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Egon Marx and Daniel Maystre

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Dyadic Green functions for the time-dependent wave equation

Egon Marx and Daniel Maystre [a)]
National Bureau of Standards, Washington, D. C. 20234

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The theory of dyadic Green functions for a transient electromagnetic field, which obeys the vector wave equation, is presented within the framework of the theory of distributions. First, the elementary solution of the scalar wave equation is derived, and then it is used to find the general solution of that equation. After establishing the equivalence between Maxwell's equations and the time-dependent vector wave equation, the dyadic elementary solution is derived and applied to solve the equation. Further properties of dyadic Green functions for the wave equation are derived within the heuristic approach to the theory of Green's functions. The paper includes a collection of formulas from the theory of distributions intended to help readers who are not familiar with the subject.

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I. INTRODUCTION

Transient electromagnetic fields obey a set of partial differential equations, Maxwell's equations, that can be reduced to the vector wave equation.

Green functions are often used in the solution of linear partial differential equations such as the wave equation. These functions correspond to solutions of differential equations with impulsive sources; sometimes they satisfy homogeneous boundary conditions and have causality or the radiation condition built into them. The source (a Dirac delta function) and some Green functions are not functions in the usual sense of the word; they are generalized functions or distributions. Delta functions were used in physics before they received a firm mathematical foundation,[1,2] and they continue to be used in a heuristic manner for most applications. Such a situation can easily lead to ambiguous or incorrect results, and a mathematically well-defined theory such as the theory of distributions should be used whenever practicable; unfortunately, many scientists and engineers are unfamiliar with the power and elegance of the theory of distributions. An excellent example of its effective use can be found in the theory of gratings.[3]

One of us[4] used the heuristic approach to derive integral equations for transient electromagnetic fields. The shortcomings of such a method were particularly evident in the determination of the free-space dyadic Green function. One of the aims of this paper is to present a proper derivation of this Green function.

Although Green functions for the scalar wave equation can be used to write integral equations for the electric and magnetic vector fields, dyadic Green functions provide a more natural way to relate a vector field to its vector sources and offer greater flexibility in solving a vector wave equation. Dyadic Green functions for the vector Helmholtz equations are developed in detail by Tai[5]; the free-space Green function for this equation is expressed in terms of the elementary solution of the scalar Helmholtz equation, which is a locally integrable function, and its derivatives. The corresponding treatment of the time-dependent equation involves derivatives of distributions. Generalized functions are also used to define the derivatives of the Green function for the scalar Helmholtz equation.[6]

We propose to introduce the basic concepts used in the computation of Green functions via the theory of distributions by presenting the analysis of the scalar wave equation in Sec. II. In Sec. III we briefly discuss the equivalence of Maxwell's equations and the vector wave equation, and then introduce dyadic Green functions. We find the elementary solution of the vector wave equation, which is essentially the free-space dyadic Green function, and show how it is applied to solve that equation.

To assist the reader who is not familiar with the theory of distributions, we collect in Appendix A some of the most important definitions and properties of distributions that we use in this paper.

There are some aspects of the heuristic approach to Green functions, such as homogeneous boundary conditions, that are not easily expressed in the language of distributions. Nevertheless, when a Green function that obeys homogeneous boundary conditions can be found, the solution of the differential equations is reduced to integrals over known functions. These Green functions and their symmetry properties are discussed in Appendix B.

We hope this paper will encourage other scientists and engineers to make wider use of a powerful tool: generalized functions or distributions.

II. THE SCALAR WAVE EQUATION

There are many kinds of phenomena in acoustics, electromagnetism, elasticity, and other branches of physics that are described by a function of space and time ψ that obeys the partial differential equation

$$\Box\psi(x) = \alpha(x), \tag{1}$$

where x stands for the four-vector $(ct,\mathbf{x})$, $\alpha(x)$ is the appropriate source term, and the d'Alembertian is defined by

$$\Box = \partial^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2. \tag{2}$$

In this paper we focus our attention on the electromagnetic

[a)] On leave from Laboratoire d'Optique Electromagnetique, Faculté des Sciences et Techniques de St. Jérôme, Rue H. Poincaré, 13397 Marseille, France.

field; applications to other fields require only minor changes.

The solutions of the wave equation can be found with the help of retarded Green functions $G_R(x,x')$. They satisfy

$$\square\, G_R(x,x') = \delta^{(4)}(x - x'), \tag{3}$$

$$G_R(x,x') = 0, \quad t < t', \tag{4}$$

where the source in Eq. (3) is the four-dimensional Dirac delta distribution and Eq. (4) expresses the physical condition of causality. Since the source of the Green function is a distribution, we should use the theory of distributions to find such a function, which actually may itself be a distribution.

Closely related to the free-space Green function is the elementary solution of Eq. (1), which satisfies, in the sense of distributions as defined in Appendix A,

$$\square\, \mathscr{G}(x) = \delta^{(4)}(x), \tag{5}$$

$$\mathscr{G}(x) = 0, \quad t < 0, \tag{6}$$

and we can find the free-space Green function from

$$G_R^{(0)}(x,x') = \mathscr{G}(x - x'). \tag{7}$$

To find $\mathscr{G}$, we assume that it is a tempered distribution and find its Fourier transform $\mathscr{K}$. Equations (A43) and (A44) allow us to reduce Eq. (5) to

$$(-\omega^2/c^2 + \mathbf{k}^2)\mathscr{K}(\mathbf{k},\omega) = (2\pi)^{-2}. \tag{8}$$

To solve for $\mathscr{G}$, we have to specify which reciprocal of $\mathbf{k}^2 - \omega^2/c^2$ we must choose. The causality condition (6), when compared to Eq. (A51), indicates that we should add a small positive imaginary part to ω, and set

$$\mathscr{K}(\mathbf{k},\omega) = (2\pi)^{-2}\,[\mathbf{k}^2 - (\omega + i\epsilon)^2/c^2]^{-1}, \tag{9}$$

if we use $\mathscr{F}_-$ for the inverse Fourier transform of the time variable; it is understood that ϵ will tend to zero after the appropriate integrations are performed, a limit that is well defined in the theory of distributions. In the complex ω plane, the poles at $\omega = \pm|\mathbf{k}|c$ are moved slightly below the real axis and, since we are using $e^{-i\omega t}$ in the inverse Fourier transform, the contour can be closed around the upper half-plane for $t < 0$ without changing the integral, which vanishes because no singularities are enclosed in the contour.

Actually, we know that $\mathscr{G}(x)$ is not a function, and, when the inverse Fourier transform is calculated by integrations, we have to use integrals that are mathematically ill defined to obtain delta distributions. For this reason, and because we can use the result in the computation of the dyadic elementary solution, we first derive the inverse Fourier transform of

$$K(\mathbf{k},\omega) = (\omega + i\epsilon)^{-2}[k^2 - (\omega + i\epsilon)^2/c^2]^{-1}$$

$$= k^{-2}(\omega + i\epsilon)^{-2} - k^{-2}[(\omega + i\epsilon)^2 - k^2c^2]^{-1}, \tag{10}$$

where $k = |\mathbf{k}|$; the transform of K is a function. Then $\mathscr{G}(x)$ can be obtained by differentiation, and the derivatives always exist for distributions. By convention we change the sign of the space part in the exponential, and compute the inverse Fourier transform

$$F(\mathbf{x},t) = (2\pi)^{-2}\int d^3k\, d\omega\, \exp[-i(\omega t - \mathbf{k}\cdot\mathbf{x})]\, K(\mathbf{k},\omega). \tag{11}$$

Equation (A52) gives

$$\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}(\omega + i\epsilon)^{-2} = -2\pi t\theta(t), \tag{12}$$

where $\theta(t)$ is the unit step function. To find the inverse transform of k^{-2}, we use Eq. (A55) to derive

$$\int_{-\infty}^{\infty} d^3k\, e^{i\mathbf{k}\cdot\mathbf{x}}\, k^{-2} = 4\pi r^{-1}\int_0^{\infty} dk\, \sin(kr)k^{-1}, \tag{13}$$

where $r = |\mathbf{x}|$. The integrand of the convergent integral on the right-hand side of Eq. (13) is an even function of k, so that we do not change the value of the integral if we extend the range of integration to $-\infty$ and divide by 2. The integrand is not singular at the origin, and so we can deform the contour or add a small negative imaginary part to k; then Eq. (A51) gives

$$\int_{-\infty}^{\infty} dk\, e^{ikr}(k - i\epsilon)^{-1} = 2\pi i\theta(r) \tag{14}$$

and Eq. (13) becomes

$$\int_{-\infty}^{\infty} d^3k\, e^{i\mathbf{k}\cdot\mathbf{x}}\, k^{-2} = 2\pi^2 r^{-1}[\theta(r) - \theta(-r)] = 2\pi^2 r^{-1}. \tag{15}$$

Similarly, we compute

$$\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\,[(\omega + i\epsilon)^2 - k^2c^2]^{-1}$$

$$= -2\pi \sin(kct)(kc)^{-1}\theta(t), \tag{16}$$

$$\int_{-\infty}^{\infty} d^3k\, e^{i\mathbf{k}\cdot\mathbf{x}}\sin(kct)k^{-3}$$

$$= 4\pi r^{-1}\int_0^{\infty} dk\, \sin(kct)\sin(kr)k^{-2}, \tag{17}$$

$$\int_{-\infty}^{\infty} dk\, \cos(ku)(k - i\epsilon)^{-2} = -\pi|u|, \tag{18}$$

$$\int_{-\infty}^{\infty} dk\, \sin(kct)\sin(kr)k^{-2} = \tfrac{1}{2}\pi(|ct + r| - |ct - r|), \tag{19}$$

whence, collecting all the necessary terms, we find

$$F(\mathbf{x},t) = \tfrac{1}{2}\pi\,\theta(t)r^{-1}(|t + r/c| - |t - r/c| - 2t), \tag{20}$$

which can be rewritten as

$$F(\mathbf{x},t) = -\pi r^{-1}(t - r/c)\,\theta(t - r/c). \tag{21}$$

By Eqs. (9) and (A43), we can set

$$\mathscr{G} = -(2\pi)^{-2}\frac{\partial^2 F}{\partial t^2}, \tag{22}$$

and we have

$$\frac{\partial F}{\partial t} = -\pi r^{-1}\,\theta\left(t - \frac{r}{c}\right), \tag{23}$$

$$\frac{\partial^2 F}{\partial t^2} = -\pi r^{-1}\delta\left(t - \frac{r}{c}\right), \tag{24}$$

where we have used Eq. (A14). Thus, the elementary solution of the wave equation is

$$\mathscr{G}(x) = (4\pi r)^{-1}\,\delta(t - r/c). \tag{25}$$

We note that $1/r$ corresponds to an integrable singularity, and we can rigorously define this distribution by its value on a test function ϕ,

$$\langle \mathscr{G},\phi \rangle = \int_0^\infty dt\,(4\pi t)^{-1} \oint_{S(ct)} dS\,\phi(\mathbf{x},t), \tag{26}$$

where we use $S(a)$ to denote the sphere of radius a centered at the origin. The support of this distribution, as defined in the Appendix A, is the future light cone given by

$$c^2t^2 - \mathbf{x}^2 = 0, \quad t \geqslant 0. \tag{27}$$

In Ref. 7 we find an elegant direct proof that the distribution $\mathscr{G}$ defined by Eq. (26) actually satisfies Eq. (5). Since the integration is only over positive values of t, $\langle \mathscr{G},\phi \rangle$ vanishes whenever the support of ϕ lies in the region $t<0$, so that the distribution $\mathscr{G}$ vanishes for $t<0$. Also in Ref. 7 we find the solution of the Cauchy initial value problem for the homogeneous wave equation.

If the source $\alpha(x)$ is given for all space and time, we can find the solution of Eq. (1) by a convolution product as defined in Eq. (A32). We write

$$\psi = \mathscr{G} * \check{\alpha}, \tag{28}$$

where we use the inverted caret to indicate that the source is a distribution that can include singular terms. We use Eqs. (A39), (5), and (A36) to verify that ψ satisfies the wave equation,

$$\Box\psi = (\Box \mathscr{G}) * \check{\alpha} = \delta * \check{\alpha} = \check{\alpha}. \tag{29}$$

The convolution product is well defined when the support of α is bounded below, but this is not a necessary condition; we assume that α is a locally integrable function that vanishes sufficiently rapidly when $t \to -\infty$ to define the integrals in what follows. We use the definitions (26) and (A32) to write

$$\langle \mathscr{G} * \alpha,\phi \rangle = \langle \mathscr{G}(\xi), \langle \alpha(\xi'),\phi(\xi+\xi')\rangle\rangle, \tag{30}$$

$$\langle \mathscr{G} * \alpha,\phi \rangle = \int_0^\infty \frac{d\tau}{4\pi\tau} \oint_{S(c\tau)} d\sigma \int_{-\infty}^\infty d^4\xi'\,\alpha(\xi')\,\phi(\xi+\xi'), \tag{31}$$

where $\xi = (c\tau,\boldsymbol{\xi})$, $\xi' = (c\tau',\boldsymbol{\xi}')$, and $d\sigma$ is the surface element in $\boldsymbol{\xi}$-space. We change the variable ξ' to $x-\xi$ and keep ξ to obtain

$$\langle \mathscr{G} * \alpha,\phi \rangle = \int_0^\infty \frac{d\tau}{4\pi\tau} \oint_{S(c\tau)} d\sigma \int_{-\infty}^\infty d^4x\,\alpha(x-\xi)\,\phi(x). \tag{32}$$

We now change the order of integration and replace ξ by $x-x'$, keeping x, and find

$$\langle \mathscr{G} * \alpha,\phi \rangle = \int_{-\infty}^\infty d^4x \left[\int_{-\infty}^t \frac{dt'}{4\pi(t-t')} \oint_{S[\mathbf{x},c(t-t')]} dS'\,\alpha(x') \right] \phi(x), \tag{33}$$

where $S(\mathbf{x},a)$ represents the sphere of radius a centered at $\mathbf{x}$. Consequently we can write Eq. (28) as

$$\psi(x) = \int_{-\infty}^t \frac{dt'}{4\pi(t-t')} \oint_{S[\mathbf{x},c(t-t')]} dS'\,\alpha(x'). \tag{34}$$

When the unknown field ψ is defined only in a region of space V bounded by a surface S, and, when the sources are given starting at a time t_0, we also have to specify the initial values of the field and its time derivative, and either the field or the normal derivative on S. In this case, we can extend the function ψ by assuming that it is zero outside V and for times before t_0. Then the given values become jumps in the function and its derivatives, and we use Eqs. (A14) and (A21) to write, if the normal $\hat{n}$ points out of V,

$$\begin{aligned} \Box\psi = \{\Box\psi\} &+ c^{-2}\,\psi(\mathbf{x},t_0)\,\delta'(t-t_0) \\ &+ \frac{1}{c^2}\frac{\partial\psi(\mathbf{x},t_0)}{\partial t}\,\delta(t-t_0) \\ &+ \nabla\cdot[\hat{n}\psi\delta(S)] + \frac{\partial\psi}{\partial n}\,\delta(S) = \check{\alpha}, \end{aligned} \tag{35}$$

where the curly brackets indicate derivatives in the sense of functions. From Eq. (28) we can derive

$$\begin{aligned} \psi = \mathscr{G} * \alpha &+ \frac{1}{c^2}\frac{\partial\mathscr{G}}{\partial t} * \psi(\mathbf{x},t_0)\,\delta(t-t_0) \\ &+ \frac{1}{c^2}\mathscr{G} * \frac{\partial\psi(\mathbf{x},t_0)}{\partial t}\,\delta(t-t_0) \\ &+ \frac{\partial\mathscr{G}}{\partial n} * \psi\delta(S) + \mathscr{G} * \frac{\partial\psi}{\partial n}\,\delta(S), \end{aligned} \tag{36}$$

which is equivalent to the well-known formula

$$\begin{aligned} \psi(x) = &\int_{t_0}^t dt' \int_V dV'\,\alpha(x')\,G_R^{(0)}(x,x') \\ &- \frac{1}{c^2}\int_V dV' \left[\psi(x')\frac{\partial G_R^{(0)}(x,x')}{\partial t'} - \frac{\partial\psi(x')}{\partial t'}\,G_R^{(0)}(x,x')\right]_{t'=t_0} \\ &- \int_{t_0}^t dt' \oint_S dS' \left[\psi(x')\frac{\partial G_R^{(0)}(x,x')}{\partial n'} - \frac{\partial\psi(x')}{\partial n'}\,G_R^{(0)}(x,x')\right]. \end{aligned} \tag{37}$$

Equation (36) or (37) can be used to compute ψ by integrations only if ψ and $\partial\psi/\partial n$ are both known on S. Since only one of these boundary values is required to determine ψ, Eq. (37) reduces to an integral equation for the function on S that is not given when we let $\mathbf{x}$ approach the surface from the outside, where ψ vanishes. If two arbitrary functions are given on S for ψ and $\partial\psi/\partial n$, and Eq. (37) is used to compute $\psi(x)$, this function does not vanish outside V unless the given functions satisfy the above-mentioned integral equation; the jumps in ψ and $\partial\psi/\partial n$ will be as given, but the boundary conditions will not be satisfied.

III. MAXWELL'S EQUATIONS AND DYADIC GREEN FUNCTIONS

The free-space Maxwell equations for the electromagnetic fields **E** and **B** are

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{38}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{39}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{40}$$

$$\nabla\times\mathbf{B} = \mu_0\,\mathbf{j} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}, \tag{41}$$

where the sources are the current density **j** and the charge density ρ, and ϵ_0 and μ_0 are the permittivity and permeability of the vacuum which are related to the speed of light by $\epsilon_0\mu_0 = c^{-2}$. These equations are consistent only if the equation that expresses charge conservation,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0, \tag{42}$$

is satisfied. Consequently, the charge density has to be given only at the initial time t_0, and we can find it at other times from

$$\rho(\mathbf{x},t) = \rho(\mathbf{x},t_0) - \int_{t_0}^{t} dt'\,\nabla\cdot\mathbf{j}\,(\mathbf{x},t'). \tag{43}$$

It should also be remembered that only Eqs. (39) and (41) are equations of motion, while (38) and (40) are constraints that the initial values of **E** and **B** have to satisfy; for later times, the equations of motion and charge conservation ensure that these constraints are satisfied.

If we restrict the fields to a region V bounded by a surface S, we need to know boundary values of the fields. The normal component of one field can be obtained from the tangential component of the other field by means of the relations

$$\frac{\partial}{\partial t}(\hat{n}\cdot\mathbf{E}) = -c^2\,\nabla_s\cdot(\hat{n}\times\mathbf{B}) - \frac{\mathbf{j}}{\epsilon_0}, \tag{44}$$

$$\frac{\partial}{\partial t}(\hat{n}\cdot\mathbf{B}) = \nabla_s\cdot(\hat{n}\times\mathbf{E}), \tag{45}$$

where $\hat{n}$ is the normal to S and ∇_s is the surface gradient operator. Furthermore, the energy balance equation,

$$\oint_S dS\,\hat{n}\cdot\mathbf{E}\times\mathbf{B} = -\frac{1}{2}\frac{d}{dt}\int_V dV\left(\mathbf{B}^2 + \frac{\mathbf{E}^2}{c^2}\right) - \mu_0\int_V dV\,\mathbf{j}\cdot\mathbf{E}, \tag{46}$$

when applied to the difference of two solutions of Maxwell's equations with the same sources, initial values, and boundary values, implies that the fields are uniquely determined by the current density, the initial values of **E** and **B** subject to the constraints, and the tangential component of either **E** or **B**.

We can express the fields in terms of the sources, initial values, and boundary values by means of the equations[4]

$$\begin{aligned}\mathbf{E}(x) = {} & \int_{t_0}^{t} dt'\int_V dV'\left[\mu_0\,\mathbf{j}(x')\frac{\partial G_R(x,x')}{\partial t'} + \frac{\rho(x')}{\epsilon_0}\nabla' G_R(x,x')\right] - \int_V dV'\left[\frac{1}{c^2}\mathbf{E}(x')\frac{\partial G_R(x,x')}{\partial t'} - \mathbf{B}(x')\times\nabla' G_R(x,x')\right]_{t'=t_0} \\ & + \int_{t_0}^{t} dt'\left[\oint_S d\,\mathbf{S}'\mathbf{E}(x')\cdot\nabla' G_R(x,x') - \oint_S d\,\mathbf{S}'\cdot\mathbf{E}(x')\,\nabla' G_R(x,x') - \oint_S d\,\mathbf{S}'\cdot\nabla' G_R(x,x')\mathbf{E}(x') - \oint_S d\,\mathbf{S}'\times\mathbf{B}(x')\frac{\partial G_R(x,x')}{\partial t'}\right],\end{aligned} \tag{47}$$

$$\begin{aligned}\mathbf{B}(x) = {} & \mu_0\int_{t_0}^{t} dt'\int_V dV'\,\mathbf{j}(x')\times\nabla' G_R(x,x') - \frac{1}{c^2}\int_V dV'\left[\mathbf{E}(x')\times\nabla' G_R(x,x') + \mathbf{B}(x')\frac{\partial G_R(x,x')}{\partial t'}\right]_{t'=t_0} \\ & + \int_{t_0}^{t} dt'\left[\frac{1}{c^2}\oint_S d\,\mathbf{S}'\times\mathbf{E}(x')\frac{\partial G_R(x,x')}{\partial t'} - \oint_S d\,\mathbf{S}'\cdot\nabla' G_R(x,x')\mathbf{B}(x') + \oint_S d\,\mathbf{S}'\mathbf{B}(x')\cdot\nabla' G_R(x,x') - \oint_S d\,\mathbf{S}'\cdot\mathbf{B}(x')\,\nabla' G_R(x,x')\right],\end{aligned} \tag{48}$$

where $G_R(x,x')$ is a Green function for the scalar wave equation.

An alternative approach to the solution of Maxwell's equations is their reduction to a single vector wave equation. We now show how this equation can be solved with the help of dyadic Green functions.

We eliminate **B** from Eqs. (39) and (41) to obtain

$$\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} + \nabla\times(\nabla\times\mathbf{E}) = -\mu_0\frac{\partial\mathbf{j}}{\partial t}. \tag{49}$$

The initial value of the time derivative of **E** is obtained from **B** through Eq. (41). The boundary conditions on the field **E** are given either as the tangential component of **E** or the tangential component of $\nabla\times\mathbf{E}$, which can be obtained from the tangential component of **B** through Eq. (39). Once the field **E** is determined, the field **B** can be found from Eq. (39) and its initial value; these fields satisfy Maxwell's equations.

The elementary solution of the vector wave equation is a dyadic Q that satisfies

$$\frac{1}{c^2}\frac{\partial^2 \mathsf{Q}(x)}{\partial t^2}+\nabla\times[\nabla\times\mathsf{Q}(x)]=\delta^{(4)}(x)\mathsf{I}, \tag{50}$$

where I is the unit 3×3 dyadic, and the causality condition

$$\mathsf{Q}(x)=0,\quad t<0. \tag{51}$$

We can find the elementary solution following the procedure in Sec. II. If U is the Fourier transform of Q, it satisfies

$$-(\omega^2/c^2)\mathsf{U}-\mathbf{k}\times(\mathbf{k}\times\mathsf{U})=(2\pi)^{-2}\mathsf{I}. \tag{52}$$

Scalar multiplication by $\mathbf{k}$ on the left allows us to find $\mathbf{k}\cdot\mathsf{U}$, which is then substituted into the expansion of the triple vector product. We again add a small positive imaginary part to ω to satisfy the causality condition (51), and solve for U to find

$$\mathsf{U}(\mathbf{k},\omega)=\frac{1}{4\pi^2}\frac{(\omega+i\epsilon)^2\mathsf{I}-c^2\mathbf{k}\,\mathbf{k}}{(\omega+i\epsilon)^2[\mathbf{k}^2-(\omega+i\epsilon)^2/c^2]}. \tag{53}$$

The denominator, apart from the numerical factor, is precisely the function $K(\mathbf{k},\omega)$ in Eq. (10). Equation (A43) then implies that

$$\mathsf{Q}(\mathbf{x},t)=\frac{1}{4\pi^2}\left[-\mathsf{I}\frac{\partial^2 F(\mathbf{x},t)}{\partial t^2}+c^2\nabla\nabla F(\mathbf{x},t)\right]. \tag{54}$$

The time derivative is given by Eq. (24), and we have to compute the gradient of the gradient of F. To handle the singularity of the function at the origin, we consider F the limit as $\epsilon\to 0$ of a function F_ϵ that vanishes inside a sphere of radius ϵ centered at the origin; we write

$$F_\epsilon(\mathbf{x},t)=F(\mathbf{x},t)\,\theta(r-\epsilon). \tag{55}$$

For a sequence of distributions T_j that tend to a limit T, it has been shown[1] that the derivatives T'_j tend to T' (which is not necessarily the case for a sequence of functions). We thus consider a sequence of positive values ϵ_j that tend to zero, compute first ∇F_ϵ, and obtain ∇F by letting ϵ tend to zero. The function F_ϵ is discontinuous on the spherical surface $S(\epsilon)$, and we use Eq. (A18) to find

$$\nabla F_\epsilon=\pi t r^{-3}\mathbf{x}\,\theta(t-r/c)\,\theta(r-\epsilon)-\pi\epsilon^{-1}(t-\epsilon/c)\hat{r}\delta[S(\epsilon)], \tag{56}$$

where $\hat{r}$ is the unit normal to $S(\epsilon)$. When $\epsilon\to 0$, the first term tends to a locally integrable function, and we only have to examine the surface integral of the coefficient of $\delta[S(\epsilon)]$. If $f(\mathbf{x})$ is a locally integrable function, we can write

$$\oint_{S(\epsilon)} dS\, f(\mathbf{x})\phi(\mathbf{x})\approx\phi(0)\oint_{S(\epsilon)} dS\, f(\mathbf{x}), \tag{57}$$

which is true to lowest order in ϵ because the continuous test function ϕ satisfies

$$|\phi(\mathbf{x})-\phi(0)|<a\epsilon. \tag{58}$$

for a constant a when $\mathbf{x}$ is on the sphere $S(\epsilon)$. The surface integral reduces to an integral over the solid angle and the limit is

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(t-\frac{\epsilon}{c}\right)\oint_{S(1)}\epsilon^2\hat{r}\,d\Omega=0. \tag{59}$$

Thus, the gradient of F is

$$\nabla F=\pi t r^{-3}\,\mathbf{x}\,\theta(t-r/c). \tag{60}$$

We use the same procedure to compute the second-order derivatives. If we multiply a component of ∇F by $\theta(r-\epsilon)$, where we choose ϵ such that $\epsilon<ct$, we obtain a function that is discontinuous on the spherical surfaces $r=\epsilon$ and $r=ct$. The second derivatives of the function F for $\epsilon<r<ct$ are given by

$$\left\{\frac{\partial^2 F}{\partial x_i\partial x_j}\right\}=\pi t\frac{\partial}{\partial x_i}\left(\frac{x_j}{r^3}\right)=\pi t\left(\frac{\delta_{ij}}{r^3}-\frac{3x_ix_j}{r^5}\right), \tag{61}$$

and, adding the singular parts, we have

$$\begin{aligned}\nabla(\nabla F)_\epsilon&=\pi t\,(r^{-3}\mathsf{I}-3r^{-5}\,\mathbf{x}\,\mathbf{x})\,\theta(r-\epsilon)\,\theta(t-r/c)\\&\quad+\pi t r^{-4}\,\mathbf{x}\,\mathbf{x}\delta[S(\epsilon)]-\pi t r^{-4}\,\mathbf{x}\,\mathbf{x}\,\delta[S(ct)].\end{aligned} \tag{62}$$

We note that the function in Eq. (61) is not Lebesgue-integrable at the origin, but the limit of the distribution must exist since those of the other terms in Eq. (62) all exist. For the coefficient of $\delta[S(\epsilon)]$ we have

$$\lim_{\epsilon\to 0+}\oint_{S(\epsilon)} dS\frac{x_ix_j}{r^4}\phi(\mathbf{x})=\frac{4\pi}{3}\delta_{ij}\,\phi(0), \tag{63}$$

and we write

$$\begin{aligned}\nabla\nabla F&=\pi t r^{-3}(\mathsf{I}-3r^{-2}\,\mathbf{x}\,\mathbf{x})\,\theta(t-r/c)\\&\quad+\tfrac{4}{3}\pi t\delta^{(3)}(\mathbf{x})\mathsf{I}-\pi c^{-1}r^{-3}\mathbf{x}\,\mathbf{x}\,\delta[S(ct)],\end{aligned} \tag{64}$$

where the first term on the right-hand side is a kind of principal value. We can use symmetry arguments to prove that the integral of $r^{-3}\delta_{ij}-3r^{-5}x_ix_j$ over a sphere vanishes; for $i\neq j$ the integrand is antisymmetric in the coordinates, and for $i=j$ we obtain three equal integrals whose sum vanishes. Finally, Eq. (54) becomes

$$\begin{aligned}\mathsf{Q}(\mathbf{x},t)&=(4\pi r)^{-1}\,\delta(t-r/c)(\mathsf{I}-r^{-2}\,\mathbf{x}\,\mathbf{x})\\&\quad+(4\pi r^3)^{-1}\,c^2t\,\theta(t-r/c)(\mathsf{I}-3r^{-2}\,\mathbf{x}\,\mathbf{x})\\&\quad+\tfrac{1}{3}c^2\,t\,\theta(t)\,\delta^{(3)}(\mathbf{x})\mathsf{I}.\end{aligned} \tag{65}$$

This distribution can be defined by an equation similar to (26) for the scalar elementary solution. For a component Q_{ij} of Q, we have

$$\begin{aligned}\langle Q_{ij},\phi\rangle&=\int_0^\infty\frac{dt}{4\pi t}\oint_{S(ct)} dS\,(\delta_{ij}-\alpha_i\alpha_j)\,\phi(\mathbf{x},t)\\&\quad+\lim_{\epsilon\to 0+}\int_{\epsilon/c}^\infty\frac{c^2tdt}{4\pi}\int_\epsilon^{ct}\frac{dr}{r^3}\oint_{S(r)} dS\,(\delta_{ij}-3\alpha_i\alpha_j)\,\phi(\mathbf{x},t)\\&\quad+\int_0^\infty\frac{c^2tdt}{3}\delta_{ij}\,\phi(0,t),\end{aligned} \tag{66}$$

where $\alpha_i=x_i/r$. The limit of the second integral must exist, as discussed after Eq. (62). Note that Q is symmetric.

To solve the inhomogeneous wave equation (49) when the sources are known for all space and for all past times, we use

$$\mathbf{E}=-\mu_0\mathsf{Q}*\cdot\frac{\partial\,\mathbf{j}}{\partial t}, \tag{67}$$

where we have both a convolution product of distributions and a dot product of the dyadic and the vector. We can rewrite this equation in the form

$$\mathbf{E}(x) = -\int_{-\infty}^{t} \frac{\mu_0 dt'}{4\pi(t-t')}\oint_{S[\mathbf{x},c(t-t')]} dS'\left[\frac{\partial \mathbf{j}(x')}{\partial t'} - \frac{\mathbf{R}}{R^2}\mathbf{R}\cdot\frac{\partial \mathbf{j}(x')}{\partial t'}\right]$$
$$-\int_{-\infty}^{t} \frac{(t-t')dt'}{4\pi\epsilon_0}\int_0^{c(t-t')}\frac{dr'}{R^3}\oint_{S(\mathbf{x},r')} dS'\left[\frac{\partial \mathbf{j}(x')}{\partial t'} - 3\frac{\mathbf{R}}{R^2}\mathbf{R}\cdot\frac{\partial \mathbf{j}(x')}{\partial t'}\right] - \int_{-\infty}^{t}\frac{(t-t')dt'}{3\epsilon_0}\frac{\partial \mathbf{j}(\mathbf{x},t')}{\partial t'}, \tag{68}$$

where $\mathbf{R} = \mathbf{x} - \mathbf{x}'$ and $R = |\mathbf{R}|$. This result has to agree, of course, with the corresponding expression obtained from Eq. (47), that is,

$$\mathbf{E}(x) = \int_{-\infty}^{t} dt'\int dV'\left[\mu_0\,\mathbf{j}(x')\frac{\partial G_R^{(0)}(x,x')}{\partial t'} + \frac{1}{\epsilon_0}\rho(x')\,\nabla' G_R^{(0)}(x,x')\right], \tag{69}$$

which is also a convolution product of distributions. To show the equivalence of these expressions, we first use Eq. (A39) to obtain

$$\mathbf{E}(x) = -\int_{-\infty}^{t} dt'\int dV'\left[\mu_0\frac{\partial \mathbf{j}(x')}{\partial t'} + \frac{1}{\epsilon_0}\nabla'\rho(x')\right] G_R^{(0)}(x,x'), \tag{70}$$

and, taking into account Eqs. (7), (22), and (42), we show that

$$\begin{aligned}\nabla\rho*\frac{\partial^2 F}{\partial t^2} &= \nabla\frac{\partial\rho}{\partial t}*\frac{\partial F}{\partial t}\\ &= -\nabla\,\nabla\cdot\mathbf{j}*\frac{\partial F}{\partial t} = -\nabla\,\nabla\cdot\frac{\partial \mathbf{j}}{\partial t}*F\\ &= -\nabla\cdot\frac{\partial \mathbf{j}}{\partial t}*\nabla F = -\frac{\partial \mathbf{j}}{\partial t}*\cdot\nabla\nabla F,\end{aligned} \tag{71}$$

and Eq. (68) becomes

$$\mathbf{E} = -\mu_0\frac{\partial \mathbf{j}}{\partial t}*\cdot\frac{-\mathsf{I}\partial^2 F/\partial t^2 + c^2\,\nabla\nabla F}{4\pi^2}, \tag{72}$$

which is the same as Eq. (67).

As done in Sec. II for the scalar wave equation, initial conditions and boundary conditions can be built into the sources of the vector wave equation. We use Eq. (A20) twice to obtain

$$\begin{aligned}&\nabla\times(\nabla\times\mathbf{E})\\ &= \{\nabla\times(\nabla\times\mathbf{E})\} + \hat{n}\times\Delta\,(\nabla\times\mathbf{E})\,\delta(S) + \nabla\times[\hat{n}\times\Delta\,\mathbf{E}\,\delta(S)],\end{aligned} \tag{73}$$

where the quantity in the square brackets is a singular distribution. We assume that the fields vanish for $t < t_0$ and outside V, and find

$$\begin{aligned}&\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} + \nabla\times(\nabla\times\mathbf{E})\\ &= -\mu_0\frac{\partial \mathbf{j}}{\partial t} + \mathbf{E}(\mathbf{x},t_0)\delta'(t-t_0) + \frac{\partial \mathbf{E}(\mathbf{x},t_0)}{\partial t}\delta(t-t_0)\\ &+ \Delta\,[\hat{n}\times(\nabla\times\mathbf{E})]\delta(S) + \nabla\times[\Delta\,(\hat{n}\times\mathbf{E})\,\delta(S)].\end{aligned} \tag{74}$$

If the initial and boundary conditions are given for $\mathbf{B}$, we use Eqs. (39) and (41) to change Eq. (74) into

$$\begin{aligned}&\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} + \nabla\times(\nabla\times\mathbf{E})\\ &= -\mu_0\frac{\partial \mathbf{j}}{\partial t} + \mathbf{E}(\mathbf{x},t_0)\,\delta'(t-t_0)\\ &+ \left[c^2\nabla\times\mathbf{B}(\mathbf{x},t_0) - \frac{\mathbf{j}(\mathbf{x},t_0)}{\epsilon_0}\right]\delta(t-t_0)\\ &- \frac{\partial}{\partial t}\Delta\,(\hat{n}\times\mathbf{B})\,\delta(S) + \nabla\times[\Delta\,(\hat{n}\times\mathbf{E})\,\delta(S)],\end{aligned} \tag{75}$$

Convolution with the dyadic elementary solution provides a formal solution of the wave equation that also satisfies the initial and the boundary conditons; we obtain

$$\begin{aligned}\mathbf{E} = &-\mu_0\frac{\partial \mathsf{Q}}{\partial t}*\cdot\,\mathbf{j} + \frac{\partial \mathsf{Q}}{\partial t}*\cdot\mathbf{E}(\mathbf{x},t_0)\delta(t-t_0)\\ &+ \mathsf{Q}*\cdot\left[c^2\,\nabla\times\mathbf{B}(\mathbf{x},t_0) - \frac{\mathbf{j}(\mathbf{x},t_0)}{\epsilon_0}\right]\delta(t-t_0)\\ &+ \mathsf{Q}*\cdot\hat{n}\times\frac{\partial\mathbf{B}}{\partial t}\delta(S) + \hat{n}\times\mathbf{E}\delta(S)*\cdot\,\nabla\times\mathsf{Q},\end{aligned} \tag{76}$$

where we assume that the given functions vanish for $t < t_0$ and outside V. This equation is equivalent to Eq. (B22) when the free-space dyadic Green function is used for G_R.

In general we do not know both the tangential components of $\mathbf{E}$ and $\mathbf{B}$ on the boundary S, and Eq. (76) leads to an integral equation for the unknown boundary value of the field. We can use other Green functions that satisfy homogeneous boundary conditions, but such a condition cannot be imposed on a distribution; this possibility is discussed further in Appendix B, where the heuristic approach to Green functions is used.

The dyadic elementary solution could also have been obtained from the elementary solution for the vector Helmholtz equation[5] that satisfies the outgoing-wave condition at infinity. These two distributions are related by a Fourier transform that leads from the time variable to the frequency variable.

IV. CONCLUDING REMARKS

In this paper we have shown how the free-space Maxwell equations can be solved, or reduced to relatively simple integral equations, within the framework of the theory of distributions.

We first found the elementary solution of the scalar wave equation and showed how it is used to find an integral equation for boundary values of the field, which can then be used to compute a solution by integration. These formulations are well known, especially in the heuristic approach to Green functions, but it is useful to have a mathematically well-defined derivation that can be generalized to more difficult problems without ambiguities.

We then followed the same procedure to find the dyadic elementary solution of the vector wave equation that was derived from Maxwell's equations, and found the corresponding expression of the field in terms of the sources, the initial values, and the boundary values. This relationship reduces to an integral equation for either the tangential component of **E** or the tangential component of **B** when the field point tends to the boundary surface. The scalar elementary solution can also be used to solve Maxwell's equations, but the boundary terms include the normal components of the fields.

There are difficulties in the definition of Green functions that obey homogeneous boundary conditions within the framework of the theory of distributions, and we had to use the heuristic approach to obtain some further results, as shown in Appendix B. These Green functions are no longer invariant under translations, and the integrals that give the fields are not convolutions. It would be useful to extend the theory of distributions to cover these subjects.

In addition to presenting new results for dyadic Green functions for the time-dependent vector wave equation, we have demonstrated how the theory of distributions can be used to obtain rigorous results in problems where the heuristic approach is hazardous.

APPENDIX A: REVIEW OF DISTRIBUTIONS

In this appendix we briefly review the main concepts from the theory of distributions,[1,2] and give enough of the results to present the equations used in the paper.

A distribution is a continuous linear functional on a space of test functions. The most general set of distributions is obtained when the test functions belong to the space $\mathscr{D}$ of infinitely differentiable functions of bounded support. A distribution T is defined when a complex number c is associated with each test function ϕ, and we use the notation

$$\langle T,\phi \rangle = c. \tag{A1}$$

For instance, the Dirac delta distribution is defined by

$$\langle \delta,\phi \rangle = \phi(0). \tag{A2}$$

A distribution T_f can be associated to any locally integrable function f by

$$\langle T_f,\phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx. \tag{A3}$$

It is customary to use f both for the function and the distribution, and in most cases relations are valid for both. When T corresponds to a function, it is called a regular distribution; otherwise, it is called singular. Although, in general, it does not make sense to talk about the value of a distribution at a point x, we often write $T(x)$ even when T does not correspond to a function mainly to indicate what variable is involved; such a liberty if often taken with the delta distribution. These concepts can be generalized to distributions in spaces of higher dimensions. For instance,

$$\langle T(\mathbf{x},t),\phi(\mathbf{x},t)\rangle = c \tag{A4}$$

defines a distribution in a four-dimensional space–time.

A linear combination of distributions is defined by

$$\langle \alpha T_1 + \beta T_2,\phi \rangle = \alpha \langle T_1,\phi \rangle + \beta \langle T_2,\phi \rangle, \tag{A5}$$

and the null distribution is given by

$$\langle T_0,\phi \rangle = 0, \tag{A6}$$

for arbitrary ϕ. Consequently, two distributions T_1 and T_2 are equal if and only if, for any test function ϕ,

$$\langle T_1,\phi \rangle = \langle T_2,\phi \rangle. \tag{A7}$$

A distribution T is zero in an open region of space if the value of the functional vanishes for all test functions ϕ whose support is in that region. The complement of the union of all regions in which T is zero is called the support of the distribution. For instance, the support of the delta distribution is the origin.

A distribution can be shifted by an amount a according to

$$\langle T(x-a),\phi(x)\rangle = \langle T(x),\phi(x+a)\rangle; \tag{A8}$$

we define the symmetrically transposed distribution by

$$\langle T(-x),\phi(x)\rangle = \langle T(x),\phi(-x)\rangle, \tag{A9}$$

and we define a change of scale by

$$\langle T(ax),\phi(x)\rangle = |a|^{-1}\langle T(x),\phi(x/a)\rangle. \tag{A10}$$

Any distribution T has a derivative $T' = dT/dx$ defined by

$$\langle T',\phi \rangle = -\langle T,\phi' \rangle, \tag{A11}$$

or, for a distribution in a four-dimensional space–time, a partial derivative is defined by

$$\left\langle \frac{\partial T(\mathbf{x},t)}{\partial t},\phi(\mathbf{x},t)\right\rangle = -\left\langle T(\mathbf{x},t),\frac{\partial \phi(\mathbf{x},t)}{\partial t}\right\rangle. \tag{A12}$$

If θ is the unit step function, Eqs. (A2), (A8), and (A11) imply that

$$\theta' = \delta. \tag{A13}$$

Consider a function $f(x)$ that has a locally integrable derivative defined almost everywhere, except at a set of points x_i where f can have finite jumps. Equations (A3) and (A10) then lead to the relation

$$\frac{df}{dx} = \left\{\frac{df}{dx}\right\} + \sum_i (\Delta_i f)\delta(x-x_i), \tag{A14}$$

where the left-hand side is the derivative of f in the sense of distributions, which always exists, the derivative in curly brackets is taken in the sense of functions, $\Delta_i f$ is the jump

$$\Delta_i f = f(x_i+0) - f(x_i-0), \tag{A15}$$

and $\delta(x - x_i)$ is the shifted delta distribution.

Equation (A14) can be generalized to functions of several variables. If $f(\mathbf{x})$ is differentiable with respect to x_1 except on a surface S,

$$\frac{\partial f}{\partial x_1} = \left\{\frac{\partial f}{\partial x_1}\right\} + n_1 \Delta f \delta(S), \tag{A16}$$

where n_1 is a component of the unit normal, Δf is the jump across S in the direction of the normal (reversing $\hat{n}$ changes the sign of Δf and leaves the product unchanged), and the definition of the singular distribution is

$$\langle u(\mathbf{x})\delta(S),\phi(\mathbf{x})\rangle = \int_S dS\, u(\mathbf{x})\phi(\mathbf{x}), \tag{A17}$$

where $u(\mathbf{x})$ is a function that needs to be defined on S only. In the usual three-dimensional space, Eq. (A16) leads to the relations

$$\nabla f = \{\nabla f\} + \hat{n}\, \Delta f \delta(S), \tag{A18}$$

$$\nabla\cdot\mathbf{f} = \{\nabla\cdot\mathbf{f}\} + \hat{n}\cdot\Delta\, \mathbf{f}\, \delta(S), \tag{A19}$$

$$\nabla\times\mathbf{f} = \{\nabla\times\mathbf{f}\} + \hat{n}\times\Delta\, \mathbf{f}\, \delta(S). \tag{A20}$$

Combining Eqs. (A18) and (A19), we obtain for the Laplacian

$$\nabla^2 f = \{\nabla^2 f\} + \Delta\frac{\partial f}{\partial n}\delta(S) + \nabla\cdot[\hat{n}\, \Delta f \delta(S)], \tag{A21}$$

where $\Delta\,(\partial f/\partial n)$ is the jump in the normal derivative that comes from the term

$$\hat{n}\cdot\Delta\,\{\nabla f\} = \Delta\,[\hat{n}\cdot\{\nabla F\}] = \Delta\frac{\partial f}{\partial n}, \tag{A22}$$

and the derivatives of the singular distribution in the last term of Eq. (A21) have to be taken according to the general definition (A12).

A generalization of the ordinary product of two functions to a product of arbitrary distributions is not possible because, for instance, the product of two locally integrable functions is not necessarily locally integrable. On the other hand, the direct product of two distributions can always be defined by

$$\langle T(x)U(y),\phi(x,y)\rangle = \langle T(x),\psi(x)\rangle, \tag{A23}$$

where

$$\psi(x) = \langle U(y),\phi(x,y)\rangle \tag{A24}$$

is an indefinitely differentiable function of x that satisfies

$$\frac{d\psi}{dx} = \left\langle U(y),\frac{\partial\phi(x,y)}{\partial x}\right\rangle. \tag{A25}$$

This product is commutative and distributive over a sum, that is,

$$T(x)U(y) = U(y)T(x), \tag{A26}$$

$$T(x)[U_1(y) + U_2(y)] = T(x)U_1(y) + T(x)U_2(y), \tag{A27}$$

where Eq. (A26) means

$$\langle T(x),\langle U(y),\phi(x,y)\rangle\rangle = \langle U(y),\langle T(x),\phi(x,y)\rangle\rangle. \tag{A28}$$

The product can be generalized to more than two factors, and it is associative. We have, for instance,

$$\delta(x)\delta(y)\delta(z) = \delta^{(3)}(\mathbf{x}), \tag{A29}$$

where the three-dimensional delta distribution is defined by

$$\langle\delta^{(3)}(\mathbf{x}),\phi(\mathbf{x})\rangle = \phi(0) = \phi(0,0,0). \tag{A30}$$

The direct product of two distributions results in another distribution in a space of higher dimension. It is also often possible to define a convolution product in the same space of the original distributions, indicated by

$$V(x) = T(x)*U(x) \tag{A31}$$

and defined by

$$\langle V,\phi\rangle = \langle T(x)U(y),\phi(x+y)\rangle \tag{A32}$$

in terms of the direct product (A23). It should be remembered that Eq. (A31) does not mean that the value of the convolution product at x is the product of the values of the factors at x, even when these quantities are defined. The convolution product of two arbitrary distributions is not always defined because, even though $\phi(x)$ has a compact support, the support of $\phi(x+y)$ in the plane is essentially the union of unbounded diagonal strips and the right-hand side of Eq. (A32) need not be defined. The convolution product exists when one of the distributions has a compact support or when both distributions have supports bounded on the same side; these are sufficient conditions, and it is not necessary for either to be satisfied. When the convolution product exists, it is commutative and distributive,

$$T*U = U*T, \tag{A33}$$

$$T*(U_1 + U_2) = T*U_1 + T*U_2. \tag{A34}$$

When T and U are functions, Eq. (A32) can be rewritten in the form

$$V(x) = \int_{-\infty}^{\infty} T(x - x')U(x')\,dx'. \tag{A35}$$

Some useful particular relations are

$$\delta*T = T, \tag{A36}$$

$$\delta'*T = T', \tag{A37}$$

$$\delta(x-a)*T(x) = T(x-a), \tag{A38}$$

$$\frac{d}{dx}(T*U) = \frac{dT}{dx}*U = T*\frac{dU}{dx}. \tag{A39}$$

The convolution product can be extended in a straightforward way to more than two factors and to distributions of several variables.

The definition of the Fourier transform of a distribution is

$$\langle\mathscr{F}(T),\phi\rangle = \langle T,\mathscr{F}(\phi)\rangle, \tag{A40}$$

but this definition is not applicable in general because the Fourier transform of a function of compact support does not have a compact support. We restrict ourselves to the space of tempered distributions where the Fourier transform is defined by enlarging the space of test functions to the space $\mathscr{S}$ of indefinitely differentiable functions that decrease rapidly at infinity. By rapidly decreasing we mean that, for arbitrary nonnegative integers m and n, $x^m\phi^{(n)}(x)$ is bounded; in other words, $\phi(x)$ tends to zero faster than any power of $1/|x|$ when

$|x| \to \infty$. The Fourier transform of such a function is also in $\mathcal{S}$, and the Fourier transform of a tempered distribution is well defined by Eq. (A40). In particular, distributions of bounded support are tempered distributions, as are the distributions that correspond to locally integrable functions that increase slowly at infinity (more slowly than some power of $|x|$).

The sign of the exponential in a Fourier transform is arbitrary, and we define

$$\mathcal{F}_{\pm}(\phi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \phi(x) \exp(\pm ikx)\, dx; \qquad \text{(A41)}$$

then, $\mathcal{F}_{-}$ is the inverse transform for $\mathcal{F}_{+}$ and vice versa. Thus, for a function f we have

$$\mathcal{F}_{\pm}(f) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x) \exp(\pm ikx)\, dx, \qquad \text{(A42)}$$

if the integral exists; the Fourier transform of a distribution T satisfies

$$\mathcal{F}_{\pm}(T') = \mp ik\, \mathcal{F}_{\pm}(T), \qquad \text{(A43)}$$

$$\mathcal{F}_{\pm}[T(x-a)] = \exp(\pm iak)\, \mathcal{F}_{\pm}[T(x)]; \qquad \text{(A44)}$$

and, for the convolution product of two distributions of compact support, we have

$$\mathcal{F}_{\pm}(T * U) = \mathcal{F}_{\pm}(T)\, \mathcal{F}_{\pm}(U), \qquad \text{(A45)}$$

since the Fourier transform of a distribution of compact support is a function.

We now give some examples of distributions and their Fourier transforms. The principal value of $1/x$, which is defined by

$$\left\langle \mathrm{P}\,\frac{1}{x}, \phi(x) \right\rangle = \lim_{\epsilon \to 0+} \left[\int_{-\infty}^{-\epsilon} \frac{\phi(x)}{x}\, dx + \int_{\epsilon}^{\infty} \frac{\phi(x)}{x}\, dx \right], \qquad \text{(A46)}$$

satisfies

$$\frac{d}{dx} \log|x| = \mathrm{P}\,\frac{1}{x}, \qquad \text{(A47)}$$

$$\mathcal{F}_{\pm}\left[\mathrm{P}\,\frac{1}{x}\right] = \pm i(\tfrac{1}{2}\pi)^{1/2}\, \mathrm{sgn}(k). \qquad \text{(A48)}$$

We also have

$$\mathcal{F}_{\pm}(\delta) = (2\pi)^{-1/2}, \qquad \text{(A49)}$$

whence

$$\mathcal{F}_{\pm}[\mathrm{P}(1/x) \pm i\pi\delta] = \pm i(2\pi)^{1/2}\, \theta(k), \qquad \text{(A50)}$$

where upper or lower signs have to be taken together. It is customary to write the argument of the Fourier transform as $(x \mp i\epsilon)^{-1}$, and we have

$$\mathcal{F}_{\pm}[(x \mp i\epsilon)^{-1}] = \pm i(2\pi)^{1/2}\, \theta(k), \qquad \text{(A51)}$$

where the limit $\epsilon \to 0+$ is implicit. The Fourier transform of the derivative of this distribution is, by Eq. (A43),

$$\mathcal{F}_{\pm}[(x \mp i\epsilon)^{-2}] = -(2\pi)^{1/2}\, k\theta(k). \qquad \text{(A52)}$$

We can write

$$(x \mp i\epsilon)^{-2} = \mathrm{F.P.}(1/x^2) \mp i\pi\delta', \qquad \text{(A53)}$$

where the derivative of $\mathrm{P}(1/x)$ is related to the finite part of Hadamard of a divergent integral,

$$\left\langle \mathrm{F.P.}\frac{1}{x^2}, \phi \right\rangle = \mathrm{F.P.} \int_{-\infty}^{\infty} \frac{\phi(x)}{x^2}\, dx = \lim_{\epsilon \to 0+} \left[\int_{-\infty}^{-\epsilon} \frac{\phi(x)}{x^2}\, dx + \int_{\epsilon}^{\infty} \frac{\phi(x)}{x^2}\, dx - \frac{2\phi(0)}{\epsilon} \right]. \qquad \text{(A54)}$$

A relationship that is useful for functions is

$$\int_{-\infty}^{\infty} d^3k\, f(k) \exp(\pm i\, \mathbf{k}\cdot\mathbf{x}) = 4\pi r^{-1} \int_0^{\infty} k\, dk\, f(k) \sin(kr), \qquad \text{(A55)}$$

where $k = |\mathbf{k}|$ and $r = |\mathbf{x}|$.

APPENDIX B: HOMOGENEOUS BOUNDARY CONDITIONS

When boundary conditions are given for the unknown field in a linear partial differential equation, it is often useful to define Green functions that obey homogeneous boundary conditions. For the scalar wave equation, we can define $G_R^{(1)}(x,x')$ which vanishes when $\mathbf{x}$ is on S, and $G_R^{(2)}$, whose normal derivative vanishes on S. When $G_R^{(0)}$ is replaced by $G_R^{(1)}$ or $G_R^{(2)}$ in Eq. (37), one of the surface integrals vanishes, and we obtain a solution by integration over known functions instead of an integral equation.

There are two problems that keep us from using the theory of distributions to handle these Green functions: They are no longer functions of $x - x'$, so that Eq. (36) no longer represents convolution products, and a distribution would have to vanish on a surface, which is not an open region of space. Thus, in this context, we use the heuristic approach to Green functions.

The Green functions $G_R^{(1)}$, and $G_R^{(2)}$ and the corresponding advanced Green functions obey symmetry relations

$$G_R(x,x') = G_A(x',x), \qquad \text{(B1)}$$

$$G_R(\mathbf{x},t;\mathbf{x}',t') = G_R(\mathbf{x}',-t';\mathbf{x},-t), \qquad \text{(B2)}$$

$$\frac{\partial G_R(x,x')}{\partial t'} = -\frac{\partial G_R(x,x')}{\partial t}, \qquad \text{(B3)}$$

but, in general,

$$\nabla G_R(x,x') \neq -\nabla' G_R(x',x). \qquad \text{(B4)}$$

When V is the half-plane $z \geqslant 0$,

$$G_R^{(1)}(x,x') = \mathcal{G}(x - x') - \mathcal{G}(x - x_I'), \qquad \text{(B5)}$$

where x_I' is the image point obtained by setting $z_I' = -z'$. Even though the translated distributions are defined through Eq. (A8), giving a precise meaning to $G_R^{(1)}$, we cannot say what is meant by $G_R^{(1)}$ vanishing when $z = 0$.

If we substitute either $G_R^{(1)}$ or $G_R^{(2)}$ in Eqs. (46) and (47), unknown fields in the surface integrals are not eliminated; this constitutes some of the motivation to consider dyadic Green functions.

The appropriate form of Green theorem for the vector wave equation is

$$\int_{t_0}^{t_1} dt \int_V dV \Big[\Big\{\frac{1}{c^2}\frac{\partial^2 \mathbf{u}}{\partial t^2} + \nabla\times(\nabla\times\mathbf{u})\Big\}\cdot\mathbf{v}$$
$$- \Big\{\frac{1}{c^2}\frac{\partial^2 \mathbf{v}}{\partial t^2} + \nabla\times(\nabla\times\mathbf{v})\Big\}\cdot\mathbf{u}\Big]$$
$$= \frac{1}{c^2}\int_V dV \Big[\mathbf{v}\cdot\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u}\cdot\frac{\partial \mathbf{v}}{\partial t}\Big]_{t_0}^{t_1}$$
$$+ \int_{t_0}^{t_1} dt \oint_S d\mathbf{S}\cdot[\mathbf{u}\times(\nabla\times\mathbf{v}) - \mathbf{v}\times(\nabla\times\mathbf{u})]. \tag{B6}$$

Retarded and advanced dyadic Green functions are defined by

$$\frac{1}{c^2}\frac{\partial^2 \mathsf{G}(x,x')}{\partial t^2} + \nabla\times[\nabla\times\mathsf{G}(x,x')] = \delta^{(4)}(x-x')\mathsf{I}, \tag{B7}$$

$$\mathsf{G}_R(x,x') = 0, \quad t < t', \tag{B8}$$

$$\mathsf{G}_A(x,x') = 0, \quad t > t'. \tag{B9}$$

The two simplest types of homogeneous boundary conditions are

$$\hat{n}\cdot\mathsf{G}^{(1)}(x,x')|_{\mathbf{x}\in S} = 0, \tag{B10}$$

$$\hat{n}\cdot\nabla\times\mathsf{G}^{(2)}(x,x')|_{\mathbf{x}\in S} = 0. \tag{B11}$$

In either case, we can use x''_μ as variables of integration in Eq. (B6), let $t_0 \to -\infty$, and substitute

$$\mathbf{u}(x'') = \mathsf{G}_A(x'',x)\cdot\mathbf{a}, \tag{B12}$$

$$\mathbf{v}(x'') = \mathsf{G}_R(x'',x')\cdot\mathbf{b}, \tag{B13}$$

where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary constant vectors. We then use Eqs. (B7)–(B9) and the boundary condition (B10) or (B11) to show the symmetry relation

$$\mathsf{G}_R(x,x') = \widetilde{\mathsf{G}}_A(x',x), \tag{B14}$$

where the tilde indicates the transpose of the dyadic. Similarly, if instead of Eq. (B12) we use

$$\mathbf{u}(x'') = \mathsf{G}_R(\mathbf{x}'', -t''; \mathbf{x}', -t')\cdot\mathbf{a}, \tag{B15}$$

we find

$$\mathsf{G}_R(\mathbf{x},t;\mathbf{x}',t') = \widetilde{\mathsf{G}}_R(\mathbf{x}', -t'; \mathbf{x}, -t), \tag{B16}$$

with the corresponding result for G_A. This equation implies that

$$\frac{1}{c^2}\frac{\partial^2 \widetilde{\mathsf{G}}(x,x')}{\partial t'^2} + \nabla'\times[\nabla'\times\widetilde{\mathsf{G}}(x,x')]$$
$$= \delta^{(4)}(x-x')\mathsf{I}, \tag{B17}$$

$$\hat{n}'\cdot\widetilde{\mathsf{G}}^{(1)}(x,x')|_{\mathbf{x}'\in S} = 0, \tag{B18}$$

$$\hat{n}'\cdot\nabla'\times\widetilde{\mathsf{G}}^{(2)}(x,x')|_{\mathbf{x}'\in S} = 0. \tag{B19}$$

If we use x'_μ as variables of integration in Eq. (B6) and set

$$\mathbf{u}(x') = \mathbf{E}(x'), \tag{B20}$$

$$\mathbf{v}(x') = \widetilde{\mathsf{G}}_R(x,x')\cdot\mathbf{a}, \tag{B21}$$

we find

$$\mathbf{E}(x) = -\mu_0 \int_{t_0}^{t} dt' \int_V dV' \,\mathsf{G}_R(x,x')\cdot\frac{\partial \mathbf{j}(x')}{\partial t'}$$
$$+ \frac{1}{c^2}\int_V dV' \Big[\mathsf{G}_R(x,x')\cdot\Big\{c^2\,\nabla'\times\mathbf{B}(x') - \frac{\mathbf{j}(x')}{\epsilon_0}\Big\}$$
$$- \frac{\partial \mathsf{G}_R(x,x')}{\partial t'}\cdot\mathbf{E}(x')\Big]_{t'=t_0}$$
$$- \int_{t_0}^{t} dt' \oint_S dS'\,\hat{n}'\cdot\Big[\mathbf{E}(x')\times\{\nabla'\times\widetilde{\mathsf{G}}_R(x,x')\}$$
$$- \frac{\partial \mathbf{B}(x')}{\partial t'}\times\widetilde{\mathsf{G}}_R(x,x')\Big]. \tag{B22}$$

If we know $\hat{n}\times\mathbf{E}$ on S, we can use $\mathsf{G}_R^{(1)}$ to eliminate the second term in the surface integral, and, if we know $\hat{n}\times\mathbf{B}$ on S, $\mathsf{G}_R^{(2)}$ serves to eliminate the first term. In either case, Eq. (B22) gives $\mathbf{E}$ as a sum of integrals over known functions when the right Green function can be found.

In practice, we probably would not be able to find $G_R^{(1)}$ or $G_R^{(2)}$. Alternatively, we can use the free-space Green function $G_R^{(0)}$ in Eq. (B22) and let $\mathbf{E}$ approach S from the outside, where $\mathbf{E}$ vanishes. We obtain an integral equation where either $\hat{n}\times\mathbf{E}$ or $\hat{n}\times\mathbf{B}$ is known and the other is to be determined. We could also use $\hat{n}\times(\nabla\times\mathbf{E})$ instead of $\hat{n}\times\mathbf{B}$ in this formulation. Once we know both tangential fields, Eq. (B22) gives $\mathbf{E}$ everywhere.

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