ON R-STRONG LIE AND JORDAN IDEALS

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LIST OF SYMBOLS AND NOTATIONS

Notation	Description
$\emptyset$	empty set
$a \in R$	a is an element of R
$A \subseteq B$	A is a subset of B
$\phi : A \to B$	mapping ϕ from A into B
$\phi(a)$	image of element a under ϕ
$a^{-1}, -a$	inverse of a
$Ker\phi$	kernel of ϕ
$H \leq G$	H is a subgroup of G
$\|G\|$	order of G
$\|G : H\|$	index of H in G
$\{e\}$	the trivial subgroup
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	group of integers
$\mathbb{Z}^+$	set of positive integers
$\mathbb{Z}_n$	additive group of integers modulo n
$\sum_{i=1}^{t} \alpha_i$	sum of α_i $(i = 1, 2, \ldots, t)$
$A \cap B$	A intersection B
$\bigcap_{t \in T} V_t$	intersection of sets V_t, $t \in T$
$char R = n$	characteristic of a ring R is n
$[x, y]$	the commutator of x and y which is equal to $xy - yx$ for $x, y \in R$
$\square$	end of proof

Chapter 1

INTRODUCTION

This chapter presents the background of the study, basic definitions, statement of the problem, objectives and significance of the study, methodology, and scope and limitation.

1.1 Background of the Study

In this paper, R will denote a noncommutative ring. A ring R is said to be a prime ring if for $a, b \in R$, $aRb = \{arb \,:\, r \in R\} = (0)$ implies $a = 0$ or $b = 0$ and it is said to be semiprime ring if for $a \in R$, $aRa = \{ara \,:\, r \in R\} = (0)$ implies $a = 0$. A prime ring is a semiprime ring but the converse is not true in general. The notation $[x, y]$ means $xy - yx$, where $x, y \in R$. A Lie ideal U of R is an additive subgroup of R, such that, $[a, r] \in U$ for all $a \in U$ and $r \in R$. A Jordan ideal J of R is an additive subgroup of R, such that, $ur + ru \in J$ for all $u \in J$ and $r \in R$.

A. Verma stated in her paper, "R-strong Jordan ideals" [9], that a Jordan ideal and Lie ideal of R are the same if $charR = 2$. Also, every ideal of R is a Jordan ideal but the converse is not necessarily true [9].

A Lie ideal L of R, is said to be *R-strong Lie ideal* of a semiprime ring

R, if $aub \in L$, for all $u \in L$ and for all $a, b \in R$. A Lie ideal U of R to be a strong lie ideal if $u \in U$ implies $u^3 \in U$. In 1972, Albert Karam discussed different classes of strong lie ideals in his paper entitled "Strong Lie Ideals" [3]. In particular, he characterized R-strong Lie ideals.

A Jordan ideal V of a prime ring R, is said to be *R-strong Jordan ideal* of R, if $avb \in V$, for all $v \in V$ and for all $a, b \in R$. In 2009, Anita Verma introduced R- strong Jordan ideals in her paper entitled "R-strong Jordan ideals" [9] and investigated its properties. This paper will be an exposition and extension to the study of Anita Verma's "R-strong Jordan ideals" [9] and a partial exposition to the study of Albert Karam's "Strong Lie Ideals" [3].

1.2 Basic Definitions

This section introduces some concepts such as rings, subrings, ring homomorphisms and ideals.

Definition 1.2.1 [1] Let G be a nonempty set with a binary operation $*$. Then G or $\langle G, * \rangle$ is called a *group* if the following axioms hold:

i. $a, b \in G$ implies that $a * b \in G$ (closure);

ii. $a, b, c \in G$ implies that $(a * b) * c = a * (b * c)$ (associativity);

iii. there exists an element $e \in G$ such that $e * g = g = g * e$ for all $g \in G$ (existence of an identity element in G);

iv. for every $g \neq e$ in G there exists $g^{-1} \in G$ such that $g * g^{-1} = e = g^{-1} * g$ (existence of inverse element in G).

The element $a * b$ in G will be denoted by ab, the product of a and b.

Definition 1.2.2 [1] A group G is said to be *abelian* if for every $a, b \in G$, $ab = ba$.

Definition 1.2.3 [2] A *ring* is a nonempty set together with two binary operations (usually denoted as addition (+) and multiplication) such that:

i. $(R, +)$ is an abelian group;

ii. $(ab)c = a(bc)$ for all $a, b, c \in R$ (associative multiplication)

iii. $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ (left and right distributive laws).

If in addition:

iv. $ab = ba$ for all $a, b \in R$,

then R is a *commutative ring*. If R contains an element 1_R such that

v. $1_R a = a 1_R = a$ for all $a \in R$,

then R is said to be a *ring with unity*.

Definition 1.2.4 [2] Let R and R' be rings. A function $\phi : R \to R'$ is a *ring homomorphism* provided that for all $a, b \in R$:

$$\phi(a + b) = \phi(a) + \phi(b) \text{ and } \phi(ab) = \phi(a)\phi(b).$$

Definition 1.2.5 [4] Let $\phi : R \to R'$ be a ring homomorphism. The *kernel* of ϕ is the set $Ker\ \phi = \{x \in R : \phi(x) = 0\}$ and the *image* of ϕ is the set $Im\,\phi = \{\phi(x) \in R' : x \in R\}$. The inverse of ϕ is the set $\phi^{-1}(Y) = \{a \in R : \phi(a) \in Y\}$.

Definition 1.2.6 [2] Let R be a ring. If there is a least positive integer n such that $na = 0$ for all $a \in R$. then R is said to have *characteristic* n. If no such n exists R is said to have *characteristic zero*. (Notation: $charR = n$).

Definition 1.2.7 [2] Let R be a ring and S a nonempty subset of R that is closed under the operations of addition and multiplication in R. If S is itself a ring under these operations then S is called a *subring* of R. A subring I of a ring R is a *left ideal* provided

$$r \in R \text{ and } x \in I \Rightarrow rx \in I;$$

I is a *right ideal* provided

$$r \in R \text{ and } x \in I \Rightarrow xr \in I;$$

I is an *ideal* if it is both left and right ideal.

Definition 1.2.8 [2] The *sum* and *product* of ideals are defined as follows. For A and B ideals of a ring R,

$$A + B = \{a + b : a \in A, b \in B\}$$

and

$$AB = \{a_1b_1 + a_2b_2 + \ldots a_nb_n : a_i \in A, b_i \in B, i = 1, 2, ..., n\}$$

1.3 Statement of the Problem

This paper will be an exposition of Anita Verma's paper entitled "R-strong Jordan ideals" [9] and a partial exposition of Albert Karam's paper entitled "Strong Lie Ideals" [3] . It aims to investigate the basic properties of Jordan ideals, Lie ideals, R-Jordan ideals, and R-strong Lie ideals. It also aims to establish new results that will characterize the ring homomorphic properties of R-Jordan ideals and R-strong Lie ideals.

1.4 Objectives of the Study

This study aims to provide the detailed proofs of the following theorems:

1. If V_1 and V_2 are two R-strong Jordan ideals of R, then $V_1 + V_2$ is also a Jordan ideal of R.

2. Let $\{V_t : t \in T$ where T is an indexed set be a family of R-Strong Jordan ideals of R. Then $\bigcap_{t \in T} V_t$ is an R-strong Jordan ideal of R.

3. Let R be a ring with untity. If V_1 and V_2 are R-strong Jordan ideals of R, then V_1V_2 is also an R-strong Jordan ideal of R.

4. If V is a R-strong Jordan ideal of R, then $A_V = \{b \in R : ab + ba \in V$ for all $a \in R$ is a R-strong Jordan ideal of R.

5. If R is a ring with $2R = R$ and V is a R-strong Jordan ideal of R, then $A_V \cap V$ is a non-zero right ideal of R.

6. If L_1 and L_2 are two R-strong Lie ideals of R, then $L_1 + L_2$ is also an R-strong Lie ideal of R.

7. If R is a ring with $2R = R$ and L is a R-strong Lie ideal of R, then $B_L \cap L$ is a two-sided ideal of R.

To further characterized R- strong Jordan ideals and R-strong Lie ideals, the present study aims to prove the following:

1. If V_1, V_2,..., V_n are R-strong Jordan ideals of R, then $\sum_{i=1}^{n} V_i$ is a Jordan ideal of R.

2. Let $\phi : R \to R'$ be a ring homomorphism. If V is a R-strong Jordan ideal of R, then $\phi(V)$ is also a Jordan ideal of R'.

3. Let $\phi : R \to R'$ be a ring homomorphism. If V is a R-strong Jordan ideal of R', then $\phi^{-1}(V) = \{a \in R | \phi(a) \in V\}$ is a R-strong Jordan ideal of R.

4. If $\phi : R \to R'$ is a ring homomorphism, then the kernel of ϕ, $Ker\phi$ is a R-strong Jordan ideal of R.

5. If L_1, L_2,..., L_n are R-strong Lie ideals of R, then $\sum_{i=1}^{n} L_i$ is also a R-Lie ideal of R.

6. Let $\phi : R \rightarrow R'$ be a ring homomorphism. If L is a R-strong Lie ideal of R, then $\phi(L)$ is also a R-strong Lie ideal of R'.

7. Let $\phi : R \rightarrow R'$ be a ring homomorphism. If L is a R-strong Lie ideal of R', then $\phi^{-1}(L) = \{a \in R | \phi(a) \in L\}$ is a Lie ideal of R.

8. If $\phi : R \rightarrow R'$ is a ring homomorphism, then the kernel of ϕ, $Ker\phi$ is a R-strong Jordan ideal of R.

1.5 Significance of the Study

In this paper we will provide detailed proofs on some of the existing theories on R-strong Jordan ideals and R-strong Lie ideals. Also, we will establish new results on the ring homomorphism properties of R-strong Jordan ideals and R-strong Lie ideals. Hence, this study will facilitate researches about R-strong Jordan ideals and R-strong Lie ideals and may interest readers to study further and come up with new applications.

1.6 Methodology

The principles of Group Theory and Ring Theory are used in the proofs of the results in this paper. Definitions and basic concepts which were taken from different references are introduced in Chapter 2. All results that are established in this paper are provided with detailed proofs.

1.7 Scope and Limitation of the Study

This study deals with the basic and ring homomorphism properties of R-strong Jordan ideals and R-strong Lie ideals . It also consider examples like ring of matrices over integers. The rings that are considered in this paper are noncommutative which also includes semiprime and prime rings.

Chapter 2

PRELIMINARIES

This chapter contains some basic definitions and known results that are referred to in the succeeding chapters. Notations and symbols are also established here.

2.1 Definition of Terms and Examples

Definition 2.1.1 [2] Given sets A and B, a *function* (or *map* or *mapping*) ϕ from A to B (written $\phi : A \to B$) assigns to each $a \in A$ exactly one element $b \in B$; b is called the value of the function at a or the *image* of a and is usually written $\phi(a)$. Set A is the *domain* of the function and B is the *range* or *codomain*. Denote the effect of the function ϕ on an element of A by $a \mapsto \phi(a)$.

Definition 2.1.2 [2] A function $\phi : A \to B$ is said to be

i. *injective* (or *one-to-one*) provided for all $a, a' \in A$, $a \neq a' \Rightarrow \phi(a) \neq \phi(a')$.

ii. *surjective* (or *onto*) provided for each $b \in B$, $b = \phi(a)$ for some $a \in A$.

iii. *bijective* (or a *bijection* or a *one-to-one correspondence*) if it is both injective and surjective.

Definition 2.1.3 [1] A *binary operation* $*$ on a set is a rule which assigns to each pair of elements of the set some element of the set and call this rule of assigning as product operation.

Definition 2.1.4 [1] If G is a group, then the *order* $|G|$ of G is the number of elements in G. A group with finite order is called a *finite group*. Otherwise, it is *infinite*.

Definition 2.1.5 [1] Let G be a group and H be a nonempty subset of G that is closed under the binary operation in G. If H is itself a group under the binary operation in G, then H is said to be a *subgroup* of G, denoted by $H \leq G$.

Definition 2.1.6 [1] If G is a group, then the subgroup consisting of G itself is the *improper subgroup* of G. All other subgroups are *proper subgroups*. The subgroup $\{e\}$ is the *trivial subgroup* of G. All other subgroups are *nontrivial*.

Definition 2.1.7 [2] An element a in a ring R is said to be *idempotent* if $a^2 = a$.

2.2 Preliminary Results

Theorem 2.2.1 [4] *Let G be a group with a binary operation $*$ and H be a nonempty subet of G. Then H is a subgroup of G if and only if for all $a, b \in H$, $a * b^{-1} \in H$.*

Theorem 2.2.2 [4] *Let ϕ be a homomorphism of a group G into a group G'. Then*

i. If H is a subgroup of G, then $\phi(H) = \{\phi(h)|h \in H\}$ is a subgoup of G'.

ii. If H_1 is a subgroup of G, then $\phi^{-1}(H_1) = \{g \in G|\phi(g) \in H\}$ is a subgoup of G.

Theorem 2.2.3 [2] *Let R be a ring. Then*

i. $0a = 0a = 0$ for all $a \in R$;

ii. $(-a)b = a(-b) = -(ab)$ for all $a, b \in R$;

iii. $(-a)(-b) = ab$ for all $a, b \in R$;

v. $(\sum_{i=1}^{n} a_i)(\sum_{i=1}^{m} b_i) = \sum_{i=1}^{n}\sum_{i=1}^{m} a_i b_i$ for all $a_i, b_i \in R$.

Theorem 2.2.4 [1] *Let ϕ be a homomorphism of a ring R into a ring R'.*

i If 0 is the additive identity in R, then $\phi(0) = 0'$ is the additive identity in R', and if $a \in R$, then $\phi(-a) = -\phi(a)$.

ii. If S is a subring of R, then $\phi[S]$ is a subring of R'.

iii. If S' is a subring of R', then $\phi^{-1}[S']$ is a subring of R.

Theorem 2.2.5 [1] *Let ϕ be a homomorphism of a ring R into a ring R'. If R has unity 1, then $\phi(1)$ is unity for $\phi[R]$.*

Lemma 2.2.6 [2] *If ϕ is a homomorphism of a ring R into a ring R', then the kernel of ϕ, $Ker\phi$, is a additive subgroup of R.*

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Chapter 3

Results and Discussions

3.1 Jordan Ideals

Definition 3.1.1 [9] Let R be a noncommutative ring. An additive subgroup J of R is said to be a *Jordan ideal* of R if $ur + ru \in J$ for all $u \in J$, $r \in R$.

Example 3.1.2 Let R be a ring such that

$$R = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in \mathbb{Z} \right\}$$

and let

$$J = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{Z}, a \neq 0 \right\}.$$

Now, let

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} \in J,$$

hence

$$\begin{bmatrix} -b & 0 \\ 0 & 0 \end{bmatrix} \in J.$$

Observe that,

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -b & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a-b & 0 \\ 0 & 0 \end{bmatrix} \in J.$$

By Theorem 2.2.1, J is an additive subgroup of the ring R. Furthermore, let

$$X = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in R \text{ and } Y = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in J.$$

Observe that,

$$\begin{aligned} XY + YX &= \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \\ &= \begin{bmatrix} xa & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} ax & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} xa + ax & 0 \\ 0 & 0 \end{bmatrix} \in V \end{aligned}$$

where $xa + ax \neq 0$. Thus, J is a Jordan ideal of R.

Lemma 3.1.3 *Every ideal of R is a Jordan ideal of R.*

Proof: Let J be an ideal of R. Let $u \in J$, then by Definition 1.2.7, $ur \in J$ and $ru \in J$ for all $r \in R$. Since J is an additive subgroup, $ur + ru \in J$. Thus, J is a Jordan ideal. □

Lemma 3.1.4 *If I_1 and I_2 are two ideals of R, then $I_1 + I_2$ is an ideal of R.*

Proof: Let I_1 and I_2 be two ideals of R. Also, let $u_1 \in I_1$ and $u_2 \in I_2$, hence, $u_1 r \in I_1$ and $u_2 r \in I_2$ for all $r \in R$. Thus, $u_1 r + u_2 r \in \mathrm{I}_1 + I_2$. Observe that $u_1 r + u_2 r = (u_1 + u_2) r \in I_1 + I_2$, for all $u_1 \in I_1$, $u_2 \in I_2$ and $r \in R$. Therefore, $I_1 + I_2$ is an ideal of R. □

Lemma 3.1.5 *Let R be a ring with unity and $2R = R$. If J is the Jordan ideal of R and $1 \in J$. Then $J = R$.*

Proof:

Case 1: $J \subseteq R$

Let $u \in J$. Then $u \in R$ since J is an additive subgroup of R. Thus, $J \subseteq R$.

Case 2: $R \subseteq J$

Let $r \in R$. Since $1 \in J$ and J is a Jordan ideal of R then $1r = r = r1$ since R is a ring with unity and by Definition 3.1.1, $1r + r1 = r + r = 2r \in J$. Since $2R = R$ then $2r = r$. This implies that $2r = r \in J$. Thus, $R \subseteq J$. Furthermore, $J = R$. $\square$

Remark 3.1.6 [9] For $x, y \in R$, by $[x, y]$, we mean $xy - yx$.

Lemma 3.1.7 *If J is a Jordan ideal of R and $b \in J$, then $[[x, y], b] \in J$, for all $x, y \in R$.*

Proof: Let J be a Jordan ideal of R and let $b \in J$. Then for $x, y \in R$, $bx + xb \in J$ and $by + yb \in J$. Again, since J is a Jordan ideal, it follows that,

$(bx + xb)y + y(bx + xb) \in J \Rightarrow bxy + xby + ybx + yxb \in J$ **(1)**

and

$(by + yb)x + x(by + yb) \in J \Rightarrow byx + ybx + xby + xyb \in J$ **(2)**.

Getting the difference of **(1)** and **(2)**,

$(bxy + xby + ybx + yxb)$ - $(byx + ybx + xby + xyb)$

the two middle term can be cancelled, hence,

$= (bxy + yxb - byx - xyb)$

regrouping,

$= (bxy - byx - xyb + yxb)$

by factoring,

$= b(xy - yx) - (xy - yx)b$

Note that $[x, y] = (xy - yx)$, thus,

$\Rightarrow b[x, y] - [x, y]b \in J$

furthermore,

$\Rightarrow [[x, y], b] \in J.$ $\square$

Lemma 3.1.8 *If J_1 and J_2 are two Jordan ideals of R, then $J_1 + J_2$ is also a Jordan ideal of R.*

Proof: Suppose J_1 and J_2 are two Jordan ideals of R. Let $j \in J_1 + J_2$, then $j = x + y$ where $x \in J_1$ and $y \in J_2$. Hence, $xr + rx \in J_1$ for all $x \in J_1$ and $r \in R$. Similarly, $yr + ry \in J_2$ for all $y \in J_2$ and $r \in R$. Now.

$$\begin{aligned} jr + rj &= (x + y)r + r(x + y) \\ &= xr + yr + rx + ry \\ &= (xr + rx) + (yr + ry) \in J_1 + J_2 \end{aligned}$$

Thus, $J_1 + J_2$ is a Jordan ideal of R. $\square$

Lemma 3.1.9 *If J_1, J_2,..., J_n are Jordan ideals of R, then $\sum_{i=1}^{n} J_i$ is also a Jordan ideal of R.*

Proof: Suppose J_1, J_2,..., J_n are Jordan ideals of R. Let $j \in \sum_{i=1}^{n} J_i$, then $j = \sum_{i=1}^{n} j_i$ where $j_i \in J_i$, $i = 1, 2, ...n$. Hence, $(j_i)r + r(j_i) \in J_i$ for all $j_i \in J_i$, $i = 1, 2, ..., n$ and $r \in R$. Now,

$$
\begin{aligned}
jr + rj &= (\sum_{i=1} j_i)r + r(\sum_{i=1} j_i) \\
&= (j_1 + j_2 + \cdots + j_n)r + r(j_1 + j_2 + \cdots + j_n) \\
&= (j_1 r + j_2 r + \cdots + j_n r) + (r j_1 + r j_2 + \cdots + r j_n) \\
&= (j_1 r + r j_1) + (j_2 r + r j_2) + \cdots + (j_n r + r j_n) \\
&\in J_1 + J_2 + \cdots + J_n \\
&= \sum_{i=1}^{n} J_i.
\end{aligned}
$$

Thus, $\sum_{i=1}^{n} J_i$ is a Jordan ideal of R. $\square$

Lemma 3.1.10 *Let R be a ring with untity. If J_1 and J_2 are Jordan ideals of R, then $V_1 V_2$ is also Jordan ideal of R.*

Lemma 3.1.11 *Let $\phi : R \rightarrow R'$ be a ring homomorphism. If J is a Jordan ideal of R, then $\phi(J)$ is also a Jordan ideal of R.*

Proof: Suppose $\phi : R \to R'$ is a ring homomorphism. Let J be a Jordan ideal of R. Let $a \in J$, then $ar + ra \in J$ for all $a \in J$ and $r \in R$. Hence, $\phi(ar + ra) \in \phi(J)$. Now, since ϕ is a homomorphism,

$$\begin{aligned} \phi(ar + ra) &= \phi(ar) + \phi(ra) \\ &= \phi(a)\phi(r) + \phi(r)\phi(a). \end{aligned}$$

Hence, $\phi(a)\phi(r) + \phi(r)\phi(a) \in \phi(J)$ for all $\phi(a) \in \phi(J)$ and $\phi(r) \in \phi(R)$. Note that $\phi(R) \subseteq R'$, hence $\phi(r) \in R'$. Also, by Theorem 2.2.2, $\phi(J)$ is an additive subgroup of R'. Thus, $\phi(J)$ is a Jordan ideal of R'. $\square$

Lemma 3.1.12 *Let $\phi : R \to R'$ be a ring homomorphism. If J is a Jordan ideal R', then $\phi^{-1}(J) = \{a \in R | \phi(a) \in J\}$ is also a Jordan ideal of R.*

Proof: Suppose $\phi : R \to R'$ is a ring homomorphism. Let J be a Jordan ideal of R'. Let $a \in \phi^{-1}(J)$ and $r \in R$. Hence, $\phi(a) \in J$ and $\phi(a)\phi(r)+\phi(r)\phi(a) \in J$ for all $\phi(r) \in R'$. Now, since ϕ is a homomorphism,

$$\begin{aligned} \phi(a)\phi(r) + \phi(r)\phi(a) &= \phi(ar) + \phi(ra) \\ &= \phi(ar + ra) \end{aligned}$$

Thus, $\phi(ar + ra) \in J$ which implies that $ar + ra \in \phi^{-1}(J)$. Also, by Theorem 2.2.2 $\phi^{-1}(J)$ is an additive subgroup of R. Thus, $\phi^{-1}(J)$ is a Jordan ideal of R. $\square$

Lemma 3.1.13 *If $\phi : R \to R'$ is a ring homomorphism, then the kernel of ϕ, $Ker\phi$ is a Jordan ideal of R.*

Proof: Let $\phi : R \to R'$ be a ring homomorphism. Let $Ker\phi$ be the kernel of ϕ. By Lemma 2.2.6, $Ker\phi$ is an additive subgroup of R. Let $a \in Ker\phi$ and $r \in R$. Now, since ϕ is a ring homomorphism,

$$\begin{aligned}\phi(ar + ra) &= \phi(ar) + \phi(ra)\\ &= \phi(a)\phi(r) + \phi(r)\phi(a)\\ &= (0)\phi(r) + \phi(r)(0)\\ &= 0 + 0\\ &= 0 \in R'.\end{aligned}$$

Hence, $ar + ra \in Ker\phi$. Thus, $Ker\phi$ is a Jordan ideal. □

3.2 R-Strong Jordan Ideals

In this section, by ring R we mean a prime ring.

Definition 3.2.1 [9] A ring R is said to be *prime* if for $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$.

Definition 3.2.2 [9] Let R be a prime ring. A Jordan ideal V of R, is said to be *R-strong Jordan ideal* of R, if $avb \in V$, for all $v \in V$ and for all $a, b \in R$.

Example 3.2.3 Consider the ring

$$R = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in \mathbb{Z} \right\}$$

and the Jordan ideal

$$V = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{Z}, a \neq 0 \right\}$$

of R from Example 1.

Let

$$X = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in R$$

and

$$Y = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in V.$$

Observe that

$$\begin{aligned} XYX &= \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \\ &= \begin{bmatrix} xa & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \\ &= \begin{bmatrix} xax & 0 \\ 0 & 0 \end{bmatrix} \in V. \end{aligned}$$

Hence, V is a R-strong Jordan ideal of R.

Remark 3.2.4 *If V is a R-strong Jordan ideal of R, then V is a Jordan ideal of R.*

Remark 3.2.5 *If V_1 and V_2 are two R-strong Jordan ideals of R, then V_1+V_2 is also a Jordan ideal of R.*

Theorem 3.2.6 *If V_1 and V_2 are two R-strong Jordan ideals of R, then V_1+V_2 is also an R-strong Jordan ideal of R.*

Proof: Suppose V_1 and V_2 are two R-strong Jordan ideals of R. Let $x \in V_1+V_2$, then $x = y + z$ where $y \in V_1$ and $z \in V_2$. Hence, $ayb \in V_1$ for all $y \in V_1$ and $a, b \in R$. Similarly, $azb \in V_2$ for all $z \in V_2$ and $a, b \in R$. Now,

$$\begin{aligned} axb &= a(y+z)b \\ &= (ay+az)b \\ &= ayb + azb \in V_1 + V_2. \end{aligned}$$

By Remark 3.2.5, $V_1 + V_2$ is a Jordan ideal of R. Thus, $V_1 + V_2$ is an R-strong Jordan ideal of R since $axb \in V_1 + V_2$. □

Remark 3.2.7 *If V_1, V_2,..., V_n are R-strong Jordan ideals of R, then $\sum_{i=1}^{n} V_i$ is a Jordan ideal of R.*

Theorem 3.2.8 *If V_1, V_2,..., V_n are R-strong Jordan ideals of R, then $\sum_{i=1}^{n} V_i$ is also a R-Jordan ideal of R.*

Proof: Suppose V_1, V_2,..., V_n are R-strong Jordan ideals of R. Let $x \in \sum_{i=1}^{n} V_i$, then $x = \sum_{i=1}^{n} v_i$ where $v_i \in V_i$, $i = 1, 2, ...n$. Hence, $av_ib \in V_i$ for all $v_i \in V_i$, $i = 1, 2, ..., n$ and $a, b \in R$. Now,

$$\begin{aligned}
axb &= a(\sum_{i=1}^{n} v_i)b \\
&= a(v_1 + v_2 + \cdots + v_n)b \\
&= (av_1 + av_2 + \cdots + av_n)b \\
&= av_1b + av_2b + \cdots + av_nb \\
&\in V_1 + V_2 + \cdots + V_n \\
&= \sum_{i=1}^{n} V_i.
\end{aligned}$$

By Remark 3.2.7, $\sum_{i=1}^{n} V_i$ is a Jordan ideal of R. Thus $\sum_{i=1}^{n} V_i$ is a R-strong Jordan ideal of R. $\square$

Theorem 3.2.9 *Let* $\{V_t : t \in T$ *where* T *is an indexed set be a family of R-Strong Jordan ideals of* R*. Then* $\bigcap_{t\in T} V_t$ *is an R-strong Jordan ideal of* R*.*

Proof: Let $V = \bigcap_{t\in T} V_t$. Let $x \in V$ and $a, b \in R$. Since $x \in V$, $x \in V_t$, for all $t \in T$. Now, $x \in V_t$ and V_t is R-strong Jordan ideal, therefore $axb \in V_t$, for all $t \in T$. Hence, $axb \in \bigcap_{t\in T} = V$. $\square$

Remark 3.2.10 *The union of two R-strong Jordan ideals need not be an R-strong need not be an R-strong Jordan ideal.*

Theorem 3.2.11 *Let R be a ring with untity. If V_1 and V_2 are R-Jordan ideals of R, then V_1V_2 is also an R-strong Jordan ideal of R.*

Proof: Let V_1 and V_2 be R-strong Jordan ideals of R. Let $x \in V_1V_2$ and $r \in R$. Then $x = \sum_{i=1}^{n} a_ib_i$, $a_i \in V_1$, $b_i \in V_2$. Now, $a_i \in V_1$, $r \in R$ and V_1 is R-strong Jordan ideal, therefore $ra_ir \in V_1$, $i = 1, 2, ..., n$. Similarly, $b_i \in V_1$, $r \in R$ and V_2 is R-strong Jordan ideal. Therefore $rb_ir \in V_2$, $i = 1, 2, ..., n$. Observe that,

$$\begin{aligned}
r(\sum_{i=1}^{n} a_ib_i)r &= r(\sum_{i=1}^{n} a_ib_i)r + \sum_{i=1}^{n}(ra_ir \cdot rb_ir) - \sum_{i=1}^{n}(ra_ir \cdot rb_ir) + \sum_{i=1}^{n}(ra_irb_ir) - \sum_{i=1}^{n}(ra_irb_ir) \\
&= \sum_{i=1}^{n}(ra_ib_ir) + \sum_{i=1}^{n}(ra_ir \cdot rb_ir) - \sum_{i=1}^{n}(ra_ir \cdot rb_ir) + \sum_{i=1}^{n}(ra_irb_ir) - \sum_{i=1}^{n}(ra_irb_ir) \\
&= \sum_{i=1}^{n}(ra_ir \cdot rb_ir) + \left[\sum_{i=1}^{n}(ra_ib_ir) + \sum_{i=1}^{n}(ra_irb_ir) - \sum_{i=1}^{n}(ra_irb_ir) - \sum_{i=1}^{n}(ra_ir \cdot rb_ir)\right] \\
&= \sum_{i=1}^{n}(ra_ir \cdot rb_ir) + \sum_{i=1}^{n}(ra_irb_ir + ra_ib_ir - ra_irb_ir - ra_irrb_ir) \\
&= \sum_{i=1}^{n}(ra_ir \cdot rb_ir) + \sum_{i=1}^{n}(ra_i - ra_ir)(b_ir + rb_ir)
\end{aligned}$$

Note that $r_1a_ir_2 \in V$ for all $r_1, r_2 \in R$ and $i = 1, 2, ..., n$. Taking $r_1 = r - 1$, $r_2 = r$, we get

$$\begin{aligned}
r_1a_ir_2 &= (r-1)(a_i)(r) \\
&= (ra_i - a_i)(r) \\
&= ra_ir - a_ir
\end{aligned}$$

Hence, $(ra_ir - a_ir) \in V_1$ since $r_1a_ir_2 \in V_1$, $i = 1, 2, ..., n$. Similarly, since $r_1b_ir_2 \in V_2$ for all r_1, $r_2 \in R$ and $i = 1, 2, ..., n$. Taking $r_1 = 1 + r$, $r_2 = r$, we get

$$r_1 b_i r_2 = (1+r)(b_i)(r)$$
$$= (b_i + rb_i)r$$
$$= b_i r + rb_i r.$$

Hence, $(b_i r + rb_i r) \in V_2$ since $r_1 b_i r_2 \in V_2$, $i = 1, 2, ..., n$. So $\sum_{i=1}^{n}(ra_i - ra_i r)(b_i r + rb_i r) \in V_1 V_2$. Now, since $\sum_{i=1}^{n}(ra_i r \cdot rb_i r) \in V_1 V_2$ and $\sum_{i=1}^{n}(ra_i - ra_i r)(b_i r + rb_i r) \in V_1 V_2$. Then $r(\sum_{i=1}^{n} a_i b_i)r \in V_1 V_2$. Also, since $V_1 V_2$ is a Jordan ideal by Theorem 3.1.10, hence $V_1 V_2$ is a R-strong Jordan ideal. □

Theorem 3.2.12 *Let V be a R-strong Jordan ideal of R, where R is a ring with unity. If $v \in V$ and $a, b \in R$, then $abv + vba \in V$.*

Proof: Suppose V is a R-strong Jordan ideal of R, where R is a ring with unity. Let $v \in V$ and a, $b \in R$. Also, let avb and $bva \in V$. Then,

$$avb + bva \in V \tag{3.1}$$

Now, since V is a Jordan ideal, $a(avb + bva) + (avb + bva)a = aavb + abva + avba + bvaa \in V$. Therefore, by (3.1), $avba + bvaa \in V$. Replacing a by $(a-1)$, we get $(avb - vb)(a-1) + (abv - bv)(a-1) \in V$. Hence,

$$(avb - vb)(a-1) + (abv - bv)(a-1)$$

$$= (avb)(a-1) - (vb)(a-1) + (abv)(a-1) - (bv)(a-1)$$

$$= avba - avb - vba + vb + abva - abv - bva + bv \in V$$

Note that by (3.1), $avba + abva \in V$ and $-avb - bva \in V$. Also, since V is a Jordan ideal, $vb + bv \in V$ where $v \in V$ and $b \in R$. Thus, $-vba - abv \in V$. Since V is an additive subgroup, $vba + abv \in V$. $\square$

Let V be a R-strong Jordan ideal of R. If $a, b \in R$, we associate V with the set $A_v = \{b \in R : ab + ba \in V$ for all $a \in R$.

Theorem 3.2.13 *If V is a R-strong Jordan ideal of R, then A_V is a R-strong Jordan ideal of R.*

Proof: Suppose V is a R-strong Jordan ideal of R. Let $x \in A_V$ and $r \in R$. Since $x \in A_V$,$xr + rx \in V$. Also, since V is a Jordan ideal of R, $(xr+rx)y + y(xr+rx) \in V$ for all $y \in R$. Hence $xr + rx \in A_V$. Thus, A_V is a Jordan ideal of R. Let $b \in A_V$, $x, y \in R$. Since $b \in A_V$, $x, y, \in R$, $xb + bx$, $yb + by$ $\in V$. Since V is a R-strong Jordan ideal, $y(by + yb)y \in V$. This implies that $yby^2 + y^2by \in V$. Similarly, $x(by + yb)y \in V$ and $y(by + yb)x \in V$. Which implies that, $xby^2 + yby \in V$ and $ybyx + y^2bx \in V$ respectively. Note that

$yby^2 + y^2by \in V$, hence, $x(yby) + (yby)x \in V$ and so $yby \in A_V$. Thus, A_V is R-strong Jordan ideal. $\square$

Theorem 3.2.14 *If R is a ring with $2R = R$ and V is a R-strong Jordan ideal of R, then $A_V \cap V$ is a non-zero right ideal of R.*

Proof: Suppose V is a R-strong Jordan ideal of R. Hence V is a Jordan ideal of R. Let $2R = R$ and also let $a \in A_V$, then $ab + ba \in V$ for all $a \in R$. Since V is a Jordan ideal, then $a \in V$. Thus, $A_V \cap V \neq \emptyset$. Let $b \in A_V \cap V$, $x, y \in R$. Then $bx + xb \in V$. So $bx + xb \in A_V$. Hence, $bx + xb \in A_V \cap V$. Now,

$$\begin{aligned} xb + bx &= xb + bx + xb - xb \\ &= bx - xb + 2xb \\ &= bx - xb + xb, \text{ since } (2R = R) \\ &= bx \in A_V \cap V. \end{aligned}$$

Since $x \in R$ is arbitrary, $bx \in A_V \cap V$, for all $x \in R$. Hence, $A_V \cap V$ is a nonzero right ideal of R. $\square$

Theorem 3.2.15 *If e is an idempotent and V is a Jordan ideal of R, then eVe is an eRe-strong Jordan ideal of R.*

Remark 3.2.16 *Let $\phi : R \to R'$ be a ring homomorphism. If V is a R-strong Jordan ideal of R, then $\phi(V)$ is also a Jordan ideal of R'.*

Theorem 3.2.17 *Let $\phi : R \to R'$ be a ring homomorphism. If V is a R-strong Jordan ideal of R, then $\phi(V)$ is also a R-strong Jordan ideal of R'.*

Proof: Supppose $\phi : R \to R'$ is a ring homomorphism. Let V be a R-strong Jordan ideal of R. Let $v \in V$, then $avb \in V$ for all $a, b \in R$. Hence, for $\phi(v) \in \phi(V)$, $\phi(avb) \in \phi(V)$ for all $\phi(a)$ and $\phi(b) \in \phi(R)$. Now, since ϕ is a homomorphism,

$$\begin{aligned}\phi(avb) &= \phi(av)\phi(b)\\ &= \phi(a)\phi(v)\phi(b)\end{aligned}$$

Thus, $\phi(a)\phi(v)\phi(b) \in \phi(V)$ for all $\phi(v) \in \phi(V)$ and $\phi(a)$ and $\phi(b) \in \phi(R)$. Note that $\phi(R) \subseteq R'$, hence, $\phi(a)$, $\phi(b) \in R'$. Also, by Remark 3.2.16 $\phi(V)$ is a Jordan ideal of R'. Therefore, $\phi(V)$ is an R-strong Jordan ideal of R'. □

Remark 3.2.18 *Let $\phi : R \to R'$ be a ring homomorphism. If V is a R-strong Jordan ideal of R', then $\phi^{-1}(V) = \{a \in R | \phi(a) \in V\}$ is a Jordan ideal of R.*

Theorem 3.2.19 *Let $\phi : R \to R'$ be a ring homomorphism. If V is a R-strong Jordan ideal of R', then $\phi^{-1}(V) = \{a \in R | \phi(a) \in V\}$ is a R-strong Jordan ideal of R.*

Proof: Suppose $\phi : R \to R'$ be a ring homomorphism. Let V be a R-strong Jordan ideal of R'. Let $v \in \phi^{-1}(V)$ and $a, b \in R$. Hence, $\phi(v) \in V$ and

$\phi(a)\phi(v)\phi(b) \in\in V$ for all $\phi(a)$, $\phi(b) \in R'$. Now, since ϕ is a homomorphism,

$$\begin{aligned}\phi(a)\phi(v)\phi(b) &= \phi(av)\phi(b)\\ &= \phi(avb)\end{aligned}$$

Thus, $\phi(avb) \in V$ which implies that $avb \in \phi^{-1}(V)$. Also by Remark 3.2.18 $\phi^{-1}(V)$ is a Jordan ideal of R. Therefore, $\phi^{-1}(V)$ is a R-strong Jordan ideal of R. $\square$

Theorem 3.2.20 *If $\phi : R \to R'$ is a ring homomorphism, then the kernel of ϕ, $Ker\phi$ is a R-strong Jordan ideal of R.*

Proof: Suppose $\phi : R \to R'$ is a ring homomorphism. Let $x \in Ker\phi$ and $r \in R$. Now, since ϕ is a ring homomorphism,

$$\begin{aligned}\phi(rxr) &= \phi(r)\phi(x)\phi(r)\\ &= (0)\phi(x)(0)\\ &= 0 \in R'.\end{aligned}$$

Hence, $rxr \in Ker\phi$. By Theorem 3.1.13, $Ker\phi$ is a Jordan ideal. Thus $Ker\phi$ is a R-strong Jordan ideal. $\square$

3.3 Lie ideals and R-strong Lie ideals

In this section, by ring R we mean a semiprime ring.

Definition 3.3.1 [9] A ring R is said to be *semiprime* if for $a \in R$, $aRa = (0)$ implies $a = 0$.

Definition 3.3.2 [9] Let R be a noncommutative ring. An additive subgroup U of R is said to be a *Lie ideal* of R if $[a, r], \in U$, that is $ar - ra \in U$, for all $a \in U$, $r \in R$.

Lemma 3.3.3 *If U_1 and U_2 are two Lie ideals of R, then $U_1 + U_2$ is also a Lie ideal of R.*

Proof: Suppose U_1 and U_2 are two Lie ideals of R. Let $a \in U_1 + U_2$, then $u = x + y$ where $x \in U_1$ and $y \in U_2$. Hence, $xr - rx \in U_1$ for all $x \in U_1$ and $r \in R$. Similarly, $yr - ry \in J_2$ for all $y \in U_2$ and $r \in R$. Now.

$$
\begin{aligned}
ur - ru &= (x + y)r - r(x + y) \\
&= xr + yr - rx - ry \\
&= (xr - rx) + (yr - ry) \in J_1 + J_2
\end{aligned}
$$

Thus, $U_1 + U_2$ is a Lie ideal of R. $\square$

Lemma 3.3.4 *If U_1, U_2,..., U_n are Lie ideals of R, then $\sum_{i=1}^{n} U_i$ is also a Lie ideal of R.*

Proof: Suppose U_1, U_2,..., U_n are Jordan ideals of R. Let $u \in \sum_{i=1}^{n} U_i$, then $u = \sum_{i=1}^{n} u_i$ where $u_i \in U_i$, $i = 1, 2, ...n$. Hence, $(u_i)r - r(u_i) \in U_i$ for all

$u_i \in U_i$, $i = 1, 2, ..., n$ and $r \in R$. Now,

$$\begin{aligned} jr + rj &= (\sum_{i=1}^{n} u_i)r - r(\sum_{i=1}^{n} u_i) \\ &= (u_1 + u_2 + \cdots + u_n)r - r(u_1 + u_2 + \cdots + u_n) \\ &= (u_1 r + u_2 r + \cdots + u_n r) - (ru_1 + ru_2 + \cdots + ru_n) \\ &= (u_1 r - ru_1) + (u_2 r - ru_2) + \cdots + (u_n r - ru_n) \\ &\in U_1 + U_2 + \cdots + U_n \\ &= \sum_{i=1}^{n} U_i. \end{aligned}$$

Thus, $\sum_{i=1}^{n} U_i$ is a Lie ideal of R. □

Lemma 3.3.5 *Let $\phi : R \to R'$ be a ring homomorphism. If U is a Lie ideal of R, then $\phi(U)$ is also a Lie ideal of R.*

Proof: Suppose $\phi : R \to R'$ is a ring homomorphism. Let U be a Lie ideal of R. Let $a \in U$, then $ar - ra \in J$ for all $a \in U$ and $r \in R$. Hence, $\phi(ar - ra) \in \phi(J)$. Now, since ϕ is a homomorphism,

$$\begin{aligned} \phi(ar - ra) &= \phi(ar + (-ra)) \\ &= \phi(ar) + \phi(-ra) \\ &= \phi(ar) - \phi(ra) \\ &= \phi(a)\phi(r) + \phi(r)\phi(a). \end{aligned}$$

Hence, $\phi(a)\phi(r) + \phi(r)\phi(a) \in \phi(U)$ for all $\phi(a) \in \phi(U)$ and $\phi(r) \in \phi(R)$. Note that $\phi(R) \subseteq R'$, hence $\phi(r) \in R'$. Also, by Theorem 2.2.2, $\phi(U)$ is an additive subgroup of R'. Thus, $\phi(U)$ is a Lie ideal of R'. $\square$

Lemma 3.3.6 *Let $\phi : R \to R'$ be a ring homomorphism. If U is a Lie ideal R', then $\phi^{-1}(U) = \{a \in R | \phi(a) \in J\}$ is also a Lie ideal of R.*

Proof: Suppose $\phi : R \to R'$ is a ring homomorphism. Let U be a Lie ideal of R'. Let $a \in \phi^{-1}(U)$ and $r \in R$. Hence, $\phi(a) \in U$ and $\phi(a)\phi(r) + \phi(r)\phi(a) \in U$ for all $\phi(r) \in R'$. Now, since ϕ is a homomorphism,

$$\begin{aligned} \phi(a)\phi(r) - \phi(r)\phi(a) &= \phi(ar) - \phi(ra) \\ &= \phi(ar) + \phi(-ra) \\ &= \phi(ar + (-ra)) \\ &= \phi(ar - ra) \end{aligned}$$

Thus, $\phi(ar - ra) \in U$ which implies that $ar - ra \in \phi^{-1}(U)$. Also, by Theorem 2.2.2 $\phi^{-1}(U)$ is an additive subgroup of R. Thus, $\phi^{-1}(U)$ is a Lie ideal of R. $\square$

Lemma 3.3.7 *If $\phi : R \to R'$ is a ring homomorphism, then the kernel of ϕ, $Ker\phi$ is a Lie ideal of R.*

Proof: Let $\phi : R \to R'$ be a ring homomorphism. Let $Ker\phi$ be the kernel of ϕ. By Lemma 2.2.6, $Ker\phi$ is an additive subgroup of R. Let $a \in Ker\phi$ and

$r \in R$. Now, since ϕ is a ring homomorphism,

$$\begin{aligned}\phi(ar - ra) &= \phi(ar) + \phi(-ra)\\ &= \phi(ar) - \phi(ra)\\ &= \phi(a)\phi(r) - \phi(r)\phi(a)\\ &= (0)\phi(r) - \phi(r)(0)\\ &= 0 - 0\\ &= 0 \in R'.\end{aligned}$$

Hence, $ar - ra \in Ker\phi$. Thus, $Ker\phi$ is a Lie ideal. $\square$

Definition 3.3.8 [3] Let R be a semiprime ring such that $2R = R$. A Lie ideal L of R, is said to be *R-strong Lie ideal* of R, if $aub \in L$, for all $u \in L$ and for all $a, b \in R$.

Let L be a R-strong Lie ideal of R. If $a, b \in R$, we associate L with the set $B_L = \{b \in R : ab + ba \in R\}$. This set is a Lie ideal of R and $u^2 \in B_L$ for all $u \in L$.

Lemma 3.3.9 *If L is a R-strong Lie ideal of R, then*

i. B_L is a R-strong Lie ideal

ii. $u^2xu^2 \in B_L \cap L$ for all $u \in L$, $x \in R$.

Theorem 3.3.10 *Let $C = B_L \cap L$, Then C is a nonzero two-sided ideal.*

Remark 3.3.11 *If L_1 and L_2 are two R-strong Lie ideals of R, then $L_1 + L_2$ is also a Lie ideal of R.*

Theorem 3.3.12 *If L_1 and L_2 are two R-strong Lie ideals of R, then $L_1 + L_2$ is also an R-strong Lie ideal of R.*

Proof: Suppose L_1 and L_2 are two R-strong Lie ideals of R. Let $x \in L_1 + L_2$, then $x = y + z$ where $y \in L_1$ and $z \in L_2$. Hence, $ayb \in L_1$ for all $y \in L_1$ and $a, b \in R$. Similarly, $azb \in L_2$ for all $z \in L_2$ and $a, b \in R$. Now,

$$\begin{aligned} axb &= a(y+z)b \\ &= (ay + az)b \\ &= ayb + azb \in L_1 + L_2. \end{aligned}$$

By Remark 3.3.11, $L_1 + L_2$ is a Lie ideal of R. Thus, $L_1 + L_2$ is an R-strong Lie ideal of R since $axb \in L_1 + L_2$. □

Remark 3.3.13 *If L_1, L_2,..., L_n are R-strong Lie ieals of R, then $\sum_{i=1}^{n} L_i$ is a Lie ideal of R.*

Theorem 3.3.14 *If L_1, L_2,..., L_n are R-strong Lie ideals of R, then $\sum_{i=1}^{n} L_i$ is also a R-Lie ideal of R.*

Proof: Suppose L_1, L_2,..., L_n are R-strong Lie ideals of R. Let $x \in \sum_{i=1}^{n} L_i$, then $u = \sum_{i=1}^{n} u_i$ where $u_i \in L_i$, $i = 1, 2, ...n$. Hence, $au_ib \in L_i$ for all $u_i \in L_i$, $i = 1, 2, ..., n$ and $a, b \in R$. Now,

$$\begin{aligned}
aub &= a(\sum_{i=1}^{n} u_i)b \\
&= a(u_1 + u_2 + \cdots + u_n)b \\
&= (au_1 + au_2 + \cdots + au_n)b \\
&= au_1b + au_2b + \cdots + au_nb \\
&\in L_1 + L_2 + \cdots + L_n \\
&= \sum_{i=1}^{n} L_i.
\end{aligned}$$

By Remark 3.3.13, $\sum_{i=1}^{n} L_i$ is a Lie ideal of R. Thus $\sum_{i=1}^{n} L_i$ is a R-strong Lie ideal of R. $\square$

Remark 3.3.15 *Let $\phi : R \to R'$ be a ring homomorphism. If L is a R-strong Lie ideal of R, then $\phi(L)$ is also a Lie ideal of R'.*

Theorem 3.3.16 *Let $\phi : R \to R'$ be a ring homomorphism. If L is a R-strong Lie ideal of R, then $\phi(L)$ is also a R-strong Lie ideal of R'.*

Proof: Supppose $\phi : R \to R'$ is a ring homomorphism. Let L be a R-strong Lie ideal of R. Let $u \in L$, then $aub \in L$ for all $a, b \in R$. Hence, for $\phi(u) \in \phi(L)$, $\phi(aub) \in \phi(L)$ for all $\phi(a)$ and $\phi(b) \in \phi(R)$. Now, since ϕ is a homomorphism,

$$\begin{aligned}\phi(aub) &= \phi(au)\phi(b)\\ &= \phi(a)\phi(u)\phi(b)\end{aligned}$$

Thus, $\phi(a)\phi(u)\phi(b) \in \phi(L)$ for all $\phi(u) \in \phi(L)$ and $\phi(a)$ and $\phi(b) \in \phi(R)$. Note that $\phi(R) \subseteq R'$, hence, $\phi(a)$, $\phi(b) \in R'$. Also, by Remark 3.3.15 $\phi(L)$ is a Jordan ideal of R'. Therefore, $\phi(L)$ is an R-strong Lie ideal of R'. $\square$

Remark 3.3.17 *Let $\phi : R \to R'$ be a ring homomorphism. If L is a R-strong Lie ideal of R', then $\phi^{-1}(L) = \{a \in R | \phi(a) \in L\}$ is a Lie ideal of R.*

Theorem 3.3.18 *Let $\phi : R \to R'$ be a ring homomorphism. If L is a R-strong Lie ideal of R', then $\phi^{-1}(L) = \{a \in R | \phi(a) \in L\}$ is a R-strong Lie ideal of R.*

Proof: Suppose $\phi : R \to R'$ be a ring homomorphism. Let L be a R-strong Lie ideal of R'. Let $u \in \phi^{-1}(L)$ and $a, b \in R$. Hence, $\phi(u) \in L$ and $\phi(a)\phi(L)\phi(b) \in\in L$ for all $\phi(a)$, $\phi(b) \in R'$. Now, since ϕ is a homomorphism,

$$\begin{aligned}\phi(a)\phi(u)\phi(b) &= \phi(au)\phi(b)\\ &= \phi(aub)\end{aligned}$$

Thus, $\phi(aub) \in L$ which implies that $aub \in \phi^{-1}(L)$. Also by Remark 3.3.17 $\phi^{-1}(L)$ is a Lie ideal of R. Therefore, $\phi^{-1}(L)$ is a R-strong Lie ideal of R. $\square$

Theorem 3.3.19 *If $\phi : R \to R'$ is a ring homomorphism, then the kernel of ϕ, $Ker\phi$ is a R-strong Jordan ideal of R.*

Proof: Suppose $\phi : R \to R'$ is a ring homomorphism. Let $a \in Ker\phi$ and $r \in R$. Now, since ϕ is a ring homomorphism,

$$\begin{aligned} \phi(rar) &= \phi(r)\phi(a)\phi(r) \\ &= (0)\phi(a)(0) \\ &= 0 \in R'. \end{aligned}$$

Hence, $rar \in Ker\phi$. By Theorem 3.3.7, $Ker\phi$ is a Lie ideal. Thus $Ker\phi$ is a R-strong Lie ideal. $\square$