Maximum Likelihood Estimator – Geometric model

Suppose that we observe a set of observations $x_1, \dots, x_n \sim Geo(\pi)$. Our model is:

$$\left\{ Geo(\pi) \; ; \pi \in [0,1] \right\}$$

The likelihood function is given by:

$$L(\underline{x} \mid \pi) = \prod_{i=1}^{n} f_\pi(x_i) = \prod_{i=1}^{n} \pi (1-\pi)^{x_i - 1}$$

$$= \pi^n (1-\pi)^{\sum_{i=1}^{n} x_i - n}$$

Log-likelihood:

$$\ell(\underline{x} \mid \pi) = n \ln(\pi) + \left(\sum_{i=1}^{n} - n \right) \ln(1-\pi)$$

derivative with respect to π:

$$\ell'(\underline{x} \mid \pi) = n/\pi - \left(\sum_{i=1}^{n} x_i - n \right) \cdot \frac{1}{1-\pi}$$

MLE: $$n/\pi = \left(\sum_{i=1}^{n} x_i - n \right) / (1-\pi)$$

$$\Leftrightarrow n - \cancel{n\pi} = \pi \sum_{i=1}^{n} x_i - \cancel{n\pi}$$

$$\Leftrightarrow \hat{\pi}_{mle} = n / \sum_{i=1}^{n} x_i = 1/\overline{x} \quad \text{(inverse link)}$$

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