Method of Moments : fitting an MA(q) model

The Method of Moments (MOM) can be used to estimate the parameters of an MA(q) model by matching the sample autocovariances to the theoretical ones.

Let us recall how a moving average of order q (MA(q)) model is defined for a time series $\{Y_t\}_{t=1}^{n}$:

$$Y_t = X_t + \theta_1 X_{t-1} + \theta_2 X_{t-2} + \dots + \theta_q X_{t-q}$$

where $\{X_t\} \sim WN(0, \sigma^2)$, i.e $E[X_t] = 0$ and $Var(X_t) = \sigma^2$

Autocovariances are given by :

$$\gamma_k = \begin{cases} \sigma^2 (1 + \sum_{j=1}^{q} \theta_j^2 & \text{for } k = 0 \\ \sigma^2 \sum_{j=k}^{q} \theta_j \theta_{j-k} & 1 \leq k \leq q \\ 0, & \text{for } k > q \end{cases}$$

The sample autocovariances are given by :

$$\hat{\gamma}_k = 1/n \sum_{t=k+1}^{n} X_t X_{t-k}$$

Example: Let $\{Y_t\} = \{1, 2, 3, 4, 5\}$ for which we assume a MA(1) model $Y_t = X_t + \theta_1 X_{t-1}$

Sample mean : $1/5 \sum_{t=1}^{5} Y_t = 3$ and $\hat{\gamma}_0 = 2$, $\hat{\gamma}_1 = 0.8$

Theoretical autocovariances : $\gamma_0 = \sigma^2 (1 + \theta_1^2)$ and $\gamma_2 = \sigma^2 \theta_1$

Then $\theta_1 = \hat{\gamma}_1 / \hat{\gamma}_0 = 0.8/2 = 4$, $\sigma^2 = \hat{\gamma}_0 / (1 + \theta_1^2) \approx 1.724$