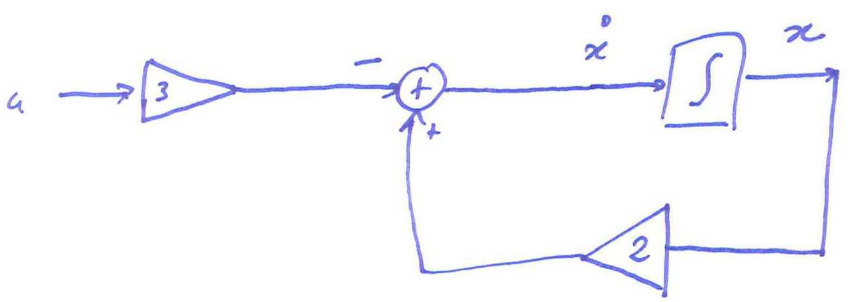
(1)

2018-02-27

$\dot{x}(t) = 2\,x(t) - 3\,u(t)$



$u_1 \to y_1$

$u_2 \to y_2$

$\left.\begin{array}{l} P_1:\ u_1 + u_2 \to y_1 + y_2 \\ P_2:\ \alpha u_1 \to \alpha y_1 \end{array}\right\} \to$ linear

$\alpha u_1 + \beta u_2 \to \alpha y_1 + \beta y_2 \to$ linear

$P_2:\ \alpha u_1 \to \alpha y_1$

$P_2:\ \beta u_2 \to \beta y_2$

$+ P_1$

$\alpha u_1 + \beta u_2 \to \alpha y_1 + \beta y_2$

2

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

slide 19 $\quad x_1 = q \quad x_2 = \dot{q} \qquad \ddot{q} + \dot{q}^3 + q = u \qquad y = q^2$

1. $\dot{x}_1 = \dot{q} = x_2$

$\dot{x}_2 = \ddot{q} = -\dot{q}^3 - q + u = -x_2^3 - x_1 + u$

$y = q^2 = x_1^2$

$\dot{x} = f(x,u) \quad f = \begin{pmatrix} x_2 \\ -x_1 - x_2^3 + u \end{pmatrix}$

$y = h(x,u) \quad h = x_1^2$

2/3. non linear + time-invariant

4. $u^* = 1, \quad \dot{q}^* = 0$ (equilibrium)

$\Downarrow$

$\ddot{q}^* = 0 \quad \Rightarrow \quad q^* = u^* = 1, \quad y^* = 1$

5. $(q^*, \dot{q}^*) = (1, 0) = (x_1^*, x_2^*)$

$z = x - x^*$

$v = u - u^*$

$w = y - y^*$

$\frac{\partial f}{\partial x}\Big|_{x^*, u^*}$

$A = \begin{vmatrix} 0 & 1 \\ -1 & -3x_2^2 \end{vmatrix}_{(1,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$B = \frac{\partial f}{\partial u}\Big|_{x^*, u^*} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$C = \frac{\partial h}{\partial x}\Big|_{x^*, u^*} = [2x_1 \quad 0]\big|_{(1,0)} = [2 \; 0]$

$D = \frac{\partial h}{\partial u}\Big|_{x^*, u^*} = 0$

<u>slide 19</u>

$$\dot{z} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

$$w = \begin{bmatrix} 2 & 0 \end{bmatrix} z + 0\, v$$

slide 20

(5)

$\dot{x} = f(x, u)$ $\qquad f = \begin{pmatrix} -x_2^2 u_1 \\ -x_1 + u_2 \\ 1 - x_2 \end{pmatrix}$ $\qquad z = x - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$v = u - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$w = y - 8$

$y = h(x, u)$ $\qquad h = x_3^2$

equilibria :

$\dot{x}_1 = 0 = -x_2^2 u_1 \quad \Rightarrow \quad u_1^* = 0$

$\dot{x}_2 = 0 = -x_1 + u_2 \quad \Rightarrow \quad x_1^* = u_2^*$

$\dot{x}_3 = 0 = 1 - x_2 \quad \Rightarrow \quad x_2^* = 1 \qquad y^* = 8$

e.g. $x^* = (1, 1, 2)$ $\quad u^* = (0, 1)$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x^*, u^*} = \begin{bmatrix} 0 & -2x_2 u_1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}_{x^*, u^*} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{x^*, u^*} = \begin{bmatrix} -x_2^2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}_{x^*, u^*} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C = \left.\frac{\partial h}{\partial x}\right|_{x^*, u^*} = \begin{bmatrix} 0 & 0 & 2x_3 \end{bmatrix}_{x^*, u^*} = \begin{bmatrix} 0 & 0 & 4 \end{bmatrix}$$

$D = 0$

$r_0 = 0$ $\qquad \dot{x} = Ax \qquad x(0)$

$$x(t) = e^{At} \cdot x(0)$$

$A \rightarrow$ V eigenvectors, linearly independent

$$\Lambda = \mathrm{diag}(\lambda_1 \dots \lambda_n)$$

$$AV = V \cdot \Lambda$$

$$A = V \cdot \Lambda \cdot V^{-1}$$

$$e^{At} = V e^{\Lambda t} V^{-1}$$

$$\hookrightarrow \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$$

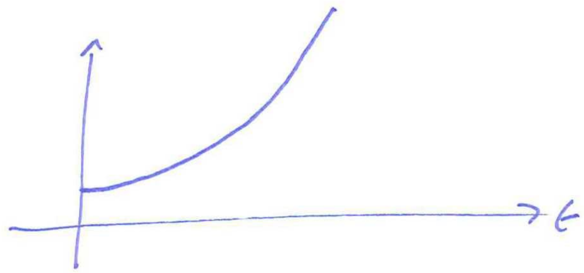
$$x_i(t) = \sum_{j=1}^{n} \alpha_{ij} e^{\lambda_j t}$$

$\hookrightarrow$ stability : eigenvalues of A

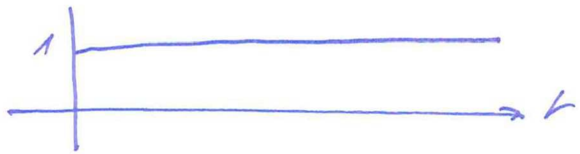
$e^{\alpha+\beta j} = e^{\alpha}(\cos\beta + j\sin\beta)$

$e^{\lambda t}$

$\lambda > 0$

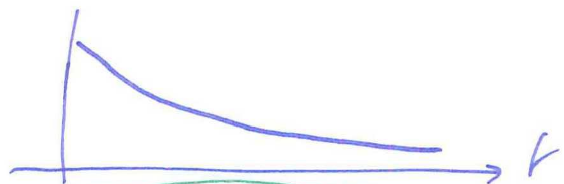


$\lambda = 0$



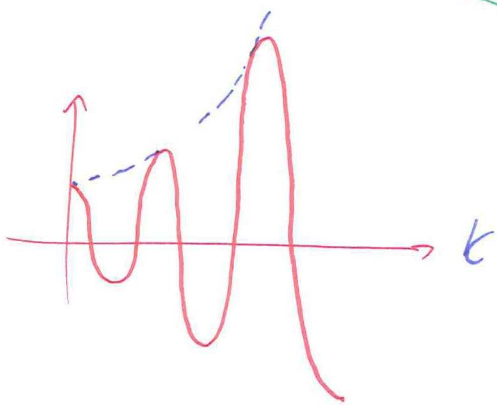
stable

$\lambda \leq 0$

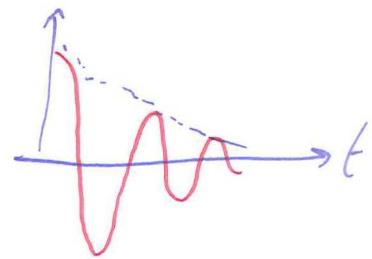


$e^{(\alpha \pm \beta j)t}$

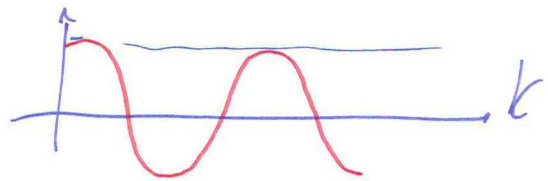
$\alpha > 0$



$\alpha < 0$



$\alpha = 0$



slide 8

8

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{vmatrix} = \lambda^2(\lambda+6) + 6 + 11\lambda = \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

roots : try divisors of 6

+ clearly λ should be negative to have $\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$

$\lambda = -1$: $-1 + 6 - 11 + 6 \stackrel{?}{=} 0$!

$\downarrow$

$\lambda - (-1)$ is a divisor

$= \lambda + 1$

$$\begin{array}{l|l} \lambda^3 + 6\lambda^2 + 11\lambda + 6 & \lambda+1 \\ \underline{\lambda^3 + \lambda^2} & \lambda^2 + 5\lambda + 6 \\ \quad 5\lambda^2 + 11\lambda & \\ \quad \underline{5\lambda^2 + 5\lambda} & \\ \qquad 6\lambda + 6 & \end{array}$$

$\ldots = (\lambda+1)(\lambda^2 + 5\lambda + 6)$

$= (\lambda+1)(\lambda+2)(\lambda+3) \Rightarrow$ roots: $-1, \; -2, \; -3$

$\rightarrow$ stable

slide 12 **Routh table**

$$\lambda^3 + 6\lambda^2 + 11\lambda + 6$$

λ^3	1	11
λ^2	6	6
λ^1	$\alpha = 11$	$\beta = 0$
λ^0	$\gamma = 6$	

$$\alpha = -\frac{1}{6}\begin{vmatrix} 1 & 11 \\ 6 & 6 \end{vmatrix} = -\frac{1}{6}(6-66) = 11$$

$$\beta = -\frac{1}{6}\begin{vmatrix} 1 & 0 \\ 6 & 0 \end{vmatrix} = 0$$

$$\gamma = -\frac{1}{\alpha}\begin{vmatrix} 6 & 6 \\ \alpha & \beta_{=0} \end{vmatrix} = -\frac{1}{\alpha}(-6\alpha) = 6$$

$\downarrow$

no sign changes

$\Downarrow$

0 eigenvalues in right half plane

$\Downarrow$

stable

Routh

$$\lambda^3 + 6\lambda^2 + 11\lambda + 6$$

$$\begin{array}{c|cc}
\lambda^3 & 1 & 11 \\
\lambda^2 & 6 & 6 \\
\hline
\lambda^1 & \overset{10}{\alpha} & \overset{0}{\beta} \\
\lambda^0 & \overset{6}{\gamma} &
\end{array}$$

sign changes

= # eigenvalues with positive real part.

$$\alpha = -\frac{1}{6}\begin{vmatrix} 1 & 11 \\ 6 & 6 \end{vmatrix} = -\frac{1}{6}(6-66) = 10$$

$$\beta = -\frac{1}{6}\begin{vmatrix} 1 & 0 \\ 6 & 0 \end{vmatrix} = 0$$

$$\gamma = -\frac{1}{\alpha}\begin{vmatrix} 6 & 6 \\ \alpha & \beta \end{vmatrix} = -\frac{1}{\alpha}(-6\alpha) = 6 \qquad \beta = 0$$

$$\lambda^6 + 4\lambda^5 + 3\lambda^4 + 2\lambda^3 + \lambda^2 + 4\lambda + 4$$

λ^6	1	3	1	4
λ^5	4	2	4	
λ^4	α	β	γ	
λ^3	a	b	c	
λ^2				
λ^1				
λ^0				

coefficients.

$$\alpha = -\frac{1}{4}\begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix}$$

$$\beta = -\frac{1}{4}\begin{vmatrix} 1 & 1 \\ 4 & 4 \end{vmatrix}$$

$$\gamma = -\frac{1}{4}\begin{vmatrix} 1 & 4 \\ 4 & 0 \end{vmatrix}$$

$$a = -\frac{1}{\alpha}\begin{vmatrix} 4 & 2 \\ \alpha & \beta \end{vmatrix}$$

$$b = -\frac{1}{\alpha}\begin{vmatrix} 4 & 4 \\ \alpha & \gamma \end{vmatrix}$$

$$c = -\frac{1}{\alpha}\begin{vmatrix} 4 & 0 \\ \alpha & 0 \end{vmatrix}$$