MSE for Binomial model

$$X \sim \mathrm{Bin}(n, \theta)$$

$$\hat{\theta}_{mle} = n^{-1} \sum_{i=1}^{n} X_i \,,$$

$$E[\hat{\theta}_{mle}] = E\left[n^{-1} \sum_{i=1}^{n} X_i\right] = \frac{1}{n} E\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\, n\, E[X_i] = \theta.$$

$$\mathrm{var}(\hat{\theta}_{mle}) = \mathrm{var}\left(n^{-1} \sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} \mathrm{var}(X_i)$$

$$= \frac{1}{n^2}\, n\, \theta(1-\theta) = \frac{\theta(1-\theta)}{n}$$

$$\mathrm{bias}(\hat{\theta}_{mle}) = E[\hat{\theta}_{mle} - \theta] = E[\hat{\theta}_{mle}] - E[\theta]$$

$$= \theta - \theta = 0$$

$$MSE(\hat{\theta}_{mle}) = \mathrm{var}(\hat{\theta}_{mle}) + \left[\mathrm{bias}(\hat{\theta}_{mle})\right]^2$$

$$= \frac{\theta(1-\theta)}{n} - 0^2 = \frac{\theta(1-\theta)}{n}.$$

The maximum likelihood estimator for θ is unbiased: it's expectation is theoretically equal to the parameter.

Reference: Introduction to Mathematical Statistics; Hogg, McKean, Craig; 2019