Bayesian inference in NDLM: Known variances

$$
\begin{aligned}
y_t &= \boldsymbol{F}_t'\boldsymbol{\theta}_t + \nu_t, \quad \nu_t \sim N(0, v_t), &(1)\\
\boldsymbol{\theta}_t &= \boldsymbol{G}_t\boldsymbol{\theta}_{t-1} + \boldsymbol{\omega}_t, \quad \boldsymbol{\omega}_t \sim N(0, \boldsymbol{W}_t), &(2)
\end{aligned}
$$

with $\boldsymbol{F}_t, \boldsymbol{G}_t, v_t$ and $\boldsymbol{W}_t$ known. We also assume a prior distribution of the form $(\boldsymbol{\theta}_0|\mathcal{D}_0) \sim N(\boldsymbol{m}_0, \boldsymbol{C}_0)$, with $\boldsymbol{m}_0, \boldsymbol{C}_0$ known.

Smoothing

For $t < T$, we have that

$$(\boldsymbol{\theta}_t|\mathcal{D}_T) \sim N(\boldsymbol{a}_T(t-T), \boldsymbol{R}_T(t-T)),$$

where

$$
\begin{aligned}
\boldsymbol{a}_T(t-T) &= \boldsymbol{m}_t - \boldsymbol{B}_t[\boldsymbol{a}_{t+1} - \boldsymbol{a}_T(t-T+1)],\\
\boldsymbol{R}_T(t-T) &= \boldsymbol{C}_t - \boldsymbol{B}_t[\boldsymbol{R}_{t+1} - \boldsymbol{R}_T(t-T+1)]\boldsymbol{B}_t',
\end{aligned}
$$

for $t = (T-1), (T-2), \ldots, 0$, with $\boldsymbol{B}_t = \boldsymbol{C}_t\boldsymbol{G}_{t+1}'\boldsymbol{R}_{t+1}^{-1}$, and $\boldsymbol{a}_T(0) = \boldsymbol{m}_T$, $\boldsymbol{R}_T(0) = \boldsymbol{C}_T$. Here $\boldsymbol{a}_t, \boldsymbol{m}_t, \boldsymbol{R}_t$, and $\boldsymbol{C}_t$ are obtained using the filtering equations as explained before.

Forecasting

For $h \geq 0$ it is possible to show that

$$
\begin{aligned}
(\boldsymbol{\theta}_{t+h} | \mathcal{D}_t) &\sim N(\boldsymbol{a}_t(h), \boldsymbol{R}_t(h)), \\
(y_{t+h} | \mathcal{D}_t) &\sim N(f_t(h), q_t(h)),
\end{aligned}
$$

with

$$
\begin{aligned}
\boldsymbol{a}_t(h) &= \boldsymbol{G}_{t+h} \boldsymbol{a}_t(h-1), \qquad & \boldsymbol{R}_t(h) &= \boldsymbol{G}_{t+h} \boldsymbol{R}_t(h-1) \boldsymbol{G}'_{t+h} + \boldsymbol{W}_{t+h}, \\
f_t(h) &= \boldsymbol{F}'_{t+h} \boldsymbol{a}_t(h), \qquad & q_t(h) &= \boldsymbol{F}'_{t+h} \boldsymbol{R}_t(h) \boldsymbol{F}_{t+h} + v_{t+h},
\end{aligned}
$$

$\boldsymbol{a}_t(0) = \boldsymbol{m}_t$, $\boldsymbol{R}_t(0) = \boldsymbol{C}_t$.