1

$$\dot{x} = Ax + Bu \qquad \longrightarrow \qquad \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u$$
$$y = Cx + Du \qquad\qquad\qquad y = \tilde{C}\tilde{x} + \tilde{D}u$$

$$\tilde{A} = V^{-1}AV \qquad \tilde{C} = CV$$
$$\tilde{B} = V^{-1}B \qquad \tilde{D} = D$$

$$\dot{x} = V\tilde{A}V^{-1}x + V\tilde{B}u$$

$$\underbrace{V^{-1}\dot{x}}_{\dot{\tilde{x}}} = \tilde{A}\underbrace{V^{-1}x}_{\tilde{x}} + \tilde{B}u \rightarrow \dot{\tilde{x}} = \tilde{A}x + \tilde{B}u$$

$$y = \tilde{C}\underbrace{V^{-1}x}_{\tilde{x}} + \tilde{D}u \rightarrow y = \tilde{C}\tilde{x} + \tilde{D}u$$

$$H(s) = C(sI - A)^{-1}B + D \qquad \tilde{H}(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}$$

(15ci)

$$\tilde{A} = V^{-1} A V \qquad \tilde{B} = V^{-1} B$$

$$\tilde{C} = C V \qquad \tilde{D} = D$$

$$\tilde{H}(s) = \tilde{C} (sI - \tilde{A})^{-1} \tilde{B} + \tilde{D}$$

$$= C V \underbrace{(sI - V^{-1} A V)}_{s V^{-1} V - V^{-1} A V}{}^{-1} V^{-1} B + D$$

$$(xyz)^{-1} = z^{-1} y^{-1} x^{-1}$$

$$\Big(\underset{x}{\underset{\shortparallel}{V^{-1}}} \underset{y}{\underset{\shortparallel}{(sI - A)}} \underset{z}{\underset{\shortparallel}{V}}\Big)^{-1}$$

$$= C V V^{-1} (sI - A)^{-1} V V^{-1} B + D$$

$$= C (sI - A)^{-1} B + D \quad = H(s)$$

$C = [\, B \;\; AB \;\; \dots \;\; A^{n-1} B \,]$

→ $x(t)$ = linear combination of columns of

$B \;\; AB \;\; A^2 B \;\; \dots$

Cayley – Hamilton

$\det(\lambda I - A) = 0 \rightarrow \lambda^n + \dots = 0$

$\lambda / A \rightarrow$ still equality

A^n = lin. combination of $I, A, A^2, \dots A^{n-1}$

→ stop at A^{n-1}

$x(t) \in \mathbb{R}^n$ can take any value if

columns of C are basis of $\mathbb{R}^n$

→ $\boxed{\text{rank}(C) = n}$ → controllable slide 22 of lecture 6

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2g} \\ 0 & 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad n = 3$$

$$C = \begin{array}{c} \begin{array}{ccc} B & AB & A^2B \end{array} \\ \begin{bmatrix} 0 & 0 & \sqrt{2g} \\ 0 & \sqrt{2g} & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ B & AB & A^2B = A(AB) \end{array} \end{array} \quad \to \quad \text{rank } C = 3$$

!!

Controllable

slide 10 of lecture 7

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2g} \\ 0 & 0 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 & 0 & \sqrt{2g} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad n = 3$$

1. $w = z_3 \Rightarrow C = [0\ 0\ 1] \Rightarrow \mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

rank $= 1 \rightarrow$ not observable

2. $w = z_1 \Rightarrow C = [1\ 0\ 0] \Rightarrow \mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2g} \end{bmatrix}$

rank $= 3 \rightarrow$ observable

3. $w = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \Rightarrow C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{2g} \\ 0 & 0 & 0 \end{bmatrix}$

$\rightarrow$ rank $= 3 \Rightarrow$ observa

or faster:
follows from 2.
as z_1 already
yields all the
required info.

7

(A, B) controllable

$\to \quad \mathcal{C} = [B \quad AB \quad \cdots \quad A^{n-1}B] \to \text{rank} = n$

$$\mathcal{C}^T = \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^T)^{n-1} \end{bmatrix} - \text{rank } n$$

G = observability matrix for $\tilde{C} = B^T \quad \tilde{A} = A^T$

$(\underbrace{B^T}_{= \tilde{C}}, \underbrace{A^T}_{= \tilde{A}})$ = observable slide 12

$$A = \begin{bmatrix} \alpha-1 & 1 \\ 0 & \alpha \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = [\alpha \;\; 0]$$

$n = 2$

$$\mathcal{C} = [B \;\; AB] = \begin{bmatrix} 1 & \alpha-1 \\ 0 & 0 \end{bmatrix} \to \text{rank} = 1$$

not controllable

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ \alpha(\alpha-1) & \alpha \end{bmatrix}$$

- $\alpha = 0 \to \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ rank $= 0$ $\to$ not observable
- $\alpha \neq 0 \to$ $\det \mathcal{O} = \alpha^2 \neq 0$
  $\downarrow$
  $\mathcal{O}$ is invertible
  $\to$ full rank
  $\to$ observable

7

zero : -2

$$H(s) = \frac{s+2}{s^2+6s+8} = \frac{s+2}{(s+2)(s+4)} = \frac{1}{s+4}$$

pole : -2

$\downarrow$

cancel

slide 20

(8)

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & -4 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad n = 3$$

$$C = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 8 & -4 \\ 0 & 1 & 0 \\ 4 & -14 & 8 \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & -1 & -4 \\ 0 & 0 & 0 \\ 1 & 3 & 8 \end{bmatrix}$$

rank $(\mathcal{C}) = 2$
$\downarrow$
not controllable
$\downarrow$
not minimal

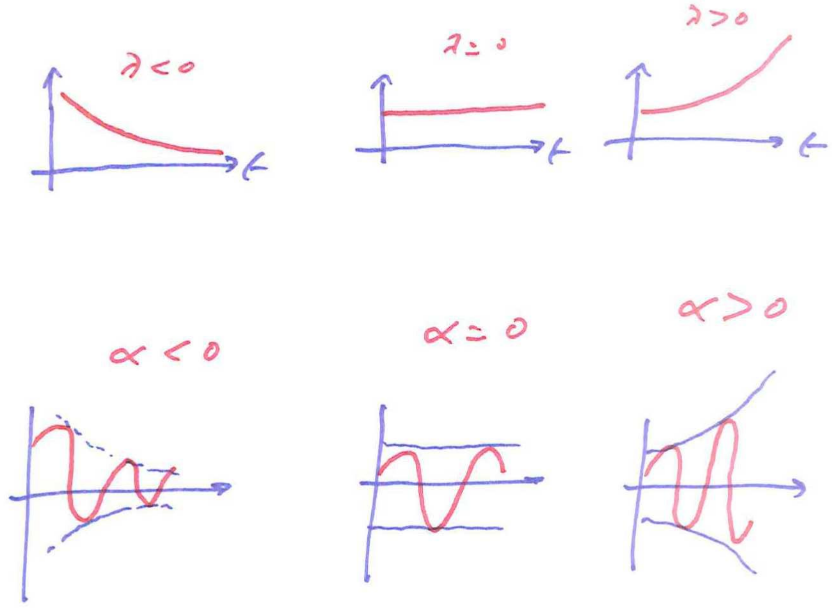
<u>note</u> $\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 2 \\ 4 & -7 & 4 \end{bmatrix}$ rank $(\mathcal{O}) = 2$?

<u>lecture 5b</u>

$$H(s) = \frac{\text{polynomial in } s}{\text{'' in } s} \quad \begin{array}{l} \rightarrow \text{zeros} \\ \rightarrow \text{poles} \end{array}$$

pole $\lambda \rightarrow e^{\lambda t}$

pole $\alpha + \beta j \rightarrow e^{\alpha t} \cos \beta t$
$\qquad\qquad\qquad\quad e^{\alpha t} \sin \beta t$



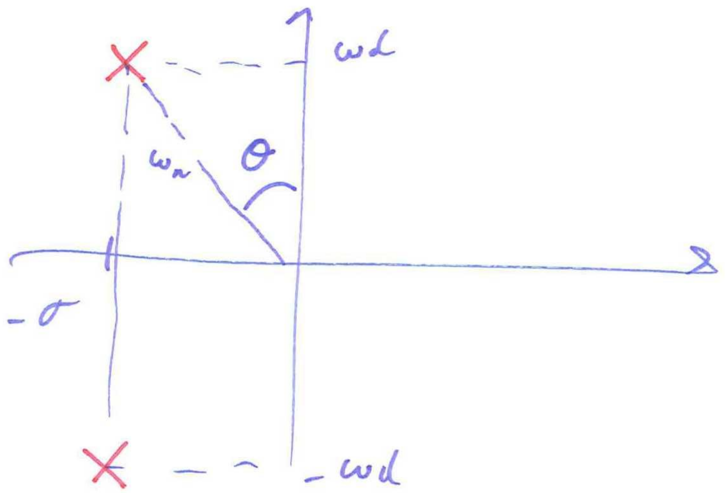
stable   stable if alg. multiplicity = geometric ''   unstable

$\boxed{\text{poles are in left-half plane}}$

$$\frac{k}{[s-(-\sigma+\omega_d i)][s-(-\sigma-\omega_d i)]} = \frac{k}{(s+\sigma)^2+\omega_d^2}$$

$$\hookrightarrow s^2+2\sigma s+\sigma^2+\omega_d^2$$

$$= \frac{k' \quad \omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$



$$\omega_n^2 = \sigma^2+\omega_d^2$$

$$\sigma = \zeta\omega_n$$

$$\zeta = \sin\theta = \cancel{\ } \sigma/\omega_n$$

ζ: damping

ω_n: undamped natural frequency