Systeem- en Regeltechniek
EE2S21

System Response + Transfer Functions

Lecture 5a

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February 19, 2015

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LTI state-space system

Suppose a system is described by the LTI model:

$$\dot{x} = Ax + Bu \qquad \text{(state equation)}$$

$$y = Cx + Du \qquad \text{(output equation).}$$

What happens to the output $y(t)$ if a certain input signal $u(t)$ is applied to the system?

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with initial condition: $x(t_0) = x_0$. If $A \in \mathbb{R}^{n \times n}$, then this system has n independent solutions denoted as $\xi_1(t), \ldots, \xi_n(t)$. Let

$$Y(t) = \big(\xi_1(t), \ldots, \xi_n(t)\big),$$

(fundamental matrix), then

$$\Phi(t, t_0) = Y(t)Y^{-1}(t_0)$$

is called **state transition matrix**.

LTI state-space system

State transition matrix describes transition from $x(t_0)$ to $x(t)$, i.e.,

$$x(t) = \Phi(t, t_0)x(t_0), \quad \Phi(t_0, t_0) = I.$$

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For LTI systems: $\Phi(t,t_0) = e^{A\cdot(t-t_0)}$, so that

$$x(t) = e^{A\cdot(t-t_0)}x(t_0),$$

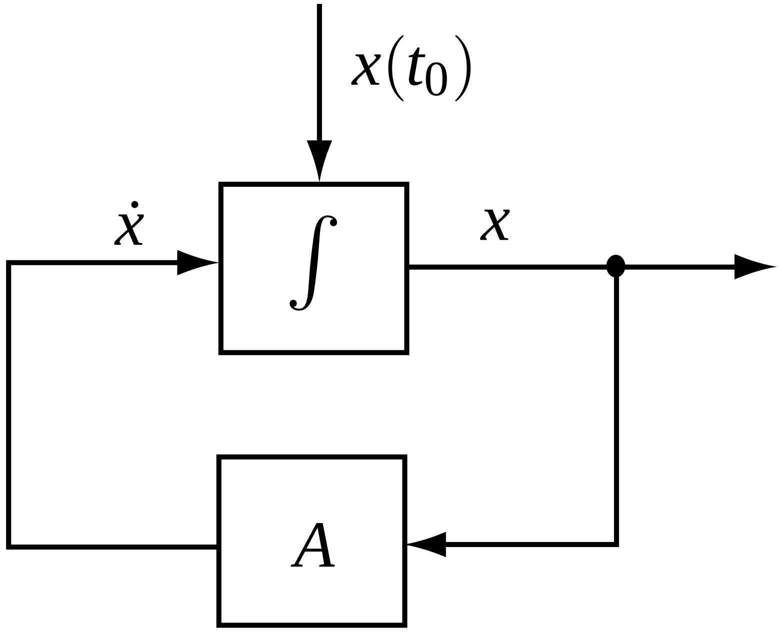
with the **matrix exponential**

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}.$$

Proof: substitute solution into $\dot{x} = Ax$...

Alternative Proof

$$x(t) = x(t_0) + \int_{t_0}^{t} \dot{x}(\sigma_1)\,d\sigma_1$$



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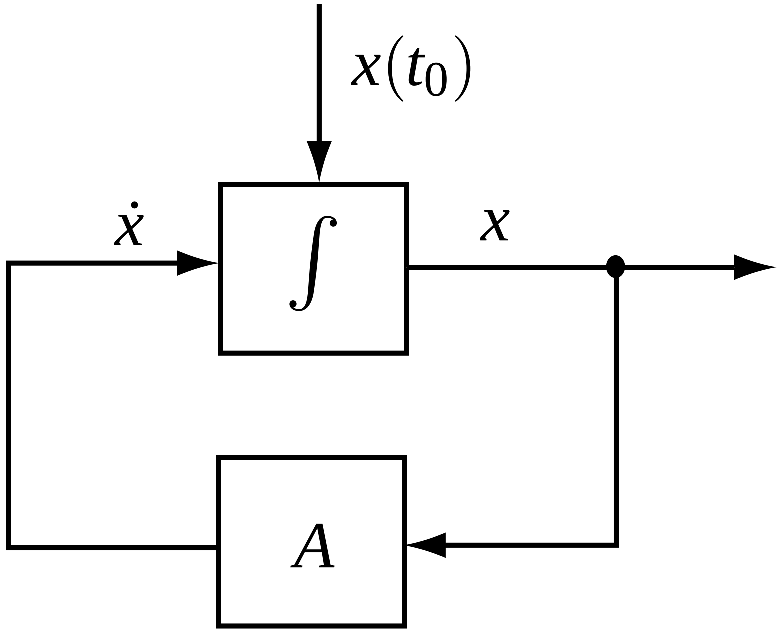

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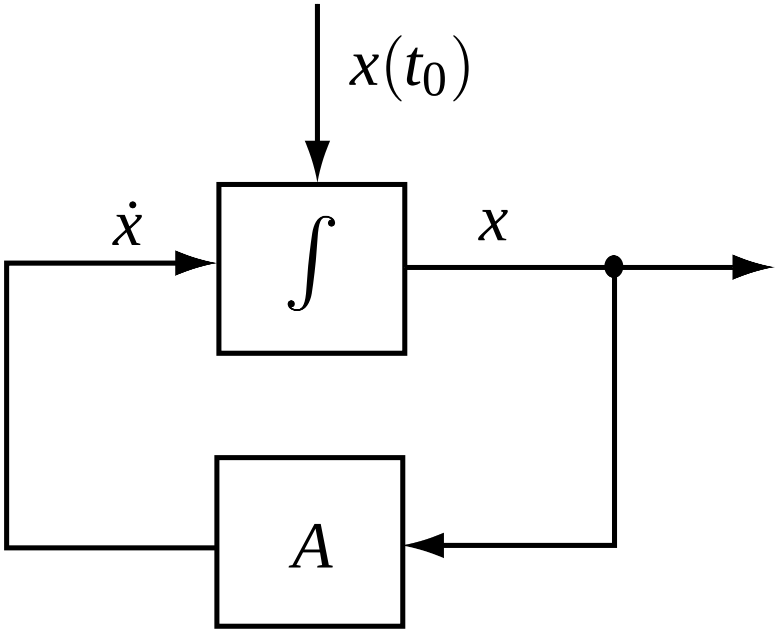

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$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} Ax(t_0)d\sigma_1 + \int_{t_0}^{t} A \int_{t_0}^{\sigma_1} Ax(t_0)d\sigma_2 d\sigma_1 \\ &\quad + \int_{t_0}^{t} A \int_{t_0}^{\sigma_1} A \int_{t_0}^{\sigma_2} Ax(t_0)d\sigma_3 d\sigma_2 d\sigma_1 + \ldots \\ &= \left[I + A \cdot (t - t_0) + \frac{A^2 \cdot (t-t_0)^2}{2!} + \cdots \right] x(t_0) \end{aligned}$$

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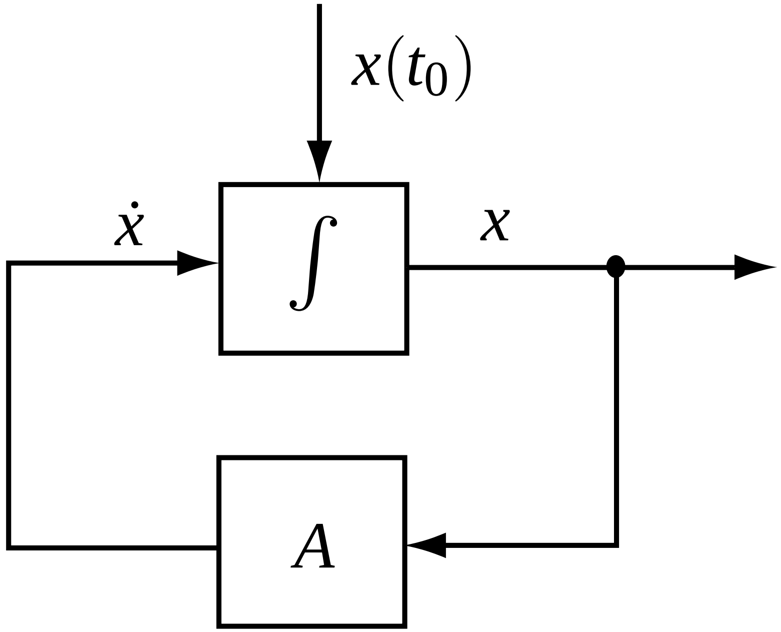

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$$= \left[I + A \cdot (t - t_0) + \frac{A^2 \cdot (t - t_0)^2}{2!} + \cdots \right] x(t_0) = \boxed{e^{A \cdot (t - t_0)} x(t_0)}$$

Non-zero input case

For $u(t) \neq 0$, we obtain

$$x(t) = e^{A(t-t_0)}x(t_0) + \boxed{\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau}, \quad t \geq t_0,$$

and for the **output response**

$$\begin{aligned} y(t) &= Cx(t) + Du(t) \\ &= Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \geq t_0. \end{aligned}$$

How to compute e^{At}???

Some Important Matrix Properties

Recall:

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- Geometric multiplicity $=$ max # of independent eigenvectors;
- Algebraic multiplicity: # of same λ's as a root of det$(\lambda I - A) = 0$.

Some Useful Properties

P1) If $A_1 A_2 = A_2 A_1$ (A_1 and A_2 commute), then

$$e^{A_1 + A_2} = e^{A_1} e^{A_2}.$$

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P3) If P is invertible, then $e^{P^{-1}AP} = P^{-1}e^{A}P$.

P4) If $\lambda_k \in \mathbb{C}$, $k = 1, \ldots, n$, then $e^{\text{diag}(\lambda_1,\ldots,\lambda_n)} = \text{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})$.

Eigenvalue Decomposition

Suppose that $A \in \mathbb{R}^{n \times n}$ has n linear independent eigenvectors forming the columns of T, then $AT = TA_D$, where $A_D = \text{diag}(\lambda_1, \ldots, \lambda_n)$.

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Hence, using properties P3 and P4, it follows that

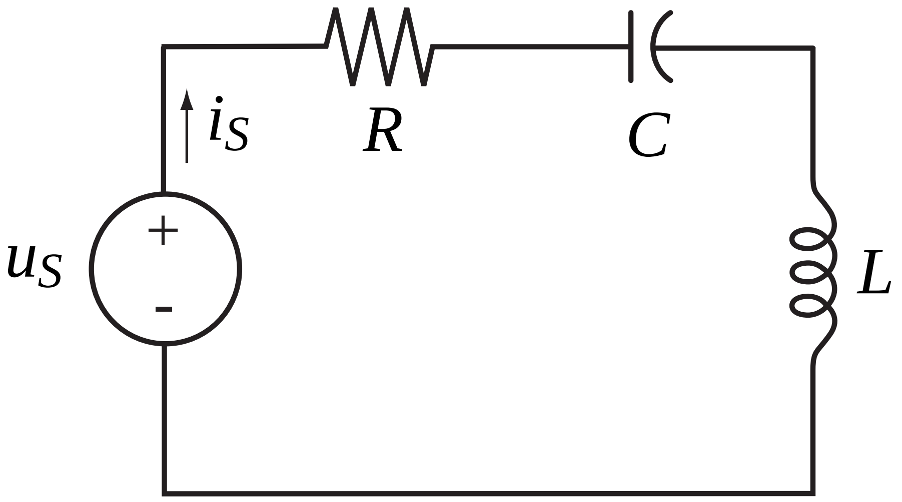
$$e^{At} = Te^{A_D t}T^{-1} = T \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} T^{-1}.$$

Example: Series RLC circuit

State: $x = \begin{pmatrix} q_C \\ \phi_L \end{pmatrix}$

Input: $u = u_S$

Output: $y = i_S$.



State space model:

$$\dot{x} = \begin{pmatrix} 0 & 1/L \\ -1/C & -R/L \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad x(0) = \begin{pmatrix} q_C(0) \\ \phi_L(0) \end{pmatrix} = 0$$

$$y = \begin{pmatrix} 0 & 1/L \end{pmatrix} x$$

Exercise

Let A be given by

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{pmatrix}.$$

Compute e^{At}.

System Response

Recall solution of $\dot{x}(t) = Ax(t) + Bu(t)$, with $x(t_0) = x_0$, is given by

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-s)}Bu(s)\,ds, \quad t \geq t_0,$$

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System Response

Let's make a few assumptions to simplify the math and emphasize the concepts...

- Let $t_0 = 0$ (start at zero) with $x(0) = 0$ (zero initial conditions);
- $D = 0$ (strictly proper system);
- SISO (single-input single output) case, i.e., $u, y \in \mathbb{R}$.

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Then the output reads

$$y(t) = \int_0^t Ce^{A(t-s)}Bu(s)\,ds \quad t \geq 0,$$

where $Ce^{A(t-s)}B \in \mathbb{R}$.

Impulse Response

Now consider input $u(t) = \delta(t)$, where $\delta(t)$ is the **delta function** having the following properties

$$\delta(t) = 0 \text{ for } t \neq 0, \quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1.$$

Or, in other words, the delta function δ is a singular function such that

$$\int_{-\infty}^{\infty} \delta(t)\varphi(t)dt = \varphi(0)$$

for any regular function φ that is continuous at 0.

Impulse Response

Hence, we obtain for $t \geq 0$

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In the MIMO (multi-input multi output) case

$$G(t) := Ce^{At}B$$

is a $p \times m$ matrix referred to as the **impulse response matrix**.

Convolution

Take as initial time $t_0 = -\infty$ and $G(t) = Ce^{At}B$ for $t \geq 0$, then

$$y(t) = \int_{-\infty}^{t} G(t-s)u(s)\,ds.$$

For physical systems, future components of $u(t)$ do not contribute to $y(t)$ at t, which means that $G(t) = 0$ for $t < 0$ (causality). Hence

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The form of this integral is known as the **convolution** integral, often denoted as

$$y(t) = G(t) * u(t).$$

Step Response

The **step response** is the output response of a system, that is initially at rest, to an input that equals a **unit step** or **Heaviside function**

$$\varepsilon(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0. \end{cases}$$

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The output $y(t) =: S(t)$ corresponding to $u(t) = \varepsilon(t)$ then reads

$$S(t) = \int_0^t Ce^{A(t-s)}B\varepsilon(s)\,ds,$$

or under the assumption that $G(t) = 0$ for $t < 0$ (causality), as

$$S(t) = \int_{-\infty}^{\infty} G(t-s)\varepsilon(s)\,ds = G(t) * \varepsilon(t).$$

Step Response vs Impulse Response

Note that $\varepsilon(t)$ is the integrated version of $\delta(t)$, i.e.,

$$\varepsilon(t) = \int_{-\infty}^{t} \delta(s) ds.$$

Indeed, since $\varepsilon(t) = 1$ for $t \geq 0$, the step response can be rewritten as (assuming $G(t) = 0$ for $t < 0$)

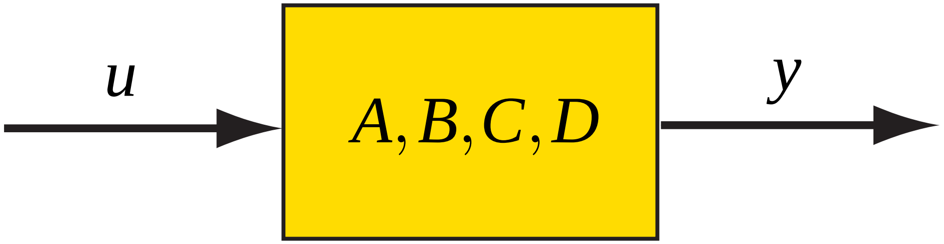
$$\begin{aligned} S(t) &= \int_{-\infty}^{\infty} G(t-s)\varepsilon(s)\, ds \\ &= \int_{0}^{\infty} G(t-s)\, ds \\ &= \int_{-\infty}^{t} G(\tau)\, d\tau = \int_{0}^{t} G(\tau)\, d\tau \quad \Rightarrow \boxed{G(t) = \frac{dS(t)}{dt}}. \end{aligned}$$

Step Response vs Impulse Response

In the MIMO (multi-input multi output) case $S(t)$ is a $p \times m$ matrix referred to as the **step response matrix**.

Recall:

SISO system (i.e., $y \in \mathbb{R}$, $u \in \mathbb{R}$):



State-space form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

How to obtain an input-output description?

Laplace transform

Laplace transform of a signal $f : [0, \infty) \to \mathbb{R}$

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st}\, dt.$$

Important properties:

- Differentiation: $\mathcal{L}\left[\frac{d}{dt} f(t)\right] = s\mathcal{L}[f(t)] - f(0);$

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- Integration: $\mathcal{L}\left[\int_0^t f(\tau)d\tau\right]=\frac{1}{s}\mathcal{L}[f(t)];$
- Convolution: $\mathcal{L}\left[\int_0^\infty f(t-\tau)g(\tau)d\tau\right]=\mathcal{L}[f(t)]\cdot\mathcal{L}[g(t)].$

Laplace transform

Some important signals:

Time signal	Laplace transform
$\delta(t)$	1
$\varepsilon(t)$	$\dfrac{1}{s}$
$\dfrac{t^n}{n!}e^{at}\varepsilon(t),\ a \in \mathbb{C},\ n \in \mathbb{N}$	$\dfrac{1}{(s-a)^{n+1}}$
$e^{at}\cos(bt)\varepsilon(t)$	$\dfrac{s-a}{(s-a)^2+b^2}$

Laplace transform

How does this work with SISO state-space systems? Consider

$$\dot{x}(t) = Ax(t) + Bu(t). \qquad (*)$$

Applying Laplace transform yields:

$$\mathcal{L}[\dot{x}(t)] = sX(s) - x(0), \quad \mathcal{L}[Ax(t) + Bu(t)] = AX(s) + BU(s).$$

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Rearrangement gives

$$X(s) = (sI_n - A)^{-1}x(0) + (sI_n - A)^{-1}BU(s).$$

Transfer Function

For the output $Y(s) = CX(s) + DU(s)$ we obtain:

$$Y(s) = C(sI_n - A)^{-1}x(0) + \Big(C(sI_n - A)^{-1}B + D\Big)U(s).$$

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Assume initial states $x(0) = 0$. Then, the function

$$\boxed{H(s) = \frac{Y(s)}{U(s)} = C(sI_n - A)^{-1}B + D}$$

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<u>Remark:</u>

- MIMO: Then $H(s)$ becomes $m \times p$ **transfer matrix**.

Exercise

Consider the system

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2g} \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u.$$

Find $H(s)$ for

a) $y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x;$

b) $y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} x;$

Transfer Function

Recall that (for $D = 0$) impulse response $G(t) = Ce^{At}B$.

Output in time-domain:

$$y(t) = \int_0^t \underbrace{Ce^{A(t-\tau)}B}_{G(t-\tau)} \cdot \delta(\tau) d\tau,$$

Convolution in time-domain is multiplication in s-domain, i.e.,

$$\mathcal{L}[Ce^{At}B] \cdot \mathcal{L}[\delta(t)] = C(sI - A)^{-1}B \cdot 1 = H(s)$$

Thus, $H(s)$ equals the impulse response!

Transition Matrix

Additionally, note that we just found another way to compute e^{At}. Indeed, the Laplace tranforms of $\dot{x} = Ax$, with $x(0) = x_0$, and $x(t) = e^{At}x_0$ are

$$X(s) = (sI - A)^{-1}x_0, \text{ and } X(s) = \mathcal{L}[e^{At}x_0] = \mathcal{L}[e^{At}]x_0.$$

respectively. This directly implies that

$$e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}],$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform.

Poles and zeros

SISO system:

$$H(s) = \frac{q(s)}{p(s)} = \frac{q_k s^k + q_{k-1} s^{k-1} + \cdots + q_1 s + q_0}{p_n s^n + p_{n-1} s^{n-1} + \cdots + p_1 s + p_0}$$
$$= c\frac{(s-b_1)(s-b_2)\cdots(s-b_k)}{(s-a_1)(s-a_2)\cdots(s-a_n)}$$

- **Gain:** $c = q_k / p_n$;
- **Zeros:** b_i's are the roots of $q(s) = 0$;
- **Poles:** a_i's are the roots of $p(s) = 0$.

Observe: $\lim_{s \longrightarrow b_i} H(s) = 0$, while $\lim_{s \longrightarrow a_i} H(s) = \infty$.

Some Properties and Definitions

- Transfer function $H(s) = \dfrac{q(s)}{p(s)}$ **proper** if

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- Transfer function $H(s) = \dfrac{q(s)}{p(s)}$ **strictly proper** if

$$\deg\{q(s)\} < \deg\{p(s)\}$$

Some Properties and Definitions

- SISO system is said to be **non-minimum phase** if at least one zero has positive real part.

  Example: Consider the system

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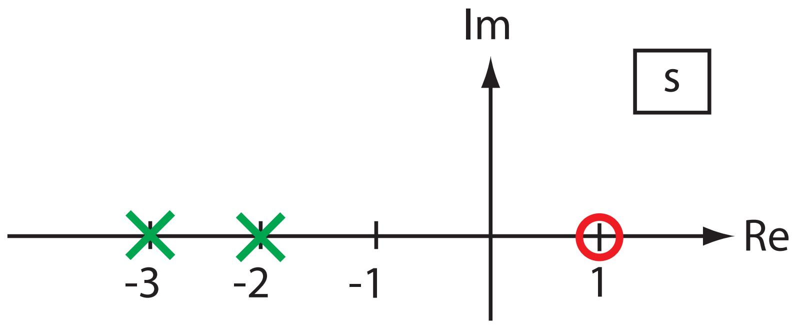

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Compute impulse response via **inverse Laplace transform**:

$$H(s) = \frac{-s+1}{(s+2)(s+3)} = \frac{a}{s+2} + \frac{b}{s+3}$$

where a and b are determined from

$$a(s+3) + b(s+2) = (a+b)s + 3a + 2b \equiv -s + 1$$

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$$H(s) = \frac{3}{s+2} - \frac{4}{s+3} \xrightarrow{\mathcal{L}^{-1}} h(t) = 3e^{-2t}\varepsilon(t) - 4e^{-3t}\varepsilon(t).$$

Example (cont'd)

Step response can be computed similarly using $\mathcal{L}[\varepsilon(t)] = \frac{1}{s}$:

$$Y(s) = H(s)U(s) = \frac{-s+1}{(s+2)(s+3)} \cdot \frac{1}{s} = \frac{\frac{1}{6}}{s} + \frac{-\frac{9}{6}}{s+2} + \frac{\frac{4}{3}}{s+3}.$$

Inverse Laplace transform yields

$$y(t) = \frac{1}{6}\varepsilon(t) - \frac{9}{6}e^{-2t}\varepsilon(t) + \frac{4}{3}e^{-3t}\varepsilon(t).$$

Example (cont'd)

Recall $H(s)$ contains RHP (right half-plane) zero $\Rightarrow$ non-minimum phase. This can also seen from time response

$$y(t) = \frac{1}{6}\varepsilon(t) - \frac{9}{6}e^{-2t}\varepsilon(t) + \frac{4}{3}e^{-3t}\varepsilon(t)$$

Take a few samples....

$$y(0) = \frac{1}{6} - \frac{9}{6} + \frac{4}{3} = 0$$

$$y(\infty) = \lim_{t\to\infty} y(t) = \frac{1}{6} > 0$$

$$\dot{y}(t) = \frac{1}{6}\delta(t) - \frac{9}{6}\left[-2e^{-2t}\varepsilon(t) + e^{-2t}\delta(t)\right] + \frac{4}{3}\left[-3e^{-3t}\varepsilon(t) + e^{-3t}\delta(t)\right]$$

$\dot{y}(0) = -1 < 0 \Rightarrow$ response first goes in opposite direction.