The Gauss method

Main idea. The set of solutions of a system of linear equations does not change if:

1) we multiply an equation by a scalar α, for $\alpha \neq 0$.

2) we change the order of the equations.

3) we add to an equation a linear combination of other equations.

The Gauss method is applied on the augmented matrix and produces a row echelon form matrix. The Gauss method proceeds as follows:

1) we exchange the order of the equations so that the coefficient of the first variable in the first row is different from zero.

2) we multiply the first equation by a scalar so that the coefficient of the first variable is one.

3) we add a combination of the first equation to all other equations so that the first variable disappears from all equations except for the first one.

4) we repeat this procedure until we obtain a row echelon form matrix.

Example of a system

Suppose that we must solve the following system

$$\left\{ \begin{aligned} 3x_1 + 2x_2 + 4x_3 &= 0 \\ x_2 + 2x_3 &= 2 \\ -x_1 + x_2 - x_3 &= 1 \end{aligned} \right.$$

In matrix form, we have

$$\underbrace{\begin{pmatrix} 3 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 1 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}}_{\mathbf{b}}$$

And the augmented matrix of the system is

$$\underbrace{\begin{pmatrix} 3 & 2 & 4 & 0 \\ 0 & 1 & 2 & 2 \\ -1 & 1 & -1 & 1 \end{pmatrix}}_{B}$$

Row echelon form of a matrix in R

```
# Create matrix A and vector b
library(matlib)

A <- matrix(c(3,2,4,0,1,2,-1,1,-1), nrow = 3, byrow=TRUE)
A
#      [,1] [,2] [,3]
# [1,]    3    2    4
# [2,]    0    1    2
# [3,]   -1    1   -1

b <- c(0,2,1)

# (reduced) Row echelon form using the Gauss method
echelon(A, b, verbose=TRUE, fractions=TRUE)
#...
# row: 3
#
# multiply row 3 by 6/5 and subtract from row 1
# [,1] [,2] [,3] [,4]
# [1,]    1    0    0 -4/3
# [2,]    0    1  1/5  3/5
# [3,]    0    0    1  7/9
#
# multiply row 3 by 1/5 and subtract from row 2
# [,1] [,2] [,3] [,4]
# [1,]    1    0    0 -4/3
# [2,]    0    1    0  4/9
# [3,]    0    0    1  7/9
```

Rank of a matrix

The rank of a matrix is the number of non-zero rows in its row echelon form.

In the previous example, we have $rank(A) = rank(B) = 3$, so the system has at least one solution. It is given by $(x_1, x_2, x_3) = \left(-\frac{4}{3}, \frac{4}{9}, \frac{7}{9} \right)$.

More generally, if $rank(A) < rank(B)$, then the system has no solution.

Existence of solutions. The linear system $A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ associated with matrix A of size $m \times n$ and having augmented matrix $B = \begin{pmatrix} & b_1 \\ A & \vdots \\ & b_m \end{pmatrix}$ has solutions if and only if $rank(A) = rank(B)$.

Rank of a matrix in R

Suppose that we have the following systems, for which we give the augmented matrices here. We will determine the rank of the matrices in R and discuss the number of solutions.

$$\begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 4 \\ 1 & -1/2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & -1 & 3 \\ -1 & 0 & 3 & -2 \\ 5 & 6 & -9 & 13 \end{pmatrix}$$

```
# example 1
A <- matrix(c(2,0,2,1,1,-1/2), nrow = 3, byrow=TRUE)
A
b <- c(1,4,0)

# rank of A and rank of the augmented matrix B
c(qr(A)$rank, qr(cbind(A,b))$rank)
# [1] 2 3 (rank(A) < rank(B): the system has no solution)

# example 2
A <- matrix(c(1,2,-1,-1,0,3,5,6,-9), nrow = 3, byrow=TRUE)
A
b <- c(3,-2,13)

# rank of A and rank of the augmented matrix B
c(qr(A)$rank, qr(cbind(A,b))$rank)
# [1] 2 2 (rank(A) = rank(B) < 3: the system has many solutions)
```

Properties of a set of solutions for homogeneous systems

The addition of two solutions of an homogeneous system $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x}^{'} = \mathbf{0}$ is another solution of the system. We have $A\mathbf{x} + A\mathbf{x}^{'} = \mathbf{0} \Leftrightarrow A(\mathbf{x}\mathbf{x}^{'}) = \mathbf{0}$.

The multiplication of a scalar $\alpha \in \mathbb{R}$ and a solution of an homogeneous system $A\mathbf{x} = \mathbf{0}$ is another solution of the system. We have $\alpha(A\mathbf{x}) = \mathbf{0} \Leftrightarrow A(\alpha\mathbf{x}) = \mathbf{0}$.

So the set of solutions of an homogeneous system $A\mathbf{x} = \mathbf{0}$ is a vector space. (It is stable for addition and multiplication).

Properties of a set of solutions for non homogeneous systems

The set of solutions of a non homogeneous system $A\mathbf{x} = \mathbf{b}$ has the form $V + w = \{v + w : v \in V\}$ where V is the vector space of the solutions of $A\mathbf{x} = \mathbf{0}$.

For example, the set of solutions of the system

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ x_1 - x_3 = 0 \end{cases}$$

is $\{(x_3, 2x_3 - 1, x_3) : x_3 \in \mathbb{R}\}$ or equivalently $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_3, x_2 = 2x_3 - 1\}$. Now, the homogeneous system

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ x_1 - x_3 = 0 \end{cases}$$

has solution $V = \{(x_3, 2x_3, x_3) : x_3 \in \mathbb{R}\}$. A solution of the non homogeneous system was $w = (0, -1, 0)$, so the set of solutions of that system is $V + w = \{(x_3, 2x_3, x_3) + (0, -1, 0) : x_3 \in \mathbb{R}\} = \{(x_3, 2x_3 - 1, x_3) : x_3 \in \mathbb{R}\}$.

Existence of solutions and conclusions

A system of linear equations has either 0,1 or an infinite number of solutions.

An homogeneous system has at least one unique solution; the trivial solution $\mathbf{x} = (0, 0, ..., 0)$.

An homogeneous system has a unique solution (the trivial solution) if $rank(A) = n$, where n is the number of unknowns in the system.

An homogeneous system has an infinite number of solutions if $rank(A) = rank(B) < n$.

A non homogeneous system has solutions if and only if $rank(A) = rank(B)$.