

Bayesian Time Series modelling and Analysis

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$$\{Y_t\} \quad \{Y_t, t = 1, 2, \dots\} \quad Y_{1:T}$$

Strong stationarity

$\{Y_t\}$ any $n > 0$, any sequence t_1, \dots, t_n
and any $h > 0$

$$(Y_{t_1}, \dots, Y_{t_n})' \quad (Y_{t_1+h}, \dots, Y_{t_n+h})'$$

Weak stationarity (second order stationarity)

$\{Y_t\}$ first and second moments of the two
sequences exist and are identical.

$$E[Y_t] = \mu, \quad \text{var}(Y_t) = \sigma^2$$

$\text{cov}(Y_t, Y_s)$ depends on $|t-s|$

Strong stationarity implies weak stationarity

autocorrelation function (ACF)

$\{Y_t\}$

autocovariance

$$\gamma(t, s) = \text{cov}(Y_t, Y_s) = E[(Y_t - \mu_t)(Y_s - \mu_s)]$$

$$\mu_t = E[Y_t], \mu_s = E[Y_s]$$

stationarity $\Rightarrow E[Y_t] = \mu$ for all t

and $\gamma(t, s) = \gamma(|t-s|)$

if $h > 0$ $\gamma(h) = \text{cov}(Y_t, Y_{t-h})$

autocorrelation

$$\rho(t, s) = \frac{\gamma(t, s)}{\sqrt{\gamma(t, t)} \sqrt{\gamma(s, s)}}$$

stationarity $\Rightarrow \rho(h) = \gamma(h) / \gamma(0)$

$$\gamma(0) = \text{var}(Y_t)$$

How to get estimates of γ and ρ ?

$$Y_{1:T}$$

Sample ACF : assuming stationarity

$$\hat{\gamma}(h) = \frac{1}{T} \sum_{t=1}^{T-h} (y_{t+h} - \bar{y})(y_t - \bar{y})$$

with $\bar{y} = \frac{1}{T} \sum_{t=1}^T y_t$,

$$\hat{\rho}(h) = \hat{\gamma}(h) / \hat{\gamma}(0)$$

The AR(1)

$$Y_t = \phi Y_{t-1} + \varepsilon_t, \quad \varepsilon_t \stackrel{iid}{\sim} N(0, \sigma^2)$$

$$\begin{aligned} Y_t &= \phi (\phi Y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ &= \phi^2 Y_{t-2} + \phi \varepsilon_{t-1} + \varepsilon_t \\ &= \dots \\ &= \phi^k Y_{t-k} + \sum_{j=0}^{k-1} \phi^j \varepsilon_{t-j} \end{aligned}$$

if $\phi \in [-1, 1]$ \Rightarrow

$$Y_t = \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j} \quad (a)$$

$$\begin{aligned}
 E[Y_t] &= 0, \quad \text{var}(Y_t) = \text{var}(a) \\
 &= \sum_{j=0}^{\infty} \phi^{2j} v \\
 &= \frac{v}{1-\phi^2} \quad \text{does not depend on } t \\
 &\quad \text{(stationary)}
 \end{aligned}$$

$$\begin{aligned}
 \gamma(h) &= E[Y_t Y_{t-h}] \\
 &= E\left[\left(\sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}\right) \left(\sum_{k=0}^{\infty} \phi^k \varepsilon_{t-h-k}\right)\right] \\
 &= E\left[\left(\varepsilon_t + \phi \varepsilon_{t-1} + \phi^2 \varepsilon_{t-2} + \dots + \phi^h \varepsilon_{t-h}\right) \right. \\
 &\quad \left. \left(\varepsilon_{t-h} + \phi \varepsilon_{t-h-1} + \phi^2 \varepsilon_{t-h-2} + \dots\right)\right]
 \end{aligned}$$

if time is not the same in the cross-products the expectation is 0 because $\text{cov}()$ and are iid.

$$\begin{aligned}
 &= E\left[\phi^h \varepsilon_{t-h}^2 + \phi^{h+1} \phi \varepsilon_{t-h-1}^2 + \phi^{h+2} \phi^2 \varepsilon_{t-h-2}^2 + \dots\right] \\
 &= v \sum_{j=0}^{\infty} \phi^{h+j} \phi^j \\
 &= v \phi^h \sum_{j=0}^{\infty} \phi^{2j} \\
 &= \frac{v \phi^h}{1-\phi^2}
 \end{aligned}$$

$$\rho(h) = \gamma(h) / \gamma(0) = \phi^h$$

In general, for any integer h , we have

$$\rho(h) = \phi^{|h|}$$

$$\gamma(h) = \frac{v \phi^{|h|}}{1 - \phi^2}$$

The larger the value of ϕ , the decay will be slower.

Maximum likelihood Estimation for AR(1)

$$Y_t = \phi Y_{t-1} + \varepsilon_t, \quad \varepsilon_t \stackrel{\text{iid}}{\sim} N(0, v)$$

(we work with the conditional likelihood)

$$\phi \in [-1, 1] \Rightarrow \{Y_t\} \text{ is stationary}$$

$$Y_1 \sim N(0, v / (1 - \phi^2))$$

$$Y_t | Y_{t-1} \sim N(\phi Y_{t-1}, v)$$

$$Y_{1:T}$$

$$\begin{aligned}
 P(Y_{1:T} | \phi, \nu) &= P(Y_1 | \phi, \nu) \prod_{t=2}^T P(Y_t | Y_{t-1}, \phi, \nu) \\
 &= \frac{1 \cdot (1 - \phi^2)^{1/2}}{(2\pi\nu)^{1/2}} \exp\left\{-\frac{Y_1^2(1 - \phi^2)}{2\nu}\right\} \\
 &\quad \frac{1}{(2\pi\nu)^{(T-1)/2}} \exp\left\{-\frac{\sum_{t=2}^T (Y_t - \phi Y_{t-1})^2}{2\nu}\right\} \\
 &\quad \quad \quad \textcolor{red}{Q^*(\phi)}
 \end{aligned}$$

$$\begin{aligned}
 Q^*(\phi) &= Y_1^2(1 - \phi^2) + \underbrace{\sum_{t=2}^T (Y_t - \phi Y_{t-1})^2}_{\textcolor{blue}{Q(\phi)}} \\
 &= \frac{(1 - \phi)^{1/2}}{(2\pi\nu)^{T/2}} \exp\left\{-\frac{Q^*(\phi)}{2\nu}\right\}
 \end{aligned}$$

This is the full likelihood.

Conditional likelihood (conditioned on the first obs)

$$\begin{aligned}
 P(Y_{2:T} | Y_1, \phi, \nu) &= \prod_{t=2}^T \frac{1}{(2\pi\nu)^{1/2}} \exp\left\{-\frac{(Y_t - \phi Y_{t-1})^2}{2\nu}\right\} \\
 &= \frac{1}{(2\pi\nu)^{(T-1)/2}} \exp\left\{-\frac{Q(\phi)}{2\nu}\right\}
 \end{aligned}$$

$$\begin{pmatrix} Y_2 \\ \vdots \\ Y_T \end{pmatrix} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_{T-1} \end{pmatrix} \phi + \begin{pmatrix} \varepsilon_2 \\ \vdots \\ \varepsilon_T \end{pmatrix}$$

\Rightarrow

$$\tilde{\underline{Y}} = \underline{X} \phi + \underline{\varepsilon}, \quad \underline{\varepsilon} \sim N(0, \nu \underline{I})$$

if \underline{X} is full rank, then

$$\hat{\phi} = (\underline{X}^T \underline{X})^{-1} \underline{X}^T \tilde{\underline{Y}}$$

$$\hat{\nu} = s^2 = \frac{(\tilde{\underline{Y}} - \underline{X} \hat{\phi})^T (\tilde{\underline{Y}} - \underline{X} \hat{\phi})}{|\dim(\tilde{\underline{Y}}) - \dim(\phi)|}$$

$$\begin{cases} \hat{\phi}_{mle} = \left(\sum_{t=2}^T y_t y_{t-1} \right) / \left(\sum_{t=2}^T y_{t-1}^2 \right) \\ s^2 = \sum_{t=2}^T (y_t - \hat{\phi}_{mle} y_{t-1})^2 / (T-2) \end{cases} \quad \text{unbiased}$$

We have to resort to numerical maximization for the AR coefficient if we work with the full likelihood.

$$Y_t = \phi Y_{t-1} + \varepsilon_t, \quad \varepsilon_t \stackrel{iid}{\sim} N(0, 1)$$

$$Y_1 | \phi \sim N\left(0, \frac{1}{(1-\phi^2)}\right)$$

$$Y_t | Y_{t-1}, \phi \sim N(\phi Y_{t-1}, 1)$$

full likelihood

$$\begin{aligned} p(Y_{1:T} | \phi) &= p(Y_1 | \phi) \cdot \prod_{t=2}^T p(Y_t | Y_{t-1}, \phi) = \\ &= \frac{(1-\phi^2)^{1/2}}{(2\pi)^{1/2}} \exp\left\{-\frac{Y_1^2 (1-\phi^2)}{2}\right\} \cdot \\ &\quad \frac{1}{(2\pi)^{(T-1)/2}} \exp\left\{-\sum_{t=2}^T \frac{(Y_t - \phi Y_{t-1})^2}{2}\right\} \\ &= \frac{(1-\phi^2)^{1/2}}{(2\pi)^{T/2}} \exp\left\{-\frac{1}{2} Q^*(\phi)\right\} \end{aligned}$$

$$\Rightarrow \log p(Y_{1:T} | \phi) = \frac{1}{2} \log(1-\phi^2) - \frac{1}{2} Q^*(\phi) + k$$

\Rightarrow numerical optimization
e.g. Newton-Raphson.

Bayesian Inference in the AR(1)