

# The Gauss method

**Main idea.** The set of solutions of a system of linear equations does not change if:

- 1) we multiply an equation by a scalar  $\alpha$ , for  $\alpha \neq 0$ .
- 2) we change the order of the equations.
- 3) we add to an equation a linear combination of other equations.

The Gauss method is applied on the augmented matrix and produces a row echelon form matrix. The Gauss method proceeds as follows:

- 1) we exchange the order of the equations so that the coefficient of the first variable in the first row is different from zero.
- 2) we multiply the first equation by a scalar so that the coefficient of the first variable is one.
- 3) we add a combination of the first equation to all other equations so that the first variable disappears from all equations except for the first one.
- 4) we repeat this procedure until we obtain a row echelon form matrix.

# Example of a system

Suppose that we must solve the following system

$$\begin{cases} 3x_1 + 2x_2 + 4x_3 = 0 \\ x_2 + 2x_3 = 2 \\ -x_1 + x_2 - x_3 = 1 \end{cases}$$

In matrix form, we have

$$\begin{pmatrix} 3 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$A \qquad \mathbf{x} \qquad \mathbf{b}$

And the augmented matrix of the system is

$$\begin{pmatrix} 3 & 2 & 4 & 0 \\ 0 & 1 & 2 & 2 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

$B$

# Row echelon form of a matrix in R

```
# Create matrix A and vector b
library(matlib)

A <- matrix(c(3,2,4,0,1,2,-1,1,-1), nrow = 3, byrow=TRUE)
A
#      [,1] [,2] [,3]
# [1,]    3    2    4
# [2,]    0    1    2
# [3,]   -1    1   -1

b <- c(0,2,1)

# (reduced) Row echelon form using the Gauss method
echelon(A, b, verbose=TRUE, fractions=TRUE)
#...
# row: 3
#
# multiply row 3 by 6/5 and subtract from row 1
# [,1] [,2] [,3] [,4]
# [1,]    1    0    0 -4/3
# [2,]    0    1 1/5 3/5
# [3,]    0    0    1 7/9
#
# multiply row 3 by 1/5 and subtract from row 2
# [,1] [,2] [,3] [,4]
# [1,]    1    0    0 -4/3
# [2,]    0    1    0 4/9
# [3,]    0    0    1 7/9
```

# Rank of a matrix

The **rank of a matrix** is the number of non-zero rows in its row echelon form.

In the previous example, we have  $\text{rank}(A) = \text{rank}(B) = 3$ , so the system has at least one solution. It is given by  $(x_1, x_2, x_3) = \left(-\frac{4}{3}, \frac{4}{9}, \frac{7}{9}\right)$ .

More generally, if  $\text{rank}(A) < \text{rank}(B)$ , then the system has no solution.

**Existence of solutions.** The linear system  $A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$  associated with matrix  $A$  of size  $m \times n$  and having augmented matrix  $B = \begin{pmatrix} & b_1 \\ A & \vdots \\ & b_m \end{pmatrix}$  has solutions if and only if  $\text{rank}(A) = \text{rank}(B)$ .

# Rank of a matrix in R

Suppose that we have the following systems, for which we give the augmented matrices here. We will determine the rank of the matrices in R and discuss the number of solutions.

$$\begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 4 \\ 1 & -1/2 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & -1 & 3 \\ -1 & 0 & 3 & -2 \\ 5 & 6 & -9 & 13 \end{pmatrix}$$

```
# example 1
```

```
A <- matrix(c(2,0,2,1,1,-1/2), nrow = 3, byrow=TRUE)
```

```
A
```

```
b <- c(1,4,0)
```

```
# rank of A and rank of the augmented matrix B
```

```
c(qr(A)$rank, qr(cbind(A,b))$rank)
```

```
# [1] 2 3 (rank(A) < rank(B): the system has no solution)
```

```
# example 2
```

```
A <- matrix(c(1,2,-1,-1,0,3,5,6,-9), nrow = 3, byrow=TRUE)
```

```
A
```

```
b <- c(3,-2,13)
```

```
# rank of A and rank of the augmented matrix B
```

```
c(qr(A)$rank, qr(cbind(A,b))$rank)
```

```
# [1] 2 2 (rank(A) = rank(B) < 3: the system has many solutions)
```

# Properties of a set of solutions for homogeneous systems

The **addition** of two solutions of an homogeneous system  $A\mathbf{x} = \mathbf{0}$  and  $A\mathbf{x}' = \mathbf{0}$  is another solution of the system. We have  $A\mathbf{x} + A\mathbf{x}' = \mathbf{0} \Leftrightarrow A(\mathbf{x} + \mathbf{x}') = \mathbf{0}$ .

The **multiplication** of a scalar  $\alpha \in \mathbb{R}$  and a solution of an homogeneous system  $A\mathbf{x} = \mathbf{0}$  is another solution of the system. We have  $\alpha(A\mathbf{x}) = \mathbf{0} \Leftrightarrow A(\alpha\mathbf{x}) = \mathbf{0}$ .

So the **set of solutions of an homogeneous system**  $A\mathbf{x} = \mathbf{0}$  is a **vector space**. (It is stable for addition and multiplication).

# Properties of a set of solutions for non homogeneous systems

The set of solutions of a non homogeneous system  $A\mathbf{x} = \mathbf{b}$  has the form  $V + w = \{v + w : v \in V\}$  where  $V$  is the vector space of the solutions of  $A\mathbf{x} = \mathbf{0}$ .

For example, the set of solutions of the system

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ x_1 - x_3 = 0 \end{cases}$$

is  $\{(x_3, 2x_3 - 1, x_3) : x_3 \in \mathbb{R}\}$  or equivalently  $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_3, x_2 = 2x_3 - 1\}$ . Now, the homogeneous system

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ x_1 - x_3 = 0 \end{cases}$$

has solution  $V = \{(x_3, 2x_3, x_3) : x_3 \in \mathbb{R}\}$ . A solution of the non homogeneous system was  $w = (0, -1, 0)$ , so the set of solutions of that system is

$$V + w = \{(x_3, 2x_3, x_3) + (0, -1, 0) : x_3 \in \mathbb{R}\} = \{(x_3, 2x_3 - 1, x_3) : x_3 \in \mathbb{R}\}.$$

# Existence of solutions and conclusions

A system of linear equations has either 0,1 or an infinite number of solutions.

An homogeneous system has at least one unique solution; the trivial solution  $\mathbf{x} = (0, 0, \dots, 0)$ .

An homogeneous system has a unique solution (the trivial solution) if  $\text{rank}(A) = n$ , where  $n$  is the number of unknowns in the system.

An homogeneous system has an infinite number of solutions if  $\text{rank}(A) = \text{rank}(B) < n$ .

A non homogeneous system has solutions if and only if  $\text{rank}(A) = \text{rank}(B)$ .