

# Polynomial trend models

## First-order polynomial

$$\begin{aligned}y_t &= \mu_t + \nu_t, \quad \nu_t \sim N(0, v_t) \\ \mu_t &= \mu_{t-1} + \omega_t, \quad \omega_t \sim N(0, w_t).\end{aligned}$$

In this case we have  $\theta_t = \mu_t$ ,  $F_t = 1$ ,  $G_t = 1$  for all  $t$ , resulting in  $\{1, 1, v_t, w_t\}$ . Forecast function  $f_t(h) = E(\mu_t | \mathcal{D}_t) = k_t$ , for all  $h > 0$ .

## Second-order polynomial

$$\begin{aligned}y_t &= \theta_{t,1} + \nu_t, \quad \nu_t \sim N(0, v_t) \\ \theta_{t,1} &= \theta_{t-1,1} + \theta_{t-1,2} + \omega_{t,1}, \quad \omega_{t,1} \sim N(0, w_{t,11}) \\ \theta_{t,2} &= \theta_{t-1,2} + \omega_{t,2}, \quad \omega_{t,2} \sim N(0, w_{t,22}),\end{aligned}$$

we can also have  $Cov(\theta_{t,1}, \theta_{t,2}) = w_{t,12} = w_{t,21}$ . This can be written as a DLM with state-space vector  $\boldsymbol{\theta}_t = (\theta_{t,1}, \theta_{t,2})'$ , and  $\{\mathbf{F}, \mathbf{G}, v_t, \mathbf{W}_t\}$  with  $\mathbf{F} = (1, 0)'$ ,

$$\mathbf{G} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{W}_t = \begin{pmatrix} w_{t,11} & w_{t,12} \\ w_{t,21} & w_{t,22} \end{pmatrix}.$$

Note that

$$\mathbf{G}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{G}^h = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix},$$

and so  $f_t(h) = (1, h)E(\boldsymbol{\theta}_t|\mathcal{D}_t) = (1, h)(k_{t,0}, k_{t,1})' = (k_{t,0} + hk_{t,1})$ . Here  $\mathbf{G} = \mathbf{J}_2(1)$  (see below). Also, we denote  $\mathbf{E}_2 = (1, 0)'$  and so, the short notation for this model is  $\{\mathbf{E}_2, \mathbf{J}_2(1), \cdot, \cdot\}$ .

## General p-th order polynomial model

We can consider a so called  $p$ -th order polynomial model. This model will have a state-space vector of dimension  $p$  and a polynomial of order  $p - 1$  forecast function on  $h$ . The model can be written as  $\{\mathbf{E}_p, \mathbf{J}_p(1), v_t, \mathbf{W}_t\}$  with  $\mathbf{F}_t = \mathbf{E}_p = (1, 0, \dots, 0)'$  and  $\mathbf{G}_t = \mathbf{J}_p(1)$ , with

$$\mathbf{J}_p(1) = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & & & \vdots \\ & & & \ddots & \ddots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

The forecast function is given by

$$f_t(h) = (k_{t,0} + k_{t,1}h + \dots k_{t,p-1}h^{p-1}).$$

There is also an alternative parameterization of this model that leads to the same algebraic form of the forecast function given by  $\{\mathbf{E}_p, \mathbf{L}_p, v_t, \mathbf{W}_t\}$ , with

$$\mathbf{L}_p = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

## Dynamic regression models

### Simple dynamic regression

$$y_t = \beta_{t,0} + \beta_{t,1}x_t + \nu_t$$

$$\beta_{t,0} = \beta_{t-1,0} + \omega_{t,0}$$

$$\beta_{t,1} = \beta_{t-1,1} + \omega_{t,1},$$

and so  $\boldsymbol{\theta}_t = (\beta_{t,0}, \beta_{t,1})'$ ,  $\mathbf{F}_t = (1, x_t)'$  and  $\mathbf{G} = \mathbf{I}_2$ . This results in a forecast function of the form  $f_t(h) = k_{t,0} + k_{t,1}x_{t+h}$ .

## General dynamic regression

$$\begin{aligned} y_t &= \beta_{t,0} + \beta_{t,1}x_{t,1} + \dots + \beta_{t,M}x_{t,M} + \nu_t \\ \beta_{t,m} &= \beta_{t-1,m} + \omega_{t,m}, \quad m = 0 : M. \end{aligned}$$

Then,  $\boldsymbol{\theta} = (\beta_{t,0}, \dots, \beta_{t,M})'$ ,  $\mathbf{F}_t = (1, x_{t,1}, \dots, x_{t,M})'$  and  $\mathbf{G} = \mathbf{I}_M$ . The forecast function is given by

$$f_t(h) = k_{t,0} + k_{t,1}x_{t+h,1} + \dots + k_{t+h,M}x_{t+h,M}.$$

A particular case is of dynamic regressions is the case of **time-varying autoregressions (TVAR)** with

$$\begin{aligned} y_t &= \phi_{t,1}y_{t-1} + \phi_{t,2}y_{t-2} + \dots + \phi_{t,p}y_{t-p} + \nu_t, \\ \phi_{t,m} &= \phi_{t-1,m} + \omega_{t,m}, \quad m = 1 : p. \end{aligned}$$