Systeem- en Regeltechniek
EE2S21

Linearization

Lecture 5c

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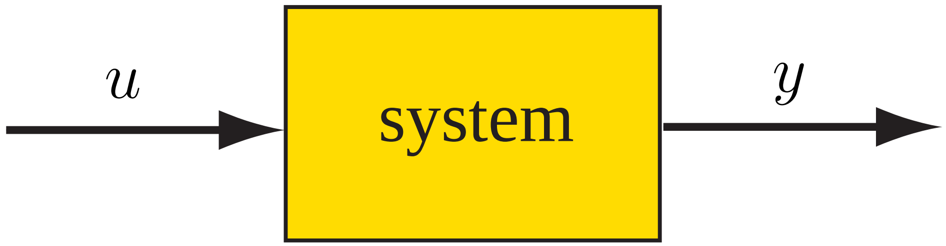
Linear versus Nonlinear

Input-output system:



Consider two possible inputs, say $u = u_1$ and $u = u_2$, each yielding an output $u_1 \to y_1$ and $u_2 \to y_2$, respectively.

Then, if for any $a, b \in \mathbb{R}$

$$a u_1 + b u_2 \to a y_1 + b y_2$$

the system is said to be linear. Otherwise: nonlinear.

Linear versus Nonlinear

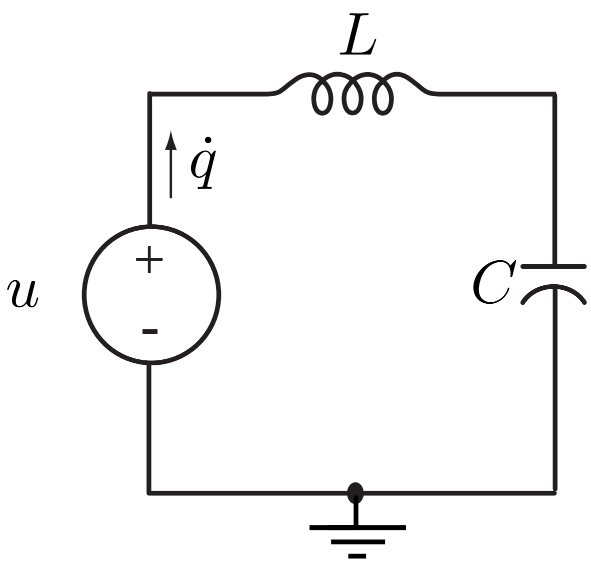
Phenomena that only occur in the presence of nonlinearities:

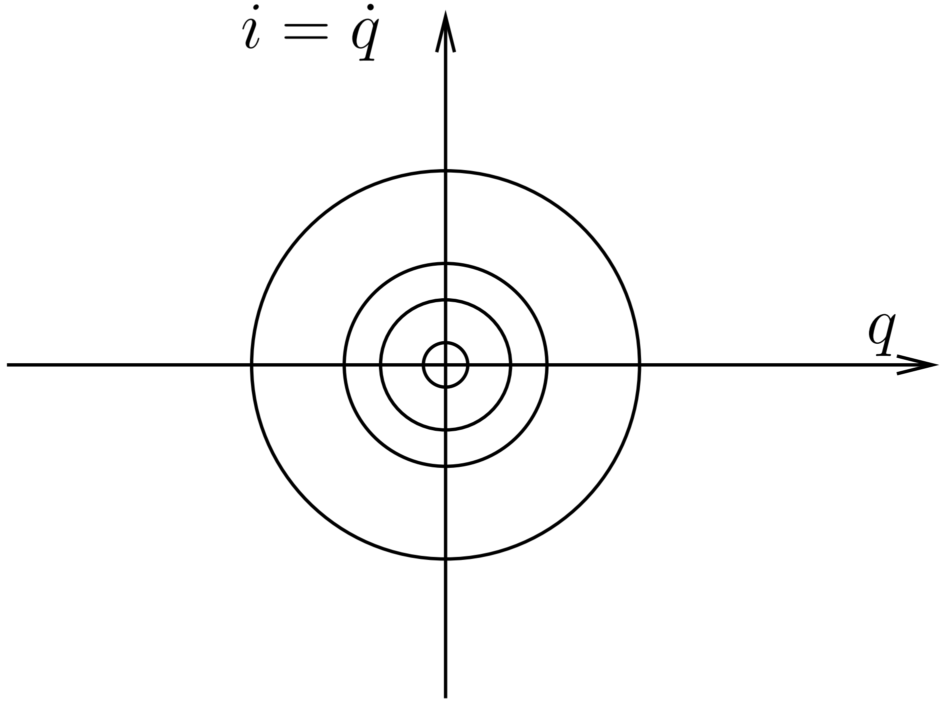
- Finite escape time, i.e., the state becomes infinite in finite time.
- Multiple isolated equilibria, stable or unstable, different characteristics. Related: domain of attraction.
- Limit cycles, i.e., an **isolated** periodic solution. See examples of linear LC network and nonlinear Van der Pol equation...

Linear versus Nonlinear

Consider phase plane of LC network

with $L = 1$ and $C = 1$:





For $u = 0 \Rightarrow \ddot{q} = -q$

Here periodic solutions are **no limit cycles!**

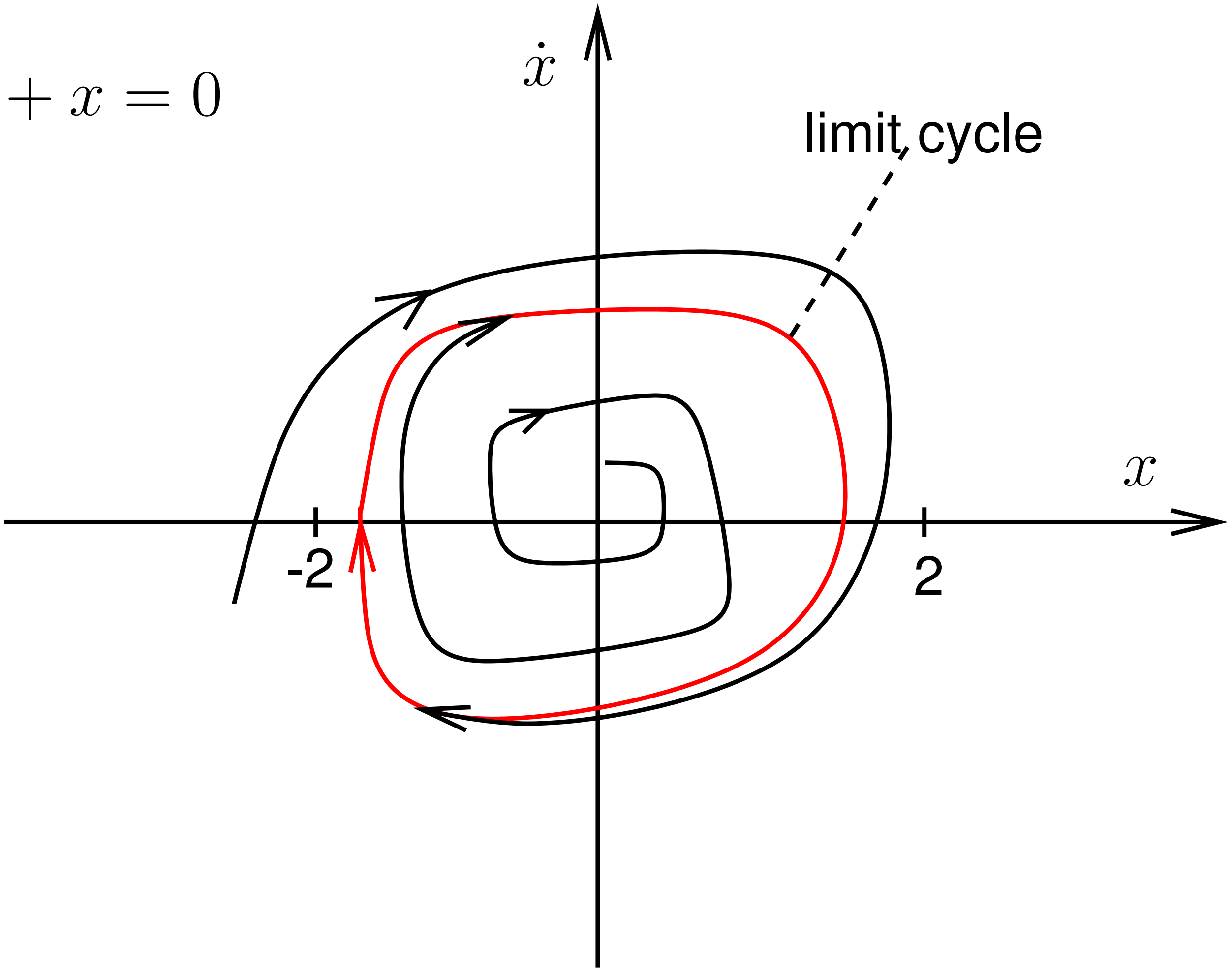
Linear versus Nonlinear

Van der Pol equation:

$$\ddot{x} + \alpha(x^2 - 1)\dot{x} + x = 0$$

Phase portrait

for $\alpha = 0.2$:



Linear versus Nonlinear

Phenomena that only occur in the presence of nonlinearities (cont'd)

- **Bifurcations**, changing parameters may change
  - number of equilibrium points
  - stability (as it does for linear systems),
  - whether or not a limit cycle occurs

    $\Rightarrow$ e.g., see Van der Pol equation if α becomes 0.
  - ...

Linear versus Nonlinear

Phenomena that only occur in the presence of nonlinearities (cont'd)

- **Chaos**, an extremely small change in the initial conditions of a system results in a very different response of the system. Hence the systems' response is extremely sensitive to small variations of the initial conditions. Chaos occurs in **deterministic systems**, i.e., don't mix it up with the concept of uncertainty in stochastic systems!

  $\Rightarrow$ Example: Double pendulum. Chua Circuit.

  More examples at:

  `http://en.wikipedia.org/wiki/Chaos_theory`

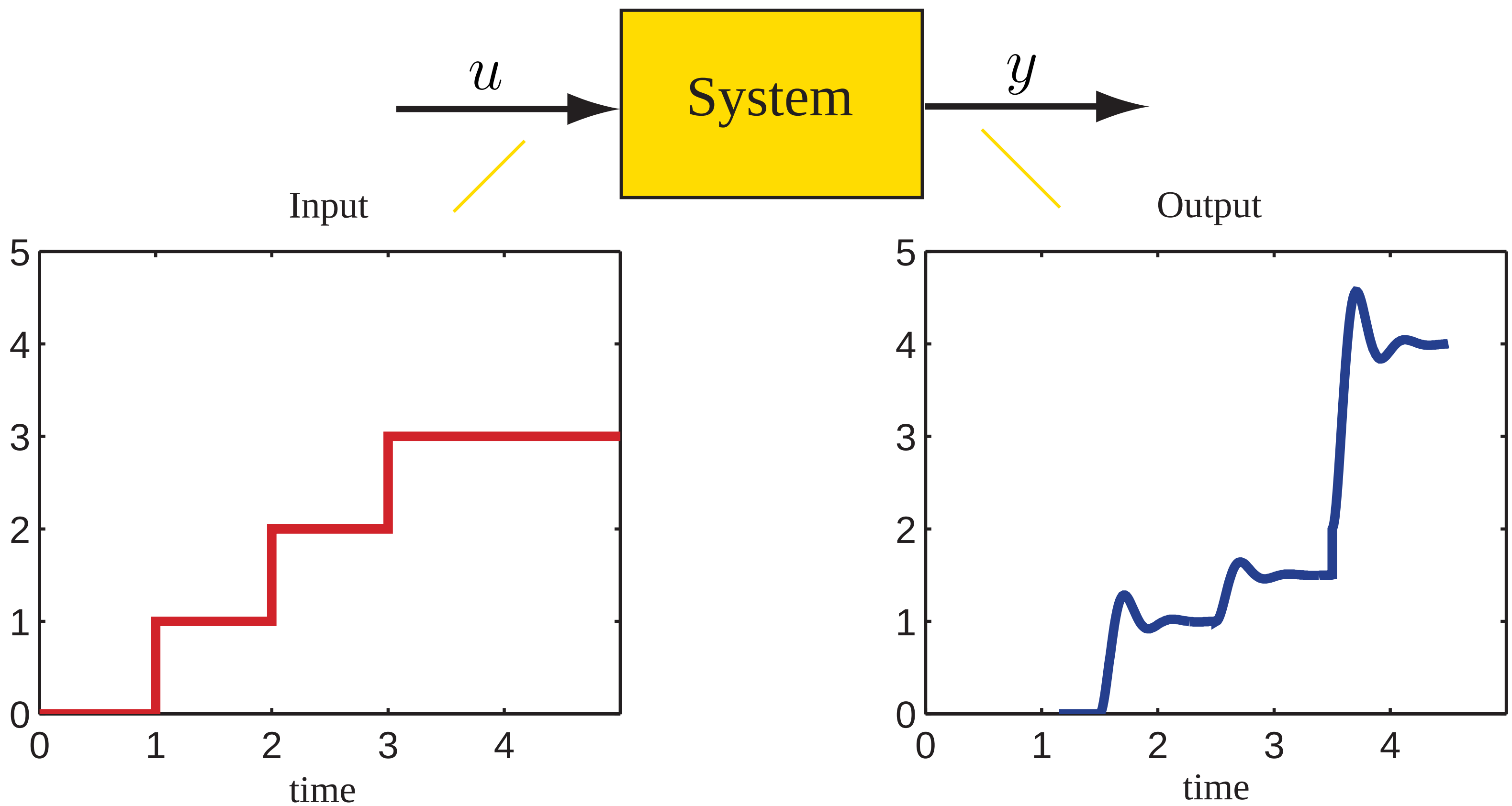
Linear versus Nonlinear

Types of nonlinearities:

- smooth nonlinearities in the dynamics (due to nonlinear constitutive relationships).
- static nonlinearities such as saturation, dead zones, relays, etc..
- two-valued nonlinearities such as hysteresis and/or backlash.
- etc...

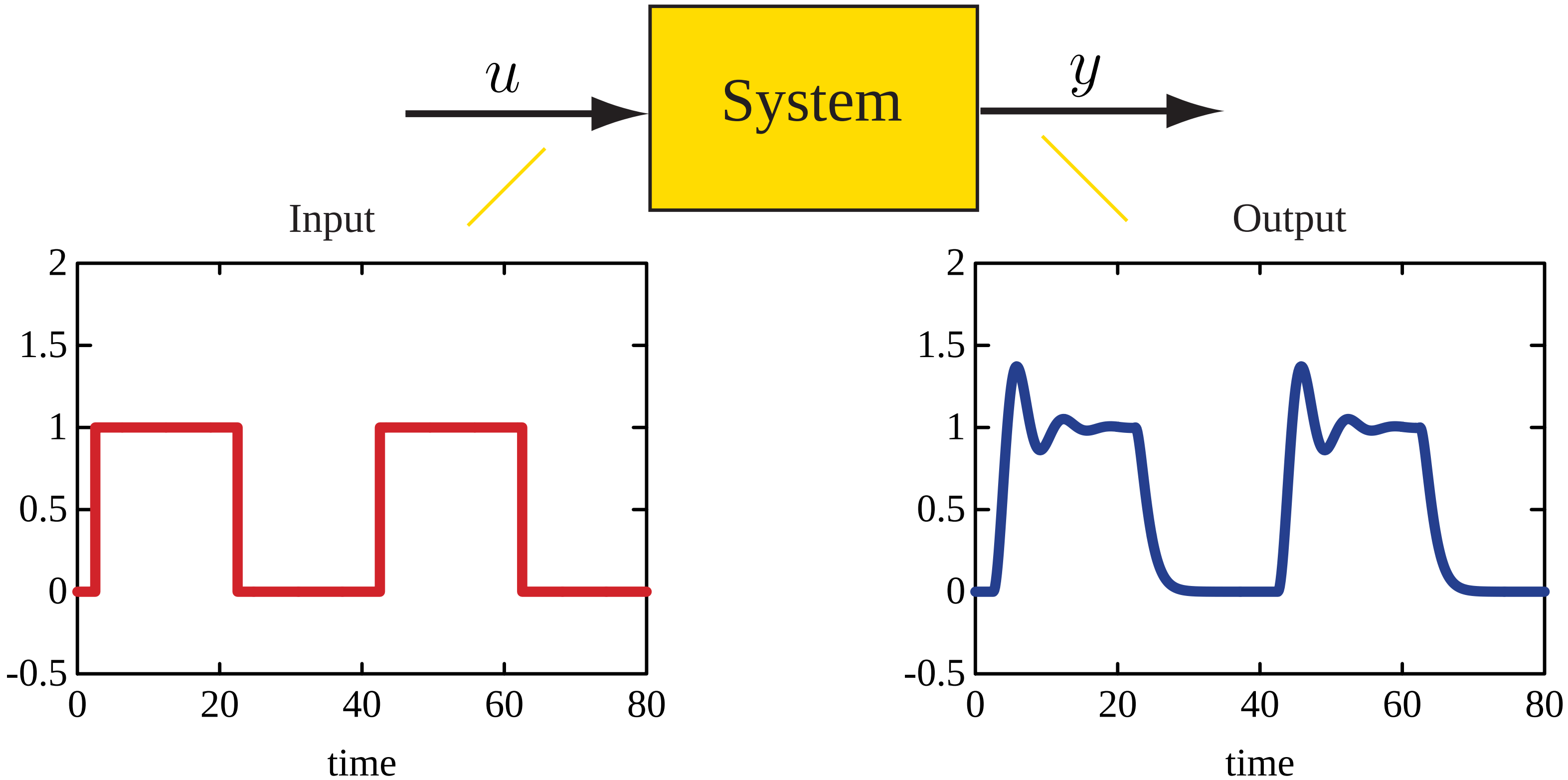
Examples...

Example 1: Response? Linear, nonlinear, dead time....?



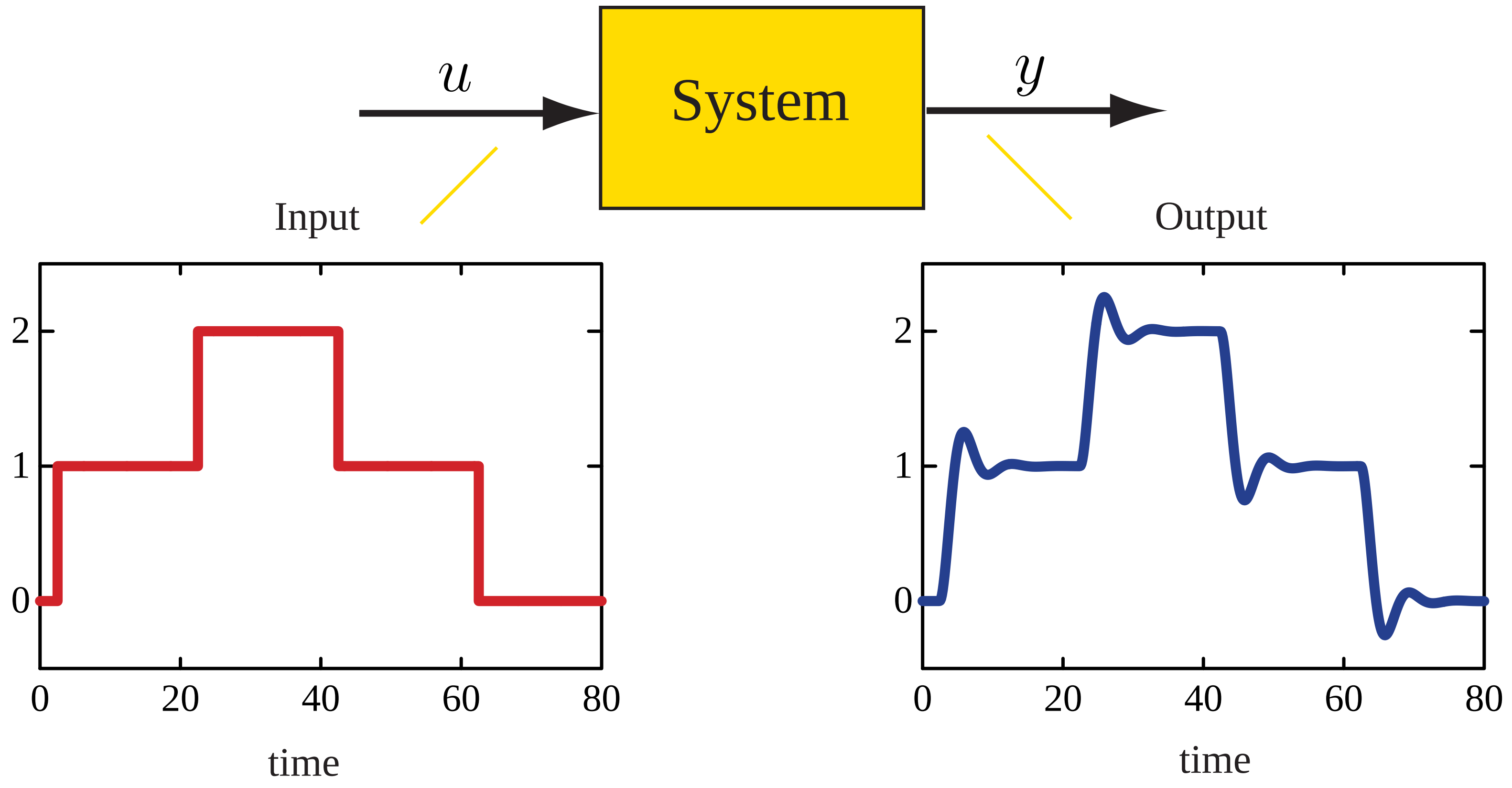
More Examples...

Example 2: Response? Linear, nonlinear, dead time....?



More Examples...

Example 3: Response? Linear, nonlinear, dead time....?



Linearization

- Despite the nonlinear phenomena, it may sometimes be useful to consider the linearization of a model of a system.
- Linearization tells something about the small neighborhood of the operating point.
- Often, the operating point is chosen to be an equilibrium.
- Based on the first order Taylor expansion.

Linearization I

Procedure:

- **Step 1:** Determine a nominal solution/operation point
- **Step 2:** Approximate the nonlinear functions around this operating point/solution
- **Step 3:** Convert the system to local coordinates

Linearization II

Determine an **solution**, say $(\tilde{x}, \tilde{u})$, for the nonlinear system

$$\dot{x} = f(x, u), \quad y = h(x, u),$$

such that $\dot{\tilde{x}} = f(\tilde{x}, \tilde{u})$, and consider a small perturbation

$$x = \tilde{x} + z, \quad u = \tilde{u} + v.$$

Taylor series expansion around $\tilde{x}, \tilde{u}$:

$$f(x, u) = f(\tilde{x}, \tilde{u}) + \left.\frac{\partial f}{\partial x}\right|_{\tilde{x},\tilde{u}} z + \left.\frac{\partial f}{\partial u}\right|_{\tilde{x},\tilde{u}} v + \text{h.o.t.}.$$

Linearization III

Since $\dot{x} = \dot{\tilde{x}} + \dot{z}$, we have that

$$\dot{\tilde{x}} + \dot{z} = f(\tilde{x}, \tilde{u}) + \left.\frac{\partial f}{\partial x}\right|_{\tilde{x},\tilde{u}} z + \left.\frac{\partial f}{\partial u}\right|_{\tilde{x},\tilde{u}} v + \text{h.o.t.}.$$

Same procedure for $y = h(x, u)$...

If $\tilde{y} = h(\tilde{x}, \tilde{u})$ and $\tilde{y} + w = h(\tilde{x} + z, \tilde{u} + v)$, then

$$\tilde{y} + w = h(\tilde{x}, \tilde{u}) + \left.\frac{\partial h}{\partial x}\right|_{\tilde{x},\tilde{u}} z + \left.\frac{\partial h}{\partial u}\right|_{\tilde{x},\tilde{u}} v + \text{h.o.t.}.$$

Linearization IV

Let

$$A = \frac{\partial f}{\partial x}\bigg|_{\tilde{x},\tilde{u}}, \; B = \frac{\partial f}{\partial u}\bigg|_{\tilde{x},\tilde{u}}, \; C = \frac{\partial h}{\partial x}\bigg|_{\tilde{x},\tilde{u}}, \; D = \frac{\partial h}{\partial u}\bigg|_{\tilde{x},\tilde{u}}.$$

Hence, since A, B, C, and D may depend on time, the linear approximation around $(\tilde{x}, \tilde{u})$ takes the form

$$\begin{aligned} \dot{z} &= A(t)z + B(t)v, \quad z(t_0) = z_0 \\ w &= C(t)z + D(t)v, \end{aligned}$$

Often, the operating point is chosen to be an equilibrium. (Does this make more sense than linearization around a non-equilibrium point?).

Linearization V

If the operating point is an equilibrium point, say $(\tilde{x}, \tilde{u}) = (x^*, u^*)$, we have that $\dot{x}^* = 0 \Rightarrow f(x^*, u^*) = 0$. Hence,

$$A = \frac{\partial f}{\partial x}\Big|_{x^*,u^*}, \; B = \frac{\partial f}{\partial u}\Big|_{x^*,u^*}, \; C = \frac{\partial h}{\partial x}\Big|_{x^*,u^*}, \; D = \frac{\partial h}{\partial u}\Big|_{x^*,u^*},$$

so that the linear approximation around (x^*, u^*) takes the form

$$\begin{aligned} \dot{z} &= Az + Bv, \quad z(t_0) = z_0 \\ w &= Cz + Dv. \end{aligned}$$

Never forget: linearized model only valid in the vicinity of the solution/equilibrium point!!!

Example: Levitated Ball System

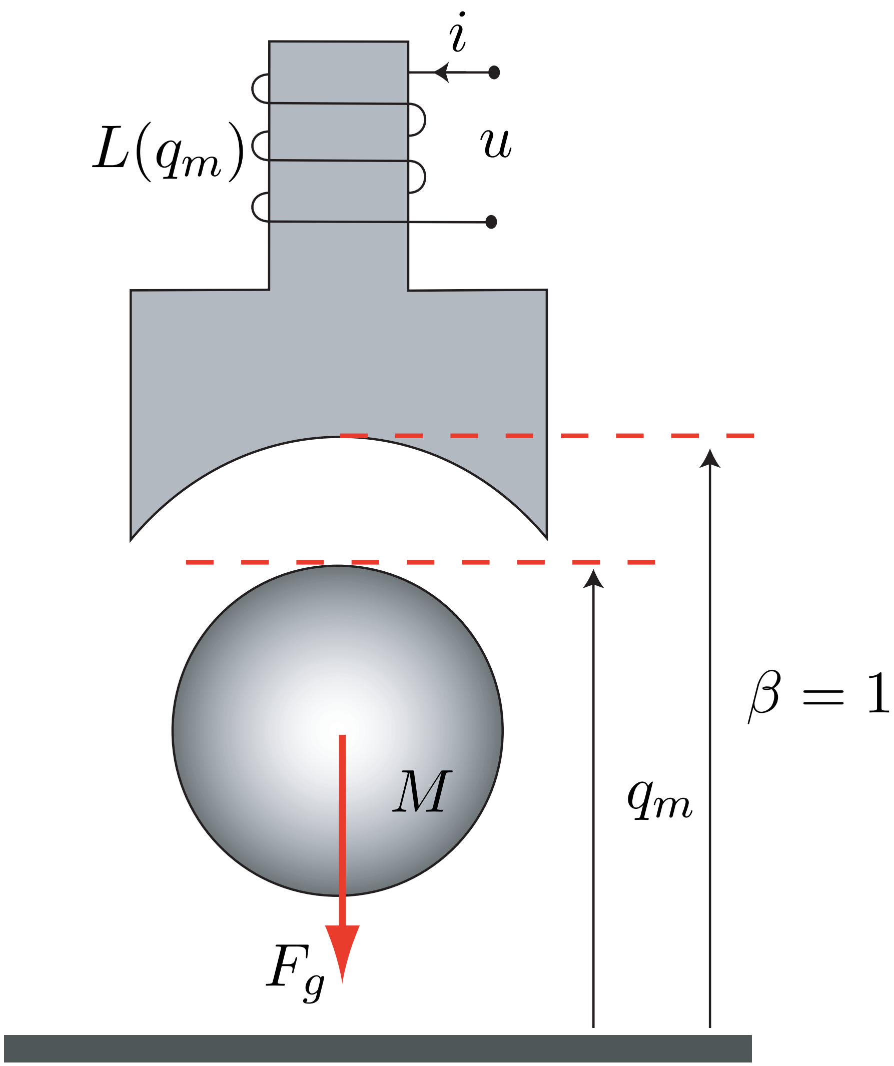
Normalized parameters:

$$L(q_m) = \frac{1}{1-q_m}, \quad M = 1, \ R = 1,$$

State-space equations:

$$\begin{aligned}\dot{q}_m &= p_m \\ \dot{p}_m &= \tfrac{1}{2}p_e^2 - Mg \\ \dot{p}_e &= u,\end{aligned}$$

with outputs $y = (q_m, (1-q_m)p_e)^T$.



Exercise

Consider the second-order system

$$\ddot{q} + \dot{q}^3 + q = u, \quad y = q^2, \qquad (*)$$

with $q, u, y \in \mathbb{R}$.

1. Write the system in state space form.
2. Is the system linear?
3. Is the system time-invariant?
4. If $u^* = 1$, show that $q^* = 1$ and $\dot{q}^* = 0$ is a solution of $(*)$.
5. Linearize the system around $(q, \dot{q}^*) = (1, 0)$.

Exercise

Consider the state space system

$$\begin{aligned}\dot{x}_1 &= -x_2^2 u_1\\ \dot{x}_2 &= -x_1 + u_2\\ \dot{x}_3 &= 1 - x_2\\ y &= x_3^2,\end{aligned}$$

with inputs $u = (u_1, u_2)^T$, output y, and state $x = (x_1, x_2, x_3)^T$.

Linearize the system around $(x_1^*, x_2^*, x_3^*) = (1, 1, 2)$

and $(u_1^*, u_2^*) = (0, 1)$.