Beta - Binomial Conjugacy

Binomial Likelihood :

$$\pi(x|p) = \binom{n}{x} p^{x}(1-p)^{1-x}$$

with $\binom{n}{x} = \dfrac{n!}{x!\,(n-x)!}$

To model the uncertainty about p, a convenient choise is a Beta prior :

$$\pi(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} \qquad \left(\text{for } \begin{array}{l}\alpha > 0,\\ \beta > 0\end{array}\right)$$

The posterior is then also Beta :

$$\pi(p|x) = \frac{\pi(x|p)\,\pi(p)}{\pi(x)}$$

$$= \underbrace{\binom{n}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}}_{\text{Do NOT depend on } p} p^{x}(1-p)^{n-x}\, p^{\alpha-1}(1-p)^{\beta-1}$$

$$\propto p^{x+\alpha-1}(1-p)^{n-x+\beta-1}$$

So $p|x \sim Be(\underline{x}+\alpha,\ n-\underline{x}+\beta)$ with $\underline{x} = \sum_{i=1}^{n} x_i$

$E[p|x] \approx \dfrac{\underline{x}+\alpha}{n+\alpha+\beta}$ $\qquad var(p|x) = \dfrac{(\underline{x}+\alpha)(n-\underline{x}+\beta)}{(n+\alpha+\beta)^2(n+\alpha+\beta+1)}$

Reference : Monte Carlo Statistical Methods ; Robert, Casella ; 2004 .