MSE for Binomial Model (Bayesian)

posterior in a conjugate model:

$$\pi(\theta \mid x) \sim \text{Beta}(\alpha + x,\ n - x + \beta)$$

with $x = \sum_{i=1}^{n} x_i$

$$E[\theta \mid x] = \frac{\alpha + x}{n + \alpha + \beta} = \frac{\alpha + E[x]}{n + \alpha + \beta} = \frac{\alpha + n\theta}{n + \alpha + \beta}$$

$$\text{var}(\theta \mid x) = \frac{(\alpha + x)(n - x + \beta)}{(n + \alpha + \beta)^2 (n + \alpha + \beta + 1)}$$

for a prior Beta$(1,1)$ and $n = 1$:

$$\text{bias}(\hat{\theta}_{bayes}) = E[\hat{\theta}_{bayes} - \theta] = E\left[\frac{1 + n\theta}{n+2}\right] - E[\theta]$$

$$= \frac{1 + n\theta}{n+2} - \theta = \frac{1 - 2\theta}{n+2}$$

$$\text{MSE}(\hat{\theta}_{bayes}) = \text{var}(\hat{\theta}_{bayes}) + \left[\text{Bias}(\hat{\theta}_{bayes})\right]^2$$

$$= \frac{(1 + n\theta)(n - n\theta + 1)}{(n+2)^2 (n+3)} + \left(\frac{1 - 2\theta}{n+2}\right)^2$$

$$= \frac{(1+\theta)(2-\theta)}{3^2 \cdot 4} + \frac{1 - 4\theta + 4\theta^2}{9}$$

$$= \frac{2 + \theta - \theta^2}{36} + \frac{4 - 16\theta + 16\theta^2}{36}$$

$$= \frac{6 - 15\theta + 15\theta^2}{36} \neq \theta - \hat{\theta}_{mle}^2$$

Reference: Introduction to Mathematical Statistics; Hogg, McKean, Craig; 2019