Moment Generating Function:

Standard Normal r.v.

$$M_Z(t) = E\left[e^{tZ}\right] \qquad \text{def. MGF}$$

$$= \int_{-\infty}^{+\infty} e^{tz} f_{\mu,\sigma}(z)\, dz \qquad \text{def. Expect.}$$

$$= \int_{-\infty}^{+\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz \qquad Z \sim N(0,1)$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z^2 - 2tz)^2}{2}}\, dz$$

$$= e^{t^2/2} \cancel{\frac{1}{\sqrt{2\pi}}} \underbrace{\int_{-\infty}^{+\infty} e^{-\frac{(z-t)^2}{2}}\, dz}_{= \cancel{\sqrt{2\pi}}}$$

$$= e^{t^2/2} \qquad \blacksquare$$

Reference: See for ex. Probability and Statistics; M. Evans and J. Rosenthal, 2009.