Seasonal Models: Fourier Representation

For any frequency $\omega \in (0, \pi)$, a model of the form $\{\boldsymbol{E}_2, \boldsymbol{J}_2(1, \omega), \cdot, \cdot\}$ with a 2-dimensional state vector $\boldsymbol{\theta}_t = (\theta_{t,1}, \theta_{t,2})'$ and

$$\boldsymbol{J}_2(1, \omega) = \begin{pmatrix} \cos(\omega) & \sin(\omega) \\ -\sin(\omega) & \cos(\omega) \end{pmatrix},$$

has a forecast function

$$f_t(h) = (1, 0)\boldsymbol{J}_2^h(1, \omega)(a_t, b_t) = a_t \cos(\omega h) + b_t \sin(\omega h), = A_t \cos(\omega h + B_t).$$

For $\omega = \pi$ the NDLM is $\{1, -1, \cdot, \cdot\}$ and has a forecast function of the form $f_t(h) = (-1)^h m_t$.

These are component Fourier models. Now, for a given period p we can build a model that contains components for the fundamental period and all the harmonics of such period using the superposition principle as follows:

Case: $p = 2m - 1$ odd

Let $\omega_j = 2\pi j/p$ for $j = 1 : (m-1)$, $\boldsymbol{F}$ a $(p-1)$-dimensional vector, or equivalently, a $2(m-1)$-dimensional vector, and $\boldsymbol{G}$ a $(p-1) \times (p-1)$ matrix with $\boldsymbol{F} = (\boldsymbol{E}_2', \boldsymbol{E}_2', \ldots, \boldsymbol{E}_2')'$, $\boldsymbol{G} = \textbf{blockdiag}[\boldsymbol{J}_2(1, \omega_1), \ldots, \boldsymbol{J}_2(1, \omega_{m-1})]$.

Case: $p = 2m$ even

In this case, $\boldsymbol{F}$ is again a $(p-1)$-dimensional vector (or, equivalently a $(2m-1)$-dimensional vector), and $\boldsymbol{G}$ is a $(p-1) \times (p-1)$ matrix such that $\boldsymbol{F} = (\boldsymbol{E}_2', \ldots, \boldsymbol{E}_2', 1)'$ and $\boldsymbol{G} =$

$\mathbf{blockdiag}[\boldsymbol{J}_2(1,\omega_1),\ldots,\boldsymbol{J}_2(1,\omega_{m-1}),-1]$.

In both cases the forecast function has the general form:

$$f_t(h)=\sum_{j=1}^{m-1}A_{t,j}\cos(\omega_j h+\gamma_{t,j})+(-1)^h A_{t,m},$$

with $A_{t,m}=0$ if p is odd.

Examples

<u>Fourier representation, $p=12$</u>: In this case $p=2\times 6$ so $\boldsymbol{\theta}_t$ is an 11-dimensional state vector, $\boldsymbol{F}=(1,0,1,0,1,0,1,0,1,0,1)'$ the Fourier frequencies are $\omega_1=2\pi/12$, $\omega_2=4\pi/12=2\pi/6$, $\omega_3=6\pi/12=2\pi/4$, $\omega_4=8\pi/12=2\pi/3$, $\omega_5=10\pi/12=5\pi/6$ and $\omega_6=12\pi/12=\pi$ the Nyquist. $\boldsymbol{G}=\mathbf{blockdiag}(\boldsymbol{J}_2(1,\omega_1),\ldots,\boldsymbol{J}_2(1,\omega_5),1)$ and the forecast function is given by

$$f_t(h)=\sum_{j=1}^{5}A_{t,j}\cos(2\pi j/12+\gamma_{t,j})+(-1)^h A_{t,6}.$$

<u>Linear trend + seasonal component with $p=4$</u>: We can use the superposition principle to build more sophisticated models. For instance, assume that we want a model with the following 2 components:

- Linear trend: $\{\boldsymbol{F}_1,\boldsymbol{G}_1,\cdot,\cdot\}$ with $\boldsymbol{F}_1=(1,0)'$,

$$\boldsymbol{G}_1=\boldsymbol{J}_2(1)=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.$$

- Full seasonal model with $p=4:\{\boldsymbol{F}_2,\boldsymbol{G}_2,\cdot,\cdot\}$ $p=2\times 2$ so $m=2$ and $\omega=2\pi/4=\pi/2$,

$\boldsymbol{F}_2 = (1, 0, 1)'$, and

$$\boldsymbol{G}_2 = \begin{pmatrix} \cos(\pi/2) & \sin(\pi/2) & 0 \\ -\sin(\pi/2) & \cos(\pi/2) & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The resulting DLM is a 5-dimensional model $\{\boldsymbol{F}, \boldsymbol{G}, \cdot, \cdot\}$ with $\boldsymbol{F} = (1, 0, 1, 0, 1)'$, and

$$\boldsymbol{G} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

and

$$f_t(h) = (k_{t,1} + k_{t,2}h) + k_{t,3}\cos(\pi h/2) + k_{t,4}\sin(\pi h/2) + k_{t,5}(-1)^h.$$