Manifolds

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Rozdziały

1 Topology

TODO: add

2 Tangent Space

2.1 Definition

Let (M, τ) be a C^k differentiable manifold, (U, ϕ) chart on M and $p \in U$. Let $\gamma_1, \gamma_2 : (-1, 1) \to U$ be two curves such that $\gamma_1(0) = \gamma_2(0) = p$ and $D_{\phi\circ\gamma_1}(x), D_{\phi\circ\gamma_2}(x) \in C^k[(-1,1), R^n]$.
Let $_\sim$T be an equivalence relation on the set of curves meeting the above conditions s.t. $\gamma_1 {}_\sim \gamma_2 \iff D_{\phi\circ\gamma_1}(\phi \circ \gamma_1)(0) = D_{\phi\circ\gamma_2}(\phi \circ \gamma_2)(0)$.
Finally, a tangent space T_pM is defined as a set of equivalence classes of curves meeting the above conditions.

$$[\gamma]_\sim = \{\gamma' : (-1,1) \to U \text{ s.t. } \gamma {}_\sim \gamma'\} \tag{1}$$
$$T_pM = \{[\gamma]_\sim : (-1,1) \to U, \phi \circ \gamma \in C^k[(-1,1), \mathbb{R}^n], \gamma(0) = p\} \tag{2}$$

Since $\gamma_1(0) = \gamma_2(0) = p \implies D_{\phi\circ\gamma_1}(0) = D_{\phi\circ\gamma_2}(0) \iff [\gamma_1]_\sim = [\gamma_2]_\sim$, it follows that
SHOW INDEPENDENCE FROM CHART.

2.2 Operations on tangent space

To define operations on the elements of T_pM, if (U, ϕ) is a chart with $p \in U$, one may define a map:

$$h_* : T_pM \to T_{\phi(p)}\mathbb{R}^n = \mathbb{R}^n, \tag{3}$$
$$h_*([\gamma]_\sim) := D_{\phi\circ\gamma}(0). \tag{4}$$
$$\tag{5}$$

Note that $D_{\phi\circ\gamma}(0)$ is a well defined $\phi \circ \gamma : \mathbb{R} \to \mathbb{R}^n$.

Then the operations on T_pM are defined as follows:

$$\text{for } u, v \in T_pM \text{ and } \lambda \in \mathbb{R} \tag{6}$$
$$u + v := h_*^{-1}(h_*(u) + h_*(v)), \tag{7}$$
$$\lambda v := h_*^{-1}(\lambda h_*(v)). \tag{8}$$
$$\tag{9}$$

2.2.1 Bijectivity

By the definition of a chart, it has to be a homeomorphism (continous, bijective) map. Thus T_pM is a vector space isomorphic to $\mathbb{R}^n$.

2.2.2 Basis

If $B = \{e_1, e_2, ..e_n\}$ is a basis of $\mathbb{R}^n$, then $B_{T_pM} = \{h_*^{-1}(e_1), h_*^{-1}(e_2), ..h_*^{-1}(e_n)\}$ is a basis of T_pM. Basis is often denoted by the following notation:

$$\frac{\partial}{\partial x^i}_\sim := h_*^{-1}(e_i) \tag{10}$$
$$\frac{\partial}{\partial x^i}_\sim \in T_pM \tag{11}$$

2.3 Differential

Let $(M_1, \tau_1)(M_2, \tau_2)$, be C^k differentiable manifolds, $f : M_1 \to M_2$ be a smooth map and $p \in U \in \tau_1$. We define a differential (or pushforward) as a map between tangent spaces as follows:

$$df : T_pM_1 \to T_{f(p)}M_2 \tag{12}$$

$$df([\gamma]_\sim) := [f \circ \gamma]_\sim \in T_{f(p)}M_2 \tag{13}$$

Note that $[f \circ \gamma]_\sim$ is a equivalence class of all curves $f \circ \gamma : (-1, 1) \to M_2$, with $(f \circ \gamma)(0) = f(p)$ and $(f \circ \gamma_1)_\sim(f \circ \gamma_2) \iff D_{\psi\circ(f\circ\gamma_1)}(0) = D_{\psi\circ(f\circ\gamma_2)}(0)$, for some ψ being a chart of M_2 on neighbourhood of $f(p)$.

2.4 Cotangent Space

Let M be a C^k differentiable manifold, $p \in M$.
If T_pM is a tangent space, then its dual space T_p^*M is called a cotangent space.

2.4.1 Basis of Cotangent Space

If $B_{T_pM} = \{b_1, b_2, ..., b_n\}$ is a basis of tangent space, then basis of its dual space $B^*_{T_p^*M} = \{b_1^*, b_2^*, ..., b_n^*\}$ can be found as follows:

$$b_i^* \in \mathcal{L}(T_pM \to \mathbb{R}), b_j \in B_{T_pM} \tag{14}$$

$$b_i^*(b_j) := \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{15}$$

3 Submersion

Let M, N be manifolds and $f : M \to N$ be a smooth map.
Its pushforward $df : T_pM \to T_{f(p)}N$ is called an immersion if it is a bijective map.

4 Tangent bundle

Let M be a C^k differentiable manifold. We define a Tangent bundle as a set consisting of all tangent spaces defined as: $TM := \bigcup_{p\in M} \{p\} \times T_pM$

4.0.1 Natural projection

Natural projection $\pi : TM \to M$ is defined as: $\pi(p, T_pM) := p$

5 Metric Tensor

Let M be a C^k differentiable manifold and $p \in M$.
A metric tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a map that is:

- Bilinear:

  - $g_p(u, \lambda v) = g_p(\lambda u, v) = \lambda g_p(u, v)$,
  - $g_p(u + w, v) = g_p(u, v) + g_p(w, v)$,
  - $g_p(u, v + w) = g_p(u, v) + g_p(u, w)$.

- Symmetric: $g_p(u, v) = g_p(v, u)$.
- Nondegenerate: $\forall v \in T_pM : v \neq 0 \implies \exists u \in T_pM : g_p(u, v) \neq 0$
- If $g_{u,v} : M \to \mathbb{R},$ with $g_{u,v}(p) := g_p(u, v)$, then
  $g_{u,v}$ is a smooth function.

6 Riemann semi-maniofld

Let M be a C^k smooth manifold and $g_p : T_pM \times T_pM \to \mathbb{R}$ be its metric tensor. We say that a tuple (M, g_p) is called a Riemann semi-maniofld.
g_p is also called a Riemann Metric.

6.1 Riemann norm

For a given metric tensor g_p, Riemann norm is defined as $\|\cdot\| : T_pM \to \mathbb{R},$ with $\|v\| := \sqrt{g_p(v, v)}$

6.2 Curve length

Let $\gamma : (a, b) \subseteq \mathbb{R} \to M$ be a parametrized smooth map.
We define the length of this curve as:
$L(\gamma) := \int_a^b \|[\gamma]_\sim\| dt = \int_a^b \sqrt{g_p([\gamma]_\sim, [\gamma]_\sim)} dt$

7 Exterior Algebra

7.1 Alternating bilinear form

Let V be a vector space over a field F. An alternating (or antisymmetric) bilinear form on V is a bilinear form $B : V \times V \to F$ such that $B(v, w) = -B(w, v)$.

7.2 Second exterior power

Let V be a finite-dimensional vector space over a field F
and $V_B = \{B : V \times V \to F : B$ is alternating bilinear form.$\}$ be a vector space of all alternating bilinear forms. The second exterior power of V, denoted with $\bigwedge^2 V$ is a dual space of V_B. i.e. $\bigwedge^2 V = V_B^*$.
Elements of $\bigwedge^2 V$ are called 2-vectors.

7.3 Exterior product

Let V be a finite-dimensional vector space over a field F and $v, u \in V$ and $\bigwedge^2 V$ be its second exterior power. Exterior product of v and u, is a linear map to F $v \wedge u \in \bigwedge^2 V$
$(v \wedge u)(B) = B(v, u)$.

From this definition, the following properties follow:

$$(u \wedge v)(B) = B(u,v) = -(u \wedge v)(B) = -B(v,u) \tag{16}$$

$$(u \wedge u)(B) = -(u \wedge u)(B) = 0 \tag{17}$$

$$\text{if } \{v_1, v_2, \ldots, v_n\} \text{ is a basis for } V\text{, then } \{v_i \wedge v_j : i,j \in \{1,2,\ldots,n\}, i<j\} \text{ is a basis for } \bigwedge^2 V. \tag{18}$$

Theorem 1. $u, v \in V, u \neq 0 \implies (u \wedge v = 0 \iff \exists_{\lambda \in F} : v = \lambda u)$

Proof. This basically mean that u, v are in the same subspace and this may be shown with the following. Let $v = \lambda u$. Then $u \wedge v = u \wedge (\lambda u) = \lambda(u \wedge u) = 0$.

□

7.4 Alternating multilinear form

Definition 7.1. Let V be a vector space over field F. An alternating multilinear form of degree p is a map: $V \times V \times \cdots \times V \to F$ such that

$$M(u_1, \ldots, u_i, \ldots, u_j, \ldots u_p) = M(u_1, \ldots, u_j, \ldots, u_i, \ldots u_p) \tag{19}$$

$$M(u_1, \ldots, \lambda u_i + w, \ldots, \ldots u_p) = \lambda M(u_1, \ldots, u_i, \ldots, \ldots u_p) + M(u_1, \ldots, w, \ldots, \ldots u_p) \tag{20}$$

The set of all such forms M is a vector space. Let $a_i : \{0, \ldots, p\} \to \{0, \ldots, n\}$ be a strictly increasing sequence, and $\{v_1, \ldots, v_n\}$ be a basis of V. Then a multilinear form M of degree p for any set of vectors in a given basis can be trivially transformed by the properties of M into a form $M(u_1, \ldots u_2) = \lambda_1 M(v_{a_1}, \ldots v_{a_p}) + \lambda_2 M(v_{a_1}, \ldots v_{a_p}) + \ldots$,M($\mathrm{v}_{a_1}, \ldots, v_{a_p}$). Since a_i is strictly increasing, if we choose p elements out of n, there are $\binom{n}{p}$ possibilities to do it. $\binom{n}{p}$ is also a dimension of the vector space of such forms M. In particular, if $p > n$, we just define a dimension to be 0.

Example 7.1. Let

$$A = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}$$

be a matrix in $\mathbb{R}^n$.
Then a map $M(v_1, v_2, \ldots, v_n) = det(A)$ is an alternating multilinear form of degree n.

7.5 Dual Operator

Definition 7.2. Let X, Y be normed vector spaces and $T : X \to Y$ be a linear operator. Dual operator $T^* : Y^* \to X^*$ (note the reversed Y^*, X^*) is defined as $T^*(f) = f \circ T$.

7.6 p-th exterior power

Definition 7.3. Let V be a finite-dimensional vector space over field F,
and $V_{AM} = \{B : V \times \cdots \times V \to F : B \text{ is alternating multilinear form.}\}$ be a vector space of all alternating multilinear forms over V. p-th exterior power of V, denoted with $\bigwedge^p V$ is a dual space of V_{AM}. i.e. $\bigwedge^p V = V_{AM}^*$. Elements of $\bigwedge^p V$ are called p-vectors.

Definition 7.4. Given $u_1, u_2 \ldots, u_n \in V$, the exterior product is a linear map
$u_1 \wedge u_2 \wedge \cdots \wedge u_n : V_{AM} \to F$ such that $(u_1 \wedge u_2 \wedge \cdots \wedge u_n)(M) = M(u_1, u_2, \ldots u_n)$.

Theorem 2. $u_1 \wedge u_2 \wedge \cdots \wedge u_n = 0 \iff$
$(\exists \lambda_1, \lambda_2, \ldots \lambda_n \in F : \exists i \in \{1, \ldots, n\} : \lambda_i \neq 0 : \lambda_1 u_1 + \lambda_2 u_2 \cdots + \lambda_n u_n = 0$ *i.e. they are linearly dependent*$)$

Proof. Without a loss of generality assume that $u_n = \lambda_1 u_1 + \lambda_2 u_2 \dots$
Then due to definition based on alternating multilinear forms, it follows that
$M(u_1, u_2, \dots, u_n) = M(u_1, u_2, \dots, \lambda_1 u_1 + \lambda_2 u_2 \dots) =$
$\lambda_1 M(u_1, u_2, \dots, u_1) + \lambda_2 M(u_1, u_2, \dots, u_2) + \dots =$
(by the antilinear property, swapping for each element a pair)(u_i, u_{n-i}),
$= -\lambda_1 M(u_1, u_2, \dots, u_1) - \lambda_2 M(u_1, u_2, \dots, u_2) - \dots$
Finishing with $a = -a \iff a = 0$ □

Exterior powers have natural properties w.r.t. the linear transformations. Given a linear operator $T : V \to W$ and a form $M : W \times W \times \dots \times W \to F_W$, we may induce an operator
$T^* M : V \times V \dots \times V \to F_W$ as $(T^* M)(v_1, v_2, \dots, v_n) := M(Tv_1, Tv_2, \dots Tv_n)$.

This defines a dual map $\bigwedge^p T : \bigwedge^p V \to \bigwedge^p W$, $(\bigwedge^p T)(v_1 \wedge v_2 \dots) := (Tv_1) \wedge (Tv_2) \dots$

Example 7.2. One of such maps is very familiar and used frequently. Take $p = n$ for some n-dimensional vector space V. Then a vector space $\bigwedge^p V$ is 1 dimensional as $\binom{n}{p} = \binom{n}{n} = 1$ and the dimension of a dual space of all the alternating linear forms, V^*_{AL} is equal to the dimension of the vector space V_{AL} itself. In fact, this map is a determinant itself.
Observe that $\bigwedge^n (v_1 \wedge \dots \wedge v_n) = Tv_1 \wedge \dots \wedge Tv_n$, ... TODO

8 1-forms

Definition 8.1. Let M_1, M_2 be two manifolds.