The Normal distribution

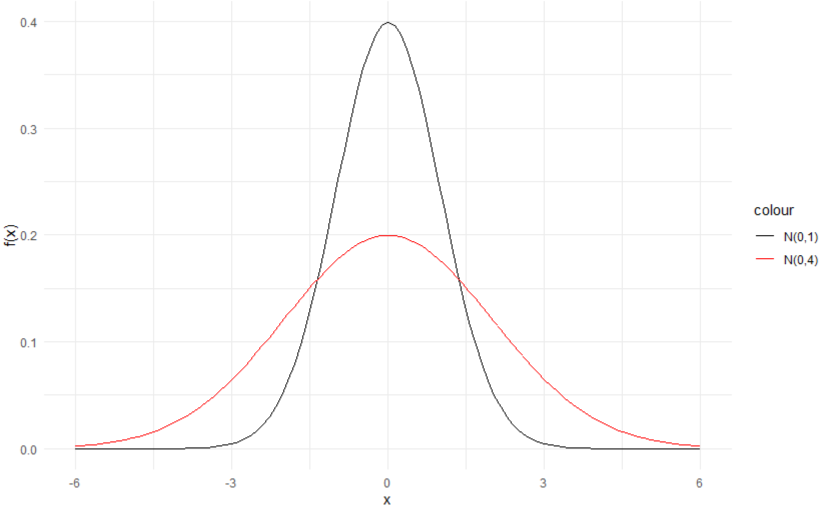
The Normal distribution is a continuous, symmetric distribution which arises when physical quantities are the sum of many independent processes, for example measurement errors or the the height of individuals. In addition, the logarithm of many variables are close to a normal distribution, for example the size of living tissues or the blood pressure. In finance, the returns in certain conditions often obey a Normal distribution or distributions related to the Normal distribution. But we have countless of other situations for which the Normal distribution appears.

Let $X \sim Normal(\mu, \sigma^2)$. Then the mean $E[X] = \mu$.

Proof of the Normal Expectation

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{+\infty} x f_{\mu,\sigma^2}(x) dx \\
&= \int_{-\infty}^{+\infty} x \frac{1}{\sqrt{2\pi}\ \sigma}\ e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx \\
&= \frac{1}{\sqrt{2\pi}\ \sigma} \int_{-\infty}^{+\infty} x\ e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx \\
&= \frac{\sqrt{2}\sigma}{\sqrt{2\pi}\ \sigma} \int_{-\infty}^{+\infty} (t\sqrt{2}\sigma + \mu)\ e^{-t^2} dt \qquad t = \frac{x-\mu}{\sqrt{2}\sigma} \Leftrightarrow x = t\sqrt{2}\sigma + \mu \\
&= \frac{1}{\sqrt{\pi}} \left(\sqrt{2}\sigma \int_{-\infty}^{+\infty} t\ e^{-t^2} dt + \mu \int_{-\infty}^{+\infty} e^{-t^2} dt \right) \\
&= \frac{1}{\sqrt{\pi}} \left(\sqrt{2}\sigma \left[-\frac{1}{2} e^{-t^2} \right]_{-\infty}^{+\infty} + \mu\sqrt{\pi} \right) \\
&= \frac{\mu\sqrt{\pi}}{\sqrt{\pi}} \\
&= \mu
\end{aligned}
$$

Visualization: Normal distribution



The Gamma distribution

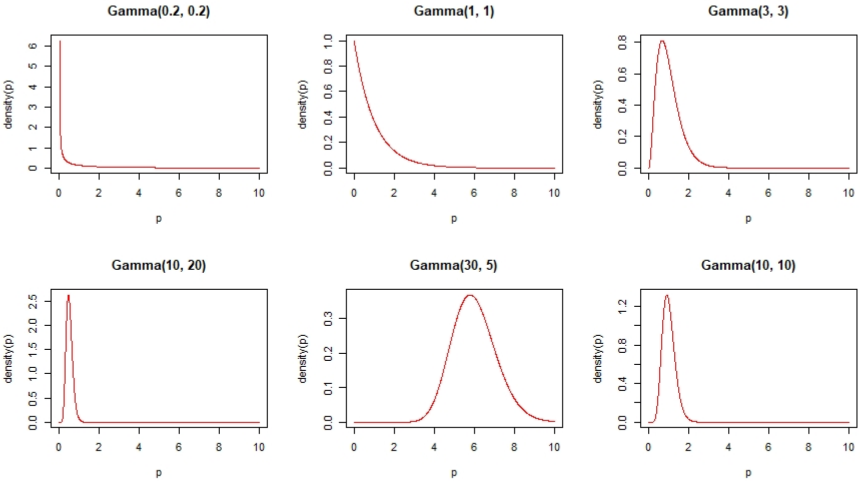
The Gamma distribution is a continuous distribution mainly used to model waiting time between events (in particular Poisson distributed events).

Let $X \sim Gamma(\alpha, \beta)$. Then the mean $E[X] = \alpha/\beta$.

Proof of the Gamma Expectation

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{+\infty} x \ f_{\alpha,\beta}(x) \ dx = \int_{0}^{\infty} x \ \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \ dx \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_{0}^{\infty} x \ x^{\alpha-1} e^{-\beta x} \ dx \quad (u = \beta x \Leftrightarrow x = u/\beta, \quad du = \beta dx \Leftrightarrow dx = du/\beta) \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_{0}^{\infty} \left(\frac{u}{\beta}\right)^\alpha e^{-u} \ \frac{du}{\beta} \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{\beta^\alpha} \frac{1}{\beta} \int_{0}^{\infty} u^\alpha e^{-u} \ du \qquad\qquad \int_{0}^{\infty} x^\alpha e^{-x} \ dx = \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{\beta^{\alpha+1}} \alpha\Gamma(\alpha) \\
&= \frac{\alpha}{\beta}
\end{aligned}
$$

Visualization: Gamma distribution



The Poisson distribution

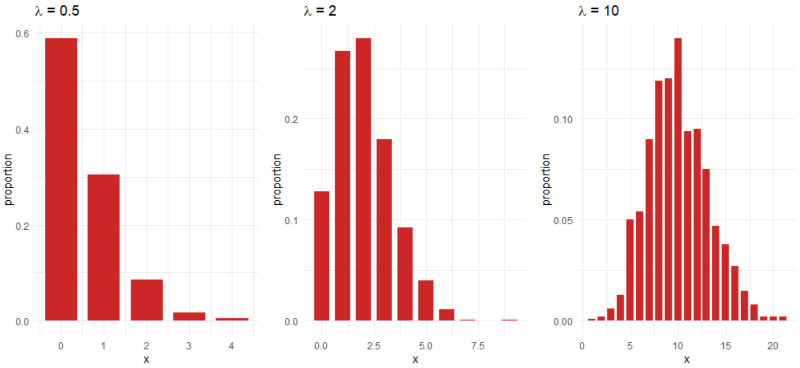
The Poisson distribution is a discrete distribution used to model a phenomenon that represents the count of events in a fixed time interval (or space unit). The r.v. X represents then the number of successes in this interval.

Let $X \sim Poisson(\lambda)$. Then the mean $E[X] = \lambda$.

Proof of Poisson Expectation

$$
\begin{aligned}
E[X] = \sum_{x=0}^{\infty} x \; P(X = x) &= \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} \\
&= e^{-\lambda} \sum_{x=1}^{\infty} x \frac{\lambda^x}{x!} \\
&= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} && \text{Let } y = x - 1 \\
&= \lambda e^{-\lambda} \underbrace{\sum_{y=0}^{\infty} \frac{\lambda^y}{y!}}_{=e^{\lambda}} && \text{Taylor series expansion for } e^{\lambda} \\
&= \lambda e^{-\lambda} e^{\lambda} = \lambda
\end{aligned}
$$

Visualization: Poisson distribution



Bayesian Questionnaire: Question 2

Which statement (only one) is correct ?

Please give a probability of correctness (a number between 0 and 100%) to all of these possible answers.

A: The Normal distribution is a discrete distribution
B: The Normal distribution is a symmetric distribution
C: The Poisson distribution is a continuous distribution
D: The Gamma distribution is a discrete distribution