Maximum Likelihood Estimation : AR(1)

model

The autoregressive (AR) model of order 1 is defined as:

$$X_t = \phi X_{t-1} + Z_t \quad , Z_t \sim N(0, \sigma^2)$$

Suppose that we observe the series $\underline{X} = (1.2, 0.8, 0.5, 0.1, -0.3)$

The likelihood function is given by:

$$L(\phi, \sigma^2 | \underline{X}) = f(X_1) \prod_{t=2}^{T} f(X_t | X_{t-1})$$

where $f(X_t | X_{t-1}) = 1/\sqrt{2\pi\sigma^2} \exp\left\{ -\frac{(X_t - \phi X_{t-1})^2}{2\sigma^2} \right\}$

The log-likelihood function is:

$$\ell(\phi, \sigma^2, \underline{X}) = \frac{T-1}{2} \log(2\pi\sigma^2) - \frac{1}{2} \sum_{t=2}^{T} (X_t - \phi X_{t-1})^2$$

and $\hat{\phi}_{mle} = \frac{\sum_{t=2}^{T} X_t X_{t-1}}{\sum_{t=2}^{T} X_{t-1}^2}$, $\hat{\sigma}^2_{mle} = \frac{1}{T-1} \sum_{t=2}^{T} (X_t - \hat{\phi}_{mle} X_{t-1})^2$

For our series, we have $\hat{\phi}_{mle} \approx 0.59$ and $\hat{\sigma}^2_{mle} \approx 0.04$.