Manifolds

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1 Topology

TODO: add

2 Tangent Space

2.1 Definition

Let (M, τ) be a C^k differentiable manifold, (U, ϕ) chart on M and $p \in U$. Let $\gamma_1, \gamma_2 : (-1, 1) \to U$ be two curves such that $\gamma_1(0) = \gamma_2(0) = p$ and $D_{\phi\circ\gamma_1}(x), D_{\phi\circ\gamma_2}(x) \in C^k[(-1, 1), R^n]$.
Let $\sim$T be an equivalence relation on the set of curves meeting the above conditions s.t. $\gamma_1 \sim \gamma_2 \iff D_{\phi\circ\gamma_1}(\phi \circ \gamma_1)(0) = D_{\phi\circ\gamma_2}(\phi \circ \gamma_2)(0)$.
Finally, a tangent space T_pM is defined as a set of equivalence classes of curves meeting the above conditions.

$$[\gamma]_\sim = \{\gamma' : (-1, 1) \to U \text{ s.t. } \gamma \sim \gamma'\} \tag{1}$$
$$T_pM = \{[\gamma]_\sim : (-1, 1) \to U, \phi \circ \gamma \in C^k[(-1, 1), \mathbb{R}^n], \gamma(0) = p\} \tag{2}$$

Since $\gamma_1(0) = \gamma_2(0) = p \implies D_{\phi\circ\gamma_1}(0) = D_{\phi\circ\gamma_2}(0) \iff [\gamma_1]_\sim = [\gamma_2]_\sim$, it follows that
SHOW INDEPENDENCE FROM CHART.

2.2 Operations on tangent space

To define operations on the elements of T_pM, if (U, ϕ) is a chart with $p \in U$, one may define a map:

$$h_* : T_pM \to T_{\phi(p)}\mathbb{R}^n = \mathbb{R}^n, \tag{3}$$
$$h_*([\gamma]_\sim) := D_{\phi\circ\gamma}(0). \tag{4}$$
$$\tag{5}$$

Note that $D_{\phi\circ\gamma}(0)$ is a well defined $\phi \circ \gamma : \mathbb{R} \to \mathbb{R}^n$.

Then the operations on T_pM are defined as follows:

$$\text{for } u, v \in T_pM \text{ and } \lambda \in \mathbb{R} \tag{6}$$
$$u + v := h_*^{-1}(h_*(u) + h_*(v)), \tag{7}$$
$$\lambda v := h_*^{-1}(\lambda h_*(v)). \tag{8}$$
$$\tag{9}$$

2.2.1 Bijectivity

By the definition of a chart, it has to be a homeomorphism (continous, bijective) map. Thus T_pM is a vector space isomorphic to $\mathbb{R}^n$.

2.2.2 Basis

If $B = \{e_1, e_2, ..e_n\}$ is a basis of $\mathbb{R}^n$, then $B_{T_pM} = \{h_*^{-1}(e_1), h_*^{-1}(e_2), ..h_*^{-1}(e_n)\}$ is a basis of T_pM. Basis is often denoted by the following notation:

$$\frac{\partial}{\partial x^i}_\sim := h_*^{-1}(e_i) \tag{10}$$

$$\frac{\partial}{\partial x^i}_\sim \in T_pM \tag{11}$$

2.3 Cotangent Space

Let M be a C^k differentiable manifold, $p \in M$.
If T_pM is a tangent space, then its dual space T_p^*M is called a cotangent space.

2.3.1 Basis of Cotangent Space

If $B_{T_pM} = \{b_1, b_2, ..., b_n\}$ is a basis of tangent space, then basis of its dual space $B^*_{T_p^*M} = \{b_1^*, b_2^*, ..., b_n^*\}$ can be found as follows:

$$b_i^* \in \mathcal{L}(T_pM \to \mathbb{R}), b_j \in B_{T_pM} \tag{12}$$

$$b_i^*(b_j) := \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{13}$$

Consider now a basis of a tangent space $\frac{\partial}{\partial x^i}_\sim := h_*^{-1}(e_i)$.
Its dual basis is given by $dx_i : T_xM \to \mathbb{R}$, $dx_i(\frac{\partial}{\partial x^j}) := \delta ij$.

2.4 Directional Derivative

Let M be a C^k differentiable manifold, $f : M \to \mathbb{R}$ be a smooth map, $p \in M$ and $v \in T_pM$.
We define a directional derivative as a map:

$$D_vf : T_pM \to \mathbb{R} \tag{14}$$

$$D_vf(w) := w(f) = D_{f(\gamma(t))}(t = 0) \text{ where } \gamma : (-1, 1) \to M \text{ s.t. } \gamma(0) = p, \gamma_\sim = v \tag{15}$$

2.5 Tangent bundle

Let M be a C^k differentiable manifold. We define a Tangent bundle as a set consisting of all tangent spaces defined as: $TM := \bigcup\limits_{p \in M} \{p\} \times T_pM$

2.6 Differential

Let $(M_1, \tau_1)(M_2, \tau_2)$, be C^k differentiable manifolds, $f : M_1 \to M_2$ be a smooth map and $p \in U \in \tau_1$.
We define a differential (or pushforward) as a map between tangent spaces as follows:

$$df : T_pM_1 \to T_{f(p)}M_2 \tag{16}$$

$$df([\gamma]_\sim) := [f \circ \gamma]_\sim \in T_{f(p)}M_2 \tag{17}$$

Note that $[f \circ \gamma]_\sim$ is a equivalence class of all curves $f \circ \gamma : (-1,1) \to M_2$, with $(f \circ \gamma)(0) = f(p)$ and $(f \circ \gamma_1)_\sim(f \circ \gamma_2) \iff D_{\psi\circ(f\circ\gamma_1)}(0) = D_{\psi\circ(f\circ\gamma_2)}(0)$, for some ψ being a chart of M_2 on neighbourhood of $f(p)$.

Given that $\{\frac{\partial}{\partial x^i}\}$ is a basis for T_pM_1 with each term corresponding to a smooth curve $\gamma_i : (-1,1) \to M$, $\gamma_i(0) = p$, i.e $\{\frac{\partial}{\partial x^i} = [\gamma_i]_\sim\}$, $\{\frac{\partial f}{\partial x^i} = [f \circ \gamma_i]_\sim\}$ is a basis for $T_{f(p)}M_2$, and is a dual basis of cotangent space $\{dx_i\}$, with $dx_i(\frac{\partial}{\partial x_j}) = \delta ij$, then

$$f(x_1, x_2, \dots x_n) = (f_1, f_2, \dots f_m) \tag{18}$$

$$df : T_pM_1 \to T_{f(p)}M_2 \tag{19}$$

$$u := \sum_i^n \lambda_i \frac{\partial}{\partial x^i}, \text{ then a differential is defined as} \tag{20}$$

$$df_p(u) = \sum_i^n \frac{\partial f}{\partial x^i}(p) dx_i(u) \tag{21}$$

$$df_p(u) = \sum_i^n \frac{\partial f}{\partial x^i}(p) dx_i \left(\sum_j^n \lambda_j \frac{\partial}{\partial x^j} \right) \tag{22}$$

$$df_p(u) = \sum_i^n \frac{\partial f}{\partial x^i}(p) dx_i \left(\lambda_i \frac{\partial}{\partial x^i} \right) \tag{23}$$

$$df_p(u) = \sum_i^n \lambda_i \frac{\partial f}{\partial x^i}(p) \tag{24}$$

$$\tag{25}$$

3 Submersion

Let M, N be manifolds and $f : M \to N$ be a smooth map.
Its pushforward $df : T_pM \to T_{f(p)}N$ is called an immersion if it is a bijective map.

3.0.1 Natural projection

Natural projection $\pi : TM \to M$ is defined as: $\pi(p, T_pM) := p$

4 Metric Tensor

Let M be a C^k differentiable manifold and $p \in M$.
A metric tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a map that is:

- Bilinear:
  - $g_p(u, \lambda v) = g_p(\lambda u, v) = \lambda g_p(u, v)$,
  - $g_p(u + w, v) = g_p(u, v) + g_p(w, v)$,
  - $g_p(u, v + w) = g_p(u, v) + g_p(u, w)$.

- Symmetric: $g_p(u,v) = g_p(v,u)$.
- Nondegenerate: $\forall v \in T_pM : v \neq 0 \implies \exists u \in T_pM : g_p(u,v) \neq 0$
- If $g_{u,v} : M \to \mathbb{R},$ with $g_{u,v}(p) := g_p(u,v)$, then
  $g_{u,v}$ is a smooth function.

5 Riemann semi-maniofld

Let M be a C^k smooth manifold and $g_p : T_pM \times T_pM \to \mathbb{R}$ be its metric tensor. We say that a tuple (M, g_p) is called a Riemann semi-maniofld.
g_p is also called a Riemann Metric.

5.1 Riemann norm

For a given metric tensor g_p, Riemann norm is defined as $\|\cdot\| : T_pM \to \mathbb{R},$ with $\|v\| := \sqrt{g_p(v,v)}$

5.2 Curve length

Let $\gamma : (a,b) \subseteq \mathbb{R} \to M$ be a parametrized smooth map.
We define the length of this curve as:
$L(\gamma) := \int_a^b \|[\gamma]_\sim\| dt = \int_a^b \sqrt{g_p([\gamma]_\sim, [\gamma]_\sim)} dt$

6 Exterior Algebra

6.1 Alternating bilinear form

Let V be a vector space over a field F. An alternating (or antisymmetric) bilinear form on V is a bilinear form $B : V \times V \to F$ such that $B(v,w) = -B(w,v)$.

6.2 Second exterior power

Let V be a finite-dimensional vector space over a field F
and $V_B = \{B : V \times V \to F : B \text{ is alternating bilinear form.}\}$ be a vector space of all alternating bilinear forms. The second exterior power of V, denoted with $\bigwedge^2 V$ is a dual space of V_B. i.e. $\bigwedge^2 V = V_B^*$.
Elements of $\bigwedge^2 V$ are called 2-vectors.

6.3 Exterior product

Let V be a finite-dimensional vector space over a field F and $v, u \in V$ and $\bigwedge^2 V$ be its second exterior power. Exterior product of v and u, is a linear map to F $v \wedge u \in \bigwedge^2 V$
$(v \wedge u)(B) = B(v,u)$.

From this definition, the following properties follow:

$$(u \wedge v)(B) = B(u,v) = -(u \wedge v)(B) = -B(v,u) \tag{26}$$
$$(u \wedge u)(B) = -(u \wedge u)(B) = 0 \tag{27}$$
$$\text{if } \{v_1, v_2, \ldots, v_n\} \text{ is a basis for } V, \text{ then } \{v_i \wedge v_j : i,j \in \{1,2,\ldots,n\}, i < j\} \text{ is a basis for } \bigwedge^2 V. \tag{28}$$

Theorem 1. $u, v \in V, u \neq 0 \implies (u \wedge v = 0 \iff \exists_{\lambda \in F} : v = \lambda u)$

Proof. This basically mean that u, v are in the same subspace and this may be shown with the following. Let $v = \lambda u$. Then $u \wedge v = u \wedge (\lambda u) = \lambda(u \wedge u) = 0$.

□

6.4 Alternating multilinear form

Definition 6.1. Let V be a vector space over field F. An alternating multilinear form of degree p is a map: $V \times V \times \cdots \times V \to F$ such that

$$M(u_1, \ldots, u_i, \ldots, u_j, \ldots u_p) = M(u_1, \ldots, u_j, \ldots, u_i, \ldots u_p) \tag{29}$$

$$M(u_1, \ldots, \lambda u_i + w, \ldots, \ldots u_p) = \lambda M(u_1, \ldots, u_i, \ldots, \ldots u_p) + M(u_1, \ldots, w, \ldots, \ldots u_p) \tag{30}$$

The set of all such forms M is a vector space. Let $a_i : \{0, \ldots, p\} \to \{0, \ldots, n\}$ be a strictly increasing sequence, and $\{v_1, \ldots, v_n\}$ be a basis of V. Then a multilinear form M of degree p for any set of vectors in a given basis can be trivially transformed by the properties of M into a form $M(u_1, \ldots u_2) = \lambda_1 M(v_{a_1}, \ldots v_{a_p}) + \lambda_2 M(v_{a_1}, \ldots v_{a_p}) + \ldots$,M($\mathrm{v}_{a_1}, \ldots, v_{a_p}$). Since a_i is strictly increasing, if we choose p elements out of n, there are $\binom{n}{p}$ possibilities to do it. $\binom{n}{p}$ is also a dimension of the vector space of such forms M. In particular, if $p > n$, we just define a dimension to be 0.

Example 6.1. Let

$$A = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}$$

be a matrix in $\mathbb{R}^n$.
Then a map $M(v_1, v_2, \ldots, v_n) = det(A)$ is an alternating multilinear form of degree n.

6.5 Dual Operator

Definition 6.2. Let X, Y be normed vector spaces and $T : X \to Y$ be a linear operator. Dual operator $T^* : Y^* \to X^*$ (note the reversed Y^*, X^*) is defined as $T^*(f) = f \circ T$.

6.6 p-th exterior power

Definition 6.3. Let V be a finite-dimensional vector space over field F,
and $V_{AM} = \{B : V \times \cdots \times V \to F : B \text{ is alternating multilinear form.}\}$ be a vector space of all alternating multilinear forms over V. p-th exterior power of V, denoted with $\bigwedge^p V$ is a dual space of V_{AM}. i.e. $\bigwedge^p V = V^*_{AM}$. Elements of $\bigwedge^p V$ are called p-vectors.

Definition 6.4. Given $u_1, u_2 \ldots, u_n \in V$, the exterior product is a linear map
$u_1 \wedge u_2 \wedge \cdots \wedge u_n : V_{AM} \to F$ such that $(u_1 \wedge u_2 \wedge \cdots \wedge u_n)(M) = M(u_1, u_2, \ldots u_n)$.

Theorem 2. $u_1 \wedge u_2 \wedge \cdots \wedge u_n = 0 \iff$
$(\exists \lambda_1, \lambda_2, \ldots \lambda_n \in F : \exists i \in \{1, \ldots, n\} : \lambda_i \neq 0 : \lambda_1 u_1 + \lambda_2 u_2 \cdots + \lambda_n u_n = 0$ *i.e. they are linearly dependent*)

Proof. Without a loss of generality assume that $u_n = \lambda_1 u_1 + \lambda_2 u_2 \ldots$
Then due to definition based on alternating multilinear forms, it follows that
$M(u_1, u_2, \ldots, u_n) = M(u_1, u_2, \ldots, \lambda_1 u_1 + \lambda_2 u_2 \ldots) =$
$\lambda_1 M(u_1, u_2, \ldots, u_1) + \lambda_2 M(u_1, u_2, \ldots, u_2) + \cdots =$
(by the antilinear property, swapping for each element a pair)(u_i, u_{n-i}),
$= -\lambda_1 M(u_1, u_2, \ldots, u_1) - \lambda_2 M(u_1, u_2, \ldots, u_2) - \ldots$
Finishing with $a = -a \iff a = 0$ □

Exterior powers have natural properties w.r.t. the linear transformations. Given a linear operator $T : V \to W$ and a form $M : W \times W \times \cdots \times W \to F_W$, we may induce an operator $T^*M : V \times V \cdots \times V \to F_W$ as $(T^*M)(v_1, v_2, \dots, v_n) := M(Tv_1, Tv_2, \dots Tv_n)$.

This defines a dual map $\bigwedge^p T : \bigwedge^p V \to \bigwedge^p W$, $(\bigwedge^p T)(v_1 \wedge v_2 \dots) := (Tv_1) \wedge (Tv_2) \dots$

Example 6.2. One of such maps is very familiar and used frequently. Take $p = n$ for some n-dimensional vector space V. Then a vector space $\bigwedge^p V$ is 1 dimensional as $\binom{n}{p} = \binom{n}{n} = 1$ and the dimension of a dual space of all the alternating linear forms, V_{AL}^* is equal to the dimension of the vector space V_{AL} itself. In fact, this map is a determinant itself.
Observe that $\bigwedge^n(v_1 \wedge \cdots \wedge v_n) = Tv_1 \wedge \cdots \wedge Tv_n$, ... TODO

7 differential forms

7.1 1-forms

Definition 7.1. Let M_1, M_2 be a n-dimensional, C^k smooth manifolds and $x \in M$.
We define a 1-form mapping $\omega_x : T_xM \to \mathbb{R}$, where $\omega_x(u)$ is a linear form.
Vector space of such 1-forms is isomorphic to a vector space $\bigwedge^1(T_xM)$.

Example 7.1. Consider a tangent space T_xM with a basis $\frac{\partial}{\partial x^i}$ and its dual basis $dx_i(\frac{\partial}{\partial x^j}) := \delta ij$. Let $\omega : T_xM \to \mathbb{R}$, $\omega(u) = f(u)dx_1(u)$. Given that $u = \sum_n^{i=1} \lambda_i \frac{\partial}{\partial x^i}$, it follows that

$$\omega(u) = f(u)dx_1\left(\sum_n^{i=1} \lambda_i \frac{\partial}{\partial x^i}\right). \text{ Then it follows, that from linearity of } w, dx_1 \tag{31}$$

$$\omega(u) = f(u)\sum_n^{i=1}\left(\lambda_i dx_1\left(\frac{\partial}{\partial x^i}\right)\right) \tag{32}$$

$$\omega(u) = f(u)\sum_n^{i=1}\lambda_i\delta_{i,1}, \text{ and finally} \tag{33}$$

$$\omega(u) = f(u)\lambda_1 \tag{34}$$

7.2 Basis of a vector space of 1-forms, $\bigwedge^1(T_xM)$

7.3 k-form

Definition 7.2. Given a vector $\omega \in \bigwedge^n(TxM)$, with $\omega = \sum \frac{\partial f}{\partial x^i}$

8 Integration of differential forms

8.1 Integration of 1-forms

Definition 8.1.

9 Tensor Space

Definition 9.1. Let $\{V_i\}$ be a set of vector spaces. The map $\tau : \prod V_i \to F$, where F is some common field for all the vector spaces, is called k-linear if its linear in each component. Set of all such maps form a vector space.

Definition 9.2. Let now $V_i = V$. We define $T^k(V) := \{\tau : \prod^k V \to F,\, \tau - \text{multilinear}\}$

Let us consider $T^1(V)$. $\tau \in T^1(V)$ takes 1 copy of V. $\tau : V \to F$. Since τ is linear in its only argument, the vector space of all such functions is simply a dual space of V. We may understand $T^k(V)$ as a generalization of dual space to product spaces.

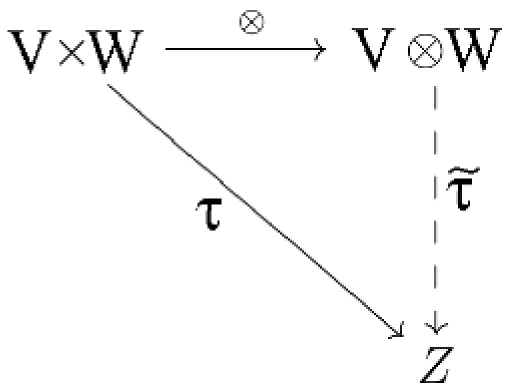
Theorem 3. *Let $T \in L(V, W)$ (L - set of all linear op. from V to W), then $\exists T^* : \tau^k(W) \to \tau^k(V)$*

Definition 9.3. Let $\tau \in T^k(V), \sigma \in T^l(V)$. We define their tensor product as follows: $\tau \otimes \sigma \in T^{k+l}$, $\tau \otimes \sigma(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+l}) = \tau(v_1, \ldots, v_k)\sigma(v_{k+1}, \ldots, v_{k+l})$

This definition yields 2 facts:
$\tau \otimes \sigma \neq \sigma \otimes \tau$ - non commutative.
$(\tau \otimes \sigma) \otimes \omega = \tau \otimes (\sigma \otimes \omega)$ - associative

9.1 Tensor Product

Let V, W be two vector spaces. We define a tensor product as a vector space A with a bilinear map $f : V \times W \to A$ with so called universal property. i.e. given a vector space Z and a bilinear map $\tau : V \times W \to Z$, $\exists! \bar{\tau} : A \to Z : \bar{\tau} \circ f = \tau$. Usually in this context A and f are denoted by $V \otimes W := A$ and $\otimes := f$.



Grafika 1: Universal property

Theorem 4. *Tensor product between 2 vector spaces always exists.*

Proof. TODO □

9.2 Tensor basis

Given 2 vector spaces V, W and their bases B_V, B_W, they induce a basis for $V \otimes W$ as follows: $B_V \otimes B_W = \{v_i \otimes w_j, \text{ where } \mathrm{v}_i \in B_V, w_j \in B_W\},$. Thus it has $\dim(V)\dim(W)$ elements.
i.e. $V \otimes W \cong \mathbb{R}^{\dim(V)\dim(W)}$.

9.2.1 Simple Tensors

Now given these bases and a scalar $a \in F$, we can define a simple tensor as $a(v_i \otimes w_j)$.
Given any tensor $T \in V \otimes W$, we can write it as a linear combination of simple tensors.

$T = \sum_{i,j} a_{ij} v_i \otimes w_j$ for some $a_{ij} \in F$. Following properties of tensor product follow:

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$$
$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$$
$$(av) \otimes w = v \otimes (aw) = a(v \otimes w)$$

9.3 Tensor type (n, k)

Definition 9.4. Let V be a vector space. We define a tensor type (n, k) as a tensor product of n copies of V and k copies of V^*. $T^n_k(V) := V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$.

9.3.1 Tensor product between type (n, k) and (m, l)

Earlier we defined how to multiply 2 tensors $\tau \in T^k(V) = T^k_0(V), \sigma \in T^l(V) = T^l_0(V)$, which correspond to the tensor types $(k, 0)$ and $(l, 0)$ respectively. i.e.

$$\tau \otimes \sigma \in T^{k+l}, \tag{35}$$
$$\tau \otimes \sigma(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+l}) = \tau(v_1, \ldots, v_k)\sigma(v_{k+1}, \ldots, v_{k+l}) \tag{36}$$

We can generalize this to any tensors of types (n, k), (m, l) as follows:

$$\tau \in T^n_k(V), \sigma \in T^m_l(V) \tag{37}$$
$$\tau \otimes \sigma \in T^{n+m}_{k+l}(V) \tag{38}$$
$$(\tau \otimes \sigma)^{i_1 \cdots i_{n+m}}_{j_1 \cdots j_{k+l}} = \tau^{i_1 \cdots i_n}_{j_1 \cdots j_k} \sigma^{i_{n+1} \cdots i_{n+m}}_{j_{k+1} \cdots j_{k+l}} \tag{39}$$

Example 9.1. T^0_0 is a scalar,
T^1_0 is a vector,
T^0_1 is a covector, linear functionals, 1-forms
T^1_1 is a linear map,

Example 9.2. An inner produce, a 2-form (or bilinear form) is a tensor of type $(0, 2)$.
A bivector is a tensor of type $(2, 0)$.
If dim(V) = 3, then a cross product is an example of a (1, 2) tensor.

10 Musical isomorphisms

Let TM be a tangent bundle, T^*M be a cotangent bundle and $\langle ., . \rangle : TM \times TM$ be an inner product. We define a musical isomorphisms $\flat : TM \to T^*M$ s.t. $\forall x, y \in TM : x^\flat(y) = \langle x, y \rangle$
Similarly its inverse is expressed by the operator $\sharp : T^*M \to TM$ s.t. $\forall \omega \in T^*M, \forall y \in TM : x(y) = \langle x^\sharp, y \rangle$