Continuous Uniform distribution

Variance

Let $X \sim U_{[a,b]}$.

PDF: $$f_{a,b}(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a,b], \\ 0 & \text{otherwise} \end{cases}$$

Then we have that $\operatorname{var}(X) = \frac{(b-a)^2}{12}$. Proof.

$$\begin{aligned}
\mathbb{E}[X^2] &= \int_{-\infty}^{+\infty} x\, f_{a,b}(x)\, dx \\
&= \frac{1}{b-a} \int_a^b x^2\, dx \\
&= \frac{1}{b-a} \left. x^3/3 \right|_a^b \\
&= \frac{1}{b-a} \frac{(b-a)(b^2-ab+a^2)}{3} \\
&= \frac{b^2+ab+a^2}{3} \quad \blacksquare
\end{aligned}$$

$$\begin{aligned}
\operatorname{var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 &= \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2 \\
&= \frac{(b-a)^2}{12} \quad \blacksquare
\end{aligned}$$

Reference: Any basic probability textbook.