MSE for Uniform model

By convention, we prefer an estimator with small MSE for all of the values of the parameter θ at once.
If for two estimators T_1 and T_2, we hav

$$E_\theta[(T_1 - g(\theta))^2] \leq E_\theta[(T_2 - g(\theta))^2] \text{ for all } \theta \in \Theta,$$

with a strict inequality for at least one value of θ, than we prefer T_1.

<u>Example</u>: Let $X_1, \dots, X_n \sim U_{[0,\theta]}$. The estimator $2\bar{X}$ is unbiased because:

$$E_\theta[2\bar{X}] = \frac{2}{n}\sum_{i=1}^{n} E[X_i] = \frac{2}{n} n \frac{\theta}{2} = \theta,$$

$$MSE_\theta(2\bar{X}) = 4\, var_\theta(\bar{X}) = \frac{4}{n^2}\sum_{i=1}^{n} var_\theta(X_i) = \frac{\theta^2}{3n}$$

The estimator $X_{(n)}$ (last ordered value) is biased because

$$E_\theta[X_{(n)}] = \int_0^\theta x n x^{n-1} \frac{1}{\theta^n}\, dx = \frac{n}{n+1}\theta$$

$$MSE_\theta(X_{(n)}) = \frac{2\theta}{(n+2)(n+1)}$$

We can cancel out the bias by the constant $\left(\frac{n+1}{n}\right)$.
However, the biased estimator $\left(\frac{n+2}{n+1}\right) X_{(n)}$ is better Than all estimators (smallest MSE).

<u>Reference</u>: An Introduction to Mathematical Statistics; Bijma, Jonker, van der Vaart; 2016.