Systeem- en Regeltechniek
EE2S21

Observability + Realization Theory

Lecture 7

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Controllability matrix

Definition:

Controllability matrix: $\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$

Theorem:

An LTI system is **controllable**

if and only if

$$\boxed{\text{rank}(\mathcal{C}) = n}$$

Controller Canonical Form

$$\dot{x}_c = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & -p_{n-2} & -p_{n-1} \end{pmatrix}}_{A_c} x_c + \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{B_c} u,$$

$$y = \underbrace{\begin{pmatrix} q_0 & q_1 & \cdots & \cdots & q_{n-1} \end{pmatrix}}_{C_c} x_c.$$

How to obtain this form?

Controller Canonical Form

Use linear combination of the columns of controllability matrix...

$$T = \begin{pmatrix} A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ p_{n-1} & 1 & \cdots & 0 & 0 \\ p_{n-2} & p_{n-1} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ p_2 & p_3 & \cdots & 1 & 0 \\ p_1 & p_2 & \cdots & p_{n-1} & 1 \end{pmatrix},$$

where the p_i's are from the characteristic equation

$$\det(\lambda I - A) = \lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0.$$

Controller Canonical Form

Hence, letting $x = Tx_c$, we get

$$\dot{x}_c = T^{-1}ATx_c + T^{-1}Bu = A_c x_c + B_c u$$
$$y = CTx_c = C_c x_c.$$

Observability

LTI state-space system:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t).
\end{aligned}
$$

Definition:

An LTI state-space system is **observable** if any initial state $x(t_0)$ is uniquely determined by the corresponding response $y(t)$ for $t_0 \leq t \leq t_f$.

Intuitively: based on the measurements (the outputs), you can reconstruct the internal variables, i.e., the state.

Example

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Output only influenced by $x_1(t)$, no coupling between $x_1(t)$ and $x_2(t)$

Conclusion: $x_2(t)$ is **not observable**

Modal form

SISO system:

$$\dot{\tilde{x}}(t) = \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}}_{\widetilde{A}} \widetilde{x}(t) + \begin{bmatrix} \widetilde{B}_{11} \\ \widetilde{B}_{21} \\ \vdots \\ \widetilde{B}_{n1} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} \widetilde{C}_{11} & \widetilde{C}_{12} & \cdots & \widetilde{C}_{1n} \end{bmatrix} \widetilde{x}(t) + Du(t)$$

Not observable if for some i: $\widetilde{C}_{1i} = 0$

Observability matrix

Definition:

Observability matrix: $\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$

Theorem:

An LTI system is **observable**

if and only if

$$\boxed{\text{rank}(\mathcal{O}) = n}$$

Exercise

Consider the system

$$\dot{z} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2g} \\ 0 & 0 & 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v.$$

1. Is the system observable for $w = z_3$
2. Is the system observable for $w = z_1$
3. Is it observable for $w = \begin{pmatrix} z_1 \\ z_3 \end{pmatrix}$?

Observability versus Controllability

Theorem:

- (A, B) controllable $\Leftrightarrow$ (B^T, A^T) observable;
- (C, A) observable $\Leftrightarrow$ (A^T, C^T) controllable.

Proof follows from

$$\left[\text{controllability matrix of } (A, B)\right]^T$$
$$= \left[\text{observability matrix of } (B^T, A^T)\right].$$

For every property that holds for controllability there exists a **dual** property in terms of observability and vice-versa.

Controllability and observability

Exercise:

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} \alpha-1 & 1 \\ 0 & \alpha \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} \alpha & 0 \end{bmatrix} x(t) \end{cases}$$

- For which $\alpha \in \mathbb{R}$ is this system controllable?
- For which $\alpha \in \mathbb{R}$ is this system observable?

State-space realization

We know that to (SISO) $\dot{x} = Ax + Bu$ and $y = Cx$ there corresponds a **strictly proper** transfer function

$$H(s) = C(sI - A)^{-1}B.$$

Similarly, to $\dot{x} = Ax + Bu$ and $y = Cx + Du$ corresponds

$$H(s) = C(sI - A)^{-1}B + D,$$

which is **proper**.

Now the reverse ...

State-space realization

State-space realization: determine a state-space description from an input-output description.

State-space realization of a system is not unique
$\Rightarrow$ linear state transformations can be applied.

Dimension of state-space realization can vary...

State-space realization

Example:
$$\begin{cases} \dot{x}_1(t) = -x_1(t) + u(t) \\ y(t) = x_1(t), \end{cases}$$

or

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + u(t) \\ \dot{x}_2(t) = -x_2(t) \\ y(t) = x_1(t). \end{cases}$$

What is the **minimal number of states** needed to describe input-output behavior (i.e., in terms of u and y)?

Minimal realization

A **minimal realization** is a state-space realization that has the smallest order (number of state variables) among all possible realizations.

Theorem:

A state-space realization is **minimal** if and only if it is **controllable and observable.**

Modal form

Recall:

$$\dot{\widetilde{x}}(t) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \widetilde{x}(t) + \begin{pmatrix} \widetilde{B}_{11} \\ \widetilde{B}_{21} \\ \vdots \\ \widetilde{B}_{n1} \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} \widetilde{C}_{11} & \widetilde{C}_{12} & \cdots & \widetilde{C}_{1n} \end{pmatrix} \widetilde{x}(t) + Du(t)$$

with $\lambda_i \neq \lambda_j$, for $i \neq j$.

- Not controllable if for some i: $\widetilde{B}_{i1} = 0$
- Not observable if for some i: $\widetilde{C}_{1i} = 0$

Jointly observable and controllable

Hence,

$$\begin{aligned} H(s) &= \widetilde{C}(sI - \widetilde{A})^{-1}\widetilde{B} + D \\ &= \sum_{i=1}^{n} \frac{\widetilde{C}_{1i}\widetilde{B}_{i1}}{s - \lambda_i} + D \end{aligned}$$

$\Rightarrow$ Not observable or controllable: **less than n terms**

$\Rightarrow$ **Pole-zero cancellations** in the transfer function $\Rightarrow$ the transfer function is reducible.

Example

System:

$$\dot{x}(t) = \begin{pmatrix} -0.2 & 0 \\ 0 & -0.3 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$$

$\Rightarrow$ Second state is not observable and not controllable!

Exercise

System:

$$\dot{x}(t) = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & -4 & 3 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} x(t)$$

Is this a minimal state-space realization?

Controller canonical form

All states are controllable:

$$H(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} = \frac{c_2 s^2 + c_1 s + c_0}{s^3 + a_2 s^2 + a_1 s + a_0} + d$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} c_0 & c_1 & c_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + du$$

Observer canonical form

All states are observable:

$$H(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} = \frac{c_2 s^2 + c_1 s + c_0}{s^3 + a_2 s^2 + a_1 s + a_0} + d$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + du$$

Canonical forms

Controller canonical form: (A_c, B_c, C_c, D_c)

Observer canonical form: (A_o, B_o, C_o, D_o)

Relation:

$$(A_o, B_o, C_o, D_o) = (A_c^T, C_c^T, B_c^T, D_c^T)$$