

Manifolds

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1 Topology

TODO: add

2 Tangent Space

2.1 Definition

Let (M, τ) be a C^k differentiable manifold, (U, ϕ) chart on M and $p \in U$. Let $\gamma_1, \gamma_2 : (-1, 1) \rightarrow U$ be two curves such that $\gamma_1(0) = \gamma_2(0) = p$ and $D_{\phi \circ \gamma_1}(x), D_{\phi \circ \gamma_2}(x) \in C^k[(-1, 1), \mathbb{R}^n]$.

Let \sim_T be an equivalence relation on the set of curves meeting the above conditions s.t. $\gamma_1 \sim_T \gamma_2 \iff D_{\phi \circ \gamma_1}(\phi \circ \gamma_1)(0) = D_{\phi \circ \gamma_2}(\phi \circ \gamma_2)(0)$.

Finally, a tangent space $T_p M$ is defined as a set of equivalence classes of curves meeting the above conditions.

$$[\gamma]_{\sim} = \{\gamma' : (-1, 1) \rightarrow U \text{ s.t. } \gamma \sim \gamma'\} \quad (1)$$

$$T_p M = \{[\gamma]_{\sim} : (-1, 1) \rightarrow U, \phi \circ \gamma \in C^k[(-1, 1), \mathbb{R}^n], \gamma(0) = p\} \quad (2)$$

Since $\gamma_1(0) = \gamma_2(0) = p \implies D_{\phi \circ \gamma_1}(0) = D_{\phi \circ \gamma_2}(0) \iff [\gamma_1]_{\sim} = [\gamma_2]_{\sim}$, it follows that
SHOW INDEPENDENCE FROM CHART.

2.2 Operations on tangent space

To define operations on the elements of $T_p M$, if (U, ϕ) is a chart with $p \in U$, one may define a map:

$$h_* : T_p M \rightarrow T_{\phi(p)} \mathbb{R}^n = \mathbb{R}^n, \quad (3)$$

$$h_*([\gamma]_{\sim}) := D_{\phi \circ \gamma}(0). \quad (4)$$

$$(5)$$

Note that $D_{\phi \circ \gamma}(0)$ is a well defined $\phi \circ \gamma : \mathbb{R} \rightarrow \mathbb{R}^n$.

Then the operations on $T_p M$ are defined as follows:

$$\text{for } u, v \in T_p M \text{ and } \lambda \in \mathbb{R} \quad (6)$$

$$u + v := h_*^{-1}(h_*(u) + h_*(v)), \quad (7)$$

$$\lambda v := h_*^{-1}(\lambda h_*(v)). \quad (8)$$

$$(9)$$

2.2.1 Bijectivity

By the definition of a chart, it has to be a homeomorphism (continuous, bijective) map. Thus $T_p M$ is a vector space isomorphic to \mathbb{R}^n .

2.2.2 Basis

If $B = \{e_1, e_2, \dots, e_n\}$ is a basis of \mathbb{R}^n , then $B_{T_p M} = \{h_*^{-1}(e_1), h_*^{-1}(e_2), \dots, h_*^{-1}(e_n)\}$ is a basis of $T_p M$.

2.3 Differential

Let $(M_1, \tau_1), (M_2, \tau_2)$ be C^k differentiable manifolds, $f : M_1 \rightarrow M_2$ be a smooth map and $p \in U \in \tau_1$. We define a differential (or pushforward) as a map between tangent spaces as follows:

$$df : T_p M_1 \rightarrow T_{f(p)} M_2 \quad (10)$$

$$df([\gamma]_{\sim}) := [f \circ \gamma]_{\sim} \in T_{f(p)} M_2 \quad (11)$$

Note that $[f \circ \gamma]_{\sim}$ is a equivalence class of all curves $f \circ \gamma : (-1, 1) \rightarrow M_2$, with $(f \circ \gamma)(0) = f(p)$ and $(f \circ \gamma_1)_{\sim} (f \circ \gamma_2) \iff D_{\psi \circ (f \circ \gamma_1)}(0) = D_{\psi \circ (f \circ \gamma_2)}(0)$, for some ψ being a chart of M_2 on neighbourhood of $f(p)$.

2.4 Cotangent Space

Let M be a C^k differentiable manifold, $p \in M$.

If $T_p M$ is a tangent space, then its dual space $T_p^* M$ is called a cotangent space.

2.4.1 Basis of Cotangent Space

If $B_{T_p M} = \{b_1, b_2, \dots, b_n\}$ is a basis of tangent space, then basis of its dual space $B_{T_p^* M} = \{b_1^*, b_2^*, \dots, b_n^*\}$ can be found as follows:

$$b_i^* \in \mathcal{L}(T_p M \rightarrow \mathbb{R}), b_j \in B_{T_p M} \quad (12)$$

$$b_i^*(b_j) := \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \quad (13)$$

3 Submersion

Let M, N be manifolds and $f : M \rightarrow N$ be a smooth map.

Its pushforward $df : T_p M \rightarrow T_{f(p)} N$ is called an immersion if it is a bijective map.

4 Tangent bundle

Let M be a C^k differentiable manifold. We define a Tangent bundle as a set consisting of all tangent spaces defined as: $TM := \bigcup_{p \in M} \{p\} \times T_p M$

4.0.1 Natural projection

Natural projection $\pi : TM \rightarrow M$ is defined as: $\pi(p, T_p M) := p$