Bayesian Computational Statistics

reference book : Bayesian Data Analysis;
Gelman et al.

Bayes' rule and its consequences

conditional probabilities

$P(A|B)$ probability of event A given that event B has occured.

$$P(A|B) = P(A \cap B) / P(B)$$

example : We have a standard 6 sided die

$$A = \{5\}, \quad B = \{1,3,5\}$$

$$P(A|B) = P(A \cap B) / P(B) = 1/3$$

$$P(A) = 1/6$$

$C = \{2,4,6\}$ so $P(A \cap C)/P(C) = 0/3 = 0$

Note: $P(A|B) \neq P(B|A)$

Bayes' rule relates $P(A|B)$ to $P(B|A)$:

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

It gives us a framework for making and updating estimates of $P(A|B)$ based on evidence.
Updating our beliefs in the face of new information.

<u>example</u>: medical testing

event A: having the disease
event B: testing positive

prior $P(A)$: 0.01 (1%)

likelihood $P(B|A)$: 0.99

false positive rate to compute the marginal 0.05

marginal: $P(B) = P(B|A) \cdot P(A) + P(B|A^c) \cdot P(A^c)$

$$= 0.99 \cdot 0.01 + 0.05 \cdot 0.99$$

$$\approx 0.0594$$

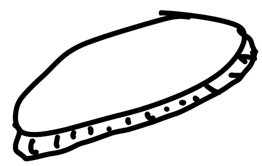
posterior $P(A|B) = \dfrac{P(B|A) \cdot P(A)}{P(B)} = \dfrac{0.99 \cdot 0.01}{0.0594} \approx \dfrac{0.1\bar{6}}{2}$

a bit surprising.

Bayesian inference

1. Start with a prior distribution
2. Collect data
3. Compute the likelihood
4. Compute the marginal probabilit
5. Compute the posterior via Bayes' rule (updated beliefs)

example: Is it a fair coin?



H_0 : It is fair

H_1 : It is biased

prior: 0.5

Data: T, T, T

$$P(H_0 \mid TTT) = \frac{P(TTT \mid H_0)\, P(H_0)}{P(TTT)}$$

$$P(TTT) = P(TTT \mid H_0) \cdot P(H_0) + P(TTT \mid H_1) \cdot P(H_1)$$

$$P(TTT \mid H_1) = \int_0^1 p^3 dp = \frac{p^4}{4}\Big|_0^1 = 1/4,$$ so

$$P(H_0 \mid TTT) = \underline{1/3}$$

Fundamentals of Bayesian Inference

Bayes' Rule

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Quizz exercise:

		prior	data
H_0 :	coin is fair	0.66	HHHHH
H_1 :	coin is biased	0.34	(5H)

likelihood :

$$P(5H | H_0) = \left(\frac{1}{2}\right)^5 = 1/32 = 0.03125$$

$$P(5H | H_1) = \int_0^1 p^5 \, dp = \left[p^6/6 \right]_0^1 = 1/6$$

marginal :

$$P(5H) = p(5H | H_0) \cdot p(H_0) + p(5H | H_1) \cdot p(H_1)$$

$$= 0.03125 \cdot 0.66 + (1/6) \cdot 0.34$$

$$\approx \underline{0.078}$$

posterior:

$$p(H_0 | 5H) = \frac{p(5H | H_0) \cdot p(H_0)}{p(5H)}$$

$$= \frac{0.03125 \cdot 0.66}{0.078} \approx \underline{0.268}$$

Bayesian Inference:

process of fitting a probability model to a set of data using Bayes' rule

<u>Notation</u>:

θ : parameter, scalars or vectors
e.g $\theta = (\beta_0, \beta_1)$

y : observed data

$\tilde{y}$: unknown but potentially observable data

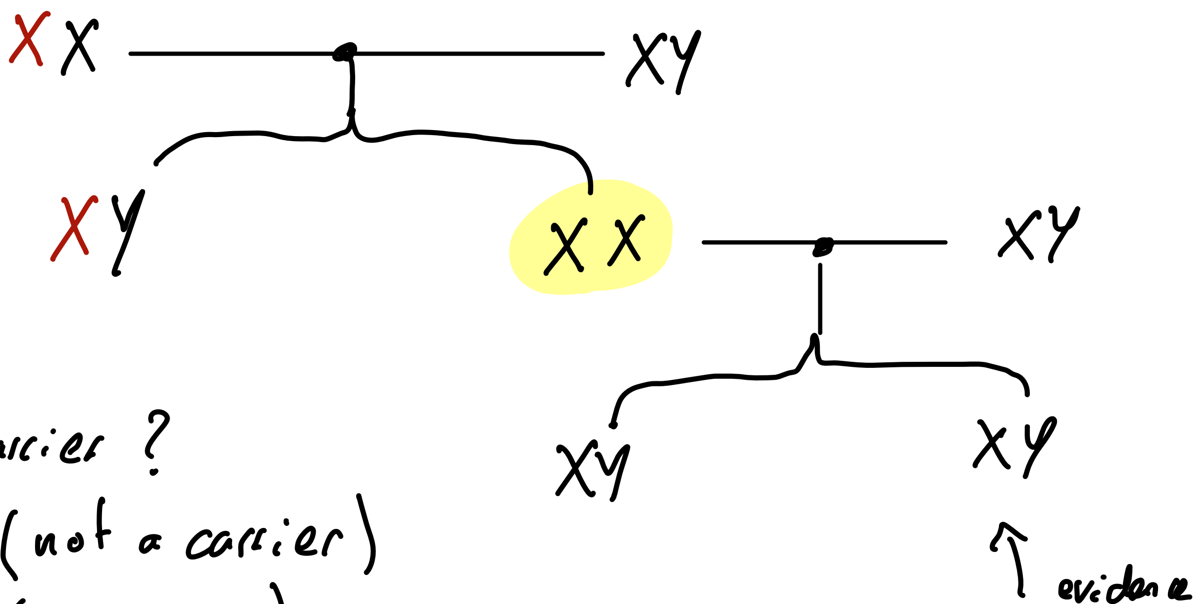
$p(x)$: pdf of x

$p(x, y)$: joint distribution of x and y

<u>Hemophilia example</u>:

from the book

χ - linked trait



Yellow is a carrier ?

$H_0 \quad \theta = 0$ (not a carrier)

$H_1 \quad \theta = 1$ (a carrier)

data : $\bar{y} = (0, 0)$

prior : $(50/50) = 0.5$

likelihood : $p(\bar{y} \mid \theta = 1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$p(\bar{y} \mid \theta = 0) = 1 \cdot 1 = 1$

marginal (for scaling) : $p(\bar{y}) = p(\bar{y} \mid \theta = 1) \cdot p(\theta = 1) + p(\bar{y} \mid \theta = 0) \cdot p(\theta = 0)$

$= \frac{1}{4} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \underline{\frac{5}{8}}$

posterior :

$p(\theta = 1 \mid \bar{y}) = \frac{1/8}{5/8} = \frac{1}{5} = \underline{0.2}$

What if there is a third child who is also XY and not afflicted. (update)

$$\tilde{y} = (0)$$

$$p(\theta=1 \mid y) = \frac{p(y \mid \theta=1) \cdot p(\theta=1)}{p(y)}$$

$$= \frac{(1/2) \cdot (1/5)}{\frac{1}{2} \cdot \frac{1}{5} + 1 \cdot 4/5} = 1/9 \approx \underline{0{,}11}$$

Exchangeability

order of observations doesn't matter

subjectivity vs objectivity

$\Rightarrow$ prior

Example from the book

Spelling corrections

'radom' $\rightarrow$ 'random'

'radom' $\rightarrow$ 'radon'

'radom' $\rightarrow$ 'radom'

data : y = radom

$$p(\theta \mid y) \propto p(\theta)\, p(y \mid \theta)$$

scaling can be done easily at the end

prior :

θ	rel. freq	prob
random	$7.6 \cdot 10^{-5}$	
radon	$6.05 \cdot 10^{-6}$	
radom	$3.12 \cdot 10^{-7}$	

$\Rightarrow$ rewriting

θ	rel. freq.	prob
random	$760 \cdot 10^{-7}$	0.923
radon	$60.5 \cdot 10^{-7}$	0.073
radom	$3.12 \cdot 10^{-7}$	0.004

likelihood

θ	$p(\text{'radom'} \mid \theta)$
random	0.00193
radon	0.000143
radom	0.975

posterior :

θ	$p(\theta)\, p(\text{'radom'} \mid \theta)$	$p(\text{'radom'} \mid \theta)$
random	$1.47 \cdot 10^{-7}$ $(1470 \cdot 10^{-10})$	~ 0.325
radon	$8.65 \cdot 10^{-10}$	~ 0.002
radom	$3.04 \cdot 10^{-7}$ $(3040 \cdot 10^{-10})$	~ 0.673

example where we don't need marginals because we can scale the results at the very end.