three or fewer agents are around. Motivated by the discussion on exactly-optimal DIC mechanisms in Section 4.3.2, the remainder of the paper is concerned with settings where jury mechanisms remain optimal with more than three agents.

5. Institutional constraints

In this section, we consider mechanisms that treat the reports of agents anonymously and that are immune to manipulations by coalitions of agents.

The literature on axiomatic peer selection has studied various notions of anonymity, and we later comment in detail on differences to results from this body of work. Anonymity is a desirable property as it distributes the power to influence the allocation evenly across agents. It also protects agents from being threatened by their peers or outside observers.

The interest in coalition-proofness arises naturally in the applications we have in mind. Recall the example from the introduction where a planner allocates a good to one of n households in a community. Here, it may be particularly easy for friends or family members to coordinate their reports.

There are two main findings. The first is that DIC is incompatible with strong but natural notions of anonymity and coalition-proofness. The second is that relaxations of these notions characterize jury mechanisms with anonymous juries. Throughout this section, we assume that all type spaces are equal.[21]

5.1. An impossibility result

Beginning with anonymity, a natural requirement is the following: Fixing two arbitrary agents, they have the same say in the winning probabilities of the others.

Definition 2. *A mechanism φ is* ***strongly anonymous (SA)*** *if all distinct agents i and j satisfy the following: For all agents k distinct from i and j, the winning probability φ_k of agent k is invariant with respect to permutations of i's and j's reports.*

Note well that SA only compares how different agents influence the allocation; it is silent on how the winning probabilities of agents compare to one another. For instance, the mechanism that always allocate to agent 1 is SA.

We next turn to manipulations by coalitions of agents.

Definition 3. *A mechanism φ is* ***strongly coalition-proof (SCP)*** *if for all non-empty subsets J of agents the probability $\sum_{i \in J} \varphi_i$ that the object is allocated to an agent in J is constant in the reports of agents in J.*

Strong coalition-proofness is a strengthening of DIC that applies to coalitions. If SCP does not hold, a group J of agents can coordinate their reports to increase the group's winning probability, and then run a lottery between themselves that strictly benefits all agents in J relative to reporting truthfully.

Unfortunately, both SA and SCP are too restrictive when combined with DIC.

Theorem 5.1. *A DIC mechanism is SA or SCP if and only if it is constant.*

The difficult part of the proof lies in showing that all SA DIC mechanisms are constant; we sketch this argument in Section 5.5 below. It is rather immediate that all SCP mechanisms are constant: Consider two arbitrary agents i and j. A change in i's report affects neither φ_i (by SCP for the coalition $\{i\}$) nor $\varphi_i + \varphi_j$ (by SCP for the coalition $\{i, j\}$). But then φ_j must be constant in i's report, too.

Theorem 5.1 has an interesting implication for environments in which the principal actually finds it optimal to "treat all agents equally." Consider the following.

[21] In an equally valid interpretation, the agents do not have a common type space but the principal is nevertheless using an indirect mechanism where all agents have the same message sets and agents are indifferent between all messages.

Assumption 1. The sets Ω_i are the same across agents i. The distribution μ is invariant with respect to permutations of the agents; that is, for all ω in Ω, all θ in Θ, and all bijections $\xi\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$ we have

$$\mu\left(\omega_1,\ldots,\omega_n,\theta_1,\ldots,\theta_n\right) = \mu\left(\omega_{\xi(1)},\ldots,\omega_{\xi(n)},\theta_{\xi(1)},\ldots,\theta_{\xi(n)}\right).$$

We emphasize that Assumption 1 severely restricts the informational content of types. For instance, an implication of the assumption is that if i and j are distinct, then for all type profiles θ_{-ij} of agents other than i and j we have

$$\mathbb{E}_{\omega_i}[\omega_i|\theta_{-ij}] = \mathbb{E}_{\omega_j}[\omega_j|\theta_{-ij}].$$

Put differently, learning the type realizations of $n-2$ of the agents has no value for discerning between the two remaining agents.

Corollary 5.2. *If Assumption 1 holds, then the principal is indifferent between all DIC mechanisms.*

Proof of Corollary 5.2. Let φ be a DIC mechanism. Let Ξ denote the set of permutations of $\{1,\ldots,n\}$. For all i and θ, let $\psi_i\colon \Theta \to [0,1]$ be defined by $\psi_i(\theta) = \frac{1}{n!}\sum_{\xi\in\Xi}\varphi_{\xi(i)}(\xi(\theta))$. Then $\psi = (\psi_1,\ldots,\psi_n)$ is a well-defined mechanism, it inherits DIC from φ, and it is SA by construction. Thus Theorem 5.1 implies that ψ is constant. Using Assumption 1 it is straightforward to verify that the principal is indifferent between φ and ψ, and indifferent between all constant mechanisms. $\square$

Theorem 5.1 gives an unsatisfying answer to the problem of designing anonymous or coalition-proof mechanisms. There are at least two ways out of this negative result.

One way involves passing to mechanisms that do not always allocate the object (our definition of a mechanism ruled out this possibility). We find that in this case there do exist non-constant DIC mechanisms that are strongly anonymous. However, we now contend with two new issues. One is that, as mentioned in Section 2, not allocating the object is impractical in the applications we have in mind. The second issue relates to the discussion from Section 4.3.2 on stochastic mechanisms. With disposal, the appropriate notion of strong anonymity is that, in addition to SA, the probability of disposing the object be permutation-invariant in the reports of the agents; call this SA*. It turns out that all non-constant SA* DIC mechanisms are stochastic. Thus the principal must be able to commit both to disposing the object and to randomizing the allocation.

The analysis of mechanisms with disposal reveals many interesting aspects of DIC mechanisms.[22] However, in view of the above issues, we relegate this analysis to Appendix D.1 of the supplementary material, instead choosing to focus on another escape route from Theorem 5.1. This route consists of relaxing SA and SCP. In the next section, we propose a pair of relaxed notions, and then go on to show that the resulting set of DIC mechanisms consists only of deterministic jury mechanisms or randomizations thereof.[23]

5.2. Weak anonymity and coalition-proofness

Our point of departure is the observation is that SA and SCP rule out DIC mechanisms that "look" anonymous and coalition-proof. Consider the following jury mechanism.

Example 1. Let $n = 4$, and let all agents i have a binary type space $\Theta_i = \{0, 1\}$. The principal uses a jury mechanism with agents 1 and 2 as jurors. If agents 1 and 2 make the same report, then agent 3 wins. Else, agent 4 wins.

This mechanism fails to be SA since a juror's report is treated differently from a candidate's report. It it is not SCP since, say, agent 1 could always report in favour of agent 3. In what sense is this mechanism nevertheless anonymous and coalition-proof?

First, note that the allocation is unchanged if one permutes the reports of the jurors, or if one permutes the reports of the candidates. The only way of affecting the allocation with a permutation involves a permutation of a juror with a candidate. Arguably, such a permutation is unreasonable since it is hard-wired into the definition of a jury mechanism that jurors and candidates play very different roles. One way of formalizing for a general mechanism the idea that agents play different roles is

[22] In Section 2 we suggested that some results extend easily to mechanisms with disposal since we could add dummy agents to the model without disposal. This trick does not work here since it is unclear how to interpret anonymity for this dummy agent. Indeed, the analysis in Appendix D.1 reveals that anonymity and the disposal option together introduce new subtleties.

[23] The idea that relaxing anonymity (rather than giving it up entirely) to obtain an interesting larger class of mechanisms is reminiscent of Bartholdi, Hann-Caruthers, Josyula, Tamuz, and Yariv (2021). They show that, in the context of May's theorem, a relaxation of symmetry (which is a notion of anonymity) allows for a rich set of voting rules that treat voters equally. This relaxation—*equitability*—is quite different from the one we consider below, though.

by considering, for each agent i, which of the other agents' winning probabilities are influenced by agent i's report. Our weaker notion of anonymity will require that two agents' reports be handled anonymously if there is overlap in whom the two influence. If there is no overlap, we impose no further requirements on how their reports affect the allocation. The above jury mechanism satisfies this notion of anonymity: For agents i and j, there is overlap in whom i and j influence if and only if both i and j are jurors (so that both influence both candidates) or both are candidates (so that both influence an empty set of others).

Formally, for a given mechanism φ, let us say agent i *influences* agent ℓ if φ_ℓ is non-constant in θ_i. Agents i and j *influence a common agent* if there exists ℓ such that i and j both influence ℓ.

Definition 4. *A mechanism φ is **weakly anonymous (WA)** if all distinct i and j satisfy the following: If i and j influence a common agent, then for all k distinct from i and j the winning probability φ_k of agent k is invariant with respect to permutations of i's and j's reports.*

If the reports of i and j are not handled anonymously in a WA mechanism, then there is no overlap in whom i and j affect with their reports. This possibility is ruled by strong anonymity: In an SA mechanism, if one of two agents i and j influence a third agent k, then both i and j must influence k.

We next turn to a weaker notion of coalition-proofness. Consider again the jury mechanism from Example 1. Although not SCP, this mechanism is coalition-proof in the weaker sense that the two jurors cannot coordinate their reports to their own benefit. Likewise, the two candidates cannot collude to improve one another's winning probability. The only coalitions that can profitably manipulate the outcome for itself must involve at least one juror and one candidate. We next argue that such a coalition may be less reasonable than others.

What are the incentives of a juror to enter a coalition with a candidate? As mentioned earlier, the juror gains when the coalition increases its overall winning probability and then (randomly) reallocating the object amongst its members. However, the juror, arguably, has little to fear from *not* entering the coalition. After all, the other prospective coalition member—a candidate—cannot affect the juror's winning probability. Therefore, if the juror decides not to enter and all others report truthfully, the candidate has no way of punishing the juror. Our relaxed notion of coalition-proofness, which applies to general mechanisms, focuses on coalitions where all coalition members can influence the winning probability of all other coalition members.

Definition 5. *Given a mechanism, a subset J of agents is* ***balanced*** *if satisfies the following: An agent is in J if and only if the agent influences all other agents in J.*

Note that, depending on the mechanism, it may be that all balanced sets are empty or contain a single agent. Indeed, this is the case in all jury mechanisms. To see this, suppose towards a contradiction that a jury mechanisms admits a balanced set with two or more agents. Neither of these agents can be a candidate since candidates influence no one. Thus all agents in the set are jurors. Since the set is balanced and contains at least two agents, there is a juror with a non-constant winning probability, which contradicts the definition of a jury mechansim.

Our relaxed notion of coalition-proofness demands immunity with respect to balanced coalitions, and nothing else. To be sure, there may be other reasonable coalitions that one would like to guard against. Including further coalitions would lead to a more restrictive notion of coalition-proofness.[24]

Definition 6. *A mechanism φ is* ***weakly coalition-proof (WCP)*** *if for all non-empty balanced subsets J of agents the probability $\sum_{i\in J}\varphi_i$ that the object is allocated to an agent in J is constant in the reports of agents in J.*

All jury mechanisms are WCP. Indeed, as noted above, all balanced sets in a jury mechanism contain at most one juror. Hence WCP follows from the fact that jurors enjoy constant winning probabilities.

5.3. A characterization of jury mechanisms

Example 1 shows that there is a non-constant DIC WA WCP jury mechanism. Are jury mechanisms the only mechanisms that are DIC, WA and WCP? The second main result of this section asserts that, up to randomizations, this is indeed the case.

Formally, let Φ^* denote the set of mechanisms that are DIC, WA, and WCP. We seek to characterize Φ^* via convex combinations of WA jury mechanisms (we already know that all jury mechanisms are WCP). As it turns out, the set Φ^* is not generally

[24] For instance, a stronger notion of coalition-proofness under which the upcoming result holds verbatim demands immunity with respect to all of the following coalitions: If i is in the coalition, then i must influence at least one other coalition member.

convex.[25] Thus we would also like to characterize the convex hull $\operatorname{co}\Phi^*$. It is easy to verify from the definitions of WA and WCP that Φ^* is a non-empty compact subset of Euclidean space. This implies that $\operatorname{co}\Phi^*$ coincides with the convex hull of extreme points of Φ^* (see Aliprantis and Border (2006, Theorem 7.67)). We characterize these extreme points. For the formal statement let us say that, in a given mechanism, an agent *influences the allocation* if the agent influences at least one other agent.

Theorem 5.3. *A mechanism is an extreme point of Φ^* if and only if it is a deterministic WA jury mechanism. A deterministic jury mechanism is WA if and only if it is invariant with respect to all permutations of those agents who influence the allocation.*

In words, WA and WCP let us accomodate non-constant DIC mechanisms, unlike SA or SCP. Further, WA and WCP exactly characterize (randomizations of) deterministic jury mechanisms in which the reports of all jurors are handled anonymously.

Corollary 5.4. *There is a deterministic WA jury mechanism that maximizes the principal's utility over* $\operatorname{co}\Phi^*$.

The remainder of this section is split into three parts. We next discuss the roles of the constraints in Theorem 5.3. Then, we sketch the proofs of Theorems 5.1 and 5.3. Lastly, we relate the theorems to results from the literature.

5.4. The roles of WA and WCP

As a characterization of jury mechanisms, it may at first seem unreasonable to impose WA and WCP simultaneously. After all, according to Theorem 5.1, the strong counterparts of WA and WCP are already maximally restrictive when applied individually. We next present several examples to argue that WA and WCP are *not* so restrictive when applied individually. These examples demonstrate that neither WA nor WCP can be dropped from the characterization in Theorem 5.3.

[25] Non-convexity is due to WA. It is easy to see, and formally verified in the proof of Theorem 5.3, that WA requires that if i and j influence a common agent, then i and j influence exactly the same set of other agents. This property need not be preserved when randomizing over WA jury mechanisms. We do not view non-convexity of Φ^* as a serious cause for concern. Theorem 5.3 shows that all mechanisms in $\operatorname{co}\Phi^*$ can be implemented by randomizing over deterministic jury mechanisms, publicly revealing the outcome of this randomization, and only then consulting the agents. In particular, the mechanism that is revealed after the initial randomization is itself DIC, WA, and WCP.

5.4.1 Weak anonymity. Consider Table 1. For $n = 4$, the table shows a deterministic DIC mechanism in which all agents influence the allocation of all others. This mechanism cannot be a jury mechanism. Yet it satisfies WCP since the only balanced set of agents is the set of all agents; this set enjoys a constant winning probability as the object is always allocated.

		θ_3		θ'_3	
		θ_1	θ'_1	θ_1	θ'_1
θ_4	θ_2	1	1	2	4
	θ'_2	4	3	2	3
θ'_4	θ_2	3	2	3	4
	θ'_2	4	2	1	1

Table 1: There are four agents, each with binary type spaces of the form $\Theta_i = \{\theta_i, \theta'_i\}$ (so that, up to relabelling, all agents have the same type spaces). The table shows a deterministic mechanism. In each cell, the integer identifies the agent who is allocated to the object at that profile. For instance, agent 1 wins at the profile $(\theta_1, \theta_2, \theta_3, \theta_4)$. It follows from inspection that the mechanism is DIC and that all agents can influence all other agents. As an aside, the mechanism shown here belongs to the class of mechanisms constructed in Proposition 2.ii of Holzman and Moulin (2013).

5.4.2 Weak coalition-proofness and disposal. We have already mentioned that Theorem 5.1 does not extend to mechanisms with disposal. As a corollary, one can show the following: If $n = 4$ and the common type space contains 7 elements, there exists a WA DIC mechanism which is stochastic and an extreme point of the set of *all* DIC mechanisms. Such a mechanism cannot be a jury mechanism; the reason is that a jury mechanism is an extreme of the set of all DIC mechanisms only if it is deterministic. Thus there are WA DIC mechanisms that are not convex combinations of jury mechanisms. See Appendix D.1 in the supplementary material for the construction of this stochastic WA DIC mechanism.[26]

As an aside, in Appendix D.1 we also show that Theorem D.1 carries over to mechansims with disposal under the following natural ammendment to WA: The probability of disposing the object is invariant with respect to all permutations con-

[26] The rough idea of the construction is to define 6 permuted copies (one for each permutation of $\{1, 2, 3\}$) of the stochastic mechanism from Section 4.2.2. By using that there are 7 types, one can show that this yields a well-defined mechanism. It is WA by construction, and the fact that it is a stochastic extreme point follows essentially from Proposition 4.3.

sidered by WA.

5.5. Ideas for the proofs of Theorem 5.1 and Theorem 5.3

We begin with the part of Theorem 5.1 that was left unexplained: All DIC SA mechanisms are constant. Let us prove here the following simpler claim.

Lemma 5.5. *If a mechanism φ is deterministic, DIC, and SA, then it is constant.*

Proof of Lemma 5.5. Consider two distinct agents i and j. Let us write $\varphi_i(t, t', \theta_{-ij})$ for i's winning probability when i reports t, j reports t', and the others report θ_{-ij}. Now consider a permutation of i's and j's reports. By SA, this permutation of has no effect on the winning probabilities of the agents other than possibly i and j. Since the object is allocated with probability one, we have $\varphi_i(t, t', \theta_{-ij}) + \varphi_j(t, t', \theta_{-ij}) = \varphi_i(t', t, \theta_{-ij}) + \varphi_j(t', t, \theta_{-ij})$. Equivalently,

$$\varphi_i(t, t', \theta_{-ij}) - \varphi_i(t', t, \theta_{-ij}) = \varphi_j(t, t', \theta_{-ij}) - \varphi_j(t', t, \theta_{-ij}). \tag{10}$$

For later reference, we note that (10) is valid for arbitrary distinct agents i and j, and all t, t' and θ_{-ij}.

Let us now argue that φ must be constant. Towards a contradiction, suppose not; that is, suppose there are distinct i and j such that j influences i's allocation at least once. By DIC, agent i's allocation depends only on the reports of j and agents other than i and j. Thus, there are reports θ_{-ij} of agents other than i and j, and reports t and t' for j such that for all reports θ_i of agent i we have

$$\varphi_i(\theta_i, t', \theta_{-ij}) - \varphi_i(\theta_i, t, \theta_{-ij}) \neq 0.$$

Since φ is deterministic, this means the difference in the above display is either 1 or -1. Suppose it is 1, the other case being similar. We thus have $\varphi_i(\theta_i, t', \theta_{-ij}) = 1$ and $\varphi_i(\theta_i, t, \theta_{-ij}) = 0$. Using DIC, we therefore also have $\varphi_i(t, t', \theta_{-ij}) - \varphi_i(t', t, \theta_{-ij}) = 1$. Hence (10) implies $\varphi_j(t, t', \theta_{-ij}) - \varphi_j(t', t, \theta_{-ij}) = 1$. Since j's winning probability is at most 1, we also find $\varphi_j(t, t', \theta_{-ij}) = 1$. To summarize, we have shown $\varphi_i(t, t', \theta_{-ij}) = \varphi_j(t, t', \theta_{-ij}) = 1$. But this means that at the profile (t, t', θ_{-ij}) both agents i and j are winning with probability 1. This is a contradiction. $\square$

How can we extend Lemma 5.5 to stochastic mechanisms? In a stochastic mechanism, we cannot conclude as in the above proof that agent i' and j's winning probabilities both change by 1. Thus we do not immediately obtain a contradiction to the fact that there is only one object to allocate at the profile (t, t', θ_{-ij}). Loosely speaking, we therefore have to consider changes in all agents' winning probabilities. The formal proof essentially proceeds by carefully summing differences of the form (10) across agents. We relegate this tedious calculation to the appendix.[27]

Let us now turn to Theorem 5.3. Most of the work goes towards arguing that if φ is in Φ^*, then it must be a convex combination of WA jury mechanisms. The idea is simple and related to our earlier impossibility result, Theorem 5.1.

Weak anonymity gives rise to an equivalence relation where two agents are equivalent if and only if they influence the same set of other agents. The equivalence relation lets us partition the set of agents into equivalence classes. The idea is now to define one jury mechanism for each of these equivalence classes; the agents in the equivalence class are the jurors in this mechanism. The main (but not the only) remaining step is to argue that agents within one equivalence class cannot influence one another; for in that case we would not have a well-defined jury mechanism.

Towards a contradiction, suppose there is an equivalence class J where one agent influences another agent in J. Using WA we show that J must be balanced. Hence WCP implies that the overall winning probability of agents in J is constant in the reports of agents in J. We can now view the winning probabilities of agents in J as defining a mechanism in a hypothetical setting where J is the set of all agents. This hypothetical mechanism is DIC and SA. Hence Theorem 5.1 implies that the hypothetical mechanism is constant. This yields a contradiction to the assumption that at least one agent in J influences another agent in J.

5.6. Related results from the literature

Recall the model of Holzman and Moulin (2013) and Mackenzie (2015), who study impartial nominations rules: Each agent reports the name of one of the others, interpreted as a nomination. The principal (randomly) selects a winner as function of the profile of nominations (and the principal must select a winner with probability 1). Agents are unable to influence their individual winning probabilities with their

[27]As an aside, notice that Theorem 5.1 implies that all extreme points of the set of DIC SA mechanisms are deterministic. This hints at an alternate and possibly simpler proof for Theorem 5.1: As a first step, argue that all extreme points of the set of DIC SA mechanisms are deterministic, and then appeal to Lemma 5.5. Unfortunately, we have not found a simple independent proof for the first step.

reports. Importantly, note that since agents cannot nominate themselves, they all have distinct message spaces. By contrast, if our agents have a common type space, we naturally find ourselves in a situation where all can send the same messages.[28]

Holzman and Moulin (2013) and Mackenzie (2015) study, among other things, ways of treating the nominations anonymously. Their notion—*anonymous ballots*—requires that the allocation depend on the profile of nominations only through the number of nominations that each agent receives. Equivalently, the allocation is unchanged if one permutes the profile in a way that does not yield self-nominations (Mackenzie, 2015, Lemma 1.1). This has no immediate analogue in our model with abstract types.

The different notions lead to different results. There are non-constant impartial nomination rules with anonymous ballots (see Mackenzie (2015, Theorem 1) for a full characterization), whereas all SA DIC mechanisms are constant.

Another difference between weak anonymity and anonymous ballots is that, for the allocation φ_i to agent i, weak anonymity only considers permutations of the agents who actually influence φ_i. This lets us escape the following negative result: All deterministic impartial nomination rules with anonymous ballots are constant (Holzman and Moulin, 2013, Theorem 3). By contrast, Example 1 presents a mechanism that is deterministic, DIC, weakly anonymous, and non-constant.

Mackenzie (2020) considers, among other things, a setting where self-nominations are allowed and the nomination rule may not return a winner. In this case, the only impartial deterministic nomination rules with anonymous ballots are constant (Mackenzie, 2020, Theorem 1). With self-nominations, anonymous ballots are a stronger form of anonymity than SA; the reason is that SA only consider permutations of the others, whereas anonymous ballots apply to arbitrary permutations of the nomination profile. We obtain Mackenzie's result as a corollary of our findings on DIC mechanisms with disposal; see Appendix D.1 in the supplementary material.

[28]The computer scientific literature on peer selection likewise assumes that agents have different message spaces. For instance, Alon et al. (2011) have agents nominate subsets of the other agents. They find that symmetric mechanisms, appropriately defined, are without loss. The fact that this is not true in our model is again explained by the fact that we have asssumed a common type space.

6. Symmetric information

In this section we argue that jury mechanisms are approximately optimal if there are many agents with access to similar information.

We again assume that the type spaces of all agents are equal. So, the distribution of the payoffs and types of n agents is over $\times_{i=1}^{n}(\Omega_i \times \Theta_i)$, and we denote this distribution by μ_n. We consider a sequence of environments satisfying the following consistency requirement: For all n, the distribution μ_n over $\times_{i=1}^{n}(\Omega_i \times \Theta_i)$ agrees with the marginal distribution of μ_{n+1} over $\times_{i=1}^{n}(\Omega_i \times \Theta_i)$. We also assume that the limit $\lim_{n\to\infty} \mathbb{E}\left[\max_{i\in\{1,\ldots,n\}} \omega_i\right]$ exists. Thus the principal's utility is bounded above as $n \to \infty$. When we say that there are n agents, we mean that the principal consults and allocates to the first n agents only.

The assumption on the environment that we now introduce captures the idea that an individual agent may be unlikely to have exclusive information, except possibly about themself.

Assumption 2. For all n, all i in $\{1, \ldots, n\}$, and all ω_i in Ω_i, we have the following: Conditional on the payoff of agent i being equal to ω_i, the distribution of $(\theta_j)_{j\in\{1,\ldots,n\}\setminus\{i\}}$ is invariant with respect to permutations of $\{1, \ldots, n\} \setminus \{i\}$.

To illustrate, consider the following environment for a fixed number n of agents.

Example 2. For all i, the type of agent i is a pair $\theta_i = (s_i, t_i)$, where s_i is an informative signal, and t_i determines the payoff from allocating i. More precisely, for some function $\hat{\omega}_i$, the payoff of allocating to i is given by $\omega_i = \hat{\omega}_i(t_i)$. The entries $t_1, \ldots, t_n$ are iid. draws from a distribution f. Conditional on $\boldsymbol{t} = (t_1, \ldots, t_n)$, the signals $s_1, \ldots, s_n$ are iid. draws from a distribution $g(\cdot|\boldsymbol{t})$. The fact that signals are conditionally iid. implies that Assumption 2 holds. Note well that we are not assuming that agents other than i are as well informed as agent i about ω_i. Indeed, agent i knows ω_i payoff whereas others do not. We are also not assuming that the distributions of payoffs ω_i are the same across i. In particular, Assumption 1 need not hold.

The main result of this section is now.

Theorem 6.1. *Let Assumption 2 hold. Let $\varepsilon > 0$. For all sufficiently large n, there is a jury mechanism with n agents that generates an expected utility within ε of the optimal expected utility with n agents.*

To understand the result, recall the two defining features of jury mechanisms: The principal only consults jurors, and only allocates to candidates. Thus a concern could be that jury mechanisms perform poorly if there are agents who would be exceptional jurors and exceptional candidates. We make two observations: First, in all DIC mechanisms, DIC precludes using an agent i's information to determine i's winning probability. Second, Assumption 2 implies that every agent is replaceable as a supplier of information about others. Taken together, the proof shows that the expected utility from a DIC mechanism with n agents can be replicated by a jury mechanism with $2n$ agents. In the $2n$-agent jury mechanism, the first n agents are candidates, and the second group of n agents are jurors whose information replaces the information elicited in the n-agent mechanism. The fact that such a replicating mechanism exists completes the proof since the principal's utility is bounded above as $n \to \infty$.[29]

Assumption 2 is stronger than what we really need for this argument. It suffices if for all groups of agents $\{1, \ldots, n\}$ there eventually comes another group that is at least as well informed as $\{1, \ldots, n\}$ about the payoffs of agents in $\{1, \ldots, n\}$, excepting each agent i's information about ω_i. This is the natural assumption that in a large group of peers there is no agent who knows something about their peers that no one else does. Assumption 3 in Appendix C formalizes this idea, and also allows for agents to have different type spaces.

For an ever-increasing number of agents, it is natural to ask whether optimal DIC mechanisms approach the principal's first-best utility; that is, the utility from a mechanism that always allocates optimally conditional on the type profile. Note that Theorem 6.1 does not claim that the principal's utility approaches first-best.

It is plainly true that the principal's first-best utility is not generally within reach: We have seen that with independent types and privately-known payoffs, or if Assumption 1 holds, there is an optimal constant mechanism. Even if non-constant mechanisms are optimal, it may be impossible to approach first-best. The reason is that the ex-post optimal decision between two agents i and j may have use to their private information about themselves. However, DIC does not allow i's winning probability to depend on the i's type. Further, the types of the others may fail to reveal i's type even as $n \to \infty$.

[29] The intuition given here recalls the result of Bulow and Klemperer (1996) that adding additional agents to a suboptimal auction may be weakly better than using an optimal auction.

When does the principal's utility then approach first-best? It suffices to answer the following: For all n_0 and ε, if n is sufficiently large, is there a DIC mechanism that generates a utility within ε of $\mathbb{E}[\max_{i\in\{1,\dots,n_0\}} \omega_i]$? This is a Bayesian inference problem in which the principal has to find the maximizer from the fixed set $\{1,\dots,n_0\}$. The principal's information is the profile of types up to agent n, where $n \to \infty$. There is a rich literature studying the asymptotic behavior of Bayesian estimators (with possibly non-iid. observations) from which one can draw to state sufficient conditions on the type distribution. We do not pursue this further here but refer the reader to Ghosal and Van Der Vaart (2007).

7. Conclusion

We found that the principal can strictly benefit from committing to random allocations. We then argued that optimal DIC mechanisms may be too complex to be practical, and that they require strong commitment power on the principal's part. This led us to the class of jury mechanisms, and we found conditions on the environment and the set of allowed mechanisms subject to which jury mechanisms are (approximately) optimal. To wrap up, we comment on many interesting directions for future work.

What are reasonable restrictions on the set of DIC mechanisms that characterize classes of interpretable mechanisms other than jury mechanisms? We explore one such restriction in Appendix D.3 of the supplementary material. Namely, we show that a strengthening of WCP characterizes a class of mechanisms that we call generalized jury mechanisms. Generalized jury mechanisms can be understood as follows: The agents are partitioned into finitely many sets $J_1,\dots,J_m$. First the agents in J_1 decide with their reports whether to allocate to someone in J_2, or whether to also consult the agents in J_2. In the second case, the agents in $J_1 \cup J_2$ decide whether to allocate to or consult J_3, and so on. In contrast to jury mechanisms, this sequential procedure allows for the jury to be determined endogenously. As such, generalized jury mechanisms seem appropriate in situations where the principal has poor information on who would even be a good juror.

It is naturally interesting to extend the analysis to settings with multiple objects, allocated simultaneously or over many periods.[30] As an aside, we acknowledge that

[30]For recent work in this direction, see De Clippel, Eliaz, Fershtman, and Rozen (2021) and Guo

a concern about jury mechanisms could be the fact that some agents are excluded ex ante from winning. In the current model, we can address this concern by randomizing over jury mechanisms. If we enrich the model by letting the principal, say, commit to future allocations, this should lead to stronger foundations for jury mechanisms—agents serving as jurors today can be promised a future spot as a candidate.

The design of optimal jury mechanism is an interesting problem in itself. We envision interesting comparative statics in models where agents who are likely to have good information are also likely to yield a high payoff to the principal. For instance, in the peer selection problem where a club has to nominate a president, a club member who is highly popular may be a suitable candidate (being well-liked for their pleasant qualities) but also have good information about others (being well-acquainted with everyone).

A further intriguing and important line of research could investigate optimal mechanisms when agents have intrinsic interests in the allocation to their peers. For motivation, consider that in a jury mechanism jurors may bias their reports if the set of candidates includes friends or family. Of course, DIC has different implications when agents care about the allocation to their peers. Nevertheless, our results provide some insight in at least two cases.

First, if we impose axiomatically that agents be unable to influence their individual winning probabilities. Second, if agents have the following lexicographic preferences: Agent i strictly prefers one allocation to another if the former has i winning with strictly higher probability. If two allocations have the same winning probability for i, agent i ranks them according to some type-dependent preference. This preference could capture i's opinion regarding which of the others is the most deserving winner if it cannot be i themself. In a jury mechanism of this model, the principal therefore also faces the problem of designing a voting rule to aggregate the information held by the jurors.

and Hörner (2021). Guo and Hörner consider a dynamic setting with a single agent, and where the principal can allocate a new unit in each period. De Clippel et al. consider a setting with multiple agents and independent types, but where the principal cannot commit.

Appendices

In Appendices A to C, respectively, we present the omitted proofs for Sections 4 to 6, respectively. Appendix D contains supplementary material.

A. The set of DIC mechanisms

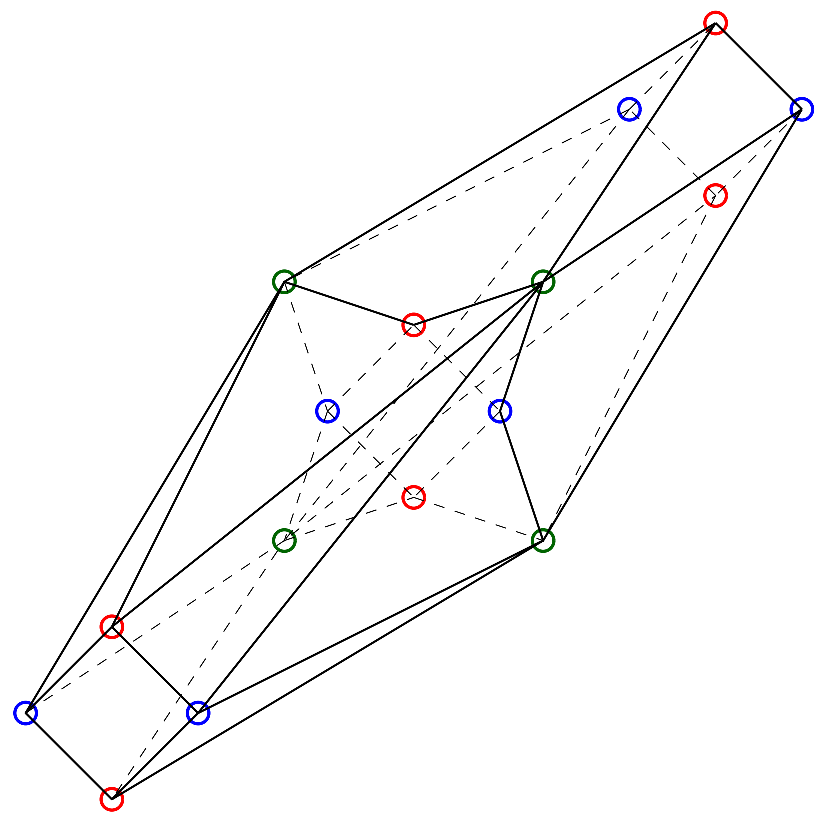


Figure 4: The feasibility graph G in an example with 3 agents. Nodes (edges) of G are depicted as circles (dashed or filled lines). There are two agents with binary types (red and blue nodes), and one agent with ternary types (green nodes). One can view this as the graph G associated with the four-agent environment of Section 4.2.2, except that all nodes of the dummy agent 4 are omitted.

A.1. Stochastic extreme points with four agents

Proof of Proposition 4.4. To keep calculations readable, it will be convient to adopt the following notation: When a DIC mechanism ψ is given, we denote

$$\begin{aligned}
&\psi_1(\theta^a_{-1}) = p^{a|b}, \quad \psi_3(\theta^c_{-3}) = p^{b|c}, \quad \psi_2(\theta^c_{-2}) = p^{c|d}, \quad \psi_3(\theta^e_{-3}) = p^{d|e},\\
&\psi_1(\theta^e_{-1}) = p^{e|f}, \quad \psi_3(\theta^f_{-3}) = p^{f|g}, \quad \psi_2(\theta^a_{-2}) = p^{g|a}.
\end{aligned}$$

In the mechanism φ^*, for example, all probabilities equal $1/2$. These probabilities do not fully describe the mechanism, but these are the only ones needed to evaluate the principal's utility. For a given value of δ, we denote the principal's utility from ψ by $V_\delta(\psi)$; it is given by

$$V_\delta(\psi) = p^{a|b} + p^{b|c} + p^{c|d} + 2p^{d|e} + p^{e|f} + p^{f|g} + p^{g|a} - \delta\left(p^{b|c} + 2p^{d|e} + p^{f|g}\right). \tag{11}$$

Direct computation shows $V_\delta(\varphi^*) = 4 - 2\delta$.

Most of the work shall go towards establishing the following auxiliary claim.

Claim A.1. *Let ψ be a DIC mechanism different from φ^*. Then $V_{1/2}(\psi) \leq V_{1/2}(\varphi^*)$. Further, there exists $\delta_\psi \in (0, 1/2)$ such that $\delta \in (0, \delta_\psi)$ implies $V_\delta(\psi) < V_\delta(\varphi^*)$.*

Proof of Claim A.1. Inspection of Figure 2 shows that ψ must satisfy the following system of inequalities:

$$\begin{aligned}
&p^{a|b} + p^{g|a} \leq 1, \quad p^{a|b} + p^{b|c} \leq 1, \quad p^{c|d} + p^{b|c} \leq 1, \quad p^{c|d} + p^{d|e} \leq 1,\\
&p^{e|f} + p^{d|e} \leq 1, \quad p^{e|f} + p^{f|g} \leq 1, \quad p^{g|a} + p^{f|g} \leq 1.
\end{aligned} \tag{12}$$

Turning to the first part of the claim, we have to show $V_{1/2}(\psi) \leq V_{1/2}(\varphi^*)$. Direct computation shows $V_{1/2}(\varphi^*) = 3$. Using (12), we can bound $V_{1/2}(\psi)$ as follows.

$$\begin{aligned}
V_{1/2}(\psi) =& p^{a|b} + p^{b|c} + p^{c|d} + 2p^{d|e} + p^{e|f} + p^{f|g} + p^{g|a} - \frac{1}{2}\left(p^{b|c} + 2p^{d|e} + p^{f|g}\right) \\
=& p^{a|b} + \frac{p^{b|c}}{2} + p^{c|d} + p^{d|e} + p^{e|f} + \frac{p^{f|g}}{2} + p^{g|a} \\
=& \underbrace{p^{a|b} + p^{g|a}}_{\leq 1} + \underbrace{\frac{p^{b|c} + p^{c|d}}{2}}_{\leq 1/2} + \underbrace{\frac{p^{c|d} + p^{d|e}}{2}}_{\leq 1/2} + \underbrace{\frac{p^{d|e} + p^{e|f}}{2}}_{\leq 1/2} + \underbrace{\frac{p^{e|f} + p^{f|g}}{2}}_{\leq 1/2} \\
\leq& 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 3.
\end{aligned}$$

Hence $V_{1/2}(\psi) \leq V_{1/2}(\varphi^*)$, as promised.

Now consider the second part of the claim. We show the contrapositive: If there exists a sequence $\{\delta_k\}_{k\in\mathbb{N}}$ in $(0, 1/2)$ that converges to 0 and such that $V_{\delta_k}(\psi) \geq 4-\delta_k$ holds for all k, then $\psi = \varphi^*$.

Let $\{\delta_k\}_{k\in\mathbb{N}}$ be such a sequence. For all δ_k, the system (12) implies the following upper bound on $V_{\delta_k}(\psi)$:

$$\begin{aligned}
V_{\delta_k}(\psi) =& \underbrace{p^{a|b} + p^{b|c}}_{\leq 1} + \underbrace{p^{c|d} + p^{d|e}}_{\leq 1} + \underbrace{p^{d|e} + p^{e|f}}_{\leq 1} + \underbrace{p^{f|g} + p^{g|a}}_{\leq 1} \\
& - \delta_k \left(p^{b|c} + 2p^{d|e} + p^{f|g}\right) \\
\leq& 4 - \delta_k \left(p^{b|c} + 2p^{d|e} + p^{f|g}\right).
\end{aligned} \tag{13}$$

Since $V_{\delta_k}(\psi) \geq 4 - 2\delta_k$ and $\delta_k > 0$, we find

$$p^{b|c} + 2p^{d|e} + p^{f|g} \leq 2. \tag{14}$$

Further, since $V_{\delta_k}(\psi) \geq 4 - 2\delta_k$ holds for all k, taking limits implies $V_0(\psi) \geq 4$. Together with the bound in (13) we get $V_0(\psi) = 4$; that is,

$$V_0(\psi) = p^{a|b} + p^{b|c} + p^{c|d} + p^{d|e} + p^{d|e} + p^{e|f} + p^{f|g} + p^{g|a} = 4 \tag{15}$$

Hence (12) and (15) imply

$$p^{a|b} + p^{b|c} = p^{c|d} + p^{d|e} = p^{d|e} + p^{e|f} = p^{f|g} + p^{g|a} = 1. \tag{16}$$

We now bound $V_0(\psi)$ a second time (the equality is by direct computation; the inequality follows from (12)):

$$V_0(\psi) = p^{a|b} + p^{g|a} + p^{b|c} + p^{c|d} + 2p^{d|e} + p^{e|f} + p^{f|g} \leq 3 + 2p^{d|e}. \tag{17}$$

Hence $V_0(\psi) = 4$ implies $p^{d|e} \geq 1/2$. We next claim $p^{d|e} = 1/2$. Towards a contradiction, suppose not, meaning $p^{d|e} > 1/2$. Hence (16) implies $p^{c|d} = p^{e|f} < 1/2$. We infer from (14) and (15) that

$$p^{a|b} + p^{c|d} + p^{e|f} + p^{g|a} \geq 2$$

holds. However, in light of (12) we have $p^{a|b} + p^{g|a} \leq 1$, and hence the previous display requires $p^{c|d} + p^{e|f} \geq 1$. This contradicts $p^{c|d} = p^{e|f} < 1/2$. Thus $p^{d|e} = 1/2$.

Let us now return to the bound derived in (17). In view of $p^{d|e} = 1/2$ and (12), we can infer from (17) that $p^{a|b} + p^{g|a} = p^{b|c} + p^{c|d} = p^{e|f} + p^{f|g} = 2p^{d|e} = 1$ holds. Together with (16), we find

$$p^{a|b} = 1 - p^{b|c} = p^{c|d} = 1 - p^{d|e} = p^{e|f} = 1 - p^{f|g} = p^{g|a}. \tag{18}$$

We already know that $p^{d|e} = 1/2$ holds. Hence all probabilities (18) must equal $1/2$. This shows that ψ agrees with φ^* at all profiles in $\Theta^* = \{\theta^a, \theta^b, \theta^c, \theta^d, \theta^e, \theta^f, \theta^g\}$. By inspecting $\Theta \setminus \Theta^*$, it is now easy to verify that ψ and φ^* also agree on $\Theta \setminus \Theta^*$. □

We now use Claim A.1 to show that all $\delta \in (0, 1/2)$ and all DIC mechanisms ψ different from φ^* satisfy $V_\delta(\psi) < V_\delta(\varphi^*)$. Fixing ψ, inspection of (11) shows that the difference $V_\delta(\psi) - V_\delta(\varphi^*)$ is an affine function of δ; that is, there exist reals x_ψ and y_ψ such that $V_\delta(\psi) - V_\delta(\varphi^*) = x_\psi + \delta y_\psi$ holds for all $\delta \in [0, 1/2]$. Let δ_ψ be as in the conclusion of Claim A.1. We already know that $V_\delta(\psi) < V_\delta(\varphi^*)$ holds if $\delta \in (0, \delta_\psi)$. Hence in what follows we assume $\delta \in [\delta_\psi, 1/2)$. We distinguish two cases. If $y_\psi \leq 0$, we have

$$V_\delta(\psi) - V_\delta(\varphi^*) = x_\psi + \delta y_\psi \leq x_\psi + \frac{\delta_\psi}{2} y_\psi = V_{\frac{\delta_\psi}{2}}(\psi) - V_{\frac{\delta_\psi}{2}}(\varphi^*) < 0,$$

and the proof is complete. If $y_\psi > 0$, then

$$V_\delta(\psi) - V_\delta(\varphi^*) = x_\psi + \delta y_\psi < x_\psi + \frac{1}{2} y_\psi = V_{1/2}(\psi) - V_{1/2}(\varphi^*) \leq 0,$$

and the proof is complete. $\square$

A.2. Sufficient conditions for all extreme points to be deterministic

Lemma A.2. *If $n \leq 3$, then all extreme DIC mechanisms are deterministic.*

Proof of Lemma A.2. If $n = 1$ or $n = 2$, it is easy to verify that all DIC mechanisms are constant. All constant mechanisms are convex combination of deterministic constant mechanisms, proving the claim. In what follows, we assume $n = 3$. Let φ be a stochastic DIC mechanism. We will find a non-zero function f such that $\varphi + f$ and $\varphi - f$ are two other DIC mechanisms. Before delving into the details, the reader may find it helpful to recall the basic idea explained in Section 4.2.3.

We use the following abbreviations. The node (i, θ_{-i}) is the *node of agent i with coordinates θ_{-i}*. A node (i, θ_{-i}) of G is an *interior node* if $\varphi_i(\theta_{-i}) \in (0, 1)$. A node $(i, , \theta_{-i})$ of G is a *0-node* (is a *1-node*) if $\varphi_i(\theta_{-i})$ is equal to 0 (equal to 1).

Most of the work will go towards proving the following auxiliary lemma.

Lemma A.3. *There are non-empty disjoint subsets R and B ("red" and "blue") of the node set V of G such that all of the following are true:*

(1) If $(i, \theta_{-i}) \in R \cup B$, then (i, θ_{-i}) is interior.

(2) For all θ in Θ, exactly one of the following is true:

   (a) The profile θ contains no node which is also in $R \cup B$. We will refer to such a profile as being uncolored.

   (b) The profile θ contains exactly one node in R, one node in B, and one node which is not in $R \cup B$. We will refer to such a profile as being two-colored*.*

Proof of Lemma A.3. It remains to prove that sets R and B with the above properties exist. Since φ is stochastic, we may assume (after possibly relabelling the agents and types) that there exists a profile θ^0 such that $(1, (\theta_2^0, \theta_3^0))$ and $(2, (\theta_1^0, \theta_3^0))$ are interior.

Let Θ_2° denote the set of types θ_2 for which $(1, (\theta_2, \theta_3^0))$ is interior. Let $\Theta_2^\partial = \Theta_2 \setminus \Theta_2^\circ$. Similarly, let Θ_1° denote the set of types θ_1 such that $(2, (\theta_1, \theta_3^0))$ is interior, and let $\Theta_1^\partial = \Theta_1 \setminus \Theta_1^\circ$. Notice that Θ_1° and Θ_2° are non-empty as, by assumption, agents 1 and 2 are enjoying interior winning probabilities at θ^0.

We consider two cases.

Case 1. Let $\Theta_1^\partial \neq \emptyset \neq \Theta_2^\partial$. We use two auxiliary claims.

Claim A.4. *If $\theta_1 \in \Theta_1^\partial$, then $(2,(\theta_1,\theta_3^0))$ is a 0-node. Similarly, if $\theta_2 \in \Theta_2^\partial$, then $(1,(\theta_2,\theta_3^0))$ is a 0-node. If $(\theta_1,\theta_2) \in (\Theta_1^\circ \times \Theta_2^\partial) \cup (\Theta_1^\partial \times \Theta_1^\circ)$, then $(3,(\theta_1,\theta_2))$ is interior.*

Proof of Claim A.4. Consider the first part of the claim. Let $\theta_1 \in \Theta_1^\partial$. Let us pick a type θ_2 in Θ_1°; by assumption, such a type exists. By definition, $(1,(\theta_2,\theta_3^0))$ is interior. By definition of Θ_1^∂, we also know that $(2,(\theta_1,\theta_3^0))$ must be either a 0- or a 1-node. But it cannot be a 1-node since $(2,(\theta_1,\theta_3^0))$ is adjacent to $(1,(\theta_2,\theta_3^0))$, and since the latter is interior. Thus $(2,(\theta_1,\theta_3^0))$ is a 0-node, as desired.

A similar argument establishes the second claim.

As for the third claim, let $(\theta_1,\theta_2) \in \Theta_1^\circ \times \Theta_2^\partial$. The previous paragraphs imply that at the profile $(\theta_1,\theta_2,\theta_3^0)$ the winning probability of agent 1 is a 0-node. Moreover, by definition of Θ_1°, the winning probabiltiy of agent 2 is interior. Thus agent 3's winning probability at this profile must be interior, meaning $(3,(\theta_1,\theta_2))$ is interior. A similar argument shows that $(3,(\theta_1,\theta_2))$ is interior whenever (θ_1,θ_2) is in $\Theta_1^\partial \times \Theta_1^\circ$. $\square$

The second auxiliary result is:

Claim A.5. *Let $\theta_3 \in \Theta_3$. If $\theta_2 \in \Theta_2^\circ$, then $(1,(\theta_2,\theta_3))$ is interior. Similarly, if $\theta_1 \in \Theta_1^\circ$, then $(2,(\theta_1,\theta_3))$ is interior.*

Proof of Claim A.5. We will prove the first part of the claim, the second being similar. Thus let $\theta_2 \in \Theta_2^\circ$. Let us find types θ_1^∂ in Θ_1^∂ and θ_2^∂ in Θ_2^∂; by assumption, such types exists. Consider the profile $(\theta_1^\partial,\theta_2^\partial,\theta_0)$. According to Claim A.4, both agent 1's and agent 2's winning probabilities at this profile equal 0. Thus $(3,(\theta_1^\partial,\theta_2^\partial))$ is a 1-node. But the node $(3,(\theta_1^\partial,\theta_2^\partial))$ is adjacent to $(2,(\theta_1^\partial,\theta_3))$. Thus agent 2's winning probability at $(2,\theta_1^\partial,\theta_3))$ must be 0. This fact implies that agent 2's winning probability at the profile $(\theta_1^\partial,\theta_2,\theta_3)$ must also be 0. Recall that θ_2 is in Θ_2° and that θ_1^∂ is in Θ_1^∂. We can now deduce from Claim A.4 that $(3,(\theta_1^\partial,\theta_2))$ is interior. Thus, at the profile $(\theta_1^\partial,\theta_2,\theta_3)$ we now know that agent 2' winning probability is 0 and that agent 3's winning probability is interior. We conclude that agent 1's winning probability at the node $(1,(\theta_2,\theta_3))$ is interior, as promised. $\square$

We are ready to define the sets R and B. We assign the following colors:

- red to all nodes of agent 1 with coordinates in $\Theta_2^\circ \times \Theta_3$,
- blue to all nodes of agent 3 with coordinates in $\Theta_1^\partial \times \Theta_2^\circ$,
- blue to all nodes of agent 2 with coordinates in $\Theta_1^\circ \times \Theta_3$,
- red to all nodes of agent 3 with coordinates in $\Theta_1^\circ \times \Theta_2^\partial$.

According to Claims A.4 and A.5, all of these nodes are interior. Moreover, all profiles are now either two-colored or uncolored: The profiles in $\Theta_1^\partial \times \Theta_2^\circ \times \Theta_3$ are two-colored via red nodes of agent 1 and blue nodes of agent 3; the profiles in $\Theta_1^\circ \times \Theta_2^\circ \times \Theta_3$ are two-colored via red nodes of agent 1 and blue nodes of agent 2; the profiles in $\Theta_1^\circ \times \Theta_2^\partial \times \Theta_3$ are two-colored via blue nodes of agent 2 and red nodes of 3; and the profiles in $\Theta_1^\partial \times \Theta_2^\partial \times \Theta_3$ are uncolored. ▲

Case 2. Suppose at least one of the sets Θ_1^∂ and Θ_2^∂ is empty. In what follows, we assume that Θ_2^∂ is empty, the other case being analogous (switch the roles of agents 1 and 2).

The assumption that Θ_2^∂ is empty means that $(1, (\theta_2, \theta_3^0))$ is interior for all θ_2. Let Θ_1^* be the set of types θ_1 such that for all $\theta_2 \in \Theta_2$ the node $(3, (\theta_1, \theta_2))$ is interior. Notice that at this point Θ_1^* may or may not be empty; we will make a case distinction further below.

We first claim that $(2, (\theta_1, \theta_3^0))$ is interior whenever θ_1 is in $(\Theta_1 \setminus \Theta_1^*)$. Towards a contradiction, suppose this were false for some $\theta_1 \in (\Theta_1 \setminus \Theta_1^*)$. This means that we can find a type $\theta_2 \in \Theta_2$ such that $(2, (\theta_1, \theta_3^0))$ and $(3, (\theta_1, \theta_2))$ both fail to be interior. But this means that at the profile $(\theta_1, \theta_2, \theta_3^0)$ only agent 1 is enjoying an interior winning probability; this is impossible.

Before proceeding further, let us assign the following colors:

- red to all nodes of agent 1 with coordinates in $\Theta_2 \times \{\theta_3^0\}$. These nodes are all interior since Θ_2^∂ is empty.
- blue to all nodes of agent 2 with coordinates in $(\Theta_1 \setminus \Theta_1^*) \times \{\theta_3^0\}$. The previous paragraph implies that these nodes are all interior.
- blue to all nodes of agent 3 with coordinates in $\Theta_1^* \times \Theta_2$. These nodes are all interior by definition of Θ_1^*.

Observe that all profiles in $\Theta_1 \times \Theta_2 \times \{\theta_3^0\}$ are now either two-colored or uncolored.

If Θ_1^* is empty, then the colors assigned above already define sets R and B with the desired properties, completing the proof. Thus suppose Θ_1^* is non-empty.

Let $\theta_3 \in \Theta_3 \setminus \{\theta_3^0\}$ be arbitrary. The fact that we have already assigned blue to the nodes of agent 3 with coordinates $\Theta_1^* \times \Theta_2$ requires us to assign some colors to the nodes of agents 1 or 2 whose 3'rd coordinate is θ_3. In this step, we will not color any further nodes of agent 3. We make a case distinction.

(1) Suppose that for all θ_1 in Θ_1^* the node $(2,(\theta_1,\theta_3))$ is interior. We assign red to all nodes of agent 2 with coordinates in $\Theta_1^* \times \{\theta_3\}$. This yields a coloring of the profiles in $\Theta_1 \times \Theta_2 \times \{\theta_3^0\}$ with the desired properties: The profiles in $\Theta_1^* \times \Theta_2 \times \{\theta_3\}$ are two-colored via red nodes of agent 2 and blue nodes of 3; the profiles in $(\Theta_1 \setminus \Theta_1^*) \times \Theta_2 \times \{\theta_3\}$ are uncolored.

(2) Suppose there exists $\tilde{\theta}_1$ in Θ_1^* such that $(2,(\theta_1,\theta_3))$ is interior. Given that $(3,(\tilde{\theta}_1,\theta_2))$ is interior for all $\theta_2 \in \Theta_2$ (recall the definition of Θ_1^*), it must be the case that, for all $\theta_2 \in \Theta_2$, the node $(1,(\theta_2,\theta_3))$ is interior.

We next claim that $(2,(\theta_1,\theta_3))$ is interior for all $\theta_1 \in (\Theta_1 \setminus \Theta_1^*)$. Suppose this were false for some $\theta_1 \in (\Theta_1 \setminus \Theta_1^*)$. The previous paragraph tells us that $(1,(\theta_2,\theta_3))$ is interior for all θ_2. Thus, if $(2,(\theta_1,\theta_3))$ fails to be interior, then $(3,(\theta_1,\theta_2))$ would have to be interior for all $\theta_2 \in \Theta_2$; this is a contradiction since θ_1 is in $(\Theta_1 \setminus \Theta_1^*)$.

We now assign red to all nodes of agent 1 with coordinates in $\Theta_2 \times \{\theta_3\}$, and assign blue to all nodes of agent 2 with coordinates in $(\Theta_1 \setminus \Theta_1^*) \times \{\theta_3\}$. The previous two paragraphs imply that all of these nodes are interior. Moreover the profiles in $\Theta_1^* \times \Theta_2 \times \{\theta_3\}$ are two-colored via red nodes of agent 1 and blue nodes of agent 3, and the profiles in $(\Theta_1 \setminus \Theta_1^*) \times \Theta_2 \times \{\theta_3\}$ are two-colored via red nodes of agent 1 and blue nodes of agent 2.

If we apply this case distinction separately to all θ_3 in $\Theta_3 \setminus \{\theta_3^0\}$, this completes the construction of R and B in Case 2. ▲

Cases 1 and 2 together complete the proof of Lemma A.3. □

We now use Lemma A.3 to complete the proof of Lemma A.2. Let

$$\varepsilon = \min_{(i,\theta_{-i}) \in R \cup B} \min\left(\varphi_i(\theta_{-i}), 1 - \varphi_i(\theta_{-i})\right)..$$

Since Θ is finite and $R \cup B$ contains only interior nodes, we have $\varepsilon > 0$. Now let, for all $i \in \{1,2,3\}$, the function $f_i \colon \Theta_{-i} \to \{-\varepsilon, 0, \varepsilon\}$ be defined as follows:

$$f_i(\theta_{-i}) = \begin{cases} -\varepsilon, & \text{if } (i, \theta_{-i}) \in R, \\ \varepsilon, & \text{if } (i, \theta_{-i}) \in B, \\ 0, & \text{if } (i, \theta_{-i}) \notin R \cup B. \end{cases}$$

Let $f = (f_1, f_2, f_3)$. The choice of ε and the assumed properties of R and B imply that $f \neq 0$ and that $\varphi + f$ is a DIC mechanism. It is also easy to verify that all $\theta \in \Theta$ satisfy $\sum_{i=1}^{n} f_i(\theta_{-i}) = 0$. Hence $\varphi - f$ is another DIC mechanism, and hence φ is not an extreme point. □

Lemma A.6. *If for all agents i we have $|\Theta_i| \leq 2$, then all extreme DIC mechanisms are deterministic.*

Proof of Lemma A.6. Let us relabel types such that we have $\Theta_i \subseteq \{0,1\}$ for all i. First, suppose we have $\Theta_i = \{0,1\}$ for all i. It is easy to see that φ is deterministic DIC mechanism if and only if it is a perfect matching of the graph that has node set $\{0,1\}^n$ and where two nodes are adjacent if and only if they differ in exactly one coordinate. Since this graph is bi-partite, Theorem 11.4 of Korte and Vygen (2018) implies that all extreme points are deterministic.

The claim for the general case, where we have $\Theta_i \subseteq \{0,1\}$ for all i, follows from the previous paragraph by viewing a DIC mechanism on Θ as a mechanism on $\{0,1\}^n$ that ignores the reports of agents i whose type spaces Θ_i are singletons. □

Lemma A.7. *If $|\{i \in \{1,\ldots,n\} \colon |\Theta_i| \geq 2\}| \leq 2$, then all extreme points of the set of DIC mechanisms are deterministic.*

Proof of Lemma A.7. We may assume that $n \geq 3$, as otherwise the claim follows from Lemma A.6. We will prove the claim for the case where $|\{i \in \{1,\ldots,n\} \colon |\Theta_i| \geq 2\}| = 2$, the other cases being simpler. After possibly relabelling the agents, suppose we have $|\Theta_1| \geq 2$ and $|\Theta_2| \geq 2$. Let φ be a stochastic DIC mechanism. Notice that at all profiles θ where either agent 1 or agent 2 is enjoying an interior winning probability, there must be an agent in $\{3,\ldots,n\}$ who is also enjoying an interior winning probability; let i_θ denote one such agent. For a number $\varepsilon > 0$ to be chosen later, consider $f \colon \Theta \to \{-\varepsilon, 0, \varepsilon\}^n$ defined for all θ as follows:

(1) If $\varphi_1(\theta) \in (0,1)$ and $\varphi_2(\theta) \in (0,1)$, let $f_1(\theta) = \varepsilon$, let $f_2(\theta) = -\varepsilon$, and let $f_i(\theta) = 0$ for all $i \notin \{1,2\}$.

(2) If $\varphi_1(\theta) \in (0,1)$ and $\varphi_2(\theta) \notin (0,1)$, let $f_1(\theta) = \varepsilon$, let $f_{i_\theta}(\theta) = -\varepsilon$, and let $f_i(\theta) = 0$ for all $i \notin \{1, i_\theta\}$.

(3) If $\varphi_1(\theta) \notin (0,1)$ and $\varphi_2(\theta) \in (0,1)$, let $f_2(\theta) = -\varepsilon$, let $f_{i_\theta}(\theta) = \varepsilon$, and let $f_i(\theta) = 0$ for all $i \notin \{2, i_\theta\}$.

Using that, for all θ, agent i_θ has a singleton type space, it is easy to see that $\varphi + f$ and $\varphi - f$ are two DIC mechanisms distinct from φ whenever ε is sufficiently small. Thus φ is not an extreme point. $\square$

A.3. Proof of Theorem 4.2

Proof of Theorem 4.2. Lemmata A.2, A.6 and A.7 imply that all extreme points are deterministic if at least one of the conditions (1) to (3) from Theorem 4.2 holds. Thus suppose conditions (1) to (3) all fail. We have to show that the set of DIC mechanisms admits a stochastic extreme point. We know from Proposition 4.3 that a stochastic extreme point exists in the hypothetical situation where $n = 4$ and the set of type profiles is $\hat{\Theta} = \{\ell, r\} \times \{u, d\} \times \{L, M, R\} \times \{0\}$. Since conditions (1) to (3) all fail, we can relabel the agents and types such that agents 1 to 4 have these sets as subsets of their respective sets of types. Using the stochastic extreme from Proposition 4.3, it is straightforward to define a stochastic extreme point for the actual set of type profiles with n agents. We omit the details. $\square$

B. Institutional constraints

We introduce some useful pieces of notation. Given a mechanism φ and an agent j, let I_j denote the set of agents i such that j's winning probability φ_j is non-constant in i's report. Let A_i denote the set of j such that $i \in I_j$. Note that DIC for agent i is equivalent to $i \notin A_i$, and equivalent to $i \notin I_i$. Agent i influences j if $j \in A_i$; equivalently, if $i \in I_j$.

B.1. Proof of Theorem 5.1

Proof of Theorem 5.1. It is clear that constant mechanisms are SA and SCP, and we have shown that SCP mechanisms are constant. Thus let φ be DIC and SA. We show φ is constant.

Let T denote the common type space. Let $N = \{1, \ldots, n\}$ denote the set of agents. Given an integer k, let T^k with generic element θ^k denote the k-fold Cartesian product of T.

We will frequently consider profiles obtained from a profile θ^{n-1} in T^{n-1} by replacing one entry of θ^{n-1}. For instance, we write (t, θ^{n-1}_{-j}) to denote the profile obtained by replacing the j'th entry of θ^{n-1} by t. Similarly, $(t, t', \theta^{n-1}_{-jj^*})$ denotes the profile obtain by replacing the j'th and j^*'th entries, respectively, of θ^{n-1} by t and t', respectively. Since for all i the winning probability φ_i is invariant with respect to permutations of $N \setminus \{i\}$, there is no need to track which of the others makes which report.

We begin the following auxiliary claim.

Claim B.1. *For all i in N, all t and t' in T, and all θ^{n-1} in T^{n-1} we have*

$$0 = \sum_{j=1}^{n-1} \left(\varphi_i(t, \theta^{n-1}_{-j}) - \varphi_i(t', \theta^{n-1}_{-j}) \right). \tag{19}$$

Proof of Claim B.1. For this proof, we find it convenient to spell out the individual winning probabilities as follows: $\varphi_i(r_i = \theta_i, r_j = \theta_j, r_{-ij} = \theta_{-ij})$ means i's winning probability when i reports θ_i, j reports θ_j, and all remaining agents report θ_{-ij}.

As in the proof of Lemma 5.5 from the main text, we can establish the following: All distinct i and j, all t and t' in T, and all θ in T^n satisfy

$$\begin{aligned} &\varphi_i(r_i = \theta_i, r_j = t, r_{-ij} = \theta_{-ij}) - \varphi_i(r_i = \theta_i, r_j = t', r_{-ij} = \theta_{-ij}) \\ =&\varphi_j(r_i = t, r_j = \theta_j, r_{-ij} = \theta_{-ij}) - \varphi_j(r_i = t', r_j = \theta_j, r_{-ij} = \theta_{-ij}). \end{aligned} \tag{20}$$

Consider summing (20) over all j in $N \setminus \{i\}$. This summation yields

$$\begin{aligned} &\sum_{j \in N \setminus \{i\}} \left(\varphi_i(r_i = \theta_i, r_j = t, r_{-ij} = \theta_{-ij}) - \varphi_i(r_i = \theta_i, r_j = t', r_{-ij} = \theta_{-ij}) \right) \\ =&\sum_{j \in N \setminus \{i\}} \left(\varphi_j(r_i = t, r_j = \theta_j, r_{-ij} = \theta_{-ij}) - \varphi_j(r_i = t', r_j = \theta_j, r_{-ij} = \theta_{-ij}) \right). \end{aligned}$$

On the right side, the profiles considered are all of the form $(r_i = t, r_{-i} = \theta_{-i})$ and $(r_i = t', r_{-i} = \theta_{-i})$, respectively. Note that by DIC we have $\varphi_i(r_i = t, r_{-i} = \theta_{-i}) - \varphi_i(r_i = t', r_{-i} = \theta_{-i}) = 0$. Hence the right side of the previous display equals

$$\sum_{j \in N} \left(\varphi_j(r_i = t, r_{-i} = \theta_{-i}) - \varphi_j(r_i = t', r_{-i} = \theta_{-i}) \right).$$

Since the object is always allocated, the term in the previous display equals 0. Hence

$$\sum_{j\in N\setminus\{i\}} (\varphi_i(r_i=\theta_i, r_j=t, r_{-ij}=\theta_{-ij}) - \varphi_i(r_i=\theta_i, r_j=t', r_{-ij}=\theta_{-ij})) = 0.$$

We now revert to our usual notation. By DIC, we may drop i's report from φ_i. Since φ_i is permutation-invariant with respect to $N\setminus\{i\}$, we may arbitrarily enumerate θ_{-i} as $\theta^{n-1}=(\theta_1,\dots,\theta_{n-1})$ and write

$$\begin{aligned}\varphi_i(r_i=\theta_i, r_j=t, r_{-ij}=\theta_{-ij}) &= \varphi_i(t,\theta^{n-1}_{-j}) \qquad \text{and}\\ \varphi_i(r_i=\theta_i, r_j=t', r_{-ij}=\theta_{-ij}) &= \varphi_i(t',\theta^{n-1}_{-j}).\end{aligned}$$

Thus we obtain the desired equality $\sum_{j=1}^{n-1}\left(\varphi_i(t,\theta^{n-1}_{-j}) - \varphi_i(t',\theta^{n-1}_{-j})\right)=0.$ □

Let us fix an arbitrary type t' in T. For each $k\in\{0,\dots,n-1\}$, let T_k denote the subset of profiles in T^{n-1} where exactly k entries differ from t'. For all i, let p_i denote i's winning probability when all other agents report t'. We will show via induction over k that i's winning probability is equal to p_i whenever the others report a profile in T_k. This completes the proof since we have $T^{n-1}=\cup_{k=0}^{n-1}T_k$.

The induction start, $k=0$, is trivial in view of the definition of p_i and the fact that T_0 contains exactly the profile where all entries equal t'.

Now let $k\geq 1$, $i\in N$, and $\theta^{n-1}\in T_k$. Suppose $\varphi_i(\hat\theta^{n-1})=p_i$ holds for all $\hat\theta^{n-1}$ in $T_0\cup\ldots\cup T_{k-1}$. Let t denote an abitrary entry of θ^{n-1} that differs from t', and let j^* denote an index such that $\theta_{j^*}=t$. Let $\tilde\theta^{n-1}=(t',\theta^{n-1}_{-j^*})$; that is, the profile $\tilde\theta^{n-1}$ is obtained from θ^{n-1} by replacing the entry of j^* by t'. Claim B.1 implies

$$\sum_{j=1}^{n-1}\varphi_i(t,\tilde\theta^{n-1}_{-j}) = \sum_{j=1}^{n-1}\varphi_i(t',\tilde\theta^{n-1}_{-j}). \tag{21}$$

Consider the profiles appearing in the sum on the left of (21). Recall that exactly k of $n-1$ entries of θ^{n-1} differ from t', one of them being j^*. We observe the following:

- Let $j\neq j^*$. Then the profile $(t,\tilde\theta^{n-1}_{-j})$ is obtained from θ^{n-1} by replacing the entries of j and j^*, respectively, with t and t', respectively. Since the entry of j^* equals t, the resulting profile equals (t',θ^{n-1}_{-j}), up to permutations.

  If j is one of the $k-1$ entries other than j^* where $\theta_j\neq t'$, then (t',θ^{n-1}_{-j}) is in

T_{k-1}. If j is one of the $n-1-k$ entries such that $\theta_j = t'$, then (t', θ^{n-1}_{-j}) is the original profile θ^{n-1}; there are $n-1-k$ such entries.

- Let $j = j^*$. Then the profile $(t, \tilde{\theta}^{n-1}_{-j})$ is obtained from θ^{n-1} by first replacing the entry of j^* by t', and then replacing the entry of j^* by t. This results in the original profile θ^{n-1}.

In view of the induction hypothesis and using that φ_i is invariant with respect to permutations, we find that the left side of (21) satisfies

$$\sum_{j=1}^{n-1} \varphi_i(t, \tilde{\theta}^{n-1}_{-j}) = (k-1)p_i + (n-k)\varphi_i(\theta^{n-1}).$$

Now consider the sum on the right of (21). In the profile $(t', \tilde{\theta}^{n-1}_{-j})$, the entries of j and j^* are both replaced by t'. Thus, for all j, the resulting profile has at most $k-1$ entries different from t'. By the induction hypothesis, therefore, the right side of (21) satisfies

$$\sum_{j=1}^{n-1} \varphi_i(t', \tilde{\theta}^{n-1}_{-j}) = (n-1)p_i.$$

We conclude from (21) and the previous two paragraphs that

$$(k-1)p_i + (n-k)\varphi_i(\theta^{n-1}) = (n-1)p_i$$

holds. Equivalently, $(n-k)(\varphi_i(\theta^{n-1}) - p_i) = 0$. Since $k \leq n-1$, we find $\varphi_i(\theta^{n-1}) = 0$, as promised. $\square$

B.2. Proof of Theorem 5.3

Proof of Theorem 5.3. Let φ be a deterministic WA jury mechanism. All jury mechanisms are WCP (see the paragraphs following Definitions 5 and 6). Thus φ is in Φ^*. Since φ is determinstic, it is an extreme point of Φ^*.

The previous paragraph establishes that all determinstic WA jury mechanisms are extreme points of Φ^*. It is also clear that a deterministic jury mechanism is WA if it is invariant with respect to all permutations of those agents who influence the allocation. To complete the proof, we show the following:

(1) If φ is a mechanism in Φ^*, then there is a convex combination of deterministic

WA jury mechanisms that equals φ.

(2) If φ is a deterministic jury mechanism in Φ^*, then it is invariant with respect to permutations of all agents who influence the allocation.

Let φ be in Φ^*. Recall the notation introduced at the beginning of Appendix B. Let $I = \cup_{i=1}^n I_i$ denote the set of agents with respect to whose reports φ is non-constant. We may assume that φ is non-constant, meaning $I \neq \emptyset$, as otherwise the proof is trivial.

We proceed along a series of claims. The first is simply a restatement of WA; the proof is omitted.

Claim B.2. *All agents i and j satisfy the following.*

(1) The allocation φ_i is invariant with respect to all permutations of I_i.

(2) If $(A_i \setminus \{j\}) \cap (A_j \setminus \{i\}) \neq \emptyset$, then $A_i \setminus \{j\} = A_j \setminus \{i\}$.

Consider the binary relation $\sim$ on I defined as follows: Given i and j in I, we have $i \sim j$ if and only if $(A_i \setminus \{j\}) \cap (A_j \setminus \{i\}) \neq \emptyset$. Claim B.2 implies that $i \sim j$ is equivalent to $\emptyset \neq A_i \setminus \{j\} = A_j \setminus \{i\}$.

Claim B.3. *Let i and j in I be such that $i \sim j$. If $i \in A_j$, then $j \in A_i$.*

Proof of Claim B.3. We will show the contrapositive. Let $j \notin A_i$. We have to show $i \notin A_j$. For this proof, let us write $\varphi_i(t, t', \theta_{-ij})$ and $\varphi_j(t, t', \theta_{-ij})$, respectively, for i's and j's winning probabilites, respectively, when i reports some type t, j reports some type t', and the others report some profile θ_{-ij}.

Let θ be an arbitrary profile. Consider the permutation where exactly the types of i and j are swapped. Claim B.2, DIC, and the definition of $\sim$ imply $A_i \setminus \{i,j\} = A_i \setminus \{j\} = A_j \setminus \{i\} = A_j \setminus \{i,j\}$. Hence $A_i^c \setminus \{i,j\} = A_j^c \setminus \{i,j\}$. By definition of A_i^c and A_j^c, the permutation does not affect the allocation of agents in $A_i^c \setminus \{i,j\} = A_j^c \setminus \{i,j\}$. WA implies that the permutation does not affect the allocation of agents in $A_i \setminus \{i,j\} = A_j \setminus \{i,j\}$. Since the object is always allocated, we find

$$\varphi_i(\theta_i, \theta_j, \theta_{-ij}) + \varphi_j(\theta_i, \theta_j, \theta_{-ij}) = \varphi_i(\theta_j, \theta_i, \theta_{-ij}) + \varphi_j(\theta_j, \theta_i, \theta_{-ij}).$$

By DIC and since $j \notin A_i$, we have that φ_j depends neither on i's nor j's report. Hence the previous equation is equivalent to

$$\varphi_i(\theta_i, \theta_j, \theta_{-ij}) = \varphi_i(\theta_j, \theta_i, \theta_{-ij}).$$

DIC implies that $\varphi_i(\theta_j, \theta_i, \theta_{-ij})$ is constant in θ_j. Hence the previous display implies that $\varphi_i(\theta_i, \theta_j, \theta_{-ij})$ must be constant in θ_j, too. This shows that φ_i is constant in j's report, meaning $i \notin A_j$. $\square$

Claim B.4. *The relation $\sim$ is an equivalence relation.*

Proof of Claim B.4. It is clear that $\sim$ is symmetric. To see that $\sim$ is reflexive, note that $i \in I$ is equivalent to $A_i \neq \emptyset$. Hence $A_i \cap A_i \neq \emptyset$, implying $i \sim i$.

Turning to transitivity, let i, j and k be agents in I such that $(A_i \setminus \{j\}) \cap (A_j \setminus \{i\}) \neq \emptyset$ and $(A_j \setminus \{k\}) \cap (A_k \setminus \{j\}) \neq \emptyset$. Claim B.2 implies $A_i \setminus \{j\} = A_j \setminus \{i\}$ and $A_j \setminus \{k\} = A_k \setminus \{j\}$. We distinguish two cases.

First, suppose there exists $\ell \in A_j \setminus \{i, k\}$. Then $j \notin A_j$ and $A_i \setminus \{j\} = A_j \setminus \{i\}$ imply $\ell \in A_i \setminus \{k\}$. Similarly, we have $\ell \in A_k \setminus \{i\}$. Thus $i \sim k$.

Second, suppose $A_j \setminus \{i, k\} = \emptyset$. Since $A_j \setminus \{i\}$ and $A_j \setminus \{k\}$ are both non-empty, we have $A_j = \{i, k\}$. Claim B.3 implies $j \in A_i$ and $j \in A_k$. In view of DIC, this implies $i \neq j \neq k$. In particular, we have $j \in A_i \setminus \{k\} \cap A_k \setminus \{i\}$. Thus $i \sim k$. $\square$

Claim B.4 implies that we may partition I into finitely-many non-empty $\sim$-equivalence classes. (Recall that I is non-empty.) Let $\mathcal{J}$ denote the collection of $\sim$-equivalence classes.

Claim B.5. *Let J and J' be distinct sets in $\mathcal{J}$. Let i and j be in J. If $j \in A_i$, then both of the following are true:*

(1) For all distinct ℓ and k in J we have $\ell \in A_k$; that is, all agents in J influence all others in J.

(2) If k is in J', then $J \cap A_k = \emptyset$; that is, no agent outside of J influences an agent in J.

Proof of Claim B.5. Consider (1). If $\ell = j$, then the claim follows from $j \in A_i$ and $i \sim k$. If $\ell \neq j$, then $\ell \sim i$ and $j \in A_i$ imply $j \in A_\ell$. Hence, by Claim B.3, we have $\ell \in A_j$. Now $j \sim k$ and $\ell \neq k$ imply $\ell \in A_k$.

Consider (2). Towards a contradiction, suppose there exists k in J' such that $\ell \in J \cap A_k$. We will show that $k \in J$; this contradicts the fact that J and J' are disjoint.

Note that $j \in A_i$ and $\ell \in A_k$ imply $i \neq j$ and $k \neq \ell$ (else DIC is contradicted). Suppose for a moment $\ell = j$. This implies $j \in (A_k \setminus \{i\}) \cap (A_i \setminus \{k\})$, and hence $i \sim k$, and hence $k \in J$. In what follows, we may thus assume $\ell \neq j$.

As an intermediate step, we claim $j \in A_\ell$. Since $i \sim \ell$, we have $A_i \setminus \{\ell\} = A_\ell \setminus \{i\}$. Using $j \in A_i$ and $\ell \neq j$, we infer $j \in A_i \setminus \{\ell\}$, and hence $j \in A_\ell$.

Claim B.3 and $j \in A_\ell$ together imply $\ell \in A_j$. Altogether, we now know that $\ell \in A_j \cap A_k$. We also have $j \neq \ell$ (by assumption) and $k \neq \ell$ (by $\ell \in A_k$ and DIC). Thus $\ell \in A_j \setminus \{k\}$ and $\ell \in A_k \setminus \{j\}$ hold. In particular, we find $j \sim k$, and hence $k \in J$. □

Claim B.6. *Let $J \in \mathcal{J}$. The allocation $(\varphi_i)_{i \in J}$ is constant in the reports of agents in J.*

Proof of Claim B.6. Towards a contradiction, suppose $(\varphi_i)_{i \in J}$ is non-constant in the reports of agents in J. Part (1) of Claim B.5 implies that all agents in J can influence all others agents in J. Part (2) of Claim B.5 implies that no agent outside of J influences an agent in J. Thus J is balanced. According to WCP, therefore, the sum $\sum_{i \in J} \varphi_i$ is constant in the reports of all agents. Let p denote this constant probability. We have $p > 0$ since else the agents in J enjoy a constant winning probability, contradicting the assumption that $(\varphi_i)_{i \in J}$ is non-constant in the reports of agents in J. Consider the functions $(\varphi_i/p)_{i \in J}$ obtained by scaling the winning probabilities of agents in J by $1/p$. These functions define a DIC mechanism in a setting where the set of agents is J. This mechanism is SA since φ is WA and since, as observed above, all agents in J influence all other agents in J. Theorem 5.1 implies that $(\varphi_i/p)_{i \in J}$ is constant; contradiction. □

Given J in $\mathcal{J}$ and i in J, notice that, by Claim B.2, the set $A_i \setminus J$ is the same for all i in J. Let us denote this common set by A_J; that is, all i in J satisfy

$$A_J = A_i \setminus J \tag{22}$$

Let $A_\emptyset$ denote the (possibly empty) set of agents i such that φ_i is constant in the reports of all agents.

Claim B.7. *The collection $\{A_J\}_{J \in \mathcal{J} \cup \{\emptyset\}}$ partitions the set of agents.*

Proof of Claim B.7. Let i be an arbitrary agent. If i has a constant winning probability, then $i \in A_\emptyset$. Else, there is another agent j such that $i \in A_j$. Since $\mathcal{J}$ partitions the agents who influence the allocation, there exists $J \in \mathcal{J}$ such that

$j \in J$. Claim B.6 and $i \in A_j$ imply $i \notin J$. Hence $i \in A_J$. Lastly, it is immediate from WA and the definition of $A_\emptyset$ that the sets $\{A_J\}_{J \in \mathcal{J} \cup \{\emptyset\}}$ are disjoint. $\square$

Let J be in $\mathcal{J}$. If an agent i is in A_J, then Claim B.7 implies that i's winning probability only depends on the reports of agents in J. Thus in what follows we write $\varphi_i(\theta_J)$ for i's winning probability. By definition of A_J, the sum $1 - \sum_{i \in A_J} \varphi_i = \sum_{i \notin A_J} \varphi_i$ is constant in the reports of J. Thus $\sum_{i \in A_J} \varphi_i$ is constant in the reports of J, too. Claim B.7 implies $A_J \cap A_{J'} = \emptyset$ whenever J' is distinct from J. Hence $\sum_{i \in A_J} \varphi_i$ is constant in the reports of agents outside of J. Thus $\sum_{i \in A_J} \varphi_i$ must be constant in all reports. We denote the constant probability by α_J.

The probability $\sum_{i \in A_\emptyset} \varphi_i$ is constant by the definition of $A_\emptyset$. We denote the constant value by $\alpha_\emptyset$. Since $\{A_J\}_{J \in \mathcal{J} \cup \{\emptyset\}}$ partitions the set of agents (Claim B.7), we have $\sum_{J \in \mathcal{J} \cup \{\emptyset\}} \alpha_J = 1$.

We next define a collection of auxiliary jury mechanisms.

For all J in $\mathcal{J} \cup \{\emptyset\}$ such that $\alpha_J > 0$, let ψ_J denote the following mechanism: For all i in A_J, agent i is allocated the object with probability $\varphi_i(\theta_J)/\alpha_J$. For all i not in A_J, agent i is allocated the object with probability 0. For all J in $\mathcal{J} \cup \{\emptyset\}$ such that $\alpha_J = 0$, let ψ_J be an arbitrary constant mechanism.

Claim B.8. *The collection $\{\psi_J\}_{J \in \mathcal{J} \cup \{\emptyset\}}$ is a collection of WA jury mechanisms satisfying $\sum_{J \in \mathcal{J} \cup \{\emptyset\}} \alpha_J \psi_J = \varphi$.*

Note that the jury mechanisms $\{\psi_J\}_{J \in \mathcal{J} \cup \{\emptyset\}}$ have not been proven to be deterministic.

Proof of Claim B.8. For all J, the mechanism ψ_J is a well-defined mechanism since α_J is the constant probability that the object is allocated to an agent in J. It inherits WA from φ. If $i \in A_J$, then $i \notin J$ (recall (22)) and i's' winning probability depends only on the reports of agents in J. This argument shows that ψ_J is a jury mechanism and that $\sum_{J \in \mathcal{J} \cup \{\emptyset\}} \alpha_J \psi_J = \varphi$ holds. $\square$

Claim B.9. *For all $J \in \mathcal{J} \cup \{\emptyset\}$, the mechanism ψ_J is a convex combination of deterministic WA jury mechanisms.*

Proof of Claim B.9. Let $J \in \mathcal{J} \cup \{\emptyset\}$. Let Ψ be the set of jury mechanisms having all following properties: For all i, the mechanism is non-constant in the report of agent i only if i is in J; the mechanism is invariant with respect to permutations of agents in J. The set Ψ is compact and convex, and it contains ψ_J. Hence the claim follows from the Krein-Milman theorem if we can show that all extreme points are deterministic. To that end, consider a stochastic mechanism in Ψ. At a profile where the mechanism randomizes, shift a small mass between two agents with interior winning probabilities; do the same at all profile obtained by permuting the reports of agents in J. Using that the mechanism is in Ψ, it easy to see that this yields two other mechanisms in Ψ, the convex hull of which contains the given stochastic one. $\square$

Claims B.8 and B.9, together with the equation $\sum_{J\in\mathcal{J}\cup\{\emptyset\}}\alpha_J = 1$ imply that φ is a convex combination of deterministic WA jury mechanisms, as promised.

Lastly, suppose that φ is a deterministic jury mechanism in Φ^*. There is nothing to prove if φ is constant. If φ is non-constant, the preceding arguments establish that $\mathcal{J}$ consists of exactly one equivalence class (else φ would not be deterministic). The claims follows easily from here. $\square$

C. Symmetric information

In this part of the appendix, we prove Theorem 6.1. As mentioned in main text, we will prove the result under a condition that is weaker than Assumption 2. When the number n of agents is clear from the context, we write μ instead of μ_n. To distinguish a random variable from its realization, we denote the former using a tilde $\sim$. For a subset of $\mathcal{N}$ of agents, we denote profile of their types by $\theta_{\mathcal{N}}$.

Assumption 3. For all n, there exists m strictly larger than n with the following property: Denoting $\mathcal{N} = \{1,\dots,n\}$ and $\mathcal{N}' = \{n+1,\dots,m\}$, there is a function $g\colon \Theta_{\mathcal{N}'}\times\Theta_{\mathcal{N}}\to\mathbb{R}_+$ with the following two properties:

(1) For all i in $\mathcal{N}$, all ω_i in Ω_i and $\theta_{\mathcal{N}\setminus\{i\}}$ in $\Theta_{\mathcal{N}\setminus\{i\}}$ we have

$$\begin{aligned}&\mu\left(\tilde{\omega}_i=\omega_i,\tilde{\theta}_{\mathcal{N}\setminus\{i\}}=\theta_{\mathcal{N}\setminus\{i\}}\right)\\ &=\sum_{\theta_{\mathcal{N}'}\in\Theta_{\mathcal{N}'}}\sum_{\theta_i\in\Theta_i} g(\theta_{\mathcal{N}'},\theta_{\mathcal{N}\setminus\{i\}},\theta_i)\mu\left(\tilde{\omega}_i=\omega_i,\tilde{\theta}_{\mathcal{N}'}=\theta_{\mathcal{N}'}\right).\end{aligned}\tag{23}$$

(2) For all $\theta_{\mathcal{N}'} \in \Theta_{\mathcal{N}'}$ we have

$$\sum_{\theta_{\mathcal{N}} \in \Theta_{\mathcal{N}}} g(\theta_{\mathcal{N}'}, \theta_{\mathcal{N}}) = 1. \tag{24}$$

Lemma C.1. *Assumption 2 implies Assumption 3.*

Proof of Lemma C.1. Let $m = 2n$. Let $\mathcal{N} = \{1, \ldots, n\}$ and $\mathcal{N}' = \{n+1, \ldots, 2n\}$, and let $\xi \colon \mathcal{N} \to \mathcal{N}'$ be a bijection. It is straightforward to verify that the function g defined as follows has the desired properties: For all $(\theta_{\mathcal{N}}, \theta_{\mathcal{N}'})$, let $g(\theta_{\mathcal{N}}, \theta_{\mathcal{N}'}) = 1$ if for all $i \in \mathcal{N}$ the types of i and $\xi(i)$ agree; else, let $g(\theta_{\mathcal{N}}, \theta_{\mathcal{N}'}) = 0$. □

Proof of Theorem 6.1. It suffices to prove claim under Assumption 3. Let $\varepsilon > 0$. For all n, let Π_n denote the principal's utility from an optimal DIC mechanism with n agents. Since $\lim_{n\to\infty} \mathbb{E}[\max_{i\in\{1,\ldots,n\}} \omega_i]$ is assumed to exist, the sequence $\{\Pi_n\}_{n\in\mathbb{N}}$ is bounded and weakly increasing; in particular, the sequence has a limit. We may thus find an integer n^* such that the utility of an optimal DIC mechanism is within ε of $\lim_{n\to\infty} \Pi_n$ whenever there are at least n^* agents. Let φ^* denote an optimal DIC mechanism for n^* agents, yielding the principal an expected utility of Π_{n^*}. Given this choice of n^*, we can complete the proof by showing that for all sufficiently large values of n, there is a jury mechanism ψ with n agents which yields the principal an expected utility equal to Π_{n^*}.

Let $\mathcal{N} = \{1, \ldots, n^*\}$. For this choice of $\mathcal{N}$, let us find m and g as in Assumption 3. Let $\mathcal{N}' = \{n+1, \ldots, m\}$. We will show that if there are at least m agents, there is a jury mechanism that generates an expected utility equal to Π_{n^*}. In fact, since the principal can always ignore the reports of additional agents, it suffices to do so for the case where there are exactly m agents. We define our candidate jury mechanism as follows: For all $i \in \mathcal{N}$, let $\psi_i \colon \Theta_{\mathcal{N}'} \to \mathbb{R}^n$ be defined by

$$\forall_{\theta_{\mathcal{N}'} \in \Theta_{\mathcal{N}'}}, \quad \psi_i(\theta_{\mathcal{N}'}) = \sum_{\theta_{\mathcal{N}} \in \Theta_{\mathcal{N}}} g(\theta_{\mathcal{N}'}, \theta_{\mathcal{N}}) \varphi_i^*(\theta_{\mathcal{N}\setminus\{i\}}).$$

For all i in $\mathcal{N}'$, let $\psi_i = 0$. Let $\psi = (\psi_1, \ldots, \psi_m)$.

Notice that ψ only depends on the reports of agents in $\mathcal{N}'$. Since $\mathcal{N}'$ is disjoint from $\mathcal{N}$, we can show that ψ is a jury mechanism for the setting with m agents by showing that ψ maps to probability distributions over $\mathcal{N}$. It is clear that φ is non-negative (as g and ψ^* are non-negative). To verify that ψ almost surely allocates to an agent in $\mathcal{N}$, we observe that for all profiles $\theta_{\mathcal{N}'}$ we have the following (the first equality is by definition of ψ; the second is from the fact that φ^* is a well-defined mechanism when the set of agents is $\mathcal{N}$; the third is from (24)):

$$\begin{aligned}\sum_{i\in\mathcal{N}}\psi_i(\theta_{\mathcal{N}'}) =&\sum_{i\in\mathcal{N}}\sum_{\theta_{\mathcal{N}}\in\Theta_{\mathcal{N}}} g(\theta_{\mathcal{N}'},\theta_{\mathcal{N}})\varphi_i^*(\theta_{\mathcal{N}\setminus\{i\}})\\ =&\sum_{\theta_{\mathcal{N}}\in\Theta_{\mathcal{N}}} g(\theta_{\mathcal{N}'},\theta_{\mathcal{N}})\\ =&1,\end{aligned}$$

as desired.

It remains to show that ψ generates an expected utility of Π_{n^*} for the principal. Recall that φ^* denotes an optimal DIC mechanism for n^* agents and generates Π_{n^*}. Thus

$$\Pi_{n^*}=\sum_{i\in\mathcal{N}}\sum_{\theta_{\mathcal{N}\setminus\{i\}}}\sum_{\omega_i}\omega_i\mu\left(\tilde{\omega}_i=\omega_i,\tilde{\theta}_{\mathcal{N}-i}=\theta_{\mathcal{N}\setminus\{i\}}\right)\varphi_i^*(\theta_{\mathcal{N}\setminus\{i\}}).$$

We infer from (23) implies that this equals

$$\begin{aligned}&\sum_{i\in\mathcal{N}}\sum_{\theta_{\mathcal{N}\setminus\{i\}}}\sum_{\omega_i}\omega_i\sum_{\theta_{\mathcal{N}'}}\sum_{\theta_i} g(\theta_{\mathcal{N}'},\theta_{\mathcal{N}\setminus\{i\}},\theta_i)\mu\left(\tilde{\omega}_i=\omega_i,\tilde{\theta}_{\mathcal{N}'}=\theta_{\mathcal{N}'}\right)\varphi_i^*(\theta_{\mathcal{N}\setminus\{i\}})\\ =&\sum_{i\in\mathcal{N}}\sum_{\omega_i}\sum_{\theta_{\mathcal{N}'}}\omega_i\mu\left(\tilde{\omega}_i=\omega_i,\tilde{\theta}_{\mathcal{N}'}=\theta_{\mathcal{N}'}\right)\sum_{\theta_{\mathcal{N}\setminus\{i\}}}\sum_{\theta_i} g(\theta_{\mathcal{N}'},\theta_{\mathcal{N}\setminus\{i\}},\theta_i)\varphi_i^*(\theta_{\mathcal{N}\setminus\{i\}})\\ =&\sum_{i\in\mathcal{N}}\sum_{\omega_i}\sum_{\theta_{\mathcal{N}'}}\omega_i\mu\left(\tilde{\omega}_i=\omega_i,\tilde{\theta}_{\mathcal{N}'}=\theta_{\mathcal{N}'}\right)\psi_i(\theta_{\mathcal{N}'}).\end{aligned}$$

This last expression is precisely the expected utility generated by ψ. □

D. Supplemental material

Appendix D.1 considers the model where the principal can dispose the object. Appendix D.2 discusses an approach to stochastic extreme points via total unimodularity. In Appendix D.3, we define and study generalized jury mechanisms.

D.1. Disposal

In this part of the appendix, we relax the principal's problem: Instead of allocating, the principal can dispose the object. An alternate interpretation is that the principal privately consumes the object. We discuss how this affects our results from the main text. Further, we show how the existence of stochastic extreme points of the set of DIC mechanisms with disposal can be related to a property of the feasibility graph G called perfection.

Beginning with the definitions, a *mechanism with disposal* is a function $\varphi\colon \Theta \to [0,1]^n$ satisfying

$$\forall_{\theta\in\Theta}, \quad \sum_{i=1}^{n} \varphi_i(\theta) \leq 1.$$

A mechanism from the main text will be referred to as a mechanism with no disposal. If there is no risk of confusion, we will drop the qualifier "with disposal" or "with no disposal". Of course, a mechanism with no disposal is also a mechanism with disposal.

A mechanism with disposal is DIC if and only if for arbitrary i the winning probability φ_i is constant in i's report. We will sometimes drop i's report θ_i from $\varphi_i(\theta_i, \theta_{-i})$.

A jury mechanism with disposal is defined in the obvious way: There is a partition of agents into jurors and candidates; jurors never win the object; candidates do not affect anyone's winning probability with their reports.

We normalize the principal's payoff from not allocating the object to 0.

A mechanism with n agents and disposal is really a mechanism with no disposal and with $n+1$ agents where agent $n+1$ has a singleton type space; the principal's payoff from allocating to $n+1$ is always 0. Likewise, if there are other agents with singleton type spaces, we can always renormalize payoffs and view allocating to one of these agents as disposing the object. In what follows, whenever considering mechanisms with disposal, let us thus simplify by assuming that no agent has a singleton type space; that is, for all agents i we have $|\Theta_i| \geq 2$.

D.1.1 Results from the main text. Here we discuss how our results change when the principal can dispose the object.

Stochastic extreme points To begin with, we have the following analogue of Theorem 4.2.

Theorem D.1. *For all agents i, let $|\Theta_i| \geq 2$. All extreme points of the set of DIC mechanisms with disposal are deterministic if and only if at least one of the following is true:*

(1) We have $n \leq 2$.

(2) For all agents i we have $|\Theta_i| = 2$.

Proof of Theorem D.1. As discussed above, a DIC mechanism with n agents and disposal is a DIC mechanism with $n+1$ agents and no disposal. The claim follows from Theorem 4.2. □

Further below, we provide an alternate proof of Theorem D.1 that does not invoke Theorem 4.2 but relies on graph-theoretic results. We emphasize that Theorem D.1 does not imply Theorem 4.2. Namely, we cannot conclude from Theorem D.1 that if $n = 3$ all extreme points of the set of DIC mechanisms with no disposal are deterministic.

Proposition 4.4 (on the suboptimality of deterministic DIC mechanisms) analogizes straightforwardly to mechanisms with disposal. Indeed, note that in our proof of Proposition 4.4 agent 4 was simply a dummy agent with payoff normalized to 0.

Corollary 4.5 (jury mechanisms with 3 agents) carries over to mechanisms with disposal in the sense that all mechanisms with disposal and 2 agents are convex combinations of deterministic jury mechanisms with disposal. Note that, according to Theorem D.1, this result does not extend to $n = 3$. With $n = 2$, a jury mechanism with disposal admits a single juror whose report determines whether or not the object is disposed or allocated to the other agent.

Institutional constraints We now turn to the results from Section 5. It is easy to see that a SCP mechanism with disposal (where SCP is defined as in the main text) must be constant. Turning to anonymous mechanisms, however, things becomes more complicated when the principal can discard the object.

In what follows, we tacitly assume the agents have a common type space, denoted T. The set of type profiles is understood to be $\Theta = \times_{i=1}^{n} T$.

We let SA be defined as in the main text—the allocation to an agent is permutation-invariant with respect to the reports of the others—and show below that there exist non-constant SA DIC mechanisms with disposal. If we strengthen this to also demand that the probability of disposing the object be permutation-invariant (which we shall refer to as SA*), this rules out non-constant *deterministic* SA* DIC mechanisms. However, there exist non-constant *stochastic* DIC mechanisms satisfying a notion of anonymity that is *even stronger than SA**. We use this to make good on a promise from the main text: We show that WCP cannot be dropped from the characterization in Theorem D.1. If we impose WCP (defined as in the main text), we restore versions of Theorem 5.1 and Theorem 5.3 for mechanisms with disposal.

Let us begin with the definitions, where some are familiar from the main text.

Definition 7. *A mechanism φ with disposal is **SA** if all distinct agents i and j satisfy the following: For all agents k distinct from i and j, the winning probability φ_k of agent k is invariant with respect to permutations of i's and j's reports.*

*A mechanism φ with disposal is **SA*** if it is SA and if the probability $1 - \sum_{i=1}^{n} \varphi_i$ that the object is not allocated is invariant with respect to permutations of $\{1, \ldots, n\}$.*

*A mechanism φ with disposal is **SA**** if all agents i, all type profiles θ, and all permutations ξ of $\{1, \ldots, n\}$ satisfy $\varphi_i(\theta) = \varphi_{\xi(i)}(\xi(\theta))$.*

*Given a mechanism with disposal, agents i and j **influence a common agent** if there exists ℓ such that i and j both influence ℓ, or if i and j both influence the probability that the object is not allocated.*

*A mechanism φ with disposal is **WA*** if all distinct i and j and k satisfy the following: If i and j influence a common agent, then the winning probability φ_k of agent k and the probability $1 - \sum_{i=1}^{n} \varphi_i$ that the object is not allocated are invariant with respect to permutations of i's and j's reports.*

*Given a mechanism with disposal, a subset J of agents is **balanced** if satisfies the following: An agent is in J if and only if the agent influences all other agents in J.*

*A mechanism φ with disposal is **WCP** if for all non-empty balanced subsets J of agents the probability $\sum_{i \in J} \varphi_i$ that the object is allocated to an agent in J is constant in the reports of agents in J.*

The interpretation of SA** is as follows: If a profile θ' is obtained from another profile θ by some permutation of the type profile, then the same permutation is applied to the agents' winning probabilities (and hence also to the probability of disposing the object). This strengthens SA* since SA* considers situations where, fixing one agent i, we only permute the types of agents other than i.

Our first result shows that SA* is compatible only with stochastic mechanisms.

Proposition D.2. *Let $n \geq 2$ and $|T| \geq 2$.*

(1) If a mechanism with disposal is non-constant, DIC and SA*, then it is stochastic.

(2) There is a non-constant DIC SA** mechanism with disposal.

(3) There is a non-constant deterministic DIC SA jury mechanism with disposal.

An important insight from Theorem 5.1 and Proposition D.2 is then as follows: To implement a non-constant DIC mechanism that is SA*, the principal must commit to random allocations and to disposing the object.

The next proposition shows via a more complicated construction that there is an SA** mechanism with disposal that is an extreme point of the set of *all* DIC mechanisms with disposal. We use this fact to show that WCP cannot be dropped from the characterization in Theorem 5.3

Proposition D.3. *Let the common type space be $T = \{1, 2, 3, 4, 5, 6, 7\}$.*

(1) If $n = 3$, then there is a mechanism with disposal that is non-constant, DIC, stochastic, SA**, and an extreme point of the set of all *DIC mechanisms with disposal.*

(2) If $n = 4$, then there is a mechanism with no disposal that is non-constant, DIC, stochastic, WA, and an extreme point of the set of all *DIC mechanisms with no disposal.*

An easy corollary is that Corollary 5.2 does not extend to mechanisms with disposal.

Corollary D.4. *Let $n = 3$ and let the common type space be $T = \{1, 2, 3, 4, 5, 6, 7\}$. There is an environment satisfying Assumption 1 in which a non-constant stochastic DIC SA** mechanism with disposal uniquely maximizes the principal's utility over the set of DIC mechanisms with disposal.*

Lastly, we show that imposing WCP leads to analogues of Theorems 5.1 and 5.3 for mechanisms with disposal.

Proposition D.5. *If a mechanism with disposal is DIC, SA*, and WCP, then it is constant. If a mechanism with disposal is DIC, WA*, and WCP, then it is a convex combination of deterministic WA* jury mechanisms with disposal.*

Symmetric information Theorem 6.1 extends to mechanisms with disposal in a straightforward way, with no changes to the proof.

D.1.2 Omitted proofs.

Proof of Proposition D.2. To prove claim (1) we show that all deterministic DIC SA* mechanisms with disposal are constant. The argument is analogous to the proof of Lemma 5.5 from the main text: In that proof, we only used the assumption that the object always be allocated when arguing that a permutation of i's and j's reports leaves the probability that the object be allocated to i or j constant. Since the present claim assumes the mechanism to be SA*, a permutation of i's and j' reports affects neither the winning probabilities of the other agents, nor the probability that the object is disposed. Thus the probability that the object is allocated to i or j is unaffected by permutations of i's and j's reports.

Next, consider claim (2). Let T^{n-1} denote the $(n-1)$-fold Cartesian product of T. Let $\hat{\theta}^{n-1}$ denote an arbitrary point in T^{n-1}, and let $\hat{\Theta}$ denote the set of profiles (of $n-1$ types) that are obtained through a permutation of $\hat{\theta}^{n-1}$. Consider φ^* defined as follows. For all agents i and profiles θ, let $\varphi_i(\theta) = 1/n$ if $\theta_{-i} \in \hat{\Theta}^{n-1}$, and let $\varphi_i(\theta) = 0$ else. It is clear that φ^* is a well-defined DIC SA* mechanism with disposal. To see that it is non-constant, it is enough to note that $\hat{\theta}^{n-1}$ is not the unique type profile in T^{n-1} given that $|T| \geq 2$.

Lastly, consider claim (3). Let $t \in T$. Consider the following jury mechanism. For i different from 1, let $\varphi_i = 0$. For all type profiles θ, let $\varphi_1(\theta) = 1$ if $t = \theta_2 = \ldots = \theta_n$, and $\varphi_1(\theta) = 0$ otherwise. The mechanism is non-constant, DIC, deterministic, and SA.[31] $\square$

Proof of Proposition D.3. Consider claim (1).

Let $T^3 = \times_{i=1}^3 T$ denote the 3-fold Cartesian product of the common type space. Let us define $T_1 = \{1, 2\}$, $T_2 = \{3, 4\}$ and $T_3 = \{5, 6, 7\}$ and $\hat{\Theta} = T_1 \times T_2 \times T_3$. In Section 4.2.2, we constructed a stochastic DIC mechanism φ^* without disposal in a setting with 4 agents, where the types of agents 1, 2, and 3, respectively, are $\{\ell, r\}$, $\{u, d\}$, $\{L, M, R\}$, respectively, and where agent 4's type is degenerate. By relabelling types, we can view φ^* as being a mechanism with disposal with 3 agents on the set

[31] As an aside, note that the mechanism just constructed is not SA*. Indeed, let t' be different from t. At the profile $(t', t, t, \ldots, t)$, the object is allocated to agent 1. Permuting the reports of agents 1 and 2 brings us to the profile $(t, t', t, \ldots, t)$ where the object is not allocated.

of type profiles $\hat{\Theta}$, and where allocating to agent 4 is identified with disposing the object. The arguments from Section 4.2.2 therefore show that, if $n = 3$ and the set of type profiles is $\hat{\Theta}$, then φ^* is an extreme point of the set of DIC mechanisms with disposal.

For later reference, we note that, at all type profiles $\theta \in \hat{\Theta}$ and all $i \in \{1, 2, 3\}$, agent i's winning probability at θ is either 0 or 1/2.

We now use φ^* to define a mechanism as in the claim.

Our candidate mechanism will be denoted ψ^*. Let Ξ denote the set of permutations of $\{1, 2, 3\}$. Let $\Theta^* = \{\xi(\theta) \colon \theta \in \hat{\Theta}, \xi \in \Xi\}$ denote the set of type profiles obtained by permuting a type profile in $\hat{\Theta}$; see Figure 3. We return to this figure later. Note that, fixing an arbitrary type profile in $\hat{\Theta}$, the types of the agents at this type profile are all distinct. Consequently, for all θ^* in Θ^* there is a unique profile θ in $\hat{\Theta}$ and ξ in Ξ such that $\theta^* = \xi(\theta)$.

For later reference, we also note the following: At an arbitrary type profile in Θ^*, the types of distinct agents must belong to distinct elements of the partition $\{T_1, T_2, T_3\}$.

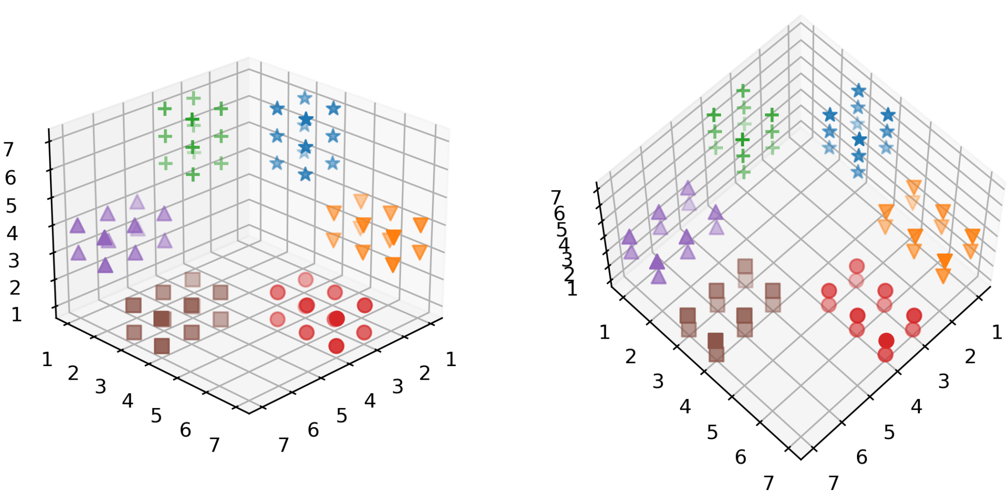


Figure 5: The set Θ^* viewed from two different angles. Each agent is associated with a distinct axes, and we have dropped the labels. Each symbol (square, circle, upward-pointing triangle, etc.) identifies a particular permutation of $\{1, 2, 3\}$. For instance, the upward-pointing triangles are obtained from the downward-pointing triangles by permuting the two agents on the horizontal axes.

We now define ψ^* as follows: For all θ^* in Θ^*, we find the unique $(\theta, \xi) \in T \times \Xi$

such that $\theta^* = \xi(\theta)$, and then let

$$(\psi_i^*(\theta^*))_{i=1}^n = \left(\varphi_{\xi(i)}^*(\xi(\theta))\right)_{i=1}^n . \tag{25}$$

In words, if θ^* is obtained from, say, θ by permuting the entries of agents 1 and 2, then $\psi^*(\theta^*)$ swaps the winning probabilities of agents 1 and 2 while leaving agent 3's winning probability unchanged. For the remaining profiles, we proceed as follows: For all agents i and profiles θ, if θ differs from at least one profile θ^* in Θ^* in agent i's type and no other agent's type, then i's winning probability at θ is set equal to i's winning probability at θ^* (which makes sense since the latter probability has already been defined in (25)); else, if no such profile θ^* in Θ^* exists, then agent i's winning probability is set equal to 0.

To complete the argument, we have to show that ψ^* is a (1) well-defined mechanism, and (2) that it is DIC, stochastic, SA**, and an extreme point of the set of DIC mechanisms with disposal. Assuming for a moment that (1) is true, it is clear that the mechanism is stochastic, and one can easily verify from the definition that it is DIC and SA**. Moreover, to show that it is an extreme point of the set of DIC mechansims, we can proceed essentially via the arguments from Section 4.2.2. Indeed, we know from Section 4.2.2 that all DIC mechanisms ψ with disposal that appear in a candidate convex combination must agree with ψ^* on $\hat{\Theta}$, and hence on Θ^*; it is then straightforward to verify that such a mechanism ψ must also agree with ψ^* on $\Theta \setminus \Theta^*$.

We now turn to (1). To show that ψ^* is a well-defined mechanism, we have to show that the winning probabilities of the agents do not sum to a number strictly above 1. Before delving into the details, consider Figure 5. The different symbols (squares, circles, upward-pointing triangles, etc.) partition Θ^* into 6 subsets (one for each permutation of $\{1, 2, 3\}$). For each of these subsets, imagine rays emanating from the subset and travelling parallel to the axes. These rays identify type profiles along which exactly one agent's is changing. Now consider the intersection of such rays originating from subsets with different symbols. The geometry of Θ^* implies that *at most* two such rays intersect simultaneously. This is one critical observation that we will use to argue that ψ^* is well-defined.

The second is that, as noted earlier, at all type profiles $\theta \in \Theta^*$ and all $i \in \{1, 2, 3\}$ agent i's winning probability under φ^* at θ is either 0 or 1/2. The mechanism ψ^*

inherits this property.

Towards a contradiction, suppose there is a profile $\theta = (\theta_1, \theta_2, \theta_3)$ in Θ where the winning probabilities under ψ^* sum to a number strictly above 1. By the previous paragraph, all three agents are therefore enjoying non-zero winning probabilities. By definition of ψ^*, we can infer the following: Since agent 1's winning probability is non-zero, there exists t_1 such that $(t_1, \theta_2, \theta_3) \in \Theta^*$. Similarly, there are t_2 and t_3 such that $(\theta_1, t_2, \theta_3) \in \Theta^*$ and $(\theta_1, \theta_2, t_3) \in \Theta^*$. Recall that $\{T_1, T_2, T_3\}$ is a partition of the common type space. Hence, for all agents i, there is a unique interger $\xi(i)$ in $\{1, 2, 3\}$ such that $\theta_i \in T_{\xi(i)}$. Let us now recall the follwing from the definition of Θ^*: If a profile is in Θ^*, then the types of distinct agents must belong two distinct elements of the partition $\{T_1, T_2, T_3\}$. Hence, we can infer from $(t_1, \theta_2, \theta_3) \in \Theta^*$ that $\xi(2) \neq \xi(3)$ holds. Similarly, from $(\theta_1, t_2, \theta_3) \in \Theta^*$ and $(\theta_1, \theta_2, t_3) \in \Theta^*$ we infer $\xi(1) \neq \xi(2)$ and $\xi(1) \neq \xi(3)$. Taken together, we infer that θ must itself be in Θ^*. Hence the vector of winning probabilities at θ is a permutation of the vector of winning probabilities at a profile θ' in $\hat{\Theta}$. At the profile θ', the winning probabilites under ψ^* agree with φ^*. Thus there is a profile where the winning probabilities under φ^* sum to a number strictly greater than 1; this is a contradiction since φ^* is a well-defined mechanism.

This completes the proof of claim (1).

Lastly, consider claim (2). Let φ^* be as in the conclusion of claim (1). The mechanism φ^* defines a mechanism ψ^* with $n+1$ agents and no disposal, where disposing the object under φ^* is viewed as allocating to agent $n+1$, and where the mechanism is constant in agent $n+1$'s report. The mechanism defined in this way is WA since φ^* is SA*; the remaining properties are inherited from φ^*. □

Proof of Corollary D.4. Let φ^* be as in the conclusion of Proposition D.3. By a separating hyperplane theorem, there are payoffs and a joint distribution such that φ^* is the unique optimal DIC mechanism with disposal. Since φ^* is SA**, we can pick these payoffs and distribution such that Assumption 1 holds. □

Proof of Proposition D.5. Consider the first claim. Towards a contradiction, suppose there is a non-constant DIC SA* WCP mechanism φ with disposal. Let J denote the non-empty set of agents who enjoy non-constant winning probabilities.

In an auxiliary step, we claim that, for arbitrary distinct i and j, agent i influences j if and only if j influences i. Indeed, a permutation of i's and j's report does not affect the total probability that the object is allocated to i or j. Hence all profiles

θ satisfy $\varphi_i(\theta_i, \theta_j, \theta_{-ij}) + \varphi_j(\theta_i, \theta_j, \theta_{-ij}) = \varphi_i(\theta_j, \theta_i, \theta_{-ij}) + \varphi_j(\theta_j, \theta_i, \theta_{-ij})$. Since the mechanism is DIC, inspection of this equality shows that i influences j if and only if j influences i.[32]

The previous paragraph implies that all agents in J influence all other agents in J, and that agents outside of J do not influence agents in J. Invoking WCP, it follows that the mechanism is a convex combination of a constant mechanism (that only allocates to agents outside of J) and a mechanism φ' with no disposal that only allocates to agents in J. Moreover, the mechanism φ' is SA in a setting where J is the set of all agents. We conclude from Theorem 5.1 that φ' is constant. Thus φ is constant; contradiction.

Now consider the second claim. Let φ be a mechanism with disposal that is DIC, WA*, and WCP. This mechanism defines a mechanism φ' with $n+1$ agents and no disposal that is constant in agent $n+1$'s report; disposing the object under φ is identified with allocating to agent $n+1$. Since φ' is constant in agent $n+1$'s report, the fact that φ is DIC, WA and WCP implies that φ' is DIC, WA, and WCP. Thus Theorem 5.3 implies that φ' is a convex combination of deterministic WA jury mechanisms with no disposal and with $n+1$ agents. By fixing the report of agent $n+1$ to an arbitrary value, we obtain a collection of deterministic WA* jury mechanisms with disposal, the convex hull of which contains φ. $\square$

D.1.3 Stochastic extreme points and the feasibility graph. In this section, we relate the existence of stochastic extreme points with disposal to the feasibility graph. We first introduce several definitions for a general graph G with nodes V and edges E.

An *induced cycle of length* k is a subset $\{v_1, \ldots, v_k\}$ of V such that, denoting $v_{k+1} = v_1$, every pair of nodes v_ℓ and $v_{\ell'}$ are adjacent if and only if $|\ell - \ell'| = 1$.

The *line graph* of G is the graph that has as node set the edge set of G; two nodes of the line graph are adjacent if and only if the two associated edges of G share a node.

A *stable set* of G is a subset of nodes of which no two are adjacent. A *clique* of G is a set of nodes such that every pair in the set are adjacent. A clique is *maximal* if it is not a strict subset of another clique. The *incidence vector* of a subset of nodes

[32]Here, $\varphi_i(t, t', \theta_{-ij})$ and $\varphi_j(t, t', \theta_{-ij})$, respectively, mean the winning probabilities of i and j, respectively, when i reports t, j reports t', and the others report θ_{-ij}.

$\hat{V}$ is the function $x\colon V \to \{0,1\}$ that equals one on $\hat{V}$ and equals zero otherwise.

With these definitions, we can view a deterministic DIC mechanism with disposal as an incidence vector of a stable set of G. Let $S(G)$ denote the set of incidence vectors belonging to some stable set of G.

The upcoming result uses another definition called *perfection*. For our purposes, it will be enough to know the following; see Korte and Vygen (2018).

Lemma D.6. *All bi-partite graphs and line graphs of bi-partite graphs are perfect. If a graph admits an induced cycle of odd length greater than* 5*, then it is not perfect.*

Our interest in perfect graphs is due to the following theorem.

Theorem D.7 (Theorem 3.1 of Chvátal (1975))**.** *A graph G with node set V and edge set E is perfect if and only if the convex hull* $\operatorname{co} S(G)$ *is equal to the set*

$$\left\{x\colon V \to [0,1]\colon \textit{all maximal cliques } X \textit{ of } G \textit{ satisfy} \sum_{v\in X} x(v) \le 1\right\}. \tag{26}$$

Consider the theorem for our particular graph G. A clique X is maximal if and only if X equals $\{(i,\theta_{-i})\}_i$ for some type profile θ. The set (26) is thus precisely the set of DIC mechanisms with free disposal. Therefore:

Lemma D.8. *All extreme points of the set of DIC mechanisms with disposal are deterministic if and only if G is perfect.*

This leads us to the following alternate proof of Theorem D.1.

Alternate proof of Theorem D.1. Let $n = 2$. Observe that the node set of G may be partitioned into the sets $\{1\} \times \Theta_2$ and $\{2\} \times \Theta_1$. By definition, two nodes (i,θ_{-i}) and (j,θ_{-j}) are adjacent only if $i \neq j$. Thus G is bi-partite. Since every bi-partite graph is perfect, the claim follows from Theorem D.7.

Suppose $|\Theta_i| = 2$ holds for all i. We may relabel the types so that $\Theta_i = \{0,1\}$ holds for all i. In this case G is the line graph of a bi-partite graph; namely the bi-partite graph with node set $\{0,1\}^n$ and where two nodes are adjacent if and only if they differ in exactly one entry. The line graph of a bi-partite graph is perfect, and so the claim again follows from Theorem D.7.

Lastly, suppose $n \geq 3$ and $|\Theta_i| > 2$ for at least one i. We will show that G admits an odd induced cycle of length seven. In view of Theorem D.7, this proves the claim.

Let us relabel the agents and types such that the type spaces contain the following subsets of types:

$$\tilde{\Theta}_1 = \{\ell, r\} \quad \text{and} \quad \tilde{\Theta}_2 = \{u, d\} \quad \text{and} \quad \tilde{\Theta}_3 = \{L, M, R\}$$

all hold. Let θ_{-123} be an arbitrary type profile of agents other than 1, 2 and 3 (assuming such agents exist). One may verify that the following is an induced cycle of length seven:

$$\begin{aligned}
(2, (\ell, M, \theta_{-123})) &\leftrightarrow (1, (d, M, \theta_{-123})) \\
&\leftrightarrow (3, (r, d, \theta_{-123})) \\
&\leftrightarrow (2, (r, R, \theta_{-123})) \\
&\leftrightarrow (3, (r, u, \theta_{-123})) \\
&\leftrightarrow (1, (u, L, \theta_{-123})) \\
&\leftrightarrow (3, (\ell, u, \theta_{-123})) \\
\leftrightarrow (2, (\ell, M, \theta_{-123}))\,.
\end{aligned}$$

□

The proof in the main text for the existence of a stochastic extreme point is slightly more elaborate than the one given above since in the former we explicitly spell out the extreme point rather than proving its existence abstractly. (The proof in the main text uses one of the agents as a dummy, and therefore also works for mechanisms with disposal.) In our view, the advantage of the more elaborate argument is that it facilitates the construction of environments with privately-known payoffs where all deterministic DIC mechanisms fail to be optimal. Moreover, we can give an interpretation as to why the principal may strictly benefit from a stochastic mechanism. That said, it is clear how the induced cycle defined in the proof of Theorem D.1 relates to the construction from the main text.

D.2. Total unimodularity

This section of the appendix discusses another sufficient condition for all extreme points to be deterministic. Our aim is to give a brief explanation for why this approach, that is based on total unimodularity, does not help us in the difficult case with three agents.

We now view a mechanism as a vector φ in $\mathbb{R}_+^{n|\Theta|}$. The vector φ has one entry for each pair of the form (i, θ). For φ to be a DIC mechanism, it should satisfy the following:

$$\begin{aligned} \forall_{i,\theta}, \quad & 1 \geq \varphi_i(\theta) \\ \forall_{i,\theta_i,\theta_i',\theta_{-i}}, \quad & 0 \geq \varphi_i(\theta_i, \theta_{-i}) - \varphi_i(\theta_i', \theta_{-i}) \geq 0 \\ \forall_{\theta}, \quad & 1 \geq \sum_i \varphi_i(\theta) \geq 1 \end{aligned} \tag{27}$$

For a suitable matrix A and vector b, the set of DIC mechanisms is then the polytope $\{\varphi \colon A\varphi \geq b, \varphi \geq 0\}$. Here, the matrix A has one row for every constraint in (27) (after splitting the constraints into one-sided inequalities). Each column of A identifies a pair of the form (i, θ).

A vector is *integral* if all of its entries are in $\mathbb{Z}$. A polytope is *integral* if all its extreme points are integral. In this language, all extreme points of the set of DIC mechanisms are deterministic if and only if the polytope $\{\varphi \colon A\varphi \geq b, \varphi \geq 0\}$ is integral.

Recall that a matrix is *totally unimodular* if all its square submatrices have a determinant equal to -1, 0, or 1. For later reference, notice that a submatrix of a totally unimodular matrix is itself totally unimodular.

Our interest in total unimodularity is due the Hoffman-Kruskal theorem; see Korte and Vygen (2018, Theorem 5.21).

Theorem D.9. *An integral matrix A is totally unimodular if and only if for all integral vectors b all extreme points of the set $\{\varphi \colon A\varphi \geq b, \varphi \geq 0\}$ are integral.*

Thus a sufficient condition for all extreme points of the set of DIC mechanisms to be determinsitic is that the constraint matrix A be totally unimodular. We will argue below that A fails to be totally unimodular whenever $n = 3$ and at least one agent has non-binary types. This explains why our approach to integrality in the three agents case case is not based on total unimodularity.[33]

[33]Note that total unimodularity of A is sufficient, but not necessary, for the polytope to be integral when b is held fixed. Therefore, the fact that A is not always totally unimodular when $n = 3$ does not imply a contradiction to the fact that, according to Theorem 4.2, all extreme points are deterministic when $n = 3$.

Lemma D.10. *For all agents i, let $|\Theta_i| \geq 2$. Let $n = 3$. If there exists i such that $|\Theta_i| \geq 3$, then A is not totally unimodular.*

Proof of Lemma D.10. Towards a contradiction, suppose A is totally unimodular. Let us consider the constraint matrix A^{fd} and vector b^{fd} that define the set of DIC mechanisms with disposal. That is, φ is a DIC mechanism with disposal if and only if $A^{fd}\varphi \geq b^{fd}$ and $\varphi \geq 0$. Notice that A^{fd} is obtained from A by dropping all rows corresponding to constraints of the form $\sum_i \varphi_i(\theta) \geq 1$; the vector b^{fd} is obtained from b by dropping the corresponding entries. In particular, the matrix A^{fd} is a submatrix of A. Hence, since A is totally unimodular, we conclude that A^{fd} is totally unimodular. We therefore infer from Theorem D.9 that all extreme points of the set $\{\varphi\colon A^{fd}\varphi \geq b^{fd}, \varphi \geq 0\}$ are integral. That is, all extreme points of the set of DIC mechanism with disposal are deterministic. Since $n = 3$, all agents have at least binary types, and at least one agent has non-binary types, we have a contradiction to Theorem D.1. □

We can also give an alternate proof of Lemma D.10 that does not require Theorem D.1. Consider the following characterization of total unimodularity due to Ghouila-Houri (1962); see Korte and Vygen (2018, Theorem 5.25).

Theorem D.11. *A matrix A with entries in $\{-1, 0, 1\}$ is totally unimodular if and only if all subsets C of columns of A satisfy the following: There exists a partition of C into subsets C^+ and C^- such that for all rows r of A we have*

$$\sum_{c \in C^+} A(r, c) - \sum_{c \in C^-} A(r, c) \in \{-1, 0, 1\}. \tag{28}$$

Alternate proof of Lemma D.10. Let us, once again, relabel the agents and types such that the type spaces contain the following subsets:

$$\tilde{\Theta}_1 = \{\ell, r\} \quad \text{and} \quad \tilde{\Theta}_2 = \{u, d\} \quad \text{and} \quad \tilde{\Theta}_3 = \{L, M, R\}$$

Fixing an arbitrary type profile θ_{-123} of agents other than 1, 2, and 3, let us define the type profiles $\{\theta^a, \theta^b, \theta^c, \theta^e, \theta^f, \theta^g\}$ as in (3) in Section 4.2.2. That is, let

$$\begin{aligned}\theta^a = (\ell, d, M, \theta_{-123}), \quad \theta^b &= (r, d, M, \theta_{-123}), \quad \theta^c = (r, d, R, \theta_{-123}),\\ \theta^d &= (r, u, R, \theta_{-123}), \quad \theta^e = (r, u, L, \theta_{-123}),\\ \theta^f &= (\ell, u, L, \theta_{-123}), \quad \theta^g = (\ell, u, M, \theta_{-123}).\end{aligned}$$

Recall that each column of A corresponds to an entry of the form (i, θ). We will argue that the following set C of columns does not admit a partition in the sense of Theorem D.11.

$$\begin{aligned}C = \{&(1, \theta^a), (1, \theta^b), (3, \theta^b), (3, \theta^c),\\ &(2, \theta^c), (2, \theta^d), (3, \theta^d), (3, \theta^e),\\ &(1, \theta^e), (1, \theta^f), (3, \theta^f), (3, \theta^g),\\ &(2, \theta^g), (2, \theta^a)\}\end{aligned}$$

Towards a contradiction, suppose C does admit a partition into sets C^+ and C^- in the sense of Theorem D.11. Let us assume $(1, \theta^a) \in C^+$, the other case being similar. Note that θ^a and θ^b differ only in the type of agent 1. Consider the row of A corresponding to the DIC constraint for agent 1 at these type profiles. By referring to (28) for this row, we deduce $(1, \theta^b) \in C^+$. Next, via a similar argument, the constraint that the object is allocated at θ^b requires $(3, \theta^b) \in C^-$. Continuing in this manner, it is easy to see that $(1, \theta^a)$ must be in C^-. Since $(1, \theta^a)$ is assumed to be in C^+, we have a contradiction to the assumption that C^+ and C^- are a partition of C. □

D.3. Generalized jury mechanisms

In this part of the appendix, we characterize a generalization of jury mechanisms. Let $N = \{1, \ldots, n\}$ denote the set of agents.

Definition 8. *A mechanism φ is a* ***generalized jury mechanism*** *if there is an integer m and a partition $\{J_1, \ldots, J_m\}$ of N satisfying the following: For all agents i and all $k \in \{1, \ldots, m\}$, we have $i \in J_k$ and only if φ_i is constant in the reports of all agents in $J_k \cup \ldots \cup J_m$*

Jury mechanisms are indeed generalized jury mechanisms: Let $m = 2$, let J_1 denote the set of jurors, and let J_2 denote the set of candidates. Further, all generalized jury mechanisms are DIC.

We can interpret a generalized jury mechanism as follows: The agents in J_1 enjoy constant winning probabilities. The agents in J_1 determine with their reports whether to immediately allocate to an agent in J_2, or whether to also consult the agents in J_2. In the second case, the object will not be allocated to an agent in J_2; instead, the agents in $J_1 \cup J_2$ determine with their reports whether to allocate to an agent in J_3, or whether to also consult the agents in J_3, and so on. The following example illustrates this idea. It also shows that generalized jury mechanisms form a larger class than jury mechanisms.

Example 3. Let $n = 6$. For all agents i, let $\Theta_i = \{0, 1\}$. If agents 1 and 2 both report 0 (report 1), the object is allocated to agent 3 (agent 4). If the reports of agents 1 and 2 disagree, and if agents 3 and 4 both report 0 (report 1), the object is allocated to agent 5 (agent 6).

This mechanism is a generalized jury mechanism with $J_1 = \{1, 2\}$, $J_2 = \{3, 4\}$ and $J_3 = \{5, 6\}$. It is not a jury mechanism since, say, agent 3 influences the allocation but is also allocated the object at some type profiles. It is also not a convex combination of jury mechanisms since it is deterministic and therefore an extreme point of the set of DIC mechanisms.

Below, we discuss two points. First, we show that generalized jury mechanisms are characterized by a notion of coalition-proofness that is stronger than weak coalition-proofness from the main text. Second, we give a simple sufficient condition for a generalized jury mechanism to reduce to a convex combination of jury mechanisms.

Recall that, given a mechanism, the set I_i denotes the set of agents with respect to whose reports agent i's winning probability is non-constant. A mechanism is DIC if and only if $i \notin I_i$.

Proposition D.12. *Let φ be a DIC mechanism. The following are equivalent:*

(1) The mechanism φ is a generalized jury mechanism.

(2) For all non-empty subsets J of N satisfying $J \subseteq \cup_{i \in J} I_i$, the total winning probability $\sum_{i \in J} \varphi_i$ of agents in J is constant in the reports of agents in J.

Proof of Proposition D.12. Beginning with the easy direction, suppose φ is a generalized jury mechanism. Let $\{J_1, \dots, J_m\}$ denote the associated partition of the set of agents. We claim that if J satisfies $J \subseteq \cup_{i \in J} I_i$, then J is empty, implying that the claim holds vaccuously. First, we observe that J cannot contain agents in J_m since the mechanism is constant in the reports of agents in J_m. Thus J is a subset of $J_1 \cup \ldots \cup J_{m-1}$. Next, we observe that J cannot contain agents in J_{m-1}; indeed, the allocation to agents in $J_1 \cup \ldots \cup J_{m-1}$ is constant in the reports of agents in J_{m-1}. Proceeding in this manner for a finite number of steps, we conclude that J is empty, as promised.

Now let φ be a DIC mechanisms such that for all non-empty subsets J satisfying $J \subseteq \cup_{i \in J} I_i$ the total winning probability $\sum_{i \in J} \varphi_i$ of agents in J is constant in the reports of agents in J. We will use the following auxiliary result.

Claim D.13. *Let J be a non-empty subset of agents. There exists $i \in J$ such that φ_i is constant in the reports of all other agents in J.*

Proof of Claim D.13. For this proof, let us use the following terminology: Given an integer ℓ and a subset $\{i_1, \ldots, i_\ell\}$ of ℓ distinct agents in J, we say the subset is a cycle of length ℓ if we have

$$\forall_{k \in 1, \ldots, \ell}, \quad i_{k+1} \in I_{i_k},$$

where we adopt the notational convention $i_1 = i_{\ell+1}$.

Now assume, towards a contradiction, that for all $i \in J$ there exists $j \in J \cap I_i$. Since J is finite, it follows that there is an integer ℓ greater than 2 and a cycle of length ℓ. Note that all agents in the cycle influence at least one other agent in the cycle. Hence the total winning probability of the cycle is constant in the reports of the agents in the cycle. Now let i be an arbitrary agent in the cycle. Since i influences at least one agent j in the cycle, the total winning probability in the cycle is constant in i's report only if i influences an agent k in the cycle distinct from j. This implies that there is a further subset of the cycle which itself forms another cycle of length, say, ℓ', where ℓ' is an integer strictly less than ℓ. By repeatedly applying this observation, we conclude that there is a cycle of length 1. If there is a cycle of length 1, DIC is violated; contradiction. $\square$

We now inductively define a partition of the set of agents. To begin, let J_1 be the set of agents whose winning probabilities are constant in the reports of all agents. We can appeal to Claim D.13 (with the set of all agents in the role of J) to infer that J_1 is non-empty. Next, assuming that for some integer k the sets $J_1, \ldots J_k$ have been defined, we define J_{k+1} as the subset of agents in $N \setminus (J_1 \cup \ldots \cup J_k)$ whose winning probabilities are constant in the reports of all agents in $N \setminus (J_1 \cup \ldots \cup J_k)$. An application of Claim D.13 with $N \setminus (J_1 \cup \ldots \cup J_k)$ in the role of J shows that J_{k+1} is non-empty. Since the set of agents is finite, this construction terminates after a finite number of steps. Thus, for some integer m, we obtain a partition $\mathcal{J}$ of N into subsets $\{J_1, \ldots, J_m\}$ satisfying the following: For all $i \in N$ and $k \in \{1, \ldots, m\}$, we have $i \in J_k$ if and only if φ_i is constant in the reports of all agents in $J_k \cup \ldots \cup J_m$. Thus the mechanism is a generalized jury mechanism. □

Lastly, we ask when a generalized mechanism reduces to a (convex combination of) jury mechanisms. One simple sufficient condition is that the mechanism be WA.

Corollary D.14. *A WA generalized jury mechanism is a convex combination of WA jury mechanisms.*

Proof of Corollary D.14. Using Proposition D.12, it is easy to see that all generalized jury mechanisms are WCP. The claim follows from Theorem 5.3 from the main text. □

Instead of invoking Theorem 5.3, one can also show directly that in a WA generalized jury mechanism each partition element enjoys a constant overall winning probability. This, in turn, is another sufficient condition for the mechanism to be a convex combination of jury mechanisms.

Proposition D.15. *Let φ be a generalized jury mechanism with partition $\{J_1, \ldots, J_m\}$. If for all $k \in \{1, \ldots, m\}$ the total winning probability $\sum_{i \in J_k} \varphi_i$ is constant in all reports, then φ is a convex combination of jury mechanisms.*

The proof is straightforward.

Proof of Proposition D.15. For all k, let α_k denote the constant probability $\sum_{i \in J_k} \varphi_i$. For all $k \in \{1, \ldots, m\}$ such that $\alpha_k > 0$, let ψ_k denote the mechanism that allocates to agent i with probability 0 if $i \notin J_k$, and allocates to i with probability $\varphi_i(\theta)/\alpha_k$ if $i \in J_k$. For all $k \in \{1, \ldots, m\}$ such that $\alpha_k = 0$, let ψ_k be an arbitrary constant mechanism. By definition, if $i \in J_k$ and $k \geq 2$, agent i's winning probability is depends only on the reports of agents in $\cup_{k'=1}^{k-1} J_{k'}$; hence ψ_k is a jury mechanism. The previous observation also implies that φ equals the convex combination $\sum_{k \in \{1, \ldots, m\}} \alpha_k \psi_k$. □

References

Aliprantis, Charalambos D. and Kim C. Border (2006). *Infinite Dimensional Analysis : A Hitchhikers Guide*. Springer Berlin Heidelberg. DOI: 10.1007/3-540-29587-9.

Alon, Noga, Felix Fischer, Ariel Procaccia, and Moshe Tennenholtz (2011). "Sum of us: Strategyproof selection from the selectors". In: *Proceedings of the 13th Conference on Theoretical Aspects of Rationality and Knowledge*, pp. 101–110.

Aziz, Haris, Omer Lev, Nicholas Mattei, Jeffrey Rosenschein, and Toby Walsh (2016). "Strategyproof peer selection: Mechanisms, analyses, and experiments". In: *Proceedings of the AAAI Conference on Artificial Intelligence*. Vol. 30. 1.

Aziz, Haris, Omer Lev, Nicholas Mattei, Jeffrey S Rosenschein, and Toby Walsh (2019). "Strategyproof peer selection using randomization, partitioning, and apportionment". In: *Artificial Intelligence* 275, pp. 295–309.

Bartholdi, Laurent, Wade Hann-Caruthers, Maya Josyula, Omer Tamuz, and Leeat Yariv (2021). "Equitable voting rules". In: *Econometrica* 89.2, pp. 563–589.

Baumann, Leonie (2018). "Self-ratings and peer review". URL: https://sites.google.com/site/leoniebaumann/.

Ben-Porath, Elchanan, Eddie Dekel, and Barton L Lipman (2014). "Optimal allocation with costly verification". In: *American Economic Review* 104.12, pp. 3779–3813.

— (2019). "Mechanisms with evidence: Commitment and robustness". In: *Econometrica* 87.2, pp. 529–566.

Bhaskar, Dhruva and Evan Sadler (2019). "Resource Allocation with Positive Externalities". In: *Available at SSRN 2853085*.

Bjelde, Antje, Felix Fischer, and Max Klimm (2017). "Impartial selection and the power of up to two choices". In: *ACM Transactions on Economics and Computation (TEAC)* 5.4, pp. 1–20.

Bloch, Francis, Bhaskar Dutta, and Marcin Dziubiński (2022). "Selecting a Winner with Impartial Referees". URL: https://www.sites.google.com/site/francisbloch1/papers.

Bloch, Francis and Matthew Olckers (2020). "Friend-based ranking". In: *arXiv preprint arXiv:1807.05093*.

Bloch, Francis and Matthew Olckers (2021). “Friend-based ranking in practice”. In: *AEA Papers and Proceedings*. Vol. 111, pp. 567–71.

Bousquet, Nicolas, Sergey Norin, and Adrian Vetta (2014). “A near-optimal mechanism for impartial selection”. In: *International Conference on Web and Internet Economics*. Springer, pp. 133–146.

Budish, Eric, Yeon-Koo Che, Fuhito Kojima, and Paul Milgrom (2013). “Designing random allocation mechanisms: Theory and applications”. In: *American economic review* 103.2, pp. 585–623.

Bulow, Jeremy and Paul Klemperer (1996). “Auctions Versus Negotiations”. In: *The American Economic Review* 86.1, pp. 180–194.

Caragiannis, Ioannis, George Christodoulou, and Nicos Protopapas (2019). “Impartial selection with additive approximation guarantees”. In: *International Symposium on Algorithmic Game Theory*. Springer, pp. 269–283.

— (2021). “Impartial selection with prior information”. In: *arXiv preprint arXiv:2102.09002*.

Chakravarty, Surajeet and Todd R Kaplan (2013). “Optimal allocation without transfer payments”. In: *Games and Economic Behavior* 77.1, pp. 1–20.

Chen, Yi-Chun, Wei He, Jiangtao Li, and Yeneng Sun (2019). “Equivalence of Stochastic and Deterministic Mechanisms”. In: *Econometrica* 87.4, pp. 1367–1390.

Chvátal, Vašek (1975). “On certain polytopes associated with graphs”. In: *Journal of Combinatorial Theory, Series B* 18.2, pp. 138–154.

Condorelli, Daniele (2009). “What money can’t buy: allocations with priority lists, lotteries and queues”. In: *Lotteries and Queues (June 16, 2009)*.

— (2012). “What money can’t buy: Efficient mechanism design with costly types.” In: *Games and Economic Behavior* 75.2, pp. 613–624.

Crémer, Jacques and Richard P McLean (1985). “Optimal selling strategies under uncertainty for a discriminating monopolist when demands are interdependent”. In: *Econometrica: Journal of the Econometric Society*, pp. 345–361.

— (1988). “Full extraction of the surplus in Bayesian and dominant strategy auctions”. In: *Econometrica: Journal of the Econometric Society*, pp. 1247–1257.

De Clippel, Geoffroy, Kfir Eliaz, Daniel Fershtman, and Kareen Rozen (2021). “On selecting the right agent”. In: *Theoretical Economics* 16.2, pp. 381–402.

De Clippel, Geoffroy, Herve Moulin, and Nicolaus Tideman (2008). “Impartial division of a dollar”. In: *Journal of Economic Theory* 139.1, pp. 176–191.

Edelman, Paul H and Attila Por (2021). “A new axiomatic approach to the impartial nomination problem”. In: *Games and Economic Behavior* 130, pp. 443–451.

Epitropou, Markos and Rakesh Vohra (2019). “Optimal On-Line Allocation Rules with Verification”. In: *International Symposium on Algorithmic Game Theory*. Springer, pp. 3–17.

Erlanson, Albin and Andreas Kleiner (2019). “A note on optimal allocation with costly verification”. In: *Journal of Mathematical Economics* 84, pp. 56–62.

Fischer, Felix and Max Klimm (2015). “Optimal impartial selection”. In: *SIAM Journal on Computing* 44.5, pp. 1263–1285.

Ghosal, Subhashis and Aad Van Der Vaart (2007). “Convergence rates of posterior distributions for noniid observations”. In: *The Annals of Statistics* 35.1, pp. 192–223.

Ghouila-Houri, Alain (1962). “Caractérisation des matrices totalement unimodulaires”. In: *Comptes Redus Hebdomadaires des Séances de l'Académie des Sciences (Paris)* 254, pp. 1192–1194.

Goldlücke, Susanne and Thomas Tröger (2020). “The multiple-volunteers principle”. In: *Available at SSRN 3753985*.

Guo, Yingni and Johannes Hörner (2021). “Dynamic allocation without money”. URL: https://hal.archives-ouvertes.fr/hal-03187506.

Hart, Sergiu and Philip J Reny (2015). “Maximal revenue with multiple goods: Nonmonotonicity and other observations”. In: *Theoretical Economics* 10.3, pp. 893–922.

He, Kevin, Fedor Sandomirskiy, and Omer Tamuz (2021). “Private private information”. In: *arXiv preprint arXiv:2112.14356*.

Holzman, Ron and Hervé Moulin (2013). “Impartial nominations for a prize”. In: *Econometrica* 81.1. Publisher: Wiley Online Library, pp. 173–196.

Kattwinkel, Deniz (2019). “Allocation with Correlated Information: Too good to be true”. URL: https://sites.google.com/view/kattwinkel.

Kattwinkel, Deniz and Jan Knoepfle (2021). “Costless Information and Costly Verification: A Case for Transparency”. URL: https://dx.doi.org/10.2139/ssrn.3426817.

Kattwinkel, Deniz, Axel Niemeyer, Justus Preusser, and Alexander Winter (2022). “Mechanisms without transfers for fully biased agents”. URL: https://arxiv.org/abs/2205.10910.

Korte, Bernhard and Jens Vygen (2018). *Combinatorial Optimization: Theory and Algorithms*. 6th ed. 2018. Algorithms and Combinatorics ; 21. Berlin, Heidelberg: Springer Berlin Heidelberg.

Laffont, Jean-Jacques and David Martimort (2009). *The theory of incentives*. Princeton University Press.

Lev, Omer, Nicholas Mattei, Paolo Turrini, and Stanislav Zhydkov (2021). "Peer Selection with Noisy Assessments". In: *arXiv preprint arXiv:2107.10121*.

Li, Yunan (2020). "Mechanism design with costly verification and limited punishments". In: *Journal of Economic Theory* 186, p. 105000.

Mackenzie, Andrew (Nov. 2015). "Symmetry and impartial lotteries". In: *Games and Economic Behavior* 94, pp. 15–28. ISSN: 0899-8256. DOI: `10.1016/j.geb.2015.08.007`. (Visited on 01/14/2022).

— (2020). "An axiomatic analysis of the papal conclave". In: *Economic Theory* 69.3, pp. 713–743.

Mattei, Nicholas, Paolo Turrini, and Stanislav Zhydkov (2020). "PeerNomination: Relaxing exactness for increased accuracy in peer selection". In: *arXiv preprint arXiv:2004.14939*.

McAfee, R. Preston and Philip J. Reny (1992). "Correlated information and mecanism design". In: *Econometrica: Journal of the Econometric Society*, pp. 395–421.

Mylovanov, Tymofiy and Andriy Zapechelnyuk (2017). "Optimal allocation with ex post verification and limited penalties". In: *American Economic Review* 107.9, pp. 2666–94.

Östergård, Patric RJ and Ville H Pettersson (2013). "Enumerating perfect matchings in n-cubes". In: *Order* 30.3, pp. 821–835.

Pycia, Marek and M Utku Ünver (2015). "Decomposing random mechanisms". In: *Journal of Mathematical Economics* 61, pp. 21–33.

Roughgarden, Tim and Inbal Talgam-Cohen (2019). "Approximately Optimal Mechanism Design". In: *Annual Review of Economics* 11.1, pp. 355–381. DOI: `10.1146/annurev-economics-080218-025607`.

Strausz, Roland (2003). "Deterministic mechanisms and the revelation principle". In: *Economics Letters* 79.3, pp. 333–337.

Tamura, Shohei and Shinji Ohseto (2014). "Impartial nomination correspondences". In: *Social Choice and Welfare* 43.1, pp. 47–54.