Oct 21 notes

Activity 2.9.3.

Compute a bunch of bases for systems of linear equations.

Consider the homogeneous system of equations

(a) Find its solution set (a subspace of $\mathbb{R}^4$)
$1x_1 + 2x_2 + 0x_3 + 1x_4 = 0$ $2x_1 + 4x_2 - x_3 - 2x_4 = 0$ $3x_1 + 6x_2 - x_3 - 1x_4 = 0$

as matrix

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 2 & 4 & -1 & -2 & 0 \\ 3 & 6 & -1 & -1 & 0 \end{bmatrix}$$

```
(Matrix(QQ,[
[1,2,0,1,0],
[2,4,-1,-2,0],
[3,6,-1,-1,0]
])).rref()
[1 2 0 1 0]
[0 0 1 4 0]
[0 0 0 0 0]
```

Corresponds to:
$1x_1 + 2x_2 + 0x_3 + 1x_4 = 0$
$1x_3 + 4x_4 = 0$

Let free variables $x_2 = a$ and $x_4 = b$
$1x_1 + 2a + 1b = 0$
$1x_3 + 4b = 0$

$1x_1 = -2a - 1b$
$1x_3 = -4b$

(b) Rewrite this solution space in set builder notation
$$\left\{ \begin{bmatrix} -2a - 1b \\ a \\ -4b \\ b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$$

(c) Rewrite this solution space in the form of a span $span = \left\{ a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, b \begin{bmatrix} -1 \\ 0 \\ -4 \\ 1 \end{bmatrix}, \right\}$

(d) Since we can say:
$$\begin{bmatrix} -2a-1b \\ a \\ -4b \\ b \end{bmatrix} = a\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 0 \\ -4 \\ 1 \end{bmatrix},$$
The basis is:

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$$

fact 2.9.3 – the solution space for a homogenous system is a basis for the solution space

Activity 2.9.5

$2x_1 + 4x_2 + 2x_3 - 4x_4 = 0$
$-2x_1 - 4x_2 + 1x_3 + 1x_4 = 0$
$3x_1 + 6x_2 - 1x_3 - 4x_4 = 0$

as matrix

```
(Matrix(QQ,[
[2,4,2,-4,0],
[-2,-4,1,1,0],
[3,6,-1,-4,0]
])).rref()
[1 2 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
```

So this means the basis is 3 equations long bc 3 pivots
$1x_1 + 2x_2 = 0$
$1x_3 = 0$
$1x_4 = 0$

when you write your solution set youd only have 3 variables Activity 2.9.6

$$x_1\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}, x_2\begin{bmatrix} 4 \\ -4 \\ 6 \end{bmatrix}, x_3\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, x_4\begin{bmatrix} -4 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
as matrix

```
(Matrix(QQ,[
[2,4,2,-4,0],
[-2,-4,1,1,0],
[3,6,-1,-4,0]
])).rref()
[1 2 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
```

So this means the basis is 3 equations long bc 3 pivots
$1x_1 + 2x_2 = 0$
$1x_3 = 0$
$1x_4 = 0$

Activity 2.9.6

Activity 2.9.7

If the solution set is 1 vector its linearly dependent for 2.9.7 the only solution is the zero vector, so the basis is just the empty set

Observation 2.9.8 if 0 vector is the only solution of a homogenous system then the basis of the solution space is the empty set
a subspace will always contain the 0 vector

Example B.1.13 VS9
Consider the homogenous system of equations:

$1x_1 + 1x_2 + 3x_3 + 1x_4 + 2x_5 = 0$
$-3x_1 + 0x_2 - 6x_3 + 6x_4 + 3x_5 = 0$
$-1x_1 + 1x_2 - 1x_3 + 1x_4 + 0x_5 = 0$
$2x_1 - 2x_2 + 2x_3 - 1x_4 + 1x_5 = 0$

(1) Find the solution space of the system

Find the RREF:

```
(Matrix(QQ,[
[1,1,3,1,2,0],
[-3,0,-6,6,3,0],
[-1,1,-1,1,0,0],
[2,-2,2,-1,1,0]
])).rref()
[1 0 2 0 1 0]
[0 1 1 0 0 0]
[0 0 0 1 1 0]
[0 0 0 0 0 0]
```

Since they correspond to the non-pivot columns, let $x_3 = a, x_5 = b$
Then we get:
$1x_1 + 2x_3 + x_5 = 0$
$1x_2 + 1x_3 = 0$
$x_3 = a\ x_4 + x_5$
$x_5 = b$

So the solution set is:

$$\left\{ \begin{bmatrix} -2a - 1b \\ -a \\ a \\ -b \\ b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$$

(2) Find the basis of the solution space.

Since $\begin{bmatrix} -2a - 1b \\ -a \\ a \\ -b \\ b \end{bmatrix} = a \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

The basis for the solution space is:

$$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$