

# Mean Squared Error of an Estimator

Mean Squared Error (MSE) of an estimator  $\theta$  is defined as

$$E_{\theta}[(\hat{\theta} - \theta)^2] = \text{var}_{\theta}(\hat{\theta}) + (\text{bias}(\hat{\theta}))^2$$

Proof :

$$\begin{aligned} \text{MSE}(\hat{\theta}) &= E[(\hat{\theta} - \theta)^2] \\ &= E[(\hat{\theta} - E[\hat{\theta}] + E[\hat{\theta}] - \theta)^2] \\ &= \underbrace{E[(\hat{\theta} - E[\hat{\theta}])^2]}_{\text{var}(\hat{\theta})} + \underbrace{E[(E[\hat{\theta}] - \theta)^2]}_{\text{constant}} \\ &\quad + \underbrace{2E[\hat{\theta} - E[\hat{\theta}]]}_{=0} + \underbrace{E[E[\hat{\theta}] - \theta]}_{\text{constant}} \\ &= \text{var}(\hat{\theta}) + (E[\hat{\theta}] - \theta)^2 + 0 \\ &= \text{var}(\hat{\theta}) + \text{bias}^2(\hat{\theta}) \end{aligned}$$

Reference : Introduction to Mathematical Statistics ; Hogg, McKean, Craig; 2019