Moment Generating Function Gamma r.v.

$$E\left[e^{tx}\right] = \int_{-\infty}^{+\infty} e^{tx} f_{\alpha,\beta}(x)\, dx$$

$$= \int_{-\infty}^{+\infty} e^{tx} \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\, dx$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_{-\infty}^{+\infty} x^{\alpha-1} e^{-(\beta-t)x}\, dx$$

$$= \frac{\beta^{\alpha}}{\cancel{\Gamma(\alpha)}} \frac{\cancel{\Gamma(\alpha)}}{(\beta-t)^{\alpha}}$$

$$= \left(\frac{\beta}{\beta-t}\right)^{\alpha}$$

The n^{th} moment is given by $E[X^n] = \left.\frac{d^n M_X(t)}{dt^n}\right|_{t=0}$

eg. First moment:

$$E[X] = M_X'(0) = \beta^{\alpha}(-\alpha)(\beta-0)^{-(\alpha-1)}(-1) = \frac{\alpha}{\beta}.$$

Reference: Introduction to Mathematical Statistics; Hogg, McKean, Craig; 2019.