

# Bayesian Computational Statistics

reference book : Bayesian Data Analysis;  
Gelman et al.

## Bayes' rule and its consequences

conditional probabilities

$P(A|B)$  probability of event A given that  
event B has occurred.

$$P(A|B) = P(A \cap B) / P(B)$$

example : We have a standard 6 sided die

$$A = \{5\}, B = \{1, 3, 5\}$$

$$P(A|B) = P(A \cap B) / P(B) = 1/3$$

$$P(A) = 1/6$$

$$C = \{2, 4, 6\} \text{ so } P(A \cap C) / P(C) = 0/3 = 0$$

Note :  $P(A|B) \neq P(B|A)$

Bayes' rule relates  $P(A|B)$  to  $P(B|A)$  :

$$P(A|B) = \frac{P(B|A) P(A)}{P(B)}$$

It gives us a framework for making and updating estimates of  $P(A|B)$  based on evidence.

Updating our beliefs in the face of new information.

example : medical testing

event A : having the disease

event B : testing positive

prior  $P(A)$  : 0.01 (1%)

likelihood  $P(B|A)$  : 0.99

false positive rate

to compute the marginal 0.05

$$\begin{aligned} \text{marginal} : P(B) &= P(B|A) \cdot P(A) + P(B|A^c) \cdot P(A^c) \\ &= 0.99 \cdot 0.01 + 0.05 \cdot 0.99 \\ &\approx 0.0594 \end{aligned}$$

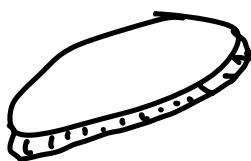
$$\text{posterior } P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)} = \frac{0.99 \cdot 0.01}{0.0594} \approx \underline{\underline{0.16}}_2$$

a bit surprising.

## Bayesian inference

1. Start with a prior distribution
2. Collect data
3. Compute the likelihood
4. Compute the marginal probability
5. Compute the posterior via Bayes' rule  
(updated beliefs)

example: Is it a fair coin?



$H_0$ : It is fair

$H_1$ : It is biased

prior: 0.5

Data: T, T, T

$$P(H_0 | TTT) = \frac{P(TTT | H_0) P(H_0)}{P(TTT)}$$

$$P(TTT) = P(TTT | H_0) \cdot P(H_0) + P(TTT | H_1) \cdot P(H_1)$$

$$P(TTT | H_1) = \int_0^1 p^3 dp = \frac{p^4}{4} \Big|_0^1 = 1/4, \text{ so}$$

$$P(H_0 | TTT) = \underline{\underline{1/3}}$$

# Fundamentals of Bayesian Inference

$$\text{Bayes' Rule} \quad P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Quizz exercise:

	<u>prior</u>	data
$H_0$ : coin is fair	0.66	HHHHH
$H_1$ : coin is biased	0.34	(5H)

Likelihood :

$$P(\text{5H} | H_0) = \left(\frac{1}{2}\right)^5 = 1/32 = 0.03125$$

$$P(\text{5H} | H_1) = \int_0^1 p^5 dp = \left[\frac{p^6}{6}\right]_0^1 = 1/6$$

marginal :  $P(\text{5H}) = P(\text{5H} | H_0) \cdot P(H_0) + P(\text{5H} | H_1) \cdot P(H_1)$

$$= 0.03125 \cdot 0.66 + \left(\frac{1}{6}\right) \cdot 0.34$$

$$\approx \underline{0.078}$$

Posterior:

$$P(H_0 | \text{5H}) = \frac{P(\text{5H} | H_0) \cdot P(H_0)}{P(\text{5H})}$$

$$= \frac{0.03125 \cdot 0.66}{0.078} \approx \underline{\underline{0.268}}$$

## Bayesian Inference:

process of fitting a probability model to  
a set of data using Bayes' rule

## Notation:

$\Theta$  : parameter, scalars or vectors  
e.g.  $\Theta = (\beta_0, \beta_1)$

$y$  : observed data

$\tilde{y}$  : unknown but potentially observable  
data

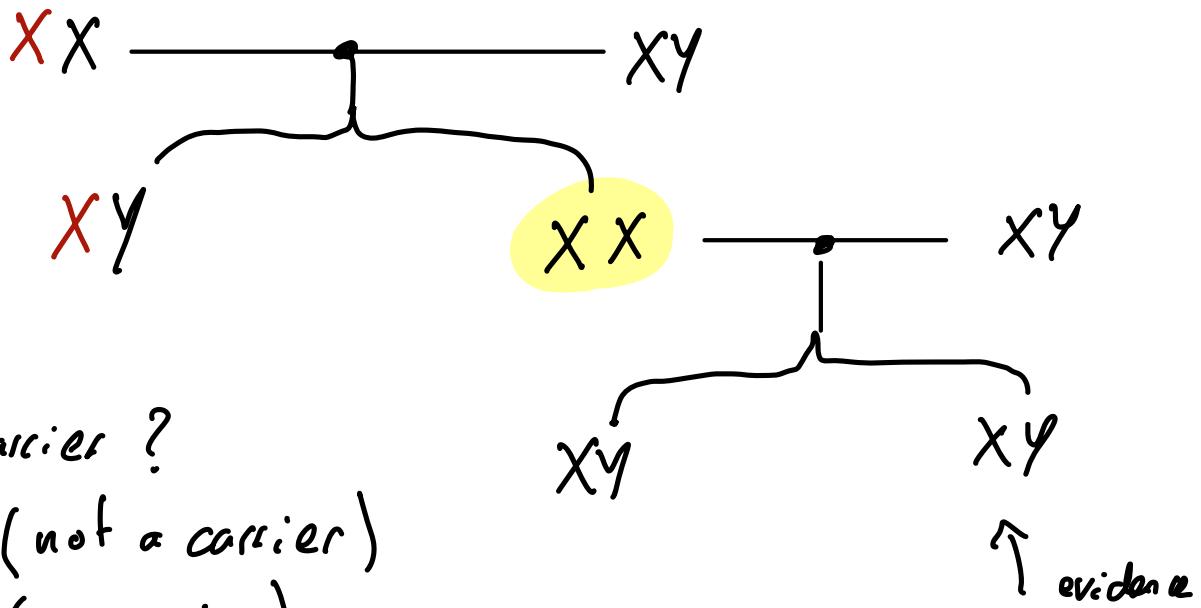
$p(x)$  : pdf of  $x$

$p(x, y)$  : joint distribution of  $x$  and  $y$

## Hemophilia example:

from the book

X-linked trait



Yellow is a carrier?

$H_0 \theta = 0$  (not a carrier)

$H_1 \theta = 1$  (a carrier)

$$\text{data : } \bar{y} = (0, 0)$$

$$\text{prior : } (50/50) = 0.5$$

$$\text{likelihood : } p(\bar{y} | \theta = 1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$p(\bar{y} | \theta = 0) = 1 \cdot 1 = 1$$

$$\begin{aligned} \text{marginal (for scaling)} : p(\bar{y}) &= p(\bar{y} | \theta = 1) \cdot p(\theta = 1) + \\ &\quad p(\bar{y} | \theta = 0) \cdot p(\theta = 0) \\ &= \frac{1}{4} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \underline{\underline{\frac{3}{8}}} \end{aligned}$$

posterior :

$$p(\theta = 1 | \bar{y}) = \frac{\frac{1}{8}}{\frac{3}{8}} = \frac{1}{5} = \underline{\underline{0.2}}$$

What if there is a third child who is also XY and not afflicted. (update)

$$\bar{y} = (0)$$
$$p(\theta=1 | y) = \frac{p(y|\theta=1) \cdot p(\theta=1)}{p(y)}$$
$$= \frac{\left(\frac{1}{2}\right) \cdot \left(\frac{1}{5}\right)}{\frac{1}{2} \cdot \frac{1}{5} + 1 \cdot \frac{4}{5}} = \frac{1}{9} \approx 0,11$$

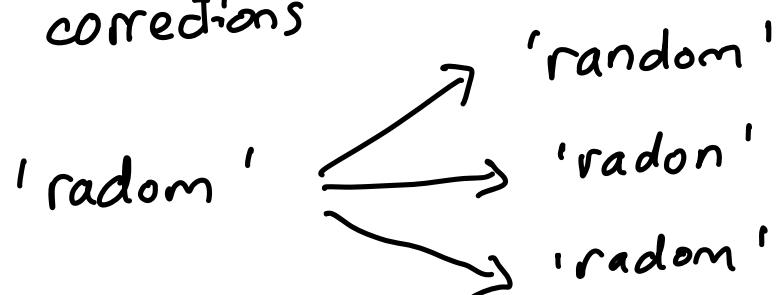
## Exchangeability

order of observations doesn't matter

Subjectivity  $\circlearrowleft$  objectivity  
=> prior

## Example from the book

Spelling corrections



data :  $y = \text{radom}$

$$p(\theta | y) \propto p(\theta) p(y | \theta)$$

Scaling can be done easily at the end

prior :

$\theta$	rel. freq	prob
random	$7.6 \cdot 10^{-5}$	
radon	$6.05 \cdot 10^{-6}$	
radom	$3.12 \cdot 10^{-7}$	

$\Rightarrow$   
rewriting

$\theta$	rel. freq.	prob
random	$760 \cdot 10^{-7}$	0.923
radon	$60.5 \cdot 10^{-7}$	0.073
radom	$3.12 \cdot 10^{-7}$	0.004

|:Likelihood

$\theta$	$p('radom'   \theta)$
random	0.00193
radon	0.000143
radom	0.975

posterior :

$\theta$	$p(\theta)$	$p('radom'   \theta)$	$p('radom'   \theta)$
random	$1.47 \cdot 10^{-7} \left( \frac{1470}{10^{10}} \right)$	$\sim 0.325$	
radon	$8.65 \cdot 10^{-10}$		$\sim 0.002$
radom	$3.04 \cdot 10^{-7} \left( \frac{3040}{10^{10}} \right)$		$\sim 0.673$

example where we don't need marginals because  
we can scale the results at the very end.

# Bayesian Computation

length in millimeters

off by -1mm or 1mm

$$\theta = 1 : y \sim N(1, 1)$$

$$\theta = -1 : y \sim N(-1, 1)$$

prior

0.5

0.5

Likelihood:

$$p(y=0.5 | \theta=1) = \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{(y-1)^2}{2}\right)} \approx 0.1405$$

marginal:

$$p(y=0.5) = p(y=0.5 | \theta=1) p(\theta=1) + p(y=0.5 | \theta=-1) p(\theta=-1) \\ \approx 0.09605$$

posterior:

$$p(\theta=1 | y=0.5) = \frac{0.07022}{0.09605} \approx 0.73$$

Stan package in R

PyStan in python

example in R : estimation of a distribution

Faulty caliper problem  
from the book

# General approach to Bayesian Computation

## Binomial and Posterior Distributions

Binary data 0, 1

Bernoulli outcomes

$$p(y|\theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$

$y$ : number of success

$\theta$ : proportion of success

$$\binom{n}{y} = \frac{n!}{y!(n-y)!}$$



Biased  $\theta = 0.75$

H = 1, T = 0

What is the probability of  
 $\overline{TTT}$ ?

$$p(y=0, n=3 | \theta = 0.75) = \binom{3}{0} 0.75^0 0.25^3 \approx \underline{0.016}$$

$$p(y=1, n=3 | \theta = 0.75) \approx \underline{0.14}$$

{TTH or THT or HTT}

example 2

$\theta$ : proportion of female birth

$y$ : number of female birth in  $n$  recorded births

$$\theta \sim U_{[0,1]}$$

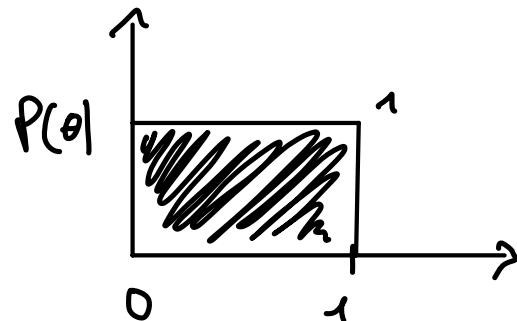
What is the posterior distribution?

Binomial likelihood.

$$p(y|\theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$

uniform prior:

$$p(\theta) = 1 \quad \text{for } \theta \in [0,1]$$



posterior:

$$p(\theta|y) \propto p(\theta) p(y|\theta)$$
$$= \binom{n}{y} \theta^y (1-\theta)^{n-y} \quad \theta \in [0,1]$$

$$p(\theta|y) \propto \theta^y (1-\theta)^{n-y}$$

once normalized

$$p(\theta|y) = \frac{\binom{n}{y}}{\alpha + \beta} \quad \alpha = y, \beta = n-y$$

Beta distribution

$\tilde{y}$  : predictions with the  $m$  next births

$$\tilde{y} \sim \text{Bin}(m, \theta)$$

so  $\theta \sim \text{Beta}$  and no longer Uniform

$$P(\tilde{y} | y)$$

$$\text{So } P(\tilde{y} | y) = \int P(\tilde{y}, \theta | y) d\theta$$

$$= \int P(\tilde{y} | \theta y) P(\theta | y) d\theta$$

$$= \left\{ \underbrace{P(\tilde{y} | \theta)}_{\text{prediction}} \underbrace{P(\theta | y)}_{\text{posterior}} \right\} d\theta$$

chain  
rule of  
probability

posterior  
predictive  
distribution

$$P(\tilde{y} | \theta) = \binom{n}{\tilde{y}} \theta^{\tilde{y}} (1-\theta)^{n-\tilde{y}}$$

$$P(\theta | y) = \text{Beta}(y+1, n-y+1)$$

$$= \frac{\Gamma(n+2)}{\Gamma(y+1) \Gamma(n-y+1)} \theta^y (1-\theta)^{n-y}$$

$$P(\tilde{y} | y) = \int \frac{\Gamma(n+2)}{\Gamma(y+1) \Gamma(n-y+1)} \binom{m}{\tilde{y}} \theta^{y+\tilde{y}} (1-\theta)^{n+m-(\tilde{y}+y)}$$

$$= \binom{m}{\tilde{y}} \frac{\Gamma(n+2)}{\Gamma(y+1) \Gamma(n-y+1)} \frac{\Gamma(y+\tilde{y}+1) \Gamma(m+n-\tilde{y}-y+1)}{\Gamma(m+n+2)}$$

$\cdot \int \frac{\Gamma(m+n+2)}{\Gamma(y+\tilde{y}+1) \Gamma(m+n-\tilde{y}+y+1)} \theta^{y+\tilde{y}} (1-\theta)^{m+n-(y+\tilde{y})} d\theta$   
= 1  
Beta distribution

$$= \binom{m}{\tilde{y}} \frac{\Gamma(n+2)}{\Gamma(y+1) \Gamma(n-y+1)} \frac{\Gamma(y+\tilde{y}+1) \Gamma(m+n-\tilde{y}-y+1)}{\Gamma(m+n+2)}$$

$$= \binom{m}{\tilde{y}} \frac{(n+1)! (y+\tilde{y})! (m+n-\tilde{y}+y)!}{y! (n-y)! \underbrace{(m+n+1)!}_{(m+n+1)(m+n)!}}$$

$$= \binom{m}{\tilde{y}} \frac{\frac{n+1}{m+n+1} \frac{n!}{y!(n-y)!}}{\frac{(y+\tilde{y})! (m+n-(y+\tilde{y}))!}{(m+n)!}}$$

$$= \binom{m}{\tilde{y}} \binom{n}{y} \left[ \frac{m+n}{y+\tilde{y}} \right]^{-1} \frac{n+1}{m+n+1}$$

$n$  = # of data points

$y$  = # of successes

$m$  = # of predicted points

$\tilde{y}$  = # of predicted successes

prediction : next birth is female ?

$$m = \tilde{y} = 1$$

$$\begin{aligned} p(\tilde{y}=1, m=1 | y) &= 1 \cdot \binom{n}{y} \left[ \frac{n+1}{y+1} \right]^{-1} \frac{n+1}{n+2} \\ &= \frac{n!}{y!(n-y)!} \cdot \frac{(y+1)!(n-y)!}{(n+1)!} \frac{n+1}{n+2} \\ &\approx \frac{y+1}{n+2} \end{aligned}$$

$\frac{y}{n}$  : data

$\frac{1}{2}$  : mean of our posterior

Beta( $\alpha, \beta$ ) have a mean of  $\frac{\alpha}{\alpha+\beta}$

$\Rightarrow$  What is the mean of the posterior distribution?

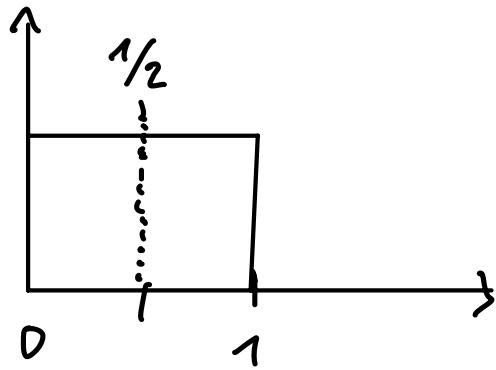
$$p(\theta | y) = \text{Beta}(y+1, n-y+1)$$

$$\mathbb{E}[\theta | y] = \frac{y+1}{n-y+1+y+1} = \frac{y+1}{n+2}$$

If  $n$  is close to 0, then the average is more influenced by the average of the posterior distribution.

As  $n$  gets large,

$$\mathbb{E}[\theta|y] \sim \frac{y}{n}$$



Given  $y$ , what is the expected value of  $\theta$ ,  
 $\mathbb{E}[\theta|y]$  in this problem?

$$\mathbb{E}[\theta|y] = \frac{y+1}{n+2}$$

If data is large

$$\mathbb{E}[\theta|y] = \frac{y}{n}$$

$$\text{var}(\theta) = \mathbb{E}[\text{var}(\theta|y)] + \text{var}(\mathbb{E}[\theta|y])$$

we observe that

$$\mathbb{E}[\text{var}(\theta|y)] < \text{var}(\theta)$$

## Priors

Reflect initial information.

# Informative vs Noninformative priors

The prior should encompass all possible values

Example:

defact rate : assumed 5% with variance 0.25%.

What could be a good informative prior?

$\Rightarrow$  defective       $\Rightarrow$  likelihood is Binomial  
 $\Rightarrow$  non defective

conjugate pair :

$$\theta \sim \text{Beta}(a, b) \quad \mu = \frac{a}{a+b}$$

$$\sigma^2 = \frac{ab}{(a+b)^2(a+b+1)}$$

## Solution :

$$\frac{a}{a+b} = 0.05 \quad \Rightarrow \quad 19a = b$$

$$0.0025 = \frac{ab}{(a+b)^2(a+b+1)}$$

$$= \frac{19}{400(20a+1)}$$

$$\Rightarrow a = \frac{18}{20} = 0.9$$

$$b = 17.1$$

So the prior is  $\theta \sim B(0.9, 17.1)$

Binomial distribution

$$p(y|\theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$

uniform prior  $p(\theta) = 1$  for  $\theta \in [0,1]$

informative prior  $Beta(a, b)$

$$p(\theta|y) \propto \underbrace{\theta^y (1-\theta)^{n-y}}_{\text{Likelihood}} \underbrace{\theta^{a-1} (1-\theta)^{b-1}}_{\text{prior}}$$

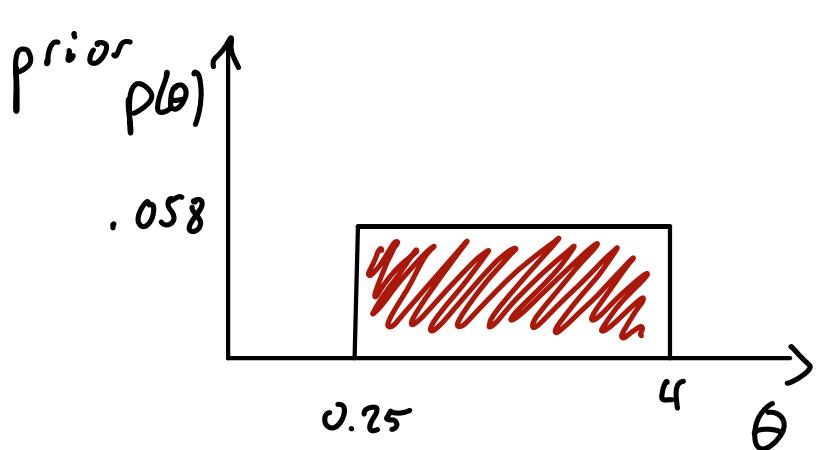
$$= \theta^{y+a-1} (1-\theta)^{n-y+b-1}$$

$$\theta|y \sim Beta(y+a, n-y+b)$$

$\Rightarrow$  conjugacy

## Nonconjugate prior distributions

Normal likelihood and Uniform prior



Uniform

$$p(\theta) = \begin{cases} 0.308 & \text{if } \theta \in [0.25, 4] \\ 0 & \text{otherwise} \end{cases}$$

posterior

$$p(\theta|y) \propto p(\theta) p(y|\theta)$$

$$\propto \begin{cases} \exp\left(-\frac{(y-\mu)^2}{2}\right) & \theta \in [0.25, 4] \\ 0 & \text{otherwise} \end{cases}$$

Truncated Normal distribution (non conjugate)

Weakly informative priors

## Quizz

$$(2) \quad p(\theta) \sim B(2, 2)$$

$$p(y|\theta) \sim \text{Bin}(10, \gamma_2)$$

$$y = 3H$$

$$y = 3 \quad n = 10 \quad a = 2, b = 2$$

$$\begin{aligned} p(\theta|y) &\sim \text{Beta}(a+y, n-y+b) \\ &\sim \text{Beta}(5, 9) \end{aligned}$$

$$(1) \quad p(\theta) \sim B(2, 2)$$

$$y < 3, n = 10$$

$$p(y|\theta) \sim \text{Bin}(n, K)$$

$$p(\theta|y) \sim \theta(1-\theta)^9 (1+8\theta+36\theta^2)$$

Other single-parameter models

example :

$$\text{Likelihood } p(y|\theta) \propto e^{-(y-\theta)^2/2\sigma^2}$$

$$\text{prior: } p(\theta) \propto e^{A\theta^2 + B\theta + C}$$

aim: show that this prior is a normal distribution

$$A\theta^2 + B\theta + C = A(\theta + \frac{B}{2A})^2 - \frac{B^2}{4A^2} + C$$

$$p(\theta) \propto e^{A\theta^2 + B\theta + C} \propto e^{A(\theta + B/2A)^2}$$

$$\text{let } A = -1/2\tau_0^2 \quad -\frac{B}{2A} = \mu_0$$

$$p(\theta) \propto e^{-\frac{1}{2}\tau_0^2(\theta - \mu_0)^2}$$

$$\theta \sim N(\mu_0, \tau_0^2)$$

prior

aim: show that the posterior is also normal

$$\begin{aligned}
 p(\theta | y) &\propto p(\theta) p(y | \theta) \\
 &= e^{-\frac{1}{2}\tau_0^2(\theta - \mu_0)^2} e^{-(y - \theta)^2 / 2\sigma^2} \\
 &= \exp \left( -\frac{1}{2} \left( \frac{(y - \theta)^2}{\sigma^2} + \frac{(\theta - \mu_0)^2}{\tau_0^2} \right) \right) \\
 &= \exp \left( -\frac{1}{2} \left( \frac{1}{\sigma^2} (y^2 - 2y\theta + \theta^2) + \frac{1}{\tau_0^2} (\theta^2 - 2\theta\mu_0 + \mu_0^2) \right) \right)
 \end{aligned}$$

$$\propto \exp\left(-\frac{1}{2}\left(\frac{1}{\sigma^2} + \frac{1}{\tau_0^2}\right)\theta^2 - 2\left(\frac{\gamma/\sigma^2 + \mu_0/\tau_0^2}{1/\sigma^2 + \gamma\tau_0^2}\right)\theta\right)$$

$$= \exp\left(-\frac{1}{2}\left(\frac{1}{\sigma^2} + \frac{1}{\tau_0^2}\right)\left(\theta^2 - 2\left(\frac{\gamma/\sigma^2 + \mu_0/\tau_0^2}{1/\sigma^2 + \gamma\tau_0^2}\right)\theta\right)\right)$$

$$\frac{1}{\tau_1^2} = \underbrace{\frac{1}{\sigma^2}}_{\text{likelihood}} + \underbrace{\frac{1}{\tau_0^2}}_{\text{prior}} \quad \mu_1 = \frac{\gamma/\sigma^2 + \mu_0/\tau_0^2}{1/\sigma^2 + \gamma\tau_0^2}$$

$$\propto \exp\left(-\frac{1}{2\tau_1^2} (\theta - \mu_1)^2\right)$$

$$\theta | y \sim N(\mu_1, \tau_1^2) \quad \text{posterior}$$

The inverse variance is called precision.

Now, for new data points  $\tilde{y}$

$$p(\tilde{y}|y) = \int p(\tilde{y}|\theta) p(\theta|y) d\theta$$

$$\propto \int \exp\left(-\frac{(\tilde{y}-\theta)^2}{2\sigma^2}\right) \exp\left(-\frac{(\theta-\mu_0)^2}{2\tau_1^2}\right) d\theta$$

$$\propto \exp(A\tilde{y} + B\tilde{y} + C)$$

$$\Rightarrow \tilde{y}|y \sim N(\cdot, \cdot)$$

$$\begin{aligned} E[\tilde{y}|y] &= E[E[\tilde{y}|\theta, y]|y] \\ &= E[E[\tilde{y}|\theta]|y] \\ &= E[\theta|y] \\ &= \mu_1 \end{aligned}$$

$$\begin{aligned} \text{var}(\tilde{y}|y) &= E[\text{var}(\tilde{y}|\theta)|y] + \text{var}(E[\tilde{y}|\theta]|y) \\ &= E[\sigma^2|y] + \text{var}(\theta|y) \\ &= \bar{v}^2 + \bar{\tau}_\theta^2 \end{aligned}$$

$$\tilde{y}|y \sim N(\mu_1, \bar{v}^2 + \bar{\tau}_\theta^2)$$

How do we handle multiple data points?

Prior:  $p(\theta) \propto \exp\left(-\frac{1}{2\bar{\tau}_0^2} (\theta - \mu_0)^2\right)$

1: Likelihood :

$$p(y|\theta) \propto \prod_{i=1}^n \exp\left(-\frac{(y_i - \theta)^2}{2\sigma^2}\right)$$
$$= \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \theta)^2\right)$$

posterior :

$$p(\theta|y) \propto p(y|\theta) p(\theta)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \theta)^2\right)$$
$$-\frac{1}{2\tau_0^2} (\theta - \mu_0)^2$$

$$= \exp\left(-\frac{1}{2\sigma^2} \left( \sum_{i=1}^n y_i^2 - 2\theta \sum_{i=1}^n y_i + n\theta^2 \right)\right)$$

$$-\frac{1}{2\tau_0^2} (\theta^2 - 2\theta\mu_0 + \mu_0^2)$$

Let  $\bar{y}$  be the sample mean, so  $n\bar{y} = \sum_{i=1}^n y_i$

$$= \exp\left(\theta^2 \left(-\frac{n}{2\sigma^2} - \frac{1}{2\tau_0^2}\right) - 2\theta \left(\frac{n\bar{y}}{\sigma^2} + \frac{1}{\tau_0^2}\right)\right)$$

$$\tau_n^2 = \left(\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}\right)^{-1}$$

$$= \exp\left(-\frac{1}{2\gamma_n^2} \left(\theta^2 - 2\tau_n^2 \left(\frac{n\bar{y}}{\sigma^2} + \frac{1}{\tau_0^2}\right)\theta\right)\right)$$

$$\mu_n = \tau_n^2 \left(\frac{n\bar{y}}{\sigma^2} + \frac{1}{\tau_0^2}\right)$$

$$\propto \exp\left(-\frac{1}{2\tau_n^2} (\theta - \mu_n)^2\right)$$

$$\Rightarrow \theta | y \sim N(\mu_n, \gamma_n^2)$$

example : screws of 5 mm with variations  
mean ? variance :  $\sigma^2 = 0.5$

Data :  $y = \{5.1, 4.9, 5.2, 5.3, 5\}$

$$n = 5 \quad \tau_0^2 = 1$$

$$\bar{y} = 5.1 \quad \mu_0 = 5$$

$$\sigma^2 = 0.5$$

$$\tau_n^2 = \left(\frac{1}{1} + \frac{5}{0.5}\right)^{-1} = \frac{1}{11} \approx 0.11$$

$$\mu_n = \frac{1}{11} \left(\frac{25.5}{0.5} + \frac{5}{1}\right) = \frac{56}{11} \approx 5.1$$

$$\theta | y \sim N(5.1, 0.1)$$

easy to compute with conjugacy

Normal distribution with unknown variance

But mean is Known

Review : Scaled inverse  $\chi^2$  distribution

$$\theta \sim \text{Inv} \chi^2(v, s^2)$$

↑                   ↑  
 deg. of freedom   scale

$$p(\theta) = \frac{(v/2)^{v/2}}{\Gamma(v/2)} s^v \theta^{-(v/2+1)} e^{-\frac{vs^2}{2\theta}}$$

$$\propto \theta^{-(v/2+1)} e^{-\frac{vs^2}{2\theta}}$$

Likelihood :

$$p(y|\sigma^2) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y_i - \theta)^2/2\sigma^2}$$

$$\propto (\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \theta)^2\right)$$

Define :  $V := \frac{1}{n} \sum_{i=1}^n (y_i - \theta)^2$  sample variance

$$p(y | \sigma^2) \propto (\sigma^2)^{-n/2} \exp\left(-\frac{nV}{2\sigma^2}\right)$$

$$\text{prior : } p(\sigma^2) \propto (\sigma^2)^{-\left(\frac{V_0}{2} + 1\right)} \exp\left(-\frac{V_0 \sigma^2}{2} \cdot \frac{1}{\sigma^2}\right)$$

posterior :

$$\begin{aligned} p(\sigma^2 | y) &\propto p(\sigma^2) p(y | \sigma^2) \\ &\propto (\sigma^2)^{-\left(\frac{V_0}{2} + 1\right)} \exp\left(-\frac{V_0 \sigma^2}{2} \cdot \frac{1}{\sigma^2}\right) \\ &\quad \cdot (\sigma^2)^{-n/2} \exp\left(-\frac{nV}{2\sigma^2}\right) \\ &= (\sigma^2)^{-\frac{1}{2}(V_0 + n + 2)} \exp\left(-\frac{1}{2\sigma^2} (V_0 \sigma^2 + nV)\right) \end{aligned}$$

$$\sigma^2 \sim \text{Inv} \chi^2(V_0 + n, \frac{V_0 \sigma^2 + nV}{V_0})$$

Poisson distributions

$\theta$  : average number of events

$y$  : actual observations

review : Gamma( $a, b$ )       $a$  : shape  
 $b$  : inverse scale

mean :  $a/b$

variance :  $a/b^2$

$$\rho(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}$$

$$\propto \theta^{a-1} e^{-b\theta}$$

poisson likelihood

$$p(y|\theta) = \prod_{i=1}^n \frac{1}{y_i!} \theta^{y_i} e^{-\theta} \propto \theta^{\bar{y}} e^{-n\theta}$$

prior :

$$\rho(\theta) \propto e^{-b_0\theta} \theta^{a_0-1}$$

posterior :  $\rho(\theta|y) \propto \rho(\theta) p(y|\theta)$

$$\propto \theta^{\bar{y} + a_0 - 1} e^{-n\theta - b_0\theta}$$

$$\theta|y \sim \text{Gamma}(a_0 + \bar{y}, b_0 + n)$$

Example:  $3/200,000$  died of asthma (1 year)  
world:  $0.6/100,000$  prior ↑ data

prior: mean of  $0.6 \Rightarrow \frac{a}{b} = 0.6$

suppose:  $a = 3, b = 5$

posterior:  $\text{Gamma}(a_0 + y, b_0 + x) = \text{Gamma}(6, 7)$

mean:  $6/7 \approx 0.86$  variance:  $6/7^2$

## Exponential distributions

$$p(y|\theta) = \theta^n e^{-n\bar{y}\theta}$$

waiting time or time between events  $\theta$ .

Conjugate prior: Gamma.

## Multiparameter models

## Nuisance parameters

example

$$y \sim N(\mu, \sigma^2)$$

Jeffrey's prior (noninformative) for  $\mu$  and  $\sigma^2$

Jeffrey's invariance principle

$$p(\theta) \propto [\bar{J}(\theta)]^{1/2} \quad \text{where}$$

$$\bar{J}(\theta) = -\in E \left[ \frac{\partial^2 \log p(y|\theta)}{\partial \theta^2} \mid \theta \right]$$

$$\theta = (\mu, \sigma^2)$$

$$L := p(y|\theta) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{1}{2\sigma^2}(y-\mu)^2\right)$$

$$\log(L) = -\frac{1}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(y-\mu)^2$$

$$= -\frac{1}{2} \log(2\pi) - \log(\sigma) - \frac{1}{2\sigma^2}(y-\mu)^2$$

$$\frac{\partial \log(L)}{\partial \mu} = \frac{1}{\sigma^2}(y-\mu)$$

$$\frac{\partial \log(L)}{\partial \sigma} = -\frac{1}{\sigma} + \frac{1}{\sigma^3}(y-\mu)^2$$

$$\frac{\partial^2 \log(\zeta)}{\partial \mu^2} = -\frac{1}{\sigma^2}$$

$$\frac{\partial \log(\zeta)}{\partial \mu \partial \sigma} = -\frac{2(y-\mu)}{\sigma^3}$$

$$\frac{\partial \log(\zeta)}{(\partial \sigma)^2} = \frac{1}{\sigma^2} - \frac{3}{\sigma^4} (y-\mu)^2$$

$$\begin{aligned} J(\theta) &= -E \left[ \det \begin{vmatrix} -\frac{1}{\sigma^2} & -\frac{2(y-\mu)}{\sigma^3} \\ -\frac{2(y-\mu)}{\sigma^3} & \frac{1}{\sigma^2} - \frac{3}{\sigma^4} (y-\mu)^2 \end{vmatrix} \right]^{1/2} \\ &= \left( E \left[ -\frac{1}{\sigma^4} + \frac{3}{\sigma^6} (y-\mu)^2 - \frac{4(y-\mu)^2}{\sigma^6} \right] \right)^{1/2} \end{aligned}$$

remember  $E[(y-\mu)^2] = \sigma^2$  so,

$$= \left( \frac{1}{\sigma^4} + \frac{1}{\sigma^4} \right)^{1/2}$$

$$= \left( \frac{2}{\sigma^4} \right)$$

$$= \frac{\sqrt{2}}{\sigma^2} \propto \frac{1}{\sigma^2}$$

$$p(\theta) \propto \frac{1}{\sigma^2}$$

Noninformative prior

posterior:

$$p(\theta | y) \propto p(y|\theta) p(\theta) \quad \theta = (\mu, \sigma^2)$$

$$\propto (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2\right) \cdot \frac{1}{\sigma^2}$$

$$\propto (\sigma^2)^{-n/2-1} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2\right)$$

$$\sum_{i=1}^n (y_i - \mu)^2 = \sum_{i=1}^n [(y_i - \bar{y}) + (\bar{y} - \mu)]^2$$

$$= \sum_{i=1}^n \left[ (y_i - \bar{y})^2 + 2(y_i - \bar{y})(\bar{y} - \mu) + (\bar{y} - \mu)^2 \right]$$

$$= \sum_{i=1}^n (y_i - \bar{y})^2 + 2(\bar{y} - \mu) \sum_{i=1}^n (y_i - \bar{y}) \cancel{+ (\bar{y} - \mu)^2} = 0$$

$$+ n(\bar{y} - \mu)^2$$

$$\Rightarrow p(\mu, \sigma^2 | y) \propto (\sigma^2)^{-n/2-1} \exp\left(-\frac{1}{2\sigma^2} \left( \sum_{i=1}^n (y_i - \bar{y})^2 + n(\bar{y} - \mu)^2 \right)\right)$$

Sample  
variance

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (y_i - \bar{y})^2$$

joint posterior:

$$p(\mu, \sigma^2 | y) \propto (\sigma^2)^{-\frac{n}{2}-1} \exp\left(-\frac{1}{2\sigma^2} \left(S^2(n-1) + n(\bar{y}-\mu)^2\right)\right)$$

We look at the conditional distribution of  $\mu$

$$p(\mu | \sigma^2, y) \propto \exp\left(-\frac{n}{2\sigma^2} (\mu - \bar{y})^2\right)$$

Normalized:

$$p(\mu | \sigma^2, y) = \sqrt{\frac{n}{2\pi\sigma^2}} \exp\left(-\frac{n}{2\sigma^2} (\mu - \bar{y})^2\right)$$

$$\mu | \sigma^2, y \sim N(\bar{y}, \frac{\sigma^2}{n}) \quad \text{conditional posterior}$$

$\Rightarrow$  marginal posterior:

$$\begin{aligned} p(\sigma^2 | y) &\propto \int_{-\infty}^{\infty} p(\mu, \sigma^2 | y) d\mu \\ &\propto \int_{-\infty}^{\infty} \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \left((n-1)S^2 + n(\bar{y}-\mu)^2\right)\right) d\mu \\ &\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} (n-1)S^2\right) \cdot \int_{-\infty}^{\infty} \exp\left(-\frac{n}{2\sigma^2} (\bar{y}-\mu)^2\right) d\mu \\ &= \sqrt{\frac{2\pi\sigma^2}{n}} \end{aligned}$$

$$p(\sigma^2 | y) \propto \sqrt{\frac{2\pi\sigma^2}{n}} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right)$$

$$\propto \sigma^{-n-1} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right) \xrightarrow{\text{Inverse Gamma distribution}}$$

$$a = \frac{n}{2} - 1, b = \frac{(n-1)s^2}{2}$$

## Normal data summary

- $y_i \sim N(\mu, \sigma^2)$ , with  $\mu, \sigma^2$  unknown
- Noninformative Jeffrey's prior  $p(\mu, \sigma^2) \propto 1/\sigma^2$
- joint posterior:  

$$p(\mu, \sigma^2 | y) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} ((n-1)s^2 + n(\bar{y}-\mu)^2)\right)$$
- conditional posterior:  

$$\mu | \sigma^2, y \sim N(\bar{y}, \frac{\sigma^2}{n})$$
- marginal distribution:  

$$\sigma^2 | y \sim \text{Inv. Gamma}\left(\frac{n}{2} - 1, \frac{(n-1)s^2}{2}\right)$$

## Posterior predictive distribution

$$p(\tilde{y}|y) = \int \int p(\tilde{y}|\mu, \sigma^2) p(\mu, \sigma^2|y) d\mu d\sigma^2$$

joint posterior:

$$p(\mu, \sigma^2|y) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} ((n-1)s^2 + n(\bar{y}-\mu)^2)\right)$$

Likelihood:

$$\tilde{y}|\mu, \sigma^2 \sim N(\mu, \sigma^2) \quad \text{so...}$$

$$\begin{aligned} p(\tilde{y}|y) &= \int_0^\infty \int_{-\infty}^\infty \sigma^{-1} \exp\left(-\frac{1}{2\sigma^2} (\tilde{y}-\mu)^2\right) \\ &\quad \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} ((n-1)s^2 + n(\bar{y}-\mu)^2)\right) \\ &= \int_0^\infty \sigma^{-n-3} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right) \int_{-\infty}^\infty \exp\left(-\frac{1}{2\sigma^2} \left((\tilde{y}-\mu)^2 + n(\bar{y}-\mu)^2\right)\right) d\mu d\sigma^2 \end{aligned}$$

$$(\tilde{y}-\mu)^2 + n(\bar{y}-\mu)^2 = \tilde{y}^2 - 2\tilde{y}\mu + \mu^2 + n\bar{y}^2 - 2n\bar{y}\mu + n\mu^2$$

$$= (n+1)\mu^2 - 2\mu(\tilde{y} + n\bar{y}) + \tilde{y}^2 + n\bar{y}^2$$

$$= (n+1)\mu^2 - 2\mu(\tilde{y} + n\bar{y}) + \frac{(n+1)\tilde{y}^2}{(n+1)} + \frac{(n+1)n\bar{y}^2}{n+1}$$

$$= (n+1)\mu^2 - 2\mu(\tilde{y} + n\bar{y}) + \frac{(n+1)\tilde{y}^2 + (n+1)n\bar{y}^2 + 2n\bar{y}\tilde{y} - 2n\bar{y}\tilde{y}}{(n+1)}$$

$$= (n+1)\mu^2 - 2\mu(\tilde{y} + n\bar{y}) + \frac{\tilde{y}^2 + 2n\bar{y}\tilde{y} + n^2\bar{y}^2 + n\bar{y}^2 - 2n\bar{y}\tilde{y} + n\tilde{y}^2}{(n+1)}$$

$$= (n+1)\mu^2 - 2\mu(\tilde{y} + n\bar{y}) + \frac{(\tilde{y} + n\bar{y})^2 + n(\bar{y} - \tilde{y})}{(n+1)}$$

$$= (n+1) \left[ \mu^2 - \frac{2\mu(\tilde{y} + n\bar{y})}{n+1} + \frac{(\tilde{y} + n\bar{y})^2}{(n+1)^2} \right] + \frac{n(\bar{y} - \tilde{y})^2}{n+1}$$

$$= (n+1) \left( \mu - \frac{(\tilde{y} + n\bar{y})^2}{n+1} + \frac{n(\bar{y} - \tilde{y})^2}{n+1} \right)$$

$$= \int_0^\infty \tau^{-n-3} \exp\left(-\frac{(n-1)\sigma^2}{2\tau^2}\right) \int_{-\infty}^\infty \exp\left(-\frac{1}{2\tau^2} \left( \underbrace{(n+1)\left(\mu - \frac{(\tilde{y} + n\bar{y})^2}{n+1} + \frac{n(\bar{y} - \tilde{y})^2}{n+1}\right)}_{*} \right) \right) d\mu d\sigma^2$$

↑  
constant w.r.t  $\mu$ .

$$= \int_0^\infty \sigma^{-n-3} \exp\left(-\frac{(n-1)s^2}{2\sigma^2} - \left(\frac{n}{n+1}\right) \frac{(\bar{y}-\tilde{y})^2}{2\sigma^2}\right) \cdot$$

$$\underbrace{\int_{-\infty}^\infty \exp\left(-\frac{n+1}{2\sigma^2} (\mu - *)\right) d\mu}_{=} \sqrt{\frac{2\pi\sigma^2}{n+1}}$$

$$= \int_0^\infty \sigma^{-n-2} \sqrt{\frac{2\pi\sigma^2}{n+1}} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right) \exp\left(-\frac{1}{2\sigma^2(n+1)}(\bar{y}-\tilde{y})^2\right) d\sigma^2$$

$$= \sqrt{\frac{2\pi\sigma^2}{n+1}} \int_0^\infty (\sigma^2)^{-\frac{n}{2}-1} \exp\left(-\frac{1}{2\sigma^2} (s^2(n-1) + \frac{n}{n+1}(\bar{y}-\tilde{y})^2)\right) d\sigma^2$$

Gamma distribution

$$\int_0^\infty x^{a-1} e^{-bx} dx = \Gamma(a) / b^a$$

$$u = 1/\sigma^2 \Rightarrow \sigma^2 = \frac{1}{u} \Rightarrow d\sigma^2 = -\frac{1}{u^2} du$$

$$= \sqrt{\frac{2\pi\sigma^2}{n+1}} \int_0^\infty u^{\frac{n}{2}+1} \exp\left(-\frac{1}{2}u(s^2(n-1) + \frac{n}{n+1}(\bar{y}-\tilde{y})^2)\right) -\frac{1}{u^2} du$$

$$= -\sqrt{\frac{2\pi\sigma^2}{n+1}} \int_0^\infty u^{n/2-1} \exp\left(-\frac{1}{2}u\left(s^2(n-1) + \frac{n}{n+1}(\bar{y}-\tilde{y})^2\right)\right) du$$

$$a = n/2, \quad b = \frac{1}{2}(s^2(n-1) + \frac{n}{n+1}(\bar{y}-\tilde{y})^2)$$

$$\tilde{y}|y \propto -\sqrt{\frac{2\pi\sigma^2}{n+1}} \frac{\Gamma(n/2)}{\left(\frac{1}{2}(s^2(n-1) + \frac{n}{n+1}(\bar{y}-\tilde{y})^2)\right)^{n/2}}$$

From here, we can show that  $\tilde{y}|y$  follows

a t distribution

$$\tilde{y}|y \sim t_{n-1}(\bar{y}, \left(\frac{n+1}{n}\right)s^2)$$

Reminder t distribution:

$$t \sim \frac{\Gamma(n/2)}{\Gamma((n-1)/2) \sqrt{(n-1)\pi\sigma^2}} \left(1 - \frac{1}{n-1} \left(\frac{\bar{y}-\tilde{y}}{s\sqrt{\frac{n+1}{n}}}\right)^2\right)^{-\frac{n}{2}}$$

$$\tilde{y}|y \propto -\sqrt{\frac{2\pi}{n+1}} \Gamma(n/2) \left(\frac{1}{2}(s^2(n-1) + \frac{n}{n+1}(\bar{y}-\tilde{y})^2)\right)^{n/2}$$

...

example: Gravity

number of measurements : 70

mean : 6.5

s.d : 2.4

Sampling from the posterior.

### Conjugate priors

Normal model with unknown mean and variance.

Multinomial model for categorical data

example: ice cream

categorical

$y = (450, 300, 250)$  data

noninformative prior : dirichlet  $\alpha = 1, 1, 1$

posterior : dirichlet .  $\alpha = (451, 301, 251)$

answer : simulation : draws and compute

$$\theta_{\text{vanilla}} - \theta_{\text{chocolate}} \sim 100\%.$$

sampling distribution for multivariate normal

$$y_1, \dots, y_n \sim N(\mu, \Sigma)$$

$$\begin{aligned} p(y_1, \dots, y_n | \mu, \Sigma) &\propto \det(\Sigma)^{-n/2} \exp\left(-\frac{1}{2}(y-\mu)\Sigma^{-1}(y-\mu)^T\right) \\ &= \det(\Sigma)^{-n/2} \exp\left(-\frac{1}{2} \text{tr}(\Sigma^{-1}S_0)\right) \end{aligned}$$

$$\text{where } S_0 = (y-\mu)(y-\mu)^T$$

likelihood

example: pen

$$\mu_0 = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad \Lambda_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{no covariance, independence}$$

$$K_0 = 1, \quad V_0 = 0$$

$$y = \left\{ \begin{pmatrix} 1.8 \\ 4.8 \end{pmatrix}, \begin{pmatrix} 2.1 \\ 5.1 \end{pmatrix}, \begin{pmatrix} 2.0 \\ 5.3 \end{pmatrix}, \begin{pmatrix} 1.9 \\ 4.9 \end{pmatrix} \right\} \quad \bar{y} = \begin{pmatrix} 1.95 \\ 5.025 \end{pmatrix}$$

$$S = \sum_{i=1}^n (y_i - \bar{y})(y_i - \bar{y})^T \approx \begin{pmatrix} 0.0167 & 0.2167 \\ 0.02167 & 0.049167 \end{pmatrix}$$

posterior:

$$\begin{aligned}
 \mu_n &= \frac{K_0}{K_0 + n} \mu_0 + \frac{n}{K_0 + n} \bar{y} \\
 &= \frac{1}{1+4} \left( \begin{matrix} 2 \\ 5 \end{matrix} \right) + \left( \frac{4}{1+4} \right) \left( \begin{matrix} 1.95 \\ 5.025 \end{matrix} \right) \\
 &= \left( \begin{matrix} 1.96 \\ 5.02 \end{matrix} \right) \quad \text{posterior mean}
 \end{aligned}$$

$$\begin{aligned}
 K_n &= 5, V_n = 4 \\
 \Lambda_n &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 5 + \frac{4}{5+4} \left( \left( \begin{matrix} 1.95 \\ 5.025 \end{matrix} \right) - \left( \begin{matrix} 2 \\ 5 \end{matrix} \right) \right) \cdot \left( \left( \begin{matrix} 1.95 \\ 5.025 \end{matrix} \right) - \left( \begin{matrix} 2 \\ 5 \end{matrix} \right) \right)^T \\
 &\approx \begin{pmatrix} 1.02 & 0.217 \\ 0.022 & 1.05 \end{pmatrix} \quad \text{posterior covariance matrix}
 \end{aligned}$$

## Bayesian models and applications

Bayesian approach to Logistic Regression

Likelihood:

$$p(y|\beta) = \exp \left( y^T X \beta - n \sum_{i=1}^K \log \left( 1 + e^{x_i^T \beta} \right) \right)$$

noninformative prior:

$$p(\beta) \propto 1$$

posterior:

$$p(\beta | y) \propto p(\beta) p(y | \beta)$$
$$\propto \exp\left(y^\top X\beta - n \sum_{i=1}^K \log(1 + e^{X_i^\top \beta})\right)$$

maximum likelihood estimation

We use MCMC to sample from the posterior

example: Bioassay analysis

comparing two proportions

$$y_1 \sim \text{Bin}(n, \theta_1)$$

$$y_2 \sim \text{Bin}(n, \theta_2)$$

$$H_0 : \theta_1 = \theta_2 \quad \text{vs} \quad H_A : \theta_1 > \theta_2$$

$$\text{Define: } z_1 = \log \frac{\theta_1}{1-\theta_1} \quad \text{and} \quad z_2 = \log \frac{\theta_2}{1-\theta_2}$$

assume

$$z_2 | z_1 \sim N(z_1, \sigma^2)$$

$$\text{and } p(z_1) \propto 1 \quad (\text{noninformative})$$

$$\text{let } v = \frac{1}{\sigma}(z_2 - z_1), \quad v \sim N(0, 1)$$

Improper joint prior:

$$p(z_1, z_2) \propto p(z_2 | z_1) p(z_1) \\ \propto \frac{1}{\sigma} \exp \left( \frac{-(z_2(\theta_2) - z_1(\theta_1))^2}{2\sigma^2} \right)$$

variable transformation trick

$$\det \begin{pmatrix} \frac{dz_1}{d\theta_1} & 0 \\ 0 & \frac{dz_2}{d\theta_2} \end{pmatrix} \dots$$

$$p(\theta_1, \theta_2) \propto \exp \left( -\frac{1}{2} v^2 \right) \theta_1^{\alpha-1} (1-\theta_1)^{\beta-1} \theta_2^{\gamma-1} (1-\theta_2)^{\delta-1} \\ \propto \exp \left( -\frac{1}{2} v^2 \right) \theta_1^{\alpha-1} (1-\theta_1)^{\beta-1} \theta_1^{\gamma-1} (1-\theta_2)^{\delta-1}$$

Howard prior w/ parameters  $\alpha, \beta, \gamma, \delta, \sigma$

The posterior distribution is:

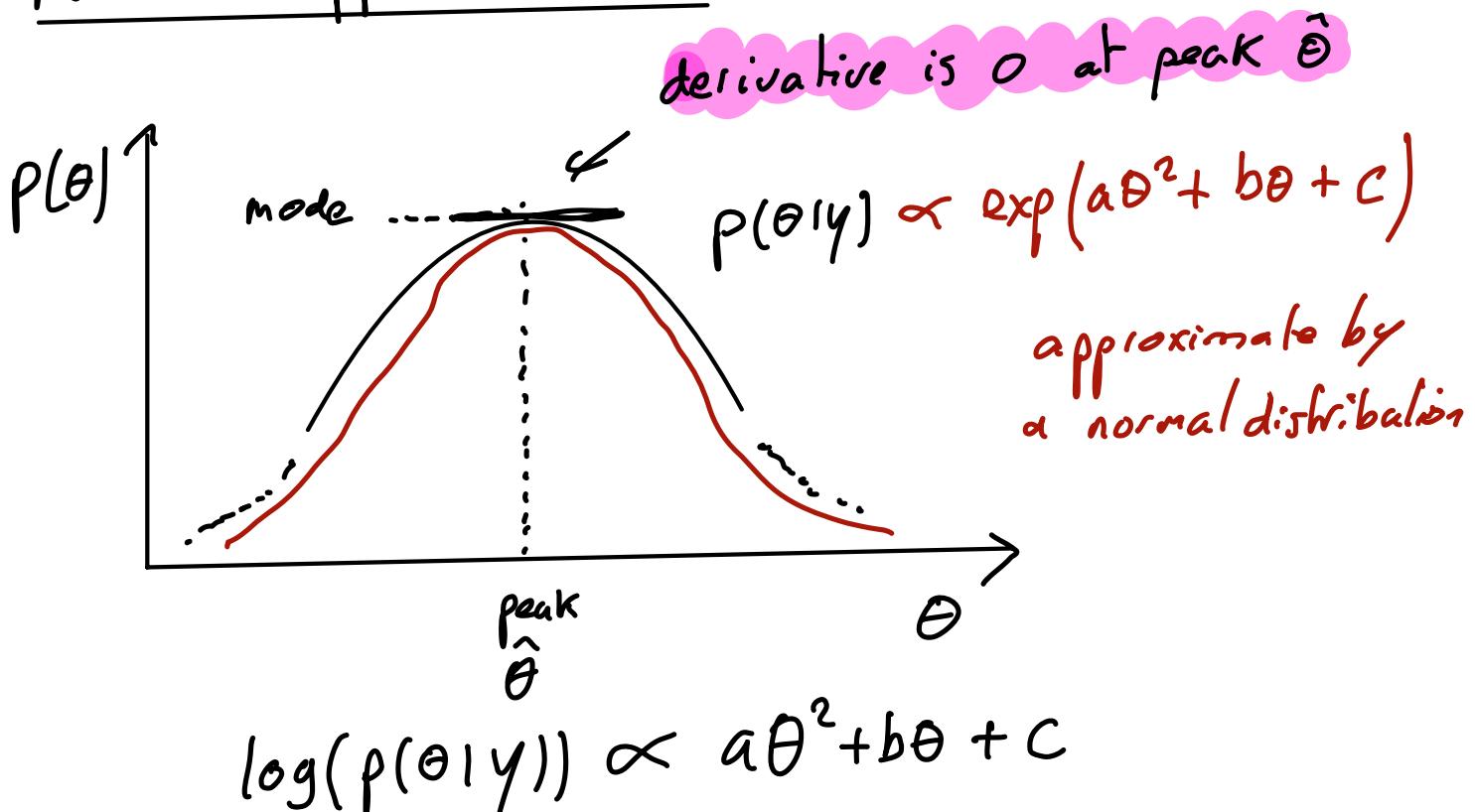
$$p(\theta_1, \theta_2 | y) \propto \theta_1^{\gamma_1} (1-\theta_1)^{n_1-y_1} \theta_2^{\gamma_2} (1-\theta_2)^{n_2-y_2} \\ \times \exp \left( -\frac{1}{2} v^2 \right) \theta_1^{\alpha-1} (1-\theta_1)^{\beta-1} \theta_1^{\gamma-1} (1-\theta_2)^{\delta-1}$$

$$\propto \exp\left(-\frac{1}{2}V^2\right) \theta_1^{y_1+\alpha-1} (1-\theta_1)^{n_1-y_1+\beta-1} \\ \times \theta_2^{y_2+V-1} (1-\theta_2)^{n_2-y_2+\delta-1}$$

$$\text{Howard}(y_1+\alpha, n_1-y_1+\beta, y_2-V, n_2-y_2+\delta)$$

Strategy for computation

Normal approximation



log posterior is roughly quadratic.

$$\frac{\partial}{\partial \theta} \log(p(\theta|y))_{\theta=\hat{\theta}} \approx 0$$

$\theta$  is likely to be multidimensional

Taylor series

$$f(\theta) \approx f(\theta_0) + \nabla f(\theta_0)^T (\theta - \theta_0) + \frac{1}{2} (\theta - \theta_0)^T H f(\theta_0) (\theta - \theta_0)$$

$$\begin{aligned} \log(p(\theta|y)) &\approx \log(p(\hat{\theta}|y)) + 0 + \frac{1}{2} (\theta - \hat{\theta})^T \\ &\quad \times \left( \frac{\partial^2}{\partial \theta^2} \log(p(\hat{\theta}|y)) \right)_{\theta=\hat{\theta}} (\theta - \hat{\theta}) \end{aligned}$$

observed Fisher information

$$I(\theta) = \frac{\partial^2}{\partial \theta^2} \log(p(\theta|y))$$

$$\begin{aligned} \log(p(\theta|y)) &\approx \log(p(\hat{\theta}|y)) + 0 + \frac{1}{2} (\theta - \hat{\theta})^T \\ &\quad \times \left( \frac{\partial^2}{\partial \theta^2} \log(p(\hat{\theta}|y)) \right)_{\theta=\hat{\theta}} (\theta - \hat{\theta}) \end{aligned}$$

$$\log(p(\theta|y)) \approx A + \frac{1}{2} (\theta - \hat{\theta})^T I(\theta) (\theta - \hat{\theta})$$

$$p(\theta|y) \approx \text{Be}^{\frac{1}{2}(\theta - \hat{\theta})^T I(\theta)(\theta - \hat{\theta})}$$

multivariate Normal  
distribution

$$p(\theta|y) \approx N(\hat{\theta}, I(\theta)^{-1})$$

example: normal distribution with unknown mean and variance.

prior: noninformative prior

$$p(\mu, \sigma^2) \propto \frac{1}{\sigma^2}$$

$$\text{show } p(\mu, \log(\sigma)) \propto 1$$

$$(\mu, \sigma^2) \mapsto (\mu, \log(\sigma))$$

$$J = \begin{pmatrix} \frac{\partial \mu}{\partial \mu} & \frac{\partial \mu}{\partial \log \sigma^2} \\ \frac{\partial \sigma^2}{\partial \mu} & \frac{\partial \sigma^2}{\partial \log \sigma} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2\sigma^2 \end{pmatrix}$$

$$\downarrow \sigma^2 = e^{2\log \sigma}$$

$$\frac{de^{2\log \sigma}}{\partial \log \sigma} = 2e^{2\log \sigma} = 2\sigma^2$$

$$|J| = 2\sigma^2 \text{ so}$$

$$p(\mu | \log \sigma) = p(\mu, \sigma^2) \cdot |J| \propto \frac{1}{\sigma^2} \cdot 2\sigma^2 \\ = 2 \propto 1$$

①

$$\theta = (\mu, \log \sigma) \quad \text{so our noninformative prior is} \\ p(\mu, \log(\sigma)) = 1$$

② find  $\hat{\theta}$

So we need to find the peak of  $p(\mu, \log \sigma | y)$

Recall from module 3:

$$p(\mu, \sigma^2 | y) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} ((n-1)s^2 + n(\bar{y}-\mu)^2)\right)$$

$\log \sigma$  due to  $|J|$

$$\Rightarrow \log(p(\mu, \log \sigma | y)) \propto -n \log \sigma - \frac{1}{2\sigma^2} ((n-1)s^2 + n(\bar{y}-\mu)^2)$$

this is the function around which we will construct our Taylor series

$\hat{\theta}$  is the point where  $\nabla \log(p(\mu, \log \sigma | y)) = \vec{0}$

$$\frac{\partial}{\partial \mu} \log(p(\mu, \log \sigma | y)) \propto \frac{n(\bar{y} - \mu)}{\sigma^2} = 0$$

$$\frac{\partial}{\partial \log \sigma} \log(p(\mu, \log \sigma | y)) \propto -n + \frac{(n-1)s^2 + n(\bar{y} - \mu)^2}{\sigma^2} = 0$$

$$\frac{n(\bar{y} - \mu)}{\sigma^2} = 0 \Rightarrow n(\bar{y} - \mu) = 0 \\ \Rightarrow \boxed{\bar{y} = \mu} = \hat{\theta}$$

$$-n + \frac{(n-1)s^2 + n(\bar{y} - \mu)^2}{\sigma^2} = 0$$

$$\Rightarrow \frac{-n\sigma^2 + (n-1)s^2}{\sigma^2} = 0$$

$$\Rightarrow (n-1)s^2 = n\sigma^2$$

$$\Rightarrow \frac{(n-1)s^2}{n} = \sigma^2 \neq \hat{\theta}_2$$

$$2 \log(\sigma) = \log\left(\frac{(n-1)s^2}{n}\right) \Rightarrow \log \sigma = \frac{1}{2} \log\left(\frac{(n-1)s^2}{n}\right)$$

So

$$\hat{\theta} = \begin{pmatrix} \bar{y} \\ \frac{1}{2} \log\left(\frac{(n-1)s^2}{n}\right) \end{pmatrix}$$

③ Find  $I(\hat{\theta})$

$$\frac{\partial^2}{(\partial \mu)^2} \log(p(\mu, \log \sigma | y)) = \frac{-n}{\sigma^2}$$

$$\frac{\partial^2}{\partial \mu \partial \log \sigma} \log(p(\mu, \log \sigma | y)) \underset{\text{at } \hat{\theta} : 0}{=} -2n(\bar{y} - \mu) e^{-2 \log \sigma}$$

$$\frac{\partial^2}{(\partial \log \sigma)^2} \log(p(\mu, \log \sigma | y)) \underset{\text{at } \hat{\theta} : -2n}{=} -2[(n-1)s^2 + n(\bar{y} - \mu)] e^{-2 \log \sigma}$$

$$\hat{\theta} = -n \cdot \frac{n}{(n-1)s^2} = \frac{-n^2}{(n-1)s^2}$$

$$\mathcal{I}(\hat{\theta}) = \begin{pmatrix} -\frac{n^2}{(n-1)s^2} & 0 \\ 0 & -2n \end{pmatrix}$$

④ Compute the normal distribution approximation

$$p(\theta | y) \approx N(\hat{\theta}, \mathcal{I}(\hat{\theta})^{-1})$$

$$p(\mu, \log \sigma | y) \approx N \left( \begin{pmatrix} \bar{Y} \\ \frac{1}{2} \log \left( \frac{(n-1)s^2}{2} \right) \end{pmatrix}, \begin{pmatrix} \frac{n^2}{(n-1)s^2} & 0 \\ 0 & 2n \end{pmatrix}^{-1} \right)$$

## Large-sample theory



$$y_1, \dots, y_n$$

$$y_i \in \{H, T\}$$

$$\begin{matrix} H=1 \\ T=0 \end{matrix}$$

$$\begin{matrix} y = \# \text{ heads} \\ n = \# \text{ of flips} \end{matrix}$$

$$\mathbb{H} = [0, 1] \quad \text{parameter space}$$

$$\mathcal{F} := \left\{ \binom{n}{y} \theta^y (1-\theta)^{n-y} \mid \theta \in \mathbb{H} \right\}$$

Case 1:  $f(y) \in \mathcal{F}$   $\exists \theta_0$  s.t.  $f(y) = \binom{n}{y} \theta_0^y (1-\theta_0)^{n-y}$

Case 2: misspecification of the model

$f(y) \notin \mathcal{F}$  but there is some function  $p(\theta_0 | y)$  that's the closest to  $f(y)$

Kullback - Leibler (KL) divergence

sketch proof 1:

Assumption:

1.  $\mathbb{H}$  is finite  $\mathbb{H} = \{\theta_0, \theta_1, \dots, \theta_n\}$
2.  $\theta_0$  is the  $KL(\theta)$  minimizer
3.  $p(\theta = \theta_0) > 0$  it's within our prior

Aim:

Show  $p(\theta = \theta_0 | y) \rightarrow 1$  as  $n \rightarrow \infty$

Let  $\theta \neq \theta_0$

We pick a value  $\theta$  that is not  $\theta_0$ , so

$$\log \left( \frac{p(\theta | y)}{p(\theta_0 | y)} \right) = \log \left( \frac{p(\theta) \prod_{i=1}^n p(y_i | \theta)}{p(\theta_0) \prod_{i=1}^n p(y_i | \theta_0)} \right)$$

$$= \log \left( \frac{p(\theta)}{p(\theta_0)} \right) + \sum_{i=1}^n \log \left( \frac{p(y_i | \theta)}{p(y_i | \theta_0)} \right)$$

observe :

$$\sum_{i=1}^n \log \left( \frac{p(y_i | \theta)}{p(y_i | \theta_0)} \right) = \log(p(y | \theta)) - \log(p(y | \theta_0))$$

$$= \log(p(y | \theta)) - \log(p(y_i)) + \log(p(y_i)) - \log(p(y_i | \theta_0))$$

$$= \log \left( \frac{p(y_i)}{p(y_i | \theta_0)} \right) - \log \left( \frac{p(y_i)}{p(y_i | \theta)} \right)$$

Recall  $y_i$  are iid. So by the law of large numbers

$$\log \left( \frac{p(\theta | y)}{p(\theta_0 | y)} \right) = \log \left( \frac{p(\theta)}{p(\theta_0)} \right) + \sum_{i=1}^n \log \left( \frac{p(y_i | \theta)}{p(y_i | \theta_0)} \right)$$

$$= \log \left( \frac{p(\theta)}{p(\theta_0)} \right) + \sum_{i=1}^n \left[ \log \left( \frac{p(y_i)}{p(y_i | \theta_0)} \right) - \log \left( \frac{p(y_i)}{p(y_i | \theta)} \right) \right]$$

$$= \log\left(\frac{p(\theta)}{p(\theta_0)}\right) + n \left[ \log\left(\frac{f(y_i)}{f(y_i|\theta_0)}\right) \right] - n \left[ \log\left(\frac{f(y_i)}{f(y_i|\theta)}\right) \right]$$

$$= \log\left(\frac{p(\theta)}{p(\theta_0)}\right) + n \underbrace{\left[ KL(\theta_0) - KL(\theta) \right]}_{\text{minimizer}}$$

Since  $\theta \neq \theta_0$  and  $\theta_0$  is a minimizer of  $KL(\cdot)$

$$\log\left(\frac{p(\theta|y)}{p(\theta_0|y)}\right) \rightarrow -\infty \quad \text{as} \quad n \rightarrow \infty$$

$$\Rightarrow \frac{p(\theta|y)}{p(\theta_0|y)} \rightarrow 0 \quad \Rightarrow \quad p(\theta|y) \rightarrow 0$$

Because probabilities sum to 1, then

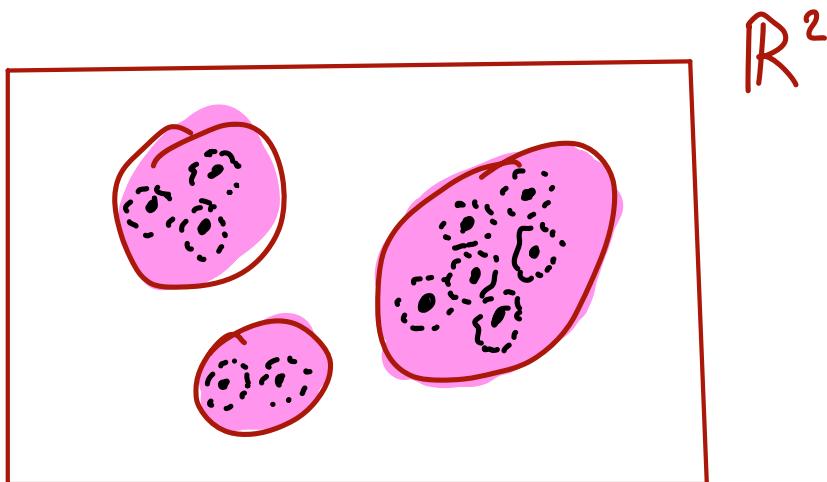
$$p(\theta_0|y) \rightarrow 1$$

Hence, the posterior distribution converges

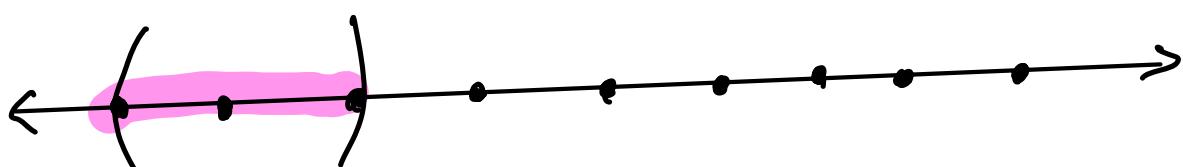
## Compact sets

A set  $(H)$  is compact if every open cover contains a finite subcover

Since  $(H) \subseteq \mathbb{R}^n$ , we picture this as closed and bounded.



Counter example :  $\mathbb{Z}$  in  $\mathbb{R}$



$$A = \{z-1, z+1 \mid z \in \mathbb{Z}\}$$

each integer is in exactly one set.

I cannot delete any set in this open cover because only one covers each point.

Sketch the proof for theorem 2

Let  $\mathbb{H}$  be a compact set and define  $\mathcal{A}$  to be an open cover such that only one set  $A_0 \in \mathcal{A}$  contains  $\theta_0$ .

Since  $\mathbb{H}$  is compact, there exists a finite subcover  $\{A_0, A_1, \dots, A_K\}$  where  $A_0$  is the set previously specified.

Use theorem 1 argument.

Let  $A \neq A_0$

$$\log \left( \frac{P(\theta \in A | y)}{P(\theta \in A_0 | y)} \right) \approx \log \left( \frac{P(\theta \in A)}{P(\theta \in A_0)} \right) +$$

$$n \in \left[ \frac{P(y_i | \theta \in A)}{P(y_i | \theta \in A_0)} \right]$$

$$P(\theta \in A | y) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

$$P(\theta \in A_0 | y) \rightarrow 1$$

### Theorem 3

#### Asymptotic Normality

#### Frequency properties

$$\theta | y \sim N(\hat{\theta}, [I(\hat{\theta})]^{-1})$$

$$(\theta - \hat{\theta}) | y \sim N(0, [I(\hat{\theta})]^{-1})$$

Now consider  $\varepsilon = [I(\hat{\theta})]^{1/2} (\theta - \hat{\theta})$

if  $x \sim N(\mu, \Sigma)$  and  $A$  is a linear transformation, then

$$Ax \sim N(A\mu, A\Sigma A^T)$$

mean:  $[I(\hat{\theta})]^{1/2} \cdot 0 = 0$

variance:  $[I(\hat{\theta})]^{1/2} (I(\hat{\theta}))^{-1} ([I(\hat{\theta})]^{1/2})^T = I$

hence

$$\varepsilon | y \sim N(0, I)$$

## hypothesis testing

$$\begin{array}{lll} \theta & y \sim N(\theta, 1) & \text{uniform prior} \\ H_0: \theta = 0 & & \\ H_A: \theta > 0 & \text{data: } y = 1 & \end{array}$$

### Frequentist approach

$$z = \frac{1-\theta}{1} = 1$$

one-sided p-value of 0.16

two-sided " 0.32

Both "fail to reject"

p-value > 0.05 threshold

### Bayesian approach

$$y = 1 \quad \text{likelihood } p(y|\theta)$$

prior  $p(\theta)$

:

posterior probability that

$\theta > 0$  is 84 %.

updating a belief with uncertainty quantified.

## Bayesian Bootstrapping

### Priors and exchangeability