Determinant of a matrix $(n = 2)$

Determinant. For a square matrix A of size n, we have that $rank(A) = n$ if and only if its determinant is non zero, that is we have $det(A) \neq 0$. Then matrix A is said to have full rank.

For $n = 2$, the determinant is computed as follows:

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} \qquad \Leftrightarrow \qquad det(A) = ad - bc$$

```
# determinant of a square matrix

# create a square matrix A of size n = 2
A <- matrix(c(1,2,3,4), nrow = 2, byrow=TRUE)
A
#      [,1] [,2]
# [1,]    1    2
# [2,]    3    4

det(A)
# [1] -2
```

Determinant of a matrix $(n > 2)$

For $n > 2$, the determinant is computed by developping with respect to a row or a column of a matrix using the following formulæ

$$det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} det(A_{ij}) \qquad \text{if we develop with respect to the } i^{th} \text{ row}$$

$$det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} det(A_{ij}) \qquad \text{if we develop with respect to the } j^{th} \text{ column}$$

For example,

$$det \begin{pmatrix} -2 & 2 & -3 \\ -1 & 1 & 3 \\ 2 & 0 & -1 \end{pmatrix} = (-1)^{2+1} \underbrace{(-1)}_{a_{21}} det \begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix} + (-1)^{2+2} \underbrace{(1)}_{a_{22}} det \begin{pmatrix} -2 & -3 \\ 2 & -1 \end{pmatrix}$$

$$+ (-1)^{2+3} \underbrace{(3)}_{a_{23}} det \begin{pmatrix} -2 & 2 \\ 2 & 0 \end{pmatrix} = (-1)(-1)(-2) + 8 + (-1)3(-4) = 18$$

Determinant of a matrix in R $(n > 2)$

```
# create a square matrix A of size n = 3
A <- matrix(c(-2,2,-3,-1,1,3,2,0,-1), nrow = 3, byrow=TRUE)
A
#      [,1] [,2] [,3]
# [1,]   -2    2   -3
# [2,]   -1    1    3
# [3,]    2    0   -1

det(A)
# [1] 18

# create a square matrix A2 of size n = 4
A2 <- matrix(c(2,0,3,1,-1,-1,-1,1,0,2,0,3,-3,-3,-2,9), nrow = 4, byrow=TRUE)
A2
#      [,1] [,2] [,3] [,4]
# [1,]    2    0    3    1
# [2,]   -1   -1   -1    1
# [3,]    0    2    0    3
# [4,]   -3   -3   -2    9
det(A2)
# [1] 0
# A2 is not invertible because det(A2) = 0.
```

Invertibility of a matrix

A square matrix A is said to be invertible if and only if its determinant is non zero, that is we have $det(A) \neq 0$. Then matrix A is full rank and can be decomposed as follows:

$$AA^{-1} = I_n$$

where A^{-1} is the inverse of matrix A and I_n is the identity matrix of size n. A matrix which is not invertible is called singular or degenerate.

One method to find the inverse of a matrix is to use the Gauss method on a square matrix A and perform the same operations on the identity matrix I_n. Once A is reduced to the identity matrix, then the identity matrix on which the operations have been applied becomes A^{-1}.

Another method to find the inverse of a square matrix A is the LU decomposition, which factors the orignial matrix into the product of a lower triangular matrix L and an upper triangular matrix U.

Matrix inversion in R

Assume that we want to invert the following matrix A

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & -1 & 3 \\ 0 & 3 & 7 \end{pmatrix}$$

In R, we can use the function *solve()*.

```
# Inverse of a matrix
# create a square matrix A of size n = 4
A <- matrix(c(1,2,-1,-1,-1,3,0,3,7), nrow = 3, byrow=TRUE)
A
#      [,1] [,2] [,3]
# [1,]    1    2   -1
# [2,]   -1   -1    3
# [3,]    0    3    7
solve(A) # returns A inverse
#      [,1] [,2] [,3]
# [1,]  -16  -17    5
# [2,]    7    7   -2
# [3,]   -3   -3    1
# check
round(A %*% solve(A))
#       [,1] [,2] [,3]
# [1,]    1    0    0
# [2,]    0    1    0
# [3,]    0    0    1
```

LU decomposition

For a square matrix A, we look for a lower triangular square matrix L and an upper triangular square matrix U such that $A = LU$.

Solving a system of the form $A\mathbf{x} = \mathbf{b}$ then comes down to solving the system $LU\mathbf{x} = \mathbf{b}$. The two matrices L and U are already in row echelon form, so that the system is easier to solve.

In addition, we have the following relationship $det(A) = det(L)det(U)$.

To find the inverse of a square matrix $A = LU$, we have to invert L and U. Then, we have that $A^{-1} = U^{-1}L^{-1}$.

LU decomposition by hand

Assume that we must apply LU decomposition on the following matrix A

$$\underbrace{\begin{pmatrix} 1 & 2 & -1 \\ -1 & -1 & 3 \\ 0 & 3 & 7 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}}_{U}$$

So, by matrix multiplication, we have that

$$\underbrace{\begin{pmatrix} 1 & 2 & -1 \\ -1 & -1 & 3 \\ 0 & 3 & 7 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{pmatrix}}_{LU}$$

Searching term by term for the coefficients of LU yields the following results

$$\underbrace{\begin{pmatrix} 1 & 2 & -1 \\ -1 & -1 & 3 \\ 0 & 3 & 7 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}}_{U}$$