Systeem- en Regeltechniek
EE2S21

Pole Placement

Lecture 8

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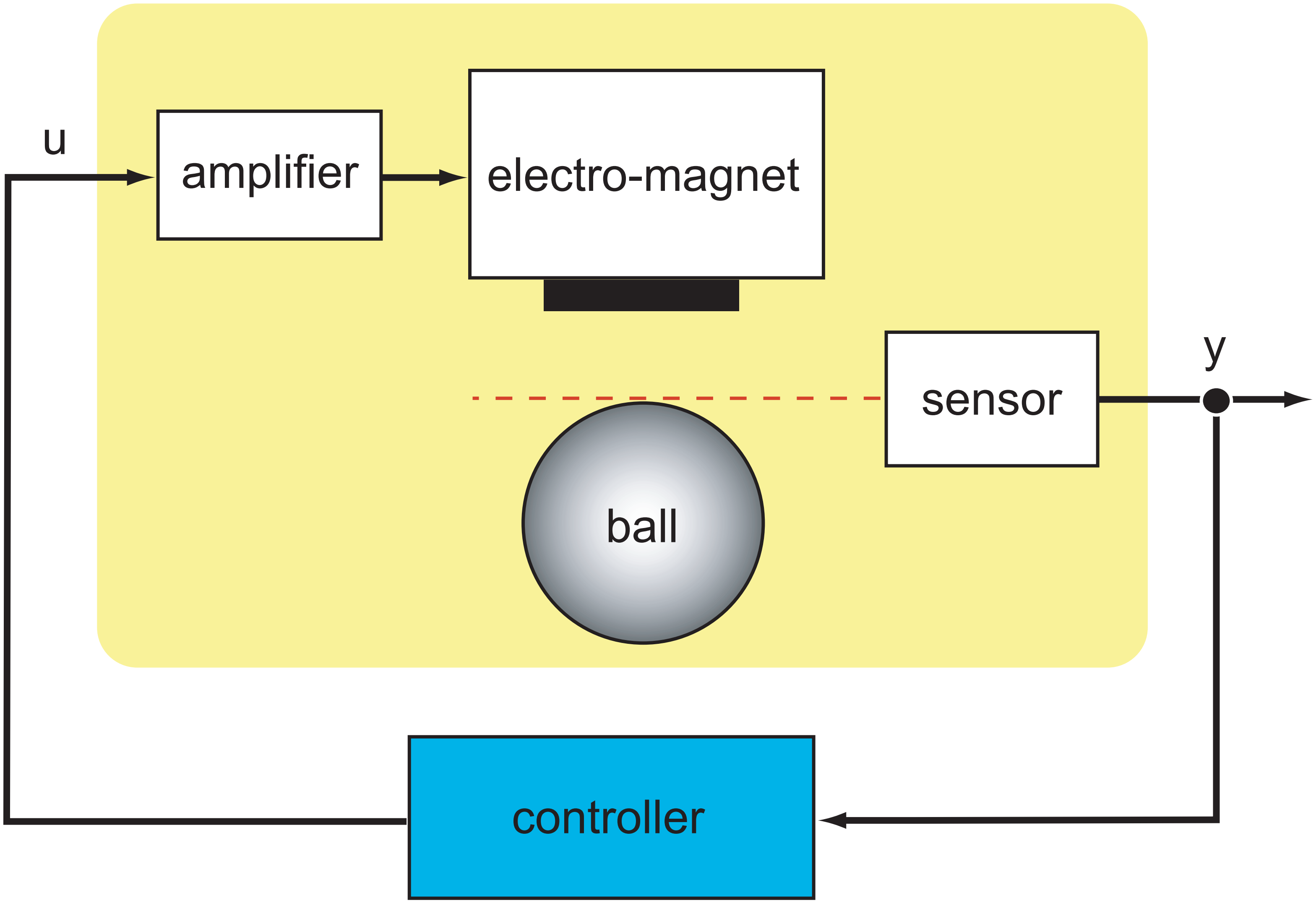
Feedback Control: Levitated Ball System



Feedback Control

Consider the LTI system

$$\dot{x} = Ax + Bu$$
$$y = Cx,$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$.

Two types of feedback:

- $u = -Fx$ (static state feedback);
- $u = -Hy$ (static output feedback).

Feedback Control

The LTI system

$$\dot{x} = Ax + Bu$$
$$y = Cx,$$

is **stabilizable** if there exists a state feedback

$$u = -Fx$$

such that $\mathrm{Re}(\lambda) < 0$ for all eigenvalues of $A - BF$.

Pole-Assignment Theorem

Theorem: The LTI system

$$\dot{x} = Ax + Bu$$
$$y = Cx,$$

is **controllable** if and only if for every polynomial

$$r(\lambda) = \lambda^n + r_{n-1}\lambda^{n-1} + \cdots + r_1\lambda + r_0$$

there exists an F such that

$$\det(\lambda I - (A - BF)) = r(\lambda).$$

Pole-Assignment Theorem

Proof: (sufficiency) Consider SISO case ($m = 1$), suppose (A, B) is controllable, and

$$\det(\lambda I - A) = \lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0.$$

Then, the set of column vectors $\{B, AB, \ldots, A^{n-1}B\}$ is linearly independent. As a result,

$$\begin{aligned}
q_n &= B \\
q_{n-1} &= AB + p_{n-1}B = Aq_n + p_{n-1}q_n \\
q_{n-2} &= A^2B + p_{n-1}AB + p_{n-2}B = Aq_{n-1} + p_{n-2}q_n \\
&\vdots \\
q_1 &= A^{n-1}B + p_{n-1}A^{n-2}B + \cdots + p_1B = Aq_2 + p_1q_n.
\end{aligned}$$

Pole-Assignment Theorem

Then, construct matrix $T = (q_1, \ldots, q_n)$ such that

$$\bar{A} = T^{-1}AT, \quad \bar{B} = T^{-1}B,$$

yielding the **controller canonical form**

$$\bar{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & -p_{n-2} & -p_{n-1} \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Pole-Assignment Theorem

Now, take

$$\bar{F} = (r_0 - p_0, r_1 - p_1, \ldots, r_{n-1} - p_{n-1}),$$

then

$$\bar{A} - \bar{B}\bar{F} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -r_0 & -r_1 & \cdots & -r_{n-2} & -r_{n-1} \end{pmatrix},$$

and therefore $\det(\lambda I - (\bar{A} - \bar{B}\bar{F})) = r(\lambda)$. ■

Controllability versus Stabilizability

- Pair (A, B) is controllable iff

$$\text{rank}(\lambda I - A \quad B) = n$$

for all $\lambda \in \mathbb{C}$.

- Pair (A, B) is stabilizable iff

$$\text{rank}(\lambda I - A \quad B) = n$$

for all $\text{Re}(\lambda) \geq 0$.

Example 1

Consider the linearized levitated ball system

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2g} \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u.$$

Design a stabilizing state feedback controller $u = -Fx$.

Example 2

Consider the LTI system

$$\dot{x} = \begin{pmatrix} -7 & 1 \\ -12 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ -\alpha \end{pmatrix} u,$$

with $\alpha \in \mathbb{R}$. Controllability matrix

$$R = [B \;\; AB] = \begin{pmatrix} 1 & -7-\alpha \\ -\alpha & -12 \end{pmatrix}$$

System uncontrollable for $\alpha = -3$ or $\alpha = -4$.

Open-loop poles: $-3, -4$. Desired closed-loop poles: $-5, -6$

Example 2 (cont'd)

Two important observations:

- Gains increase as α approaches either -3 or -4, the values where controllabilty is lost. In other words, the control effort increases as controllabity slips away.
- The further the poles are moved, the larger the required gains.