Systeem- en Regeltechniek
EE2S21

Constitutive relationships
Analogies

Lecture 2

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Introduction into modeling issues

System: Object or set of objects from which we want to study the properties.

Model of a system: A tool we use to answer questions about the system without having to do an experiment.

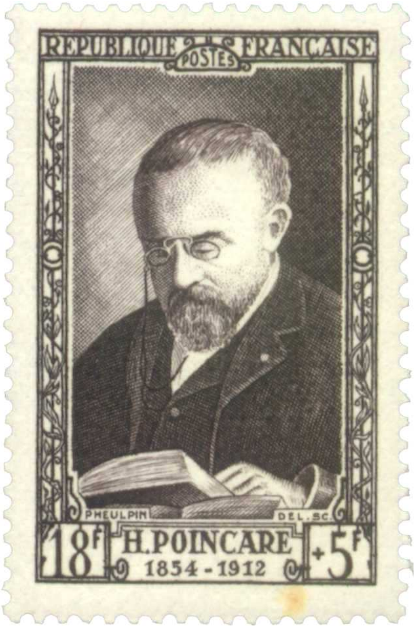
Types of models:

- mental: intuition and experience, verbal: if..., then...
- physical: scale models, laboratory set-ups
- **mathematical**: equations that describe relation between quantities that are important for behaviour of system, e.g., laws of nature. **Focus of EE2S21!**

Models for multi-physics systems

Restriction to **continuous-time** modeling based on our knowledge of the laws of nature in a structured way.

Deterministic dynamical **lumped-parameter** systems (ODE's) of the form



$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}\big(\boldsymbol{x}(t), \boldsymbol{u}(t), t\big), \quad \dot{\boldsymbol{x}}(t) = \frac{d\boldsymbol{x}(t)}{dt}$$
$$\boldsymbol{y}(t) = \boldsymbol{h}\big(\boldsymbol{x}(t), \boldsymbol{u}(t), t\big)$$

with $\boldsymbol{u} \in \mathbb{R}^m$, $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{y} \in \mathbb{R}^p$, $t \in \mathbb{R}$. Furthermore,

$$\boldsymbol{f} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n, \quad \boldsymbol{h} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^p.$$

Such set of ODE's is called a **state space model** with state x.

Models for multi-physics systems

Important subclass:

- **Time-Invariant**: $\dot{\boldsymbol{x}}(t) = \boldsymbol{f}\big(\boldsymbol{x}(t), \boldsymbol{u}(t)\big), \boldsymbol{y}(t) = \boldsymbol{h}\big(\boldsymbol{x}(t), \boldsymbol{u}(t)\big).$
- **Linear and Time-Invariant (LTI)**:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= C\boldsymbol{x}(t) + D\boldsymbol{u}(t). \end{aligned}$$

How to obtain such models? Consider following domains:

- **Electrical**
- **Mechanical translational**
- **Mechanical rotational**
- **Hydraulical**
- **Thermo-dynamical**

Models for multi-physics systems

In discussing a lumped-parameter system, we deal with:

- Sets of **elements** such as **ideal** springs, masses, dampers, inductors, capacitors, resistors, tubes, tanks, transformers, gyrators, etc..
- Sets of **variables** (signals) such as forces, velocities, voltages, currents, pressure, temperature, etc..
- Sets of **relationships** between variables.
- **Interactions** between elements.

Models for multi-physics systems

Two type of elements:

- **Dynamic** (mass, spring, inductor, capacitor, etc.)

  $\Rightarrow$ **energetic** (energy storage)

- **Static** (resistor, damper, transformer, etc.)

  $\Rightarrow$ **non-energetic** (dissipation, scaling)

Two type of element relationships:

- **Constitutive** relationships (all elements)
- **Dynamical** relationships (dynamic elements)

Models for multi-physics systems

Other types of relationships:

- **Interconnective** relationships. Interconnection of elements (Kirchhoff laws, D'Alembert, etc.)
- Combination of constitutive and dynamical relationships yields the **component** relationships.

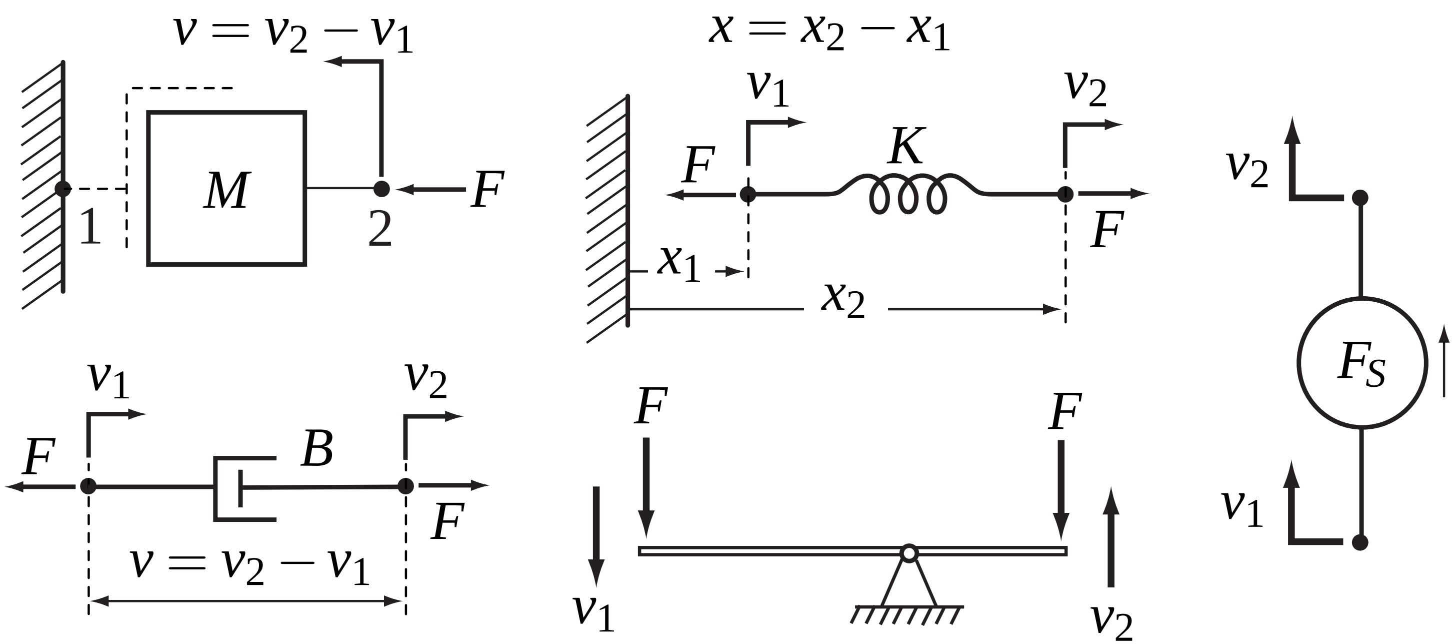
These relationships define our

$$\boxed{\boxed{\textbf{dynamical system model.}}}$$

$\Rightarrow$ This course will mainly focus on **electro-mechanical** systems.

Mechanical Translation

Symbols:



Important physical variables:

- Position $x(t)$ [m]
- Impulse momentum $p(t)$ [Ns]
- Force $F(t)$ [N]
- Velocity $v(t)$ [m/s]

Mechanical Translation

- Rate of change of position yields velocity, i.e.,

$$v(t) = \frac{dx(t)}{dt},$$

  or in integral form $x(t) = \int_{-\infty}^{t} v(\tau)d\tau$.

- Rate of change of momentum yields force, i.e.,

$$F(t) = \frac{dp(t)}{dt},$$

  or in integral form $p(t) = \int_{-\infty}^{t} F(\tau)d\tau$.

Mechanical Translation

For linear and time-invariant (LTI) elements, we know

- **Mass:** Newton's 2nd law

$$F = Ma \quad (M = \text{constant})$$

- **Spring:** Hooke's law

$$F = Kx \quad (K = \text{constant})$$

- **Damper:** mechanical 'Ohm's law'

$$F = Bv \quad (B = \text{constant})$$

Mechanical Translation

Ok, but what if M is changing with time? Can we model the behavior by just extending $F = Ma$ to $F = M(t)a$???

The answer is of course: NO!

$\Rightarrow$ We have been looking at the wrong variables in the first place.

Claim: Mass should be described by a relationship between p and v.

In general, we may define

- $p = \hat{p}(v,t)$ (velocity-controlled mass), or
- $v = \hat{v}(p,t)$ (momentum-controlled mass)

Mechanical Translation

Claim: Newton's 2nd law is $F = \frac{dp}{dt}$.

- LTI case: $p = Mv$, hence

$$F = \frac{dp}{dt} = \frac{dMv}{dt} = M\frac{dv}{dt} = Ma$$

$\Rightarrow F = Ma$ is just a consequence of a more fundamental law and belongs to the class of component relationships.

- LTV case: $p = M(t)v$, hence

$$F = \frac{dp}{dt} = M(t)\frac{dv}{dt} + \frac{dM(t)}{dt}v$$

Mechanical Translation

Note that this also works for the relativistic masses. In that case

$$p = \frac{m_0 v}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \hat{p}(v),$$

with c the speed of light.

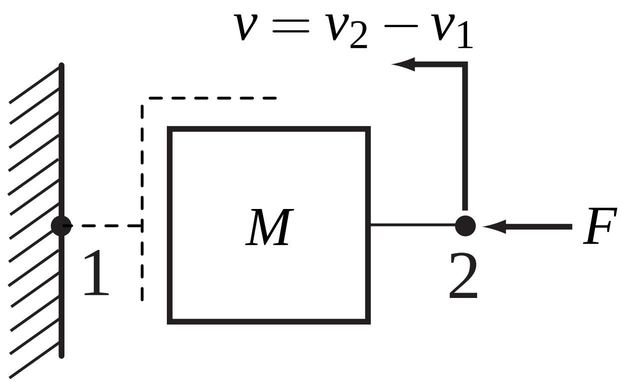
Hence relativistic mass is an example of a (velocity-controlled) nonlinear time-invariant (NLTI) mass.

Mechanical Translation

Moral of the story:

- the constitutive relationship of a mass is a curve defined in the p-v plane.
- Newton's 2nd law $F = \frac{dp}{dt}$ belongs to the class of dynamical relationships.
- $F = Ma$ is the associated component relationship.

Mechanical Translation



Mass M: (non-relativistic)

- Constitutive relationship:

$$p = Mv.$$

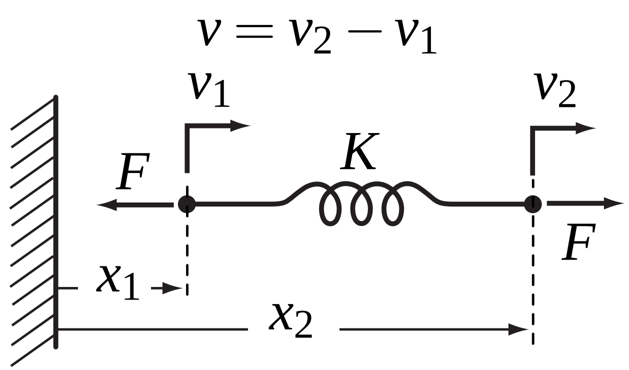
- Dynamical relationship:

$$F = \frac{dp}{dt}, \text{ or } \; p = p(t_0) + \int_{t_0}^{t} F(\tau) d\tau.$$

- Component relationship:

$$\frac{dp}{dt} = \frac{dMv}{dt} = M\frac{dv}{dt} = Ma.$$

Mechanical Translation



Translational spring K:

- Constitutive relationship:

$$x = \hat{x}(F) \quad \left[\text{linear: } x = K^{-1}F\right]$$

$$F = \hat{F}(x) \quad \left[\text{linear: } F = Kx\right].$$

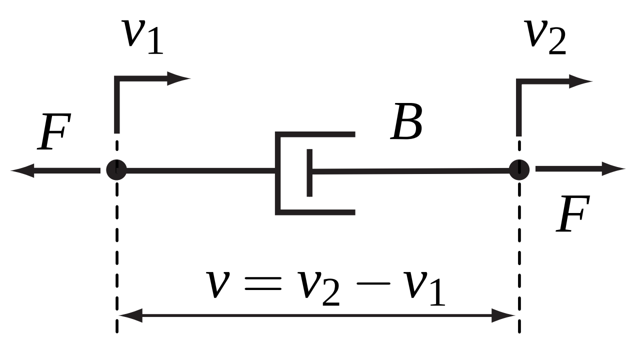
- Dynamical relationship:

$$v = \frac{dx}{dt}, \text{ or } x = x(t_0) + \int_{t_0}^{t} v(\tau)d\tau.$$

- Component relationship:

$$\frac{dx}{dt} = \frac{d\hat{x}(F)}{dt} = \frac{d\hat{x}(F)}{dF}\frac{dF}{dt} = \underbrace{K^{-1}(F)}_{\text{incremental stiffness}}\frac{dF}{dt} \quad \left[\text{linear: } v = K^{-1}\frac{dF}{dt}\right].$$

Mechanical Translation



Translational damper B:

Constitutive relationship:

- Velocity-controlled damper

$$F = \hat{F}(v) \quad [\text{linear: } F = Bv].$$

- Force-controlled damper

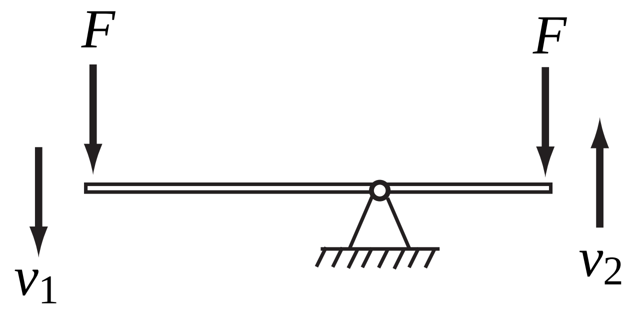
$$v = \hat{v}(F) \quad [\text{linear: } v = B^{-1}F].$$

Mechanical Translation

Transformer Tr: (no power loss: $v_1 F_1 = v_2 F_2$)

- Constitutive relationships:

$$
\begin{aligned}
F_1 &= nF_2 \\
v_1 &= \frac{v_2}{n}.
\end{aligned}
$$

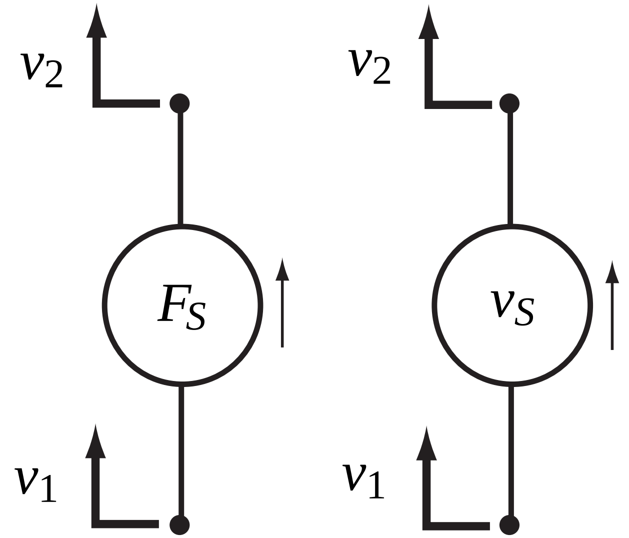


Sources S:

- Constitutive relationships:

force source: $F(t) = F_S(t)$

velocity source: $v(t) = v_S(t)$.



Mechanical Translation

Interconnective Relationships:

- Force balance:

$$\sum_k F_k = 0.$$

(d'Alembert's principle, law of conservation of impulse momenta

$\Rightarrow$ Translational mechanical "Kirchhoff" law.)

- Velocity balance not explicitly used.

Example: LTI mass-spring-damper system

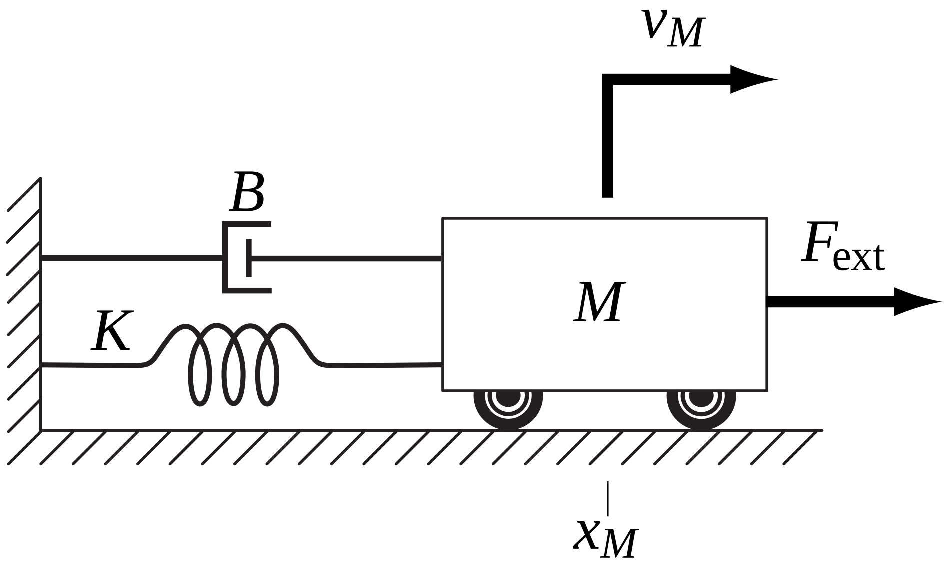
- Constitutive rel's:

$$F_B = Bv_B$$

$$F_S = F_{\text{ext}}$$

$$p_M = Mv_M$$

$$F_K = Kx_K.$$



- Dynamical rel's:

$$F_M = \frac{dp_M}{dt},$$

$$v_K = \frac{dx_K}{dt},$$

Example: mass-spring-damper system

- Interconnective rel.:

$$F_M + F_K + F_B = F_S$$

$$v_M = v_K = v_B = v_S$$

- Component rel's:

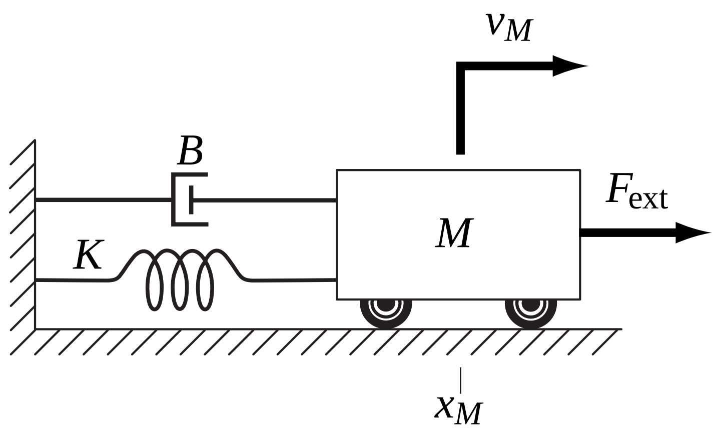
$$F_M = M\frac{dv_M}{dt}$$

$$v_K = \frac{1}{K}\frac{dF_K}{dt}$$

Example: mass-spring-damper system

Combining dynamical and interconnective relationships, yields set of **first-order ODE's**:

$$\frac{dx_K(t)}{dt} = \frac{p_M(t)}{M}$$
$$\frac{dp_M(t)}{dt} = F_{\text{ext}}(t) - Kx_K(t) - B\frac{p_M(t)}{M}.$$



If initial position $x_K(t_0)$ and initial momentum $p_M(t_0)$ are known, together with $F_{\text{ext}}(t)$, $t \geq t_0$, then further evolution of $x_K(t)$ and $p_M(t)$ is fully determined.

$x_K(t), p_M(t) \Rightarrow$ **state variables** $\Rightarrow$ **state space system**.

Example: mass-spring-damper system

Since the elements are linear and time-invariant (LTI), we can rewrite the system in terms of a system matrix A and an input matrix B as

$$\begin{bmatrix} \dot{x}_K \\ \dot{p}_M \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{M} \\ -K & -\frac{B}{M} \end{bmatrix}}_{A} \begin{bmatrix} x_K \\ p_M \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} F_{\text{ext}}.$$

The output matrix C and the direct feedthrough matrix D depend on what variable(s) are considered as output(s). In case we take the velocity of the mass as the output, we have

$$y = \underbrace{\begin{bmatrix} 0 & \frac{1}{M} \end{bmatrix}}_{C} \begin{bmatrix} x_K \\ p_M \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} F_{\text{ext}} = v_M.$$

Example: mass-spring-damper system

If we are interested in the force acting on the mass, we select

$$y = \underbrace{\begin{bmatrix} -K & -\frac{B}{M} \end{bmatrix}}_{C} \begin{bmatrix} x_K \\ p_M \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{D} F_{\text{ext}} = F_M.$$

Another possibility would be to take both the velocity of the mass and the displacement of the spring as outputs. In this case $y \in \mathbb{R}^2$ and takes the form

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{M} \end{bmatrix}}_{C} \begin{bmatrix} x_K \\ p_M \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{D} F_{\text{ext}} = \begin{bmatrix} x_K \\ v_M \end{bmatrix}.$$

Example: mass-spring-damper system

Note that in case that the spring and the damper are nonlinear and time-variant, with for example

$$F_K = \hat{F}_K(x_K) = K_0 x_K + K_1 x_K^3 \text{ (hardening spring for } K_1 > 0),$$
$$F_B = \hat{F}_B(v_B, t) = B(t) v_B^3,$$

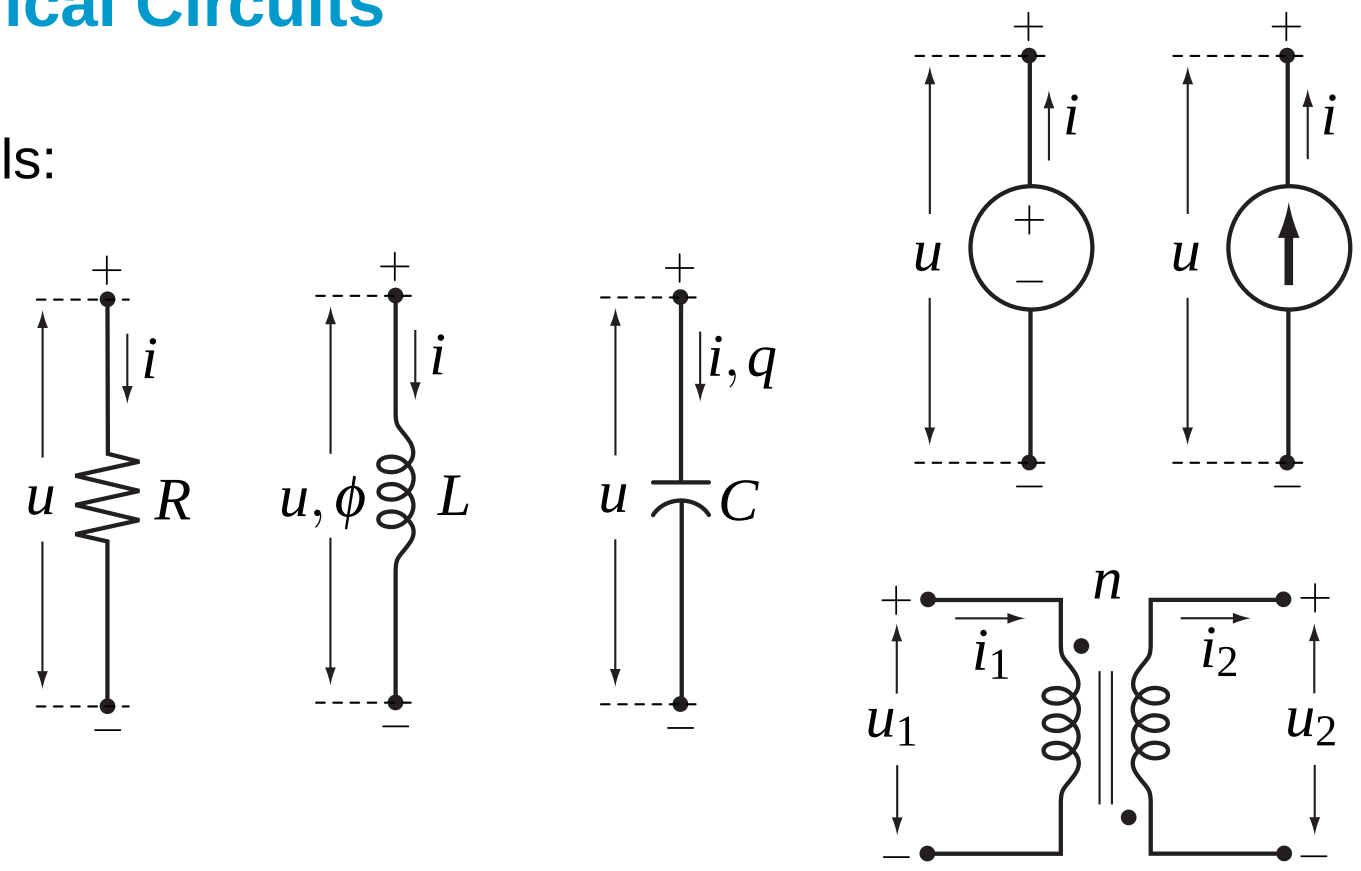
then the state equations look like

$$\dot{x}_K = \frac{p_M}{M} = f_1(p_M)$$
$$\dot{p}_M = F_{\text{ext}} - K_0 x_K - K_1 x_K^3 - B(t)\left(\frac{p_M}{M}\right)^3 = f_2(x_K, p_M, F_{\text{ext}}, t),$$

which is of the form $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t)$.

Electrical Circuits

Symbols:



Important physical variables:

- Charge $q(t)$ [C]
- Flux-linkage $\phi(t)$ [Wb]
- Voltage $u(t)$ [V]
- Current $i(t)$ [A]

Electrical Circuits

- Rate of change of charge yields current, i.e.,

$$i(t) = \frac{dq(t)}{dt},$$

or in integral form $q(t) = \int_{-\infty}^{t} i(\tau)d\tau$.

- Rate of change of flux yields voltage (Faraday's law), i.e.,

$$u(t) = \frac{d\phi(t)}{dt},$$

or in integral form $\phi(t) = \int_{-\infty}^{t} u(\tau)d\tau$.

Electrical Circuits

Linear circuit theory and physics courses tell us that

- **Resistor:** $u = Ri$ (Ohm's law)
- **Inductor:** $u = L\frac{di}{dt}$
- **Capacitor:** $i = C\frac{du}{dt}$

Claim: $u = Ri$ is a constitutive relationship, but the other two are not.

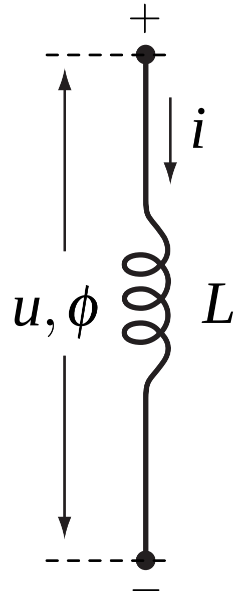
Electrical Circuits

Circuit theory revisited:

- **Resistor:** $$\begin{cases} u = \hat{u}(i,t) & \text{(current-controlled)} \\ i = \hat{i}(u,t) & \text{(voltage-controlled)} \end{cases}$$
- **Inductor:** $$\begin{cases} \phi = \hat{\phi}(i,t) & \text{(current-controlled)} \\ i = \hat{i}(\phi,t) & \text{(flux-controlled)} \end{cases}$$
- **Capacitor:** $$\begin{cases} q = \hat{q}(u,t) & \text{(voltage-controlled)} \\ u = \hat{u}(q,t) & \text{(charge-controlled)} \end{cases}$$

Electrical Circuits

Inductor L:



- Constitutive relationship:

$$\phi = \hat{\phi}(i) \quad [\text{linear: } \phi = Li]$$

$$i = \hat{i}(\phi) \quad [\text{linear: } i = L^{-1}\phi]$$

- Dynamical relationship:

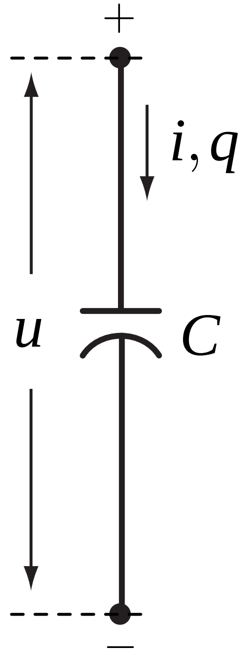
$$u = \frac{d\phi}{dt}, \text{ or } \phi = \phi(t_0) + \int_{t_0}^{t} u(\tau)d\tau.$$

- Component relationship:

$$\frac{d\phi}{dt} = \frac{d\hat{\phi}(i)}{dt} = \frac{d\hat{\phi}(i)}{di}\frac{di}{dt} = \underbrace{L(i)}_{\text{incremental inductance}}\frac{di}{dt} \quad \left[\text{linear: } u = L\frac{di}{dt}\right].$$

Electrical Circuits

Capacitor C:



- Constitutive relationship:

$$q = \hat{q}(u) \quad [\text{linear: } q = Cu]$$

$$u = \hat{u}(q) \quad [\text{linear: } u = C^{-1}q]$$

- Dynamical relationship:

$$i = \frac{dq}{dt}, \text{ or } q = q(t_0) + \int_{t_0}^{t} i(\tau)d\tau.$$

- Component relationship:

$$\frac{dq}{dt} = \frac{d\hat{q}(u)}{dt} = \frac{d\hat{q}(u)}{du}\frac{du}{dt} = \underbrace{C(u)}_{\text{incremental capacitance}}\frac{du}{dt} \quad \left[\text{linear: } i = C\frac{du}{dt}\right].$$

Electrical Circuits

Resistor R:

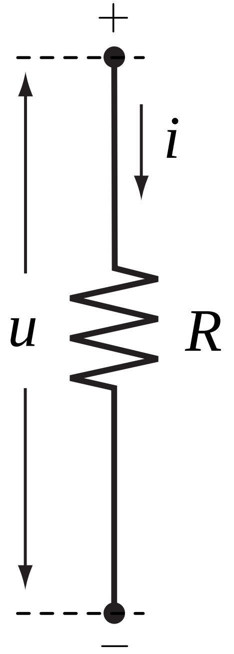
Constitutive relationship:

- Current-controlled resistor

$$u = \hat{u}(i) \quad [\text{linear: } u = Ri].$$

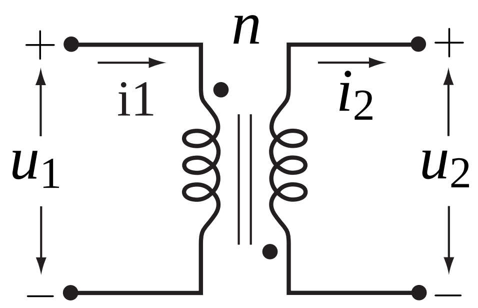
- Voltage-controlled resistor

$$i = \hat{i}(u) \quad [\text{linear: } i = Gu,\ G = R^{-1}].$$



Electrical Circuits

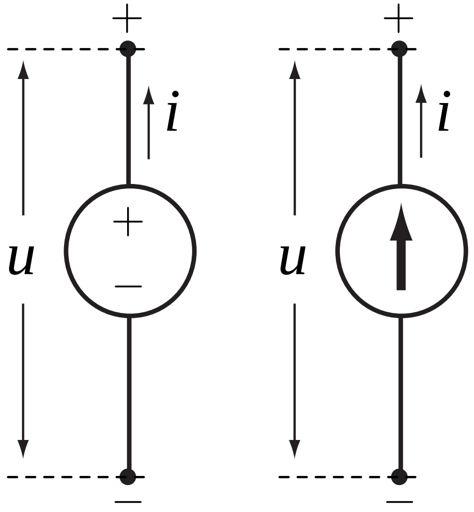
Transformer Tr: (no power loss: $i_1 u_1 = i_2 u_2$)



- Constitutive relationships:

$$u_1 = n u_2$$
$$i_1 = \frac{i_2}{n}.$$

Sources S:



- Constitutive relationships:

voltage source: $u = u_S$

current source: $i = i_S.$

TU Delft

Electrical Circuits

Interconnective Relationships:

- Kirchhoff's Current Law (KCL):

$$\sum_k i_k = 0.$$

- Kirchhoff's Voltage Law (KVL):

$$\sum_k u_k = 0.$$

Example: Linear RLC circuit

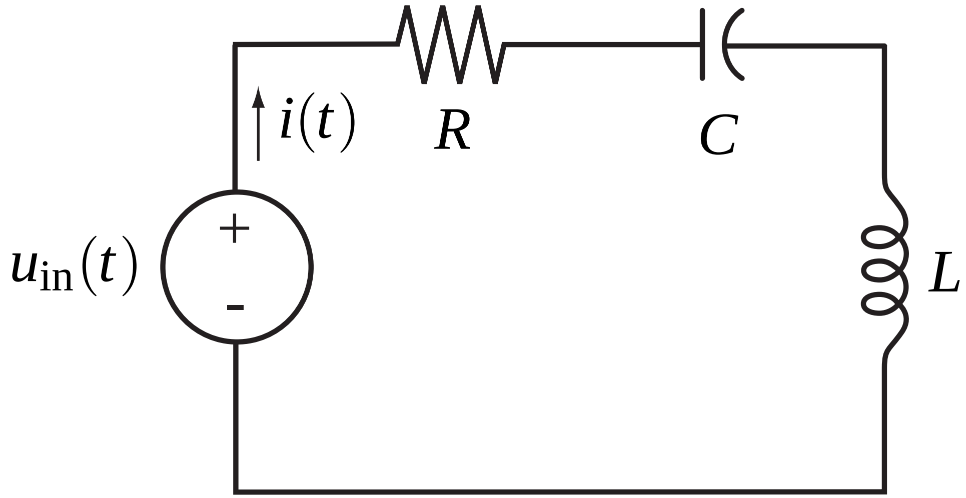
- Constitutive rel's:

$$u_R = Ri_R$$

$$u_S = u_{\text{in}}$$

$$\phi_L = Li_L$$

$$q_C = Cu_C.$$



- Dynamical rel's:

$$u_L = \frac{d\phi_L}{dt},$$

$$i_C = \frac{dq_C}{dt}.$$

Example: Linear RLC circuit

- Interconnective rel's:
  - KCL: $i_L = i_C = i_R = i_S$
  - KVL: $u_L + u_C + u_R = u_S$
- Component rel's:

$$u_L = L\frac{di_L}{dt}$$

$$i_C = C\frac{du_C}{dt}$$

Electrical vs Mechanical

- Electrical system:

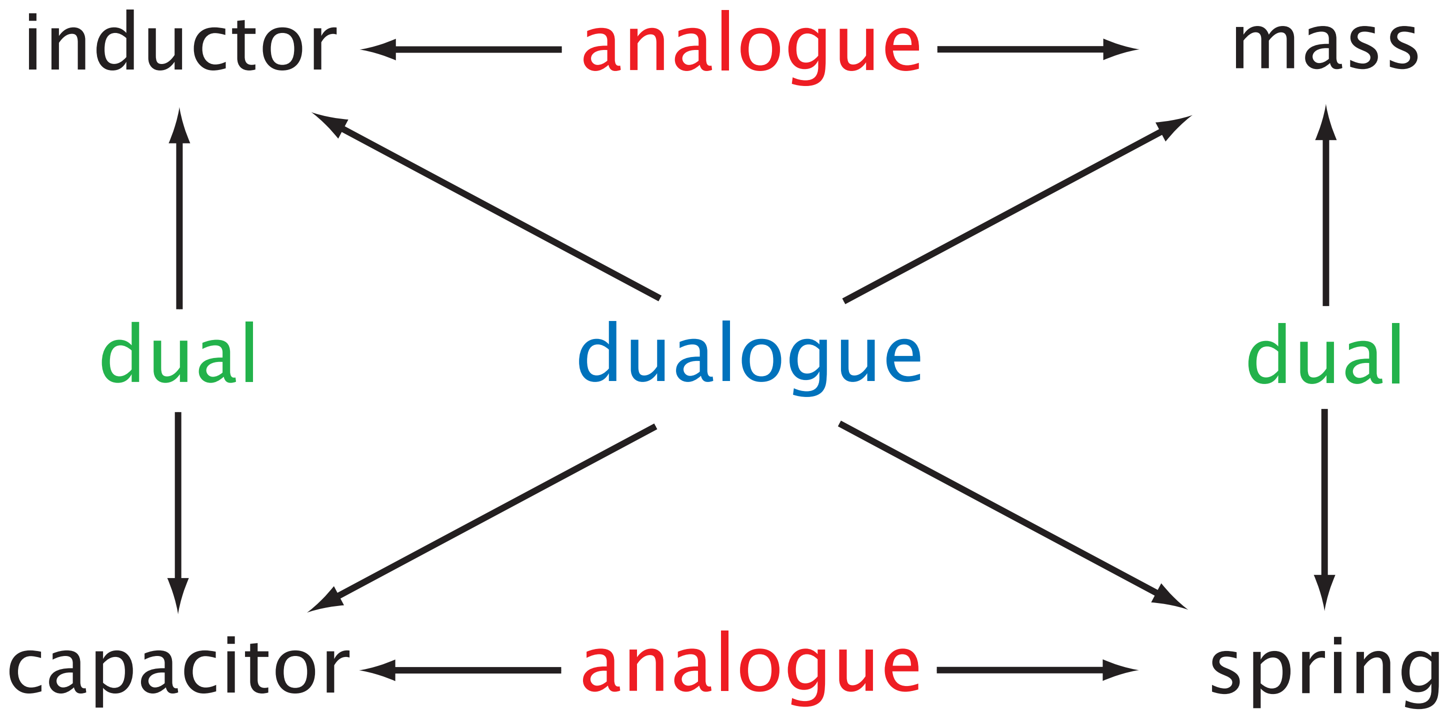
$$\frac{dq_C}{dt} = \frac{\phi_L}{L}$$
$$\frac{d\phi_L}{dt} = u_{\text{in}} - \frac{q_C}{C} - R\frac{\phi_L}{L}.$$

- Mechanical system:

$$\frac{dx_K}{dt} = \frac{p_M}{M}$$
$$\frac{dp_M}{dt} = F_{\text{ext}} - Kx_K - B\frac{p_M}{M}.$$

Any resemblance? $\Rightarrow L = M, C = K^{-1}, R = B, u_{\text{in}} = F_{\text{ext}}.$

Classical force-voltage or mass-inductance analogy:



	Effort	Flow	Gen. Position	Gen. Momentum
	e	f	q	p
Electric	voltage u [V]	current i [A]	charge q [C]	flux ϕ [Vs]
Translation	force F [N]	velocity v [m/s]	displ. x [m]	mom. p [Ns]
Rotation	torque τ [Nm]	angular vel. ω [rad/s]	angl. displ. θ [rad]	rot. mom. L [Nms]
Hydraulic	pressure p [N/m^2]	vol. flow Q [m^3/s]	volume V [m^3]	press. mom. Γ [Ns/m^2]
Thermo-dynamic	temp. T [K]	entropy flow f_T [WK^{-1}]	entropy S [J/K]	-

Generalized Elements:

- "**I**" elements: $f = \hat{f}(p)$ or $p = \hat{p}(f)$

  $\Rightarrow$ masses, inductors, etc.

- "**C**" elements: $e = \hat{e}(q)$ or $q = \hat{q}(e)$

  $\Rightarrow$ springs, capacitors, etc.

- "**R**" elements: $e = \hat{e}(f)$ or $f = \hat{f}(e)$

  $\Rightarrow$ dampers, resistors, etc.

- "**S**" elements: $e = \hat{e}(f)$ or $f = \hat{f}(e)$

  $\Rightarrow$ supplied forces, voltage source, etc.

- "**TF**" and "**GY**" elements: $e_1 = \hat{e}_1(f_1)$ and $f_2 = \hat{f}_2(e_2)$

  $\Rightarrow$ transformers and gyrators.

Generalized Dynamical Relationships:

- $f(t) = \dfrac{dq(t)}{dt}$, or $q(t) = q(t_0) + \int_{t_0}^{t} f(\tau) d\tau$

- $e(t) = \dfrac{dp(t)}{dt}$, or $p(t) = p(t_0) + \int_{t_0}^{t} e(\tau) d\tau$

Note: Generalized component relationships follow in similar way. Generalized interconnective relationships are called **junctions** (treated later with **bond graphs**).