Oct 19 notes

Fluency exam next week - October 27th.
Fact 2.8.1: Every vector space with finite dimension (every vector space V with a basis of the form $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$) is said to be isomorphic to a Euclidean space $\mathbb{R}^n$ (For the purposes of this class, isomorphic just means have bases of the same size.)

Observation 2.8.2: We have already seen this in action by converting polynomials and matrices into Euclidean vectors. Since $\mathcal{P}_3$ and $M_{2,2}$ are both four-dimensional:

$$4x^3 + 0x^2 - 1x + 4 \leftrightarrow \begin{bmatrix} 4 \\ 0 \\ -1 \\ 5 \end{bmatrix} \leftrightarrow \begin{bmatrix} 4 & 0 \\ -1 & 5 \end{bmatrix}$$

Fact 2.4.11 If S is any subset of a vector space V , then since span S collects all possible linear combinations, span S is automatically a subspace of V . In fact, span S is always the smallest subspace of V that contains all the vectors in S

Activity 2.8.3

Suppose W is a subspace of $\mathcal{P}^8$ and you know that the set $S = \left\{x^3 + x, x^2 + 1, x^4 - x\right\}$ is a linearly independent subset of W. What can you conclude about W?
A. the dimension is 3 or less
B. the dimension of W is 3 exactly
C. the dimension of W is 3 or more

- W is a subspace of $\mathcal{P}^8$, S is a sub**set** of W.
- We want to know about the dimension of W and the dimension of a vector space is equal to the size of any basis for the vector space (Definition 2.7.10).
- We can't say anything about the basis of W, but we can say something about the basis of the vectors S (which are a subset of W).
- Since we know the vectors of S are linearly independent, we also know every column of the RREF of S will have a pivot (Observation 2.5.8).
- The easiest basis describing a span S is the set of vectors in S given by the pivot columns of the RREF – to compute a basis for a subspace, compute the RREF and remove non-pivot columns (Fact 2.7.3)
- **That means S has 3 dimensions, so W has at least that many (C).**

Activity 2.8.3 continued
Confirming the answer by computing RREF
$0x^4 + x^3 + 0x^2 + 1x + 0$
$0x^4 + 0x^3 + x^2 + 0x + 1$
$x^4 + x^3 + 0x^2 - 1x + 0$

As matrix, like exponents on same row:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

```
(Matrix(QQ,[
[0,0,1],
[1,0,1],
[0,1,0],
[1,0,-1],
[0,1,0]
])).rref()
[1 0 0]
[0 1 0]
[0 0 1]
[0 0 0]
[0 0 0]
```

- Confirms that W has at least 3 dimensions.

Activity 2.8.4
Suppose W is a subspace of $\mathcal{P}^8$ and you know that W is spanned by the set $S = \left\{x^4 - x, x^3 + x, x^3 + x + 1, x^4 + 2x, x^3, 2x + 1\right\}$. What can you conclude about W?
A. the dimension is 3 or less
B. the dimension of W is 3 exactly
C. the dimension of W is 3 or more

- We want to know about the dimension of W and the dimension of a vector space is equal to the size of any basis for the vector space (Definition 2.7.10).
- We can't say anything about the basis of W, but we can say something about the basis of the vectors S (which span W).
- The easiest basis describing a span S is the set of vectors in S given by the pivot columns of the RREF – to compute a basis for a subspace S, compute the RREF and remove non-pivot columns (Fact 2.7.3)
- Since S spans W we have a pivot in every row of the RREF of S (Observation 2.5.8).
- Since like-exponents go on a row, and we have 4 unique exponent-types, we

should have 4 pivot rows, each with their own pivot columns.
- In other words, if we have a pivot on every row, we have a pivot for every term.
- Since we have 4 terms (no x^2 term), we have 4 pivots and 4 dimensions.

Confirm by finding RREF:
As equations w zeroes filled in for clarity

$1x^4 + 0x^3 + 0x^2 - 1x + 0$
$0x^4 + 1x^3 + 0x^2 + 1x + 0$
$0x^4 + 1x^3 + 0x^2 + 1x + 1$
$1x^4 + 0x^3 + 0x^2 + 2x + 0$
$0x^4 + 1x^3 + 0x^2 + 0x + 0$
$0x^4 + 0x^3 + 0x^2 + 2x + 1$

As matrix, like exponents on same row:
$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

```
(Matrix(QQ,[
[1,0,0,1,0,0],
[0,1,1,0,1,0],
[0,0,0,0,0,0],
[-1,1,1,2,0,2],
[0,0,1,0,0,1]
])).rref()
[   1    0    0    0  1/3 -2/3]
[   0    1    0    0    1   -1]
[   0    0    1    0    0    1]
[   0    0    0    1 -1/3  2/3]
[   0    0    0    0    0    0]
```

- Confirms that we have 4 pivot rows/columns, - If we have a pivot on every row, we have a pivot for every term.
- Since we have 4 terms (no x^2 term), we have 4 pivots and 4 dimensions.

Observation 2.8.5: The space of polynomials of any degree has the basis of all its exponent terms, so it is a natural example of an infinite dimensional vector space and can't be treated as an isomorphic finite-dimensional Euclidean space.

Definition 2.9.1: – a homogeneous system of linear equations is one of form:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= 0 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= 0 \\ &\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= 0 \end{aligned}$$

Which is equivalent to the vector equation:
$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$

...And the augmented matrix:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} & | & 0 \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} & | & 0 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} & | & 0 \end{bmatrix}$$

Activity 2.9.2:

Note that if $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ are solutions to $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$,

then so is $\begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$, since $a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}$ and $b_1\vec{v}_1 + \cdots + b_n\vec{v}_n = \vec{0}$.

This implies that $(a1 + b_1)\vec{v}_1 + \cdots + (a_n + b_n)\vec{v}_n = \vec{0}$

Similarly, if $c \in \mathbb{R}$, $\begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$ is a solution.

Thus, the solution set of a homogenous system is:

A. A basis for $\mathbb{R}^n$,
B. The subspace of $\mathbb{R}^n$,
or
C. The empty set

Answer is B because all of the exposition in this problem just described how we are closed to vector addition and scalar multiplication, so we can say that we do have a subspace. We don't know if its linearly dependent and spans, and I'm not sure why it isn't the empty set.
The first part is basically saying that if you multiply $\vec{v}_1 \dots \vec{v}_n$ by either $\vec{a}_1 \dots \vec{a}_n$ or $\vec{b}_1 \dots \vec{b}_n$, you'd get a vector of zeroes ($\vec{0}$) and so they are a solution to the system.

Since that is true, then even if we add a+b and multiply by v, it's also still a a solution (v(a+b) = 0).

As an example, let's say our vectors are as follows:

$v_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_3 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

What values could we have for vectors a, b, c where $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{0}$ would hold true?

$x_1 = 0, x_2 = 0, x_3 = 0$
$x_1 = 2, x_2 = 0, x_3 = -1$
$x_1 = 4, x_2 = 0, x_3 = -2$

And so on... such that $\left\{ \begin{bmatrix} -2ac \\ 0c \\ ac \end{bmatrix} \middle| a, c \in \mathbb{R} \right\}$ and $\vec{0}$ is also a solution.

subspace is a set of vectors that's closed under vector addition and scalar multiplication