Oct 17 notes

Definition 2.6.3 A basis is a linearly independent set that spans a vector space.

Fact 2.7.3 to compute a basis for the subspace span S, simply remove the vectors corresponding to the non-pivot columns of RREF

Fact 2.7.9 Any non-trivial real vector space has infinitely-many different bases, but all the bases for a given vector space are exactly the same size (non-trivial means that it isn't just 0 vector).

Definition 2.7.10 The dimension of a vector space is equal to the size of any basis for the vector space (number of vectors in basis).

Activity 2.7.4

$$\text{Let } W = span\left\{\begin{bmatrix}1\\3\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\\2\end{bmatrix}, \begin{bmatrix}4\\5\\3\\0\end{bmatrix}, \begin{bmatrix}3\\2\\2\\1\end{bmatrix}\right\}. \text{ Find a basis for } W.$$

Calculate the RREF of the matrix:

```
W=Matrix(QQ,[ [1,2,4,3],[3,-1,5,2],[1,1,3,2], [-1,2,0,1] ])
W.rref()
[1 0 2 1]
[0 1 1 1]
[0 0 0 0]
[0 0 0 0]
```

First two are pivot columns, the second two are free variables, so the first two columns are the 'basis' of the subspace W.

subspace $W = span\left\{\begin{bmatrix}1\\3\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\\2\end{bmatrix}, \begin{bmatrix}4\\5\\3\\0\end{bmatrix}, \begin{bmatrix}3\\2\\2\\1\end{bmatrix}\right\}$ has $\left\{\begin{bmatrix}1\\3\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\\2\end{bmatrix}\right\}$ as a basis.

Activity 2.7.5

Let W be the subspace of $\mathcal{P}$ given by
$\mathrm{W} = \mathrm{span}\left\{x^3+3x^2+x-1, 2x^3-x^2+x+2, 4x^3+5x^2+3x, 3x^3+2x^2+2x+1\right\}$,
find a basis for W.

x^3+3x^2+x-1
$2x^3-x^2+x+2$
$4x^3+5x^2+3x+0$
$3x^3+2x^2+2x+1$

Matrices of polynomial equations should be set up such that all the like-exponents are on the same row, so re-group like exponents.

$x^3+2x^3+4x^3+3x^3$
$3x^2-x^2+5x^2+2x^2$
$x+x+3x+2x$
$-1+2+0+1$

As matrix:

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 3 & -1 & 5 & 2 \\ 1 & 1 & 3 & 2 \\ -1 & 2 & 0 & 1 \end{bmatrix}$$

```
Matrix(QQ,[
[1,2,4,3],
[3,-1,5,2],
[1,1,3,2],
[-1,2,0,1]
]).rref()
[1 0 2 1]
[0 1 1 1]
[0 0 0 0]
[0 0 0 0]
```

First two are pivots, so the basis of the subspace is:

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 1 & 1 \\ -1 & 2 \end{bmatrix}$$

And when we put that back into the format of the original queston:
$\left\{x^3+3x^2+x-1, 2x^3-x^2+x+2\right\}$

Activity 2.7.6

Let W be the subspace of $M_{2,2}$ given by $W = span = \left\{ \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 5 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \right\}$. Find a basis for W.

Multiply each matrix by some scalar (x1, x2, x3, x4)

$$\begin{bmatrix} 1x_1 & 3x_1 \\ 1x_1 & -1x_1 \end{bmatrix}, \begin{bmatrix} 2x_2 & -1x_2 \\ 1x_2 & 2x_2 \end{bmatrix}, \begin{bmatrix} 4x_3 & 5x_3 \\ 3x_3 & 0x_3 \end{bmatrix}, \begin{bmatrix} 3x_4 & 2x_4 \\ 2x_4 & 1x_4 \end{bmatrix}$$

expand:

$$\begin{bmatrix} (1x_1 + 2x_2 + 4x_3 + 3x_4) & (3x_1 - 1x_2 + 5x_3 + 2x_4) \\ (1x_1 + 1x_2 + 3x_3 + 2x_4) & (-1x_1 + 2x_2 + 0x_3 + 1x_4) \end{bmatrix}$$

as matrix:

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 3 & -1 & 5 & 2 \\ 1 & 1 & 3 & 2 \\ -1 & 2 & 0 & 1 \end{bmatrix}$$

```
W=Matrix(QQ,[
[1,2,4,3],
[3,-1,5,2],
[1,1,3,2],
[-1,2,0,1]
])
W.rref()
[1 0 2 1]
[0 1 1 1]
[0 0 0 0]
[0 0 0 0]
```

The first two pivots are the basis of the subspace:

$$\left\{ \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \right\}$$

Activity 2.7.7

S = $\left\{ \begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix} \right\}$ T = $\left\{ \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\\1\end{bmatrix} \right\}$

Find a basis for span S and span T

Span S

```
Matrix(QQ,[
[2,2,2,1],
[3,0,-3,5],
[0,1,2,-1],
[1,-1,3,0]
]).rref()
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

Basis of span S: $\left\{ \begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix} \right\}$

Basis of span T

```
Matrix(QQ,[
[2,2,1,2],
[0,-3,5,3],
[1,2,-1,0],
[-1,-3,0,1]
]).rref()
[ 1  0  0  2]
[ 0  1  0 -1]
[ 0  0  1  0]
[ 0  0  0  0]
```

Basis of T

$$\left\{ \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix} \right\}$$

Activity 2.7.11 Find the dimension of each subspace of R^3 by finding for each corresponding matrix.

$$A = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

```
A=Matrix(QQ,[
[2,2,4,-3],
[3,0,3,0],
[0,0,0,1],
[-1,3,2,3]
])
A.rref()
[1 0 1 0]
[0 1 1 0]
[0 0 0 1]
[0 0 0 0]
```

No pivot in column 3, so remove it to get basis:

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$B = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

```
Matrix(QQ,[
[2,2,3,-1,4],
[3,0,13,10,3],
[0,0,7,7,0],
[-1,3,16,14,2]
]).rref()
[ 1  0  0 -1  1]
[ 0  1  0 -1  1]
[ 0  0  1  1  0]
[ 0  0  0  0  0]
```

$$\text{basis of } B = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$$

$$C = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \\ 5 \end{bmatrix} \right\}$$

```
Matrix(QQ,[
[2,4,-3,3],
[3,3,0,6],
[0,0,1,1],
[-1,2,3,5]
]).rref()
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

$$\text{basis of } C = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \\ 5 \end{bmatrix} \right\}$$

$$D = \left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right\}$$

```
Matrix(QQ,[
[5,-2,4],
[3,1,5],
[0,0,1],
[-1,3,3]
]).rref()
[1 0 0]
[0 1 0]
[0 0 1]
[0 0 0]
```

$$\text{Basis of } D = \left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right\}$$