Systeem- en Regeltechniek
EE2S21

Performance Specifications

Lecture 5b

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DIAM / DCSC

TU Delft
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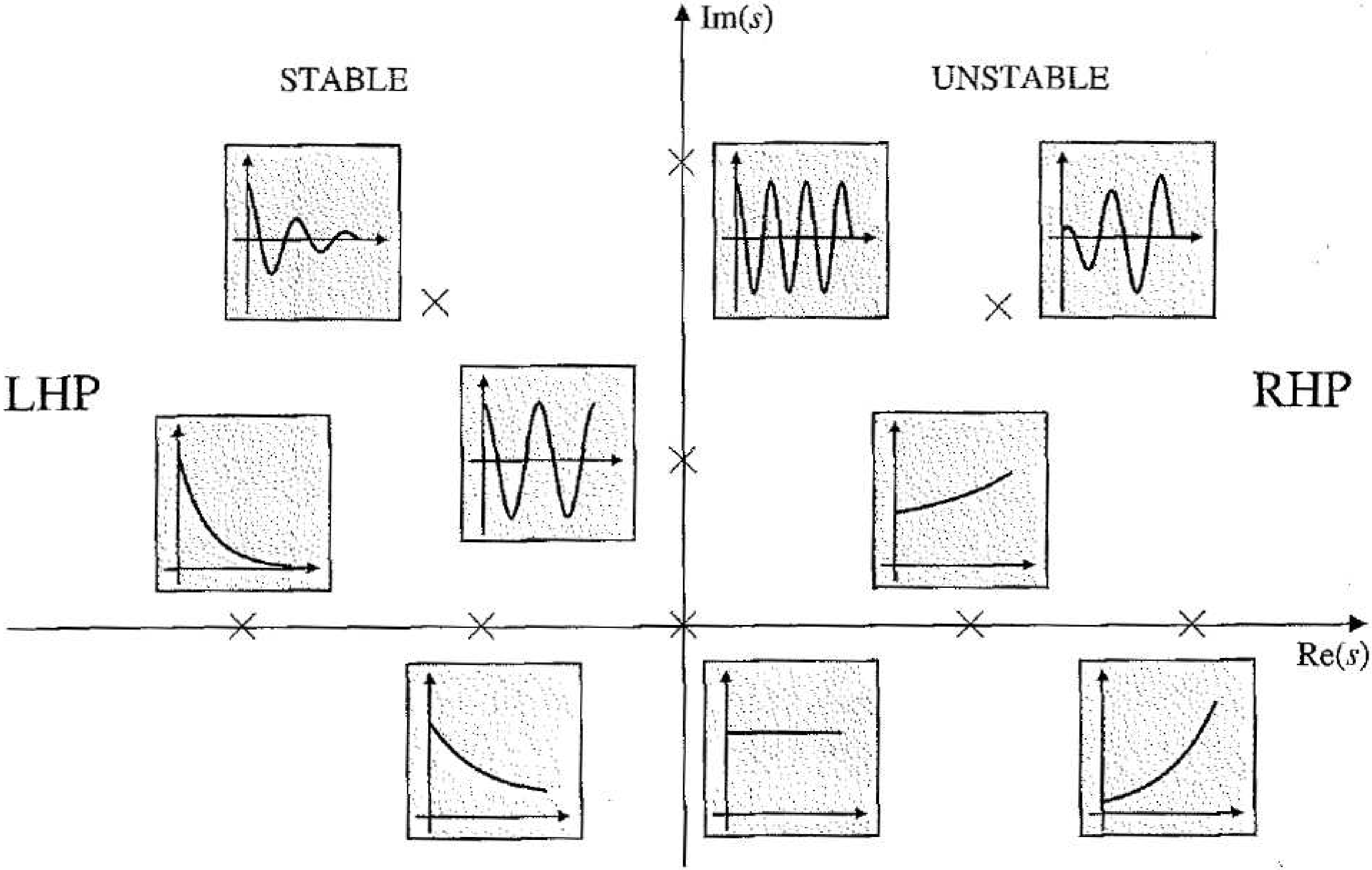
Dynamic Response



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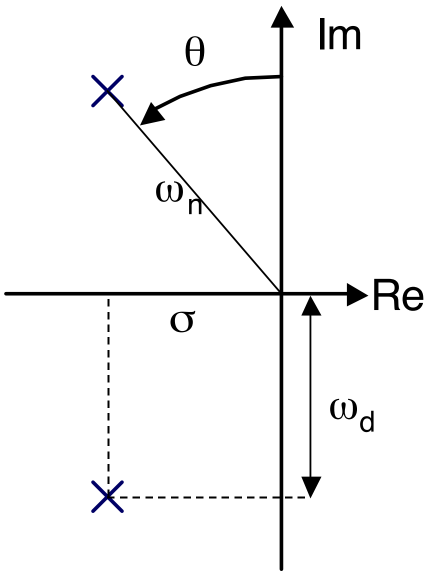
Second-Order Systems Characterization

Consider the transfer function of a 2nd-order system:

$$H(s) = \frac{k}{(s+\sigma+i\omega_d)(s+\sigma-i\omega_d)} = \frac{k}{(s+\sigma)^2+\omega_d^2},$$

This can be rewritten as



$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

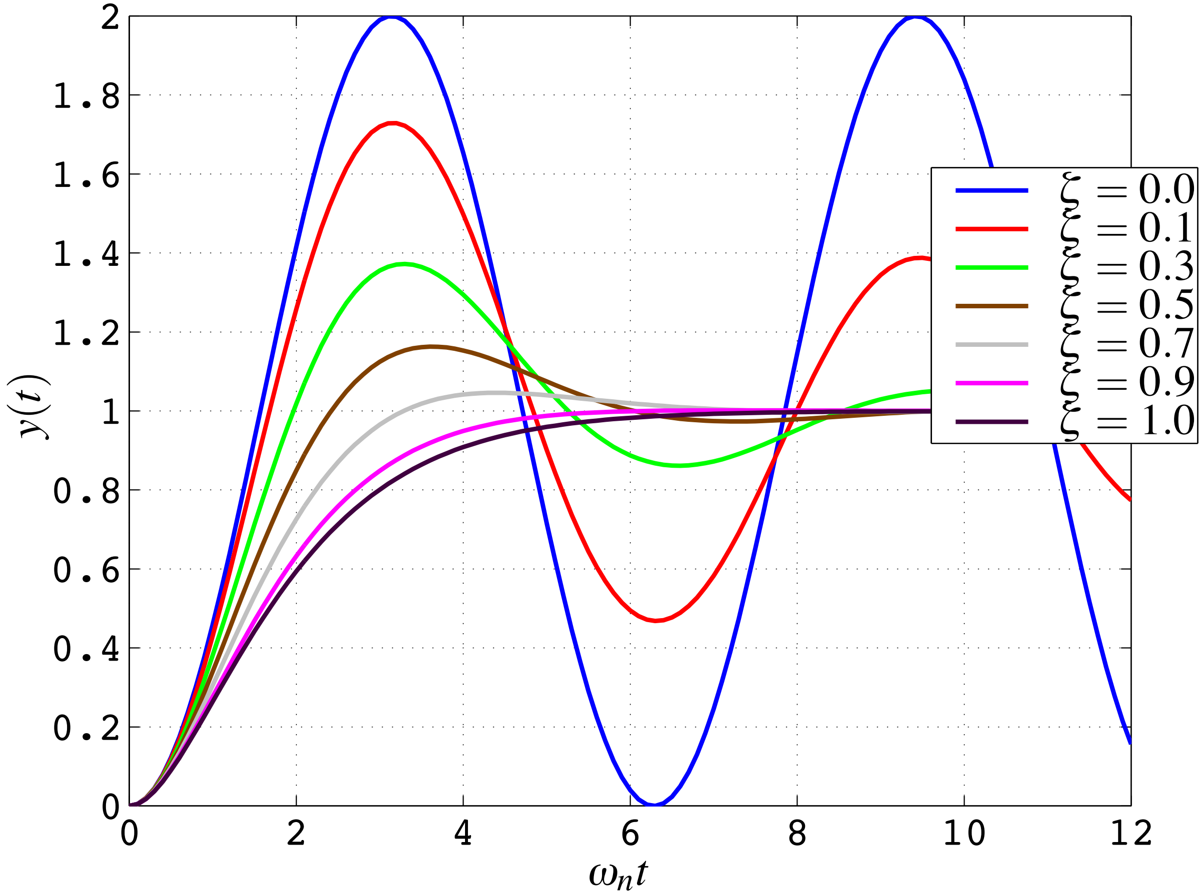
$$\sigma = \zeta\omega_n$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\theta = \sin^{-1}\zeta$$

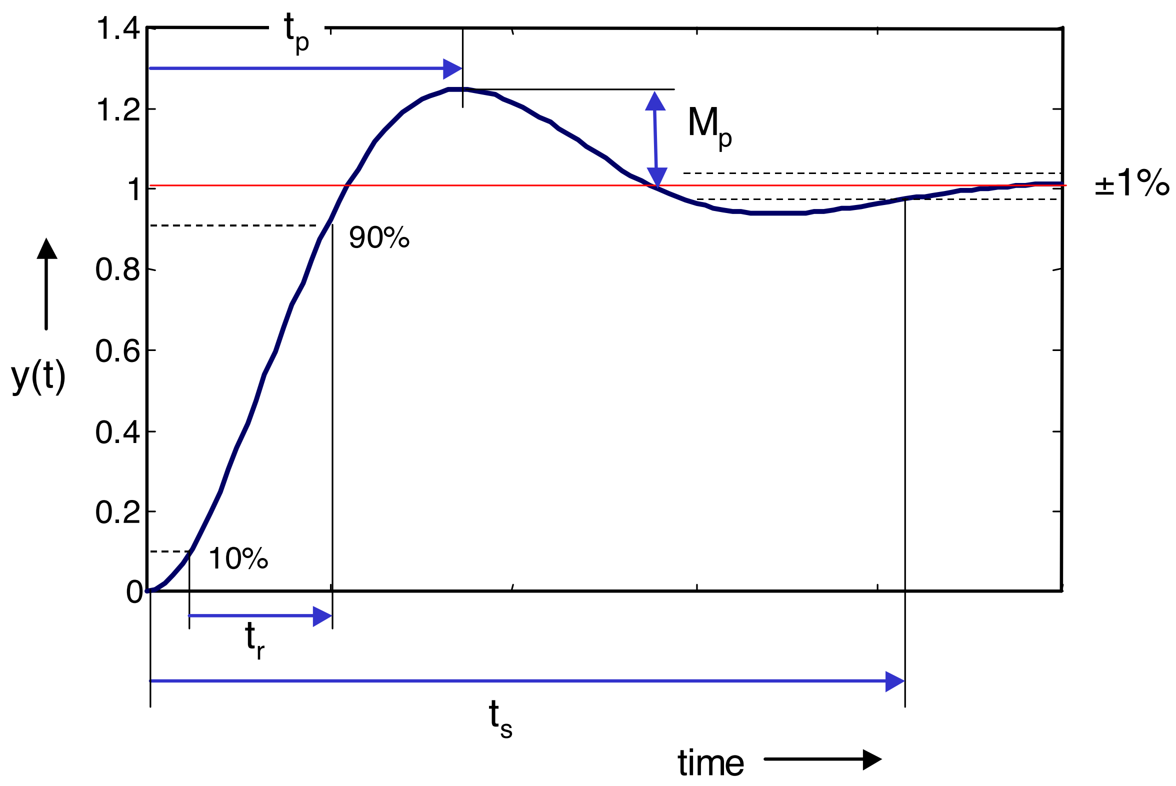
with ζ the **damping** and ω_n the **undamped natural frequency**.

Step Response 2nd-order System with Complex Poles



$$H(s)\cdot\frac{1}{s} = \frac{1}{s\left[\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1\right]} \xrightarrow{\mathcal{L}^{-1}} y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

Step Response Characterization



t_r	rise time	t_p	peak-time
t_s	settling time	M_p	overshoot

Specifications for Second-Order Systems

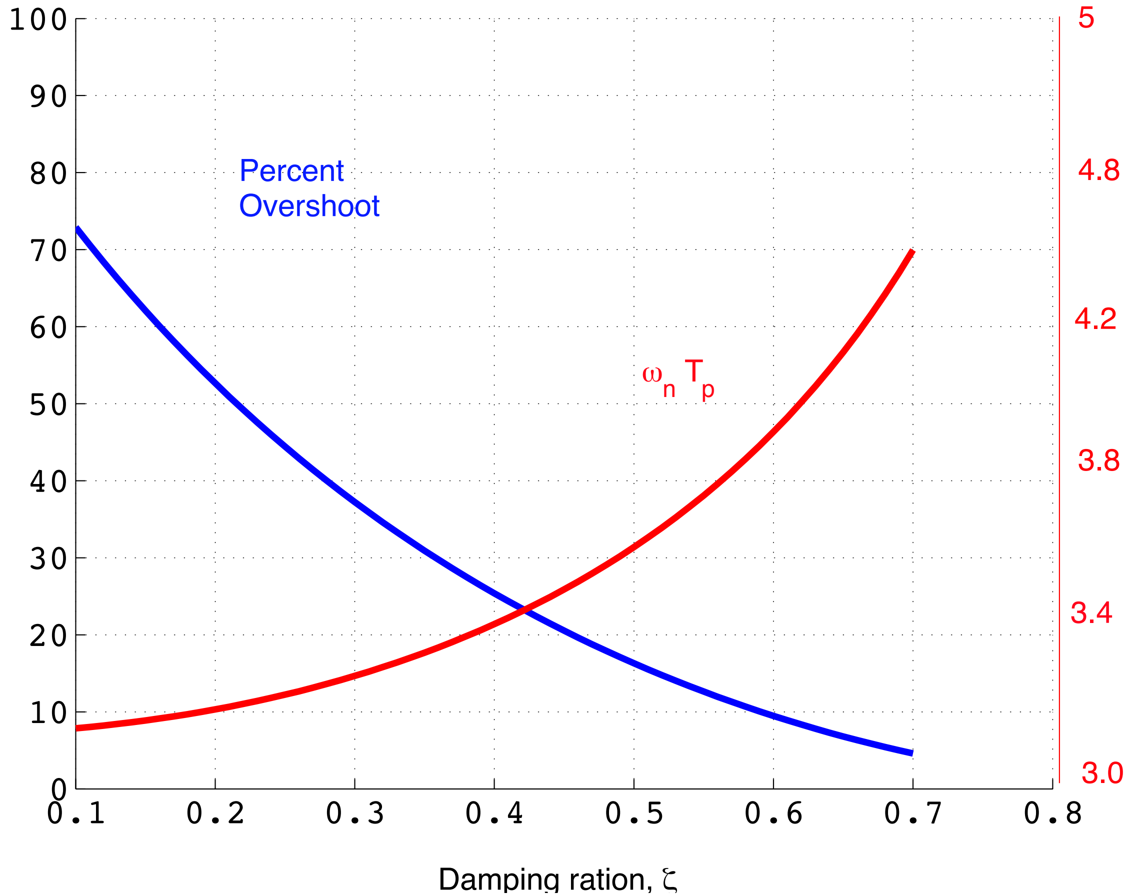
$$t_r = \frac{1.8}{\omega_n}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$t_s = \frac{4.6}{\zeta\omega_n} \quad \text{for} \quad \pm 1\%$$

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Trade-Off Speed-of-Response — Overshoot

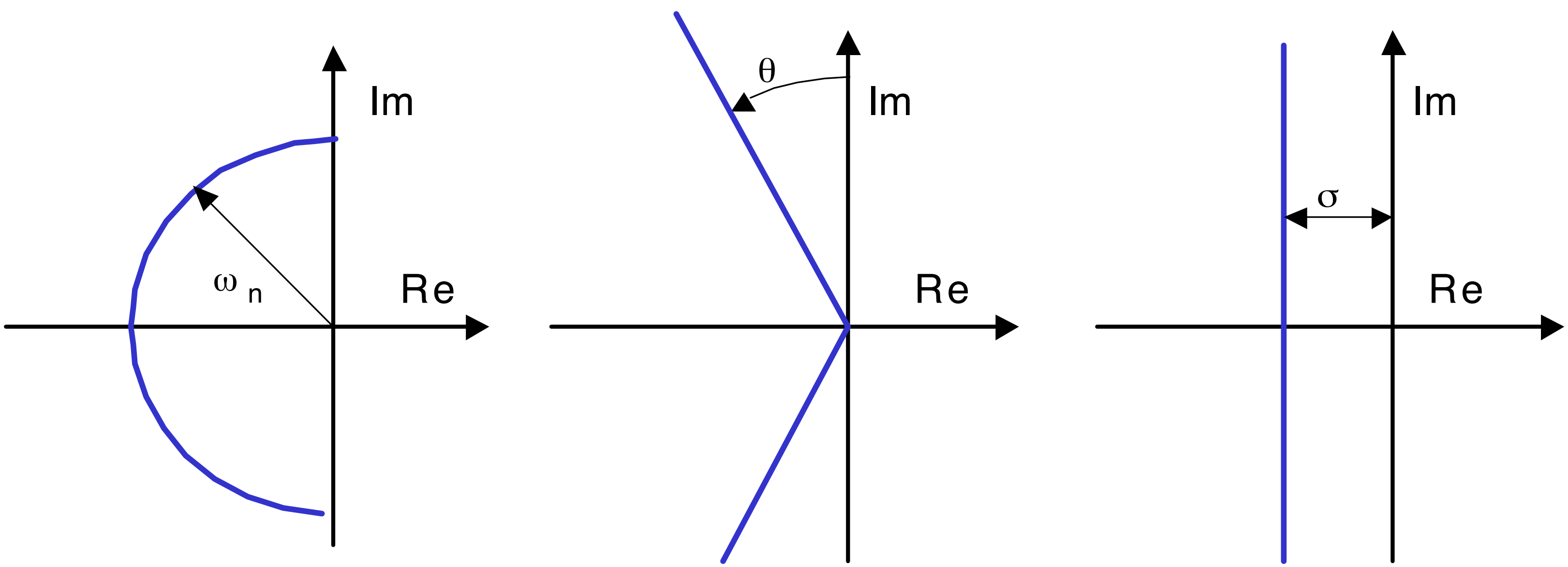


$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \qquad \omega_n T_p = \frac{\pi}{\sqrt{1-\zeta^2}}$$

Specifications

$t_r \quad t_s \quad M_p$

$$\left\{\begin{array}{l} \omega_n \geq \dfrac{1.8}{t_r} \\ \zeta \geq \zeta\left(M_p\right) \\ \sigma \geq \dfrac{4.6}{t_s} \end{array}\right.$$



Specifications in the s-plane (example)

$t_r \le 0.6\ s.$

$M_p \le 10\,\%$

$t_s \le 3\ s.$

$$\omega_n \ge \overset{3}{\cancel{2.8}}\ rad/s.$$

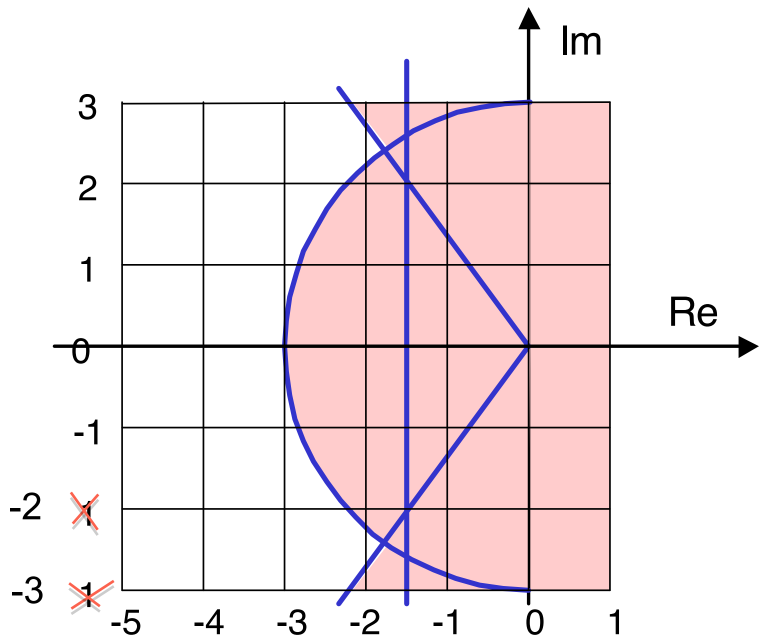
$\zeta \ge 0.6$

$\sigma \ge 1.5$

$$\omega_n \ge \frac{1.8}{t_r}$$

$$\zeta \ge \zeta\left(M_p\right)$$

$$\sigma \ge \frac{4.6}{t_s}$$



Additional pole in 2nd order all-pole system

$$H(s) = \frac{1}{\left((\frac{s}{\omega_n})^2 + 2\zeta(\frac{s}{\omega_n}) + 1 \right) \left(\gamma \frac{s}{\omega_n} + 1 \right)}$$

— —- *EffectAddPole.m* — —-

Additional pole in 2nd order all-pole system

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—–- *EffectAddPole.m* —–-

Conclusion: Provided $|\frac{\omega_n}{\gamma}| \geq 5\zeta\omega_n$ ($|\frac{1}{\gamma}| \geq 5\zeta$), the 3rd order system can be accurately approximated by an all-pole second order system.

Additional zero in 2nd-order all-pole system

$$H(s) = \frac{1}{(\frac{s}{\omega_n})^2 + 2\zeta(\frac{s}{\omega_n}) + 1} \cdot \left(\frac{s/\omega_n}{\alpha\zeta} + 1 \right)$$

Additional zero in $s = -\alpha\sigma$.

Consider normalized situation: $s/\omega_n \rightarrow s$ (or $\omega_n = 1$):

$$H(s) = \underbrace{\frac{1}{s^2 + 2\zeta s + 1}}_{H_0} + \frac{1}{\alpha\zeta} \cdot \underbrace{\frac{s}{s^2 + 2\zeta s + 1}}_{H_d}$$

Additional zero in 2nd-order all-pole system

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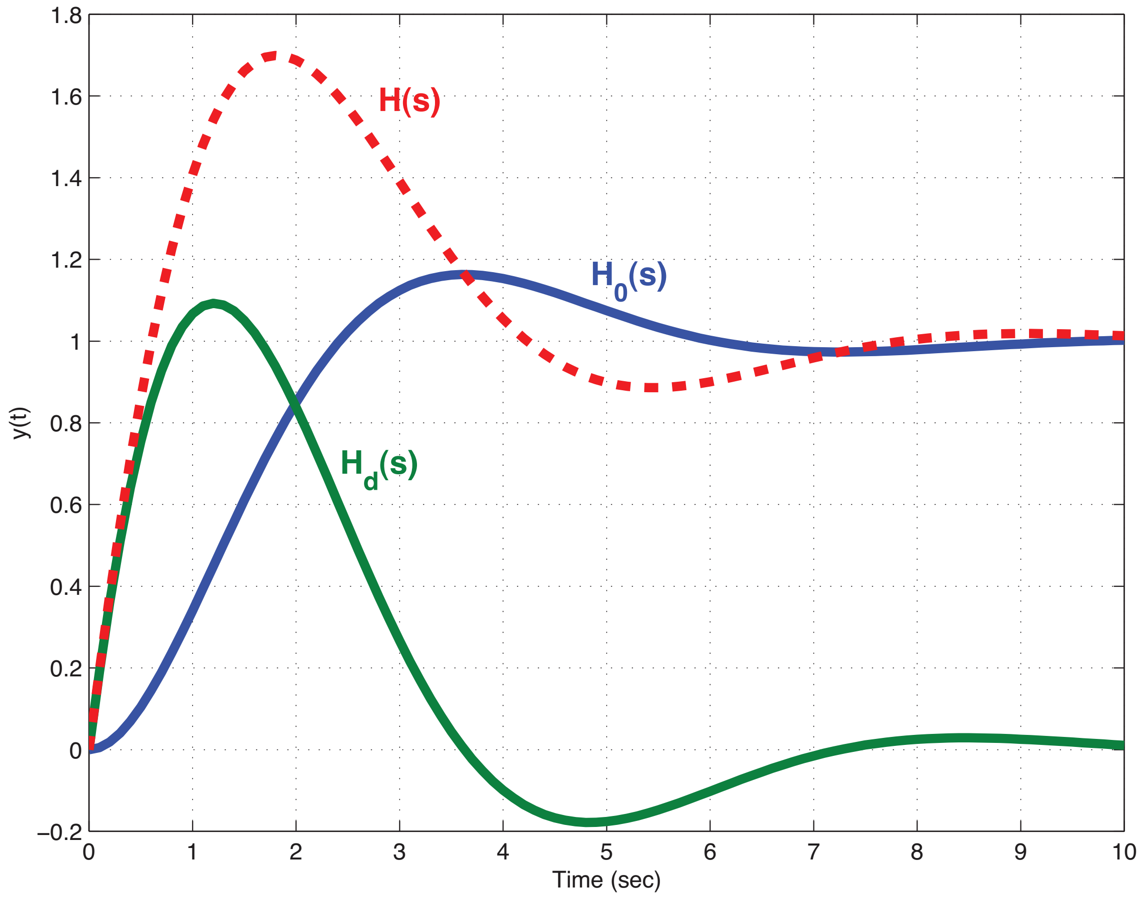
Time response of H_d is derivative of response of H_0.

Step Response Change due to extra zero: $s = -\alpha\sigma$



Typical consequence of extra zero

Let

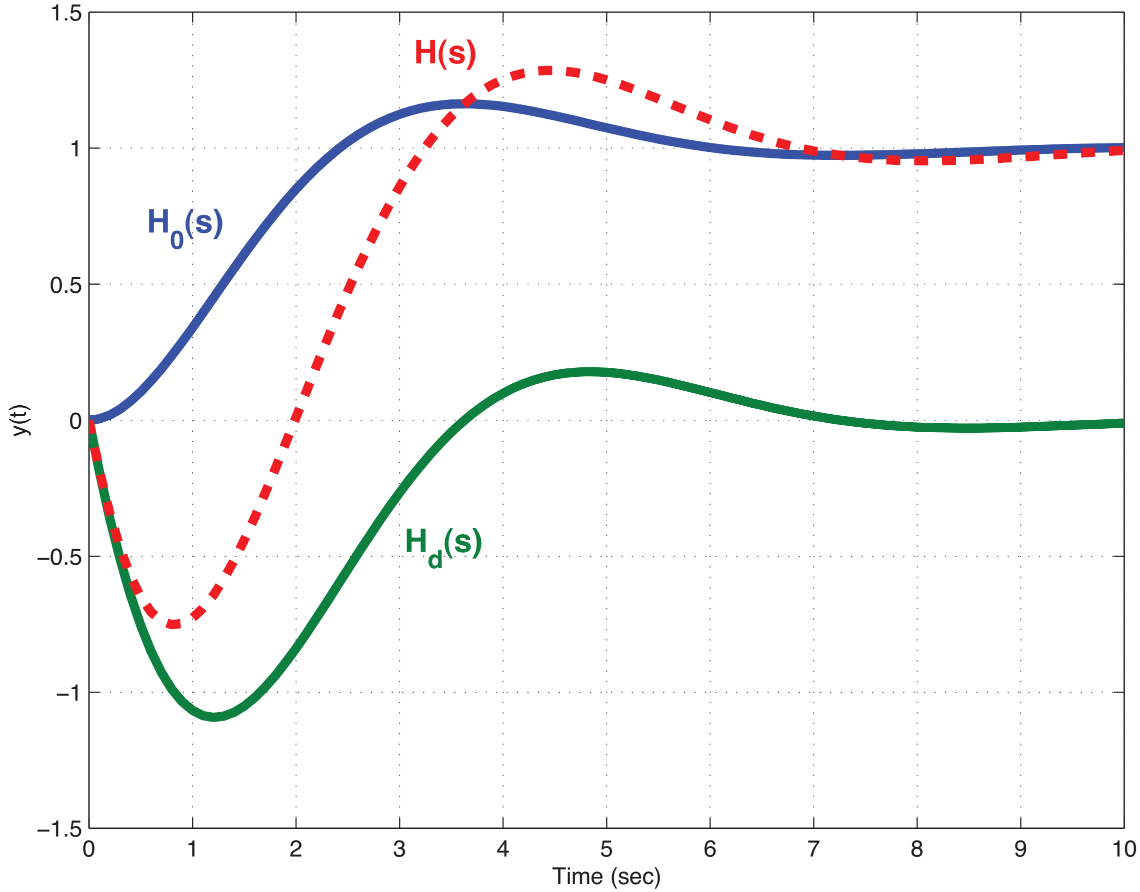
$$H(s) = \underbrace{\frac{1}{s^2+2\zeta s+1}}_{H_0} + \frac{1}{\alpha\zeta} \cdot \underbrace{\frac{s}{s^2+2\zeta s+1}}_{H_d}$$

If α not large enough $\longrightarrow$ overshoot M_p increases

If $\alpha < 0$ (zero in RHP) initial respons can become negative

Extra zero in right half plane: $s = \alpha\sigma$

Then $H(s)$ is called non-minimum phase and has step response



Summary

For second-order systems:

- Rise time $t_r \cong \frac{1.8}{\omega_n}$
- Overshoot $M_p \cong \begin{cases} 5\% & \zeta = 0.7 \\ 16\% & \zeta = 0.5 \\ 35\% & \zeta = 0.3 \end{cases}$
- Settling time $t_s \cong \frac{4.6}{\sigma}$

An additional zero causes:

overshoot $\Uparrow$ if zero $< 4\sigma$

undershoot (overshoot) $\Downarrow$ if zero in right half plane

Frequency response

Recall: **Transfer function:**

$$H(s) = \frac{Y(s)}{U(s)}$$

Frequency response:

$$H(i\omega) = H(s)\Big|_{s=i\omega} \quad \Rightarrow \quad H(i\omega) = \frac{Y(i\omega)}{U(i\omega)}$$

$\Rightarrow H(i\omega)$ is Fourier transform of impulse response $G(t)$.

Complex frequency response

Frequency response is complex-valued

Real and Imaginary part

$$H(i\omega) = \mathrm{Re}\{H(i\omega)\} + i\mathrm{Im}\{H(i\omega)\}$$

In **polar form:** $H(i\omega) = |H(i\omega)|e^{i\phi(i\omega)}$

Magnitude: $|H(i\omega)| = \sqrt{(\mathrm{Re}\{H(i\omega)\})^2 + (\mathrm{Im}\{H(i\omega)\})^2}$

Phase angle: $\phi(i\omega) = \angle H(i\omega) = \tan^{-1}\left(\dfrac{\mathrm{Im}\{H(i\omega)\}}{\mathrm{Re}\{H(i\omega)\}}\right)$

Sinusoidal frequency response

Steady state-response (for a stable system)

$$y_{ss}(t) = \lim_{t\to\infty} y(t) = \lim_{t\to\infty}\Big(y_h(t) + y_p(t)\Big) = \lim_{t\to\infty} y_p(t)$$

Sinusoidal input: $u(t) = A\sin(\omega t + \psi)$

Particular solution: $y_p(t) = B\sin(\omega t + \zeta)$

Response of an LTI system to a sinusoidal input is a sinusoidal output with same frequency but possibly different amplitude and phase.

Sinusoidal frequency response

Sinusoidal input:

$$\begin{aligned} u(t) &= A\sin(\omega t + \psi) \\ &= \frac{A}{2i}\left(e^{i(\omega t+\psi)} - e^{-i(\omega t+\psi)}\right). \end{aligned}$$

Two components:

$$u_1(t) = \frac{A}{2i}e^{i(\omega t+\psi)} \quad \text{and} \quad u_2(t) = -\frac{A}{2i}e^{-i(\omega t+\psi)}.$$

Superposition: $y_{ss}(t) = y_{ss1}(t) + y_{ss2}(t)$.

Sinusoidal frequency response

Hence,

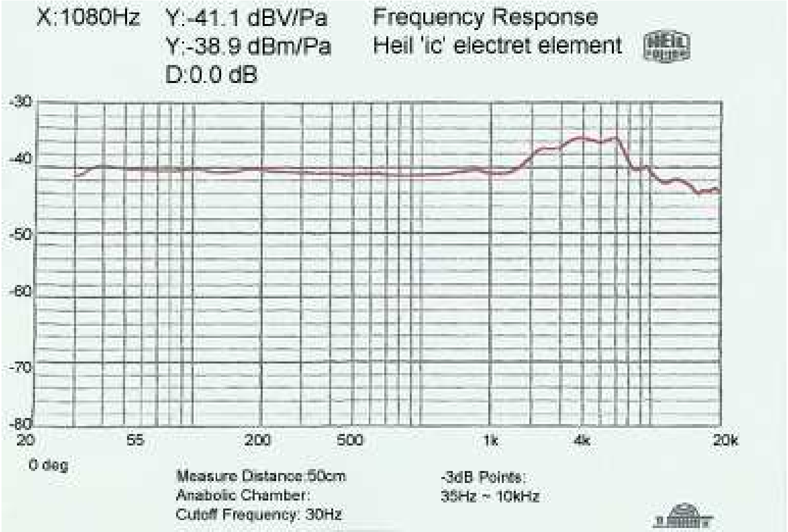
$$\begin{aligned} y_{ss}(t) &= y_{ss1}(t) + y_{ss2}(t) \\ &= \frac{A}{2i} H(i\omega) e^{i(\omega t + \psi)} - \frac{A}{2i} H(-i\omega) e^{-i(\omega t + \psi)}, \end{aligned}$$

or with $H(i\omega)$ in polar form $H(i\omega) = |H(i\omega)| e^{i\phi(i\omega)}$:

$$\begin{aligned} y_{ss}(t) &= \frac{A}{2i} |H(i\omega)| \left(e^{i(\omega t + \psi)} e^{i\phi(i\omega)} - e^{-i(\omega t + \psi)} e^{-i\phi(i\omega)} \right) \\ &= A|H(i\omega)| \frac{1}{2i} \left(e^{i(\omega t + \psi + \phi(i\omega))} - e^{-i(\omega t + \psi + \phi(i\omega))} \right) \\ &= A|H(i\omega)| \sin\Big(\omega t + \psi + \phi(i\omega)\Big). \end{aligned}$$

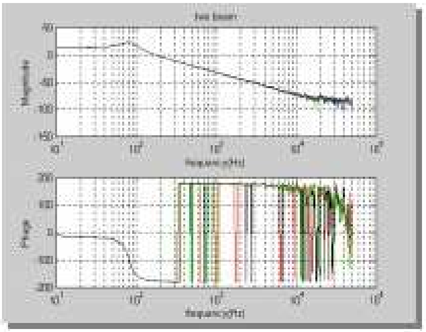
Frequency response

Microphone



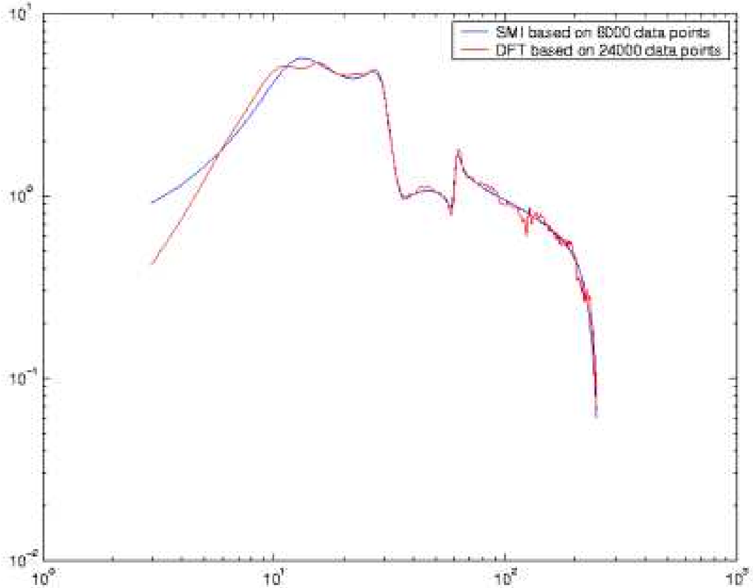
Hard disk/CD/DVD

pickup cartridge



Frequency response

Seat test rig



How about: $H(j\omega) = \dfrac{2j\omega + 3}{-\omega^2 + 3j\omega + 2}$?