

Beta - Binomial Conjugacy

Binomial Likelihood:

$$\pi(x|p) = \binom{n}{x} p^x (1-p)^{1-x}$$

$$\text{with } \binom{n}{x} = \frac{n!}{x!(n-x)!}$$

To model the uncertainty about p , a convenient choice is a Beta prior:

$$\pi(p) = \frac{\Gamma(\alpha, \beta)}{\Gamma(\alpha) \Gamma(\beta)} p^{\alpha-1} (1-p)^{\beta-1} \quad \left(\begin{array}{l} \text{for } \alpha > 0 \\ \beta > 0 \end{array} \right)$$

The posterior is then also Beta:

$$\begin{aligned} \pi(p|x) &= \frac{\pi(x|p) \pi(p)}{\pi(x)} \\ &= \binom{n}{x} \underbrace{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \Gamma(\beta)}}_{\text{Do NOT depend on } p} p^x (1-p)^{n-x} p^{\alpha-1} (1-p)^{\beta-1} \end{aligned}$$

$$\begin{aligned} \text{So } p|x &\sim \text{Be}(x+\alpha, n-x+\beta) \text{ with } x = \sum_{i=1}^n x_i \\ E[p|x] &\approx \frac{x+\alpha}{n+\alpha+\beta} \quad \text{var}(p|x) = \frac{(x+\alpha)(n-x+\beta)}{(n+\alpha+\beta)^2 (n+\alpha+\beta+1)} \end{aligned}$$

Reference: Monte Carlo Statistical Methods;
Robert, Casella; 2004.