Mean and Variance of Exponential Family models (3/3)

Given a model with location parameter θ and scale parameter ϕ, with log likelihood function $\ell(\theta, \phi, y)$ and having density or mass function

$$f(y;\theta,\phi) = \exp\left(\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\theta)\right)$$

wher $a(\cdot)$, $b(\cdot)$ and $c(\cdot)$ are specific functions.

We can prove that $\text{var}_y\left(\frac{\partial \ell}{\partial \theta}\right) = b''(\theta)a(\phi)$, indeed:

$$\text{var}\left(\frac{\partial \ell}{\partial \theta}\right) = -E\left[\frac{\partial^2 \ell}{\partial^2 \theta^2}\right] \quad \text{by definition}$$

$$\iff \text{var}\left(\frac{y - b'(\theta)}{a(\phi)}\right) = -E\left[\frac{-b''(\theta)}{a(\phi)}\right]$$

$$\iff \frac{1}{a^2(\phi)}\text{var}(Y) = \frac{b''(\theta)}{a(\phi)}$$

$$\iff \text{var}(Y) = b''(\theta)\,a(\phi)$$

So the variance of the variable of interest Y is the second derivative of the function containin only the location parameter θ times the function containing the scale parameter ϕ.