Manifolds

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Rozdziały

1 Topology

TODO: add

2 Tangent Space

2.1 Definition

Let (M, τ) be a C^k differentiable manifold, (U, ϕ) chart on M and $p \in U$. Let $\gamma_1, \gamma_2 : (-1, 1) \to U$ be two curves such that $\gamma_1(0) = \gamma_2(0) = p$ and $D_{\phi\circ\gamma_1}(x), D_{\phi\circ\gamma_2}(x) \in C^k[(-1, 1), R^n]$.
Let $\sim$T be an equivalence relation on the set of curves meeting the above conditions s.t. $\gamma_1 {\sim} \gamma_2 \iff D_{\phi\circ\gamma_1}(\phi \circ \gamma_1)(0) = D_{\phi\circ\gamma_2}(\phi \circ \gamma_2)(0)$.
Finally, a tangent space T_pM is defined as a set of equivalence classes of curves meeting the above conditions.

$$[\gamma]_\sim = \{\gamma' : (-1, 1) \to U \text{ s.t. } \gamma {\sim} \gamma'\} \tag{1}$$

$$T_pM = \{[\gamma]_\sim : (-1, 1) \to U, \phi \circ \gamma \in C^k[(-1, 1), \mathbb{R}^n], \gamma(0) = p\} \tag{2}$$

Since $\gamma_1(0) = \gamma_2(0) = p \implies \mathrm{D}_{\phi\circ\gamma_1}(0) = D_{\phi\circ\gamma_2(0)}\phi \circ \gamma_2'(0) \iff [\gamma_1]_\sim = [\gamma_2]_\sim$, it follows that
SHOW INDEPENDENCE FROM CHART.

2.2 Differential

Let $(M_1, \tau_1)(M_2, \tau_2)$, be C^k differentiable manifolds, $f : M_1 \to M_2$ be a smooth map and $p \in U \in \tau_1$.
We define a differential (or pushforward) as a map between tangent spaces as follows:

$$df : T_pM \to T_{f(p)}M \tag{3}$$

$$df([\gamma]_\sim) := D_{f\circ\gamma}(0) \tag{4}$$

$$\tag{5}$$

2.3 Operations on tangent space

To define operations on the elements of T_pM, if (U, ϕ) is a chart with $p \in U$, one may define a differential being a bijection:

$$h_* : T_pM \to T_{\phi(p)}\mathbb{R}^n = \mathbb{R}^n \tag{6}$$

$$h_*([\gamma]_\sim) := D_{\phi\circ\gamma}(0) \tag{7}$$

$$\tag{8}$$

Then the operations on T_pM are defined as follows:

$$\text{for } u, v \in T_pM \text{ and } \lambda \in \mathbb{R} \tag{9}$$

$$u + v := h_*^{-1}(h_*(u) + h_*(v)) \tag{10}$$

$$\lambda u := h_*^{-1}(\lambda h_*(v)) \tag{11}$$

$$\tag{12}$$

Thus T_pM is a vector space isomorphic to $\mathbb{R}^n$.

2.3.1 Basis

If $B = \{e_1, e_2, ..e_n\}$ is a basis of $\mathbb{R}^n$, then $B^* = \{h_*^{-1}(e_1), h_*^{-1}(e_2), ..h_*^{-1}(e_n)\}$ is a basis of T_pM.