Overview on the Development of Fuzzy Random Variables [1]

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Abstract

This paper presents a backward analysis on the interpretation, modelling and impact of the concept of fuzzy random variable. After some preliminaries, the situations modelled by means of fuzzy random variables as well as the main approaches to model them are explained. We also summarize briefly some of the probabilistic studies concerning this concept as well as some statistical applications.

Key words: d_∞-metric, fuzzy arithmetic, fuzzy values, fuzzy random variables, Hausdorff metric, random sets, (W, φ)-metric.

1 Introduction

Fuzzy random variables were introduced to model and analyze 'imprecisely-valued' measurable functions associated with the sample space of a random experiment, when the imprecision in values of these functions is formalized in terms of fuzzy sets.

Different approaches to this concept have been developed in the literature, the most widely considered being that introduced by Kwakernaak (1978, 1979) and Kruse and Meyer (1987), and the one by Puri and Ralescu (1986) (and Klement, Puri and Ralescu, 1986).

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In this paper we recall these main approaches and refer to some of the relevant studies in Probability and Statistics involving them.

The structure of the paper is as follows: in Section 2 we present some preliminaries on the fuzzy values we will deal with; in Section 3 the two main approaches to the concept of fuzzy random variable are given and illustrated; Section 4 gathers some probabilistic results concerning fuzzy random variables; finally, Section 5 discusses statistical problems with fuzzy random variables.

2 Preliminaries on fuzzy values

Consider the p-dimensional Euclidean space $\mathbb{R}^p$ and let $\|\cdot\|$ denote the usual associated norm. Let $\mathcal{K}(\mathbb{R}^p)$ denote the class of the nonempty compact subsets of $\mathbb{R}^p$ and let $\mathcal{K}_c(\mathbb{R}^p)$ be the subclass of convex sets in $\mathcal{K}(\mathbb{R}^p)$.

The largest *class of fuzzy values* we will deal with in this paper is the following:

$$\mathcal{F}(\mathbb{R}^p) = \{U : \mathbb{R}^p \to [0,1] \mid U_\alpha \in \mathcal{K}(\mathbb{R}^p) \text{ for all } \alpha \in [0,1]\}$$

(where U_α denotes the α-level of fuzzy set U for all $\alpha \in (0,1]$, and U_0 being the closure of the support of U, that is, $U_0 = \mathrm{cl}(\operatorname{supp} U)$). In other words, $\mathcal{F}(\mathbb{R}^p)$ represents the class of the upper semicontinuous functions in $[0,1]^{\mathbb{R}^p}$ with compact closure of the support.

Concerning the *arithmetic of fuzzy values*, the basic operations are the sum and the multiplication by scalars, which are defined as follows:

The space $\mathcal{K}(\mathbb{R}^p)$ can be endowed with a semilinear structure by means of the Minkowski sum and the product by a scalar $K+K' = \{k+k' \mid k \in K,\ k' \in K'\}$, $\lambda K = \{\lambda k \mid k \in K\}$, for $K, K' \in \mathcal{K}(\mathbb{R}^p)$ and $\lambda \in \mathbb{R}$.

In a similar way, $\mathcal{F}(\mathbb{R}^p)$ can be endowed with a semilinear structure by means of the sum and the product by a scalar based on Zadeh's extension principle (1975). The application of this principle for elements in $\mathcal{F}(\mathbb{R}^p)$ states that

$$(U+V)(x) = \sup_{y+z=x} \min\{U(y), V(z)\},$$

$$(\lambda U)(x) = \sup_{\lambda y = x} U(y) = \begin{cases} U(\lambda^{-1}x) & \text{if } \lambda \neq 0, \\ \mathbf{1}_{\{0\}}(x) & \text{if } \lambda = 0, \end{cases}$$

for all $U, V \in \mathcal{F}(\mathbb{R}^p)$ and $\lambda \in \mathbb{R}$. Under the conditions imposed on fuzzy sets we are working with, and inspired by the results in Nguyen (1978), one can prove for all $\alpha \in [0,1]$ that

$$(U+V)_\alpha = U_\alpha + V_\alpha, \qquad (\lambda U)_\alpha = \lambda U_\alpha,$$

whence the absence of linearity of $\mathcal{K}(\mathbb{R}^p)$ with the considered operations from the set-valued arithmetic (i.e., there is no inverse for the Minkowski addition) is inherited by $\mathcal{F}(\mathbb{R}^p)$ with the usual fuzzy set-valued arithmetic.

Regarding *metrics between fuzzy values*, we first recall the well-known *Hausdorff metric* on $\mathcal{K}(\mathbb{R}^p)$, defined so that for any $K, K' \in \mathcal{K}(\mathbb{R}^p)$

$$d_H(K, K') = \max\left\{\sup_{k\in K}\inf_{k'\in K'} |k-k'|, \sup_{k'\in K'}\inf_{k\in K} |k-k'|\right\}.$$

The Hausdorff metric is fundamental for the measurability of fuzzy set-valued mappings in Puri and Ralescu's definition. However, to analyze probabilistic and statistical aspects involving fuzzy random variables, other metrics have also been considered.

In this way, for the purpose of discussing some limit theorems for fuzzy random variables, the d_∞*-metric* introduced by Puri and Ralescu (1981, 1983) is defined so that for $U, V \in \mathcal{F}(\mathbb{R}^p)$

$$d_\infty(U, V) = \sup_{\alpha\in[0,1]} d_H(U_\alpha, V_\alpha).$$

On the other hand, for the purpose of some inferential statistical procedures involving fuzzy random variables, the D_W^φ*-metric* introduced for the one-dimensional convex case by Bertoluzza *et al.* (1995) (and recently generalized to the p-dimensional convex case by Körner and Näther, 2002) is defined so that for $U, V \in \mathcal{F}_c(\mathbb{R})$

$$D_W^\varphi(U, V) = \sqrt{\int_{[0,1]}\int_{[0,1]} [d_{(\lambda)}^{(\alpha)}(U, V)]^2 d\,W(\lambda)\, d\,\varphi(\alpha)},$$

with

$$d_{(\lambda)}^{(\alpha)}(U, V) = |\mathrm{f}_U(\alpha, \lambda) - \mathrm{f}_U(\alpha, \lambda)|\,, \quad \mathrm{f}_U(\alpha, \lambda) = \lambda \sup U_\alpha + (1-\lambda) \inf U_\alpha,$$

and W and φ being identified with probability measures on the measurable space $([0,1], \mathcal{B}[0,1])$, W being associated with a non-degenerate distribution while φ has a distribution function which is strictly increasing on $[0,1]$. Conditions for W and φ are imposed to guarantee D_W^φ is in fact a metric. Nevertheless, it should be noted that although W and φ are identified with probability measures, the distance D_W^φ does not have a probabilistic meaning.

Remark 1 Sometimes the notion of fuzzy random variable has been formalized in a more general setting, either by considering a separable Banach space $\mathbb{B}$ instead of a Euclidean one (see, for instance, Puri and Ralescu, 1991), or by removing the compactness assumed for variable values (see, for instance, Ogura and Li, 2001, Krätschmer, 2004).

3 Fuzzy random variables: motivations and approaches

In the literature combining Fuzzy Logic and Probability Theory we can find several notions and models, like fuzzy information systems (Okuda *et al.*, 1978), fuzzy probabilities (see, for instance, Zadeh, 1986, Ralescu, 1995, Nguyen *et al.*, 1999), probabilistic sets (Hirota, 1992), and so on.

Fuzzy random variables represent a well-formalized concept underlying many recent probabilistic and statistical studies involving data obtained from a random experiment when these data are assumed to be fuzzy set-valued. Fuzzy Random Variables have been considered in the setting of a random experiment to model

- either a fuzzy perception/observation of a mechanism (the so-called 'original random variable') associating a real value with each experimental outcome,
- or an essentially fuzzy-valued mechanism, that is, a mechanism associating a fuzzy value with each experimental outcome.

For the first situation, Kwakernaak (1978, 1979) introduced a mathematical model which has been later formalized in a clear way by Kruse and Meyer (1987). In ***Kwakernaak/Kruse and Meyer's approach***, a fuzzy random variable is viewed as a fuzzy perception/observation/report of a classical real-valued random variable. The model is stated as follows:

Definition 3.1 *Given a probability space* $(\Omega, \mathcal{A}, P)$*, a mapping*

$$\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R})$$

is said to be a ***fuzzy random variable*** *(or FRV for short) if for all* $\alpha \in [0,1]$ *the two real-valued mappings*

$$\inf \mathcal{X}_\alpha : \Omega \to \mathbb{R}, \qquad \sup \mathcal{X}_\alpha : \Omega \to \mathbb{R}$$

(defined so that for all $\omega \in \Omega$ *we have that* $\mathcal{X}_\alpha(\omega) = [\inf (\mathcal{X}(\omega))_\alpha, \sup (\mathcal{X}(\omega))_\alpha]$*) are real-valued random variables.*

Probabilistic and statistical studies for FRVs in Kwakernaak/Kruse and Meyer's approach usually concern either 'crisp' parameters of the 'original' random variable or fuzzy-valued parameters defined on the basis of Zadeh's extension principle. More precisely, Kruse and Meyer (1987) have defined

Definition 3.2 *If* $\theta(X)$ *is a real-valued parameter of a random variable* $X : \Omega \to \mathbb{R}$ *associated with the probability space* $(\Omega, \mathcal{A}, P)$*, and* $\mathcal{E}(\Omega, \mathcal{A}, P)$ *denotes the class of all possible 'originals' of a FRV* $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R})$ *associated with* $(\Omega, \mathcal{A}, P)$*, then the* ***induced fuzzy parameter*** *of* $\mathcal{X}$ *corresponds to*

$$\theta(\mathcal{X}) : \mathbb{R} \to [0,1]$$

such that for all $t \in \mathbb{R}$

$$\theta(\mathcal{X})(t) = \sup_{X \in \mathcal{E}(\Omega,\mathcal{A},P) \mid \theta(X)=t} \inf_{\omega \in \Omega} \{\mathcal{X}(\omega)(X(\omega))\}.$$

As an example of an induced fuzzy parameter, we mention that if $\theta(X) = E(X|P)$, then $\theta(\mathcal{X})$ corresponds to the so-called *fuzzy expected value* of $\mathcal{X}$ which is the fuzzy set in $\mathcal{F}_c(\mathbb{R})$ such that for each $\alpha \in [0,1]$

$$(\theta(\mathcal{X}))_\alpha = [E(\inf \mathcal{X}_\alpha|P), E(\sup \mathcal{X}_\alpha|P)].$$

It should be emphasized that the notion of (induced) fuzzy parameter is essentially different from the classical one, since classical parameters are involved explicitly in the expression for the distribution of the associated variable, whereas the fuzzy-valued parameters are not necessarily included in it.

Example 3.1 Consider the random experiment in which a given researcher chooses at random a conference including Fuzzy Logic within its scope and being held in the period 2004-2005.

Assume that we consider the random variable $\mathcal{X}$ = 'perception about the registration fee of the chosen conference for EUSFLAT members'.

This situation can be modelled by means of the concept of FRV in Kwakernaak/Kruse and Meyer's approach. Thus, the probability space corresponds to $(\Omega, \mathcal{A}, P)$:

$$\Omega = \{\text{conferences to be held 2004 to 2005}$$
$$\text{and involving Fuzzy Logic among their covered topics}\},$$
$$\mathcal{A} = \mathcal{P}(\Omega) = \text{power set of } \Omega$$

and, because of the choice of the conference is made at random,

$$P(\omega) = \frac{1}{\text{card}(\Omega)} \quad \text{for all } \omega \in \Omega.$$

On the other hand, the perception of the registration fee for EUSFLAT members can be viewed as a FRV $\mathcal{X}$ associated with $(\Omega, \mathcal{A}, P)$ and taking on the values

$$\widetilde{x}_1 = \text{'very cheap'}, \quad \widetilde{x}_2 = \text{'cheap'}, \quad \widetilde{x}_3 = \text{'rather cheap'}, \quad \widetilde{x}_4 = \text{'moderate'},$$
$$\widetilde{x}_5 = \text{'rather expensive'}, \quad \widetilde{x}_6 = \text{'expensive'}, \quad \widetilde{x}_7 = \text{'very expensive'}$$

which can be described, for instance, by means of trapezoidal fuzzy sets like those in Figure 1.

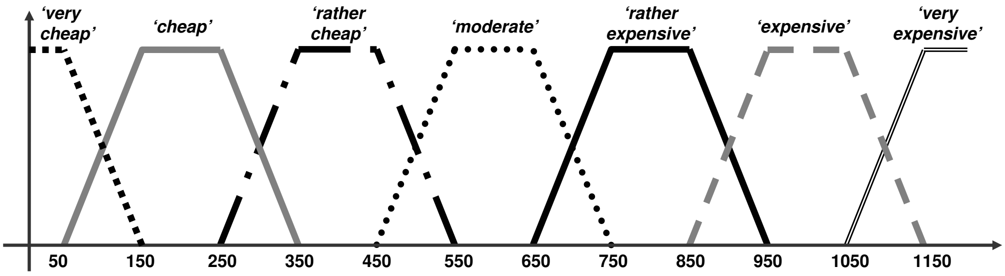

Figure 1. Values of the ‘perception of the registration fee’ of a conference (the abscissa represents the registration fee in euros)

In this example there is an underlying ‘original random variable’ which is

$$X^0 : \Omega \to \mathbb{R}$$

with

$$X^0(\omega) = \text{true registration fee for EUSFLAT members in Conference } \omega.$$

If, on the basis of the registration fees of conferences in Ω, we can get the probability distribution

$$P(\{\omega \in \Omega \,|\, \mathcal{X}(\omega) = \widetilde{x}_1\}) = .04,\ P(\{\omega \in \Omega \,|\, \mathcal{X}(\omega) = \widetilde{x}_2\}) = .14,$$
$$P(\{\omega \in \Omega \,|\, \mathcal{X}(\omega) = \widetilde{x}_3\}) = .25,\ P(\{\omega \in \Omega \,|\, \mathcal{X}(\omega) = \widetilde{x}_4\}) = .34,$$
$$P(\{\omega \in \Omega \,|\, \mathcal{X}(\omega) = \widetilde{x}_5\})\}) = .19,\ P(\{\omega \in \Omega \,|\, \mathcal{X}(\omega) = \widetilde{x}_6\}) = .03,$$
$$P(\{\omega \in \Omega \,|\, \mathcal{X}(\omega) = \widetilde{x}_7\}) = .01,$$

then if $\theta(\mathcal{X}) = E(X|P)$ we obtain

$$\inf\left(\theta(\mathcal{X})\right)_\alpha = E(\inf \mathcal{X}_\alpha|P)$$
$$= .04 \inf(\widetilde{x}_1)_\alpha + .14 \inf(\widetilde{x}_2)_\alpha + .25 \inf(\widetilde{x}_3)_\alpha + .34 \inf(\widetilde{x}_4)_\alpha$$
$$+.19 \inf(\widetilde{x}_5)_\alpha + .03 \inf(\widetilde{x}_6)_\alpha + .01 \inf(\widetilde{x}_7)_\alpha,$$

$$\sup\left(\theta(\mathcal{X})\right)_\alpha = E(\sup \mathcal{X}_\alpha|P)$$
$$= .04 \sup(\widetilde{x}_1)_\alpha + .14 \sup(\widetilde{x}_2)_\alpha + .25 \sup(\widetilde{x}_3)_\alpha + .34 \sup(\widetilde{x}_4)_\alpha$$
$$+.19 \sup(\widetilde{x}_5)_\alpha + .03 \sup(\widetilde{x}_6)_\alpha + .01 \sup(\widetilde{x}_7)_\alpha,$$

whence the fuzzy expected value is the one represented in Figure 2.

For the second situation to be modelled, Puri and Ralescu (1986) introduced another mathematical approach. In ***Puri and Ralescu’s approach***, a fuzzy random variable is viewed as a mechanism associating a fuzzy set in $\mathcal{F}(\mathbb{R}^p)$ with each experimental outcome. The model is stated as follows:

Figure 2. Expected value of the 'perception of the registration fee' of a conference

Definition 3.3 *Given a probability space* $(\Omega, \mathcal{A}, P)$*, a mapping*

$$\mathcal{X} : \Omega \to \mathcal{F}(\mathbb{R}^p)$$

is said to be a ***fuzzy random variable*** *(also called* ***random fuzzy set****, and referred to as FRV for short) if for all* $\alpha \in [0, 1]$ *the set-valued mappings*

$$\mathcal{X}_\alpha : \Omega \to \mathcal{K}(\mathbb{R}^p),$$

defined so that for all $\omega \in \Omega$

$$\mathcal{X}_\alpha(\omega) = (\mathcal{X}(\omega))_\alpha,$$

are random sets (that is, Borel-measurable mappings with the Borel σ*-field generated by the topology associated with the Hausdorff metric on* $\mathcal{K}(\mathbb{R}^p)$*).*

Klement *et al.* (1986) have slightly modified the definition above by considering also FRVs as an extension of random sets but formalizing the measurability for $\mathcal{F}(\mathbb{R}^p)$-valued mappings as the Borel-measurability with respect to the supremum metric d_∞. As concluded by Colubi *et al.* (2001, 2002a) and Kim (2002), this measurability condition is stronger than the one in Puri and Ralescu's definition (which is equivalent to considering the Borel-measurability with respect to the Skorokhod metric on $\mathcal{F}(\mathbb{R}^p)$). In this special issue, Terán gives a necessary and sufficient condition for the equivalence of the two measurability conditions.

Example 3.2 Consider the random experiment in which a client of a Savings Bank is chosen.

Consider the random variable $\mathcal{X}$ = 'classification of the chosen client in accordance with his/her degree of aversion to investment'.

The key difference between this situation and the one in Example 3.1 is that there is no underlying exact degree of aversion to investment. The situation in this example can be modelled by means of the concept of FRV in Puri and Ralescu's approach. Thus, the probability space corresponds to $(\Omega, \mathcal{A}, P)$:

$$\Omega = \{\text{clients of the considered Savings Bank}\}, \quad \mathcal{A} = \mathcal{P}(\Omega)$$

and, because of the choice of the client is supposed to be made at random,

$$P(\omega) = \frac{1}{\text{card}(\Omega)} \quad \text{for all } \omega \in \Omega.$$

On the other hand, the classification of clients with respect to their degree of aversion to investment can be viewed as an FRV $\mathcal{X}$ associated with $(\Omega, \mathcal{A}, P)$ and, for instance, taking the values

$$\widetilde{x}_1 = \text{'}\textit{very low degree of aversion}\text{'}, \quad \widetilde{x}_2 = \text{'}\textit{low degree of aversion}\text{'},$$

$$\widetilde{x}_3 = \text{'}\textit{medium degree of aversion}\text{'}, \quad \widetilde{x}_4 = \text{'}\textit{high degree of aversion}\text{'},$$

$$\widetilde{x}_5 = \text{'}\textit{very high degree of aversion}\text{'}$$

which can be described, for instance, by means of S-curve fuzzy sets like those in Figure 3.

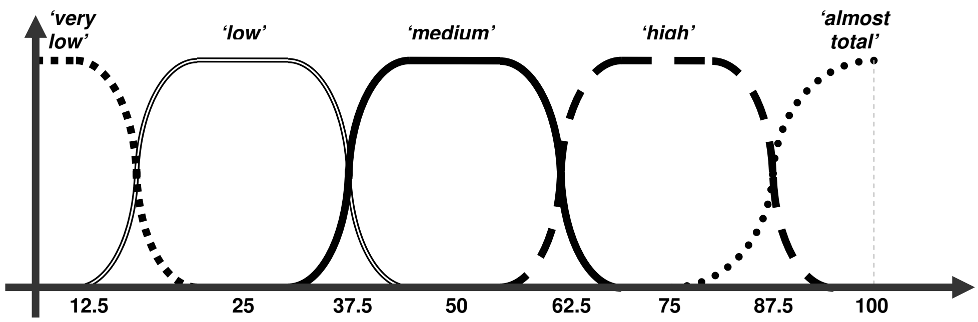

Figure 3. Values of the 'classification in accordance with the degree of aversion to investment'

Probabilistic and statistical studies for FRVs in Puri and Ralescu's approach usually concern parameters extending those from the set-valued case. The best known is the 'fuzzy expected value' which is defined (Puri and Ralescu 1986) as follows:

Definition 3.4 *Given a probability space $(\Omega, \mathcal{A}, P)$, if $\mathcal{X} : \Omega \to \mathcal{F}(\mathbb{R}^p)$ is a fuzzy random variable which is integrably bounded in the sense that $d_H(\{0\}, \mathcal{X}_0) \in L^1(\Omega, \mathcal{A}, P)$, then the* ***fuzzy expected value*** *of $\mathcal{X}$ corresponds to the unique $E(\mathcal{X} \mid P) \in \mathcal{F}(\mathbb{R}^p)$ such that for all $\alpha \in [0, 1]$*

$$(E(\mathcal{X}|P))_\alpha = \textit{Aumann's integral of the random set } \mathcal{X}_\alpha$$

$$= \{E(X|P) \mid X : \Omega \to \mathbb{R}^p,\ X \in L^1(\Omega, \mathcal{A}, P),\ X \in \mathcal{X}_\alpha \ a.s.\ [P]\}.$$

Example 3.3 Consider a probability space $(\Omega, \mathcal{A}, P)$ and a FRV $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R}^2)$ whose values are truncated cones with bases centered in $(0,0)$, the truncation point being a real-valued random variable $\beta : \Omega \to \mathbb{R}$ having uniform distribution $\mathcal{U}_{[0,1]}$ and the radius being also a real-valued random variable $\varrho : \Omega \to \mathbb{R}$ having a chi-square distribution with 1 degree of freedom (see Figure 4).

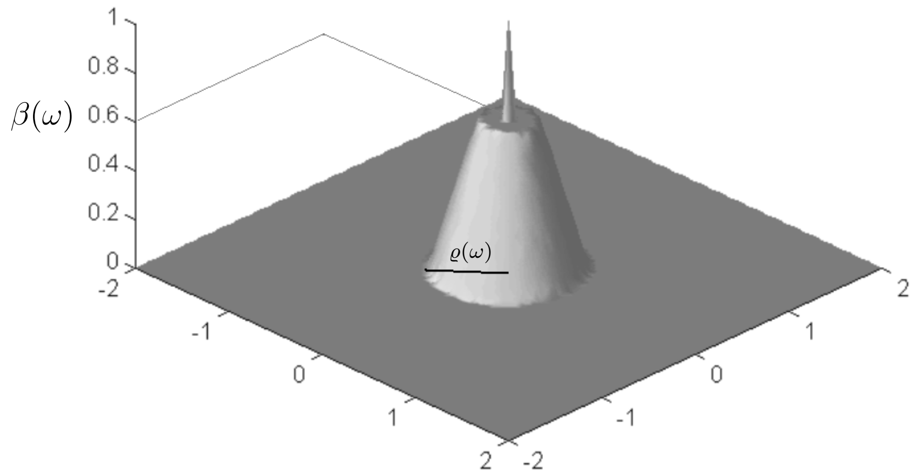

Figure 4. Value $\mathcal{X}(\omega)$ of FRV $\mathcal{X}$ in Example 3.3

In this case, the (fuzzy) value of FRV $\mathcal{X}$ in ω is given by

$$\mathcal{X}(\omega)(x,y) = \begin{cases} 0 & \text{if } x^2+y^2 > [\varrho(\omega)]^2 \\ \gamma(\omega) & \text{if } x^2+y^2 = [\gamma(\omega)]^2 \text{ with } \varrho(\omega)(1-\beta(\omega)) \le \gamma(\omega) \le \varrho(\omega) \\ \beta(\omega) & \text{if } 0 < x^2+y^2 < [\varrho(\omega)]^2[1-\beta(\omega)]^2 \\ 1 & \text{if } x^2+y^2 = 0. \end{cases}$$

and, as a consequence, the α-level mapping $\mathcal{X}_\alpha$ is given by

$$\mathcal{X}_\alpha(\omega) = \begin{cases} \{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2 \le [\varrho(\omega)]^2(1-\alpha)^2\} & \text{if } 0 \le \alpha \le \beta(\omega) \\ \{(0,0)\} & \text{if } \alpha > \beta(\omega). \end{cases}$$

If we now compute the Aumann expectation of the α-level mapping, then (see Colubi 2000, and Colubi *et al.* 2002b), it corresponds to the circle of center $(0,0)$ and radius $1-\alpha$, that is, the fuzzy expected value of $\mathcal{X}$ is the one in Figure 5.

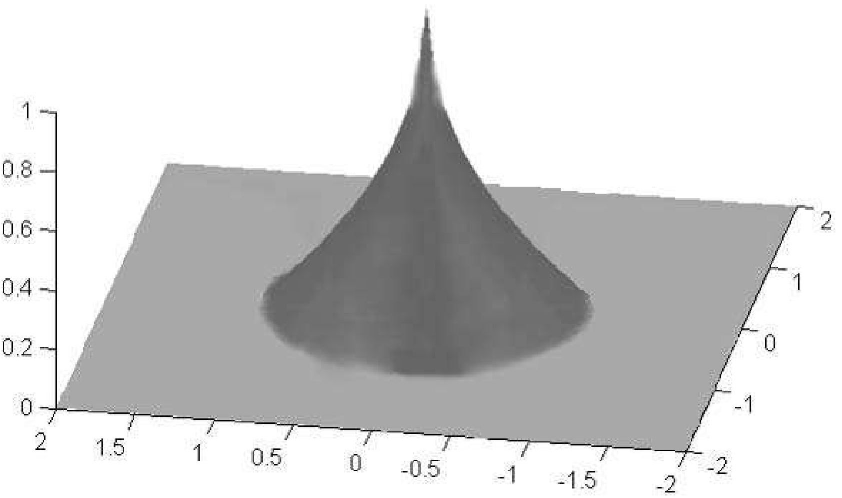

Figure 5. Fuzzy expected value of FRV $\mathcal{X}$ in Example 3.3

The two approaches for the concept of fuzzy random variable we have just recalled can be related as follows:

Proposition 3.1 *Given a probability space* $(\Omega, \mathcal{A}, P)$, *and a mapping* $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R})$, *then,* $\mathcal{X}$ *is a FRV in Kwakernaak/Kruse and Meyer's sense if, and only if, it is a FRV in Puri and Ralescu's sense.*

Proposition 3.2 *Given a probability space* $(\Omega, \mathcal{A}, P)$ *and a mapping* $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R})$, *then, if the parameter* $\theta(X)$ *of the 'original' real-valued random variable is the expected value* $E(X|P)$, *then the induced parameter* $\theta(\mathcal{X})$ *in Kruse and Meyer's sense (Definition 3.2) coincides with the fuzzy expected value* $E(\mathcal{X}|P)$ *in Puri and Ralescu's sense (Definition 3.4).*

4 Some probabilistic and statistical studies involving fuzzy random variables

For the last two decades many studies dealing with fuzzy random variables have been carried out. Most of them concern ***probabilistic aspects and results*** like, for instance,

- *integration* and *differentiation* of FRVs in probabilistic settings (cf. Puri and Ralescu 1983, López-Díaz and Gil 1998, Román-Flores and Rojas-Medar 1998, Gong and Wu 2002, Rodríguez-Múñiz *et al.* 2003ab, Krätschmer, 2004, etc.);
- *fuzzy martingales*, *sub-* and *super-martingales* (cf. Puri and Ralescu 1991, Stojaković 1994, Li *et al.* 2001, Feng 2000, Li and Ogura 2003, Terán 2004, etc.) and other stochastic processes. For a review, see also Li *et al.* 2002.

Several authors have examined Weak and Strong *Laws of Large Numbers* for FRVs to support the suitability of the fuzzy expected value concept by Puri and Ralescu (in Definition 3.4) as the limit of sample means of independent and identically distributed FRVs (see, for instance, Klement *et al.* 1984, 1986, Inoue and Taylor 1995). In particular, in the last years different approaches have been developed to state the Strong Law of Large Numbers for FRVs in terms of the d_∞ metric (cf. Colubi *et al.* 1999, Molchanov 1999, Proske and Puri 2002, etc. and more recently Terán 2005a, for a Marcinkiewicz-Zygmund type of SLLN).

Conditions and implications in Colubi *et al.* (1999) are stated as follows:

Theorem 4.1 *Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\mathcal{X}_1 : \Omega \to \mathcal{F}(\mathbb{R}^p)$ be a fuzzy random variable such that $E(d_\infty(\mathcal{X}_1, \mathbf{1}_{\{0\}})|P) < \infty$. If we suppose that $\{\mathcal{X}_n\}_n$, $\mathcal{X}_n : \Omega \to \mathcal{F}(\mathbb{R}^p)$, is a sequence of pairwise independent and identically distributed fuzzy random variables and $S_n = \mathcal{X}_1 + \ldots + \mathcal{X}_n$, then*

$$\lim_{n\to\infty} d_\infty \left(\frac{1}{n}\, S_n, E(\mathcal{X}_1 \,|\, P) \right) = 0 \quad a.s.[P],$$

where $\left(\frac{1}{n}\; S_n \right)_\alpha = \frac{1}{n}\,(S_n)_\alpha$.

Conversely, if $\{\mathcal{X}_n\}_n$, $\mathcal{X}_n : \Omega \to \mathcal{F}(B)$, is a sequence of pairwise independent and identically distributed fuzzy random variables and there exists $U \in \mathcal{F}(\mathbb{R}^p)$ so that $\lim_{n\to\infty} d_\infty \left(\frac{1}{n}\, S_n, U \right) = 0$ a.s.$[P]$, then $E(d_\infty(\mathcal{X}_1, \mathbf{1}_{\{0\}})|P) < \infty$ and $U = E(\mathcal{X}_1 \,|\, P)$.

The problem of developing limit theorems of the central type has also received attention. Nevertheless, the lack of an appropriate operational model like the normal distribution for real-valued random variables entails important differences and inconveniences. Thus, although Puri and Ralescu (1985) have stated a concept of normality for FRVs, its use becomes quite limited in practice since (fuzzy) variable values are forced to have the same shape with different location. Several authors (see, for instance, Klement *et al.* 1984 -where the first proofs for SLLN and CLT with FRVs were given-, Körner, 2000, Li *et al.* 2003, Ogura and Li 2004, Terán 2005b) have proven Central Limit Theorems with different metrics, but the essence of the classical CLTs has been lost in some sense. Among them, Li *et al.* (2003) have stated the following result:

Theorem 4.2 *Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\mathcal{X}_1 : \Omega \to \mathcal{F}(\mathbb{R}^p)$ be a fuzzy random variable such that $E([d_\infty(\mathcal{X}_1, \mathbf{1}_{\{0\}})]^2|P) < \infty$. If we suppose that $\{\mathcal{X}_n\}_n$, $\mathcal{X}_n : \Omega \to \mathcal{F}(\mathbb{R}^p)$, is a sequence of pairwise independent and identically distributed fuzzy random variables, then*

$$\sqrt{n}\, d_\infty \left(\frac{1}{n}\, S_n, E(\mathcal{X}_1 \,|\, P) \right) \text{ converges in law (i.e., weakly) to } d_\infty(\underline{Y}, \mathbf{1}_{\{0\}})$$

as $n \to \infty$, where $\underline{Y}$ is a tight Gaussian process in $l^\infty\left([0,1] \times B(\mathbb{R}^p)\right)$, $B(\mathbb{R}^p)$ denoting the closed unit ball in $\mathbb{R}^p$.

In spite of many papers having been devoted to studies of Statistics in a fuzzy environment, only a small part of the literature refers to statistical problems involving FRVs. The best known contribution is that developed by Kruse and Meyer (1987) in which most of inferences and arguments were based on the original real-valued random variables.

The studies involving FRVs in Puri and Ralescu's sense (which in fact are applicable to both approaches) have been mainly published in the last few years. Among these studies we can mention, for instance,

- statistical decision problems with fuzzy set-valued utility/loss function (cf. Gil and López-Díaz 1996, López-Díaz and Gil 1998, Rodríguez-Muñiz *et al.* 2005), in which the utility/loss function has been modelled by means of FRVs;
- regression analysis (especially in which concerns estimation problems) in which output/input data have been modelled in terms of FRVs and parameters have been supposed to be real- or fuzzy-valued (cf. Diamond 1990, Körner and Näther 1998); descriptive/probabilistic analysis of FRVs by defining measures of absolute and relative variation, like *dispersion* and *inequality indices* (e.g., Körner 1997, Gil *et al.* 1998, Lubiano *et al.* 2000, Körner and Näther 2002).

Asymptotic *hypothesis testing methods concerning real-valued measures/parameters* associated with FRVs have been developed (see, for instance, Lubiano, 1999, Lubiano *et al.* 2000).

On the other hand, recently some procedures to *test hypothesis about the fuzzy expected value of FRVs* have been established (see Körner 2000, Montenegro *et al.* 2001, 2004ab, Gil *et al.* 2005) by considering the Large Sample Theory for classical Statistics. The results by Körner (2000) are theoretically applicable to a wide class of FRVs, although in practice the methodology cannot be directly applied. On the contrary, the results by Montenegro *et al.* (2001, 2004ab), Gil *et al.* (2005), based on the D_W^{φ} metric, are immediately applicable to FRVs having a finite number of different values (the so-called simple FRVs). However, the last constraint is scarcely restrictive in practice.

As an example of an asymptotic procedure for the fuzzy expected value of an FRV we can mention the one obtained to test the two-sample problem (Montenegro *et al.* 2001) as follows:

Theorem 4.3 *For $i \in \{1, 2\}$, let $(\Omega_i, \mathcal{A}_i, P_i)$ be a probability space, and let $\mathcal{X}_i : \Omega_i \to \mathcal{F}_c(\mathbb{R})$ be a simple FRV associated with this probability space. Let $\widetilde{x}_{ik_i} \in \mathcal{F}_c(\mathbb{R})$, $k_i \in \{1, \ldots, r_i\}$ be the different values FRV $\mathcal{X}_i$ takes on Ω_i, and let $\mathbf{p}_i = (p_{i1}, \ldots, p_{i(r_i-1)})$ be the vector-valued parameter representing the population distribution of $\mathcal{X}_i$ (i.e., $p_{ik_i} = P\{\omega_i \in \Omega_i \,|\, \mathcal{X}_i(\omega_i) = \widetilde{x}_{ik_i}\}$). Let $E(\mathcal{X}_i|\mathbf{p}_i)$ denote the population fuzzy expected value of $\mathcal{X}_i$.*

For $i \in \{1,2\}$, consider a simple random sample $\mathcal{X}_{i1},\ldots,\mathcal{X}_{in_i}$ from $\mathcal{X}_i$ (i.e., $\mathcal{X}_{i1},\ldots,\mathcal{X}_{in_i}$ are independent and identically distributed, the common distribution being that of $\mathcal{X}_i$). Let $\mathbf{f}_{n_i}=(f_{n_i1},\ldots,f_{n_i(r_i-1)})$ be the vector-valued parameter representing the sample distribution of $\mathcal{X}_i$, and let $\overline{\mathcal{X}}_i=\dfrac{1}{n_i}(\mathcal{X}_{i1}+\ldots+\mathcal{X}_{in_i})$ denote the sample fuzzy expected value of $\mathcal{X}_i$.

To test at the nominal significance level $\alpha\in[0,1]$ the null hypothesis $\mathrm{H}_0: E(\mathcal{X}_1|\mathbf{p}_1)=E(\mathcal{X}_2|\mathbf{p}_2)$ against the alternative $\mathrm{H}_1: E(\mathcal{X}_1|\mathbf{p}_1)\neq E(\mathcal{X}_2|\mathbf{p}_2)$, the hypothesis H_0 should be rejected whenever

$$2(n_1+n_2)\Big[D_W^{\varphi}(\overline{\mathcal{X}}_1,\overline{\mathcal{X}}_2)\Big]^2>\tau_\alpha,$$

where τ_α is the $100(1-\alpha)$ fractile of the linear combination of chi-square independent variables $\widehat{\lambda_1}\chi^2_{1,1}+\ldots+\widehat{\lambda_q}\chi^2_{1,q}$, with $\widehat{\lambda_1},\ldots,\widehat{\lambda_q}$ ($q\leq r_1+r_2-2$) being the non-null eigenvalues of the $(r_1+r_2-2)\times(r_1+r_2-2)$ matrix

$$\mathbf{B}(\mathbf{f}_{n_1},\mathbf{f}_{n_2})\,\mathbf{H}\left(\Big[D_W^{\varphi}(\overline{\mathcal{X}}_1,\overline{\mathcal{X}}_2)\Big]^2\right)\mathbf{B}(\mathbf{f}_{n_1},\mathbf{f}_{n_2})^t$$

where

$$\mathbf{B}(\mathbf{f}_{n_1},\mathbf{f}_{n_2})\mathbf{B}(\mathbf{f}_{n_1},\mathbf{f}_{n_2})^t=\Big[I^F_{\mathcal{X}}(\mathbf{f}_{n_1},\mathbf{f}_{n_2})\Big]^{-1}$$

$$=\left(\begin{array}{c|c}\dfrac{n_1+n_2}{n_1}\Big[I^F_{\mathcal{X}_1}(\mathbf{f}_{n_1})\Big]^{-1} & \mathbf{0}\\ \hline \mathbf{0} & \dfrac{n_1+n_2}{n_2}\Big[I^F_{\mathcal{X}_2}(\mathbf{f}_{n_2})\Big]^{-1}\end{array}\right),$$

$$\Big[I^F_{\mathcal{X}_i}(\mathbf{f}_{n_i})\Big]^{-1}=[f_{ik_i}(\delta_{k_il_i}-f_{il_i})]\qquad i=1,2,\qquad \delta_{..}=\text{ Kronecker delta}$$

and the sample Hessian matrix is given by

$$\mathbf{H}\left(\Big[D_W^{\varphi}(\overline{\mathcal{X}}_1,\overline{\mathcal{X}}_2)\Big]^2\right)=\left(\begin{array}{c|c}[\widehat{h}_{k_1l_1}] & [\widehat{h}_{k_1l_2}]\\ \hline [\widehat{h}_{k_2l_1}] & [\widehat{h}_{k_2l_2}]\end{array}\right)\quad \textit{with}\quad \widehat{h}_{k_il_j}=\frac{\partial^2\Big[D_W^{\varphi}(\overline{\mathcal{X}}_1,\overline{\mathcal{X}}_2)\Big]^2}{\partial f_{ik_i}\partial f_{jl_j}}$$

$$=(-1)^{1-\delta_{ij}}2\,E\left([\mathrm{f}_{\widetilde{x}_{ik_i}}(\cdot,\cdot)-\mathrm{f}_{\widetilde{x}_{ir_i}}(\cdot,\cdot)][\mathrm{f}_{\widetilde{x}_{jl_j}}(\cdot,\cdot)-\mathrm{f}_{\widetilde{x}_{jr_j}}(\cdot,\cdot)]\,\Big|\,W\otimes\varphi\right)$$

(which in fact is a matrix irrespective of the sample).

Furthermore, the probability of rejecting H_0 under H_1 converges to 1 as $n\to\infty$.

Asymptotic results in this theorem and similar ones for one- and k-sample problems are easily applicable and lead to suitable inferences when large samples are available.

For general samples, *bootstrap techniques* have been used to approximate the asymptotic tests above (see Montenegro 2003, Montenegro *et al.* 2004b, Gil *et al.* 2005). In this special issue González-Rodríguez *et al.* present a summarized review on the approximation of the asymptotic tests for general $\mathcal{F}_c(\mathbb{R}^p)$-valued FRVs in the one-sample case.

Montenegro and others have introduced the bootstrap methodology to deal with FRVs, and they have shown the *adequacy of the bootstrap techniques in the setting of FRVs* because of two arguments, namely:

- the lack of suitable operational models for the distribution of FRVs;
- the good behavior shown in the comparative studies developed in previous papers (Montenegro *et al.* 2001, 2004b, Gil *et al.* 2005).

To perform bootstrap techniques and especially to compare different statistical approaches as well as to show empirically some probabilistic results, some introductory simulation studies have been developed (see Colubi *et al.* 2002b). Thus, by simulating some special types of FRVs studies have been carried out

- either to corroborate the convergence of the arithmetic mean of the simulated values to the population fuzzy expected value of the considered FRV,
- or to compare different techniques constructed to test hypotheses on parameters of FRVs (in particular, asymptotic and bootstrap methods);
- or to compare the probabilities of errors and the power function of the analogue tests based on real-valued and fuzzy-valued sample information.

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