# Some definitions

**System of equations.** A system of equations is a set of equations having the same unknowns. A solution must satisty all the equations of the system. A system is said to be linear if all equations are linear, that is polynomial of order 1.

**Matrix.** A rectangular table of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where the entries are numbers is called a matrix. $A = (a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ is of size $m \times n$.

# Product of a matrix and a vector

The product of a matrix $A = (a_{ij})$, of size $m \times n$ and a vector $\mathbf{x}$ of size $n$ results in a vector having $m$ entries. It is given by

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}}_{A\mathbf{x}}$$

```r
# create matrix A and vector x
A = matrix(c(1,0,-1,0,0,0,2,0,-2,0,0,1,-1,0,-1,0,4,-4,-1,-2), nrow = 4, byrow = TRUE)
A
#      [,1] [,2] [,3] [,4] [,5]
# [1,]    1    0   -1    0    0
# [2,]    0    2    0   -2    0
# [3,]    0    1   -1    0   -1
# [4,]    0    4   -4   -1   -2
x = c(1,1,1,1,1)

# product matrix A by vector x
A %*% x
#      [,1]
# [1,]    0
# [2,]    0
# [3,]   -1
# [4,]   -3
```

# Systems of equations

The system of equations

$$\begin{cases} 3x_1 - 2x_2 + x_3 = 1 \\ \quad x_1 + x_2 - x_3 = 3 \end{cases}$$

is written in matrix form as follows:

$$\underbrace{\begin{pmatrix} 3 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}_{\mathbf{b}}$$

We have $A\mathbf{x} = \mathbf{b}$. Moreover, $B$ is called the augmented matrix of the system and is given by

$$\underbrace{\begin{pmatrix} 3 & -2 & 1 & 1 \\ 1 & 1 & -1 & 3 \end{pmatrix}}_{B}$$

# Solutions of a system of equations

A system of equations of the form $A\mathbf{x} = \mathbf{b}$ has a solution if $\mathbf{b}$ is a linear combination of the columns of $A$. That is we have

$$\mathbf{b} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} x_2 + \ldots + \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} x_n$$

Moreover, a system of equations is said to be compatible if it has at least one solution. Otherwise, it is not compatible.

Eventually, a system is said to be homogeneous if $\mathbf{b} = \mathbf{0}$. In this case, $\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is one solution of the system.

# Systems of equations in R (1)

For the general case of non-square matrices, we can apply Singular Value Decomposition (SVD) to solve the systems of equations.

```
# solve a linear system of equations (for non square matrices)
A = matrix(c(3,-2,1,1,1,-1), nrow = 2, byrow = TRUE)
A
#      [,1] [,2] [,3]
# [1,]    3   -2    1
# [2,]    1    1   -1
b = matrix(c(1,3), nrow = 2, byrow = TRUE)
b
#      [,1]
# [1,]    1
# [2,]    3

# Solve the linear system Ax = b
sol <- svd(A)$v %*% diag(1/(svd(A)$d)) %*% t(svd(A)$u) %*% b
sol
#            [,1]
# [1,]  1.2142857
# [2,]  0.8571429
# [3,] -0.9285714

# check
A[1,1] * sol[1] + A[1,2] * sol[2] + A[1,3] * sol[3]
# [1] 1
A[2,1] * sol[1] + A[2,2] * sol[2] + A[2,3] * sol[3]
# [1] 3
```

# Systems of equations in R (2)

For the particular case of square matrices, we can apply the function *solve()*.

```
# solve a linear system of equations (square matrices)
A2 = matrix(c(3,-2,1,1), nrow = 2, byrow = TRUE)
A2
#      [,1] [,2]
# [1,]    3   -2
# [2,]    1    1
b = matrix(c(1,3), nrow = 2, byrow = TRUE)
b
#      [,1]
# [1,]    1
# [2,]    3

# Solve the linear system A2x = b
sol <- solve(A2,b)
sol
#      [,1]
# [1,]  1.4
# [2,]  1.6

# check
A2[1,1] * sol[1] + A2[1,2] * sol[2]
# [1] 1
A2[2,1] * sol[1] + A2[2,2] * sol[2]
# [1] 3
```

# Example: linear equations in Chemistry

Copper $(Cu)$ and sullfuric acid $(H_2SO_4)$ $\Longleftrightarrow$ Water $(H_2O)$, Sulfur dioxide $(SO_2)$ and Copper sulfate $(CuSO_4)$.

We set

$$x_1Cu + x_2H_2SO_4 = x_3H_2O + x_4SO_2 + x_5CuSO_4$$

We get the following system of equations

$$\begin{cases} x_1 = x_5 & \leftarrow Cu \\ 2x_2 = 2x_3 & \leftarrow H \\ x_2 = x_4 + x_5 & \leftarrow S \\ 4x_2 = x_3 + 2x_4 + 4x_5 & \leftarrow O \end{cases}$$

One solution of the system is $x_1 = 1, x_2 = 2, x_3 = 2, x_4 = 1, x_5 = 1$. This yields the reaction $Cu + 2H_2SO_4 = 2H_2O + SO_2 + CuSO_4$.