Oct 14 notes

Activity 2.6.2:

$\vec{e}_1 = \hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e}_3 = \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

a) $\vec{v}$ can be expressed as a linear combination of $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ because we have a pivot in every row, we can set i, j, and k to be any scalar to get any values of $\vec{v}$

b) If $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}$ can't be expressed as a linear combination of $\vec{e}_1$, $\vec{e}_2$, and $\vec{w}$ because we don't have a pivot in row 3, so not spanning set.

c) All vectors in $\mathbb{R}^3$ can be written as linear combinations of $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ because we have a pivot in every row and column - linearly independent vectors that span the set

Observation 2.6.3: A basis is a linearly independent set that spans a vector space. In example 2.6.2, $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ is an example of the standard basis vectors of $\mathbb{R}^3$.

Observation 2.6.4 A basis may be thought of as a collection of building blocks for a vector space, since every vector in the space can be expressed as a unique linear combination of basis vectors.

Observation 2.5.8 says:
- A set of vectors is **linearly independent** if and only if the RREF has all pivot columns.
- A set of $\mathbb{R}^m$ vectors **spans** $\mathbb{R}^m$ if the RREF has all pivot rows

Definition 2.6.3 says:
- A basis is a linearly independent set that spans a vector space.
(This means it should have pivots in every row and column.)

Activity 2.6.5
Label each of the sets as: spanning $\mathbb{R}^4$, linearly independent, or a basis for $\mathbb{R}^4$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 2 & 4 & -3 \\ 3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 3 & 2 & 3 \end{bmatrix}, C = \begin{bmatrix} 2 & 2 & 3 & -1 & 4 \\ 3 & 0 & 13 & 10 & 3 \\ 0 & 0 & 7 & 7 & 0 \\ -1 & 3 & 16 & 14 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 4 & -3 & 3 \\ 3 & 3 & 0 & 6 \\ 0 & 0 & 1 & 1 \\ -1 & 2 & 3 & 5 \end{bmatrix}, E = \begin{bmatrix} 5 & -2 & 4 \\ 3 & 1 & 5 \\ 0 & 0 & 1 \\ -1 & 3 & 3 \end{bmatrix}$$

```
B=Matrix(QQ,[ [2,2,4,-3],[3,0,3,0],[0,0,0,1], [-1,3,2,3] ])
B.rref()
C=Matrix(QQ,[ [2,2,3,-1,4],[3,0,13,10,3],[0,0,7,7,0], [-1,3,16,14,2] ])
C.rref()
D=Matrix(QQ,[ [2,4,-3,3],[3,3,0,6],[0,0,1,1], [-1,2,3,5] ])
D.rref()
E=Matrix(QQ,[ [5,-2,4],[3,1,5],[0,0,1], [-1,3,3] ])
E.rref()
```

$$RREF(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, RREF(B) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$RREF(C) = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$RREF(D) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, RREF(E) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Span (all pivot rows in the RREF): A, D
Linearly Independent (RREF has all pivot columns): A, D, E
Basis (all pivot rows and columns in the RREF): A, D
None of the above: B, C

Activity 2.6.6

If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for R^4 that means RREF would be:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This would be the identity matrix, with pivots in all rows and columns.

Fact 2.6.7
A basis for R^n must have n vectors and its square matrix must reduce to the identity matrix containing all 0s except for 1 along the diagonal pointing down and to the right.

Observation 2.7.1 - subspace of a vector space is a subset that is itself a vector space.

One easy way to construct a subspace is to take the span of set, but a linearly dependent set contains "redundant" vectors. For example, only two of the three vectors in the following image are needed to span the planar subspace.

Activity 2.7.2 Consider the subspace of $\mathbb{R}^4$ given by:

$$W = span = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}$$

(a) Mark the parts of the RREF of the matrix to show what parts show that W's spanning set is linearly dependent.

```
W=Matrix(QQ,[ [2,2,2,1],[3,0,-3,5],[0,1,2,-1], [1,-1,-3,0] ])
W.rref()
```

RREF(W):

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Linearly Independent means RREF has all pivot columns, but column 3 does not have a pivot so this is a linearly dependent system.

(b) Find a basis for W by removing a vector from its spanning set to make it linearly independent.

Drop 3rd vector because it isn't a pivot column.

```
W=Matrix(QQ,[ [2,2,1],[3,0,5],[0,1,-1], [1,-1,0] ])
W.rref()
```

RREF(W):

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Fact 2.7.3: To compute a basis for the subspace, simply remove the vectors corresponding to the non-pivot columns of the RREF.