likelihood $p(y|\theta) \propto e^{-(y-\theta)^2/2\sigma^2}$

Prior $p(\theta) \propto e^{A\theta^2 + B\theta + C}$

Aim: this prior is a normal distribution

$$A\theta^2 + B\theta + C = A(\theta + B/2A)^2 - B^2/4A^2 + C$$

$$p(\theta) \propto e^{A\theta^2 + B\theta + C} \propto e^{A(\theta + B/2A)^2}$$

let $A = -\frac{1}{2\tau_0^2}$ $\quad \frac{-B}{2A} = \mu_0$

prior

$$\boxed{p(\theta) \propto e^{-\frac{1}{2\tau_0^2}(\theta - \mu_0)^2} \qquad \theta \sim N(\mu_0, \tau_0^2)}$$

Aim: Posterior is also normal

$$p(\theta|y) \propto p(\theta)\, p(y|\theta)$$

$$= e^{-\frac{1}{2\tau_0^2}(\theta-\mu_0)^2} \; e^{-(y-\theta)^2/2\sigma^2}$$

$$= \exp\left[-\frac{1}{2}\left(\frac{(y-\theta)^2}{\sigma^2} + \frac{(\theta-\mu_0)^2}{\tau_0^2}\right)\right]$$

$$= \exp\left[-\frac{1}{2}\left(\frac{1}{\sigma^2}(y^2 - 2\theta y + \theta^2) + \frac{1}{\tau_0^2}(\theta^2 - 2\theta\mu_0 + \mu_0^2)\right)\right]$$

$$\propto \exp\left[-\frac{1}{2}\left[\left(\frac{1}{\sigma^2} + \frac{1}{\tau_0^2}\right)\theta^2 - 2\left(\frac{y}{\sigma^2} + \frac{\mu_0}{\tau_0^2}\right)\theta\right]\right]$$

$$= \exp\left[\frac{-1}{2}\left(\frac{1}{\sigma^2}+\frac{1}{\tau_0^2}\right)\left(\theta^2 - 2\left(\frac{y/\sigma^2 + \mu_0/\tau_0^2}{1/\sigma^2 + 1/\tau_0^2}\right)\theta\right)\right]$$

$$\boxed{\frac{1}{\tau_1^2} = \frac{1}{\sigma^2}+\frac{1}{\tau_0^2}} \qquad \boxed{\mu_1 = \frac{y/\sigma^2 + \mu_0/\tau_0^2}{1/\sigma^2 + 1/\tau_0^2}}$$

$$\propto \exp\left[\frac{-1}{2\tau_1^2}(\theta-\mu_1)^2\right]$$

$$\theta \mid y \sim N(\mu_1, \tau_1^2)$$

$$p(\tilde{y}|y) = \int p(\tilde{y}|\theta)\,p(\theta|y)\,d\theta$$

$$\propto \int \exp\left(-\frac{(\tilde{y}-\theta)^2}{2\sigma}\right)\exp\left(-\frac{(\theta-\mu_1)^2}{2\tau_1^2}\right)d\theta$$

$$\propto \exp\left(A\tilde{y}^2 + B\tilde{y} + c\right)$$

$$\tilde{y}|y \sim N(\cdot\ ,\ \cdot)$$

$$E(\tilde{y}|y) = E\left(E(\tilde{y}|\theta, y)|y\right) = E\left(E(\tilde{y}|\theta)|y\right) = E(\theta|y) = \mu_1$$

$$\mathrm{Var}(\tilde{y}|y) = E\left(\mathrm{var}(\tilde{y}|\theta)|y\right) + \mathrm{Var}\left(E(\tilde{y}|\theta)|y\right)$$

$$= E(\sigma^2|y) + \mathrm{Var}(\theta|y)$$

$$= \sigma^2 + \tau_1^2$$

$$\boxed{\tilde{y}|y \sim N(\mu_1,\ \sigma^2 + \tau_1^2)}$$

How do we handle multiple data points?

<u>Prior:</u> $p(\theta) \propto \exp\left(-\frac{1}{2\tau_0^2}(\theta-\mu_0)^2\right)$

<u>likelihood:</u> $p(y|\theta) \propto \prod_{i=1}^{n} \exp\left(-\frac{(y_i-\theta)^2}{2\sigma^2}\right) = \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\theta)^2\right)$

<u>Posterior:</u>

$$p(\theta|y) \propto p(y|\theta)\,p(\theta)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\theta)^2 - \frac{1}{2\tau_0^2}(\theta-\mu_0)^2\right)$$

$$= \exp\left(\frac{-1}{2\sigma^2}\left(\sum_{i=1}^{n} y_i^2 - 2\theta\sum_{i=1}^{n} y_i + n\theta^2\right) - \frac{1}{2\tau_0^2}\left(\theta^2 - 2\theta\mu_0 + \mu_0^2\right)\right)$$

Let $\bar{y}$ be the sample mean, so $n\bar{y} = \sum_{i=1}^{n} y_i$

$$= \exp\left(\theta^2\left(\frac{-n}{2\sigma^2} - \frac{1}{2\tau_0^2}\right) + \theta\left(\frac{n\bar{y}}{\sigma^2} + \frac{1}{\tau_0^2}\right)\right)$$

$$\tau_n^2 = \left(\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}\right)^{-1}$$

$$= \exp\left(\frac{-1}{2\tau_n^2}\left(\theta^2 - 2\tau_n^2\left(\frac{n\bar{y}}{\sigma^2} + \frac{1}{\tau_0^2}\right)\theta\right)\right)$$

$$\mu_n = \tau_n^2\left(\frac{n\bar{y}}{\sigma^2} + \frac{1}{\tau_0^2}\right)$$

$$\propto \exp\left(\frac{-1}{2\tau_n^2} (\theta - \mu_n)^2 \right)$$

$$\boxed{\theta | y \sim N(\mu_n, \tau_n^2)}$$

Example Screws

Data $\quad y = \{5.1, 4.9, 5.2, 5.3, 5.0\}$

$n = 5 \quad \bar{y} = 5.1 \quad \sigma^2 = 0.5 \quad \tau_0^2 = 1.0 \quad \mu_0 = 5.0$

$$\tau_n^2 = \left(\frac{1}{1} + \frac{5}{0.5} \right)^{-1} = \frac{1}{11} \approx 0.11$$

$$\mu_n = \frac{1}{11}\left(\frac{25.5}{0.5} + \frac{5.0}{1.0} \right) = \frac{1}{11}(51 + 5) = \frac{56}{11} \approx 5.1$$

$$\boxed{\theta | y \sim N(5.1, 0.1)}$$

Review: Scaled inverse χ^2 distribution

$$\theta \sim \text{Inv-}\chi^2(\underset{\substack{\uparrow \\ \text{deg. of freedom}}}{\nu},\ \underset{\substack{\uparrow \\ \text{scale}}}{s^2})$$

$$p(\theta) = \frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)} s^{\nu} \theta^{-(\nu/2+1)} e^{\frac{-\nu s^2}{2\theta}}$$

$$\propto \theta^{-(\nu/2+1)} e^{\left(-\frac{\nu s^2}{2} \cdot \frac{1}{\theta}\right)}$$

Unknown variance σ^2 but known mean θ

likelihood:

$$p(y|\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y_i-\theta)^2/2\sigma^2}$$

$$\propto (\sigma^2)^{-n/2} \exp\left(\frac{-1}{2\sigma^2} \sum_{i=1}^{n} (y_i-\theta)^2\right)$$

Define $\quad v := \frac{1}{n}\sum_{i=1}^{n}(y_i-\theta)^2 \quad \leftarrow$ sample variance.

$$p(y|\sigma^2) \propto (\sigma^2)^{-n/2} \exp\left(-\frac{nv}{2\sigma^2}\right)$$

Prior:

$$p(\sigma^2) \propto (\sigma^2)^{-(\frac{\nu_0}{2}+1)} \exp\left(\frac{-\nu_0\sigma_0^2}{2} \cdot \frac{1}{\sigma^2}\right)$$

Posterior:

$$p(\sigma^2|y) \propto p(\sigma^2)\, p(y|\sigma^2)$$

$$\propto (\sigma^2)^{-(\nu_0/2+1)} \exp\left(\frac{-\nu_0\sigma_0^2}{2} \cdot \frac{1}{\sigma^2}\right) (\sigma^2)^{-n/2} \exp\left(-\frac{nv}{2\sigma^2}\right)$$

$$= (\sigma^2)^{-\frac{1}{2}(\nu_0+n+2)} \exp\left(-\frac{1}{2\sigma^2}(\nu_0\sigma_0^2+nv)\right)$$

$$\boxed{\sigma^2 \sim \text{Inv-}\chi^2\left(\nu_0+n,\ \frac{\nu_0\sigma_0^2+nv}{\nu_0+n}\right)}$$

Review: Gamma (a, b)

a - shape

b - inverse scale

$$P(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}$$

$$\propto \theta^{a-1} e^{-b\theta}$$

mean: a/b

variance: a/b^2

Poisson Likelihood:

$$p(y|\theta) = \prod_{i=1}^{n} \frac{1}{y!} \theta^{y_i} e^{-\theta} \propto \theta^{n\bar{y}} e^{-n\theta}$$

Prior:

$$p(\theta) \propto e^{-b_0\theta} \theta^{a_0-1}$$

Posterior:

$$p(\theta|y) \propto p(\theta)\, p(y|\theta) \propto \theta^{n\bar{y}+a_0-1} e^{-n\theta-b_0\theta}$$

$$\boxed{\theta|y \sim \text{Gamma}(a_0 + n\bar{y},\ b_0 + n)}$$

$3/200{,}000$ died of asthma $\rightarrow$ 1 year $\leftarrow$ data

$\hookrightarrow 1.5/100{,}000$ $x = 2.0$

World data $0.6/100{,}000$ $\leftarrow$ prior

Prior 0.6 mean $\frac{a}{b} = 0.6$

Suppose $a = 3.0$ and $b = 5.0$

Posterior Gamma $(a_0 + y, \; b_0 + x) =$ Gamma $(6.0, 7.0)$

mean $= \frac{6}{7} \approx 0.86$ variance $= \frac{6}{7^2} < \frac{3}{5^2}$

$3/200,000$ died of asthma $\rightarrow$ 1 year $\leftarrow$ data

$\hookrightarrow 1.5/100,000$ Exposure $\underline{x = 2.0}$

World data : $0.6/100,000$ $\leftarrow$ prior

Prior mean 0.6 and a Gamma distribution

$\text{Gamma}(\alpha_0, \beta_0)$ $\quad$ mean $= \dfrac{\alpha_0}{\beta_0} = 0.6$

suppose $\alpha_0 = 3.0$ $\quad \beta_0 = 5.0$

Posterior $\text{Gamma}(\alpha_0 + y, \beta_0 + x) = \text{Gamma}(6.0, 7.0)$

Variance $\dfrac{6}{7^2} < \dfrac{3}{5^2}$ why?

mean $= \dfrac{6}{7} \approx 0.86$