Systeem- en Regeltechniek
EE2S21

Stability + Controllability

Lecture 6

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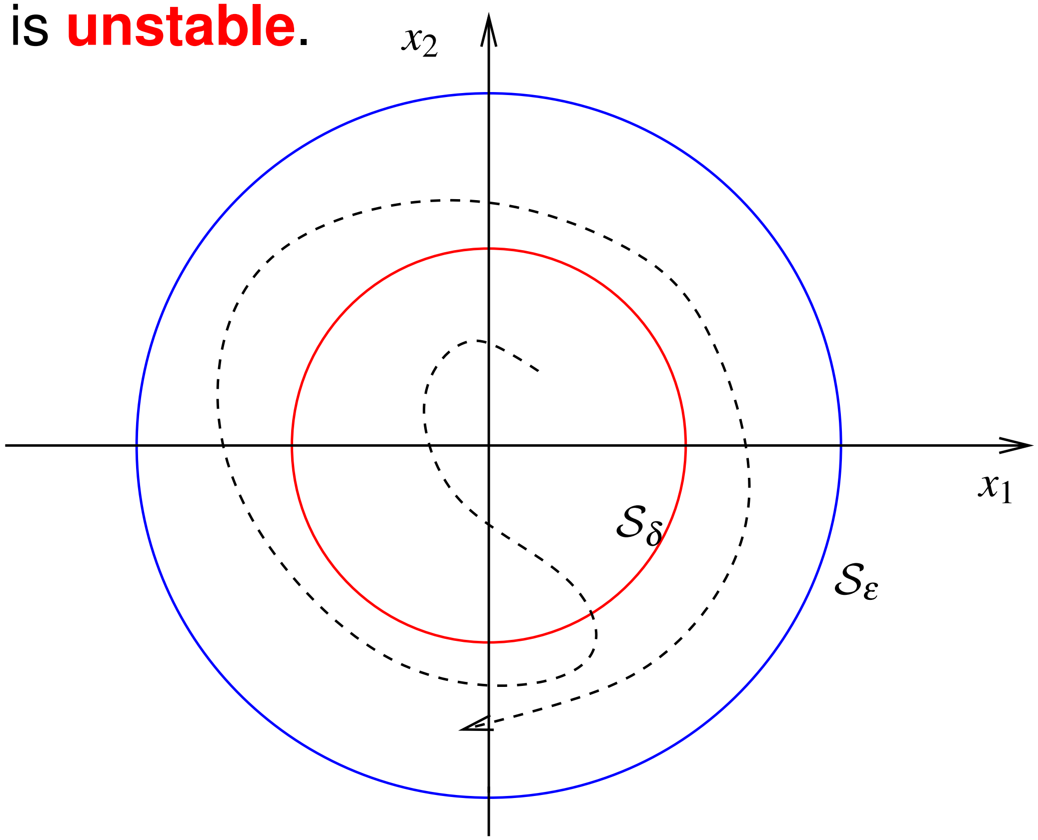
Concepts of stability

Equilibrium point $x^* = 0$ is **stable** if **for any** $\varepsilon > 0$, **there exists** $\delta(\varepsilon) > 0$ such that if $\|x(0)\| < \delta$, then $\|x(t)\| < \varepsilon$, for all $t \geq 0$.

Otherwise, the equilibrium point is **unstable**.

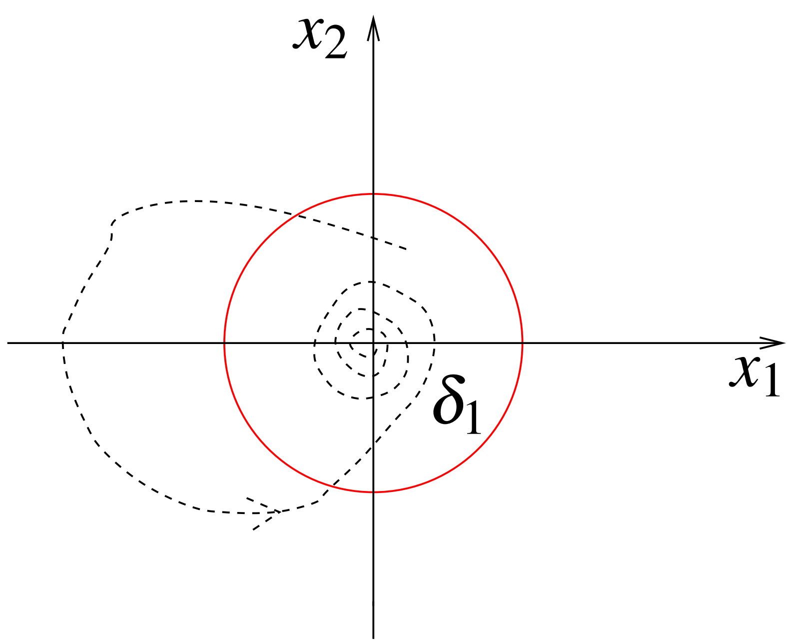
Stability means that the system trajectory $x(t)$ can be kept **arbitrarily close** to $x^* = 0$ by starting **sufficiently close** to it.



Concepts of stability

Equilibrium point **asymptotically stable** if it is stable and if $\exists \delta_1 > 0$ s.t. $\|x(0)\| < \delta_1 \Rightarrow \lim_{t\to\infty} x(t) = 0$.

Meaning: system eventually converges to eq. point.



Internal stability

Autonomous LTI system:

$$\dot{x}(t) = Ax(t).$$

Stable if for any t_0 and $\delta > 0$ there exists a $\varepsilon > 0$ s.t. $||x(t_0)|| < \delta \Rightarrow ||x(t)|| < \varepsilon$ for all $t \geq t_0$, with

$$\|x(t)\| = \sqrt{x_1^2(t) + x_2^2(t) + \cdots + x_n^2(t)}.$$

Recall that solution is given by

$$x(t) = \Phi(t, t_0)x(t_0) = \boxed{e^{A\cdot(t-t_0)}x(t_0).}$$

$\Rightarrow$ internal stability depends on the A matrix!

Internal stability

Characteristic polynomial of A matrix

$$\det(\lambda_i I_n - A) = 0 \;\Leftrightarrow\; (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n) = 0.$$

If $\lambda_i v_i = A v_i$ has n independent v_i's, then convert state-space system into **modal form** via eigenvalue decomposition: $A = T \Lambda T^{-1}$.

Modal decomposition:

$$\begin{aligned} x(t) &= \Phi(t, t_0) x(t_0) = e^{A(t-t_0)} x(t_0) \\ &= T e^{\Lambda(t-t_0)} T^{-1} x(t_0) = \boxed{\sum_{i=1}^{n} \mu_i e^{\lambda_i (t-t_0)} v_i} \end{aligned}$$

$\Rightarrow$ Stability properties depend on **eigenvalues** and **eigenvectors** of the A matrix.

Internal stability

- **Eigenvalues with positive real part**

  $\Rightarrow$ keep growing over time;

- **Eigenvalues with negative real part**

  $\Rightarrow$ go to zero over time;

- **Eigenvalues with zero real part**

  $\Rightarrow$ bounded state only if they correspond to a complete set of linearly independent eigenvectors.

How about stability of $\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} x$?

Internal stability

Theorem:

LTI state-space system

$$\dot{x}(t) = Ax(t)$$

is **stable** if and only if all the eigenvalues of the matrix A have **nonpositive real part** and to any eigenvalue with a **zero real part** and algebraic multiplicity k there correspond k **linearly independent eigenvectors**, i.e., algebraic multiplicity $\equiv$ geometric multiplicity.

Theorem:

Asymptotically stable if and only if all the eigenvalues of the matrix A have **negative real part**.

Exercise

Consider the LTI system

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix} x.$$

Is this system stable?

Routh's Criterion

For asymptotic stability all eigenvalues must be in the open left half plane. This can be done by *explicitly* solving

$$\det(\lambda I - A) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

for the λ's and checking the *sign* of $\mathrm{Re}(\lambda)$.

Computationally less expensive approach is via **Routh's criterion**. Routh's criterion only checks whether the eigenvalues lie in the right or left half plane — not the precise locations.

Routh table:

For $a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$, with $a_n \neq 0$

$$\begin{array}{cccc}
a_n & a_{n-2} & a_{n-4} & \cdots \\
a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
\hline
b_{n-2} & b_{n-4} & b_{n-6} & \cdots \\
c_{n-3} & c_{n-5} & c_{n-7} & \cdots \\
d_{n-4} & d_{n-6} & d_{n-8} & \cdots \\
\vdots & \vdots & \vdots &
\end{array}$$

with $(\bullet)_{-1} = 0$, and

$$b_{n-2} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}, \; b_{n-4} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}, \; c_{n-3} = \frac{-1}{b_{n-2}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-2} & b_{n-4} \end{vmatrix}, \text{ etc.}$$

Routh's criterion:

The roots of the polynomial

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0,$$

with $a_n \neq 0$, all have a negative real part if and only if the Routh table consists of $n+1$ rows and all the elements in the first column of the table have the same sign, i.e., all elements of this column are either positive or negative.

Exercise

Recall the system

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix} x.$$

Is this system stable?

We already found

$$\det(\lambda I - A) = \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0.$$

Fill the Routh table...

BIBO stability

Bounded-input bounded-output stability of LTI state-space system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t),$$

with

$$y(t) = Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

related to internal stability as follows:

Theorem:

If all eigenvalues of the matrix A have **negative real part**, then the system (A,B,C,D) is **BIBO stable**.

BIBO stability

Consider a SISO system and the convolution integral.

$$y(t) = \int_{-\infty}^{\infty} G(t-\tau)u(\tau)\,d\tau,$$

with $G(t)$ the impulse response.

BIBO stable if the impulse response $G(t)$ satisfies

$$\int_{-\infty}^{\infty} |G(t)|\,dt < \infty.$$

Controllability

LTI state-space system:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t).\end{aligned}$$

Definition:

An LTI state-space system is **controllable** if given any initial state $x(t_0)$ there exists an input signal $u(t)$ for $t_0 \leq t \leq t_f$ such that $x(t_f) = 0$ for some $t_f \geq t_0$.

Example

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Input only influences $x_1(t)$, no coupling between $x_1(t)$ and $x_2(t)$

Conclusion: $x_2(t)$ is **not controllable**.

How to determine controllability of a general LTI system?

Modal form

Recall eigenvalue decomposition: $A = V\Lambda V^{-1}$

State transformation V:

$$\widetilde{A} = V^{-1}AV; \qquad \widetilde{B} = V^{-1}B;$$
$$\widetilde{C} = CV; \qquad \widetilde{D} = D.$$

$\Rightarrow A$ matrix becomes **diagonal**: $\widetilde{A} = V^{-1}AV = \Lambda$, and

$$\begin{aligned} V^{-1}\dot{x}(t) &= V^{-1}AVV^{-1}x(t) + V^{-1}Bu(t) \\ y(t) &= CVV^{-1}x(t) + Du(t). \end{aligned}$$

Hence, by letting $\widetilde{x}(t) = V^{-1}x(t)$, we obtain...

Modal form

...SISO system:

$$\dot{\widetilde{x}}(t) = \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}}_{\widetilde{A}} \widetilde{x}(t) + \begin{bmatrix} \widetilde{B}_{11} \\ \widetilde{B}_{21} \\ \vdots \\ \widetilde{B}_{n1} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} \widetilde{C}_{11} & \widetilde{C}_{12} & \cdots & \widetilde{C}_{1n} \end{bmatrix} \widetilde{x}(t) + Du(t)$$

Not controllable if for some i: $\widetilde{B}_{i1} = 0$

What if modal form does not exist?

What about MIMO?

Controllability matrix

Definition:

Controllability matrix: $\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$

Theorem:

An LTI system is **controllable**

if and only if

$$\boxed{\text{rank}(\mathcal{C}) = n}$$

Exercise

Consider the system

$$\dot{z} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2g} \\ 0 & 0 & 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v.$$

Is the system controllable?