PACF of the AR(1) process

It is possible to show that the PACF of an autoregressive process of order one is zero after the first lag. We can use the Durbin-Levinson recursion to show this.

For lag $n = 0$ we have $\phi(0,0) = 0$.

For lag $n = 1$ we have:

$$\phi(1,1) = \rho(1) = \phi.$$

For lag $n = 2$ we compute $\phi(2,2)$ as

$$\begin{aligned}\phi(2,2) &= \frac{(\rho(2) - \phi(1,1)\rho(1))}{(1 - \phi(1,1)\rho(1))} \\ &= \frac{(\phi^2 - \phi^2)}{(1 - \phi^2)} = 0\end{aligned}$$

and we also obtain

$$\phi(2,1) = \phi(1,1) - \phi(2,2)\phi(1,1) = \phi.$$

For lag $n = 3$ we compute $\phi(3,3)$ as

$$\begin{aligned}
\phi(3,3) &= \frac{(\rho(3) - \sum_{h=1}^{2} \phi(2,h)\rho(3-h))}{(1 - \sum_{h=1}^{2} \phi(2,h)\rho(h))} \\
&= \frac{(\phi^3 - \phi(2,1)\rho(2) - \phi(2,2)\rho(1))}{(1 - \phi(2,1)\rho(1) - \phi(2,2)\rho(2))} \\
&= \frac{(\phi^3 - \phi^3 - 0)}{(1 - \phi^2)} = 0,
\end{aligned}$$

and we also obtain

$$\begin{aligned}
\phi(3,1) &= \phi(2,1) - \phi(3,3)\phi(2,2) = \phi \\
\phi(3,2) &= \phi(2,2) - \phi(3,3)\phi(2,1) = 0.
\end{aligned}$$

We can prove by induction that in the case of an AR(1), for any lag n, $\phi(n,h) = 0$, $\phi(n,1) = \phi$ and $\phi(n,h) = 0$ for $h \geq 2$ and $n \geq 2$.

Then, the PACF of an AR(1) is zero for any lag above 1 and the PACF coefficient at lag 1 is equal to the AR coefficient ϕ.