Jeffreys prior for Poisson model

Jeffreys prior: $\pi_j(\theta) = \sqrt{I(\theta)}$

with $I(\theta) = -E\left[\frac{\partial^2 \ln L(\theta|X)}{\partial\theta^2}\right]$ (Fisher inform. of the sample)

$$\frac{\partial^2 \ln L(\theta|X)}{\partial\theta^2} = -\sum_{i=1}^{n} x_i / \theta^2$$

and we note that $E\left[\sum_{i=1}^{n} x_i\right] = E[n\bar{x}] = nE[\bar{x}] = n\theta$

So we have that:

$$\begin{aligned} I(\theta) &= E\left[\sum_{i=1}^{n} x_i\right] / \theta^2 \\ &= n\theta / \theta^2 \\ &= n / \theta \\ &\propto \theta^{-1} \end{aligned}$$

(Fisher information for 1 observation)

So $\pi_j(\theta) = \sqrt{I(\theta)} \propto \theta^{-1/2}$

$$\sim \text{Gamma}(1/2, 0)$$

Reference: Monte Carlo Statistical Methods; Robert, Casella; 2004.