Maximum Likelihood Estimators – Gamma model

Suppose that we observe a set of random variables $x_1, \dots, x_n \sim Ga(\alpha, \beta)$. The model is:

$$\{ Ga(\alpha,\beta) \, ; \, \alpha, \beta > 0 \}$$

α, β are the unknown parameters.

The likelihood function is:

$$L(\underline{x} \mid \alpha, \beta) = \prod_{i=1}^{n} f_{\alpha,\beta}(x_i) = \prod_{i=1}^{n} \frac{\beta^\alpha}{\Gamma(\alpha)} x_i^{\alpha-1} e^{-\beta x_i}$$

$$= \left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right)^n x_i^{n(\alpha-1)} e^{-\beta \sum_{i=1}^{n} x_i}$$

$$= \left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} e^{-\beta \sum_{i=1}^{n} x_i} \mathbb{1}_{\mathbb{R}^+}(x_i)$$

log-likelihood:

$$\ell(\underline{x} \mid \alpha, \beta) = n\left[\alpha \ln(\beta) - \ln(\Gamma(\alpha))\right] - (\alpha-1) \sum_{i=1}^{n} \ln(x_i) - \beta \sum_{i=1}^{n} x_i + \underbrace{\sum_{i=1}^{n} \ln\left(\mathbb{1}_{\mathbb{R}^+}(x_i)\right)}_{\text{does not contain } \alpha \text{ or } \beta.}$$

$$\frac{\partial \ell(x \mid \alpha, \beta)}{\partial \alpha} = n \ln(\beta) - n \underbrace{\frac{\partial \ln(\Gamma(\alpha))}{\partial \alpha}}_{\text{digamma function; cannot be solved analytically}} - \sum_{i=1}^{n} \ln(x_i)$$

$$\frac{\partial \ell(x \mid \alpha, \beta)}{\partial \beta} = n\alpha \frac{1}{\beta} - \sum_{i=1}^{n} x_i \iff \hat{\beta}_{mle} = \alpha \Big/ \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$= \alpha / \overline{x}$$

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