Formula sheet EE2S21

Efforts and flows

	Effort	Flow	Generalized position	Generalized momentum
	e	f	q	p
Electric	voltage u [V]	current i [A]	charge q [C]	flux ϕ [Vs]
Translation	force F [N]	velocity v [m/s]	displacement x [m]	moment p [Ns]
Rotation	torque τ [Nm]	angular velocity ω [rad/s]	angular displ. θ [rad]	rotational mom. m [Nms]
Hydraulic	pressure p [N/m^2]	volume flow Q [m^3/s]	volume V [m^3]	pressure mom. Γ [Ns/m^2]
Thermo-dynamic	temperature T [K]	entropy flow f_T [WK^{-1}]	entropy S [J/K]	-

Bond graphs

Ideal transformer: $e_2 = ne_1$, $f_2 = \frac{1}{n} f_1$ Ideal gyrator: $e_2 = rf_1$, $f_2 = \frac{1}{r} e_1$

$$\xrightharpoondown[f_1]{e_1} \underset{n}{\overset{}{\mathrm{TF}}} \xrightharpoondown[f_2]{e_2} \qquad \xrightharpoondown[f_1]{e_1} \underset{r}{\mathrm{GY}} \xrightharpoondown[f_2]{e_2}$$

System response

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

Linearization in operation point (x^*, u^*)

For $x = x^* + z, u = u^* + v$: $f(x) \approx f(x^*) + \dfrac{df}{dx}(x^*)z$

$$g(x,u) \approx g(x^*,u^*) + \frac{\partial g}{\partial x}(x^*,u^*)z + \frac{\partial g}{\partial u}(x^*,u^*)v$$

Laplace transforms

Impulse response $H(s)$: $Y(s) = H(s)U(s)$ $H(s) = C(sI-A)^{-1}B + D$

Properties: $\mathscr{L}\left(\dfrac{df}{dt}(t)\right) = sF(s) - f(0^+)$ $\lim_{t\to+\infty} f(t) = \lim_{s\to 0^+} sF(s)$

Table with Laplace transforms

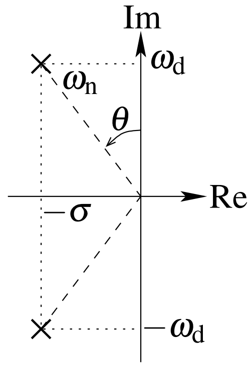
$f(t)$ $(t \geqslant 0)$	$F(s)$	$f(t)$ $(t \geqslant 0)$	$F(s)$	$f(t)$ $(t \geqslant 0)$	$F(s)$
$\delta(t)$	1	e^{-at}	$\dfrac{1}{s+a}$	$\sin bt$	$\dfrac{b}{s^2+b^2}$
$1(t)$	$\dfrac{1}{s}$	te^{-at}	$\dfrac{1}{(s+a)^2}$	$\cos bt$	$\dfrac{s}{s^2+b^2}$
t	$\dfrac{1}{s^2}$	t^2e^{-at}	$\dfrac{2!}{(s+a)^3}$	$e^{-at}\sin bt$	$\dfrac{b}{(s+a)^2+b^2}$
t^2	$\dfrac{2!}{s^3}$	$t^m e^{-at}$	$\dfrac{m!}{(s+a)^{m+1}}$	$e^{-at}\cos bt$	$\dfrac{s+a}{(s+a)^2+b^2}$
t^m	$\dfrac{m!}{s^{m+1}}$				

Partial fraction expansion: $$\frac{1}{(s+a)^n} \quad \rightarrow \quad \frac{A_1}{s+a}+\frac{A_2}{(s+a)^2}+\cdots+\frac{A_n}{(s+a)^n}$$

$$\frac{1}{(s^2+bs+c)^m} \quad \rightarrow \quad \frac{B_1 s+C_1}{s^2+bs+c}+\frac{B_2 s+C_2}{(s^2+bs+c)^2}+\cdots+\frac{B_m s+C_m}{(s^2+bs+c)^m}$$

if $b^2-4c<0$ (i.e., no real zeros for s^2+bs+c)

1st and 2nd-order systems



Standard form: $$H_1(s)=\frac{K}{\tau s+1}$$

$$H_2(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{K\omega_n^2}{(s+\sigma)^2+\omega_d^2}$$

with $\sigma=\zeta\omega_n$, $\omega_d=\omega_n\sqrt{1-\zeta^2}$ and $\sin\theta=\zeta$. Poles: $-\sigma \pm j\omega_d$

Performance criteria

Rise time (10→90%): $t_r \approx \dfrac{1.8}{\omega_n}$

Peak time: $t_p=\dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

Settling time (±1%): $t_s=\dfrac{4.6}{\zeta\omega_n}=\dfrac{4.6}{\sigma}$

Overshoot: $M_p=\exp\left(\dfrac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$

Stability

Continuous-time system is stable if all poles have a strictly negative real part.

Routh table for $s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n$:

$$\begin{array}{rcllll}
s^n & : & 1 & a_2 & a_4 & \ldots \\
s^{n-1} & : & a_1 & a_3 & a_5 & \ldots \\
s^{n-2} & : & b_1 & b_2 & b_3 & \ldots \\
s^{n-3} & : & c_1 & c_2 & c_3 & \ldots \\
\vdots & & & & & \\
s^0 & : & * & & &
\end{array}
\qquad \text{with } b_1 = -\frac{1}{a_1}\begin{vmatrix} 1 & a_2 \\ a_1 & a_3 \end{vmatrix},\ b_2 = -\frac{1}{a_1}\begin{vmatrix} 1 & a_4 \\ a_1 & a_5 \end{vmatrix}, \ldots$$

$$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix},\ c_2 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix}, \ldots$$

$\rightarrow$ system is stable if <u>all</u> number in the 1st column with numbers are positive.

PID controllers

$$D(s) = K\left(1 + \frac{1}{T_\mathrm{i} s} + T_\mathrm{d} s\right)$$

Ziegler-Nichols 1 (*quarter decay*):	Ziegler-Nichols 2 (*ultimate gain*):
static gain K_s, delay τ_d, time constant τ	ultimate gain K_u, period P_u
P: $K = \frac{1}{K_\mathrm{s}} \cdot \frac{\tau}{\tau_\mathrm{d}}$	P: $K = 0.5 K_\mathrm{u}$
PI: $K = \frac{0.9}{K_\mathrm{s}} \cdot \frac{\tau}{\tau_\mathrm{d}}$ $T_\mathrm{i} = \frac{\tau_\mathrm{d}}{0.3}$	PI: $K = 0.45 K_\mathrm{u}$ $T_\mathrm{i} = \frac{1}{1.2} P_\mathrm{u}$
PID: $K = \frac{1.2}{K_\mathrm{s}} \cdot \frac{\tau}{\tau_\mathrm{d}}$ $T_\mathrm{i} = 2\tau_\mathrm{d}$ $T_\mathrm{d} = 0.5\tau_\mathrm{d}$	PID: $K = 0.6 K_\mathrm{u}$ $T_\mathrm{i} = \frac{1}{2} P_\mathrm{u}$ $T_\mathrm{d} = \frac{1}{8} P_\mathrm{u}$

Root locus

$1 + KL(s) = 0$ with $K \geqslant 0$ with loop gain $L(s) = \dfrac{\prod_{i=1}^{m}(s - z_i)}{\prod_{j=1}^{n}(s - p_j)}$; branches start in poles & end in zeros or asymptotically go to ∞ at angles $\dfrac{180^\circ + 360^\circ(k-1)}{n-m}$ for $k = 1, \ldots, n-m$ and radiating out from point $\alpha = \dfrac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n-m}$ on real axis

Frequency domain

$G(j\omega) = G_{re} + jG_{im}$:

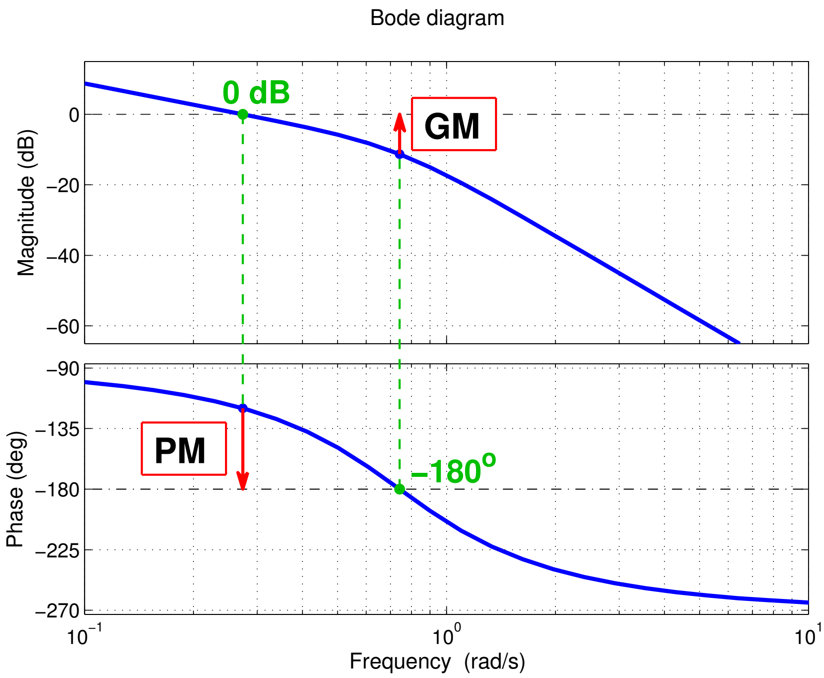
magnitude: $A = \sqrt{G_{re}^2 + G_{im}^2}$

phase: $\phi = \arctan \frac{G_{im}}{G_{re}}$

dB $\leftrightarrow 20 \log_{10} A$

Phase margin (PM):
(phase at 0 dB) + 180°

Gain margin (GM):
−(magnitude in dB at −180°)
or 1 / (magnitude at −180°)



Nyquist stability criterion

$$\text{\# encirclements of } L(s) \text{ around -1 } (N) = \text{\# poles of } \frac{L(s)}{1+L(s)} \text{ in right half-plane } (Z)$$
$$- \text{\# poles of } L(s) \text{ in right half-plane } (P)$$

with $L(s)$ the loop gain. For encirclements a clockwise encirclement adds 1 and a counter-clockwise encirclement subtracts 1.

Trigonometric functions

$$\tan(30^\circ) = \frac{\sqrt{3}}{3} \qquad \tan(45^\circ) = 1 \qquad \tan(60^\circ) = \sqrt{3}$$