

Bayesian inference in NDLM: Known variances

Consider a NDLM given by

$$y_t = \mathbf{F}_t' \boldsymbol{\theta}_t + \nu_t, \quad \nu_t \sim N(0, v_t), \quad (1)$$

$$\boldsymbol{\theta}_t = \mathbf{G}_t \boldsymbol{\theta}_{t-1} + \boldsymbol{\omega}_t, \quad \boldsymbol{\omega}_t \sim N(0, \mathbf{W}_t), \quad (2)$$

with $\mathbf{F}_t, \mathbf{G}_t, v_t$ and \mathbf{W}_t known. We also assume a prior distribution of the form $(\boldsymbol{\theta}_0 | \mathcal{D}_0) \sim N(\mathbf{m}_0, \mathbf{C}_0)$, with $\mathbf{m}_0, \mathbf{C}_0$ known.

Filtering

We are interested in finding $p(\boldsymbol{\theta}_t | \mathcal{D}_t)$ for all t . Assume that the posterior at $t - 1$ is such that

$$(\boldsymbol{\theta}_{t-1} | \mathcal{D}_{t-1}) \sim N(\mathbf{m}_{t-1}, \mathbf{C}_{t-1}).$$

Then, we can obtain the following:

1. *Prior at time t* : $(\boldsymbol{\theta}_t | \mathcal{D}_{t-1}) \sim N(\mathbf{a}_t, \mathbf{R}_t)$, with

$$\mathbf{a}_t = \mathbf{G}_t \mathbf{m}_{t-1}$$

and

$$\mathbf{R}_t = \mathbf{G}_t \mathbf{C}_{t-1} \mathbf{G}_t' + \mathbf{W}_t.$$

2. *One-step forecast:* $(y_t|\mathcal{D}_{t-1}) \sim N(f_t, q_t)$, with

$$f_t = \mathbf{F}_t' \mathbf{a}_t, \quad q_t = \mathbf{F}_t' \mathbf{R}_t \mathbf{F}_t + v_t$$

3. *Posterior at time t :* $(\boldsymbol{\theta}_t|\mathcal{D}_t) \sim N(\mathbf{m}_t, \mathbf{C}_t)$, with

$$\mathbf{m}_t = \mathbf{a}_t + \mathbf{R}_t \mathbf{F}_t q_t^{-1} (y_t - f_t),$$

$$\mathbf{C}_t = \mathbf{R}_t - \mathbf{R}_t \mathbf{F}_t q_t^{-1} \mathbf{F}_t' \mathbf{R}_t.$$

Now, denoting $e_t = (y_t - f_t)$ and $\mathbf{A}_t = \mathbf{R}_t \mathbf{F}_t q_t^{-1}$, we can rewrite the equations above as

$$\mathbf{m}_t = \mathbf{a}_t + \mathbf{A}_t e_t,$$

$$\mathbf{C}_t = \mathbf{R}_t - \mathbf{A}_t q_t \mathbf{A}_t'.$$