Eigenvalues and eigenvectors

Let A be a square matrix of size n. λ_i is an eigenvalue of matrix A if it is a solution of the equation

$$det(A - \lambda I) = 0$$

A square matrix of size n has n eigenvalues. In addition, the trace of a square matrix is always equal to the sum of its eigenvalues and the determinant of a square symmetric matrix is always equal to the product of its eigenvalues.

The vector $\mathbf{u}_i \neq 0$ is an eigenvector of A associated to the eigenvalue λ_i if

$$A\mathbf{u}_i = \lambda_i \mathbf{u}_i$$

If A is a square, real, symmetric matrix, then there exist n normed, orthogonal eigenvectors.

Diagonalization of a square matrix

Let A be a square matrix. We are looking for a square matrix P such that $P^{-1}AP$ is diagonal, that is its extradiagonal entries are zeros. To diagonalize A, we proceed as follows:

1) We compute the roots of the characteristic polynomial of $det(A - \lambda I) = 0$. In other words, we compute the eigenvalues of A, that is λ_i.

2) For each eigenvalue λ_i, we determine the corresponding eigenspace E_{λ_i}.

3) For each eigenspace we give a basis. If we have n vectors (if the sum of the dimensions of the eigenspaces is n), then we create the matrix P which has as columns the vectors of the basis of all eigenspaces E_{λ_i}.

Then, the matrix $P^{-1}AP = \Lambda$ is diagonal and each λ_i appears in the diagonal a number of times equal to $dim(E_{\lambda_i})$.

Example of diagonalization

We consider the following matrix A that we want to diagonalize, that is to find the diagonal matrix Λ and the square matrix P such that $P^{-1}AP = \Lambda$.

$$\text{A} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Its eigenvalues are found by computing $det(A - \lambda I) = 0$. We have

Eigenvalues of example matrix

$$det(A - \lambda I) = det \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$= (-1)^{1+1}(2-\lambda)det \begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} +(-1)^{2+1} + (-1) + det \begin{pmatrix} -1 & -1 \\ -1 & 2-\lambda \end{pmatrix}$$

$$+ (-1)^{1+3}(-1)det \begin{pmatrix} -1 & -1 \\ -1 & 2-\lambda \end{pmatrix}$$

$$\begin{aligned} &= (2-\lambda)\Big((2-\lambda)^2 - 1\Big) + (-1)(2-\lambda) - 1(-1)(2-\lambda) \\ &= -\lambda^3 + 6\lambda^2 - 9\lambda \\ &= -\lambda(\lambda - 3)^2 \end{aligned}$$

So the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 3$ and $\lambda_3 = 3$.

Basis of eigenspace E_0

First, we set

$$(A - \lambda_1 I)\mathbf{x} = 0$$

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

In row echelon form, we get

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So $dim(E_0) = 1$ and a basis is given by

$$E_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 2x_2 - x_3, x_2 = x_3\} =< (1, 1, 1) >$$

Basis of eigenspace E_3

First, we set

$$(A - \lambda_2 I)\mathbf{x} = 0$$

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

In row echelon form, we get

$$\begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So $dim(E_3) = 2$ and bases are given by

$$E_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = -x_2 - x_3\} =< (-1, 1, 0), (-1, 0, 1) >$$

Diagonalization

Since $dim(E_0) + dim(E_3) = 1 + 2 = 3$, A can be diagonalized and we have

$$\Lambda = P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

With P being given by

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

It is the matrix containing the bases of the eignenspaces as columns.

Diagonalization in R

```
# Diagonalization of a matrix

# Create matrix A
A <- matrix(c(2,-1,-1,-1,2,-1,-1,-1,2), nrow = 3, byrow=TRUE)
A
#       [,1] [,2] [,3]
# [1,]     2   -1   -1
# [2,]    -1    2   -1
# [3,]    -1   -1    2

P <- eigen(A)$vectors
Lambda <- diag(eigen(A)$values)
round(solve(P) %*% A %*% P)
#       [,1] [,2] [,3]
# [1,]     3    0    0
# [2,]     0    3    0
# [3,]     0    0    0
```