Fourier Analysis

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Chapter Contents

1 Cauchy Riemann

Let $f : \Omega \to \mathbb{C}$ be a complex-valued function of a complex variable with $f(z) = u(z) + iv(z)$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ is a function defined as $g(x, y) = (u'(x, y), v'(x, y))^T$, with $u(z) = u'(x, y), v(z) = v'(x, y)$. Jacobian matrix of g is given by

$$J_g = \begin{bmatrix} \frac{\partial u'}{\partial x} & \frac{\partial u'}{\partial y} \\ \frac{\partial v'}{\partial x} & \frac{\partial v'}{\partial y}. \end{bmatrix} \tag{1}$$

The property we would like to achieve is that we somehow treat an $\mathbb{R}^2$ vector as a scalar in $\mathbb{C}$, so that linear transformation given by J_g is orthogonal (with appropriate scalar). Multiplication of 2 complex numbers may only scale/rotate coordinates given by the $(Re(z), Im(z))$ and so we want our matrix multiplication to follow that same property. Finally what we get is:

$$J_g = \begin{bmatrix} \frac{\partial u'}{\partial x} & \frac{\partial u'}{\partial y} \\ \frac{\partial v'}{\partial x} & \frac{\partial v'}{\partial y}. \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a. \end{bmatrix} \tag{2}$$

And so, finally, we get the Cauchy-Riemann equations:

$$\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} = 0, \frac{\partial u'}{\partial x} - \frac{\partial v'}{\partial y} = 0. \tag{3}$$

These are the conditions under which we say that f is complex differentiable at some z_0. Moreover, if f is complex differentiable in its domain, then we say that f is holomorphic.

2 Bounded variation

Let $f : [a, b] \to \mathbb{C}$ be a complex-valued function of a real variable. We say that f is of bounded variation if the quantity

$$V_a^b(f) = \sup_P \sum_{i=1}^{n} {}_A |f(x_i) - f(x_{i-1})|, \tag{4}$$

where $P \in \{A = \{x_0, x_1, \ldots, x_{n_A}\} :$ - partition of (a, b) $\}$.