Exponential distribution : introduction

The Exponential distribution is mainly used to model waiting time untill events or lifetime duration. In survival analysis, the exponential distribution is commonly used to model the time until an event occurs, such as time to failure in engineering, time to death in medical studies or time to customer churn in business.

Let us recall the probability density function (pdf) of the exponential distribution (called here Exponential I) and an alternative parametrization for the exponential distribution (Exponential II).

Exponential I $\quad f_\lambda(x) = \lambda e^{-\lambda x} \, \mathbb{1}_{\mathbb{R}_+}(x)$

Exponential II $\quad f_\lambda(x) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \, \mathbb{1}_{\mathbb{R}_+}(x)$

Exponential I: expectation

We consider the **Exponential I** parametrization. Let $X \sim Exp(\lambda)$. Then $E[X] = \frac{1}{\lambda}$ and $var(X) = \frac{1}{\lambda^2}$. Proof.

Fist moment (expectation):

$$E[X] = \int_{-\infty}^{+\infty} x \ f_\lambda(x) \ dx = \int_0^{\infty} x \ \lambda e^{-\lambda x} \ dx$$

$$= \int_0^{\infty} u \ e^{-u} \ \frac{du}{\lambda} \qquad (u = \lambda x \Leftrightarrow x = u/\lambda, \quad du = \lambda dx \Leftrightarrow dx = du/\lambda)$$

$$= \frac{1}{\lambda} \underbrace{\int_0^{\infty} u \ e^{-u} \ du}_{=\Gamma(2)=1\Gamma(1)=1} = \frac{1}{\lambda}$$

Exponential I: second moment and variance

Second moment:

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 \ f_\lambda(x) \ dx = \int_0^\infty x^2 \ \lambda e^{-\lambda x} \ dx = \frac{1}{\lambda} \int_0^\infty u^2 \ e^{-u} \ \frac{du}{\lambda}$$

$$= \frac{1}{\lambda} \frac{1}{\lambda} \underbrace{\int_0^\infty u^2 \ e^{-u} \ du}_{=\Gamma(3)=2\Gamma(2)=2*1=2} = 2 * \frac{1}{\lambda^2} = \frac{2}{\lambda^2}$$

By definition, the variance is:

$$var(X) = E[X^2] - \left(E[X]\right)^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

Working example

A manufacturing company is interested in analyzing the time it takes for their products to fail after being put into service. They assume that the product failure time obeys an exponential distribution with an average lifetime of 500 hours. What is the probability that a randomly chosen product will fail within 300 hours?

Using the CDF: $F_\lambda(x) = 1 - e^{-\lambda x}$. We are looking for $F_\lambda(300) = 1 - e^{-\frac{1}{500}300} = 1 - e^{-3/5} = 0.4512$, or slightly above 45%.

```
mean = 500; rate = 1 / mean; time_to_fail = 300
# probability that the product fails withing 300 hours
1 - exp(- rate * time_to_fail)
# [1] 0.4511884
```

```
import scipy.stats as stats
mean = 500; rate = 1 / mean; time_to_fail = 300
# probability that the product fails withing 300 hours
stats.expon.cdf(time_to_fail, scale = 1/rate)
# 0.4511883639059735
```

Memorylessness

The Exponential distribution is the only continuous **memoryless** probability distribution. The memorylessness property is a key characteristic of the exponential distribution. For an exponentially distributed r.v. X, the conditional probability of X exceeding a specific time $t+s$ (future time) given that it has already survived up to time t (past time) is equal to the unconditional probability of X exceeding s. In other words, the future behavior of the random variable is independent of its past behavior.

$$P(X > x) = 1 - P(X \leq x) = 1 - F_\lambda(x) = 1 - (1 - e^{-\lambda x}) = e^{-\lambda x}$$

$$P(X > t + s) = e^{-\lambda(t+s)} = e^{-\lambda t}e^{-\lambda s} = P(X > t)P(X > s)$$

Exponential II: expectation

We consider the **Exponential II** parametrization. Let $X \sim Exp(\lambda)$. Then $E[X] = \lambda$ and $var(X) = \lambda^2$. Proof.

Fist moment (expectation):

$$E[X] = \int_{-\infty}^{+\infty} x \ f_\lambda(x) \ dx = \int_0^\infty x \ \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \ dx$$

$$= \int_0^\infty u \ e^{-u} \ \lambda du \qquad\qquad (u = \frac{x}{\lambda} \Leftrightarrow x = \lambda u, \quad du = \frac{dx}{\lambda} \Leftrightarrow dx = \lambda du)$$

$$= \lambda \underbrace{\int_0^\infty u \ e^{-u} \ du}_{=\Gamma(2)=1\Gamma(1)=1} = \lambda$$

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Exponential II: second moment and variance

Second moment:

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 \ f_\lambda(x) \ dx = \int_{0}^{\infty} x^2 \ \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \ dx = \lambda \int_{0}^{\infty} u^2 \ e^{-u} \ \lambda du$$

$$= \lambda \ \lambda \underbrace{\int_{0}^{\infty} u^2 \ e^{-u} \ du}_{=\Gamma(3)=2\Gamma(2)=2*1=2} = 2 * \lambda^2 = 2\lambda^2$$

By definition, the variance is:

$$var(X) = E[X^2] - \left(E[X]\right)^2 = 2\lambda^2 - \left(\lambda\right)^2 = \lambda^2$$

References

R.V.Hogg and E.A.Tanis: Probability and Statistical Inference, Sixth Edition, Prentice Hall, Upper Saddle River, N.J., 2001.

Ross, S., A First Course in Probability, Eighth Edition, Pearson, 2010

The R Project for Statistical Computing:
https://www.r-project.org/

Python:
https://www.python.org/

course notes