MSE for Binomial Model (Bayesian)

posterior in a conjugate model:

$$\pi(\theta \mid x) \sim \text{Beta}(\alpha + x,\ n - x + \beta)$$

with $x = \sum_{i=1}^{n} x_i$

$$E[\theta \mid x] = \frac{\alpha + x}{n + \alpha + \beta} = \frac{\alpha + E[x]}{n + \alpha + \beta} = \frac{\alpha + n\theta}{n + \alpha + \beta}$$

$$\text{var}(\theta \mid x) = \frac{(\alpha + x)(n - x + \beta)}{(n + \alpha + \beta)^2 (n + \alpha + \beta + 1)}$$

for a prior Beta$(1,1)$ and $n = 1$:

$$\text{bias}(\hat{\theta}_{bayes}) = E[\hat{\theta}_{bayes} - \theta] = E\left[\frac{1 + x}{n+2}\right] - E[\theta]$$

$$= \frac{1}{n+2} E[1 + \theta] - \theta = \frac{n - 2\theta}{n+2}$$

$$MSE(\hat{\theta}_{bayes}) = \text{var}(\hat{\theta}_{bayes}) + [\text{Bias}(\hat{\theta}_{bayes})]^2$$

$$= \frac{1}{(n+2)^2} n\theta(1-\theta) + \left(\frac{1 - 2\theta}{n+2}\right)^2$$

$$= \frac{n\theta - n\theta^2}{(n+2)^2} + \frac{1 - 2\theta - 2\theta^2}{(n+2)^2}$$

$$= \frac{1 + 2\theta - 2\theta^2 + n\theta - n\theta^2}{(n+2)^2}$$

$$= \frac{1 + 3\theta - 3\theta^2}{9} \neq \theta - \hat{\theta}_{mle}$$

Reference: Introduction to Mathematical Statistics; Hogg, McKean, Craig; 2019