Proof that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

For a nonnegative integer α, the gamma function is $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$

We have

$$\Gamma(\alpha+1) = \int_0^\infty x^{(\alpha+1)-1} e^{-x}\, dx$$

$$= \int_0^\infty x^{\alpha} e^{-x}\, dx \qquad \begin{pmatrix} u = x^\alpha, & du = \alpha x^{\alpha-1} dx \\ v = -e^{-x}, & dv = e^{-x} dx \end{pmatrix}$$

$$= \left[x^\alpha e^{-x}\right]_0^\infty + \int_0^\infty \alpha x^{\alpha-1} e^{-x}\, dx$$

$$= 0 + \alpha \underbrace{\int_0^\infty x^{\alpha-1} e^{-x}\, dx}_{=\Gamma(\alpha)}$$

$$= \alpha\Gamma(\alpha)$$

∎

So we proved that important property of the gamma function stating that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.