

Manifolds

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1 Topology

TODO: add

2 Tangent Space

2.1 Definition

Let (M, τ) be a C^k differentiable manifold, (U, ϕ) chart on M and $p \in U$. Let $\gamma_1, \gamma_2 : (-1, 1) \rightarrow U$ be two curves such that $\gamma_1(0) = \gamma_2(0) = p$ and $D_{\phi \circ \gamma_1}(x), D_{\phi \circ \gamma_2}(x) \in C^k[(-1, 1), \mathbb{R}^n]$.

Let \sim_T be an equivalence relation on the set of curves meeting the above conditions s.t. $\gamma_1 \sim_T \gamma_2 \iff D_{\phi \circ \gamma_1}(\phi \circ \gamma_1)(0) = D_{\phi \circ \gamma_2}(\phi \circ \gamma_2)(0)$.

Finally, a tangent space $T_p M$ is defined as a set of equivalence classes of curves meeting the above conditions.

$$[\gamma]_{\sim} = \{\gamma' : (-1, 1) \rightarrow U \text{ s.t. } \gamma \sim \gamma'\} \quad (1)$$

$$T_p M = \{[\gamma]_{\sim} : (-1, 1) \rightarrow U, \phi \circ \gamma \in C^k[(-1, 1), \mathbb{R}^n], \gamma(0) = p\} \quad (2)$$

Since $\gamma_1(0) = \gamma_2(0) = p \implies D_{\phi \circ \gamma_1}(0) = D_{\phi \circ \gamma_2}(0) \iff [\gamma_1]_{\sim} = [\gamma_2]_{\sim}$, it follows that
SHOW INDEPENDENCE FROM CHART.

2.2 Operations on tangent space

To define operations on the elements of $T_p M$, if (U, ϕ) is a chart with $p \in U$, one may define a map:

$$h_* : T_p M \rightarrow T_{\phi(p)} \mathbb{R}^n = \mathbb{R}^n, \quad (3)$$

$$h_*([\gamma]_{\sim}) := D_{\phi \circ \gamma}(0). \quad (4)$$

$$(5)$$

Note that $D_{\phi \circ \gamma}(0)$ is a well defined $\phi \circ \gamma : \mathbb{R} \rightarrow \mathbb{R}^n$.

Then the operations on $T_p M$ are defined as follows:

$$\text{for } u, v \in T_p M \text{ and } \lambda \in \mathbb{R} \quad (6)$$

$$u + v := h_*^{-1}(h_*(u) + h_*(v)), \quad (7)$$

$$\lambda v := h_*^{-1}(\lambda h_*(v)). \quad (8)$$

$$(9)$$

2.2.1 Bijectivity

By the definition of a chart, it has to be a homeomorphism (continuous, bijective) map. Thus $T_p M$ is a vector space isomorphic to \mathbb{R}^n .

2.2.2 Basis

If $B = \{e_1, e_2, \dots, e_n\}$ is a basis of \mathbb{R}^n , then $B_{T_p M} = \{h_*^{-1}(e_1), h_*^{-1}(e_2), \dots, h_*^{-1}(e_n)\}$ is a basis of $T_p M$. Basis is often denoted by the following notation:

$$\frac{\partial}{\partial x^i} \sim := h_*^{-1}(e_i) \quad (10)$$

$$\frac{\partial}{\partial x^i} \sim \in T_p M \quad (11)$$

2.3 Cotangent Space

Let M be a C^k differentiable manifold, $p \in M$.

If $T_p M$ is a tangent space, then its dual space $T_p^* M$ is called a cotangent space.

2.3.1 Basis of Cotangent Space

If $B_{T_p M} = \{b_1, b_2, \dots, b_n\}$ is a basis of tangent space, then basis of its dual space $B_{T_p^* M} = \{b_1^*, b_2^*, \dots, b_n^*\}$ can be found as follows:

$$b_i^* \in \mathcal{L}(T_p M \rightarrow \mathbb{R}), b_j \in B_{T_p M} \quad (12)$$

$$b_i^*(b_j) := \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \quad (13)$$

Consider now a basis of a tangent space $\frac{\partial}{\partial x^i} \sim := h_*^{-1}(e_i)$.
Its dual basis is given by $dx_i : T_x M \rightarrow \mathbb{R}$, $dx_i(\frac{\partial}{\partial x^j}) := \delta_{ij}$.

2.4 Tangent bundle

Let M be a C^k differentiable manifold. We define a Tangent bundle as a set consisting of all tangent spaces defined as: $TM := \bigcup_{p \in M} \{p\} \times T_p M$

2.5 Differential

Let $(M_1, \tau_1), (M_2, \tau_2)$, be C^k differentiable manifolds, $f : M_1 \rightarrow M_2$ be a smooth map and $p \in U \in \tau_1$. We define a differential (or pushforward) as a map between tangent spaces as follows:

$$df : T_p M_1 \rightarrow T_{f(p)} M_2 \quad (14)$$

$$df([\gamma]_{\sim}) := [f \circ \gamma]_{\sim} \in T_{f(p)} M_2 \quad (15)$$

Note that $[f \circ \gamma]_{\sim}$ is a equivalence class of all curves $f \circ \gamma : (-1, 1) \rightarrow M_2$, with $(f \circ \gamma)(0) = f(p)$ and $(f \circ \gamma_1)_{\sim} (f \circ \gamma_2) \iff D_{\psi \circ (f \circ \gamma_1)}(0) = D_{\psi \circ (f \circ \gamma_2)}(0)$, for some ψ being a chart of M_2 on neighbourhood of $f(p)$.

Given that $\{\frac{\partial}{\partial x^i}\}$ is a basis for $T_p M_1$ with each term corresponding to a smooth curve $\gamma_i : (-1, 1) \rightarrow M$, $\gamma_i(0) = p$, i.e $\{\frac{\partial}{\partial x^i} = [\gamma_i]_{\sim}\}$, $\{\frac{\partial f}{\partial x^i} = [f \circ \gamma_i]_{\sim}\}$ is a basis for $T_{f(p)} M_2$, and is a dual basis of cotangent space $\{dx_i\}$, with $dx_i(\frac{\partial}{\partial x^j}) = \delta_{ij}$, then

$$f(x_1, x_2, \dots, x_n) = (f_1, f_2, \dots, f_m) \quad (16)$$

$$df : T_p M_1 \rightarrow T_{f(p)} M_2 \quad (17)$$

$$u := \sum_i^n \lambda_i \frac{\partial}{\partial x^i}, \text{ then a differential is defined as} \quad (18)$$

$$df_p(u) = \sum_i^n \lambda_i \frac{\partial f}{\partial x^i} dx_i \quad (19)$$

3 Submersion

Let M, N be manifolds and $f : M \rightarrow N$ be a smooth map.

Its pushforward $df : T_p M \rightarrow T_{f(p)} N$ is called an immersion if it is a bijective map.

3.0.1 Natural projection

Natural projection $\pi : TM \rightarrow M$ is defined as: $\pi(p, T_p M) := p$

4 Metric Tensor

Let M be a C^k differentiable manifold and $p \in M$.

A metric tensor $g_p : T_p M \times T_p M \rightarrow \mathbb{R}$ is a map that is:

- Bilinear:
 - $g_p(u, \lambda v) = g_p(\lambda u, v) = \lambda g_p(u, v)$,
 - $g_p(u + w, v) = g_p(u, v) + g_p(w, v)$,
 - $g_p(u, v + w) = g_p(u, v) + g_p(u, w)$.
- Symmetric: $g_p(u, v) = g_p(v, u)$.
- Nondegenerate: $\forall v \in T_p M : v \neq 0 \implies \exists u \in T_p M : g_p(u, v) \neq 0$
- If $g_{u,v} : M \rightarrow \mathbb{R}$, with $g_{u,v}(p) := g_p(u, v)$, then $g_{u,v}$ is a smooth function.

5 Riemann semi-manifold

Let M be a C^k smooth manifold and $g_p : T_p M \times T_p M \rightarrow \mathbb{R}$ be its metric tensor. We say that a tuple (M, g_p) is called a Riemann semi-manifold.

g_p is also called a Riemann Metric.

5.1 Riemann norm

For a given metric tensor g_p , Riemann norm is defined as $\|\cdot\| : T_p M \rightarrow \mathbb{R}$, with $\|v\| := \sqrt{g_p(v, v)}$

5.2 Curve length

Let $\gamma : (a, b) \subseteq \mathbb{R} \rightarrow M$ be a parametrized smooth map.

We define the length of this curve as:

$$L(\gamma) := \int_a^b \|\dot{\gamma}\| dt = \int_a^b \sqrt{g_p(\dot{\gamma}, \dot{\gamma})} dt$$

6 Exterior Algebra

6.1 Alternating bilinear form

Let V be a vector space over a field F . An alternating (or antisymmetric) bilinear form on V is a bilinear form $B : V \times V \rightarrow F$ such that $B(v, w) = -B(w, v)$.

6.2 Second exterior power

Let V be a finite-dimensional vector space over a field F and $V_B = \{B : V \times V \rightarrow F : B \text{ is alternating bilinear form.}\}$ be a vector space of all alternating bilinear forms. The second exterior power of V , denoted with $\bigwedge^2 V$ is a dual space of V_B . i.e. $\bigwedge^2 V = V_B^*$. Elements of $\bigwedge^2 V$ are called 2-vectors.

6.3 Exterior product

Let V be a finite-dimensional vector space over a field F and $v, u \in V$ and $\bigwedge^2 V$ be its second exterior power. Exterior product of v and u , is a linear map to F $v \wedge u \in \bigwedge^2 V$
 $(v \wedge u)(B) = B(v, u)$.

From this definition, the following properties follow:

$$(u \wedge v)(B) = B(u, v) = -(u \wedge v)(B) = -B(v, u) \quad (20)$$

$$(u \wedge u)(B) = -(u \wedge u)(B) = 0 \quad (21)$$

$$\text{if } \{v_1, v_2, \dots, v_n\} \text{ is a basis for } V, \text{ then } \{v_i \wedge v_j : i, j \in \{1, 2, \dots, n\}, i < j\} \text{ is a basis for } \bigwedge^2 V. \quad (22)$$

Theorem 1. $u, v \in V, u \neq 0 \implies (u \wedge v = 0 \iff \exists \lambda \in F : v = \lambda u)$

Proof. This basically mean that u, v are in the same subspace and this may be shown with the following. Let $v = \lambda u$. Then $u \wedge v = u \wedge (\lambda u) = \lambda(u \wedge u) = 0$. □

6.4 Alternating multilinear form

Definition 6.1. Let V be a vector space over field F . An alternating multilinear form of degree p is a map: $V \times V \times \dots \times V \rightarrow F$ such that

$$M(u_1, \dots, u_i, \dots, u_j, \dots, u_p) = M(u_1, \dots, u_j, \dots, u_i, \dots, u_p) \quad (23)$$

$$M(u_1, \dots, \lambda u_i + w, \dots, \dots, u_p) = \lambda M(u_1, \dots, u_i, \dots, \dots, u_p) + M(u_1, \dots, w, \dots, \dots, u_p) \quad (24)$$

The set of all such forms M is a vector space. Let $a_i : \{0, \dots, p\} \rightarrow \{0, \dots, n\}$ be a strictly increasing sequence, and $\{v_1, \dots, v_n\}$ be a basis of V . Then a multilinear form M of degree p for any set of vectors in a given basis can be trivially transformed by the properties of M into a form $M(u_1, \dots, u_p) = \lambda_1 M(v_{a_1}, \dots, v_{a_p}) + \lambda_2 M(v_{a_1}, \dots, v_{a_p}) + \dots, M(v_{a_1}, \dots, v_{a_p})$. Since a_i is strictly increasing, if we choose p elements out of n , there are $\binom{n}{p}$ possibilities to do it. $\binom{n}{p}$ is also a dimension of the vector space of such forms M . In particular, if $p > n$, we just define a dimension to be 0.

Example 6.1. Let

$$A = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$$

be a matrix in \mathbb{R}^n .

Then a map $M(v_1, v_2, \dots, v_n) = \det(A)$ is an alternating multilinear form of degree n .

6.5 Dual Operator

Definition 6.2. Let X, Y be normed vector spaces and $T : X \rightarrow Y$ be a linear operator. Dual operator $T^* : Y^* \rightarrow X^*$ (note the reversed Y^*, X^*) is defined as $T^*(f) = f \circ T$.

6.6 p -th exterior power

Definition 6.3. Let V be a finite-dimensional vector space over field F , and $V_{AM} = \{B : V \times \cdots \times V \rightarrow F : B \text{ is alternating multilinear form.}\}$ be a vector space of all alternating multilinear forms over V . p -th exterior power of V , denoted with $\bigwedge^p V$ is a dual space of V_{AM} . i.e. $\bigwedge^p V = V_{AM}^*$. Elements of $\bigwedge^p V$ are called p -vectors.

Definition 6.4. Given $u_1, u_2, \dots, u_n \in V$, the exterior product is a linear map $u_1 \wedge u_2 \wedge \cdots \wedge u_n : V_{AM} \rightarrow F$ such that $(u_1 \wedge u_2 \wedge \cdots \wedge u_n)(M) = M(u_1, u_2, \dots, u_n)$.

Theorem 2. $u_1 \wedge u_2 \wedge \cdots \wedge u_n = 0 \iff (\exists \lambda_1, \lambda_2, \dots, \lambda_n \in F : \exists i \in \{1, \dots, n\} : \lambda_i \neq 0 : \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n = 0 \text{ i.e. they are linearly dependent})$

Proof. Without a loss of generality assume that $u_n = \lambda_1 u_1 + \lambda_2 u_2 + \cdots$

Then due to definition based on alternating multilinear forms, it follows that

$$M(u_1, u_2, \dots, u_n) = M(u_1, u_2, \dots, \lambda_1 u_1 + \lambda_2 u_2 + \cdots) =$$

$$\lambda_1 M(u_1, u_2, \dots, u_1) + \lambda_2 M(u_1, u_2, \dots, u_2) + \cdots =$$

$$(\text{by the antilinear property, swapping for each element a pair})(u_i, u_{n-i}),$$

$$= -\lambda_1 M(u_1, u_2, \dots, u_1) - \lambda_2 M(u_1, u_2, \dots, u_2) - \cdots$$

$$\text{Finishing with } a = -a \iff a = 0$$

□

Exterior powers have natural properties w.r.t. the linear transformations. Given a linear operator $T : V \rightarrow W$ and a form $M : W \times W \times \cdots \times W \rightarrow F_W$, we may induce an operator $T^*M : V \times V \cdots \times V \rightarrow F_W$ as $(T^*M)(v_1, v_2, \dots, v_n) := M(Tv_1, Tv_2, \dots, Tv_n)$.

This defines a dual map $\bigwedge^p T : \bigwedge^p V \rightarrow \bigwedge^p W$, $(\bigwedge^p T)(v_1 \wedge v_2 \cdots) := (Tv_1) \wedge (Tv_2) \cdots$

Example 6.2. One of such maps is very familiar and used frequently. Take $p = n$ for some n -dimensional vector space V . Then a vector space $\bigwedge^p V$ is 1 dimensional as $\binom{n}{p} = \binom{n}{n} = 1$ and the dimension of a dual space of all the alternating linear forms, V_{AL}^* is equal to the dimension of the vector space V_{AL} itself. In fact, this map is a determinant itself.

Observe that $\bigwedge^n(v_1 \wedge \cdots \wedge v_n) = Tv_1 \wedge \cdots \wedge Tv_n, \dots$ TODO

7 1-forms

Definition 7.1. Let M_1, M_2 be a n -dimensional, C^k smooth manifolds and $x \in M$.

We define a 1-form mapping $\omega_x : T_x M \rightarrow \mathbb{R}$, where $\omega_x(u)$ is a linear form.

Vector space of such 1-forms is isomorphic to a vector space $\bigwedge^1(T_x M)$.

Example 7.1. Consider a tangent space $T_x M$ with a basis $\frac{\partial}{\partial x^i}$ and its dual basis $dx_i(\frac{\partial}{\partial x^j}) := \delta_{ij}$. Let $\omega : T_x M \rightarrow \mathbb{R}$, $\omega(u) = f(u)dx_1(u)$. Given that $u = \sum_{i=1}^n \lambda_i \frac{\partial}{\partial x^i}$, it follows that

$$\omega(u) = f(u)dx_1\left(\sum_{i=1}^n \lambda_i \frac{\partial}{\partial x^i}\right). \text{ Then it follows, that from linearity of } \omega, dx_1 \quad (25)$$

$$\omega(u) = f(u) \sum_{i=1}^n \left(\lambda_i dx_1\left(\frac{\partial}{\partial x^i}\right) \right) \quad (26)$$

$$\omega(u) = f(u) \sum_{i=1}^n \lambda_i \delta_{i,1}, \text{ and finally} \quad (27)$$

$$\omega(u) = f(u)\lambda_1 \quad (28)$$

7.1 Basis of a vector space of 1-forms, $\bigwedge^1(T_x M)$

Given a smooth function $f : M_1 \rightarrow M_2$ and a basis of tangent space $\frac{\partial}{\partial x^i}$, we define a

Definition 7.2. Let M_1, M_2 be two manifolds and $f : M_1 \rightarrow M_2$ be a smooth function.