# Simple Allocation with Correlated Types*

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This version: September 10, 2022

**Abstract**

A single indivisible object is to be allocated to one of $n$ agents who all desire the object. The efficient allocation depends on the private types of the agents. Types are correlated, meaning that each agent may have information about the types of the others. Monetary transfers are unavailable. We study optimal dominant-strategy incentive-compatible (DIC) mechanisms. Our main results make a case for *jury mechanisms*. A jury mechanism splits the agents into a set of jurors and a set of candidates. The jury decides which of the candidates wins the object; jury members never win the object. Jury mechanisms are optimal when $n \leq 3$, approximately optimal in symmetric environments with many agents, and the only deterministic DIC mechanisms satisfying an anonymity axiom. Exactly-optimal DIC mechanisms may require commitment to random allocations.

**Keywords**: Allocation, Peer selection, Correlation, Mechanisms without transfers

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*We thank Kailin Chen, Gregorio Curello, Günnur Ege Destan, Francesc Dilmé, Deniz Kattwinkel, Jan Knoepfle, Daniel Krähmer, Thomas Kohler, Patrick Lahr, Stephan Lauermann, Benny Moldovanu, Georg Nöldeke, Paul Schäfer, Ludvig Sinander, Omer Tamuz as well as audiences at Bonn, Yale, CMiD 2022, and CRC TR 224 YRW for their comments. Niemeyer was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – EXC 2126/1 – 390838866. Preusser gratefully acknowledges funding by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through CRC TR 224 (Project B04). Preusser is grateful for the hospitality of Yale University, where parts of this paper were written.

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# 1 Introduction

The problem of allocating a scarce resource is a basic problem in economics. We consider a situation where a single indivisible object is allocated to a group of agents who all desire the object. The efficient allocation depends on the agents' private information, but prices cannot be used to elicit this information. What are good rules for allocating the object?

The following are some concrete examples of the kind of settings we have in mind. Consider a planner who allocates a social grant to a group of households. The planner would like to give it to whomever needs it most. Neighboring households may have better information about one another than the external planner. However, simply asking each household whether they need the good more than their neighbor creates incentives for overstating one's own need. Or consider a group of agents that has to pick a president. Each agent knows who in their circle of friends would best serve the group as a whole. However, being president is prestigious, and hence each agent may selfishly claim that they themself are the most suitable candidate. (Here, the "object" that is to be allocated is the presidency of the group.) Lastly, consider the allocation of funds within a firm. Management would like to allocate to the division of the firm with the highest marginal return. If returns are correlated across divisions via some underlying state of the world, each division can make an informed prediction about the others. However, if each division wants to maximize its own allocation, there is again an incentive to overstate one's own productivity.

A common feature of these examples is that each agent has valuable information about their peers. An abstract rule for allocating the object—a *mechanism*—designates a winner as a function of the reported information. Simply asking the agents who should win the object, however, risks that all agents selfishly nominate themselves. The mechanism design literature has thus suggested to use *impartial* mechanisms. Impartiality entails that one's report does not affect one's own winning probability. Suppose for simplicity that each agent gets a positive payoff from winning the object, but is otherwise indifferent to who else gets it. Impartial mechanisms then correspond exactly to mechanism where reporting one's information truthfully is a dominant-strategy; that is, mechanisms that are dominant-strategy incentive-compatible (DIC). This shall also be our approach.

In our model, there are $n$ agents. Allocating to agent $i$ generates a payoff $\omega_i$. In the above examples, this payoff may reflect the group's value for $i$ as a president, or household $i$'s valuation for the good, or the marginal return on the dollar generated by division $i$. We refer to $\omega_i$ as the payoff of the principal who designs the mechanism. As often in mechanism design, the principal need not be taken literally but may simply be a metaphor to capture the objective of, say, picking as president whoever best serves the group as a whole.

The payoffs $\omega_1, \ldots, \omega_n$ are initially unknown to all agents and the principal. Each agent $i$ has a private type $\theta_i$. For the most part, we impose no structure on the joint distribution of payoffs and types $\omega_1, \theta_1, \ldots, \omega_n, \theta_n$. Types and payoffs may thus be correlated, which lets us capture the idea that each agent $i$ may have information about all payoffs $\omega_1, \ldots, \omega_n$.

A mechanism asks agents to report their types, and then allocates the object to one of them, possibly using a lottery. The principal evaluates the mechanism via the expected payoffs from the allocation.[1]

Our first contribution is to show that optimal DIC mechanisms may require the use of lotteries. Outside of special cases, which we fully characterize, there are stochastic DIC mechanisms that cannot be implemented by randomizing over deterministic ones. An implication is that all deterministic DIC mechanisms may fail to be optimal.

Optimal DIC mechanisms may thus be complicated and involve commitment to lotteries. However, there is a remarkable case where this is not so. When there are three or fewer agents ($n \leq 3$), there is an optimal DIC mechanism that is deterministic and takes the following simple form: The set of agents is partitioned into *jurors* and *candidates*. The allocation only depends on the reports of jurors, and the object is always allocated to one of the candidates. We call this a *jury mechanisms*. All jury mechanisms are DIC since jurors cannot award the object to themselves.[2]

Our remaining results strengthen the case for jury mechanisms. We show that jury mechanisms are approximately optimal when there are many agents and, roughly speaking, the agents are symmetrically informed about the vector of payoffs. In this

[1]We refer to this problem as the *simple allocation problem*, loosely defined as the mechanism design problem of allocating an indivisible object without the use of money and to maximize the expected payoffs of the principal. We borrow the name from Ben-Porath et al. (2019) who, among other things, study a version of the problem where the types of the agents are verifiable.

[2]It is a known result that if $n \leq 3$, then all *deterministic* DIC mechanisms are jury mechanisms (Holzman and Moulin, 2013; Kato and Ohseto, 2002). We extend this result by showing that if $n \leq 3$, then *all* DIC mechanisms are convex combinations of jury mechanisms. This extension requires a substantial argument.

case, it is also without loss to process the jurors' reports in a way that preserves the anonymity of the jurors. For our final result, we show that jury mechanisms are actually the only deterministic DIC mechanisms satisfying an anonymity axiom.

In summary, our contribution is to characterize when the set of DIC mechanisms is fully described by deterministic ones, and to identify restrictions on the environment or the set of permissible mechanisms such that jury mechanisms are (approximately) optimal. Jury mechanisms have other properties that are intuitively desirable: They are simple to implement in that it is easy to explain to each agent their role in the mechanism. Since jurors are impartial to which of the candidates wins the object, jurors will unaninmously agree on what is efficient conditional on the collective information of the jurors. Hence the principal never has to act against the will of the jury.

To gain an intuition for some of these results, let us explain the basic trade-off the principal faces. Fix a DIC mechanism and a type profile $\theta$. Suppose agent $i$ wins deterministically (with probability 1) at $\theta$. By DIC, agent $i$'s reported type does not affect $i$'s winning probability, meaning that $i$ also wins at all type profiles $\theta'$ obtained by a unilateral change of $i$'s type. If agent $i$'s type contains information about the others, the change in $i$'s type may reveal that at $\theta'$ it is much better to allocate to an agent other than $i$. Thus there is a tension between allocating to agent $i$ and using $i$'s information. By contrast, if the allocation is random and, say, agent $i$ only wins with probability $\frac{1}{2}$ at the two profiles, the principal can let agent $i$'s type inform how to split the remaining probability mass of $\frac{1}{2}$ among the other agents. We follow this intuition to construct an example where the optimal DIC mechanism, which we spell out explicitly, is stochastic and unique.

We can understand the approximate optimality of jury mechanisms via the same intuition. This result concerns environments where, for all agents $i$, all agents other than $i$ are exchangeable in terms of informing about the payoff $\omega_i$ from allocating to $i$. When agents are exchangeable, increasing the number of agents loosens the aforementioned tension between allocating to an agent and using that agent's information. The principal incurs essentially no loss when not consulting the agents who are sometimes allocated the object—this is the defining property of a jury mechanism.

The paper is organized as follows. The next section discusses related work. Section 3 presents the model. In Section 4, we characterize when deterministic DIC mechanisms suffice for the principal's problem, and we introduce jury mechanisms.

In Section 5, we ask when jury mechanisms are approximately optimal. Section 6 considers anonymous mechanisms. Section 7 concludes. All proofs are in Appendices A to C. Appendices D and E contains supplementary material.

## 2 Related literature

Holzman and Moulin (2013) study desirable axioms for deterministic DIC mechanisms (there called impartial nomination rules) that allocate an object without the use of money. They mostly focus on situations where each agent nominates one of the others. Relative to the literature following Holzman and Moulin, we consider optimal DIC mechanisms and advance the understanding of the set of DIC mechanisms (including stochastic ones) when agents report abstract types.[3] This abstraction is important since nominations do not generally let agents express all their private information. As already noted by Holzman and Moulin, many of their axioms have no clear counterparts with abstract types. Jury mechanisms capture the spirit of some of their axioms, though. For instance, jury mechanisms satisfy *negative unanimity*, which requires that an agent receiving no nominations does not win, in the sense that the jury's decision is unanimous. They also satisfy *positive unanimity*, which requires that an agent receiving all nominations does win, in the same loose sense. Jury mechanisms however fail their *no dummy* axiom, which requires that all agents have a say in the allocation, and their *no exclusion* axiom, which requires that all agents have a chance at winning.

Alon et al. (2011) initiated a literature on approximately-optimal DIC mechanisms (there called strategy-proof mechanisms) for the following problem.[4] In the model, each agent's type specifies a subset of the others, interpreted as the set of others which the given agent approves as a winner of the object. The principal wishes to select the agent with the most approvals. Mechanisms are ranked according to approximation ratios rather than according to expected payoffs.[5] This leads to optimal mechanisms

[3] Further contributions include Edelman and Por (2021), Mackenzie (2015, 2020), and Tamura and Ohseto (2014). See also De Clippel et al. (2008).

[4] Further contributions to this literature include Aziz et al. (2016, 2019), Bjelde et al. (2017), Bousquet et al. (2014), Caragiannis et al. (2019), Fischer and Klimm (2015), Lev et al. (2021), and Mattei et al. (2020). Many of these papers consider the more difficult problem of allocating multiple objects.

[5] Given $\alpha \in [0, 1]$, a mechanism has an approximation ratio of $\alpha$ if it guarantees a fraction $\alpha$ of some benchmark value. The guarantee is computed across all realizations of the type profile; that

qualitatively different from ours, and hence our results provide a novel perspective on the problem. For example, while jury mechanisms can be optimal in our model, the 2-*partition mechanism* of Alon et al. (2011), which is perhaps the most natural analogue of our jury mechanisms in their model, is not optimal for their problem.[6]

Within the literature following Alon et al. (2011), the recent work of Caragiannis et al. (2021) is perhaps closest to ours. Their setup is that of Alon et al. (2011), except that there is a prior over agents' approval sets and mechanisms are evaluated via expected payoffs. Caragiannis et al. (2021) study a particular DIC mechanism—*approval voting with default*—focusing on its performance as the number of agent diverges.

Other related papers study non-monetary instruments for screening the agents.[7] Closest to us are recent papers that study how correlation betweeen types can be used to screen the agents via their beliefs (Bloch et al., 2022; Kattwinkel, 2019; Kattwinkel and Knoepfle, 2021; Kattwinkel et al., 2022). Our model has no instruments for providing strict incentives. We focus on the core economic problem of aggregating agents' impartial reports.

The papers of Baumann (2018) and Bloch and Olckers (2021, 2022) study related setting but focus on different questions. For instance, Bloch and Olckers (2022) ask whether the principal can reconstruct the ordinal ranking of agents from their reports when agents prefer that the principal assign them a high rank.

There exist several papers on the problem of aggregating the opinions of a jury. See Amorós (2020) for recent work and further references. This problem is not our focus (as jurors impartially report their information to the principal). Our results provide a foundation for using jury mechanisms.

is, across all possible approval sets. The benchmark value at a particular realization is the maximal number of approvals across agents.

[6]The 2-partition mechanism randomly splits the agents into two subsets, and then selects an agent from the first subset with the most approvals from agents in the second subset. Alon et al. (2011, Theorem 4.1) show that the 2-partition mechanism has an approximation ratio of $\frac{1}{4}$. Fischer and Klimm (2015) give a mechanism that achieves the optimal ratio of $\frac{1}{2}$.

[7]Examples of such instruments include promises of future allocations (Guo and Hörner, 2021), delaying the allocation (Condorelli, 2009), costly verification (Ben-Porath et al., 2014, 2019; Epitropou and Vohra, 2019; Erlanson and Kleiner, 2019), costly signaling (Chakravarty and Kaplan, 2013; Condorelli, 2012), allocative externalities (Bhaskar and Sadler, 2019; Goldlücke and Tröger, 2020), or ex-post punishments (Li, 2020; Mylovanov and Zapechelnyuk, 2017).

# 3 Model

A principal allocates a single indivisible object to one of $n$ agents. For each agent $i$, let $\Omega_i$ be a finite set of reals representing the possible payoffs to the principal from allocating to agent $i$; let $\Theta_i$ be a finite set representing agent $i$'s possible private types. Let $\Omega = \times_{i=1}^n \Omega_i$ and $\Theta = \times_{i=1}^n \Theta_i$. Payoffs and types are distributed according to a joint distribution $\mu$ over $\Omega \times \Theta$. Each agent $i$ privately observes $\theta_i \in \Theta_i$. Payoffs $\omega_1, \ldots, \omega_n$ are unobserved by the agents and the principal (but types may be informative about payoffs through $\mu$).

On the side of the agents, we assume that, for all $i$ and at all type profiles, agent $i$ strictly prefers winning the object to not winning it; agent $i$ is indifferent to which of the others is allocated the object.

A *mechanism* is a function $\varphi\colon \Theta \to [0,1]^n$ such that all $\theta \in \Theta$ satisfy $\sum_{i=1}^n \varphi_i(\theta) = 1$. Here $\varphi_i(\theta)$ denotes the probability that agent $i$ is allocated the object when the reported type profile is $\theta$. Since the object is allocated to one of the agents, these probabilites sum to 1.[8]

A mechanism $\varphi$ is *dominant-strategy incentive-compatible (DIC)* if truthfully reporting one's type is a dominant strategy. For the assumed preferences of the agents, this means agents cannot influence their own winning probabilities with their reports. That is, for all agents $i$, and all type profiles $\theta$ and $\theta'$ that differ only in $i$'s type, the mechanism satisfies $\varphi_i(\theta) = \varphi_i(\theta')$.[9] The principal evaluates a DIC mechanism $\varphi$ via the expected payoff from the allocation, which is given by $\mathbb{E}_{\omega,\theta}\left[\sum_{i=1}^n \varphi_i(\theta)\omega_i\right]$. The Revelation Principle implies that DIC mechanisms are without loss.

To conclude the model presentation, we give a simple benchmark where the principal *cannot* elicit information.

**Example 1.** Suppose that for all $i$ and $\theta_{-i}$ we have $\mathbb{E}_{\omega_i}[\omega_i|\theta_{-i}] = \mathbb{E}_{\omega_i}[\omega_i]$; that is,

[8]The requirement that the principal always allocate the object keeps with earlier work in the literature (e.g. Alon et al. (2011) and Holzman and Moulin (2013)). One motivation is that not allocating may be Pareto inefficient in the applications we have in mind. In the examples from the introduction, the planner destroys the valuable good, the group has to go on without a president, management burns or privately consumes funds, etc. In Appendix D, we relax this requirement.

[9]To see this in detail, let $u_i(\theta)$ denote the payoff to agent $i$ when $i$ is allocated the object at a type profile $\theta$. We normalize $i$'s payoff when not allocated the object to 0, and we assume $u_i > 0$. A mechanism $\varphi$ is DIC if and only if all $i, \theta_i, \theta_i', \theta_{-i}, \theta_{-i}'$ satisfy $u_i(\theta_i, \theta_{-i})\varphi_i(\theta_i, \theta_{-i}') \geq u_i(\theta_i, \theta_{-i})\varphi_i(\theta_i', \theta_{-i}')$. Since $u_i > 0$ and since $\theta_i$ and $\theta_i'$ are arbitrary, we must have $\varphi_i(\theta_i, \theta_{-i}') = \varphi_i(\theta_i', \theta_{-i}')$. That is, agent $i$'s report never affects $\varphi_i$.

the types of agents other than $i$ are uninformative about the payoff from allocating to $i$. The only source of information about $\omega_i$ is therefore $i$'s own type. If the principal would let this type inform $i$'s allocation, agent $i$ would misreport it. Hence there is an optimal mechanism that ignores all reports. Formally, let $\varphi$ be a DIC mechanism. By DIC, we may drop $i$'s type from $\varphi$. We have $\mathbb{E}_{\omega,\theta}\left[\sum_{i=1}^{n}\varphi_i(\theta)\omega_i\right] = \sum_{i=1}^{n}\mathbb{E}_{\theta_{-i}}[\varphi_i(\theta_{-i})\mathbb{E}_{\omega_i}[\omega_i|\theta_{-i}]] = \sum_{i=1}^{n}\mathbb{E}_{\theta_{-i}}[\varphi_i(\theta_{-i})\mathbb{E}_{\omega_i}[\omega_i]] \leq \max_{i=1}^{n}\mathbb{E}_{\omega_i}[\omega_i]$. This upper bound can be obtained by constantly allocating to an agent in $\arg\max_{i=1}^{n}\mathbb{E}_{\omega_i}[\omega_i]$.

# 4 Random allocations and extreme points

In this section, we ask whether the set of DIC mechanisms is fully characterized by deterministic ones.

**Definition 1.** A mechanism is **deterministic** if it maps to a subset of $\{0,1\}^n$. A mechanism is **stochastic** if it is not deterministic.

One way of constructing a stochastic DIC mechanism is by randomizing over two distinct deterministic DIC mechanisms. Our interest is in stochastic DIC mechanisms that cannot be constructed in this way. That is, we are asking whether the set of DIC mechanisms admits stochastic *extreme points*.[10]

The upcoming Theorem 4.2 fully characterizes when stochastic extreme points exist; existence depends on the number of agents and the cardinalities of the type spaces. In this model, all extreme points are candidates for optimal mechanisms (Appendix E.1), and hence Theorem 4.2 implies that deterministic DIC mechanisms are not generally sufficient for solving the principal's problem.

Before presenting the result, we give a simple example of a stochastic extreme point. We moreover construct an environment in which this extreme point is uniquely optimal. This exercise sheds light on the principal's basic trade-off and gives an economic intuition for why randomization is beneficial.

[10] Recall that a point $x$ in a subset $X$ of Euclidean space is an extreme point of $X$ if $x$ cannot be written as a convex combination of two other points in $X$. According to the Krein-Milman theorem (Aliprantis and Border, 2006, Theorem 7.68), a non-empty, convex, compact subset of Euclidean space coincides with the convex hull of its extreme points. The set of DIC mechanisms is convex and compact, and therefore coincides with the convex hull of its extreme points.

### 4.1 An illustrative example

Suppose there are four agents and that their type spaces are labelled as follows.

$$\Theta_1 = \{\ell, r\}, \quad \Theta_2 = \{u, d\}, \quad \Theta_3 = \{f, c, b\}, \quad \Theta_4 = \{0\}. \tag{4.1}$$

Figure 1 shows (among other things that are not yet relevant) the type profiles of agents 1, 2, and 3; the degenerate type of agent 4 is omitted. The types of agents 1, 2, and 3 span a 3-dimensional box. Each edge of the box represents a set of type profiles along which exactly one agent's type is changing. Hence DIC requires that the winning probability of this agent be constant along the edge. Formally, we identify such an edge by a pair $(i, \theta_{-i})$, where $i$ indicates the agent whose type is changing, and $\theta_{-i}$ indicates the fixed types of the others.

Let $\Theta^* = \{\theta^a, \theta^b, \theta^c, \theta^d, \theta^e, \theta^f, \theta^g\}$ be the set of type profiles shown in Figure 1. Formally, these are the profiles

$$\begin{aligned}
&\theta^a = (\ell, d, c, 0), \quad \theta^b = (r, d, c, 0), \quad \theta^c = (r, d, b, 0), \\
&\theta^d = (r, u, b, 0), \quad \theta^e = (r, u, f, 0), \quad \theta^f = (\ell, u, f, 0), \\
&\theta^g = (\ell, u, c, 0).
\end{aligned} \tag{4.2}$$

Let $V^*$ denote the edges passing through the profiles in $\Theta^*$; these are the highlighted edges in Figure 1. Formally, let

$$\begin{aligned}
V^* = \{&(1, \theta^a_{-1}), (3, \theta^c_{-3}), (2, \theta^c_{-2}), (3, \theta^e_{-3}), \\
&(1, \theta^e_{-1}), (3, \theta^f_{-3}), (2, \theta^a_{-2})\}.
\end{aligned}$$

Our candidate stochastic extreme point $\varphi^*$ is defined as follows: For all $i \in \{1, 2, 3\}$ and $\theta \in \Theta$, let

$$\varphi^*_i(\theta) = \begin{cases} \frac{1}{2}, & \text{if } (i, \theta_{-i}) \in V^*, \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

Further, for all $\theta \in \Theta$ let

$$\varphi^*_4(\theta) = 1 - \sum_{i \in \{1,2,3\}} \varphi^*_i(\theta). \tag{4.4}$$

The winning probabilities of agents 1, 2 and 3 are depicted in Figure 1 by the probabilities of $\frac{1}{2}$ attached to various edges of the box. It is easy to verify from the figure that $\varphi^* = (\varphi_1^*, \varphi_2^*, \varphi_3^*, \varphi_4^*)$ is a well-defined DIC mechanism.

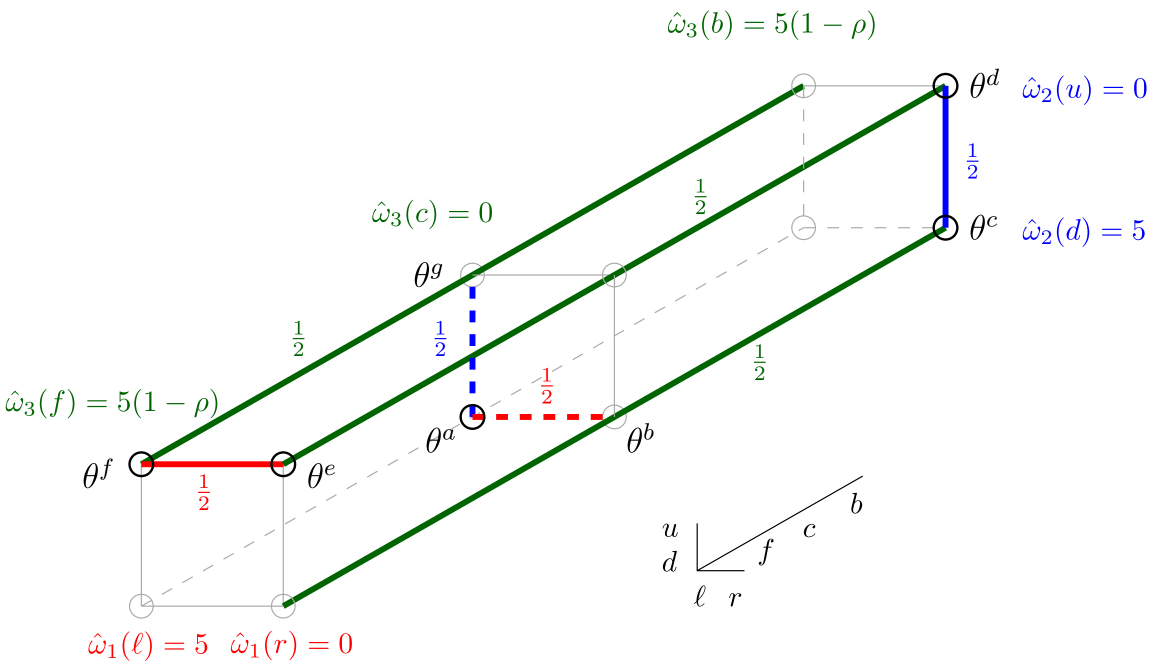



Figure 1: The set of types of agents 1, 2, and 3. The probabilities $\frac{1}{2}$ attached to the edges of the box represent the relevant values of the mechanism $\varphi^*$. The payoffs are as defined in (4.6). The distribution $\mu$ assigns probability $\frac{1}{5}$ to the profiles $\{\theta^a, \theta^c, \theta^d, \theta^e, \theta^f\}$. All other profiles have probability 0.

Further below we specify payoffs $\Omega$ and a distribution $\mu$ such that $\varphi^*$ is the unique optimal DIC mechanism. This implies that $\varphi^*$ is an extreme point of the set of DIC mechanisms. Since the proof for uniqueness is somewhat involved, we next present a simple self-contained argument showing that $\varphi^*$ is an extreme point.

Let $\varphi$ be a DIC mechanism that receives non-zero weight in a convex combination that equals $\varphi^*$. We show $\varphi = \varphi^*$. For all profiles $\theta \in \Theta^*$, there are exactly two agents $i$ and $j$ such that $(i, \theta_{-i})$ and $(j, \theta_{-j})$ both belong to $V^*$; these are the two highlighted edges of the box that intersect at $\theta$. Hence, by definition of $\varphi^*$, in this situation $\varphi^*$ randomizes evenly between $i$ and $j$. Since $\varphi$ is part of a convex combination that equals $\varphi^*$, it follows that $\varphi$ also randomizes between $i$ and $j$ (though at this point we do not know using which probabilities). Repeatedly applying this observation shows:

$$\begin{aligned}\varphi_1(\theta^a) = 1 - \varphi_3(\theta^c) = \varphi_2(\theta^c) &= 1 - \varphi_3(\theta^e) \\ &= \varphi_1(\theta^e) \\ &= 1 - \varphi_3(\theta^f) = \varphi_2(\theta^a) = 1 - \varphi_1(\theta^a).\end{aligned} \tag{4.5}$$

In particular, we have $\varphi_1(\theta^a) = 1 - \varphi_1(\theta^a)$, implying $\varphi_1(\theta^a) = \frac{1}{2}$. Hence all probabilities in (4.5) equal $\frac{1}{2}$. Hence $\varphi$ agrees with $\varphi^*$ at all profiles in $\Theta^*$. By inspecting $\Theta \setminus \Theta^*$, we may easily convince ourselves that $\varphi$ and $\varphi^*$ also agree on $\Theta \setminus \Theta^*$. Thus $\varphi^*$ is an extreme point.

What is it about the number $n$ of agents and the cardinalities of the type spaces that permits the construction of the stochastic extreme point $\varphi^*$? The dummy agent 4 is used to ensure that the mechanism is well-defined (recall that the object must be allocated). The cardinalities of the type spaces matter for the following reason. Informally speaking, the argument given in the previous paragraph rests on the fact that the profiles $\theta^a, \dots, \theta^g$ constitute a cycle of *odd* length. The fact that the cycle length is odd is what leads to the critical equality $\varphi_1(\theta^a) = \varphi_2(\theta^a)$ in (4.5). One can still find such an odd cycle if we added more agents or more types to the environment. Conversely, it becomes more difficult to construct such a cycle for smaller type spaces.[11]

We now turn to the problem of specifying payoffs and a joint distribution such that $\varphi^*$ is uniquely optimal. To that end, we define a distribution $\mu$ over type profiles, and for all agents $i$ a function $\hat{\omega}_i \colon \Theta_i \to \mathbb{R}$. The interpretation is that, conditional on $i$'s type being $\theta_i$, the payoff from allocating to $i$ is $\omega_i = \hat{\omega}_i(\theta)$ with probability 1.

Our candidate payoffs are parametrized by $\rho \in [0, \frac{1}{2}]$. Let $\hat{\omega}_1, \dots \hat{\omega}_4$ be defined as

[11] There is a deeper graph-theoretic connection between the existence of stochastic extreme points and the existence of certain odd cycles. See Appendix D. We should clarify here that the results in this appendix concern the relaxed problem where the principal is not required to always allocate the object. The resulting characterization of extreme points is implied by the upcoming Theorem 4.2, but not vice versa.

follows (see Figure 1):

$$\begin{aligned} \hat{\omega}_1(r) = \hat{\omega}_2(u) = \hat{\omega}_3(c) &= 0 \\ \hat{\omega}_1(\ell) = \hat{\omega}_2(d) &= 5 \\ \hat{\omega}_3(f) = \hat{\omega}_3(b) &= 5(1-\rho) \\ \hat{\omega}_4 &= 0. \end{aligned} \tag{4.6}$$

Lastly, let $\mu$ be defined by

$$\forall_{\theta\in\Theta}, \quad \mu(\theta) = \begin{cases} \frac{1}{5}, & \text{if } \theta \in \{\theta^a, \theta^c, \theta^d, \theta^e, \theta^f\} \\ 0, & \text{else.} \end{cases} \tag{4.7}$$

**Proposition 4.1.** *If $\rho \in (0, \frac{1}{2})$, then $\varphi^*$ is the unique optimal DIC mechanism.*

In the introduction we intuited that the principal has a trade-off between allocating to an agent and using that agent's information, and that this trade-off underlies the benefits from randomizing the allocation. In this example, this trade-off depends on the parameter $\rho$ in agent 3's payoff. For an intuition, let $\rho = 0$. Now, allocating to agent 3 is ex-post optimal at *all except one* of the five profiles in the support of $\mu$. The constant mechanism that always allocates to agent 3 is in fact an optimal DIC mechanism when $\rho = 0$. To explain why $\varphi^*$ is another optimal mechanism when $\rho = 0$, note that agent 3's type $\theta_3$ contains information: when $\theta_3 = c$, the type profile must be $\theta^a$, which is the unique type profile in the support of $\mu$ where allocating to agent 3 is not optimal. The mechanism $\varphi^*$ uses this information by allocating to agents 1 and 2 at $\theta^a$.

In summary, for $\rho = 0$, there are at least two optimal mechanisms: Always allocate to agent 3, or use the mechanism $\varphi^*$ that both allocates to agent 3 and uses agent 3's information. It follows that when $\rho$ is non-zero, then $\varphi^*$ does strictly better than always allocating to agent 3. Since $\rho$ decreases the payoff from allocating to agent 3, it is now intuitive that $\varphi^*$ is uniquely optimal when $\rho$ is strictly positive but close to 0. If we increase $\rho$ further and further, then $\varphi^*$ eventually ceases to be optimal since the principal optimally avoids allocating to agent 3 altogether. The critical value turns out to be $\rho = \frac{1}{2}$, where $\varphi^*$ is an optimal DIC mechanism, but not the only one. For instance there is another optimal DIC mechanism that allocates to agent 1 at $\theta^a, \theta^b, \theta^e$ and $\theta^f$, and allocates to agent 2 at $\theta^c$ and $\theta^d$ (and otherwise allocates to agent 4). Since the principal's utility is affine in $\rho$, it follows from here the $\varphi^*$ is uniquely optimal for $\rho \in (0, \frac{1}{2})$.

**Remark 1.** Chen et al. (2019) establish an equivalence result between stochastic and deterministic mechanisms for certain mechanism design problems. Their Theorem 1 and Remark 2 imply the following (for our setup): *Let the type profile follow an atomless distribution in Euclidean space. For all mechanisms $\varphi$, there is a deterministic mechanism $\varphi'$ such that all $i$ and $\theta_i$ satisfy $\mathbb{E}_{\theta_{-i}}[\varphi_i(\theta_i, \theta_{-i})|\theta_i] = \mathbb{E}_{\theta_{-i}}[\varphi_i'(\theta_i, \theta_{-i})|\theta_i]$.* In light of this result, one may not have expected that $\varphi^*$ is uniquely optimal. After all, in the above example the contribution of allocating to agent $i$ at $\theta_i$, namely $\hat{\omega}_i(\theta_i)\mathbb{E}_{\theta_{-i}}[\varphi_i(\theta_i, \theta_{-i})|\theta_i]$, depends on $\varphi$ only through the interim-expected allocation probability $\mathbb{E}_{\theta_{-i}}[\varphi_i(\theta_i, \theta_{-i})|\theta_i]$. The results are reconciled by noting that the mechanism $\varphi'$ is not guaranteed to be DIC even if $\varphi$ is DIC.

**Remark 2.** In the example, agent $i$'s type $\theta_i$ pins down the payoff $\omega_i = \hat{\omega}_i(\theta_i)$. That is, each agent $i$ privately knows $\omega_i$. The example therefore demonstrates that stochastic extreme points may be uniquely optimal in environments where the correlation between types and payoffs is limited. Although we could have used a separating hyperplane theorem to construct an environment where $\varphi^*$ is uniquely optimal, this would not have guaranteed that the resulting environment has such privately-known payoffs. Indeed, the following is another environment where $\varphi^*$ is uniquely optimal but payoffs are not privately known: the distribution of type profiles is uniform over $\Theta^*$; for all agents $i$ and profiles $\theta$, the payoff of allocating to $i$ conditional on $\theta$ is 1 if $(i, \theta_{-i}) \in V^*$, and 0 otherwise.

## 4.2 Full characterization

We now fully characterize when the set of DIC mechanisms admits a stochastic extreme point. In a nutshell, all extreme points are deterministic if $n \leq 3$; stochastic extreme points exist if $n \geq 4$ and type spaces are not too small.

**Theorem 4.2.** *Fix $n$ and $\Theta_1, \ldots, \Theta_n$. All extreme points of the set of DIC mechanisms are deterministic if and only if at least one of the following is true:*

*(1) There are at most three agents; that is $n \leq 3$.*

*(2) All agents have at most two types; that is, for all $i$ we have $|\Theta_i| \leq 2$.*

*(3) At least $(n-2)$-many agents have a degenerate type; that is, we have*

$$|\{i \in \{1, \dots, n\} \colon |\Theta_i| = 1\}| \geq n-2.$$

If conditions (1) to (3) all fail, we extend the example from Section 4.1 to construct a stochastic extreme point. Proving that (1) is sufficient for all extreme points to be deterministic turns out to be involved. Sufficiency of (2) is related to a generalization of the well-known Birkhoff-von Neumann theorem. Sufficiency of (3) is neither technically nor economically very interesting, but must be included for completeness.[12]

To better interpret stochastic extreme points, let us give an example of a class of DIC mechanisms that are stochastic but *not* extreme points. Consider an indirect mechanism with a dominant-strategy equilibrium. This equilibrium implements a DIC (direct) mechanism $\varphi$. If $\varphi$ is stochastic, the randomness can stem from randomness in the outcome function of the indirect mechanism, or the strategies of the agents. It turns out that if $\varphi$ is a stochastic extreme point, then the indirect mechanism must have a random outcome function (see Appendix E.2).

The fact that the principal may strictly benefit from committing to random allocations has practical implications. Deterministic mechanisms are preferable to stochastic ones. The reason is that if a stochastic mechanism is used the principal must commit to honoring the outcome of a random process. This point is discussed in greater length in (Budish et al., 2013; Chen et al., 2019; Laffont and Martimort, 2009; Pycia and Ünver, 2015). To see the commitment issue in a concrete example, consider again the environment from Section 4.1. At the profile $\theta^e$, a coin flip determines whether agent 1 or 3 wins the object. Yet, at this profile, the payoff of allocating to agent 3 is strictly higher than the payoff of allocating to agent 1. Thus the principal must be able to commit to honoring the outcome of the coin flip. In fact, we note that in this example the principal commits to flipping a coin at all five type profiles in the support of the distribution. For $\rho \in (0, \frac{1}{2})$, the principal is indifferent to the outcome of the coin flip only at one of these profiles. Thus the commitment issue does not arise in a low probability event.

[12] The reader may wonder whether one can prove sufficiency of (1) to (3) by viewing the set of DIC mechanisms as the set of solutions to a linear system of inequalities, checking for total unimodularity of the constraint matrix, and then invoking the Hoffman-Kruskal theorem (Korte and Vygen, 2018, Theorem 5.21). This approach works for the case where all type spaces are binary; our proof uses a result which can itself be derived from the Hoffman-Kruskal theorem. However, in the difficult case with three agents, the constraint matrix is *not* generally totally unimodular (see Appendix E.5).

While optimal DIC mechanisms may thus involve commitment to lotteries, we next use Theorem 4.2 and a known result from the literature to identify a special case where optimal mechanisms are deterministic and simple. When $n \leq 3$, the set of DIC mechanisms is fully characterized by so-called deterministic *jury mechanisms*.

### 4.3 Jury mechanisms

Given a mechanism $\varphi$ and an agent $i$, we say $i$ *influences* $\varphi$ if $\varphi$ is non-constant in $i$'s report.[13]

**Definition 2.** A mechanism $\varphi$ is a **jury mechanism** if for all agents $i$ that influence $\varphi$ we have $\varphi_i = 0$; that is, agent $i$ never wins the object.

Given a jury mechanism, it is natural to refer to the agents who influence the allocation as *jurors* (and the set of jurors as the *jury*); the remaining agents are *candidates*. In words then, the principal nominates a subset of agents as jurors. The collective opinion of the jury determines the allocation to the candidates, and jurors never win. Observe that all jury mechanisms are DIC.

Theorem 4.2 implies that if $n \leq 3$, then all extreme points are deterministic. It is a known result that if $n \leq 3$, then all deterministic DIC mechanisms are jury mechanisms (Holzman and Moulin, 2013, Proposition 2.i).[14] Hence:

**Corollary 4.3.** *Let $n \leq 3$. A mechanism is DIC if and only if it is a convex combination of deterministic jury mechanisms. In particular, there is a deterministic jury mechanism that maximizes the principal's utility over the set of DIC mechanisms.*

With three or fewer agents, a jury mechanism admits a single juror who deliberates between the other two agents. Hence the content of the result is that all DIC mechanisms with three or fewer agents can be implemented by nominating a juror (according to some distribution over the set of agents), and then asking the juror to pick one of the others as a winner of the object. Optimally, the principal ignores the information of at least two of the agents.

[13] That is, there are distinct profiles $\theta$ and $\theta'$ that differ only in the type of agent $i$ and are such that $\varphi(\theta) \neq \varphi(\theta')$.

[14] Holzman and Moulin (2013) note that this result is essentially due to Kato and Ohseto (2002). Neither of these papers refer to these mechanisms as jury mechanisms, but their definitions are equivalent to ours when $n \leq 3$. For a broader discussion of the relationship to Kato and Ohseto (2002), we refer the reader to Holzman and Moulin (2013, Section 1.4).

# 5 Approximate optimality of jury mechanisms

In this section, we identify conditions under which jury mechanisms are approximately optimal if the number of agents is large. The following example conveys the basic idea.

**Example 2.** Suppose that each agent $i$ observes $\omega_i$, and that $(\omega_1, \ldots, \omega_n)$ are independently and identically distributed. Additionally, there is a *public* signal $s$ about $(\omega_1, \ldots, \omega_n)$ observed by all agents. Each agent's type is therefore $\theta_i = (\omega_i, s)$. Let $\varphi$ be a DIC mechanism. Now suppose a new agent $n+1$ joins the group, and suppose agent $n+1$ also observes the public signal $s$. Agent $n+1$ may also observe some additional information, but this will not be relevant here. We claim there is a jury mechanism that only uses agent $n+1$ as juror and that leaves the principal as well off as $\varphi$. To see this, note that by ignoring the reports of agents 1 to $n$, the principal does not lose the information in the public signal since agent $n+1$ also observe this signal. The only information that is potentially lost is the first $n$ agents' knowledge of their own payoffs $\omega_1, \ldots, \omega_n$. However, each $\omega_i$ is uninformative about the other payoffs (by independence). Moreover, DIC of the original mechanism $\varphi$ implies that $\omega_i$ could not have been used to determine $i$'s own allocation. Thus the principal actually does not lose information at all when ignoring the reports of agents 1 to $n$.

The main result of this section generalizes the previous example as follows. Under an assumption on the distribution of types and payoffs, an arbitrary DIC mechanism with $n$ agents can be replicated by a jury mechanism when additional agents are around. If the principal's utility remains bounded in $n$, an implication is that jury mechanisms become approximately optimal as $n \to \infty$.

We introduce new notation to accomodate the growing number of agents. The agents share a common finite type space ($\Theta_1 = \Theta_i$ for all $i$). The prior distribution of payoffs and types is now a Borel-probability measure $\mu$ on $\times_{i \in \mathbb{N}}(\Omega_i \times \Theta_i)$,[15] where each $\Omega_i$ is a finite set of reals.

The following assumption captures the idea that if $i$, $j$ and $k$ are three distinct agents, then $i$ and $j$ have can provide the same information about $\omega_k$. This is the case in Example 2 where the public signal is the only information that $i$ and $j$ have

[15]The set $\times_{i \in \mathbb{N}}(\Omega_i \times \Theta_i)$ is equipped with the product metric.

about $\omega_k$. We will, however, not assume that agents other than $i$ and $j$ have the same information as $k$ about $\omega_k$.

**Assumption 1.** For all $n \in \mathbb{N}$, all $i \in \{1, \ldots, n\}$, and all $\omega_i \in \Omega_i$, we have the following: Conditional on the payoff of agent $i$ being equal to $\omega_i$, the distribution of $(\theta_j)_{j \in \{1,\ldots,n\} \setminus \{i\}}$ is invariant with respect to permutations of $\{1, \ldots, n\} \setminus \{i\}$.

When there are $n$ agents (meaning the principal only consults and allocates to the first $n$ agents), let $\Pi_n$ denote the principal's utility from an optimal DIC mechanism. In the same situation, let $\Pi_n^J$ denote the utility from a jury mechanism that is optimal among jury mechanisms.

**Theorem 5.1.** *Let Assumption 1 hold. For all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\Pi_n \leq \Pi_{n+m}^J$. If, additionally, the sequence $\{\Pi_n\}_{n \in \mathbb{N}}$ converges, then $\lim_{n \to \infty}(\Pi_n - \Pi_n^J) = 0$.*[16]

In words, if $m$ new agents are added to the group, the principal is as well off with a jury mechanism with $n + m$ agents as with an arbitrary DIC mechanism $\varphi$ with $n$ agents. The proof shows this claim for a jury mechanism that has the new $m$ agents as jurors, and the old $n$ agents as candidates, and where $m = n$. The second claim in the theorem is an immediate implication of the first.

Assumption 1 is stronger than what we really need for this argument. It suffices if, informally, for all groups of agents $\{1, \ldots, n\}$ there eventually comes a disjoint group of agents that is at least as well informed as $\{1, \ldots, n\}$ about the payoffs of agents in $\{1, \ldots, n\}$, excepting each agent $i$'s information about $\omega_i$. Assumption 2 in Appendix B formalizes this idea, and also allows for agents to have different type spaces.

**Remark 3.** Theorem 5.1 does not assert that DIC mechanisms become approximately ex-post optimal conditional on the type profile. For example, in Example 2, the only information that is used in the allocation is the public signal. The public signal may fail to reveal the entire profile of payoffs.

**Remark 4.** While the form of Assumption 1 stated above is the one that is most convenient for our proofs, the reader may wonder exactly which environments satisfy

[16] The assumption that the sequence $\{\Pi_n\}_{n \in \mathbb{N}}$ converges is mild, in our view. It holds if, say, the limit $\lim_{n \to \infty} \mathbb{E}\left[\max_{i=1}^n \mathbb{E}[\omega_i | \theta_1, \ldots, \theta_n]\right]$ exists. Each of these iterated expectations is the principal's utility from the allocation that is ex-post optimal conditional on the type profile when there $n$ agents.

Assumption 1. To answer this, one can use de Finetti's theorem to characterize, for each $i$ and $\omega_i$, the conditional distribution of types of agents other than $i$; see Hewitt and Savage (1955).

# 6 Anonymous juries

In this section, we study DIC mechanisms that process the agents' reports anonymously. Anonymity is a desirable property of practical mechanisms as it protects agents from threats by their peers or outside observers. The previous section actually identifies a setting where anonymity arises out of the principal's maximization problem. Consider the following notion of anonymity for jury mechanisms.

**Definition 3.** A jury mechanism has an **anonymous jury** if the mechanism is invariant with respect to permutations of the reports of the jurors.[17]

If Assumption 1 holds, then for all jury mechanisms $\varphi$ there exists a deterministic jury mechanism that has an anonymous jury and that weakly improves on $\varphi$.[18] Theorem 5.1 thus tells us that deterministic jury mechanisms with anonymous juries are approximately optimal under Assumption 1 and when the number of agents is large.

In this section, we consider two notions of anonymity. The only DIC mechanisms satisfying the stronger notion turn out to be constant mechanisms. All deterministic DIC mechanisms satisfying the weaker notion are jury mechanisms with anonymous juries.

Throughout this section, we assume that the agents share a common type space. For ease of comparison, we present our two notions of anonymity side-by-side.

**Definition 4.** Let $\varphi$ be a mechanism.

(1) Given $i$, $j$, and $k$ that are all distinct, agent $i$ and $j$ are **exchangeable for** $k$ if $\varphi_k$ is invariant with respect to all permutations of $i$'s and $j$'s reports.[19]

[17] That is, for all $(\theta_i)_{i\in J}$ and all permutations $\xi$ of $J$ we have $\varphi((\theta_i)_{i\in J}) = \varphi((\theta_{\xi(i)})_{i\in J})$. Here we omit at a slight abuse of notation the types of the candidates since these do not influence the allocation.

[18] Given a type profile $(\theta_i)_{i\in J}$ of the jury $J$, Assumption 1 implies that the set of candidates that are optimal conditional on $(\theta_i)_{i\in J}$ is invariant with respect to permutations of $(\theta_i)_{i\in J}$. Hence, by breaking ties among the candidates according to some fixed order, selecting an optimal candidate at each type profile of the jury yields a deterministic jury mechanism with an anonymous jury. This mechanism does weakly better than all other jury mechanisms with the same jury.

[19] That is, for all profiles $\theta$ and $\theta'$ such that $\theta$ is obtained from $\theta'$ by permuting the types of $i$ and $j$, we have $\varphi_k(\theta) = \varphi_k(\theta')$.

(2) The mechanism is **anonymous** if for all $i$, $j$, and $k$ that are all distinct, agents $i$ and $j$ are exchangeable for $k$.

(3) Given distinct $i$ and $k$, agent $i$ **influences** $k$ if $\varphi_k$ is non-constant in $i$'s report.[20]

(4) The mechanism is **relatively anonymous** if for all $i$, $j$, and $k$ that are all distinct and are such that $i$ and $j$ influence $k$, agents $i$ and $j$ are exchangeable for $k$.

Verbally, anonymity requires that $k$'s winning probability does not change if one permutes the reports of the others. A consequence is that either all agents in $\{1, \ldots, n\} \setminus \{k\}$ or none of the agents in $\{1, \ldots, n\} \setminus \{k\}$ influence $k$. Relative anonymity relaxes anonymity as follows: When testing whether $k$'s winning probability changes through some permutations, we now only consider permutations of those agents who actually influence agent $k$. The set of agents who influence $k$ may thus be a proper subset of $\{1, \ldots, n\} \setminus \{k\}$. Towards the goal of protecting agents from threats by their peers, relative anonymity is arguably as reasonable as anonymity: if it is common knowledge that $i$ does not influence $k$, it seems unnecessary to protect agent $i$ from threats by $k$.

**Theorem 6.1.** *All anonymous DIC mechanisms are constant.*

Note well that anonymity does *not* demand that $i$ and $j$ be exchangeable for $i$'s own winning probability (exchangeability concerns three distinct agents $i$, $j$ and $k$). If we did demand this, the result would follow rather trivially from DIC. Theorem 6.1 is instead more subtly related to the fact that the principal always allocates the object. To gain an intuition, consider the claim for deterministic mechanisms. By anonymity, permuting the reports of two agents $i$ and $j$ does not affect the winning probabilities of the others. Since the object is always allocated, it follows that the permutation does not affect $\varphi_i + \varphi_j$. Using this fact, one can show that if a change in $i$'s report can increase $j$'s winning probability, then the same change in $j$'s report also increases $i$'s winning probability. Since the mechanism is deterministic, this implies the existence of a profile where $i$ and $j$ both win, which is impossible. Generalizing this argument to stochastic mechanisms requires a careful summation over changes in winning probabilities.

[20] That is, there exist two type profiles $\theta$ and $\theta'$ that differ only in $i$'s type and are such that $\varphi_k(\theta) \neq \varphi_k(\theta')$.

Theorem 6.1 implies that a non-constant DIC mechanism must admit some asymmetry in how it processes the reports of different agents. This brings us to relative anonymity. We offer the following characterization for deterministic mechanisms.

**Theorem 6.2.** *A mechanism is deterministic, relatively anonymous, and DIC if and only if it is a deterministic jury mechanism with an anonymous jury.*

It is easy to verify that a jury mechanism with an anonymous jury is relatively anonymous. For the proof of the other direction, fix a deterministic relatively anonymous DIC mechanism. We define a binary relation on the set of agents where two agents $i$ and $j$ are related if and only if there is a third agent $k$ that $i$ and $j$ both influence. The proof shows that this is an equivalence relation. Further, two agents in the same equivalence class do not influence one another, but influence the same (possibly empty) set of agents outside the class. These arguments follow the intuition given for deterministic mechanisms in Theorem 6.1, but require additional work. Next we show that cannot be multiple equivalence classes; else, there is a profile where two distinct classes allocate the object to two distinct agents, which is impossible. Finally, the unique equivalence class defines an anonymous jury.

**Remark 5.** The requirement that the object always be allocated cannot we dropped from Theorem 6.1. If the requirement is relaxed, we can construct a stochastic anonymous DIC mechanism that is an extreme point of the set of *all* DIC mechanisms. The construction proceeds by "symmetrizing" the stochastic extreme point from Section 4.1; see Appendix D. A related point turns out to be the restriction to deterministic mechanisms in Theorem 6.2. In Appendix E.3, we show that a simple modification of the "symmetrized" extreme point yields a relatively anonymous DIC mechanism (that always allocates the object and) that is not a convex combination of jury mechanisms.[21]

**Remark 6.** Theorem 6.1 implies that the principal cannot elicit information in environments in which the principal actually finds it without loss to "treat all agents symmetrically." See Appendix E.4 for a formal statement.

Holzman and Moulin (2013) and Mackenzie (2015) previously considered a notion of anonymity for DIC mechanisms when each agent nominates one of the others.

[21] In Appendix E.3, we show that an analogue of Theorem 6.2 does obtain for stochastic mechanisms if one strengthens relative anonymity and, additionally, demands that the mechanism be immune to certain coalitional manipulations.

Let us keep with the terminology of Holzman and Moulin by refering to these mechanisms as *impartial nomination rules*. Their notion of anonymity for a nomination rule—*anonymous ballots*—requires that the winning probabilities depend only on the number of nominations received by each agent.[22] This notion has no immediate analogue in our model with abstract types. Importantly, note that in a nomination rule agents cannot nominate themselves, and hence they all have distinct message spaces. By contrast, we have assumed in this section that agents have the same type space. Hence our notion of anonymity does not nest anonymous ballots as a special case.

The different notions lead to a different characterizations. While all anonymous DIC mechanisms are constant (Theorem 6.1), there are non-constant impartial rules with anonymous ballots. In one easy example for the latter, the principal picks one of the agents uniformly at random, and then that agent's nomination determines a winner. Mackenzie (2015, Theorem 1) fully characterizes impartial nomination rules with anonymous ballots. We note however that all *deterministic* impartial nomination rules with anonymous ballots are constant (Holzman and Moulin, 2013, Theorem 3).

In summary, there are at least three escape routes from the impossibility result Theorem 6.1: Relaxing anonymity (to, say, relative anonymity), relaxing the requirement that the principal always allocate the object (Remark 5), or restricting to message spaces with some inherent asymmetry across agents (such as prohibiting self-nominations in a nomination rule),

# 7 Conclusion

We found that the principal may strictly benefit from committing to random allocations. We further identified conditions on the environment and the set of allowed mechanisms subject to which jury mechanisms are (approximately) optimal.

For future work, it is naturally interesting to extend the analysis to settings with multiple objects, allocated simultaneously or over many periods.[23] If the principal

[22]Equivalently, the allocation is unchanged if one permutes the profile in a way that does not yield self-nominations (Mackenzie, 2015, Lemma 1.1). Mackenzie uses the name *voter anonymity* instead of anonymous ballots.

[23]For recent work in this direction, see De Clippel et al. (2021) and Guo and Hörner (2021). Guo and Hörner consider a dynamic setting with a single agent, and where the principal can allocate a new unit in each period. De Clippel et al. consider a setting with multiple agents and independent types, but where the principal cannot commit. The literature following Alon et al. (2011) has also studied settings with multiple objects.

can commit to future allocations, this should lead to stronger foundations for jury mechanisms. Agents serving as jurors today can be promised a future spot as a candidate, and this may help justify excluding jurors as potential winners in the present.

The problem of finding an optimal composition of the jury is an interesting problem in itself. We envision interesting comparative statics in environments where agents who are likely to have good information are also likely to yield a high payoff to the principal. For instance, in the example from the introduction where a group chooses a persident, an agent who is popular with others may be a suitable candidate (being well-liked for their pleasant qualities) but also have good information about others (being well-acquainted with everyone).

A further important line of research could investigate optimal DIC mechanisms when the agents have intrinsic interests in the allocation to their peers. In a jury mechanism, jurors may bias their reports if the set of candidates includes family or friends. Of course, DIC has different implications when agents care about the allocation to their peers. Our results nevertheless provide insight in some cases. Firstly, if we impose axiomatically that agents be unable to influence their individual winning probabilities. Secondly, if agents have the following lexicographic preferences: Agent $i$ strictly prefers one allocation to another if the former has $i$ winning with strictly higher probability. If two allocations have the same winning probability for $i$, agent $i$ ranks them according to some type-dependent preference. This preference could capture $i$'s opinion regarding which of the others is the most deserving winner if it cannot be $i$ themself; it may even coincide with the principal's preferences. Fixing a jury of agents, the principal therefore also faces the problem of incentivizing information of the jury.

# Appendices

In Appendices A to C, respectively, we present the omitted proofs for Sections 4 to 6, respectively. Appendix D studies the model where the principal can dispose the object. Appendix E contains results that were previously mentioned in passing.

# Appendix A Random allocations and extreme points

In this part of the appendix, we prove Theorem 4.2 and Proposition 4.1.

## A.1 Proof of Proposition 4.1

*Proof of Proposition 4.1.* To keep calculations readable, it will be convient to adopt the following notation: When a DIC mechanism $\varphi$ is given, we denote

$$\begin{aligned}&\varphi_1(\theta^a) = p^{a|b}, \quad \varphi_3(\theta^c) = p^{b|c}, \quad \varphi_2(\theta^c) = p^{c|d}, \quad \varphi_3(\theta^e) = p^{d|e},\\ &\varphi_1(\theta^e) = p^{e|f}, \quad \varphi_3(\theta^f) = p^{f|g}, \quad \varphi_2(\theta^a) = p^{g|a}.\end{aligned}$$

In the mechanism $\varphi^*$, for example, all probabilities equal $\frac{1}{2}$. These probabilities do not fully describe the mechanism, but these are the only ones needed to evaluate the principal's utility. For a given value of $\rho$, we denote the principal's utility from $\varphi$ by $V_\rho(\varphi)$. Direct computation shows

$$V_\rho(\varphi) = p^{a|b} + p^{b|c} + p^{c|d} + 2p^{d|e} + p^{e|f} + p^{f|g} + p^{g|a} - \rho\left(p^{b|c} + 2p^{d|e} + p^{f|g}\right). \quad \text{(A.1)}$$

Direct computation shows $V_\rho(\varphi^*) = 4 - 2\rho$.

Most of the work shall go towards establishing the following auxiliary claim.

**Claim A.1.** *Let $\varphi$ be a DIC mechanism different from $\varphi^*$. Then $V_{\frac{1}{2}}(\varphi) \leq V_{\frac{1}{2}}(\varphi^*)$. Further, there exists $\rho_\varphi \in (0, \frac{1}{2})$ such that $\rho \in (0, \rho_\varphi)$ implies $V_\rho(\varphi) < V_\rho(\varphi^*)$.*

*Proof of Claim A.1.* Inspection of Figure 1 shows that $\varphi$ must satisfy the following system of inequalities:

$$\begin{aligned}&p^{a|b} + p^{g|a} \leq 1, \quad p^{a|b} + p^{b|c} \leq 1, \quad p^{c|d} + p^{b|c} \leq 1, \quad p^{c|d} + p^{d|e} \leq 1,\\ &p^{e|f} + p^{d|e} \leq 1, \quad p^{e|f} + p^{f|g} \leq 1, \quad p^{g|a} + p^{f|g} \leq 1.\end{aligned} \quad \text{(A.2)}$$

Turning to the first part of the claim, we have to show $V_{\frac{1}{2}}(\varphi) \leq V_{\frac{1}{2}}(\varphi^*)$. Direct computation shows $V_{\frac{1}{2}}(\varphi^*) = 3$. Using (A.2), we can bound $V_{\frac{1}{2}}(\varphi)$ as follows.

$$\begin{aligned}
V_{\frac{1}{2}}(\varphi) =& p^{a|b} + p^{b|c} + p^{c|d} + 2p^{d|e} + p^{e|f} + p^{f|g} + p^{g|a} - \frac{1}{2}\left(p^{b|c} + 2p^{d|e} + p^{f|g}\right) \\
=& p^{a|b} + \frac{p^{b|c}}{2} + p^{c|d} + p^{d|e} + p^{e|f} + \frac{p^{f|g}}{2} + p^{g|a} \\
=& \underbrace{p^{a|b} + p^{g|a}}_{\leq 1} + \underbrace{\frac{p^{b|c} + p^{c|d}}{2}}_{\leq \frac{1}{2}} + \underbrace{\frac{p^{c|d} + p^{d|e}}{2}}_{\leq \frac{1}{2}} + \underbrace{\frac{p^{d|e} + p^{e|f}}{2}}_{\leq \frac{1}{2}} + \underbrace{\frac{p^{e|f} + p^{f|g}}{2}}_{\leq \frac{1}{2}} \\
\leq& 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \\
=& 3.
\end{aligned}$$

Hence $V_{\frac{1}{2}}(\varphi) \leq V_{\frac{1}{2}}(\varphi^*)$, as promised.

Now consider the second part of the claim. We show the contrapositive: If there exists a sequence $\{\rho_k\}_{k\in\mathbb{N}}$ in $(0, \frac{1}{2})$ that converges to 0 and such that $V_{\rho_k}(\varphi) \geq V_{\rho_k}(\varphi^*)$ holds for all $k$, then $\varphi = \varphi^*$. Let $\{\rho_k\}_{k\in\mathbb{N}}$ be such a sequence. For all $\rho_k$, the system (A.2) implies the following upper bound on $V_{\rho_k}(\varphi)$:

$$\begin{aligned}
V_{\rho_k}(\varphi) =& \underbrace{p^{a|b} + p^{b|c}}_{\leq 1} + \underbrace{p^{c|d} + p^{d|e}}_{\leq 1} + \underbrace{p^{d|e} + p^{e|f}}_{\leq 1} + \underbrace{p^{f|g} + p^{g|a}}_{\leq 1} \\
& - \rho_k\left(p^{b|c} + 2p^{d|e} + p^{f|g}\right) \\
\leq& 4 - \rho_k\left(p^{b|c} + 2p^{d|e} + p^{f|g}\right).
\end{aligned} \tag{A.3}$$

Since $V_{\rho_k}(\varphi) \geq V_{\rho_k}(\varphi^*) = 4 - 2\rho_k$ and $\rho_k > 0$, we find

$$p^{b|c} + 2p^{d|e} + p^{f|g} \leq 2. \tag{A.4}$$

Further, since $V_{\rho_k}(\varphi) \geq 4 - 2\rho_k$ holds for all $k$, taking limits implies $V_0(\varphi) \geq 4$. Together with the bound in (A.3) we get $V_0(\varphi) = 4$; that is,

$$V_0(\varphi) = p^{a|b} + p^{b|c} + p^{c|d} + p^{d|e} + p^{d|e} + p^{e|f} + p^{f|g} + p^{g|a} = 4 \tag{A.5}$$

Hence (A.2) and (A.5) imply

$$p^{a|b} + p^{b|c} = p^{c|d} + p^{d|e} = p^{d|e} + p^{e|f} = p^{f|g} + p^{g|a} = 1. \tag{A.6}$$

We now bound $V_0(\varphi)$ a second time (the equality is by direct computation; the inequality follows from (A.2)):

$$V_0(\varphi) = p^{a|b} + p^{g|a} + p^{b|c} + p^{c|d} + 2p^{d|e} + p^{e|f} + p^{f|g} \leq 3 + 2p^{d|e}. \tag{A.7}$$

Hence $V_0(\varphi) = 4$ implies $p^{d|e} \geq \frac{1}{2}$. We next claim $p^{d|e} = \frac{1}{2}$. Towards a contradiction, suppose not, meaning $p^{d|e} > \frac{1}{2}$. Hence (A.6) implies $p^{c|d} = p^{e|f} < \frac{1}{2}$. Now, we also know from (A.4) and (A.5) that

$$p^{a|b} + p^{c|d} + p^{e|f} + p^{g|a} \geq 2$$

holds. However, in light of (A.2) we have $p^{a|b} + p^{g|a} \leq 1$, and hence the previous display requires $p^{c|d} + p^{e|f} \geq 1$. This contradicts $p^{c|d} = p^{e|f} < \frac{1}{2}$. Thus $p^{d|e} = \frac{1}{2}$.

Let us now return to the bound derived in (A.7). In view of $p^{d|e} = \frac{1}{2}$ and (A.2), we can infer from (A.7) that $p^{a|b} + p^{g|a} = p^{b|c} + p^{c|d} = p^{e|f} + p^{f|g} = 2p^{d|e} = 1$ holds. Together with (A.6), we find

$$p^{a|b} = 1 - p^{b|c} = p^{c|d} = 1 - p^{d|e} = p^{e|f} = 1 - p^{f|g} = p^{g|a}. \tag{A.8}$$

We already know that $p^{d|e} = \frac{1}{2}$ holds. Hence all probabilities (A.8) must equal $\frac{1}{2}$. This shows that $\varphi$ agrees with $\varphi^*$ at all profiles in $\Theta^* = \{\theta^a, \theta^b, \theta^c, \theta^d, \theta^e, \theta^f, \theta^g\}$. By inspecting $\Theta \setminus \Theta^*$, it is now easy to verify that $\varphi$ and $\varphi^*$ also agree on $\Theta \setminus \Theta^*$. $\square$

We now use Claim A.1 to show that all $\rho \in (0, \frac{1}{2})$ and all DIC mechanisms $\varphi$ different from $\varphi^*$ satisfy $V_\rho(\varphi) < V_\rho(\varphi^*)$. Fixing $\varphi$, inspection of (A.1) shows that the difference $V_\rho(\varphi) - V_\rho(\varphi^*)$ is an affine function of $\rho$; that is, there exist reals $a_\varphi$ and $b_\varphi$ such that $V_\rho(\varphi) - V_\rho(\varphi^*) = a_\varphi + b_\varphi \rho$ holds for all $\rho \in [0, \frac{1}{2}]$. Let $\rho_\varphi \in (0, \frac{1}{2})$ be as in the conclusion of Claim A.1.

If $\rho \in (0, \rho_\varphi)$, the choice of $\rho_\varphi$ implies $V_\rho(\varphi) < V_\rho(\varphi^*)$, and so we are done. Hence in what follows we assume $\rho \in [\rho_\varphi, \frac{1}{2})$. We distinguish two cases.

If $b_\varphi \leq 0$, then

$$V_\rho(\varphi) - V_\rho(\varphi^*) = a_\varphi + b_\varphi \rho \leq a_\varphi + b_\varphi \frac{\rho_\varphi}{2} = V_{\frac{\rho_\varphi}{2}}(\varphi) - V_{\frac{\rho_\varphi}{2}}(\varphi^*).$$

Now $\frac{\rho_\varphi}{2} \in (0, \rho_\varphi)$ and the choice of $\rho_\varphi$ imply $V_{\frac{\rho_\varphi}{2}}(\varphi) - V_{\frac{\rho_\varphi}{2}}(\varphi^*) < 0$, and we are done.

If $b_\varphi > 0$, then

$$V_\rho(\varphi) - V_\rho(\varphi^*) = a_\varphi + b_\varphi \rho < a_\varphi + b_\varphi \frac{1}{2} = V_{\frac{1}{2}}(\varphi) - V_{\frac{1}{2}}(\varphi^*).$$

Now Claim A.1 implies $V_{\frac{1}{2}}(\varphi) - V_{\frac{1}{2}}(\varphi^*) \leq 0$, and we are done. □

## A.2 Sufficient conditions for all extreme points to be deterministic

In this part of the appendix, we show that conditions (1) to (3) in Theorem 4.2 are each sufficient for all extreme points of the set of DIC mechanisms to be deterministic.

**Lemma A.2.** *If $n \leq 3$, then all extreme points of the set of DIC mechanisms are deterministic.*

*Proof of Lemma A.2.* If $n = 1$ or $n = 2$, it is easy to verify that all DIC mechanisms are constant. All constant mechanisms are convex combination of deterministic constant mechanisms, proving the claim. In what follows, let $n = 3$. Given an arbitrary stochastic DIC mechanism $\varphi$, we will find a non-zero function $f$ such that $\varphi + f$ and $\varphi - f$ are two other DIC mechanisms.

Before delving into the details, let us explain the basic idea. If at some type profile one agent is enjoying an interior winning probability, then, since the object is always allocated, there must be at least one other agent who also enjoys an interior winning probability at that type profile. The function $f$ represents a shift of a small probability mass between these two agents. Note that an agent's winning probability appears in the feasibility constraint of many different type profiles, and that we want $\varphi + f$ and $\varphi - f$ to be DIC. Hence we have to shift masses in a manner that is consistent with DIC across profiles. What makes this difficult is that the identities of the agents who enjoy an interior winning probability may change from one profile to the next.

In what follows, we fix a stochastic DIC mechanisms $\varphi$.

Let us agree to the following terminology. In view of DIC, we drop $i$'s type from $\varphi_i$. Given a profile $\theta$, we refer to the equation $\sum_{i \in \{1,2,3\}} \varphi_i(\theta_{-i}) = 1$ as the *feasibility constraint* at profile $\theta$. We refer to $(i, \theta_{-i})$ as the *node of agent $i$ with coordinates $\theta_{-i}$*. Lastly, when we say $\varphi_i(\theta_{-i})$ is interior we naturally mean $\varphi_i(\theta_{-i}) \in (0, 1)$.

Most of the work will go towards proving the following auxiliary claim.

**Claim A.3.** *There are non-empty disjoint subsets $R$ and $B$ ("red" and "blue") of $\cup_{i\in\{1,2,3\}}(\{i\} \times \Theta_{-i})$ such that all of the following are true:*

*(1) If $(i, \theta_{-i}) \in R \cup B$, then $\varphi_i(\theta_{-i})$ is interior.*

*(2) For all $\theta \in \Theta$, exactly one of the following is true:*

*(a) There does not exist $i \in \{1, 2, 3\}$ such that $(i, \theta_{-i}) \in R \cup B$.*

*(b) There exists exactly one $i \in \{1, 2, 3\}$ such that $(i, \theta_{-i}) \in R$, exactly one $j \in \{1, 2, 3\}$ such that $(j, \theta_{-j}) \in B$, and exactly one $k \in \{1, 2, 3\}$ such that $(k, \theta_{-k}) \notin R \cup B$.*

Before proving Claim A.3, let us use it to complete the proof of Lemma A.2. For a number $\varepsilon$ to be chosen in a moment, let $f : \Theta \to \{-\varepsilon, 0, \varepsilon\}^3$ be defined as follows:

$$\forall_{\theta\in\Theta}, \quad f_i(\theta) = \begin{cases} -\varepsilon, & \text{if } (i, \theta_{-i}) \in R, \\ \varepsilon, & \text{if } (i, \theta_{-i}) \in B, \\ 0, & \text{if } (i, \theta_{-i}) \notin R \cup B. \end{cases}$$

By finiteness of $\Theta$ and Claim A.3, if we choose $\varepsilon > 0$ sufficiently close to 0, then $\varphi + f$ and $\varphi - f$ are two DIC mechanisms. Since $f$ is non-zero, it follows that $\varphi$ is not an extreme point. It remains to prove Claim A.3.

*Proof of Claim A.3.* Given candidate sets $R$ and $B$, let us say a profile $\theta$ is *uncolored* if it falls into case (2.a) of Claim A.3. A profile *two-colored* if it falls into case (2.a) of Claim A.3. In this terminology, our goal is to construct sets $R$ and $B$ such that all $(i, \theta_{-i}) \in R \cup B$ satisfy $\varphi_i(\theta_{-i}) \in (0, 1)$, and such that all type profiles are either uncolored or two-colored.

Since $\varphi$ is stochastic, we may assume (after possibly relabelling the agents and types) that there exists a profile $\theta^0$ such that $\varphi_1(\theta_2^0, \theta_3^0)$ and $\varphi_2(\theta_1^0, \theta_3^0)$ are interior.

Let $\Theta_2^\circ$ denote the set of types $\theta_2$ for which $\varphi_1(\theta_2, \theta_3^0)$ is interior. Let $\Theta_2^\partial = \Theta_2 \setminus \Theta_2^\circ$. Similarly, let $\Theta_1^\circ$ denote the set of types $\theta_1$ such that $\varphi_2(\theta_1, \theta_3^0)$ is interior, and let $\Theta_1^\partial = \Theta_1 \setminus \Theta_1^\circ$. Notice that $\Theta_1^\circ$ and $\Theta_2^\circ$ are non-empty as, by assumption, agents 1 and 2 are enjoying interior winning probabilities at $\theta^0$.

We consider two cases.

**Case 1.** Let $\Theta_1^\partial \neq \emptyset$ and $\Theta_2^\partial \neq \emptyset$.

We establish two auxiliary claims.

**Claim A.4.** *If $\theta_1 \in \Theta_1^\partial$, then $\varphi_2(\theta_1, \theta_3^0) = 0$. Similarly, if $\theta_2 \in \Theta_2^\partial$, then $\varphi_1(\theta_2, \theta_3^0) = 0$. If $(\theta_1, \theta_2) \in (\Theta_1^\circ \times \Theta_2^\partial) \cup (\Theta_1^\partial \times \Theta_1^\circ)$, then $\varphi_3(\theta_1, \theta_2)$ is interior.*

*Proof of Claim A.4.* Consider the first part of the claim. Let $\theta_1 \in \Theta_1^\partial$. Let us find a type $\theta_2$ in $\Theta_1^\circ$; by assumption of Case 1, such a type exists. By definition, $\varphi_1(\theta_2, \theta_3^0)$ is interior. By definition of $\Theta_1^\partial$, we also know that $\varphi_2(\theta_1, \theta_3^0)$ must either equal 0 or 1. But it cannot equal 1 since $\varphi_2(\theta_1, \theta_3^0)$ and $\varphi_1(\theta_2, \theta_3^0)$ both appear in the feasibility constraint at the profile $(\theta_1, \theta_2, \theta_3^0)$, and since $\varphi_1(\theta_2, \theta_3^0)$ is interior. Thus $\varphi_2(\theta_1, \theta_3^0) = 0$, as desired.

A similar argument establishes the second claim.

As for the third claim, let $(\theta_1, \theta_2) \in \Theta_1^\circ \times \Theta_2^\partial$. The previous two paragraphs imply that at the profile $(\theta_1, \theta_2, \theta_3^0)$ the winning probability of agent 1 is 0. Moreover, by definition of $\Theta_1^\circ$, the winning probabiltiy of agent 2 is interior. Thus agent 3's winning probability at this profile must be interior, meaning $\varphi_3(\theta_1, \theta_2)$ is interior. A similar argument shows that $\varphi_3(\theta_1, \theta_2)$ is interior whenever $(\theta_1, \theta_2)$ is in $\Theta_1^\partial \times \Theta_1^\circ$. □

The second auxiliary result is:

**Claim A.5.** *Let $\theta_3 \in \Theta_3$. If $\theta_2 \in \Theta_2^\circ$, then $\varphi_1(\theta_2, \theta_3)$ is interior. Similarly, if $\theta_1 \in \Theta_1^\circ$, then $\varphi_2(\theta_1, \theta_3)$ is interior.*

*Proof of Claim A.5.* We will prove the first part of the claim, the second being similar. Thus let $\theta_2 \in \Theta_2^\circ$. By assumption of Case 1, we may find $\theta_1^\partial \in \Theta_1^\partial$ and $\theta_2^\partial \in \Theta_2^\partial$. We make two auxiliary observations.

First, consider the profile $(\theta_1^\partial, \theta_2^\partial, \theta_3^0)$. According to Claim A.4, both agent 1's and agent 2's winning probabilities at this profile equal 0. Thus $\varphi_3(\theta_1^\partial, \theta_2^\partial) = 1$. But $\varphi_3(\theta_1^\partial, \theta_2^\partial)$ and $\varphi_2(\theta_1^\partial, \theta_3)$ both appear in the feasibility constraint at the profile $(\theta_1^\partial, \theta_2^\partial, \theta_3)$. Hence $\varphi_2(\theta_1^\partial, \theta_3) = 0$.

Second, since $\theta_1^\partial \in \Theta_1^\partial$ and $\theta_2 \in \Theta_2^\circ$, we infer from Claim A.4 that $\varphi_3(\theta_1^\partial, \theta_2)$ is interior.

The previous two observations imply that at the profile $(\theta_1^\partial, \theta_2, \theta_3)$ agent 2's winning probability is 0 and that agent 3's winning probability is interior. Hence $\varphi_1(\theta_2, \theta_3)$ is interior, as promised. □

We are ready to define the sets $R$ and $B$. We assign the following colors (recall the terminology introduced in the paragraph before Claim A.3):

- red to all nodes of agent 1 with coordinates in $\Theta_2^\circ \times \Theta_3$,

- blue to all nodes of agent 3 with coordinates in $\Theta_1^\partial \times \Theta_2^\circ$,
- blue to all nodes of agent 2 with coordinates in $\Theta_1^\circ \times \Theta_3$,
- red to all nodes of agent 3 with coordinates in $\Theta_1^\circ \times \Theta_2^\partial$.

According to Claims A.4 and A.5, all of these nodes are interior. Moreover, all profiles are now either two-colored or uncolored: The profiles in $\Theta_1^\partial \times \Theta_2^\circ \times \Theta_3$ are two-colored via red nodes of agent 1 and blue nodes of agent 3; the profiles in $\Theta_1^\circ \times \Theta_2^\circ \times \Theta_3$ are two-colored via red nodes of agent 1 and blue nodes of agent 2; the profiles in $\Theta_1^\circ \times \Theta_2^\partial \times \Theta_3$ are two-colored via blue nodes of agent 2 and red nodes of 3; and the profiles in $\Theta_1^\partial \times \Theta_2^\partial \times \Theta_3$ are uncolored. ▲

**Case 2.** Suppose at least one of the sets $\Theta_1^\partial$ and $\Theta_2^\partial$ is empty. In what follows, we assume that $\Theta_2^\partial$ is empty, the other case being analogous (switch the roles of agents 1 and 2).

The assumption that $\Theta_2^\partial$ is empty means that $\varphi_1(\theta_2, \theta_3^0)$ is interior for all $\theta_2$. Let $\Theta_1^*$ be the set of types $\theta_1$ such that for all $\theta_2 \in \Theta_2$ the probability $\varphi_3(\theta_1, \theta_2)$ is interior. Notice that at this point $\Theta_1^*$ may or may not be empty; we will make a case distinction further below.

We first claim that if $\theta_1 \in (\Theta_1 \setminus \Theta_1^*)$, then $\varphi_2(\theta_1, \theta_3^0)$ is interior. Towards a contradiction, suppose this were false for some $\theta_1 \in (\Theta_1 \setminus \Theta_1^*)$. This means that we can find a type $\theta_2 \in \Theta_2$ such that $\varphi_2(\theta_1, \theta_3^0)$ and $\varphi_3(\theta_1, \theta_2)$ both fail to be interior. Recall from the previous paragraph that $\varphi_1(\theta_2, \theta_3^0)$ is interior for all $\theta_2$. Hence at the profile $(\theta_1, \theta_2, \theta_3^0)$ only agent 1 is enjoying an interior winning probability; this is impossible.

Before proceeding further, let us assign the following colors:

- red to all nodes of agent 1 with coordinates in $\Theta_2 \times \{\theta_3^0\}$. These nodes are all interior since $\Theta_2^\partial$ is empty.
- blue to all nodes of agent 2 with coordinates in $(\Theta_1 \setminus \Theta_1^*) \times \{\theta_3^0\}$. The previous paragraph implies that these nodes are all interior.
- blue to all nodes of agent 3 with coordinates in $\Theta_1^* \times \Theta_2$. These nodes are all interior by definition of $\Theta_1^*$.

Observe that all profiles in $\Theta_1 \times \Theta_2 \times \{\theta_3^0\}$ are now either two-colored or uncolored.

If $\Theta_1^*$ is empty, then the colors assigned above already define sets $R$ and $B$ with the desired properties, completing the proof. Thus suppose $\Theta_1^*$ is non-empty.

Let $\theta_3 \in \Theta_3 \setminus \{\theta_3^0\}$ be arbitrary. The fact that we have already assigned blue to the nodes of agent 3 with coordinates $\Theta_1^* \times \Theta_2$ requires us to assign some colors to the nodes of agents 1 or 2 whose 3'rd coordinate is $\theta_3$. In this step, we will not color any further nodes of agent 3. We make a case distinction.

(1) Suppose that for all $\theta_1$ in $\Theta_1^*$ the probability $\varphi_2(\theta_1, \theta_3)$ is interior. We assign red to all nodes of agent 2 with coordinates in $\Theta_1^* \times \{\theta_3\}$. This yields a coloring of the profiles in $\Theta_1 \times \Theta_2 \times \{\theta_3^0\}$ with the desired properties: The profiles in $\Theta_1^* \times \Theta_2 \times \{\theta_3\}$ are two-colored via red nodes of agent 2 and blue nodes of 3; the profiles in $(\Theta_1 \setminus \Theta_1^*) \times \Theta_2 \times \{\theta_3\}$ are uncolored.

(2) Suppose there exists $\tilde{\theta}_1 \in \Theta_1^*$ such that $\varphi_2(\theta_1, \theta_3)$ is interior. Given that $\varphi_3(\tilde{\theta}_1, \theta_2)$ is interior for all $\theta_2 \in \Theta_2$ (recall the definition of $\Theta_1^*$), it must be the case that, for all $\theta_2 \in \Theta_2$, the probability $\varphi_1(\theta_2, \theta_3)$ is interior.

We next claim that $\varphi_2(\theta_1, \theta_3)$ is interior for all $\theta_1 \in (\Theta_1 \setminus \Theta_1^*)$. Suppose this were false for some $\theta_1 \in (\Theta_1 \setminus \Theta_1^*)$. The previous paragraph tells us that $\varphi_1(\theta_2, \theta_3)$ is interior for all $\theta_2$. Thus, if $\varphi_2(\theta_1, \theta_3)$ fails to be interior, then $\varphi_3(\theta_1, \theta_2)$ would have to be interior for all $\theta_2 \in \Theta_2$; this is a contradiction since $\theta_1 \in (\Theta_1 \setminus \Theta_1^*)$.

We now assign red to all nodes of agent 1 with coordinates in $\Theta_2 \times \{\theta_3\}$, and assign blue to all nodes of agent 2 with coordinates in $(\Theta_1 \setminus \Theta_1^*) \times \{\theta_3\}$. The previous two paragraphs imply that all of these nodes are interior. Moreover the profiles in $\Theta_1^* \times \Theta_2 \times \{\theta_3\}$ are two-colored via red nodes of agent 1 and blue nodes of agent 3, and the profiles in $(\Theta_1 \setminus \Theta_1^*) \times \Theta_2 \times \{\theta_3\}$ are two-colored via red nodes of agent 1 and blue nodes of agent 2.

If we apply this case distinction separately to all $\theta_3$ in $\Theta_3 \setminus \{\theta_3^0\}$, this completes the construction of $R$ and $B$ in Case 2. ▲

Cases 1 and 2 together complete the proof of Claim A.3. □

□

**Lemma A.6.** *If for all agents $i$ we have $|\Theta_i| \leq 2$, then all extreme points of the set of DIC mechanisms are deterministic.*

*Proof of Lemma A.6.* Let us relabel types such that we have $\Theta_i \subseteq \{0, 1\}$ for all $i$. First, suppose we have $\Theta_i = \{0, 1\}$ for all $i$. It is easy to see that $\varphi$ is deterministic DIC mechanism if and only if it is a perfect matching of the graph that has node set $\{0, 1\}^n$ and where two nodes are adjacent if and only if they differ in exactly one coordinate. Since this graph is bi-partite, Theorem 11.4 of Korte and Vygen (2018) implies that all extreme points are deterministic.

The claim for the general case, where we have $\Theta_i \subseteq \{0,1\}$ for all $i$, follows from the previous paragraph by viewing a DIC mechanism on $\Theta$ as a mechanism on $\{0,1\}^n$ that ignores the reports of agents $i$ whose type spaces $\Theta_i$ are singletons. □

**Lemma A.7.** *If $|\{i \in \{1,\dots,n\}\colon |\Theta_i| \geq 2\}| \leq 2$, then all extreme points of the set of DIC mechanisms are deterministic.*

*Proof of Lemma A.7.* We may assume $n \geq 3$, as otherwise the claim follows from Lemma A.6. We will prove the claim for the case where $|\{i \in \{1,\dots,n\}\colon |\Theta_i| \geq 2\}| = 2$, the other cases being simpler. After possibly relabelling the agents, suppose we have $|\Theta_1| \geq 2$ and $|\Theta_2| \geq 2$. Let $\varphi$ be a stochastic DIC mechanism. Notice that at all profiles $\theta$ where either agent 1 or agent 2 but not both is enjoying an interior winning probability, there must be an agent in $\{3,\dots,n\}$ who is also enjoying an interior winning probability; let $i_\theta$ denote one such agent. For a number $\varepsilon > 0$ to be chosen later, consider $f\colon \Theta \to \{-\varepsilon, 0, \varepsilon\}^n$ defined for all $\theta$ as follows:

(1) If $\varphi_1(\theta) \in (0,1)$ and $\varphi_2(\theta) \in (0,1)$, let $f_1(\theta) = \varepsilon$, let $f_2(\theta) = -\varepsilon$, and let $f_i(\theta) = 0$ for all $i \notin \{1,2\}$.
(2) If $\varphi_1(\theta) \in (0,1)$ and $\varphi_2(\theta) \notin (0,1)$, let $f_1(\theta) = \varepsilon$, let $f_{i_\theta}(\theta) = -\varepsilon$, and let $f_i(\theta) = 0$ for all $i \notin \{1, i_\theta\}$.
(3) If $\varphi_1(\theta) \notin (0,1)$ and $\varphi_2(\theta) \in (0,1)$, let $f_2(\theta) = -\varepsilon$, let $f_{i_\theta}(\theta) = \varepsilon$, and let $f_i(\theta) = 0$ for all $i \notin \{2, i_\theta\}$.

Using that, for all $\theta$, agent $i_\theta$ has a singleton type space, it is easy to see that $\varphi + f$ and $\varphi - f$ are two DIC mechanisms distinct from $\varphi$ whenever $\varepsilon$ is sufficiently small. Thus $\varphi$ is not an extreme point. □

## A.3 Proof of Theorem 4.2

*Proof of Theorem 4.2.* Lemmata A.2, A.6 and A.7 imply that all extreme points are deterministic if at least one of the conditions (1) to (3) from Theorem 4.2 holds. Thus suppose conditions (1) to (3) all fail. We have to show that the set of DIC mechanisms admits a stochastic extreme point. We know from Section 4.1 that a stochastic extreme point exists in the hypothetical situation where $n = 4$ and the set of type profiles is $\hat{\Theta} = \{\ell, r\} \times \{u, d\} \times \{f, c, b\} \times \{0\}$. Since conditions (1) to (3) all fail, we can relabel the agents and types such that agents 1 to 4 have these sets as subsets of their respective sets of types. Using the stochastic extreme from

Section 4.1, it is straightforward to define a stochastic extreme point for the actual set of type profiles with $n$ agents. We omit the details. $\square$

# Appendix B Approximate optimality of jury mechanisms

In this part of the appendix, we prove Theorem 5.1. To distinguish a random variable from its realization, we denote the former using a tilde $\sim$. Given a set $N$ of agents, we denote the profile of their types by $\theta_N$, and the set of these profiles by $\Theta_N$. For example, given $i \in N$, $\omega_i \in \Omega_i$, and $\theta_{N\setminus\{i\}} \in \Theta_{N\setminus\{i\}}$, we write $\mu\left(\tilde{\omega}_i = \omega_i, \tilde{\theta}_{N\setminus\{i\}} = \theta_{N\setminus\{i\}}\right)$ to mean the probability of the event that $i$'s payoff is $\omega_i$ and the types of the other agents in $N$ are $\theta_{N\setminus\{i\}}$.

**Assumption 2.** For all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ with the following property: Denoting $N = \{1, \dots, n\}$ and $N' = \{n+1, \dots, n+m\}$, there is a function $g\colon \Theta_{N'} \times \Theta_N \to \mathbb{R}_+$ with the following two properties:

(1) For all $i \in N$, all $\omega_i \in \Omega_i$ and $\theta_{N\setminus\{i\}} \in \Theta_{N\setminus\{i\}}$ we have

$$\begin{aligned} &\mu\left(\tilde{\omega}_i = \omega_i, \tilde{\theta}_{N\setminus\{i\}} = \theta_{N\setminus\{i\}}\right) \\ &= \sum_{\theta_{N'} \in \Theta_{N'}} \sum_{\theta_i \in \Theta_i} g(\theta_{N'}, \theta_{N\setminus\{i\}}, \theta_i)\mu\left(\tilde{\omega}_i = \omega_i, \tilde{\theta}_{N'} = \theta_{N'}\right). \end{aligned} \tag{B.1}$$

(2) For all $\theta_{N'} \in \Theta_{N'}$ we have

$$\sum_{\theta_N \in \Theta_N} g(\theta_{N'}, \theta_N) = 1. \tag{B.2}$$

**Lemma B.1.** *Assumption 1 implies Assumption 2.*

*Proof of Lemma B.1.* Let $m = n$. Let $N = \{1, \dots, n\}$ and $N' = \{n+1, \dots, 2n\}$, and let $\xi\colon N \to N'$ be a bijection. It is straightforward to verify that the function $g$ defined as follows has the desired properties: For all $(\theta_N, \theta_{N'})$, let $g(\theta_N, \theta_{N'}) = 1$ if for all $i \in N$ the types of $i$ and $\xi(i)$ agree; else, let $g(\theta_N, \theta_{N'}) = 0$. $\square$

*Proof of Theorem 5.1.* The second part of the claim is immediate from the first. For the first part, let $\varphi$ be an arbitrary DIC mechanism with $n$ agents. Let $N =$

$\{1,\ldots,n\}$. For this choice of $N$, we invoke Lemma B.1 to find $m$ and $g$ as in Assumption 2. Let $N' = \{n+1,\ldots,n+m\}$. We define our candidate jury mechanism as follows: For all $i \in N$, let $\psi_i \colon \Theta_{N'} \to \mathbb{R}^n$ be defined by

$$\forall_{\theta_{N'} \in \Theta_{N'}}, \quad \psi_i(\theta_{N'}) = \sum_{\theta_N \in \Theta_N} g(\theta_{N'}, \theta_N) \varphi_i^*(\theta_{N\setminus\{i\}}).$$

For all $i \in N'$, let $\psi_i = 0$. Let $\psi = (\psi_1, \ldots, \psi_m)$.

Notice that $\psi$ only depends on the reports of agents in $N'$. Since $N'$ is disjoint from $N$, we can show that $\psi$ is a jury mechanism in the setting with $n+m$ agents by showing that $\psi$ maps to probability distributions over $N$. It is clear that $\varphi$ is non-negative (as $g$ and $\psi^*$ are non-negative). To verify that $\psi$ almost surely allocates to an agent in $N$, we observe that for all profiles $\theta_{N'}$ we have the following (the first equality is by definition of $\psi$; the second is from the fact that $\varphi^*$ is a well-defined mechanism when the set of agents is $N$; the third is from (B.2)):

$$\sum_{i\in N} \psi_i(\theta_{N'}) = \sum_{i\in N} \sum_{\theta_N \in \Theta_N} g(\theta_{N'}, \theta_N) \varphi_i^*(\theta_{N\setminus\{i\}}) = \sum_{\theta_N \in \Theta_N} g(\theta_{N'}, \theta_N) = 1,$$

as desired. We complete the proof by verifying that $\varphi$ and $\psi$ lead to the same utility of the principal. We write the utility from $\varphi$ as follows (the first equality follows from (B.1); the remaing equalities obtain by rearranging):

$$\begin{aligned}
&\sum_{i\in N} \sum_{\theta_{N\setminus\{i\}}} \sum_{\omega_i} \omega_i \mu\left(\tilde{\omega}_i = \omega_i, \tilde{\theta}_{N-i} = \theta_{N\setminus\{i\}}\right) \varphi_i^*(\theta_{N\setminus\{i\}}) \\
=&\sum_{i\in N} \sum_{\theta_{N\setminus\{i\}}} \sum_{\omega_i} \omega_i \sum_{\theta_{N'}} \sum_{\theta_i} g(\theta_{N'}, \theta_{N\setminus\{i\}}, \theta_i) \mu\left(\tilde{\omega}_i = \omega_i, \tilde{\theta}_{N'} = \theta_{N'}\right) \varphi_i^*(\theta_{N\setminus\{i\}}) \\
=&\sum_{i\in N} \sum_{\omega_i} \sum_{\theta_{N'}} \omega_i \mu\left(\tilde{\omega}_i = \omega_i, \tilde{\theta}_{N'} = \theta_{N'}\right) \sum_{\theta_{N\setminus\{i\}}} \sum_{\theta_i} g(\theta_{N'}, \theta_{N\setminus\{i\}}, \theta_i) \varphi_i^*(\theta_{N\setminus\{i\}}) \\
=&\sum_{i\in N} \sum_{\omega_i} \sum_{\theta_{N'}} \omega_i \mu\left(\tilde{\omega}_i = \omega_i, \tilde{\theta}_{N'} = \theta_{N'}\right) \psi_i(\theta_{N'}).
\end{aligned}$$

This last expression is precisely the expected utility from $\psi$. $\square$

# Appendix C Anonymous juries

In this part of the appendix we prove Theorems 6.1 and 6.2.

## C.1 Proof of Theorem 6.1

*Proof of Theorem 6.1.* Let $\varphi$ be DIC and anonymous.

The following notation is useful. Let $T$ denote the common type space. Let $T^{n-1}$ with generic element $\theta^{n-1}$ denote the $(n-1)$-fold Cartesian product of $T$. We will frequently consider profiles obtained from a profile $\theta^{n-1}$ in $T^{n-1}$ by replacing one entry of $\theta^{n-1}$. For instance, we write $(t, \theta^{n-1}_{-j})$ to denote the profile obtained by replacing the $j$'th entry of $\theta^{n-1}$ by $t$.

By DIC, for all $i$, we may drop $i$'s type from $i$'s winning probability. Thus we write $\varphi_i(\theta^{n-1})$ for $i$'s winning probability when the types of the others are $\theta^{n-1} \in T^{n-1}$. Anonymity implies that $\varphi_i(\theta^{n-1})$ is invariant to permutations of $\theta^{n-1}$.

We use the following auxiliary claim.

**Claim C.1.** *Let $i \in \{1, \dots, n\}$, $t \in T$, $t' \in T$, and $\theta^{n-1} \in T^{n-1}$. Then*

$$\sum_{j=1}^{n-1} \left(\varphi_i(t, \theta^{n-1}_{-j}) - \varphi_i(t', \theta^{n-1}_{-j})\right) = 0. \tag{C.1}$$

*Proof of Claim C.1.* Let us arbitrarily label $\theta^{n-1}$ as $(\theta_j)_{j \in N \setminus \{i\}}$. Let us also fix an arbitrary type $\theta_i \in T$.

In an intermediate step, let $j$ be distinct from $i$. For clarity, we spell out winning probabilities as follows: $\varphi_i(r_i = t, r_j = t', r_{-ij} = \theta_{-ij})$ means $i$'s winning probability when $i$ reports $t$, $j$ reports $t'$, and all remaining agents report $\theta_{-ij}$. A permutation of $i$'s and $j$'s reports does not change the winning probabilities of the agents other than $i$ and $j$. Since the object is allocated with probability one, we have

$$\begin{aligned}&\varphi_i(r_i = t, r_j = t', r_{-ij} = \theta_{-ij}) + \varphi_j(r_i = t, r_j = t', r_{-ij} = \theta_{-ij})\\ =&\varphi_i(r_i = t', r_j = t, r_{-ij} = \theta_{-ij}) + \varphi_j(r_i = t', r_j = t, r_{-ij} = \theta_{-ij}).\end{aligned}$$

By rearranging the previous display, and by DIC, we obtain

$$\begin{aligned}&\varphi_i(r_i = t, r_j = t', r_{-ij} = \theta_{-ij}) - \varphi_i(r_i = t', r_j = t, r_{-ij} = \theta_{-ij})\\ =&\varphi_j(r_i = t', r_j = \theta_j, r_{-ij} = \theta_{-ij}) - \varphi_j(r_i = t, r_j = \theta_j, r_{-ij} = \theta_{-ij}).\end{aligned} \tag{C.2}$$

Now consider summing (C.2) over all $j \in \{1, \ldots, n\} \setminus \{i\}$. This summation yields

$$\sum_{j\colon j\neq i} (\varphi_i(r_i = t, r_j = t', r_{-ij} = \theta_{-ij}) - \varphi_i(r_i = t', r_j = t, r_{-ij} = \theta_{-ij})) \tag{C.3}$$

$$= \sum_{j\colon j\neq i} (\varphi_j(r_i = t', r_j = \theta_j, r_{-ij} = \theta_{-ij}) - \varphi_j(r_i = t, r_j = \theta_j, r_{-ij} = \theta_{-ij}))\,. \tag{C.4}$$

In (C.4), the profiles considered are all of the form $(r_i = t', r_{-i} = \theta_{-i})$ and $(r_i = t, r_{-i} = \theta_{-i})$, respectively. Note that by DIC we have $\varphi_i(r_i = t', r_{-i} = \theta_{-i}) - \varphi_i(r_i = t, r_{-i} = \theta_{-i}) = 0$. Hence (C.4) equals

$$\sum_{j=1}^{n} (\varphi_j(r_i = t', r_{-i} = \theta_{-i}) - \varphi_j(r_i = t, r_{-i} = \theta_{-i}))\,.$$

Since the object is always allocated, the term in the previous display equals 0. Hence the sum in (C.3) equals

$$\sum_{j\colon j\neq i} (\varphi_i(r_i = \theta_i, r_j = t', r_{-ij} = \theta_{-ij}) - \varphi_i(r_i = \theta_i, r_j = t, r_{-ij} = \theta_{-ij})) = 0.$$

We now revert to our usual notation. By DIC, we may drop $i$'s report from $\varphi_i$. Since $\varphi_i$ is permutation-invariant with respect to $N \setminus \{i\}$, we may also write

$$\begin{aligned}\varphi_i(r_i = \theta_i, r_j = t', r_{-ij} = \theta_{-ij}) &= \varphi_i(t', \theta^{n-1}_{-j}) \qquad \text{and}\\ \varphi_i(r_i = \theta_i, r_j = t, r_{-ij} = \theta_{-ij}) &= \varphi_i(t, \theta^{n-1}_{-j}).\end{aligned}$$

Thus we obtain the desired equality $\sum_{j=1}^{n-1} \left(\varphi_i(t', \theta^{n-1}_{-j}) - \varphi_i(t, \theta^{n-1}_{-j})\right) = 0.$ □

In what follows, let $i$ be an arbitrary agent. We show $i$'s winning probability is constant in the reports of others. To that end, let us fix an arbitrary type $t^* \in T$. For all $k \in \{0, \ldots, n-1\}$, let $T^{n-1}_k$ denote the subset of profiles in $T^{n-1}$ where exactly $k$-many entries are distinct from $t^*$. Let $p_i$ denote $i$'s winning probability when all other agents report $t^*$. We will show via induction over $k$ that $i$'s winning probability is equal to $p_i$ whenever the others report a profile in $T_k^{n-1}$. This completes the proof since $T^{n-1} = \cup_{k=0}^{n-1} T_k^{n-1}$ holds.

*Base case* $k = 0$. The base case $k = 0$ is immediate from the definitions of $p_i$ and $T_0$.

*Induction step.* Let $k \geq 1$. Let all $\hat{\theta}^{n-1} \in \cup_{\ell=0}^{k-1} T_\ell^{n-1}$ satisfy $\varphi_i(\hat{\theta}^{n-1}) = p_i$. Letting $\theta^{n-1} \in T_k^{n-1}$ be arbitrary, we show $\varphi_i(\theta^{n-1}) = p_i$.

By anonymity, we may assume that exactly the first $k$ entries of $\theta^{n-1}$ are distinct from $t^*$. That is, there exist types $t_1, \ldots, t_k$ all distinct from $t^*$ such that $\theta^{n-1} = (t_1, \ldots, t_k, t^*, \ldots, t^*)$.

Let $\tilde{\theta}^{n-1} = (t_1, \ldots, t_{k-1}, t^*, \ldots, t^*)$. This profile is obtained from $\theta^{n-1}$ by replacing $t_k$ by $t^*$. We now invoke Claim C.1 to infer

$$\sum_{j=1}^{n-1} \varphi_i(t_k, \tilde{\theta}_{-j}^{n-1}) = \sum_{j=1}^{n-1} \varphi_i(t^*, \tilde{\theta}_{-j}^{n-1}). \tag{C.5}$$

Consider the profiles appearing in the sum on the left of (C.5) as $j$ varies from 1 to $n-1$.

(1) Let $j \leq k-1$. Since exactly the first $k-1$ entries of $\tilde{\theta}$ are distinct from $t^*$, it follows that $(t_k, \tilde{\theta}_{-j}^{n-1})$ is another profile where exactly $k-1$ entries differ from $t^*$. Hence the induction hypothesis implies $\varphi_i(t_k, \tilde{\theta}_{-j}^{n-1}) = p_i$.

(2) Let $j > k-1$. In the profile $(t_k, \tilde{\theta}_{-j}^{n-1})$, the first $k-1$ entries are $t_1, \ldots, t_{k-1}$, the $j$'th entry is $t_k$, and all remaining entries are $t^*$. Hence $(t_k, \tilde{\theta}_{-j}^{n-1})$ is a permutation of $\theta^{n-1}$. Anonymity implies $\varphi_i(t_k, \tilde{\theta}_{-j}^{n-1}) = \varphi_i(\theta^{n-1})$.

Hence the sum on the left of (C.5) equals $\sum_{j=1}^{n-1} \varphi_i(t, \tilde{\theta}_{-j}^{n-1}) = (k-1)p_i + (n-k)\varphi_i(\theta^{n-1})$

Now consider the sum on the right of (C.5). For all $j$, a moment's thought reveals that the profile $(t^*, \tilde{\theta}_{-j}^{n-1})$ contains at most $(k-1)$-many entries different from $t^*$. By the induction hypothesis, therefore, the sum on the right of (C.5) equals $(n-1)p_i$.

The previous two paragraphs and (C.5) imply $(k-1)p_i + (n-k)\varphi_i(\theta^{n-1}) = (n-1)p_i$. Equivalently, $(n-k)(\varphi_i(\theta^{n-1}) - p_i) = 0$. Since $k \leq n-1$, we find $\varphi_i(\theta^{n-1}) = p_i$, as promised. $\square$

## C.2 Proof of Theorem 6.2

*Proof of Theorem 6.2.* We omit the straightforward verification that a jury mechanism with an anonymous jury is relatively anonymous.

For the converse, let $\varphi$ be deterministic, relatively anonymous, and DIC. Let $N$ denote the set of agents, and let $T$ denote the common type space. For this proof, we write $\varphi(\theta)$ to mean the agent who wins at profile $\theta$; this makes sense since $\varphi$ is deterministic.

Let $I_i$ denote the set of agents that influence agent $i$'s winning probability. For all $j \in N$, let $A_j = \{i \in N : j \in I_i\}$ be the set of agents that are influenced by $j$. Let $I = \{i \in N : A_i \neq \emptyset\}$. We may assume that $\varphi$ is non-constant, meaning $I \neq \emptyset$, as otherwise the proof is trivial.

Given two agents $i$ and $j$, let $D_{i-j} = A_i \setminus A_j$, and $D_{j-i} = A_j \setminus A_i$, and $C_{ij} = A_j \cap A_i$, and $N_{ij} = N \setminus (A_i \cup A_j)$. Note that, by DIC, the set $C_{ij}$ contains neither $i$ nor $j$. Hence relative anonymity implies that for all $k \in C_{ij}$ the winning probability of $k$ is invariant with respect to permutations of $i$ and $j$.

When $i$, $j$, and $k$ are given, we write $(t, t', t'', \theta_{-ijk})$ to mean the profile where $i$, $j$, and $k$, respectively, report $t$, $t'$, and $t''$, respectively, and all others report $\theta_{-ijk}$.

**Claim C.2.** *Let $i$ and $j$ be distinct. Let $\theta_{-ij} \in \Theta_{-ij}$. If there exists $\theta_i, \theta_j \in T$ such that $\varphi(\theta_i, \theta_j, \theta_{-ij}) \in D_{i-j}$, then all $\theta'_i, \theta'_j \in T$ satisfy $\varphi(\theta'_i, \theta'_j, \theta_{-ij}) \in D_{i-j}$.*

*Proof of Claim C.2.* We drop the fixed type profile $\theta_{-ij}$ of the others from the notation. To show $\varphi(\theta'_i, \theta'_j) \in D_{i-j}$, it suffices to show $\varphi(\theta'_i, \theta_j) \in D_{i-j}$ since if the latter is true then definition of $D_{i-j}$ implies $\varphi(\theta'_i, \theta'_j) = \varphi(\theta'_i, \theta_j)$.

We first claim $\varphi(\theta_j, \theta_i) \in D_{i-j}$. If $\varphi(\theta_j, \theta_i) \in N_{ij}$, then $\varphi(\theta_j, \theta_i) = \varphi(\theta_i, \theta_j)$, and we have a contradiction to $\varphi(\theta_i, \theta_j) \in D_{i-j}$. If $\varphi(\theta_j, \theta_i) \in C_{ij}$, then relative anonymity implies $\varphi(\theta_i, \theta_j) \in C_{ij}$, and we have another contradiction to $\varphi(\theta_i, \theta_j) \in D_{i-j}$. If $\varphi(\theta_j, \theta_i) \in D_{j-i}$, then $\varphi(\theta_j, \theta_i) = \varphi(\theta_i, \theta_i) \in D_{j-i}$. However, from $\varphi(\theta_i, \theta_j) \in D_{i-j}$ we know $\varphi(\theta_i, \theta_j) = \varphi(\theta_i, \theta_i) \in D_{i-j}$; contradiction. Thus $\varphi(\theta_j, \theta_i) \in D_{i-j}$.

We next claim $\varphi(\theta'_i, \theta_j) \in (D_{i-j} \cup C_{ij})$. Towards a contradiction, suppose not. Then $\varphi(\theta'_i, \theta_j) \in (D_{j-i} \cup N_{ij})$, and hence $\varphi(\theta'_i, \theta_j) = \varphi(\theta_i, \theta_j) \notin D_{i-j}$. This contradicts the assumption $\varphi(\theta_i, \theta_j) \in D_{i-j}$.

In view of the previous paragraph, we can complete the proof by showing $\varphi(\theta'_i, \theta_j) \notin C_{ij}$. Towards a contradiction, let $\varphi(\theta'_i, \theta_j) \in C_{ij}$. Relative anonymity implies $\varphi(\theta_j, \theta'_i) \in$

$C_{ij}$. We have shown earlier that $\varphi(\theta_j, \theta_i) \in D_{i-j}$ holds. Hence $\varphi(\theta_j, \theta'_i) \in D_{i-j}$, and this contradicts $\varphi(\theta_j, \theta'_i) \in C_{ij}$. Thus $\varphi(\theta'_i, \theta_j) \notin C_{ij}$ and the proof is complete. $\square$

**Claim C.3.** *Let $i, j, k$ be distinct. Let $\theta_k \in T$ and $\theta_{-ijk} \in \Theta_{-ijk}$ be such that all $\theta'_i, \theta'_j \in T$ satisfy $\varphi(\theta'_i, \theta'_j, \theta_k, \theta_{-ijk}) \in (C_{ij} \cup N_{ij})$. Then, all $\theta'_i, \theta'_j, \theta'_k \in T$ satisfy $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in (C_{ij} \cup N_{ij})$.*

*Proof of Claim C.3.* Towards a contradiction, suppose $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in (D_{i-j} \cup D_{j-i})$. Suppose $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in D_{i-j}$, the other case being similar. The inclusions $\varphi(\theta'_i, \theta'_j, \theta_k, \theta_{-ijk}) \in (C_{ij} \cup N_{ij})$ and $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in D_{i-j}$ together imply $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in A_k$. Hence $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in D_{k-j}$. We now invoke Claim C.2 to infer $\varphi(\theta'_i, \theta'_j, \theta_k, \theta_{-ijk}) \in D_{k-j}$. Since we also have $\varphi(\theta'_i, \theta'_j, \theta_k, \theta_{-ijk}) \in (C_{ij} \cup N_{ij})$, we infer $\varphi(\theta'_i, \theta'_j, \theta_k, \theta_{-ijk}) \in N_{ij}$. In particular, we have $\varphi(\theta'_i, \theta'_j, \theta_k, \theta_{-ijk}) \notin A_i$. Hence $\varphi(\theta'_i, \theta'_j, \theta_k, \theta_{-ijk}) \in D_{k-i}$. We now invoke Claim C.2 to infer $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in D_{k-i}$. In particular, we have $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \notin A_i$. This contradicts the assumption $\varphi(\theta'_i, \theta'_j, \theta'_k, \theta_{-ijk}) \in D_{i-j}$. $\square$

**Claim C.4.** *If $C_{ij} \neq \emptyset$, then $D_{i-j} \cup D_{j-i} = \emptyset$.*

*Proof of Claim C.4.* Let $k \in C_{ij}$. We may find a profile $\theta$ such that $\varphi(\theta) = k$ as else $k$'s winning probability is constantly 0 (which would contradict $k \in C_{ij}$). Denoting by $\theta_{-ij}$ the types of agents other than $i$ and $j$ at $\theta$, we appeal to Claim C.2 to infer that all $\theta'_i, \theta'_j \in T$ satisfy $\varphi(\theta'_i, \theta'_j, \theta_{-ij}) \in (C_{ij} \cup N_{ij})$. Repeatedly applying Claim C.3 implies that all profiles $\theta'$ satisfy $\varphi(\theta') \in (C_{ij} \cup N_{ij})$. It follows that all agents in $D_{i-j} \cup D_{j-i}$ enjoy a winning probability that is constantly equal to 0. Recalling the definitions $D_{i-j} = A_i \setminus A_j$, and $D_{j-i} = A_j \setminus A_i$, it follows that $D_{i-j} \cup D_{j-i}$ is empty. $\square$

Recall the definition $I = \{i \in N \colon A_i \neq \emptyset\}$. Consider the binary relation $\sim$ on $I$ defined as follows: Given $i$ and $j$ in $I$, we let $i \sim j$ if and only if $C_{ij} \neq \emptyset$.

**Claim C.5.** *The relation $\sim$ is an equivalence relation. For all $i, j \in I$, if $i \sim j$, then $i \notin A_j$ and $A_i = A_j$.*

*Proof of Claim C.5.* It is clear that $\sim$ is symmetric. As for reflexivity, note that $i \in I$ implies $A_i = C_{ii} \neq \emptyset$. Turning to transitivity, suppose $i \sim j$ and $j \sim k$. Hence $C_{ij} \neq \emptyset$ and $C_{jk} \neq \emptyset$. Let $\ell \in C_{jk}$. Claim C.4 and $C_{ij} \neq \emptyset$ together imply $D_{j-i} = \emptyset$. Hence $\ell \in C_{jk}$ implies $\ell \in C_{ij}$. Hence $\ell \in C_{jk} \cap C_{ij}$, implying $\ell \in C_{ik}$. Hence $i \sim k$.

As for the second part of the claim, let $i \sim j$. Thus $C_{ij} \neq \emptyset$. Claim C.4 implies $D_{j-i} = D_{i-j} = \emptyset$. This immediately implies $A_i = A_j$. Together with DIC, we also infer $i \notin A_j$. $\square$

Claim C.5 implies that we may partition $I$ into finitely-many non-empty $\sim$-equivalence classes. (Recall that $I$ is non-empty.) We now claim that there is exactly one $\sim$-equivalence class. Towards a contradiction, suppose not. In view of Claim C.5, this means that there are distinct $i$ and $j$ such that $A_i \cap A_j = \emptyset$ and $A_i \neq \emptyset \neq A_j$. Let $J_i$ and $J_j$, respectively, denote the equivalence classes containing $i$ and $j$, respectively. Let $k \in A_i$ and $\ell \in A_j$. Claim C.5 implies $k \notin J_i$ and $\ell \notin J_j$ and $k \neq \ell$. Since $k \in A_i$ and $\varphi$ is deterministic, there is a type profile $\theta$ such that $\varphi(\theta) = k$; there must be another type profile $\theta'$ such that $\varphi(\theta') = \ell$. However, the definition of equivalence classes implies that $k$'s winning probability depends only on the types of agents in $J_i$, and that $\ell$'s winning probability depends only on the types of agents in $J_j$. Hence there is a type profile where both $k$ and $\ell$ are winning with probability 1 (such a type profile is obtained by changing at the profile $\theta$ the types of agents in $J_j$ to their respective types at $\theta'$, and keeping all other types fixed). Contradiction.

Now, Claim C.5 implies that the members of the unique $\sim$-equivalence class do not influence one another, and that they influence the same set of others. By relative anonymity, it follows $\varphi$ that is a deterministic jury mechanism with an anonymous jury. $\square$

# Appendix D Supplementary material: Disposal

In this part of the appendix, we relax the principal's problem: Instead of allocating, the principal can dispose the object. An alternative interpretation is that the principal privately consumes it. We discuss how this affects our results from the main text (Appendix D.1). Further, we show how the existence of stochastic extreme points of the set of DIC mechanisms with disposal can be related to stable set polytope of a certain graph (Appendix D.3).

Beginning with the definitions, a *mechanism with disposal* is a function $\varphi \colon \Theta \to$

$[0,1]^n$ satisfying

$$\forall_{\theta\in\Theta}, \quad \sum_{i=1}^{n} \varphi_i(\theta) \leq 1.$$

A mechanism from the main text will be referred to as a mechanism with no disposal. If there is no risk of confusion, we will drop the qualifier "with disposal" or "with no disposal". Of course, a mechanism with no disposal is also a mechanism with disposal.

A mechanism with disposal is DIC if and only if for arbitrary $i$ the winning probability $\varphi_i$ is constant in $i$'s report. We will sometimes drop $i$'s report $\theta_i$ from $\varphi_i(\theta_i, \theta_{-i})$.

A jury mechanism with disposal is defined as in the basic model: For all $i$, if agent $i$ influences the allocation, then $i$ never wins the object.

We normalize the principal's payoff from not allocating the object to 0.

A mechanism with $n$ agents and disposal can be viewed as a mechanism with no disposal and with $n+1$ agents where agent $n+1$ has a singleton type space; the principal's payoff from allocating to $n+1$ is always 0. Likewise, if there are other agents with singleton type spaces, we can always renormalize payoffs and view allocating to one of these agents as disposing the object. In what follows, whenever considering mechanisms with disposal, let us thus simplify by assuming that no agent has a singleton type space; that is, for all agents $i$ we have $|\Theta_i| \geq 2$.

## D.1 Results from the main text

Here we discuss how our results change when the principal can dispose the object.

To begin with, we have the following analogue of Theorem 4.2.

**Theorem D.1.** *Fix $n$ and $\Theta_1, \dots, \Theta_n$. For all agents $i$, let $|\Theta_i| \geq 2$. All extreme points of the set of DIC mechanisms with disposal are deterministic if and only if at least one of the following is true:*

*(1) We have $n \leq 2$.*

*(2) For all agents $i$ we have $|\Theta_i| = 2$.*

*Proof of Theorem D.1.* As discussed above, a DIC mechanism with $n$ agents and disposal is a DIC mechanism with $n+1$ agents and no disposal. The claim follows from Theorem 4.2. □

Further below, we provide an alternative proof of Theorem D.1 that does not invoke Theorem 4.2 but relies on graph-theoretic results. We emphasize that Theorem D.1 does not imply Theorem 4.2. Namely, we cannot conclude from Theorem D.1 that if $n = 3$ all extreme points of the set of DIC mechanisms with no disposal are deterministic.

Proposition 4.1 (on the suboptimality of deterministic DIC mechanisms) analogizes straightforwardly to mechanisms with disposal. Indeed, note that in our proof of Proposition 4.1 agent 4 was simply a dummy agent with payoff normalized to 0.

Corollary 4.3 (jury mechanisms with 3 agents) carries over to mechanisms with disposal in the sense that all mechanisms with disposal and 2 agents are convex combinations of deterministic jury mechanisms with disposal. Note that, according to Theorem D.1, this result does not extend to $n = 3$. With $n = 2$, a jury mechanism with disposal admits a single juror whose report determines whether or not the object is disposed or allocated to the other agent.

Theorem 5.1 (approximate optimality of jury mechanisms under Assumption 1 and large $n$) extends to mechanisms with disposal in a straightforward way, with no changes to the proof.

We now turn to the results from Section 6. In what follows, we tacitly assume the agents have a common type space, denoted $T$. The set of type profiles is understood to be $\Theta = \times_{i=1}^{n} T$.

We first show that Theorem 6.1 does not extend to mechanisms with disposal, even under the following stronger notion of anonymity. For convenience, we also repeat the definition of anonymity from the main text.

**Definition 5.** Let $\varphi$ be a mechanism with diposal.

(1) The mechanism is is **strongly anonymous** if all agents $i$, all type profiles $\theta$, and all permutations $\xi$ of $\{1, \ldots, n\}$ satisfy $\varphi_i(\theta) = \varphi_{\xi(i)}(\xi(\theta))$.

(2) Given $i$, $j$, and $k$ that are all distinct, agent $i$ and $j$ are **exchangeable for** $k$ if $\varphi_k$ is invariant with respect to all permutations of $i$'s and $j$'s reports.

(3) The mechanism is **anonymous** if for all $i$, $j$, and $k$ that are all distinct, agents $i$ and $j$ are exchangeable for $k$.

The interpretation of strong anonymity is as follows: If a profile $\theta'$ is obtained from another profile $\theta$ by some permutation of the type profile, then the same permutation is applied to the agents' winning probabilities (and hence also to the probability of disposing the object). This strengthens anonymity since anonymity considers situations where, fixing one agent $i$, we only permute the types of agents other than $i$.

The following result shows that Theorem 6.1 does not extend to mechanisms with disposal. We later use this result to show that Theorem 6.2 does not extend to stochastic relatively anonymous DIC mechanisms without disposal.

**Proposition D.2.** *Let $n = 3$ and let the common type space be $T = \{1, 2, 3, 4, 5, 6, 7\}$. There is a mechanism with disposal that is non-constant, DIC, stochastic, strongly anonymous, and an extreme point of the set of* all *DIC mechanisms with disposal.*

The proof follows below.

We next turn to the question of whether Theorem 6.2 extends to mechanisms with disposal. We provide an affirmative answer under the following strengthening of relative anonymity. Given a mechanism, let $\varphi_0 = 1 - \sum_{i=1}^{n} \varphi_i$ denote the probability that the object is not allocated.

**Definition 6.** Let $\varphi$ be a mechanism with disposal. Let $N = \{1, \ldots, n\}$ and $N_0 = N \cup \{0\}$.

(1) Given distinct $i \in N$ and $k \in N_0$, agent $i$ **influences** $k$ if $\varphi_k$ is non-constant in $i$'s report.
(2) The mechanism is **relatively $*$-anonymous** if for all $i \in N$, $j \in N$, and $k \in N_0$ that are all distinct and are such that $i$ and $j$ influence $k$, agents $i$ and $j$ are exchangeable for $k$.

In words, relative anonymity is strengthened by demanding that the disposal probability $\varphi_0$ is permutation-invariant with respect to those agents who influence $\varphi_0$.

It follows from Theorem 6.2 that a deterministic relatively $*$-anonymous DIC mechanism with disposal is a deterministic jury mechanism with an anonymous jury. To see this, let us view disposing the object as allocating to agent 0. Now, agent 0 does not have the same type space as the other agents. Since it was a maintained assumption of Section 6, we cannot yet appeal to Theorem 6.2. But, we can simply view the mechanism as a mechanism where agent 0's type space is also $T$ and 0's report is always ignored. By now appealing to Theorem 6.2, the claim follows.

## D.2 Proof of Proposition D.2

*Proof of Proposition D.2.* Let $T^3 = \times_{i=1}^3 T$ denote the 3-fold Cartesian product of the common type space. Let us define $T_1 = \{1,2\}$, $T_2 = \{3,4\}$ and $T_3 = \{5,6,7\}$ and $\hat{\Theta} = T_1 \times T_2 \times T_3$. In Section 4.1, we constructed a stochastic DIC mechanism $\varphi^*$ without disposal in a setting with 4 agents, where the types of agents 1, 2, and 3, respectively, are $\{\ell, r\}$, $\{u, d\}$, $\{f, c, b\}$, respectively, and where agent 4's type is degenerate. By relabelling types, we can view $\varphi^*$ as being a mechanism with disposal with 3 agents on the set of type profiles $\hat{\Theta}$, and where allocating to agent 4 is identified with disposing the object. The arguments from Section 4.1 show that, if $n = 3$ and the set of type profiles is $\hat{\Theta}$, then $\varphi^*$ is an extreme point of the set of DIC mechanisms with disposal.

For later reference, we note that, at all type profiles $\theta \in \hat{\Theta}$ and all $i \in \{1,2,3\}$, agent $i$'s winning probability at $\theta$ is either 0 or 1/2.

We now use $\varphi^*$ to define a mechanism as in the claim.

Our candidate mechanism will be denoted $\psi^*$. Let $\Xi$ denote the set of permutations of $\{1,2,3\}$. Let $\Theta^* = \{\xi(\theta) \colon \theta \in \hat{\Theta}, \xi \in \Xi\}$ denote the set of type profiles obtained by permuting a type profile in $\hat{\Theta}$; see Figure 2. We return to this figure later. Note that, fixing an arbitrary type profile in $\hat{\Theta}$, the types of the agents at this type profile are all distinct. Consequently, for all $\theta^*$ in $\Theta^*$ there is a unique profile $\theta$ in $\hat{\Theta}$ and $\xi$ in $\Xi$ such that $\theta^* = \xi(\theta)$.

For later reference, we also note the following: At an arbitrary type profile in $\Theta^*$, the types of distinct agents must belong to distinct elements of the partition $\{T_1, T_2, T_3\}$.

We now define $\psi^*$ as follows: For all $\theta^*$ in $\Theta^*$, we find the unique $(\theta, \xi) \in T \times \Xi$ such that $\theta^* = \xi(\theta)$, and then let

$$(\psi_i^*(\theta^*))_{i=1}^n = \left(\varphi^*_{\xi(i)}(\xi(\theta))\right)_{i=1}^n . \tag{D.1}$$

In words, if $\theta^*$ is obtained from, say, $\theta$ by permuting the entries of agents 1 and 2, then $\psi^*(\theta^*)$ swaps the winning probabilities of agents 1 and 2 while leaving agent 3's winning probability unchanged. For the remaining profiles, we proceed as follows: For all agents $i$ and profiles $\theta$, if $\theta$ differs from at least one profile $\theta^*$ in $\Theta^*$ in agent $i$'s type and no other agent's type, then $i$'s winning probability at $\theta$ is set equal to $i$'s winning probability at $\theta^*$ (which makes sense since the latter probability has already

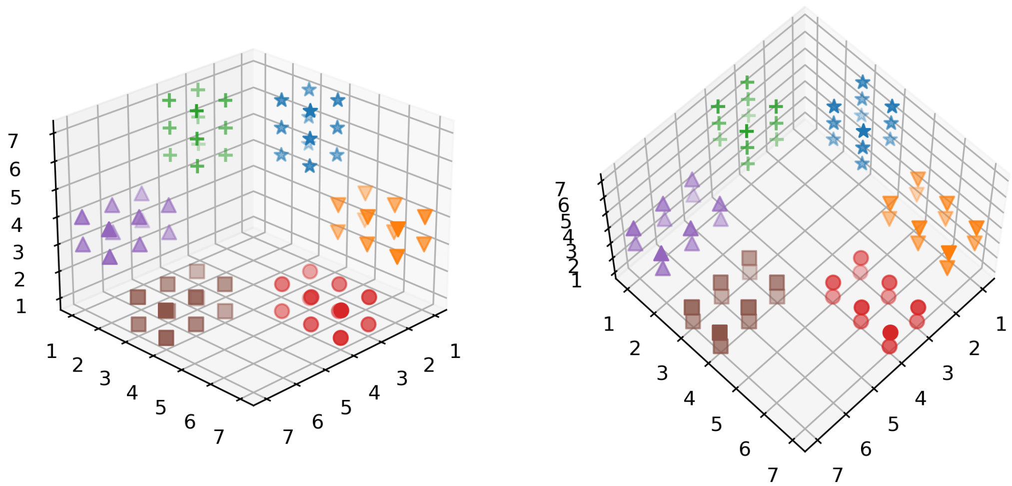


Figure 2: The set $\Theta^*$ viewed from two different angles. Each agent is associated with a distinct axes, and we have dropped the labels. Each symbol (square, circle, upward-pointing triangle, etc.) identifies a particular permutation of $\{1, 2, 3\}$. For instance, the upward-pointing triangles are obtained from the downward-pointing triangles by permuting the two agents on the horizontal axes.

been defined in (D.1)); else, if no such profile $\theta^*$ in $\Theta^*$ exists, then agent $i$'s winning probability is set equal to 0.

To complete the argument, we have to show that $\psi^*$ is a (1) well-defined mechanism, and (2) that it is DIC, stochastic, strongly anonymous, and an extreme point of the set of DIC mechanisms with disposal. Assuming for a moment that (1) is true, it is clear that the mechanism is stochastic, and one can easily verify from the definition that it is DIC and strongly anonymous. Moreover, to show that it is an extreme point of the set of DIC mechansims, we can proceed essentially via the arguments from Section 4.1. Indeed, we know from Section 4.1 that all DIC mechanisms $\psi$ with disposal that appear in a candidate convex combination must agree with $\psi^*$ on $\hat{\Theta}$, and hence on $\Theta^*$; it is then straightforward to verify that such a mechanism $\psi$ must also agree with $\psi^*$ on $\Theta \setminus \Theta^*$.

We now turn to (1). To show that $\psi^*$ is a well-defined mechanism, we have to show that the winning probabilities of the agents do not sum to a number strictly above 1. Before delving into the details, consider Figure 2. The different symbols (squares, circles, upward-pointing triangles, etc.) partition $\Theta^*$ into 6 subsets (one for each permutation of $\{1, 2, 3\}$). For each of these subsets, imagine rays emanating from the subset and travelling parallel to the axes. These rays identify type profiles along which exactly one agent's is changing. Now consider the intersection of such rays originating from subsets with different symbols. The geometry of $\Theta^*$ implies that *at most* two such rays intersect simultaneously. This is one critical observation that we will use to argue that $\psi^*$ is well-defined.

The second is that, as noted earlier, at all type profiles $\theta \in \Theta^*$ and all $i \in \{1, 2, 3\}$ agent $i$'s winning probability under $\varphi^*$ at $\theta$ is either 0 or 1/2. The mechanism $\psi^*$ inherits this property.

Towards a contradiction, suppose there is a profile $\theta = (\theta_1, \theta_2, \theta_3)$ in $\Theta$ where the winning probabilities under $\psi^*$ sum to a number strictly above 1. By the previous paragraph, all three agents are therefore enjoying non-zero winning probabilities. By definition of $\psi^*$, we can infer the following: Since agent 1's winning probability is non-zero, there exists $t_1$ such that $(t_1, \theta_2, \theta_3) \in \Theta^*$. Similarly, there are $t_2$ and $t_3$ such that $(\theta_1, t_2, \theta_3) \in \Theta^*$ and $(\theta_1, \theta_2, t_3) \in \Theta^*$. Recall that $\{T_1, T_2, T_3\}$ is a partition of the common type space. Hence, for all agents $i$, there is a unique interger $\xi(i)$ in $\{1, 2, 3\}$ such that $\theta_i \in T_{\xi(i)}$. Let us now recall the follwing from the definition of $\Theta^*$: If a profile is in $\Theta^*$, then the types of distinct agents must belong two distinct elements of the partition $\{T_1, T_2, T_3\}$. Hence, we can infer from $(t_1, \theta_2, \theta_3) \in \Theta^*$ that $\xi(2) \neq \xi(3)$ holds. Similarly, from $(\theta_1, t_2, \theta_3) \in \Theta^*$ and $(\theta_1, \theta_2, t_3) \in \Theta^*$ we infer $\xi(1) \neq \xi(2)$ and $\xi(1) \neq \xi(3)$. Taken together, we infer that $\theta$ must itself be in $\Theta^*$. Hence the vector of winning probabilities at $\theta$ is a permutation of the vector of winning probabilities at a profile $\theta'$ in $\hat{\Theta}$. At the profile $\theta'$, the winning probabilites under $\psi^*$ agree with $\varphi^*$. Thus there is a profile where the winning probabilities under $\varphi^*$ sum to a number strictly greater than 1; this is a contradiction since $\varphi^*$ is a well-defined mechanism. $\square$

## D.3 Stochastic extreme points and perfect graphs

In this section, we relate the existence of stochastic extreme points with disposal to a graph-theoretic property called perfection.

### D.3.1 Preliminaries

We first introduce several definitions for a general graph $G$ with nodes $V$ and edges $E$. All graphs are understood to be simple and undirected.

An *induced cycle of length* $k$ is a subset $\{v_1, \ldots, v_k\}$ of $V$ such that, denoting $v_{k+1} = v_1$, two nodes $v_\ell$ and $v_{\ell'}$ in the subset are adjacent if and only if $|\ell - \ell'| = 1$.

The *line graph* of $G$ is the graph that has as node set the edge set of $G$; two nodes of the line graph are adjacent if and only if the two associated edges of $G$ share a node in $G$.

A *clique* of $G$ is a set of nodes such that every pair in the set are adjacent. A clique is *maximal* if it is not a strict subset of another clique. A *stable set* of $G$ is a subset of nodes of which no two are adjacent. The *incidence vector* of a subset of nodes $\hat{V}$ is the function $x\colon V \to \{0,1\}$ that equals one on $\hat{V}$ and equals zero otherwise. Let $S(G)$ denote the set of incidence vectors belonging to some stable set of $G$.

The upcoming result uses another definition called *perfection*. For our purposes, it will be enough to know the following.

**Lemma D.3.** *All bi-partite graphs and line graphs of bi-partite graphs are perfect. If a graph admits an induced cycle of odd length greater than five, then it is not perfect.*

These facts may be found in Korte and Vygen (2018).

Our interest in perfect graphs is due to the following theorem due to Chvátal (1975, Theorem 3.1); one may also find it in Korte and Vygen (2018, Theorem 16.21).

**Theorem D.4.** *A graph $G$ with node set $V$ and edge set $E$ is perfect if and only if the convex hull $\operatorname{co} S(G)$ is equal to the set*

$$\left\{ x\colon V \to [0,1]\colon \textit{all maximal cliques } X \textit{ of } G \textit{ satisfy} \sum_{v \in X} x(v) \leq 1 \right\}. \tag{D.2}$$

The set $\operatorname{co} S(G)$ is the *stable set polytope* of $G$. The set in (D.2) is the *clique-constrained stable set polytope* of $G$.

We next define a graph $G$ such that the set of deterministic DIC mechanisms with disposal corresponds to $S(G)$, and such that the set of all DIC mechanisms with disposal coincides with the clique-constrained stable set polytope of $G$. In view of Theorem D.4, the question of whether all extreme points are deterministic thus reduces to checking whether $G$ is a perfect graph.

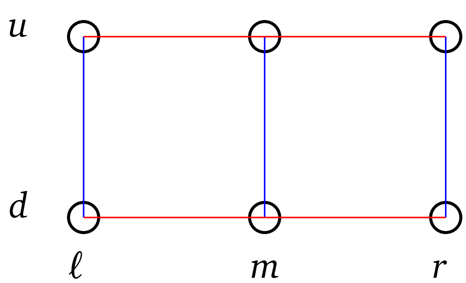
### D.3.2 The feasibility graph

Consider the following graph $G$ with node set $V$ and edge set $E$. Let $V = \cup_{i=1}^{n} (\{i\} \times \Theta_{-i})$, and let two nodes $(i, \theta_{-i})$ and $(j, \theta'_{-j})$ be adjacent if and only if $i \neq j$ and there is a type profile $\hat{\theta}$ satisfying $\hat{\theta}_{-i} = \theta_{-i}$ and $\hat{\theta}_{-j} = \theta'_{-j}$. We refer to $G$ as the *feasibility graph*.

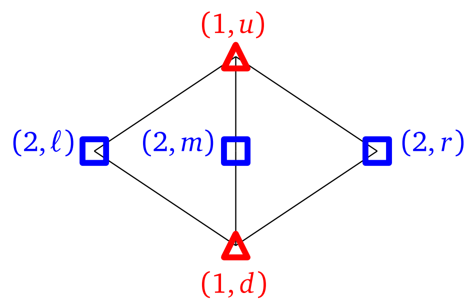
Informally, a node $(i, \theta_{-i})$ is the index for agent $i$'s winning probability when the type profile of the others is $\theta_{-i}$. Two nodes are adjacent if and only if there is a profile $\hat{\theta}$ such that the associated winning probabilities simultaneously appear in the feasibility constraint

$$\sum_{i=1}^{n} \varphi_i(\hat{\theta}_{-i}) \leq 1. \tag{D.3}$$

Figure 3 shows the feasibility graph in an example with two agents; Figure 4 shows it in an example with three agents




(a) The set of type profiles $\Theta$. Circles represent type profiles.




(b) The graph $G$. Red triangles represent nodes of $G$ that are associated with agent 1. Blue squares represent nodes associated with agent 2.

Figure 3: There are two agents with types $\Theta_1 = \{\ell, m, r\}$ and $\Theta_2 = \{u, d\}$.

Given a node $v = (i, \theta_{-i})$ of $G$, let us write $\varphi(v) = \varphi_i(\theta_{-i})$. Note that a clique in the feasibility graph is a subset of nodes of $V$ such that the winning probabilities associated with these nodes all appear in the same feasibility constraint (D.3). It follows that there is a one-to-one mapping between maximal cliques of $G$ and type profiles. For a DIC mechanism with disposal, the feasibility constraint (D.3) may thus be equivalently stated as follows: For all maximal cliques of $X$ of $G$, we have $\sum_{v \in X} \varphi(v) \leq 1$. Thus the set of DIC mechanisms with disposal coincides with the set (D.2). One may similarly verify that the set of deterministic DIC mechanisms with disposal coincides with $S(G)$. In view of Theorem D.4, we deduce:

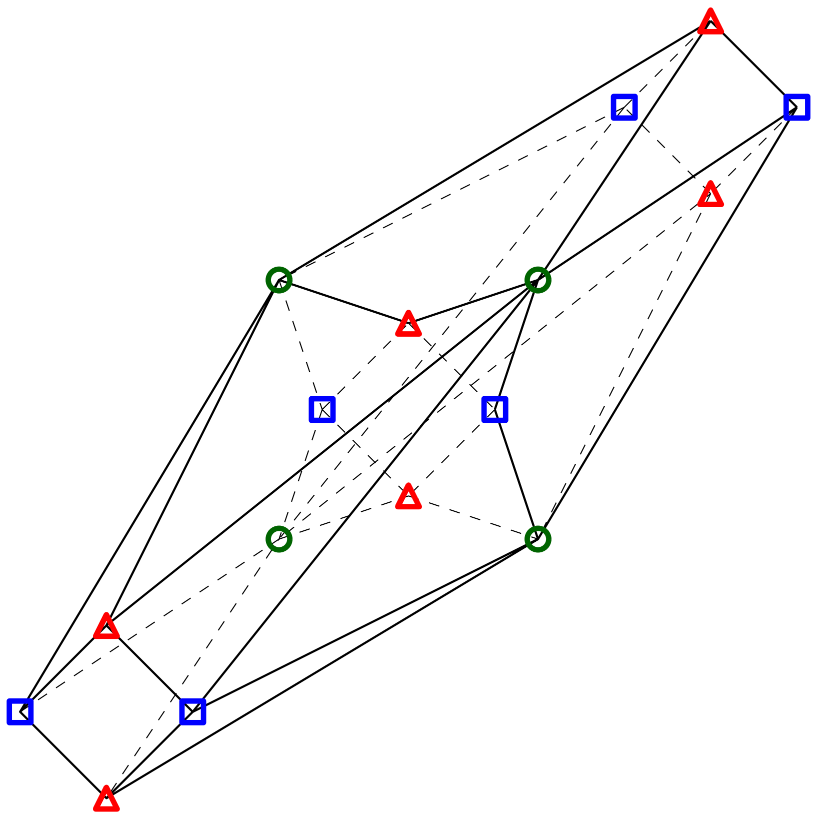


Figure 4: The feasibility graph $G$ in an example with three agents. Agents 1 and 2 each have two possible types. The nodes of $G$ associated with agents 1 and 2, respectively, are depicted by red triangles and blue squares, respectively. Agent 3 has three possible types; the associated nodes are depicted by green circles. One may view this as the graph $G$ associated with the four-agent environment of Section 4.1, except that all nodes of the dummy agent 4 are omitted.

**Lemma D.5.** *All extreme points of the set of DIC mechanisms with disposal are deterministic if and only if $G$ is perfect.*

This leads us to the following alternative proof of Theorem D.1.

*Alternative proof of Theorem D.1.* Let $n = 2$. Observe that the node set of $G$ may be partitioned into the sets $\{1\} \times \Theta_2$ and $\{2\} \times \Theta_1$. By definition, two nodes $(i, \theta_{-i})$ and $(j, \theta_{-j})$ are adjacent only if $i \neq j$. Thus $G$ is bi-partite. Since every bi-partite graph is perfect (Lemma D.3), the claim follows from Theorem D.4.

Suppose $|\Theta_i| = 2$ holds for all $i$. We may relabel the types so that $\Theta_i = \{0, 1\}$ holds for all $i$. In this case $G$ is the line graph of a bi-partite graph; namely the bi-partite graph with node set $\{0, 1\}^n$ and where two nodes are adjacent if and only if they differ in exactly one entry. The line graph of a bi-partite graph is perfect (Lemma D.3), and so the claim again follows from Theorem D.4.

Lastly, suppose $n \geq 3$ and $|\Theta_i| > 2$ for at least one $i$. We will show that $G$ admits an odd induced cycle of length seven. In view of Lemma D.3 and Theorem D.4, this proves that there exists a stochastic extreme point. Let us relabel the agents and types such that the type spaces contain the following subsets of types:

$$\tilde{\Theta}_1 = \{\ell, r\} \quad \text{and} \quad \tilde{\Theta}_2 = \{u, d\} \quad \text{and} \quad \tilde{\Theta}_3 = \{f, c, b\}$$

all hold. Let $\theta_{-123}$ be an arbitrary type profile of agents other than 1, 2 and 3 (assuming such agents exist). One may verify that the following is an induced cycle of length seven:

$$\begin{aligned}
(2, (\ell, c, \theta_{-123})) &\leftrightarrow (1, (d, c, \theta_{-123})) \\
&\leftrightarrow (3, (r, d, \theta_{-123})) \\
&\leftrightarrow (2, (r, b, \theta_{-123})) \\
&\leftrightarrow (3, (r, u, \theta_{-123})) \\
&\leftrightarrow (1, (u, f, \theta_{-123})) \\
&\leftrightarrow (3, (\ell, u, \theta_{-123})) \\
\leftrightarrow (2, (\ell, c, \theta_{-123}))\,.
\end{aligned}$$

□

The proof in the main text for the existence of a stochastic extreme point is slightly more elaborate than the one given above since in the former we explicitly spell out the extreme point. (The proof in the main text uses one of the agents as a dummy, and therefore also works for mechanisms with disposal.) In our view, the advantage of the more elaborate argument is that it facilitates the construction of environments where all deterministic DIC mechanisms fail to be optimal. This let us give an interpretation as to why the principal may strictly benefit from a stochastic DIC mechanism. That said, it is clear how the induced cycle defined in the proof of Theorem D.1 relates to the construction from the main text.

# Appendix E Supplementary material: Additional results

## E.1 All extreme points are candidates for optimality

**Lemma E.1.** *Let $n \in \mathbb{N}$. Let $\Theta_1, \dots, \Theta_n$ be finite sets, and let $\Theta = \times_{i=1}^n \Theta_i$. If $\varphi$ is an extreme point of the set of DIC mechanisms when there are $n$ agents and the set of type profiles is $\Theta$, then there exists a set $\Omega$ of payoff profiles and a distribution $\mu$ over $\Omega \times \Theta$ such that in the environment $(n, \Omega, \Theta, \mu)$ the mechanism $\varphi$ is the unique optimal DIC mechanism.*

*Proof of Lemma E.1.* The set of DIC mechanisms is a polytope in Euclidean space (being the set of solutions to a finite system of linear inequalities). Hence all its extreme points are exposed (Aliprantis and Border, 2006, Corollary 7.90). In particular, since $\varphi$ is an extreme point, there is a function $p\colon \{1, \dots, n\} \times \Theta \to \mathbb{R}$ such that for all DIC mechanisms $\varphi'$ different from $\varphi$ we have $\sum_{i,\theta} p_i(\theta)(\varphi_i(\theta) - \varphi_i'(\theta)) > 0$. By suitably choosing $\Omega$ and $\mu$, the function $p$ represents the principal's objective function. For example, one possible choice of $\Omega$ and $\mu$ is as follows: Let the marginal of $\mu$ on $\Theta$ be uniform; for all agents $i$, let $\Omega_i$ be the image of $p_i$; for all $\theta$, conditional on the type profile realizing as $\theta$, let the payoff of allocating to agent $i$ be $|\Theta| p_i(\theta)$. $\square$

## E.2 Implementation with deterministic outcome functions

An indirect mechanism specifies a tuple $M = (M_1, \dots, M_n)$ of finite message sets, and an outcome function $g\colon \times_i M_i \to \Delta\{0, \dots, n\}$. (Given a finite set $X$, we denote by $\Delta X$ the set of distributions over $X$.) The outcome function is *deterministic* if for all $m$ the distribution $g(m)$ is degenerate. A *strategy* of agent $i$ is a function $\sigma_i\colon \Theta \to \Delta M_i$. A DIC mechanism $\varphi$ is *implementable* via $(M, g)$ if there is a dominant-strategy equilibrium $(\sigma_1, \dots, \sigma_n)$ of $(M, g)$ such that all profiles $\theta$ satisfy $\varphi(\theta) = \sum_m g(m) \prod_i \sigma_i(m_i|\theta_i)$.

**Lemma E.2.** *If a stochastic DIC mechanism $\varphi$ is implementable via an indirect mechanism with a deterministic outcome function, then $\varphi$ is not an extreme point of the set of DIC mechanisms.*

*Proof of Lemma E.2.* Let $(M, g, \sigma)$ implement $\varphi$. We may assume $g_i$ does not depend on $i$'s message; the reason is that for $\sigma$ to be a dominant-strategy equilibrium, agent $i$'s strategy must be supported on messages that give $i$ the same winning probability. Let us abbreviate $\sigma(m|\theta) = \prod_i \sigma_i(m_i|\theta_i)$. Let $\Sigma$ denote the set of functions from $\Theta$ to $\Delta(\times_i M_i)$, and notice that $\Sigma$ contains $\sigma$. The extreme points of $\Sigma$ are deterministic; that is, they are mappings from $\Theta$ to $\times_i M_i$. Since $\Sigma$ is also compact, convex and non-empty, the Krein-Milman theorem implies that $\sigma$ is a convex combination of functions from $\Theta$ to $\times_i M_i$. Denote these functions by $\{\hat{\sigma}_k\}_k$, and the weights in the combination by $\{\alpha_k\}_k$. For all $k$, define the mechanism $\varphi_k$ for all $\theta$ by $\varphi_k(\theta) = \sum_m \hat{\sigma}_k(m|\theta)g(m)$. Since $g$ and $\hat{\sigma}_k$ are deterministic, it follows that $\varphi_k$ is deterministic. Moreover $\varphi_k$ is DIC since for all $i$ we have that $g_i$ is constant in $i$'s message. Since for all $\theta$ we have $\varphi(\theta) = \sum_m \sigma(m|\theta)g(m) = \sum_m \sum_k \alpha_k \hat{\sigma}_k(m|\theta)g(m) = \sum_k \alpha_k \varphi_k(\theta)$, we conclude that $\varphi$ is a convex combination of deterministic DIC mechanisms. Since $\varphi$ is stochastic, it follows that $\varphi$ is not an extreme point of the set of DIC mechanisms. $\square$

## E.3 Balanced coalitions and anonymous juries

In this part of the appendix, we extend Theorem 6.2 to stochastic mechanisms under an additional axiom that requires immunity against certain coalitional manipulations, and a strengthening of relative anonymity.

### E.3.1 Balanced coalitions

As in the main text, given a mechanism $\varphi$, we say agent $i$ *influences* agent $\ell$ if $\varphi_\ell$ is non-constant in $i$'s report

**Definition 7.** Let $\varphi$ be a mechanism.

A subset $J$ of agents is **balanced** if it satisfies the following: An agent is in $J$ if and only if the agent influences all other agents in $J$.

The mechanism is **immune against balanced coalitions** if for all non-empty balanced subsets $J$ of agents the probability $\sum_{i \in J} \varphi_i$ that the object is allocated to an agent in $J$ is constant in the reports of agents in $J$.

All jury mechanisms are immune against balanced coalitions. Indeed, in a jury mechanism all balanced subsets contain at most one agent, and hence immunity follows from DIC.

Balancedness captures the idea that some agents may find it easier to cooperate than others. For instance, suppose that, in some given mechanism, agent $j$ cannot influence agent $i$. Arguably, agent $i$ now has little to fear when *not* entering a coalition with agent $j$. After all, if the others report truthfully, agent $j$ has no way of punishing agent $i$ within the mechanism. Thus there is a sense in which coalitions where some members cannot influence other members are less plausible than balanced coalitions. Demanding immunity with respect to a larger set of coalitions would lead to a more restrictive notion of coalition-proofness. Indeed, the only mechanisms immune against *all* coalitions are constant ones.[24]

### E.3.2 Relative anonymity and common agents

We now turn to a strengthening of relative anonymity. Recall the following definitions from the main text for a given mechanism $\varphi$ and distinct agents $i$, $j$ and $k$: Agents $i$ and $j$ are *exchangeable for $k$* if $\varphi_k$ is invariant with respect to all permutations of $i$'s and $j$'s reports. Agent $i$ *influences agent $k$* if $\varphi_k$ is non-constant in $i$'s report.

**Definition 8.** Let $\varphi$ be a mechanism.

(1) Let $i$ and $j$ be distinct. Agents $i$ and $j$ **influence a common agent** if there exists $\ell$ such that $i$ and $j$ both influence $\ell$.

(2) The mechanism $\varphi$ is **strongly relatively anonymous** if for all $i$, $j$ and $k$ that are all distinct, if $i$ and $j$ influence a common agent, then $i$ and $j$ are exchangeable for $k$.

This notion strengthens relative anonymity from the main text (Definition 4). Recall that relative anonymity demands that $i$ and $j$ be exchangeable for $k$ if $i$ and $j$ both influence $k$. Strong relative anonymity also demands that $i$ and $j$ be exchangeable for $k$ as soon as $i$ and $j$ both influence *some* agent $\ell$, where $\ell$ may differ from $k$. We observe, however, that strong relative anonymity does not imply that all agents influence all others. It may be the case that agents $i$ and $j$ both influence $\ell$ and both fail to influence agent $k$. The content of the axiom is that if $i$ and $j$ influence a common agent, then one of $i$ and $j$ influences $k$ if and only if both of them influence

[24] To briefly verify this, fix a mechanism $\varphi$ that is immune against all coalitions. Consider two arbitrary agents $i$ and $j$. A change in $i$'s report affects neither $\varphi_i + \varphi_j$ (by immunity against the coalition $\{i, j\}$) nor $\varphi_i$ (by immunity against $\{i\}$). Hence $\varphi_j$ is also constant in $i$'s report. Hence $\varphi$ is constant.

$k$ (and, if they influence $k$, then they are also exchangeable for $k$). We formalize this fact in the proof of the upcoming result (see Claim E.4).

An implication of Theorem 6.2 is that relative anonymity and strong relative anonymtiy are equivalent for deterministic DIC mechanisms. They are not equivalent for stochastic DIC mechanisms, as the following example shows.

**Example 3.** There are fives agents with common type space $\{0, 1\}$. Let $\varphi_1$ be the jury mechanism that allocates to agent 3 if $\theta_1 = 0$, and else allocates to agent 5. Let $\varphi_2$ be the mechanism that allocates to agent 4 if $\theta_2 = 0$, and else allocates to agent 5. The convex combination $\frac{\varphi_1+\varphi_2}{2}$ is DIC and relatively anonymous. However, it is not strongly relatively anonymous: Agents 1 and 2 influence a common agent (agent 5), but they are not exchangeable for agent 3 nor agent 4.

### E.3.3 Characterization

Let $\Phi^*$ denote the set of mechanisms that are DIC, strongly relatively anonymous, and immune against balanced coalitions. The next result characterizes the extreme points of $\Phi^*$. One may verify that $\Phi^*$ is compact (as a subset of Euclidean space), and hence $\Phi^*$ is contained in the convex hull of its extreme points.[25]

**Theorem E.3.** *A mechanism is an extreme point of $\Phi^*$ if and only if it is a deterministic jury mechanism with an anonymous jury.*

In words, all mechanisms in $\Phi^*$ (or randomizations thereof) can be implemented by randomizing over deterministic jury mechanisms with anonymous juries. Hence the extreme poitns of $\Phi^*$ are exactly the mechanisms characterized by Theorem 6.2.

Let us give a brief sketch of the proof (the formal proof follows below in Appendix E.3.5). Strong relative anonymity implies a partition of the agents into equivalence class, where two agents are equivalent if and only if they influence the same set of others.[26] These equivalence classes are the candidates for the juries of the jury mechanisms that we eventually define. The main step for the remainder of the argument is then to show that agents within one equivalence class cannot influence one

[25] The set $\Phi^*$ is not generally convex. Example 3 actually shows this. The mechanisms $\varphi_1$ and $\varphi_2$ are both in $\Phi^*$, but the convex combination $\frac{\varphi_1+\varphi_2}{2}$ fails to be strongly relatively anonymous.

[26] The reader may recall that we defined a similar equivalence relation in the proof of Theorem 6.2. Relative to that proof, verifying that the relation is indeed an equivalence relation is simpler due to the stronger notion of anonymity.

another (for else these would not be well-defined juries). This is the only step where we appeal to immunity against balanced coalitions, and where the impossibility result Theorem 6.1 turns out to be useful.

#### E.3.4 Discussion of the axioms in Theorem E.3

Here we provide two examples with the following features.

(1) Example 4 presents a mechanism that is relatively anonymous, stochastic, DIC, immune against balanced coalitions, and an extreme point of the set of *all* DIC mechanisms.

(2) Example 5 presents a mechanism that is strongly relatively anonymous, stochastic, DIC, and an extreme point of the set of *all* DIC mechanisms.

Both of these mechanisms fail to be convex combinations of jury mechanisms. Hence the examples demonstrate that in Theorem E.3 one cannot relax strong relative anonymity to relative anonymity,[27] and one cannot drop immunity against balanced coalitions.

To construct these examples, recall from Appendix D that a *mechanism with disposal* means a function $\varphi\colon \Theta \to [0,1]^n$ such that all $\theta$ satisfy $\sum_{i=1}^n \varphi_i(\theta) \leq 1$. A mechanism $\varphi$ with disposal is DIC if for all $i$ the probability $\varphi_i$ is constant in $i$'s report. Proposition D.2 from Appendix D shows the following: If $n = 3$ and the common type space of the agents is $T = \{1,2,3,4,5,6,7\}$, then the set of DIC mechanisms with disposal admits a stochastic extreme point $\varphi^*$ with the following properties:

(1) For all $i \in \{1,2,3\}$, agent $i$ is influenced by all agents in $\{1,2,3\} \setminus \{i\}$, and $\varphi_i^*$ is invariant with respect to all permutations of $\{1,2,3\} \setminus \{i\}$.

(2) The probability $1 - \sum_{i=1}^3 \varphi_i^*$ is non-constant and invariant with respect to all permutations of $\{1,2,3\}$.

We next use $\varphi^*$ to construct the two aforementioned examples.

For our first example, we modify $\varphi^*$ to obtain a mechanism that is relatively anonymous, stochastic, DIC, immune against balanced coalitions, and an extreme point of the set of all DIC mechanisms (without disposal).

[27] We already know from Example 3 that relative anonymity and strong relative anonymity are not generally equivalent. However, the mechanism from this example is a convex combination of deterministic jury mechanisms with anonymous juries, and hence it cannot make the point that one cannot relax strong relative anonymity in Theorem E.3.

**Example 4.** Let $n = 5$. Let the common type space be $T = \{1, 2, 3, 4, 5, 6, 7\}$. The mechanism will ignore the reports of agents 4 and 5; we drop their reports from the notation. Now, for all $(\theta_1, \theta_2, \theta_3) \in T \times T \times T$, let

$$
\begin{aligned}
\tilde{\varphi}_1(\theta_1, \theta_2, \theta_3) &= \varphi_1^*(\theta_1, \theta_2, \theta_3) \\
\tilde{\varphi}_2(\theta_1, \theta_2, \theta_3) &= \varphi_2^*(\theta_1, \theta_2, \theta_3) \\
\tilde{\varphi}_3(\theta_1, \theta_2, \theta_3) &= 0 \\
\tilde{\varphi}_4(\theta_1, \theta_2, \theta_3) &= 1 - \sum_{i=1}^{3} \varphi_i^*(\theta_1, \theta_2, \theta_3) \\
\tilde{\varphi}_5(\theta_1, \theta_2, \theta_3) &= \varphi_3^*(\theta_1, \theta_2, \theta_3).
\end{aligned}
$$

In words, the winning probabilities of agents 1 and 2 are unchanged. Whenever agent 3 was given the object under $\varphi^*$, the object is now given to agent 5. Agent 4's winning probability is the probability that the object is not allocated under $\varphi^*$.

The properties of $\varphi^*$ imply that $\tilde{\varphi}$ is a well-defined DIC mechanism and a stochastic extreme point of the set of DIC mechanisms. Further, $\tilde{\varphi}$ is not a jury mechanism (agents 1 and 2 enjoy non-constant winning probabilities but also influence the allocation). Consequently, $\tilde{\varphi}$ is *not* a convex combination of jury mechanisms. We next claim that $\tilde{\varphi}$ is relatively anonymous and immune against balanced coalitions.

We begin with immunity against balanced coalitions. Towards a contradiction, suppose there is a balanced coalition of at least two agents. Note that the coalition cannot include agent 3 (as $\tilde{\varphi}_3$ is constant) nor agents 4 and 5 (as neither of these influence another agents). Hence the coalition contains exactly agents 1 and 2. But agent 3 influences agents 1 and 2, and hence balancedness is contradicted. Thus all balanced coalitions contain at most one agent. Immunity against balanced coalitions follows from DIC.

Relative anonymity follows immediately from the properties of $\varphi^*$.

For completeness, let us also verify that $\tilde{\varphi}$ is not strongly relatively anonymous. Agents 2 and 3 influence a common agent, namely agent 1. However, the reports of agent 2 and 3 are not exchangeable for all agents distinct from 2 and 3. Namely, agent 2 influences agent 5, whereas agent 3 does not (since $\varphi_3^*$ is constant in $\theta_3$).

We now present a further minor modification of $\varphi^*$ that delivers a mechanism that is strongly relatively anonymous, stochastic, DIC, and an extreme point of the set of all DIC mechanisms.

**Example 5.** Let $n = 4$. Let the common type space be $T = \{1, 2, 3, 4, 5, 6, 7\}$. Loosely speaking, our candidate mechanism $\hat{\varphi}$ is exactly as above except that agent 3 and agent 5 now coincide. That is, for all $(\theta_1, \theta_2, \theta_3) \in T \times T \times T$ let

$$\begin{aligned}
\hat{\varphi}_1(\theta_1, \theta_2, \theta_3) &= \varphi_1^*(\theta_1, \theta_2, \theta_3) \\
\hat{\varphi}_2(\theta_1, \theta_2, \theta_3) &= \varphi_2^*(\theta_1, \theta_2, \theta_3) \\
\hat{\varphi}_3(\theta_1, \theta_2, \theta_3) &= \varphi_3^*(\theta_1, \theta_2, \theta_3) \\
\hat{\varphi}_4(\theta_1, \theta_2, \theta_3) &= 1 - \sum_{i=1}^{3} \varphi_i^*(\theta_1, \theta_2, \theta_3).
\end{aligned}$$

As in Example 4, we have that $\varphi^*$ is a well-defined DIC mechanism and a stochastic extreme point of the set of DIC mechanisms. Yet $\hat{\varphi}$ fails immunity against balanced coalitions since the coalition $\{1, 2, 3\}$ is balanced but has a non-constant winning probability. Strong relative anonymity of $\hat{\varphi}$ follows immediately from the properties of $\varphi^*$.

Let us briefly explain why $\hat{\varphi}$ but not $\tilde{\varphi}$ satisfies strong relative anonymity. In $\hat{\varphi}$, agent 3's winning probability is not permutation-invariant with respect to the reports of 2 and 3 even though 2 and 3 influence a common agent. However, this is not a violation of strong relative anonymity since the definition strong relative anonymity (Definition 8) considers distinct agents $i$, $j$ and $k$. Turning to $\tilde{\varphi}$, agent 3's winning probability becomes the winning probability of agent 5, and hence we have now have a violation of strong relative anonymity.

#### E.3.5 Proof of Theorem E.3

*Proof of Theorem E.3.* Let $\varphi$ be a deterministic jury mechanism with an anonymous jury. All jury mechanisms are immune against balanced coalitions. Thus $\varphi$ is in $\Phi^*$. Since $\varphi$ is determinstic, it is an extreme point of $\Phi^*$.

Now let $\varphi \in \Phi^*$. We complete the proof by finding a convex combination of deterministic jury mechanisms with anonymous juries that equals $\varphi$.

For all agents $i$, let $I_i$ denote the set of agents that influence $i$. Let $A_i$ denote the set of agents that are influence by $i$. Let $I = \cup_{i=1}^{n} I_i$ denote the set of agents with respect to whose reports $\varphi$ is non-constant. We may assume that $\varphi$ is non-constant, meaning $I \neq \emptyset$, as otherwise the proof is trivial.

We proceed along a series of claims. The first of these is essentially a restatement of strong relative anonymity.

**Claim E.4.** *All agents $i$ and $j$ satisfy the following.*

*(1) The allocation $\varphi_i$ is invariant with respect to all permutations of $I_i$.*

*(2) If $(A_i \setminus \{j\}) \cap (A_j \setminus \{i\}) \neq \emptyset$, then $A_i \setminus \{j\} = A_j \setminus \{i\}$.*

*Proof of Claim E.4.* Consider (1). The agents in $I_i$ influence a common agents, namely $i$. Hence $\varphi_i$ is invariant with respect to pairwise permutations of $I_i$. Hence $\varphi_i$ is invariant with respect to all permutations of $I_i$.

Consider (2). If $(A_i \setminus \{j\}) \cap (A_j \setminus \{i\}) \neq \emptyset$, then $i$ and $j$ influence a common agent. Hence, if $k$ is distinct from $i$ and $j$, then $\varphi_k$ is invariant with respect to permutations of $i$ and $j$. Thus $i$ influences $k$ if and only if $j$ influences $k$. □

Consider the binary relation $\sim$ on $I$ defined as follows: Given $i$ and $j$ in $I$, we have $i \sim j$ if and only if $(A_i \setminus \{j\}) \cap (A_j \setminus \{i\}) \neq \emptyset$. Claim E.4 implies that $i \sim j$ is equivalent to $\emptyset \neq A_i \setminus \{j\} = A_j \setminus \{i\}$.

**Claim E.5.** *Let $i$ and $j$ in $I$ be such that $i \sim j$. If $i \in A_j$, then $j \in A_i$.*

*Proof of Claim E.5.* We will show the contrapositive. Let $j \notin A_i$. We have to show $i \notin A_j$. For this proof, let us write $\varphi_i(t, t', \theta_{-ij})$ and $\varphi_j(t, t', \theta_{-ij})$, respectively, for $i$'s and $j$'s winning probabilites, respectively, when $i$ reports some type $t$, $j$ reports some type $t'$, and the others report some profile $\theta_{-ij}$.

Let $\theta$ be an arbitrary profile. Consider the permutation where exactly the types of $i$ and $j$ are swapped. Claim E.4, DIC, and the definition of $\sim$ imply $A_i \setminus \{i, j\} = A_i \setminus \{j\} = A_j \setminus \{i\} = A_j \setminus \{i, j\}$. Hence $A_i^c \setminus \{i, j\} = A_j^c \setminus \{i, j\}$. By definition of $A_i^c$ and $A_j^c$, the permutation does not affect the allocation of agents in $A_i^c \setminus \{i, j\} = A_j^c \setminus \{i, j\}$. Anonymity implies that the permutation does not affect the allocation of agents in $A_i \setminus \{i, j\} = A_j \setminus \{i, j\}$. Since the object is always allocated, we find

$$\varphi_i(\theta_i, \theta_j, \theta_{-ij}) + \varphi_j(\theta_i, \theta_j, \theta_{-ij}) = \varphi_i(\theta_j, \theta_i, \theta_{-ij}) + \varphi_j(\theta_j, \theta_i, \theta_{-ij}).$$

By DIC and since $j \notin A_i$, we have that $\varphi_j$ depends neither on $i$'s nor $j$'s report. Hence the previous equation is equivalent to

$$\varphi_i(\theta_i, \theta_j, \theta_{-ij}) = \varphi_i(\theta_j, \theta_i, \theta_{-ij}).$$

DIC implies that $\varphi_i(\theta_j, \theta_i, \theta_{-ij})$ is constant in $\theta_j$. Hence the previous display implies that $\varphi_i(\theta_i, \theta_j, \theta_{-ij})$ must be constant in $\theta_j$, too. This shows that $\varphi_i$ is constant in $j$'s report, meaning $i \notin A_j$. □

**Claim E.6.** *The relation $\sim$ is an equivalence relation.*

*Proof of Claim E.6.* It is clear that $\sim$ is symmetric. To see that $\sim$ is reflexive, note that $i \in I$ is equivalent to $A_i \neq \emptyset$. Hence $A_i \cap A_i \neq \emptyset$, implying $i \sim i$.

Turning to transitivity, let $i$, $j$ and $k$ be agents in $I$ such that $(A_i \setminus \{j\}) \cap (A_j \setminus \{i\}) \neq \emptyset$ and $(A_j \setminus \{k\}) \cap (A_k \setminus \{j\}) \neq \emptyset$. Claim E.4 implies $A_i \setminus \{j\} = A_j \setminus \{i\}$ and $A_j \setminus \{k\} = A_k \setminus \{j\}$. We distinguish two cases.

First, suppose there exists $\ell \in A_j \setminus \{i, k\}$. Then $j \notin A_j$ and $A_i \setminus \{j\} = A_j \setminus \{i\}$ imply $\ell \in A_i \setminus \{k\}$. Similarly, we have $\ell \in A_k \setminus \{i\}$. Thus $i \sim k$.

Second, suppose $A_j \setminus \{i, k\} = \emptyset$. Since $A_j \setminus \{i\}$ and $A_j \setminus \{k\}$ are both non-empty, we have $A_j = \{i, k\}$. Claim E.5 implies $j \in A_i$ and $j \in A_k$. In view of DIC, this implies $i \neq j \neq k$. In particular, we have $j \in A_i \setminus \{k\} \cap A_k \setminus \{i\}$. Thus $i \sim k$. □

Claim E.6 implies that we may partition $I$ into finitely-many non-empty $\sim$-equivalence classes. (Recall that $I$ is non-empty.) Let $\mathcal{J}$ denote the collection of $\sim$-equivalence classes.

**Claim E.7.** *Let $J$ and $J'$ be distinct sets in $\mathcal{J}$. Let $i$ and $j$ be in $J$. If $j \in A_i$, then both of the following are true:*

*(1) For all distinct $\ell$ and $k$ in $J$ we have $\ell \in A_k$; that is, all agents in $J$ influence all others in $J$.*

*(2) If $k$ is in $J'$, then $J \cap A_k = \emptyset$; that is, no agent outside of $J$ influences an agent in $J$.*

*Proof of Claim E.7.* Consider (1). If $\ell = j$, then the claim follows from $j \in A_i$ and $i \sim k$. If $\ell \neq j$, then $\ell \sim i$ and $j \in A_i$ imply $j \in A_\ell$. Hence, by Claim E.5, we have $\ell \in A_j$. Now $j \sim k$ and $\ell \neq k$ imply $\ell \in A_k$.

Consider (2). Towards a contradiction, suppose there exists $k$ in $J'$ such that $\ell \in J \cap A_k$. We will show that $k \in J$; this contradicts the fact that $J$ and $J'$ are disjoint.

Note that $j \in A_i$ and $\ell \in A_k$ imply $i \neq j$ and $k \neq \ell$ (else DIC is contradicted). Suppose for a moment $\ell = j$. This implies $j \in (A_k \setminus \{i\}) \cap (A_i \setminus \{k\})$, and hence $i \sim k$, and hence $k \in J$. In what follows, we may thus assume $\ell \neq j$.

As an intermediate step, we claim $j \in A_\ell$. Since $i \sim \ell$, we have $A_i \setminus \{\ell\} = A_\ell \setminus \{i\}$. Using $j \in A_i$ and $\ell \neq j$, we infer $j \in A_i \setminus \{\ell\}$, and hence $j \in A_\ell$.

Claim E.5 and $j \in A_\ell$ together imply $\ell \in A_j$. Altogether, we now know that $\ell \in A_j \cap A_k$. We also have $j \neq \ell$ (by assumption) and $k \neq \ell$ (by $\ell \in A_k$ and DIC). Thus $\ell \in A_j \setminus \{k\}$ and $\ell \in A_k \setminus \{j\}$ hold. In particular, we find $j \sim k$, and hence $k \in J$. □

**Claim E.8.** *Let $J \in \mathcal{J}$. The allocation $(\varphi_i)_{i \in J}$ is constant in the reports of agents in $J$.*

*Proof of Claim E.8.* Towards a contradiction, suppose $(\varphi_i)_{i\in J}$ is non-constant in the reports of agents in $J$. Part (1) of Claim E.7 implies that all agents in $J$ can influence all others agents in $J$. Part (2) of Claim E.7 implies that no agent outside of $J$ influences an agent in $J$. Thus $J$ is balanced. Since $\varphi$ is immune against balanced coalitions, therefore, the sum $\sum_{i \in J} \varphi_i$ is constant in the reports of all agents. Let $p$ denote this constant probability. We have $p > 0$ since else the agents in $J$ enjoy a constant winning probability, contradicting the assumption that $(\varphi_i)_{i\in J}$ is non-constant in the reports of agents in $J$. Consider the functions $(\varphi_i/p)_{i\in J}$ obtained by scaling the winning probabilities of agents in $J$ by $1/p$. These functions define a DIC mechanism in a setting where the set of agents is $J$. This mechanism satifies the hypotheses of Theorem 6.1 since $\varphi$ is DIC and strongly relatively anonymous, and since, as observed above, all agents in $J$ influence all other agents in $J$. Thus $(\varphi_i/p)_{i\in J}$ is constant; contradiction. □

Given $J$ in $\mathcal{J}$ and $i$ in $J$, Claims E.4 and E.8 imply that the set $A_i$ is disjoint from $J$ and the same across all $i \in J$. Let us denote this common set by $A_J$. Let $A_\emptyset$ denote the (possibly empty) set of agents $i$ such that $\varphi_i$ is constant in the reports of all agents. The following fact is immediate

**Claim E.9.** *The collection $\{A_J\}_{J \in \mathcal{J} \cup \{\emptyset\}}$ partitions the set of agents.*

Let $J$ be in $\mathcal{J}$. If an agent $i$ is in $A_J$, then Claim E.9 implies that $i$'s winning probability only depends on the reports of agents in $J$. Thus in what follows we write $\varphi_i(\theta_J)$ for $i$'s winning probability. By definition of $A_J$, the sum $1 - \sum_{i \in A_J} \varphi_i = \sum_{i \notin A_J} \varphi_i$ is constant in the reports of $J$. Thus $\sum_{i \in A_J} \varphi_i$ is constant in the reports of $J$, too. Claim E.9 implies $A_J \cap A_{J'} = \emptyset$ whenever $J'$ is distinct from $J$. Hence $\sum_{i \in A_J} \varphi_i$ is constant in the reports of agents outside of $J$. Thus $\sum_{i \in A_J} \varphi_i$ must be constant in all reports. We denote the constant probability by $\alpha_J$.

The probability $\sum_{i \in A_\emptyset} \varphi_i$ is constant by the definition of $A_\emptyset$. We denote the constant value by $\alpha_\emptyset$. Since $\{A_J\}_{J \in \mathcal{J} \cup \{\emptyset\}}$ partitions the set of agents (Claim E.9), we have $\sum_{J \in \mathcal{J} \cup \{\emptyset\}} \alpha_J = 1$.

We next define a collection of auxiliary jury mechanisms.

For all $J$ in $\mathcal{J} \cup \{\emptyset\}$ such that $\alpha_J > 0$, let $\psi_J$ denote the following mechanism: For all $i$ in $A_J$, agent $i$ is allocated the object with probability $\varphi_i(\theta_J)/\alpha_J$. For all $i$ not in $A_J$, agent $i$ is allocated the object with probability 0. For all $J$ in $\mathcal{J} \cup \{\emptyset\}$ such that $\alpha_J = 0$, let $\psi_J$ be an arbitrary constant mechanism.

**Claim E.10.** *The collection $\{\psi_J\}_{J \in \mathcal{J} \cup \{\emptyset\}}$ is a collection of jury mechanisms with anonymous juries satisfying $\sum_{J \in \mathcal{J} \cup \{\emptyset\}} \alpha_J \psi_J = \varphi$.*

Note that the jury mechanisms $\{\psi_J\}_{J \in \mathcal{J} \cup \{\emptyset\}}$ have not been proven to be deterministic.

*Proof of Claim E.10.* For all $J$, the mechanism $\psi_J$ is a well-defined mechanism since $\alpha_J$ is the constant probability that the object is allocated to an agent in $J$. Anonymity of the jury is inherited from strong relative anonymity of $\varphi$. If $i \in A_J$, then $i \notin J$ (recall the discussion preceding Claim E.9) and $i$'s' winning probability depends only on the reports of agents in $J$. This argument shows that $\psi_J$ is a jury mechanism and that $\sum_{J \in \mathcal{J} \cup \{\emptyset\}} \alpha_J \psi_J = \varphi$ holds. $\square$

The following claim is easily verified using the Krein-Milman theorem.

**Claim E.11.** *For all $J \in \mathcal{J} \cup \{\emptyset\}$, the mechanism $\psi_J$ is a convex combination of deterministic jury mechanisms with anonymous juries.*

*Proof of Claim E.11.* Let $J \in \mathcal{J} \cup \{\emptyset\}$. Let $\Psi$ be the set of jury mechanisms having all following properties: For all $i$, the mechanism is non-constant in the report of agent $i$ only if $i$ is in $J$; the mechanism is invariant with respect to permutations of agents in $J$. The set $\Psi$ is compact and convex, and it contains $\psi_J$. Hence the claim follows from the Krein-Milman theorem if we can show that all extreme points are deterministic. To that end, consider a stochastic mechanism in $\Psi$. At a profile where the mechanism randomizes, shift a small mass between two agents with interior winning probabilities; do the same at all profile obtained by permuting the reports of agents in $J$. Using that the mechanism is in $\Psi$, it easy to see that this yields two other mechanisms in $\Psi$, the convex hull of which contains the given stochastic one. $\square$

Claims E.10 and E.11, together with the equation $\sum_{J\in\mathcal{J}\cup\{\emptyset\}}\alpha_J = 1$ imply that $\varphi$ is a convex combination of deterministic jury mechanisms with anonymous juries. $\square$

## E.4 No information elicitation in fully-symmetric environments

Here we state and prove an interesting corollary of Theorem 6.1 for environments in which the principal actually finds it optimal to "treat all agents equally." Consider environments satisfying the following:

**Assumption 3.** The sets $\Omega_i$ are the same across agents $i$. The sets $\Theta_i$ are the same across agents $i$. The distribution $\mu$ is invariant with respect to permutations of the agents; that is, for all $\omega \in \Omega$, all $\theta \in \Theta$, and all bijections $\xi\colon \{1,\dots,n\} \to \{1,\dots,n\}$ we have

$$\mu\left(\omega_1,\dots,\omega_n,\theta_1,\dots,\theta_n\right) = \mu\left(\omega_{\xi(1)},\dots,\omega_{\xi(n)},\theta_{\xi(1)},\dots,\theta_{\xi(n)}\right).$$

Assumption 3 severely restricts the informational content of types. For instance, an implication of the assumption is that if $i$ and $j$ are distinct, then for all type profiles $\theta_{-ij}$ of agents other than $i$ and $j$ we have

$$\mathbb{E}_{\omega_i}[\omega_i|\theta_{-ij}] = \mathbb{E}_{\omega_j}[\omega_j|\theta_{-ij}].$$

Put differently, learning the type realizations of $n-2$ of the agents has no value for discerning between the two remaining agents. Observe that this assumption is far more restrictive than Assumption 1. The latter makes no assumption about how the distribution of payoffs $\omega_i$ differ across $i$.

**Corollary E.12.** *If Assumption 3 holds, then the principal is indifferent between all DIC mechanisms.*

*Proof of Corollary E.12.* Let $\varphi$ be a DIC mechanism. Let $\Xi$ denote the set of permutations of $\{1, \ldots, n\}$. For all $i$, let $\psi_i \colon \Theta \to [0, 1]$ be defined by

$$\forall_{\theta \in \Theta}, \quad \psi_i(\theta) = \frac{1}{n!} \sum_{\xi \in \Xi} \varphi_{\xi(i)}(\xi(\theta)),$$

and let $\psi = (\psi_1, \ldots, \psi_n)$. Then $\psi$ is a well-defined mechanism, it inherits DIC from $\varphi$, and it is anonymous by construction. Thus Theorem 6.1 implies that $\psi$ is constant. Using Assumption 3 it is straightforward to verify that the principal is indifferent between $\varphi$ and $\psi$, and indifferent between all constant mechanisms. Since $\psi$ is constant and $\varphi$ was arbitrary, the proof is complete.

□

We note that Corollary E.12 does not extend to mechanisms with disposal (as defined in Appendix D). Indeed, the following is an easy corollary of Proposition D.2.

**Corollary E.13.** *Let $n = 3$ and let the common type space be $T = \{1, 2, 3, 4, 5, 6, 7\}$. There is an environment satisfying Assumption 3 in which a non-constant stochastic DIC strongly anonymous mechanism with disposal uniquely maximizes the principal's utility over the set of DIC mechanisms with disposal.*

*Proof of Corollary E.13.* Let $\varphi^*$ be as in the conclusion of Proposition D.2. Since the set of DIC mechanisms with disposal is a polytope in Euclidean space, there is a function $p \colon \{1, \ldots, n\} \times \Theta \to \mathbb{R}$ such that for all DIC mechanisms $\varphi$ different from $\varphi^*$ we have $\sum_{i,\theta} p_i(\theta)(\varphi_i^*(\theta) - \varphi_i(\theta)) > 0$. Since $\varphi^*$ is strongly anonymous, we may assume that all agents $i$, all type profiles $\theta$ and all permutations $\xi$ satisfy $p_i(\theta) = p_{\xi(i)}(\xi(\theta))$. The function $p$ defines an environment that satisfies Assumption 3 and in which $\varphi^*$ is the unique optimal DIC mechanism. □

## E.5 Total unimodularity

This section of the appendix discusses another potential approach for showing that all extreme points are deterministic. Our aim is to give a brief explanation for why this approach, that is based on total unimodularity, does not actually help us for the proof of Theorem 4.2 in the difficult case with three agents.

For a function $\varphi\colon \Theta \to [0,1]^n$ to be a DIC mechanism, it should satisfy the following:

$$\begin{aligned} \forall_{i,\theta}, &\quad 1 \geq \varphi_i(\theta) \\ \forall_{i,\theta_i,\theta_i',\theta_{-i}}, &\quad 0 \geq \varphi_i(\theta_i,\theta_{-i}) - \varphi_i(\theta_i',\theta_{-i}) \geq 0 \\ \forall_{\theta}, &\quad 1 \geq \sum_i \varphi_i(\theta) \geq 1 \end{aligned} \tag{E.1}$$

For a suitable matrix $A$ and vector $b$, the set of DIC mechanisms is then the polytope $\{\varphi\colon A\varphi \geq b, \varphi \geq 0\}$. Here, the matrix $A$ has one row for every constraint in (E.1) (after splitting the constraints into one-sided inequalities). Each column of $A$ identifies a pair of the form $(i, \theta)$.

A matrix or a vector is *integral* if its entries are all in $\mathbb{Z}$. A polytope is *integral* if all its extreme points are integral. In this language, all extreme points of the set of DIC mechanisms are deterministic if and only if the polytope $\{\varphi\colon A\varphi \geq b, \varphi \geq 0\}$ is integral.

Recall that a matrix is *totally unimodular* if all its square submatrices have a determinant equal to $-1$, 0, or 1. For later reference, notice that a submatrix of a totally unimodular matrix is itself totally unimodular.

Our interest in total unimodularity is due the Hoffman-Kruskal theorem; see Korte and Vygen (2018, Theorem 5.21).

**Theorem E.14.** *An integral matrix $A$ is totally unimodular if and only if for all integral vectors $b$ all extreme points of the set $\{\varphi\colon A\varphi \geq b, \varphi \geq 0\}$ are integral.*

Thus a sufficient condition for all extreme points of the set of DIC mechanisms to be deterministic is that the constraint matrix $A$ be totally unimodular. Unfortunately:

**Lemma E.15.** *For all agents $i$, let $|\Theta_i| \geq 2$. Let $n = 3$. If there exists $i$ such that $|\Theta_i| \geq 3$, then $A$ is not totally unimodular.*

This explains why our approach to integrality in the case with three agents is not based on total unimodularity.[28]

[28]Note that total unimodularity of $A$ is sufficient, but not necessary, for the polytope to be integral when $b$ is held fixed. Therefore, the fact that $A$ is not always totally unimodular when $n = 3$ does not imply a contradiction to the fact that, according to Theorem 4.2, all extreme points are deterministic when $n = 3$.

*Proof of Lemma E.15.* Towards a contradiction, suppose $A$ is totally unimodular. Let us consider the constraint matrix $\tilde{A}$ and vector $\tilde{b}$ that define the set of DIC mechanisms with disposal (as defined in Appendix D). That is, $\varphi$ is a DIC mechanism with disposal if and only if $\tilde{A}\varphi \geq \tilde{b}$ and $\varphi \geq 0$. Notice that $\tilde{A}$ is obtained from $A$ by dropping all rows corresponding to constraints of the form $\sum_i \varphi_i(\theta) \geq 1$; the vector $\tilde{b}$ is obtained from $b$ by dropping the corresponding entries. In particular, the matrix $\tilde{A}$ is a submatrix of $A$. Hence, since $A$ is totally unimodular, we conclude that $\tilde{A}$ is totally unimodular. We therefore infer from Theorem E.14 that all extreme points of the set $\{\varphi\colon \tilde{A}\varphi \geq \tilde{b}, \varphi \geq 0\}$ are integral. That is, all extreme points of the set of DIC mechanism with disposal are deterministic. Since $n = 3$, all agents have at least binary types, and at least one agent has non-binary types, we have a contradiction to Theorem D.1. $\square$

We can also give an alternative proof of Lemma E.15 that does not require Theorem D.1. Consider the following characterization of total unimodularity due to Ghouila-Houri (1962); see Korte and Vygen (2018, Theorem 5.25).

**Theorem E.16.** *A matrix $A$ with entries in $\{-1, 0, 1\}$ is totally unimodular if and only if all subsets $C$ of columns of $A$ satisfy the following: There exists a partition of $C$ into subsets $C^+$ and $C^-$ such that for all rows $r$ of $A$ we have*

$$\left(\sum_{c \in C^+} A(r,c) - \sum_{c \in C^-} A(r,c)\right) \in \{-1, 0, 1\}. \tag{E.2}$$

*Alternative proof of Lemma E.15.* Let us, once again, relabel the agents and types such that the type spaces contain the following subsets:

$$\tilde{\Theta}_1 = \{\ell, r\} \quad \text{and} \quad \tilde{\Theta}_2 = \{u, d\} \quad \text{and} \quad \tilde{\Theta}_3 = \{f, c, b\}$$

Fixing an arbitrary type profile $\theta_{-123}$ of agents other than 1, 2, and 3, let us define the type profiles $\{\theta^a, \theta^b, \theta^c, \theta^e, \theta^f, \theta^g\}$ as in (4.2) in Section 4.1. That is, let

$$\begin{aligned} \theta^a = (\ell, d, c, \theta_{-123}), \quad \theta^b &= (r, d, c, \theta_{-123}), \quad \theta^c = (r, d, b, \theta_{-123}), \\ \theta^d &= (r, u, b, \theta_{-123}), \quad \theta^e = (r, u, f, \theta_{-123}), \\ \theta^f &= (\ell, u, f, \theta_{-123}), \quad \theta^g = (\ell, u, c, \theta_{-123}). \end{aligned}$$

Recall that each column of $A$ corresponds to an entry of the form $(i, \theta)$. We will argue that the following set $C$ of columns does not admit a partition in the sense of Theorem E.16.

$$
\begin{aligned}
C = \{&(1, \theta^a), (1, \theta^b), (3, \theta^b), (3, \theta^c), \\
&(2, \theta^c), (2, \theta^d), (3, \theta^d), (3, \theta^e), \\
&(1, \theta^e), (1, \theta^f), (3, \theta^f), (3, \theta^g), \\
&(2, \theta^g), (2, \theta^a)\}
\end{aligned}
$$

Towards a contradiction, suppose $C$ does admit a partition into sets $C^+$ and $C^-$ in the sense of Theorem E.16. Let us assume $(1, \theta^a) \in C^+$, the other case being similar. Note that $\theta^a$ and $\theta^b$ differ only in the type of agent 1. Consider the row of $A$ corresponding to the DIC constraint for agent 1 at these type profiles. By referring to (E.2) for this row, we deduce $(1, \theta^b) \in C^+$. Next, via a similar argument, the constraint that the object is allocated at $\theta^b$ requires $(3, \theta^b) \in C^-$. Continuing in this manner, it is easy to see that $(1, \theta^a)$ must be in $C^-$. Since $(1, \theta^a)$ is assumed to be in $C^+$, we have a contradiction to the assumption that $C^+$ and $C^-$ are a partition of $C$. $\square$

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