Manifolds

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1 Topology

TODO: add

2 Tangent Space

2.1 Definition

Let (M, τ) be a C^k differentiable manifold, (U, ϕ) chart on M and $p \in U$. Let $\gamma_1, \gamma_2 : (-1, 1) \to U$ be two curves such that $\gamma_1(0) = \gamma_2(0) = p$ and $D_{\phi\circ\gamma_1}(x), D_{\phi\circ\gamma_2}(x) \in C^k[(-1, 1), R^n]$.
Let $\sim$T be an equivalence relation on the set of curves meeting the above conditions s.t. $\gamma_1{\sim}\gamma_2 \iff D_{\phi\circ\gamma_1}(\phi \circ \gamma_1)(0) = D_{\phi\circ\gamma_2}(\phi \circ \gamma_2)(0)$.
Finally, a tangent space T_pM is defined as a set of equivalence classes of curves meeting the above

conditions.

$$[\gamma]_\sim = \{\gamma' : (-1,1) \to U \text{ s.t. } \gamma \sim \gamma'\} \tag{1}$$
$$T_pM = \{[\gamma]_\sim : (-1,1) \to U, \phi \circ \gamma \in C^k[(-1,1), \mathbb{R}^n], \gamma(0) = p\} \tag{2}$$

Since $\gamma_1(0) = \gamma_2(0) = p \implies D_{\phi\circ\gamma_1}(0) = D_{\phi\circ\gamma_2}(0) \iff [\gamma_1]_\sim = [\gamma_2]_\sim$, it follows that
SHOW INDEPENDENCE FROM CHART.

2.2 Operations on tangent space

To define operations on the elements of T_pM, if (U, ϕ) is a chart with $p \in U$, one may define a map:

$$h_* : T_pM \to T_{\phi(p)}\mathbb{R}^n = \mathbb{R}^n, \tag{3}$$
$$h_*([\gamma]_\sim) := D_{\phi\circ\gamma}(0). \tag{4}$$
$$\tag{5}$$

Note that $D_{\phi\circ\gamma}(0)$ is a well defined $\phi \circ \gamma : \mathbb{R} \to \mathbb{R}^n$.

Then the operations on T_pM are defined as follows:

$$\text{for } u, v \in T_pM \text{ and } \lambda \in \mathbb{R} \tag{6}$$
$$u + v := h_*^{-1}(h_*(u) + h_*(v)), \tag{7}$$
$$\lambda v := h_*^{-1}(\lambda h_*(v)). \tag{8}$$
$$\tag{9}$$

2.2.1 Bijectivity

By the definition of a chart, it has to be a homeomorphism (continous, bijective) map. Thus T_pM is a vector space isomorphic to $\mathbb{R}^n$.

2.2.2 Basis

If $B = \{e_1, e_2, ..e_n\}$ is a basis of $\mathbb{R}^n$, then $B_{T_pM} = \{h_*^{-1}(e_1), h_*^{-1}(e_2), ..h_*^{-1}(e_n)\}$ is a basis of T_pM.

2.3 Differential

Let $(M_1, \tau_1)(M_2, \tau_2)$, be C^k differentiable manifolds, $f : M_1 \to M_2$ be a smooth map and $p \in U \in \tau_1$. We define a differential (or pushforward) as a map between tangent spaces as follows:

$$df : T_pM_1 \to T_{f(p)}M_2 \tag{10}$$
$$df([\gamma]_\sim) := [f \circ \gamma]_\sim \in T_{f(p)}M_2 \tag{11}$$

Note that $[f \circ \gamma]_\sim$ is a equivalence class of all curves $f \circ \gamma : (-1,1) \to M_2$, with $(f \circ \gamma)(0) = f(p)$ and $(f \circ \gamma_1) \sim (f \circ \gamma_2) \iff D_{\psi\circ(f\circ\gamma_1)}(0) = D_{\psi\circ(f\circ\gamma_2)}(0)$, for some ψ being a chart of M_2 on neighbourhood of $f(p)$.

2.4 Cotangent Space

Let M be a C^k differentiable manifold, $p \in M$.
If T_pM is a tangent space, then its dual space T_p^*M is called a cotangent space.

2.4.1 Basis of Cotangent Space

If $B_{T_pM} = \{b_1, b_2, ..., b_n\}$ is a basis of tangent space, then basis of its dual space $B^*_{T_p^*M} = \{b_1^*, b_2^*, ..., b_n^*\}$ can be found as follows:

$$b_i^* \in \mathcal{L}(T_pM \to \mathbb{R}), b_j \in B_{T_pM} \tag{12}$$

$$b_i^*(b_j) := \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{13}$$

3 Submersion

Let M, N be manifolds and $f : M \to N$ be a smooth map.
Its pushforward $df : T_pM \to T_{f(p)}N$ is called an immersion if it is a bijective map.

4 Tangent bundle

Let M be a C^k differentiable manifold. We define a Tangent bundle as a set consisting of all tangent spaces defined as: $TM := \bigcup\limits_{p \in M} \{p\} \times T_pM$

4.0.1 Natural projection

Natural projection $\pi : TM \to M$ is defined as: $\pi(p, T_pM) := p$

5 Metric Tensor

Let M be a C^k differentiable manifold and $p \in M$.
A metric tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a map that is:

- Bilinear:
  - $g_p(u, \lambda v) = g_p(\lambda u, v) = \lambda g_p(u, v)$,
  - $g_p(u + w, v) = g_p(u, v) + g_p(w, v)$,
  - $g_p(u, v + w) = g_p(u, v) + g_p(u, w)$.
- Symmetric: $g_p(u, v) = g_p(v, u)$.
- Nondegenerate: $\forall v \in T_pM : v \neq 0 \implies \exists u \in T_pM : g_p(u, v) \neq 0$
- If $g_{u,v} : M \to \mathbb{R}$, with $g_{u,v}(p) := g_p(u, v)$, then
  $g_{u,v}$ is a smooth function.

6 Riemann semi-maniofld

Let M be a C^k smooth manifold and $g_p : T_pM \times T_pM \to \mathbb{R}$ be its metric tensor. We say that a tuple (M, g_p) is called a Riemann semi-maniofld.
g_p is also called a Riemann Metric.

6.1 Riemann norm

For a given metric tensor g_p, Riemann norm is defined as $\|\cdot\| : T_pM \to \mathbb{R},$ with $\|v\| := \sqrt{g_p(v, v)}$

6.2 Curve length

Let $\gamma : (a, b) \subseteq \mathbb{R} \to M$ be a parametrized smooth map.
We define the length of this curve as:
$L(\gamma) := \int_a^b \|[\gamma]_\sim\| dt = \int_a^b \sqrt{g_p([\gamma]_\sim, [\gamma]_\sim)} dt$