Binary logistic Regression $E[y_i | \underline{x}_{ij}]$

Consider n independent observations $y_1, \dots, y_n \sim B(p)$ conditionally on a set of p categorical or numerical predictors x_j, $j = 1, \dots, p$. The model is

$$g\left(E[y_i | \underline{x}_{ij}]\right) = g(\pi_i) = \beta_0 + \beta_1 x_{i1} + \dots + \beta_p x_{ip} = \underline{x}_i^T \underline{\beta}$$

canonical link function:

$$g\left(E[y_i | x_{ij}]\right) = \ln\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \sum_{j=1}^{p} \beta_j \cdot x_{ij} = \underline{x}_i^T \underline{\beta}$$

It follows naturally that

$$E[y_i | x_{ij}] = \pi_i = \frac{e^{\underline{x}_i^T \underline{\beta}}}{1 + e^{\underline{x}_i \underline{\beta}}}$$

Indeed, we have that

$$\begin{aligned}
e^{\ln(\pi_i/(1-\pi_i))} &= e^{\underline{x}_i^T \underline{\beta}} \\
\pi_i / (1-\pi_i) &= e^{\underline{x}_i^T \underline{\beta}} \\
\pi_i &= e^{\underline{x}_i^T \underline{\beta}} (1-\pi_i) \\
\pi_i + \pi_i e^{\underline{x}_i^T \underline{\beta}} &= e^{\underline{x}_i^T \underline{\beta}}
\end{aligned}$$

$$\Leftrightarrow \pi_i = 1 / (1 + e^{-\underline{x}_i^T \underline{\beta}}) = \frac{e^{\underline{x}_i^T \underline{\beta}}}{1 + e^{\underline{x}_i^T \underline{\beta}}}$$

$\blacksquare$