Series Representations of Pi: Math Seminar 2014

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Abstract

In this paper I will examine series aspects of estimates of π. Namely,

1. Gregory's Series
2. Machin's Formula
3. Takano's Series
4. An examination of rates of convergence of the series
5. Proof of and use of Euler's Summation formula
6. The Zeta function
7. Probabilistic aspects

The items that were not covered in class, or were considered as items "additonal" to the minimum list will be enoted with a *.

1 Gregory's Series

Gregory's series was discovered by Scottish mathematician James Gregory in 1671[1]. The series is also known as the Gregory Liebniz series as it was also discovered, independently by Liebniz. This discovery preceeded the development of calculus and was developed by a growing trend toward the study of power series. Gregory seems to have been influenced by a mathematician named Mercator's proof that

$$log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ...$$

Later, in Janurary 1671, Gregory sent a letter to a mathematician named Collins[1] that may have been the beginning of his discovery of the series approximation of π in which he seems to have been working out the Taylor series for $\tan x$. Leibniz was heavily influenced by Blaise Pascal and his work with infinitesimal triangles, and this work, along with Mercator's influence, led to his approximation of π as an infinite series. This same series was discovered independenty by Indian mathematician Kerala Gargya Nilakantha. Although Gregory's series was an ingenious break through in the approximation of π It's convergence was too slow. I will use modern mathematics to examine Gregory's Series.[11]

Proof. We begin our proof of the power series expansion of the inverse tangent function by noting that

$$\text{If } y = \tan^{-1} x \to y' = \frac{1}{1+x^2}$$

this gives

$$\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\, dt$$

Recall that a power series expansion is given by

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ with } |x| < 1.$$

We can rewrite

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

and so

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

from this we obtain

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 \dots$$

Recall the radius of convergence for a power series is $|x| < 1$ so we have

$$|-x^2| < 1 \to |x|^2 < 1 \to |x| < 1$$

Now,

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

and since our series is convergent by the geometric series test, we may integrate term by term

$$\int \frac{1}{1+x^2}\, dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\, dx$$

and so we have

$$\int \frac{1}{1+x^2}\, dx = \int 1 - x^2 + x^4 - x^6 + x^8 \dots$$

Upon integrating we obtain (while suppressing the constant of integration)

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} \dots$$

This says

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \text{ when } |x| < 1.$$

The radius of convergence is given by

$$|x| < 1$$

and checking the end points via the alternating series test the interval of convergence is

$$-1 \leq x \leq 1$$

$$\tan^{-1}(1) = \frac{\pi}{4} \text{ and } \tan^{-1}(-1) = -\frac{\pi}{4}$$

which gives the interesting series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \ldots$$

$\square$

The next step in approximating π is to multiply by 4. In doing so we obtain

$$\pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$$

Sir Isaac Newton commented that this would have been enough to prove that Gregory was among the most gifted mathematicians of his time even if he had never done any other mathematics[1].By the Alternating Series Test [12], we can examine the partial sums to determine what sorts of rational approximations of π we can obtain. This shows that while this series is a first, powerful step in the right direction, it is not "efficient" in yielding approximations of π. An inspection of the rates of convergence of this and the following Machin's formula can be found in section 3.

2 Machin's Formula

John Machin (1680-1752),in 1706, used what have become known as "Machin Formulas" to find 100 decimals places of π. [2] There are two well-known derivations of Machin's Formula. One of them involves complex numbers and the other involves trigonometric identities. Machin's Formulas is

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} \tag{1}$$

Machin's formula attempts to create a series for π that converges very quickly.He was succesful. First we will look at a derivation that involves complex numbers.

Complex numbers are number of the form [8]

$$a + bi \mid a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}$$

In the complex plane, a is the horizontal component and bi the vertical component of any point. We can find the angle swept out from the horizontal axis to any point by first observing the relation

$$\tan\theta = \frac{b}{a} \tag{2}$$

From this point, it can be seen that

$$\tan^{-1}\frac{b}{a} = \theta$$

We know from trigonometry that

$$tan^{-1}\frac{b}{a} = 1 \text{ whenever } a = b$$

This is because

$$\tan\frac{\pi}{4} = 1 \rightarrow \frac{\pi}{4} = \tan^{-1}(1)$$

Taking our Machin's formula from above (1) and observing the relation (2) We can observe

$$(5+i)^4(-239+i) = (-114244 - 114244i)$$

Since upon converting our terms into complex numbers we may multiply them as an equivalence to adding angles together. So, we see that $a = b$ as required. We now have a series approximation for π that will converge much more quickly than Gregory's series, namely:

$$\pi = 16\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)5^{2n-1}} - 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)239^{2n-1}}$$

We will examine the rates of convergence compared with Gregory's series in section 3. Before we look some other "Machin like" formulas, let us look at a deirvation that would have been of the type used by Machin, since complex numbers were not really in style at the time.[10]

Proof. First observe the familiar formula

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

In the case where $a = b$, we have

$$\tan 2a = \frac{2\tan a}{1 - \tan^2 a}$$

We can now make use of the inverse tangent function and say

$$\tan(a) = \frac{1}{5} \to a = \tan^{-1}\frac{1}{5}$$

$$\tan 2a = \frac{\frac{2}{5}}{1-(1/5)^2} = \frac{5}{12}$$

Now take

$$\tan 4a = \frac{\frac{10}{12}}{1-(5/12)^2} = \frac{120}{119}$$

But, we are interested in values of tan that are equal to 1, so it makes sense to compute the difference between our result and $\frac{\pi}{4}$

$$\tan a - \frac{\pi}{4} = \frac{\tan 4a - \tan\frac{\pi}{4}}{1+\tan 4a \tan\frac{\pi}{4}} = \frac{1}{239}$$

And finally, we have

$$4a - \frac{\pi}{4} = \tan^{-1}\frac{1}{239} \to \frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}.$$

□

2.1 *Takano's Formula

Evidently, there was a competition between the United State and Japan to calculate π.[4]Takano is a high school teacher, poet and mathematician from Japan. Takano's formula is

$$\frac{\pi}{4} = 12\tan^{-1}\frac{1}{49} + 32\tan^{-1}\frac{1}{57} - 5\tan^{-1}\frac{1}{239} + 12\tan^{-1}\frac{1}{110443}$$

This gives the series

$$4\sum_{n=0}^{\infty}\frac{32(-1)^n(57)^{-(2n+1)}}{2n+1} - \frac{5(-1)^n(239)^{-(2n+1)}}{2n+1} + \frac{12(-1)^n(49)^{-(2n+1)}}{2n+1} + \frac{(-1)^n(110443)^{-(2n+1)}}{2n+1}$$

In 2002, Yasumasa Kenada, a Japanese mathematician who held several world records for calculating π, used Takano's Formula to calculate, on a Hitachi SR8000 1 trillion, 2411 hundred million digits of π.[4]

3 An examination of rates of convergence

We get an idea of the rate of convergence of two of our aforementioned series by taking a look at their respective partial sums, i.e. we can choose some $N \in \mathbb{N}$ and obtain a partial sum. This result is due to the Alternating Series Estimation Theorem. The theorem say a partial sum of any convergent series can be used as an approximation to the total sum (with some error). For our purposes we are not really concerned with the error, since we are just going to compare each of the series for different values of N. We will take a look at only one value for N First, we will evaluate each for $N = 300$ in Gregory's Series and then for Machin's Formula. We will abstain from computing using Takano's Formula since it was computed in hexidecimal.

$$N = 300$$

Gregory-Leibniz-Nilakantha Series

$$4\sum_{n=1}^{300} \frac{(-1)^n}{2n+1} = 3.1326000000000149416$$

Machin's Formula

$$16\sum_{n=1}^{300} \frac{(-1)^{n-1}}{(2(n)-1)5^{2(n)-1}} - 4\sum_{n=1}^{300} \frac{(-1)^{n-1}}{(2(n)-1)239^{2(n)-1}} \approx 3.1415937000000\overline{0}$$

As it can be seen, after 300 terms, Gregory's Series still has not converged to even 3.14, while Machin's Formula has obtained several correct decimal places. In fact, for N well over $1,000$ Gregory's Series gets nearer to reaching 4 correct decimal places, but not quite.

4 Euler's Summation Formula

Euler's Summation formula was introduced by Euler as well as Maclaurin. They used it for different reasons. Euler was using it to make series converge faster and Maclauring was using it to take integrals. Today, we use the products of this formula to illustrate the relationship between summations of areas and definite integrals in the form of "Formal Right Riemann's Sums."The Euler's summation uses **Bernoulli Numbers** The Bernoulli numbers are given by the power series

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}$$

The even Bernoulli numbers also have an asymptotic series reprsentation given by

$$B_2 n \approx (-1)^n 4\sqrt{\pi n}\left(\frac{n}{\pi e}\right)^{2n}$$

(an asymptotic series representation is a series representation of a function that is divergent or convergent and whose partials sums can be made as good an approximation as one would like)[5]. The odd Bernoulli numbers after B_1 are 0. So, π makes an appearance in the Bernoulli numbers [7], which is not surpising as they can be seen in the series expanstions of trigonometric functions.

This derivation of Euler's Summation Formula comes from Apostol [2] and Goodey [5]. While I cannot claim I understand every detail, I do have the gist of the formula and derivation. We will need to observe that the periodic Bernoulli functions are defined

$$P_k(x) = B_k(x - [x]).$$

Theorem Let ϕ be a function such that ϕ has continuous derivatives of order $2m+1$ on the interval [1,n], then

$$\sum_{i=1}^{n} \phi(i) = \int_1^n \phi(x)dx + \frac{1}{(2m+1)!}\int_1^n P_{2m+1}(x)\phi^{(2m+1)}(x)dx +$$

$$\sum_{r=1}^{m} \frac{b_2 r}{(2r)!}\left(\phi^{2r-1}(n) - \phi^{2r-1}(1)\right) + \frac{1}{2}(\phi(n)+\phi(1))$$

Proof. This proof will be by induction. Let $m = 0$ then we have

$$\int_1^n P_1(x)\phi'(x)dx = \sum_{k=1}^{n-1}\left(\int_k^{k+1} x\phi'(x)dx - \left(k+\frac{1}{2}\right)(\phi(k+1)-\phi(k))\right)$$

$$\sum_{k=1}^{n-1}\left([x\phi(x)]_k^{k+1} - \int_k^{k+1}\phi(x)dx - \left(k+\frac{1}{2}\right)(\phi(k+1)-\phi(k))\right)$$

$$\sum_{k=1}^{n-1}\left(\left(\frac{1}{2}\right)(\phi(k+1)-\phi(k))\right)\int_k^{k+1}\phi(x)dx$$

$$= \sum_{i=1}^{n} - \int_1^n \phi(x)dx - \frac{1}{2}(\phi(1)+\phi(n))$$

Now, we assume that ϕ has $\phi'^{(2m+1)}$ that is continuous on [1,n]. Then

$$\int_n^1 P_{(2m+1)}(x)\phi^{(2m-1)}(x)dx = \frac{1}{2m}\int_1^n P'_{2m}(x)\phi^{2m-1}(x)dx$$

$$= \frac{1}{2m}\left(\left[\phi^{(2m-1)}(x)P_2m(x)\right] - \int_1^n P_{2m}(x)\phi^{2m}(x)dx\right)$$

$$= \frac{1}{2m}\left(b_2m(\phi^{2m-1}(n) - \phi^{2m-1}(1)) - \int_1^n P_2m(x)\phi^{(2m)}(x)dx\right)$$

From above, we observe the fact odd Bernoulli numbers are zero,

$$= \frac{1}{2m}\left(b_2m(\phi^{(2m-1)}(n) - \phi^{(2m-1)}(1)) + \frac{1}{2m+1}\int_1^n P_{2m+1}(x)\phi^{(2m+1)}(x)dx\right)$$

Therefore,

$$\sum_{i=1}^{n}\phi(i) = \int_1^n \phi(x)dx + \frac{1}{(2m-1)!}\int_1^n P_{2m-1}(x)\phi^{(2m-1)}(x)dx$$

$$+ \sum_{r=1}^{(m-1)} \frac{b_2 r}{(2r)!}\left(\phi^{(2r-1)}(n) - \phi^{(2r-1)}(1)\right) + \frac{1}{2}(\phi(n)+\phi(1))$$

$$= \int_1^n \phi(x)dx + \frac{1}{(2m+1)!}\int_1^n P_{2m+1}(x)\phi^{(2m+1)}(x)dx + \frac{b_{2m}}{(2m!)}\left(\phi^{(2m-1)}(n) - \phi^{(2m-1)}(1)\right)$$

$$+ \sum_{r=1}^{(m-1)} \frac{b_2 r}{(2r)!}\left(\phi^{(2r-1)}(n) - \phi^{(2r-1)}(1)\right)$$

□

One notable application allows us to derive summation formulas for powers of integers. One such examples is

$$\sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

5 The Zeta Function

The Riemann Zeta Function, as derived in [9], in terms of an integral is

$$\zeta(x) \equiv \frac{1}{\gamma(x)}\int_0^\infty \frac{u^x - 1}{e^u - 1}du,$$

if $x > 1$ and if $x \in \mathbb{Z}$, then

$$\frac{u^{n-1}}{e^{u-1}} = \frac{e^{-u}u^{n-1}}{1 - e^{-u}} = e^{-u}u^{n-1}\sum_{k=0}^{\infty} e^{-ku} = \sum_{k=1}^{\infty} e^{-ku}u^{n-1},$$

then

$$\int_0^\infty \frac{u^n - 1}{e^u - 1}du = \sum_{k=1}^{\infty}\int_0^\infty e^{-ku}u^{n-1}du$$

Then, assume $y \equiv ku$ and $dy = kdu$ and

$$\zeta(n) = \frac{1}{\gamma(n)}\sum_{k=1}^{\infty}\int_0^\infty e^{-ku}u^{n-1}du =$$

$$\frac{1}{\gamma(n)}\sum_{k=1}^{\infty}\int_0^\infty e^{-y}\left(\frac{y}{k}\right)^{n-1}\frac{dy}{k}$$

$$\frac{1}{\gamma(n)}\frac{1}{k^n}\sum_{k=1}^{\infty}\int_0^\infty e^{-y}y^{n-1}dy$$

From class $\gamma(n)$ is the Gamma Function. And, when we integrate the last expression we obtain

$$\zeta(n) = \sum_{k=1}^{\infty}\frac{1}{k^n}.$$

From this we may obtain some interesting results. I will not take the time to show the calculation, but we have the following results

$$\zeta(2) = \frac{\pi^2}{6}$$

and further

$$\zeta(4) = \frac{\pi^4}{90}$$

The Riemann Zeta function serves to+ extend the domain of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

Additionally, it is very important to those who are interested in solving the "Riemann Hypothesis", which claims, in my understanding, that the distribution of the zeros of the zeta function have a relationship to the distribution of primes.

6 Probabilistic Aspects

Euler did derive a formula directly related to primes based on the zeta fucntion. This formula is known as the **Euler Product Formula** and is:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \frac{1}{1-p^{-s}}$$

A simple derivation, from [8] is as follows:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^2}...$$

If we multiply both sides by a factor of $\frac{1}{2^s}$ we obtain

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^2}\dots$$

Subtracting the second term we obtain

$$(1-\frac{1}{2^s})\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s}...$$

$$(1-\frac{1}{3^s})(1-\frac{1}{2^s})\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{11^s} + \frac{1}{13^s}...$$

Then, we continue in this manner until we have

$$\prod_{p\ prime}(1-\frac{1}{p^s})\zeta(s)=1$$

e And now we divide to obtain Euler's Product Formula

$$\zeta(s)=\prod_{p\ prime}\left(1-\frac{1}{p^{-s}}\right)$$

This formula will tell us the probablity that s randomly chosen natural numbers are relatively prime. Subsequently, a formula that can give us the probabliity that s randomly chosen natural numbers share a common factor is given by

$$1-\zeta(s)$$

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