Gamma-Poisson Conjugacy

Poisson (likelihood):

$$\pi(x|\lambda) = \frac{\lambda^{\Sigma x_i} e^{-n\lambda}}{\prod_{i=1}^{n}(x_i!)} = \frac{\lambda^{n\bar{x}} e^{-n\lambda}}{\prod_{i=1}^{n}(x_i!)}$$

Gamma prior on λ:

$$\pi(\lambda) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda\beta}$$

with

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$$

Gamma posterior:

$$\pi(\lambda|x) = \pi(x|\lambda)\,\pi(\lambda) \,/\, \pi(x)$$

$$= \frac{\lambda^{n\bar{x}} e^{-n\lambda}}{\underline{\prod(x_i!)}} \boxed{\frac{\beta^{\alpha}}{\Gamma(\alpha)}} \lambda^{\alpha-1} e^{-\lambda\beta}$$

Do Not depend on λ

$$\propto \lambda^{\alpha+n\bar{x}-1} e^{-(n+\beta)\lambda}$$

$\left(\text{Gamma}(\alpha+n\bar{x}, n+\beta)\right)$

and we have that

$$E[\lambda|x] = \frac{\alpha+n\bar{x}}{n+\beta}, \quad var(\lambda|x) = \frac{\alpha+n\bar{x}}{(n+\beta)^2}$$

where $n\bar{x} = \sum_{i=1}^{n} x_i$; sum of counts.

Reference: Monte Carlo Statistical Methods; Robert, Casella ; 2004.