The Poisson-Gamma conjugacy

Poisson PMF: $f_\lambda(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$

Likelihood function:

$$\pi(\underline{x}|\lambda) = \frac{\lambda^{\sum_{i=1}^n x_i} e^{-n\lambda}}{\prod_{i=1}^n x_i!} \quad \text{with } \sum_{i=1}^n x_i = n\bar{x}$$

Gamma prior for λ:

$$\pi(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda\beta} \quad , \quad \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

(Gamma function)

Posterior:

$$\pi(\lambda|\underline{x}) = \frac{\pi(\underline{x}|\lambda)\pi(\lambda)}{\pi(x)}$$

$$= \frac{\lambda^{n\bar{x}} e^{-n\lambda}}{\prod_{i=1}^n (x_i!)} \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda\beta}$$

$$= \underbrace{\frac{1}{\prod_{i=1}^n (x_i!)} \frac{\beta^\alpha}{\Gamma(\alpha)}}_{\text{Does NOT depend on } \lambda} \lambda^{n\bar{x}} e^{-n\lambda} \lambda^{\alpha-1} e^{-\lambda\beta}$$

$$\propto \lambda^{\alpha+n\bar{x}-1} e^{-(n+\beta)\lambda}$$

So we have that

$$\lambda \mid \underset{\sim}{x} \sim \text{Gamma}\left(\alpha + n\bar{x}, \beta + n\right)$$

Posterior mean and variance:

$$E\left[\lambda \mid \underline{x}\right] = \frac{\alpha + n\bar{x}}{n + \beta},$$

$$\text{var}\left(\lambda \mid \underline{x}\right) = \frac{\alpha + n\bar{x}}{(n+\beta)^2},$$

with $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, sum of the counts divided by the sample size "n".

Reference: Monte Carlo Statistical Methods; Robert, Casella; 2004