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Poverty: fuzzy measurement and crisp ordering

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Abstract In this paper we make several contributions to the literature of poverty measurement. We first identify a plausible source for the fuzziness in poverty identification which renders a natural and direct estimation of the poverty membership function; we then provide an axiomatic characterization for an important class of fuzzy poverty measures; and we finally derive a set of crisp dominance conditions for fuzzy partial poverty orderings. We also illustrate the results with the US CPS data.

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1 Introduction

In 2013, an individual living alone in the continental USA is regarded as poor if his annual income is less than \$11,490. This threshold is referred to as the official poverty line which has been in existence since 1965. Over the years, the dichotomous nature of the poverty line coupled with the official formula of poverty evaluation has generated much criticism and debate. Sen (1976) pointed out that the official headcount ratio—the percentage of people below the poverty line—does not distinguish between a poor person who is close to the poverty line and a poor person who is at the bottom of income distribution. In this official approach, the drawing of the poverty line is of paramount importance in painting the picture of poverty for the society. Sen (1976) argued that a poverty measure should be sensitive not only to the "incidence of poverty"—whether a person is poor or not—but also to the level of his income and to the distribution of income among the poor. Such a "distribution-sensitive" poverty measure would give

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less weight to a poor income next to the poverty line than to a poor income further below the poverty line.

The seminal work of Sen (1976) focused on the aggregation issue of poverty measurement—aggregating all poor's poverty values into a single number—the issue of poverty identification or "who is poor?" was not particularly addressed. While the use of a single poverty line seems to solve the problem of poverty identification, the issue actually is much more complicated and the study of it over the last two decades has given rise to new directions in the research on poverty measurement. One new direction of the research focuses on the dichotomous nature of the poverty line and questions whether we can always unequivocally classify a person as poor or non-poor. It is argued that while there are many cases where we can quite confidently determine a person's poverty status (such as a street beggar—poor; and an oil baron—non-poor), we often run into situations where poverty-status classification is not so clear-cut. If we accept that an individual's poverty status has some level of ambiguity, then we should measure poverty as a fuzzy quantity. The new research in this direction applies the tools developed in the fuzzy set theory to poverty measurement and, to distinguish it from poverty measurement with a single poverty line, the new literature is referred to as fuzzy poverty measurement.[1]

This paper contributes further to the literature of fuzzy poverty measurement. Specifically, we investigate issues on poverty fuzzification (i.e., allowing poverty to be a fuzzy concept), characterize a class of decomposable (additively separable) fuzzy poverty measures and derive a set of exact dominance conditions for fuzzy poverty orderings. The early literature on fuzzy poverty measurement concentrated on specifying poverty membership functions which use a real number between zero (definitely non-poor) and one (definitely poor) to indicate an individual's poverty status. For example, Cerioli and Zani (1990) introduced, for a given dimension such as income, a straight-line membership function that links between one and zero and decreases linearly with income. Chakravarty (2006) generalizes Cerioli and Zani's membership function to allow it to change non-linearly. While Cerioli and Zani's approach requires a specification of two income levels such that an individual becomes "definitely poor" or "definitely not poor," Cheli and Lemmi (1995) presented a "totally" fuzzy and relative approach in which the degree of poverty membership depends on an individual's relative rank in the distribution.

The specification of a poverty membership function is content-dependent, i.e., it depends on the source of the vagueness/fuzziness. In everyday life there are many different sources of vagueness. Hisdal (1986) documented a dozen sources of fuzziness but enlisted three as the main sources. Interpreted within the content of poverty

[1] Two other new directions are multidimensional poverty measurement and partial poverty orderings, respectively. Multidimensional poverty measurement is based on the argument that an individual's welfare is essentially multidimensional and hence an individual's poverty status must also be evaluated from a multidimensional perspective. Contributions to this research include Tsui (2002), Bourguignon and Chakravarty (2003), Duclos et al. (2006), and Alkire and Foster (2011). The research on partial poverty orderings accepts the use of a single poverty line but the exact location of the line is uncertain and may vary within a range. The research seeks conditions under which poverty rankings remain valid for a range of poverty lines. Contributions to this area of research include Atkinson (1987), Foster and Shorrocks (1988), Jenkins and Lambert (1997), and Zheng (1999).

measurement, the three sources of fuzziness are: the fuzziness due to the imprecise estimation of income although an exact poverty line can be established; the fuzziness due to the fact that poverty is essentially a multidimensional concept but it is determined by using income alone; and the fuzziness due to the fact that different people may have different ideas about the income level below which an individual is poor. The first source of fuzziness has most commonly been used to justify fuzzy set theoretic approach to poverty measurement. For example, Chakravarty (2006) stated that "it is often impossible to acquire sufficiently detailed information on income and consumption of different basic needs and hence the poverty status of a person is not always clear cut" (p. 51). Shorrocks and Subramanian (1994) also invoked a similar line of justification but noting that "it does not conform with the strict definition of a fuzzy set."

The shape of a membership function has been a topic of immense interest in the fuzzy-set theory literature. The "aesthetically pleasing" (Beliakov 1996) S-shape curve has been found to be a common property for many proposed membership functions (Dombi 1990). Shorrocks and Subramanian (1994) also believed that the poverty membership function "is likely to be " an inverted-S-shape function—"continuous and non-increasing in income" and its slope "first increases and then declines." For the three sources of fuzziness described above, Beliakov (1996) proved that every source gives rise to an averaging process and, consequently, yields a S-shape membership function.

In this paper, we register with the third source of fuzziness as the source of fuzziness in the poverty line. That is, people (voters or social evaluators) have different perceptions about what constitutes poverty. This interpretation allows us to introduce a meaningful "density function" for a poverty membership function—which is akin to the density function for a cumulative distribution function. With this type of poverty membership density function, we are able to extend uniquely any crisp decomposable (additively separable) poverty measure to a fuzzy decomposable poverty measure that Shorrocks and Subramanian (1994) characterized. Our extension does not depend on the hard-to-justify "m-linearity condition" that Shorrocks and Subramanian relied upon; it also clearly avoids the common confusion between "fuzzy measures of poverty" and "measures of the depth of poverty" (Qizilbash 2006). For such a decomposable fuzzy poverty measure, the ranking of income distributions obviously depends on the poverty membership function used. Although the fuzzy set theory literature is filled with methods for membership function construction, no effort has been made to derive a poverty membership function from empirical data. While the approach we take allows us to construct empirical membership functions, in this paper, we do not attempt to specify a poverty membership function beyond the S-shape requirement. What we are interested in achieving in this paper is to derive dominance conditions such that one distribution has no more poverty than another distribution for all possible membership functions with the S-shape. The advantage of this exercise is clear: if we are interested in ordinal poverty ranking and a dominance relationship is observed then there is no need to estimate membership functions. One set of dominance conditions we derive turn out to be an interesting device of bifurcated Lorenz dominance.

The rest of the paper is organized as follows. The next section characterizes poverty as a fuzzy concept and specifies the general properties for a poverty membership function. It also discusses the sources of fuzziness that have been considered. Accepting the third source of fuzziness and the corresponding membership functions, Sect. 3 axiomatically characterizes additively separable (decomposable) fuzzy poverty measures. Section 4 derives a set of dominance conditions for all possible poverty membership functions and/or for all poverty measures. We also apply a set of dominance conditions to the US income data. Section 5 concludes.

2 Fuzzy poverty set and poverty membership functions

Consider a continuous income distribution represented by a cumulative distribution function $F\colon \mathfrak{R}_+ \to [0, 1]$. A crisp poverty set of F contains individuals who are unambiguously identified as poor. This is achieved by employing a singular poverty line z and anyone whose income below z is poor. Denote the poverty set by $A =: \{x|x \in \mathfrak{R}_+ \text{ and } x < z\}$, then the poverty status of an individual can be expressed using the following two-valued poverty characteristic function (1 is poor and 0 is nonpoor):

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \in \bar{A} \end{cases}. \tag{2.1}$$

The practice of making a sharp distinction between the poor and the nonpoor has been criticized as "difficult" (Shorrocks and Subramanian 1994) and "nonrealistic" (Cerioli and Zani 1990). Sen (1981, p. 13) also pointed out that "while the concept of a nutritional requirement is a rather loose one, there is no reason to suppose that the concept of poverty is clear cut and sharp...a certain amount of vagueness is implicit in both the concepts." Over the last two decades, many poverty measurement researchers have argued that the distinction between the two groups is vague and the notion of being poor is a fuzzy one. They have suggested that the tools developed in the fuzzy set theory should be employed to measure poverty.

Many linguistic variables such as "tall" and "bald" are considered vague predicates. A vague or fuzzy predicate such as "tall" , according to Keefe and Smith (1996) and Qizilbash (2006), must possess three characteristics: it involves "borderline" situations where one cannot decide unequivocally "tall" and "not tall" ; one cannot establish a sharp boundary between "tall" and "not tall" ; and the presence of a Sorites paradox—making a "tall" person a millimeter shorter keeps the person "tall" but repeating the same step many times eventually will make the person "not tall" or "short." Qizilbash (2006) argues that the notion of poverty meets all three criteria and hence poverty is indeed a fuzzy predicate: there are cases where a person cannot be obviously classified as "poor" or "nonpoor" ; there is no clear-cut poverty line to separate the poor from the nonpoor; and any poor person can be made nonpoor by repeatedly increasing his income by a penny but each penny alone does not change his poverty status.

The fuzziness in poverty is characterized via a poverty membership function. To do so, we are required to specify two income levels z_l and z_u ($z_l < z_u$) so that any individual with an income below z_l is definitely counted as poor while any individual with an income above z_u will definitely be counted as nonpoor. An immediate criticism of this exercise is: If one cannot specify precisely a value for the poverty line, how could one be expected to specify two exact values for the upper and lower bounds of the fuzzy poverty region? This argument is the very reason behind Cheli and Lemmi's "total fuzzy approach" and is also the justification for considering "higher-order vagueness" (Sainsbury 1991)—there is even a vagueness about the degree of membership. A simple response to this criticism, besides the standard responses from the fuzzy-set theories (Qizilbash 2006, p. 14), is that failing to specify two such boundaries, we are unable to proclaim that a street beggar is definitely poor and a person owning a Ferrari is definitely non-poor! Certainly the exact specification of z_l and z_u has a flavor of arbitrariness but the source of fuzziness we subscribe to will provide a way to determine their values.

Definition 2.1 The membership that income x is in poverty is represented by a differentiable function $m(x)$ such that

$$m(z_l) = 1, \quad m(z_u) = 0 \text{ and } \quad 0 \leq m(x) \leq 1 \text{ for all } z_l < x < z_u \tag{2.2}$$

for some finite values z_l and z_u with $z_l < z_u$.

Clearly, for $z_l < x < z_u$, $m(x)$ must also be decreasing in x to reflect the fact that an increase in income makes the individual less likely to be poor. This amounts to assuming $m'(x) < 0$. Another commonly assumed property is that $m(x)$ is of an inverted-S shape (the solid curve in Fig. 1). That is, there exists a point ξ between z_l and z_u such that

$$\begin{gathered} m''(x) < 0 \text{ for } z_l < x < \xi \text{ and } m''(x) > 0 \text{ for } \xi < x < z_u; \\ \text{and } m'(x) = 0 \text{ for } x \leq z_l \text{ or } x \geq z_u. \end{gathered} \tag{2.3}$$

Considering all possible membership functions possessing these properties, we can form two sets of membership functions as follows:

$$\mathfrak{M}_1 =: \left\{m | m'(x) < 0 \text{ for } z_l < x < z_u\right\}$$

and

$$\mathfrak{M}_2 =: \left\{m | m(x) \text{ is in } \mathfrak{M}_1 \text{ and satisfies (2.3)}\right\}.$$

In Sect. 4, we will derive crisp poverty dominance conditions for all membership functions in $\mathfrak{M}_1$ and $\mathfrak{M}_2$.

An example of poverty membership function is

$$m_1(x) = \begin{cases} 1 & \text{if } x \leq z_l \\ \left(\frac{z_u - x}{z_u - z_l}\right)^\theta & \text{if } z_l < x < z_u \\ 0 & \text{if } x \geq z_u \end{cases}$$

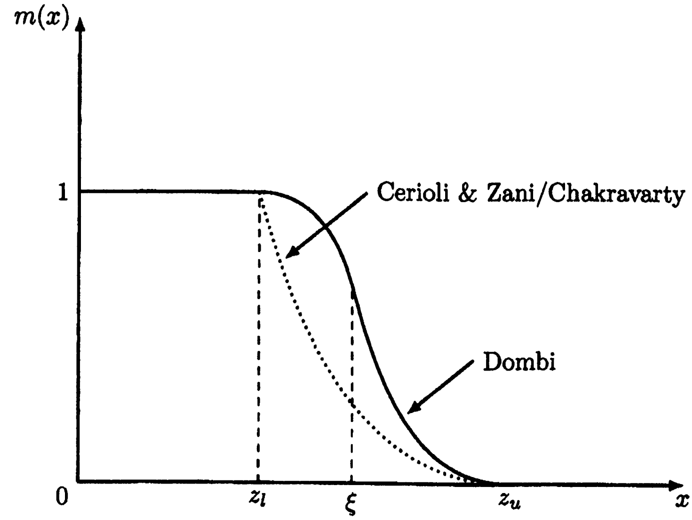

Fig. 1 Poverty membership functions

that Cerioli and Zani (1990) introduced ($\theta = 1$) and Chakravarty (2006) generalized ($\theta > 1$). Another example is

$$m_2(x) = \begin{cases} 1 & \text{if } x \leq z_l \\ \frac{(z_u - x)^2}{(x - z_l)^2 + (z_u - x)^2} & \text{if } z_l < x < z_u \\ 0 & \text{if } x \geq z_u \end{cases}$$

which is a member of the class that Dombi constructed (1990, Theorem 2). Both membership functions are non-increasing in x (and strictly decreasing in x over (z_l, z_u)) and hence are members of $\mathfrak{M}_1$. But $m_1(x)$ is not of an inverted-S shape and thus does not belong to $\mathfrak{M}_2$. It is easy to check that $m_2(x)$ is of an inverted-S shape and is a member of $\mathfrak{M}_2$. Both membership functions are plotted above in Fig. 1.

What is the meaning of the m function? The cases of $m = 1$ and 0 are straightforward, but $m = 0.8$? In fact, the difficulty in interpreting a number such as 0.8 is one of the main criticisms about the degree of membership and about the fuzzy set theory in general. The first two sources of fuzziness documented in the Introduction do not seem to provide a sensible interpretation. The third source of fuzziness—different people may perceive poverty differently—does lead to a meaningful interpretation. To state the interpretation, we need to introduce a "density function"—akin to that for a cumulative distribution function—for a poverty membership function.

Definition 2.2 For a poverty membership function $m(x)$, its density function is

$$\rho(x) = |m'(x)| \quad \text{for all } z_l < x < z_u \tag{2.4}$$

and $\rho(x) = 0$ for any $x \leq z_l$ and for any $x \geq z_u$.

Since poverty is a social concern, it is reasonable that members of the society should decide what is considered living in poverty. This understanding has been reflected in survey data such as the world bank's living standards measurement study (LSMS) and the European income and living conditions (SILC) data. For example, in SILC survey participants are asked "In your opinion, what is the very lowest net monthly income that your household would have to have in order to make ends meet?" Suppose we interpret this monthly income as the poverty line that each survey participant perceives and denote it x. Now tally up the answers by the value of x. Let $\rho(x)$ denote the proportion of voters in the society who have elected x as the "reasonable poverty line." It follows immediately that $1 - m(x)$—as a cumulation of $\rho(x)$—is *the proportion of voters in the society believing the poverty line is at or less than* x; $m(x)$ *is the proportion of voters believing the poverty line is at least* x. These interpretations flow naturally from the definition of the third source of fuzziness. The approach also leads to a natural determination of the values of z_l and z_u: z_l is set at the minimum value of x that voters have elected and z_u is the maximum value of x that voters have chosen. Given the one-to-one correspondence between $m(x)$ and $\rho(x)$, in the rest of the paper, we will use both terms interchangeably.

3 Decomposable fuzzy poverty measures

For a crisp poverty set, issues in poverty measurement have been thoroughly researched and a large volume of literature has been established following Sen's 1976 seminal work. For a given poverty line $z \in \Re_{++}$, a poverty measure $P(F; z)$ indicates the poverty level associated with distribution F. An additively separable (decomposable) poverty measure is

$$P(F; z) = \int_0^z p(x, z) dF(x) \tag{3.1}$$

where $p(x, z)$ is the individual deprivation function with $p(x, z) > 0$ for $x < z$ and $p(x, z) = 0$ for $x \geq z$. We further require that $p(x, z)$ be twice differentiable with respect to x for $x < z$ with $p_x = \frac{\partial p(x,z)}{\partial x} \leq 0$ and $p_{xx} = \frac{\partial^2 p(x,z)}{\partial x^2} \geq 0$.[2] Examples of additively separable poverty measures are the Watts measure with $p(x, z) = \ln z - \ln x$ for $x > 0$, the Clark et al. second measure with $p(x, z) = \frac{1}{\epsilon}[1-(x/z)^{\epsilon}]$ and $0 < \epsilon < 1$, the Foster et al. measure with $p(x, z) = (1 - x/z)^{\alpha}$ and $\alpha \geq 2$, and the Kolm-type constant-distribution-sensitivity (CDS) measure with $p(x, z) = e^{\sigma(z-x)} - 1$ and $\sigma > 0$.

For a fuzzy poverty set, the level of poverty is determined not by any single poverty line but instead by a poverty membership function m. To distinguish between these two categories of poverty measures, we follow the literature and label the poverty measures for a crisp poverty set as *crisp poverty measures* and the measures for a fuzzy poverty set *fuzzy poverty measures*. We further denote a fuzzy poverty measure

[2] These two conditions ensure that a poverty measure satisfies the monotonicity axiom (poverty is reduced if the income of a poor individual is increased) and the transfer axiom (poverty is reduced by a progressive transfer of income among the poor). For surveys on poverty axioms, poverty measures and poverty orderings, see Zheng (1997, 2000), Lambert (2001) and, more recently, Chakravarty (2009).

$\Pi(F; m)$. Since a characterization function $\chi_A(x)$ is a degenerated form of poverty membership function, a crisp poverty measure can be written as $\Pi(F; \chi_A)$.

Conceptually, each crisp poverty measure such as the Sen measure can be extended to be a fuzzy poverty measure, and each fuzzy poverty measure should collapse to a crisp poverty measure if the membership function $m(x)$ degenerates to $\chi_A(x)$. The extension from a crisp poverty measure to a fuzzy poverty measure, however, as Shorrocks and Subramanian (1994) demonstrated, is not straightforward and is not even unique. With the exception of Shorrocks and Subramanian (1994), all the extensions [e.g., Cerioli and Zani (1990), Cheli and Lemmi (1995) and Chakravarty (2006)] in the literature have focused on the fuzzification of the headcount ratio and the fuzzy headcount ratio is simply the sum of poverty memberships:

$$\Pi(F; m) = \int_0^{\infty} m(x)dF(x). \tag{3.2}$$

For a general crisp poverty measure such as (3.1) with a poverty deprivation function $p(x, z)$, since it can be written as

$$\Pi(F; \chi_A) = \int_0^{\infty} p(x, z)\chi_A(x)dF(x)$$

a seemingly natural fuzzy poverty measure could be obtained by replacing $\chi_A(x)$ with $m(x)$. But such a generalization may fail to serve as a reasonable fuzzy poverty measure as Shorrocks and Subramanian (1994) pointed out.

In an important (yet still unpublished) contribution to fuzzy poverty measurement, Shorrocks and Subramanian (1994) proved that for every crisp poverty measure $P(F; z)$, under certain conditions, there is a unique generalization to fuzzy poverty measure

$$\Pi(F; m) = \int_0^{\infty} P(F; z)d\big[1 - m(z)\big]. \tag{3.3}$$

A critical assumption employed in the derivation is the m-linearity condition which require $\Pi(F; m)$ to be linear in m, i.e.,

$$\Pi[F; \alpha m_1 + (1 - \alpha)m_2] = \alpha\Pi(F; m_1) + (1 - \alpha)\Pi(F; m_2)$$

for all membership functions m_1 and m_2. But "there is no compelling reason for imposing this requirement" as Shorrocks and Subramanian (1994) recognized and, thus, their characterization of the fuzzy poverty measures (3.3) carries an ad hoc flavor.

In this section, we provide an alternative yet more elementary and intuitive characterization to (3.3) when $P(F; z)$ is additively separable or decomposable. This characterization is based on the source of fuzziness that we subscribe to: voters have different perceptions regarding what constitutes being in poverty. Since $P(F; z)$ is decomposable, by (3.1), we only need to characterize the fuzzy counterpart of individual poverty deprivation function $p(x; z)$. This characterization is presented below.

Recall that we view poverty as a social concern, any individual's poverty status and his level of poverty should be judged and decided by the public or the voters. Suppose a group of voters believe that the "true" poverty line should be z, then an individual with income x will be judged by this group to have the poverty level of $p(x; z)$. Assume the size of this voting group is $\rho(z)$. Considering all groups of voters, we have a distribution of poverty levels represented by $\{p(x; z), \rho(z)\}$ for all $z \in [z_l, z_u]$. For ease of presentation and derivation, we assume that there are $k > 1$ voter groups and denote the fuzzy poverty level voted by the public for the individual with income x as $\pi(x; m)$. This amounts to assuming that there are k discrete poverty lines—a necessary deviation from our continuous poverty-space setting. Denote the k voted poverty lines as $z_1 < z_2 < \cdots < z_k$, also $p_i = p(x, z_i)$ and $\rho_i = \rho(z_i)$ for $i = 1, 2, \ldots, k$. The relationship between $\pi(x; m)$ and $\{p(x; z), \rho(z)\}$ can be represented by

$$\pi(x; m) = f(p_1, p_2, \ldots, p_k; \rho_1, \rho_2, \ldots, \rho_k) \tag{3.4}$$

for some differentiable function f.

If $\pi(x; m)$ is voted by the public, what will be the maxims governing the voting process. Below we consider a reasonable and intuitive "equal voting-right" axiom.

The poverty equal voting-right axiom Suppose a fraction ϵ of ρ_j is shifted to ρ_i and denote the new poverty membership function as $\tilde{m}$, then the resulting change in the poverty level of the individual with income x,

$$\Delta\pi = \pi(x; \tilde{m}) - \pi(x; m),$$

does not depend on any $\rho_i, i = 1, 2, \ldots, k$.

The essence of the axiom is "one man one vote"—all votes carry the same weight in determining the poverty level, it does not matter which group the voter may be in and whether the group size is large or small. This seems to be a reasonable requirement for a fair voting process. When a voter changes his vote of poverty line from z_j to z_i with $z_i < z_j$, the poverty membership function changes from m to $\tilde{m}$ and the individual poverty level π should decrease. The amount of decrease $\Delta\pi$, however, should not depend on ρ_i or ρ_j; it can only depend on $p_i, i = 1, 2, \ldots, k$. Otherwise, if $\Delta\pi$ depends on the size of ρ_i or ρ_j, then it means that the voter's vote carries different weights in different groups—it depends upon which coalition he chooses to join; in a sense he is no longer an independent voter.[3] Thus, the equal voting-right axiom also means that there is no gain from strategic coalition-formation among the voters. To state and prove the implication of this axiom for $\pi(x; m)$, we also need the following normalization axiom on individual fuzzy poverty deprivation.

The normalization axiom If for some i, $\rho_i = 1$ (thus $\rho_j = 0$ for all $j \neq i$), then $\pi(x; m) = p_i$.

[3] In some elections such as the US presidential election, this axiom is violated as voters in different states carry different weights. In the US gubernatorial and other local elections, however, this axiom is satisfied.

The idea behind this normalization axiom is pretty straightforward: if the entire society agrees upon a single poverty line z_i, then the individual's poverty level π is simply $p_i = p(x; z_i)$.

Proposition 3.1 *A differentiable individual fuzzy poverty deprivation function $\pi(x; m)$ satisfies the poverty equal voting-right axiom and the normalization axiom if and only if it is of the form*

$$\pi(x; m) = \sum_{i=1}^{k} p_i \rho_i. \tag{3.5}$$

Proof First consider a shift of ϵ proportion of voters from ρ_k to ρ_{k-1}, $\epsilon < \rho_k$. That is, some voters change their votes from z_k to z_{k-1}. Denote

$$\tilde{f}(\rho_1, \ldots, \rho_2, \ldots, \rho_k) = f(p_1, p_2, \ldots, p_k; \rho_1, \ldots, \rho_2, \ldots, \rho_k), \tag{3.6}$$

then the change in the individual poverty value of π is

$$\Delta\pi_k = \tilde{f}(\rho_1, \ldots, \rho_{k-1} + \epsilon, \rho_k - \epsilon) - \tilde{f}(\rho_1, \rho_2, \ldots, \rho_k).$$

The poverty equal voting-right axiom states that $\Delta\pi_k$ can only be a function of ϵ. That is,

$$\tilde{f}(\rho_1, \ldots, \rho_{k-1} + \epsilon, \rho_k - \epsilon) - \tilde{f}(\rho_1, \rho_2, \ldots, \rho_k) = h(\epsilon) \tag{3.7}$$

for some differentiable function h.

Differentiate (3.7) with respect to ϵ,

$$\tilde{f}_{k-1}(\rho_1, \ldots, \rho_{k-1} + \epsilon, \rho_k - \epsilon) - \tilde{f}_k(\rho_1, \ldots \rho_{k-1} + \epsilon, \rho_k - \epsilon) = h'(\epsilon).$$

Let $\epsilon \to 0$, we have

$$\tilde{f}_{k-1}(\rho_1, \ldots, \rho_{k-1}, \rho_k) - \tilde{f}_k(\rho_1, \ldots \rho_{k-1}, \rho_k) = h'(0)$$

or

$$\hat{f}_{k-1}(\rho_{k-1}, \rho_k) - \tilde{f}_k(\rho_{k-1}, \rho_k) = h'(0) \equiv c_k \tag{3.8}$$

where $\hat{f}(\rho_{k-1}, \rho_k) = \tilde{f}(\rho_1, \ldots, \rho_{k-1}, \rho_k)$.

Using the method of Lagrange (e.g., Dennemeyer 1968, p. 17) to the quasi-linear partial differentiation equation (3.8), we solve the following subsidiary equations

$$\frac{d\rho_{k-1}}{1} = \frac{d\rho_k}{-1} = \frac{du}{c_k}$$

which yield the following intergrals

$$\rho_{k-1} + \rho_k = v \text{ and } c_k \rho_k + u = w$$

for some constants v and w. The general solution to (3.8) is

$$s(v, w) = s(\rho_{k-1} + \rho_k, c_k \rho_k + u) = 0 \tag{3.9}$$

for some arbitrary function s. Solving (3.9) for u, we obtain

$$\hat{f} = u = r(\rho_{k-1} + \rho_k) - c_k \rho_k \tag{3.10}$$

for some differentiable function r. Restoring $\tilde{f}$, we have

$$\tilde{f}(\rho_1, \ldots, \rho_k) = \tilde{r}(\rho_1, \ldots, \rho_{k-1} + \rho_k) - c_k \rho_k \tag{3.10a}$$

for some differentiable function $\tilde{r}$.

Now consider a further shift of $\epsilon (< \rho_{k-1})$ proportion of voters from ρ_{k-1} to ρ_{k-2}. Similarly we have

$$\hat{f}_{k-2}(\rho_{k-2}, \rho_{k-1} + \rho_k) - \hat{f}_{k-1}(\rho_{k-2}, \rho_{k-1} + \rho_k) = c_{k-1} \tag{3.8a}$$

where $\hat{f}$ is similarly defined as above and c_{k-1} is a constant. The general solution to (3.8a) is

$$\tilde{f}(\rho_1, \ldots, \rho_k) = \tilde{r}(\rho_1, \ldots, \rho_{k-2} + \rho_{k-1} + \rho_k) - c_{k-1}\rho_{k-1} - c_k \rho_k \tag{3.10b}$$

for some differentiable function $\tilde{r}$ and some constants c_{k-1} and c_k. Repeating this process of voter shifting from ρ_{k-2} to ρ_{k-3}, from ρ_{k-3} to ρ_{k-4}, ..., and finally from ρ_2 to ρ_1, we eventually arrive at

$$\begin{aligned}
\tilde{f}(\rho_1, \ldots, \rho_k) &= \tilde{r}(\rho_1 + \rho_2 + \cdots + \rho_{k-1} + \rho_k) - c_2\rho_2 - \cdots - c_{k-1}\rho_{k-1} - c_k\rho_k \\
&= \tilde{r}(1) - c_2\rho_2 - \cdots - c_{k-1}\rho_{k-1} - c_k\rho_k \\
&= \tilde{r}(1)(\rho_1 + \rho_2 + \cdots + \rho_{k-1} + \rho_k) - c_2\rho_2 - \cdots - c_{k-1}\rho_{k-1} - c_k\rho_k \\
&= \tilde{r}(1)\rho_1 + (1 - c_2)\rho_2 + \cdots + (1 - c_k)\rho_k. \qquad (3.10c)
\end{aligned}$$

Here we have used the fact that $\rho_1 + \rho_2 + \cdots + \rho_{k-1} + \rho_k = 1$. Let $\tilde{c}_1 = \tilde{r}(1)$ and $\tilde{c}_i = (1 - c_i)$ for $i = 2, \ldots, k$ and restoring f as defined in (3.6), we have

$$f(p_1, p_2, \ldots, p_k; \rho_1, \ldots, \rho_2, \ldots, \rho_k) = \sum_{i=1}^{k} \tilde{c}_i(p_1, p_2, \ldots, p_k)\rho_i$$

for some continuous functions $\tilde{c}_i(p_1, p_2, \ldots, p_k), i = 1, 2, \ldots, k$. Finally, for each i, let $\rho_i = 1$ and thus $\rho_j = 0$ for $j \neq i$, the normalization axiom entails

$$\tilde{c}_i(p_1, p_2, \ldots, p_k) = p_i, \quad i = 1, 2, \ldots, k$$

which leads to (3.5). This proves the necessity of the proposition. The sufficiency is obvious. □

Translated back into the continuous space, the individual fuzzy poverty deprivation is

$$\pi(x; m) = \int_0^\infty p(x, z)\rho(z)dz \tag{3.11}$$

and the aggregate decomposable fuzzy poverty measure is thus

$$\begin{aligned}\Pi(F; m) &= \int_0^\infty \pi(x; m)dF(x) = \int_0^\infty \int_0^\infty p(x, z)\rho(z)dzdF(x) \\ &= \int_0^\infty \left[\int_0^\infty p(x, z)dF(x)\right]\rho(z)dz = \int_0^\infty P(F; z)\rho(z)dz \end{aligned} \tag{3.12}$$

which is the same as that derived by Shorrocks and Subramanian (1994).

4 Crisp poverty orderings

The fuzzy poverty measure characterized in the previous section involves a crisp poverty measure and a poverty membership function. In (crisp) poverty comparisons, it is recognized that the use of any single poverty measure is arbitrary and multiple poverty measures should be consulted. In the literature, the uncertainty in choosing "proper" poverty measures has led to the development of the literature of poverty-measure partial poverty orderings. Once a dominance relationship is found to hold between a pair of income distributions, all poverty measures satisfying certain conditions (or in a class) will agree on the poverty ranking. For fuzzy poverty measurement, there is also a similar issue of poverty-measure partial poverty ordering. That is, for a given poverty membership function $m(x)$, one may wish to know the conditions under which all fuzzy poverty measures will reach the same conclusion on poverty orderings.

Conceptually, the specification of a poverty membership function should be less arbitrary and more "scientific" than the choice of a poverty measure. Over the last half-century since Zadeh (1965) introduced the notion of a fuzzy set, various techniques to quantify fuzziness have been developed [see, for example, Chapter 10 of Klir and Yuan (1995)]. These techniques have found numerous applications in a wide range of areas such as in the designing and manufacturing of washing machine and rice cooker [see, for example, a recent survey on industrial applications of fuzzy control by Precup and Hellendoorn (2011)]. Researchers, however, do not appear to agree on the existence of a precise membership function. Dubois and Prade (1985, p. 2), for example, believed that "precise membership values do not exist by themselves, they are tendency indices that are subjectively assigned by an individual or a group" and "the grades of membership reflect an 'ordering' of the objects in the universe." The entire school of "higher-order vagueness" (e.g., Sainsbury 1991) also doubts the existence of a precise membership function on the philosophical grounds. In fuzzy poverty measurement, the voter-frequency membership approach we follow in this paper may appear to be free of the arbitrariness that Dubois and Prade alluded to, but the fact that voters *subjectively* decide what constitutes poverty suggests that some degree of non-uniqueness may remain in computing the membership values. This non-uniqueness points to the usefulness in checking the robustness of poverty rankings by different poverty membership functions. This is a new type of partial poverty orderings and it is the main focus of this section.

Before presenting any new dominance conditions, we need to define two types of additively separable poverty measures: absolute poverty measures and relative poverty measures. Later on, we will establish dominance conditions for these two types of measures. An additively separable (crisp) poverty measure defined by (3.1) is absolute if

$$P(F; z) = \int_0^z \tilde{p}(z - x) dF(x) \tag{4.1}$$

and relative if

$$P(F; z) = \int_0^z \tilde{p}\left(\frac{x}{z}\right) dF(x) \tag{4.2}$$

for some differentiable function $\tilde{p}$. That is, an absolute poverty measure remains unaffected if all incomes and the poverty line are changed by the same dollar amount; a relative poverty measure remains unaffected if all incomes and the poverty line are changed by the same percentage. Of the poverty measure examples given immediately after (3.1), the first three (Watts, Clark et al. and Foster et al.) are relative and the last one (the Kolm-type) is absolute.

For ease of reference in later presentation, we further specify four subclasses of absolute and relative poverty measures:

$$\begin{aligned}
\mathfrak{P}_1^a &=: \left\{P(\cdot; \cdot\}|P \text{ is (4.1) with } \frac{\partial \tilde{p}}{\partial x} \le 0\right\} \\
\mathfrak{P}_2^a &=: \left\{P(\cdot; \cdot\}|P \text{ is (4.1) with } \frac{\partial \tilde{p}}{\partial x} \le 0 \text{ and } \frac{\partial^2 \tilde{p}}{\partial x^2} \ge 0\right\} \\
\mathfrak{P}_1^r &=: \left\{P(\cdot; \cdot\}|P \text{ is (4.2) with } \frac{\partial \tilde{p}}{\partial x} \le 0\right\} \\
\mathfrak{P}_2^r &=: \left\{P(\cdot; \cdot\}|P \text{ is (4.2) with } \frac{\partial \tilde{p}}{\partial x} \le 0 \text{ and } \frac{\partial^2 \tilde{p}}{\partial x^2} \ge 0\right\}.
\end{aligned}$$

4.1 Poverty-membership partial ordering for a given (crisp) poverty measure

Recall that we consider two classes of membership functions $\mathfrak{M}_1$ and $\mathfrak{M}_2$—members in $\mathfrak{M}_1$ are non-increasing in income (i.e., $m'(x) \le 0$ or $\rho(x) = -m'(x) \ge 0$) and members in $\mathfrak{M}_2$ are non-increasing and are of inverted-S shape (i.e., there exists a point $\xi \in (z_l, z_u)$ such that $m''(x) \leq 0$ for $x \in (z_l, \xi)$ and $m''(x) \geq 0$ for $x \in (\xi, z_u)$; and $m'(z_l) = m'(z_u) = 0$). The following proposition establishes the partial poverty ordering conditions for all membership functions in each of the two classes.

Proposition 4.1 *For any crisp poverty measure $P(\cdot;\cdot)$ and any two income distributions F and G that have the same poverty membership function $m(x)$, F has no more poverty then G, i.e., $\Pi(F; m) \leq \Pi(G; m)$, for all membership functions in $\mathfrak{M}_1$ if and only if*

$$P(F; z) \leq P(G; z) \quad \textit{for all } z \in [z_l, z_u]; \tag{4.3}$$

and $\Pi(F; m) \leq \Pi(G; m)$ for all membership functions in $\mathfrak{M}_2$ and a given ξ if and only if

$$\begin{aligned} &\int_z^{\xi} P(F; t)dt \leq \int_z^{\xi} P(G; t)dt \textit{ for all } z \in [z_l, \xi] \textit{ and} \\ &\int_{\xi}^{z} P(F; t)dt \leq \int_{\xi}^{z} P(G; t)dt \textit{ for all } z \in [\xi, z_u]. \end{aligned} \tag{4.4}$$

Proof For a fuzzy poverty measure $\Pi(F; m)$ with crisp poverty measure $P(\cdot;\cdot)$, via (3.13), distribution F has no more poverty than distribution G if and only if

$$\int_{z_l}^{z_u} P(F; z)\rho(z)dz \leq \int_{z_l}^{z_u} P(G; z)\rho(z)dz$$

or

$$\Pi(F; m) - \Pi(G; m) = \int_{z_l}^{z_u} \big[P(F; z) - P(G; z)\big]\rho(z)dz \leq 0. \tag{4.5}$$

For a membership function $m(x)$ in $\mathfrak{M}_1$, the only requirement on $\rho(x)$ is $\rho(x) \geq 0$. The sufficiency of condition (4.3) is obvious. The necessity of (4.3) can be easily verified using the method of contradiction that Atkinson (1987) employed. That is, if (4.3) is false and $P(F; z) > P(G; z)$ over some open interval (a, b), then we can choose

$$\rho(z) = \frac{1}{b-a} \quad \text{for } a < z < b \text{ and } \rho(z) = 0 \text{ elsewhere.}$$

It follows that

$$\Pi(F; m) - \Pi(G; m) = \frac{1}{b-a}\int_a^b \big[P(F; z) - P(G; z)\big]dz > 0$$

which contradicts (4.5).

To prove (4.4), denote $\Delta P(z) \equiv P(F;z) - P(G;z)$, $\Delta\Pi(m) \equiv \Pi(F;m) - \Pi(G;m)$ and further expand $\Delta\Pi(m)$ as follows:

$$\begin{aligned}
\Delta\Pi(m) &= \int_{z_l}^{\xi} \Delta P(z)\rho(z)dz + \int_{\xi}^{z_u} \Delta P(z)\rho(z)dz \\
&= \int_{z_l}^{\xi} \rho(z)d\left[\int_{\xi}^{z} \Delta P(t)dt\right] + \int_{\xi}^{z_u} \rho(z)d\left[\int_{z}^{\xi} \Delta P(t)dt\right] \\
&= \left\{\rho(z)d\left[\int_{\xi}^{z} \Delta P(t)dt\right]\Bigg|_{z_l}^{\xi} - \int_{z_l}^{\xi} \rho'(z)\left[\int_{\xi}^{z} \Delta P(t)dt\right]dz\right\} \\
&\quad + \left\{\rho(z)d\left[\int_{z}^{\xi} \Delta P(t)dt\right]\Bigg|_{\xi}^{z_u} - \int_{\xi}^{z_u} \rho'(z)\left[\int_{z}^{\xi} \Delta P(t)dt\right]dz\right\}.
\end{aligned}$$

Since $\rho(z_l) = \rho(z_u) = 0$ by the inverted-S-shape assumption for an $m(x)$ in $\mathfrak{M}_2$, we have

$$\rho(z)d\left[\int_{\xi}^{z} \Delta P(t)dt\right]\Bigg|_{z_l}^{\xi} = 0 \text{ and } \rho(z)d\left[\int_{z}^{\xi} \Delta P(t)dt\right]\Bigg|_{\xi}^{z_u} = 0.$$

Therefore, we have

$$\begin{aligned}
\Delta\Pi(m) &= -\int_{z_l}^{\xi} \rho'(z)\left[\int_{\xi}^{z} \Delta P(t)dt\right]dz - \int_{\xi}^{z_u} \rho'(z)\left[\int_{z}^{\xi} \Delta P(t)dt\right]dz \\
&= \int_{z_l}^{\xi} \rho'(z)\left[\int_{z}^{\xi} \Delta P(t)dt\right]dz + \int_{\xi}^{z_u} [-\rho'(z)]\left[\int_{z}^{\xi} \Delta P(t)dt\right]dz.
\end{aligned}$$

Since $\rho'(z) \geq 0$ (or $m''(z) \leq 0$) over (z_u, ξ) and $\rho'(z) \leq 0$ (or $m''(z) \geq 0$) over (ξ, z_u), the sufficiency of (4.4) is straightforward. Following Atkinson's proof of his condition IIA, the necessity of (4.4) could be proved by using a well-known result that a convex (or a concave) function can be uniformly approximated by piecewise linear functions. We can also construct a contradiction directly as follows. Suppose the first part of (4.4) is false and $\int_z^{\xi} \Delta P(t)dt > 0$ for $z \in (c,d) \subset (z_l, z_u)$, then we can construct a $\rho(z)$ such that [4]

[4] An example of $\rho(z)$ is

$$\rho(z) = \begin{cases} \gamma c + \theta & \text{if } z_l < z < c \\ \gamma z + \theta & \text{if } c < z < d \\ \gamma d + \theta & \text{if } d < z < z_u \end{cases}$$

where $\gamma > 0$ and $\theta > 0$ satisfy

$$w\gamma + (z_u - z_l)\theta = 1$$

with $w = \frac{1}{2}(c^2 - d^2) - cz_l + dz_u > 0$. It is easy to verify that such a $\rho(z)$ is a "density function" for a poverty membership function. Note that the definition excludes a small neighborhood at z_l and z_u to enable $\rho(z_l) = \rho(z_u) = 0$.

$$\rho'(z) = \gamma > 0 \text{ for } c < z < d \text{ and } \rho'(z) = 0 \text{ for } z \in (z_l, z_u)\backslash(c, d).$$

It follows that $\Delta\Pi(m) = \gamma \int_c^d \left[\int_z^\xi \Delta P(t)dt\right] dz > 0$ - the required contradiction. □

The conditions derived in Proposition 4.1 are valid for any crisp poverty measure. Condition (4.3) can be viewed as a first-order condition while condition (4.4) can be regarded as of second-order nature. Clearly, the first-order condition implies the second-order conditions. The first-order condition requires a dominance of crisp poverty values over the entire poverty border region $[z_l, z_u]$. This result is very intuitive: if any value $\tilde{z}$ within $[z_l, z_u]$ can be a poverty line, then by choosing a membership function such that $\rho(z) > 0$ over $(\tilde{z}-\varepsilon, \tilde{z}+\varepsilon)$ and $\rho(z) = 0$ elsewhere, (4.5) collapses to $P(F; \tilde{z}) \leq P(G; \tilde{z})$ as $\varepsilon \to 0$.

The second-order condition (4.4) is interesting: it is a bifurcated Lorenz-type dominance condition (Fig. 2). The device is constructed as follows: from the switching point ξ, poverty values are cumulated toward both ends. Since $P(\cdot; z)$ generally increases as z increases, the left panel of the device is a concave curve while the right panel is a convex one. Distribution F has less poverty than distribution G if and only if the device of F lies below that of G.

The switching point ξ is significant in the condition. It can be viewed as the center of the poverty border region (i.e., between z_l and z_u). For example, in 2013, the official poverty line of \$11,490 can be regarded as the center and 25 % above the value (i.e., \$14,362.5) is the upper bound z_u, and 25 % below the value (i.e., \$8,617.5) is the lower bound z_l. In condition (4.4), if ξ approaches either end of the poverty border region, only one part of the condition survives and it becomes a simple cumulation of the first-order condition (4.3). For example, if $\xi \to z_l$, the second-order condition becomes

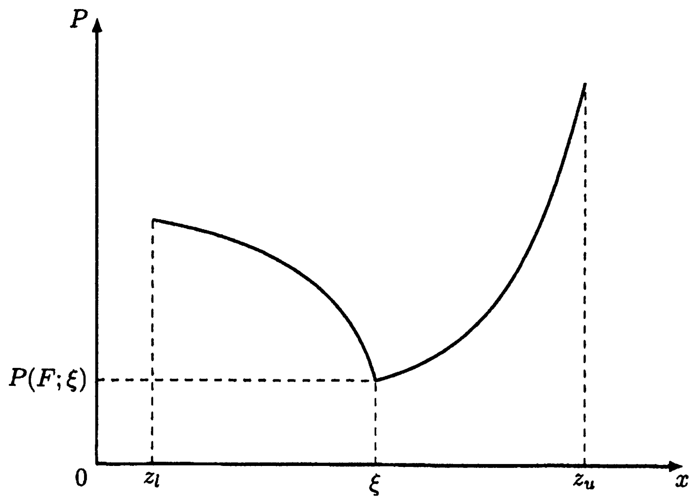

Fig. 2 Bifurcated Lorenz curve

$$\int_{z_l}^{z} P(F;t)dt \le \int_{z_l}^{z} P(G;t)dt \text{ for all } z \in [z_l, z_u].$$

If one is unsure about the location of ξ or one allows ξ to vary over the entire poverty border region, then the second-order condition may collapse to the first-order condition. To see this, in the first condition of (4.4), let $z = \xi - \varepsilon$ for some small $\varepsilon > 0$ and dividing both sides by ϵ, we have

$$\frac{1}{\varepsilon}\int_{\xi-\varepsilon}^{\xi} P(F;t)dt \le \frac{1}{\varepsilon}\int_{\xi-\varepsilon}^{\xi} P(G;t)dt. \tag{4.6}$$

Let $\varepsilon \to 0$ and taking limits of the both sides, we would obtain $P(F;\xi) \le P(G;\xi)$ and condition (4.3) ensues as ξ varies over $[z_l, z_u]$. This result is summarized in the following corollary.

Corollary 4.1 *The second-order condition* (4.4) *becomes the first-order condition* (4.3) *if ξ can take any value in* $[z_l, z_u]$.

4.2 Poverty-measure partial ordering for a given membership function

As afore-suggested, it would be useful to extend poverty-measure partial oderings to fuzzy poverty measurement. Here a poverty membership function is given and we wish to establish dominance conditions for one distribution to have unambiguously less poverty than another distribution as judged by all possible fuzzy poverty measures. Similar to the practice in the literature of crisp-poverty-measure partial orderings, we limit our derivations to additively separable (decomposable) fuzzy poverty measures.[5]

For an additively separable (decomposable) fuzzy poverty measure, distribution F has no more poverty than distribution G if and only if

$$\begin{aligned}\Pi(F;m) &= \int_0^\infty \int_0^\infty p(x,z)\rho(z)dzdF(x) \le \int_0^\infty \int_0^\infty p(x,z)\rho(z)dzdG(x)\\ &= \Pi(G;m)\end{aligned}$$

which becomes (by specifying the poverty border region)

$$\int_{z_l}^{z_u} \rho(z)\left[\int_0^z p(x,z)dF(x)\right]dz \le \int_{z_l}^{z_u} \rho(z)\left[\int_0^z p(x,z)dG(x)\right]dz. \tag{4.7}$$

In crisp partial poverty orderings ($z_u \to z_l$), the poverty line is the same in all poverty deprivation computations and the comparisons of poverty values are equivalent to the comparisons of income shortfalls from the (same) poverty line. In fuzzy poverty orderings (4.7), however, poverty values computed from different poverty lines are

[5] Note that the dominance conditions derived for crisp decomposable measures often are only sufficient for rank-based poverty measures such as the Sen measure—see Zheng (2000) for a detailed exposition.

compared (i.e., $p(x, z)$ is compared with $p(x', z')$ for $x \neq x'$ and $z \neq z'$) and, consequently, a general dominance condition for (4.7) with all possible poverty deprivation functions seems elusive to be established. To gain some tractability, in what follows we limit our investigation to the two afore-defined types of poverty measures, namely the absolute classes ($\mathfrak{P}_1^a$ and $\mathfrak{P}_2^a$) and the relative classes ($\mathfrak{P}_1^r$ and $\mathfrak{P}_2^r$). Either type of poverty measures enables the distances between incomes and the poverty lines to be quantified (i.e., $z-x$ or x/z) and compared.[6] Below we first consider absolute poverty measures.

Proposition 4.2 *For a given membership density function $\rho(z)$ with $z \in [z_l, z_u]$, condition (4.7) holds for all poverty measures in $\mathfrak{P}_1^a$ if and only if*

$$\int_{z_l}^{z_u} \rho(z) F_z(z-y)dz \leq \int_{z_l}^{z_u} \rho(z) G_z(z-y)dz \text{ for all } y \in [0, z_u] \tag{4.8}$$

where $F_z(x)$ is $F(x)$ censored at z: $F_z(x) = F(x)$ for $x < z$ and $F_z(x) = 1$ for $x \geq z$ ($G_z(x)$ is similarly defined); and condition (4.7) holds for all poverty measures in $\mathfrak{P}_2^a$ if and only if

$$\int_0^y \int_{z_l}^{z_u} \rho(z) F_z(z-t)dzdt \leq \int_0^y \int_{z_l}^{z_u} \rho(z) G_z(z-t)dzdt \text{ for all } y \in [0, z_u]. \tag{4.9}$$

Proof Denote $y \equiv z - x$ for $x < z$ (and $y \equiv 0$ for $x \geq z$) and denote y's cumulative distribution functions $\tilde{F}_z(y)$ and $\tilde{G}_z(y)$, respectively. Then $\tilde{F}_z(y)$ is related to $F(x)$ via

$$\begin{aligned} \tilde{F}_z(y_0) &= P\{y \leq y_0\} = P\{z - x \leq y_0\} = P\{x \geq z - y_0\} \\ &= 1 - F_z(z - y_0) \end{aligned} \tag{4.10}$$

where $F_z(x) = F(x)$ for $x < z$ and $F_z(x) = 1$ for $x \geq z$. We then have

$$\begin{aligned} \Pi(F; m) &= \int_{z_l}^{z_u} \rho(z) \left[\int_0^z \tilde{p}(z-x)dF(x)\right] dz = \int_{z_l}^{z_u} \rho(z) \left[\int_z^0 \tilde{p}(y)dF(z-y)\right] dz \\ &= \int_{z_l}^{z_u} \rho(z) \left[\int_z^0 \tilde{p}(y)d\{1 - \tilde{F}_z(y)\right] dz \text{ (by 4.10)} \\ &= \int_{z_l}^{z_u} \rho(z) \left[\int_0^z \tilde{p}(y)d\tilde{F}_z(y)\right] dz \\ &= \int_{z_l}^{z_u} \int_0^{z_u} \rho(z)\tilde{p}(y)\tilde{f}_z(y)dydz \end{aligned}$$

[6] One could also consider a more general intermediate poverty measure, $\tilde{p}(\frac{z-x}{z^\eta})$, which contains both relative and absolute measures as special cases ($\eta = 1$ and 0, respectively). For a general treatment of intermediate poverty measures, see Zheng (2007).

where $\tilde{f}_z(y)$ is the density function of $\tilde{F}_z(y)$ with $\tilde{f}_z(y) = 0$ for $y \geq z$. In the last step, we have also replaced the upper bound z with z_u which is the maximum value of z. Reverse the order of integration,

$$\begin{aligned}\Pi(F; m) &= \int_0^{z_u} \int_{z_l}^{z_u} \rho(z)\tilde{p}(y)\tilde{f}_z(y)dzdy \\ &= \int_0^{z_u} \tilde{p}(y) \int_{z_l}^{z_u} \rho(z)\tilde{f}_z(y)dzdy \\ &= \int_0^{z_u} \tilde{p}(y)d\left[\int_{z_l}^{z_u} \rho(z)\tilde{F}_z(y)dz\right]\end{aligned}$$

and (4.7) becomes

$$\int_0^{z_u} \tilde{p}(y)d\left[\int_{z_l}^{z_u} \rho(z)\tilde{F}_z(y)dz\right] \leq \int_0^{z_u} \tilde{p}(y)d\left[\int_{z_l}^{z_u} \rho(z)\tilde{G}_z(y)dz\right]. \tag{4.11}$$

For all poverty measures in $\mathfrak{P}_1^a$, $\tilde{p}(0) = 0$ and $\tilde{p}'(y) = -\frac{\partial \tilde{p}}{\partial x} > 0$, the necessary and sufficient condition for (4.7a) can be shown (again using the Atkinson method of contradiction) to be

$$\int_{z_l}^{z_u} \rho(z)\tilde{F}_z(y)dz \geq \int_{z_l}^{z_u} \rho(z)\tilde{G}_z(y)dz \text{ for all } y \in [0, z_u]. \tag{4.12}$$

For all poverty measures in $\mathfrak{P}_2^a$, $\tilde{p}(0) = 0$, $\tilde{p}'(y) > 0$ and $\tilde{p}''(y) < 0$, the necessary and sufficient condition for (4.7a) can also be shown to be

$$\int_0^y \int_{z_l}^{z_u} \rho(z)\tilde{F}_z(t)dzdt \geq \int_0^y \int_{z_l}^{z_u} \rho(z)\tilde{G}_z(t)dzdt \text{ for all } y \in [0, z_u]. \tag{4.13}$$

Substituting (4.10) into (4.12), we have

$$\int_{z_l}^{z_u} \rho(z)[1 - F_z(z - y)]dz \geq \int_{z_l}^{z_u} \rho(z)[1 - G_z(z - y)]dz \text{ for all } y \in [0, z_u] \tag{4.11a}$$

or equivalently (because $\int_{z_l}^{z_u} \rho(z)dz = 1$),

$$\int_{z_l}^{z_u} \rho(z)F_z(z - y)dz \leq \int_{z_l}^{z_u} \rho(z)G_z(z - y)dz \text{ for all } y \in [0, z_u]. \tag{4.11b}$$

Condition (4.9) is obtained similarly. □

Conditions (4.8) and (4.9) generalize the partial ordering conditions for crisp poverty measures to fuzzy poverty measurement. It is well known that the first-order dominance condition for all poverty measures (or the relative and the absolute subclasses) with $p_x(x, z) < 0$ and for a single poverty line z is $F(x) \leq G(x)$ over $x \in [0, z]$. Here for poverty orderings with fuzzy poverty measures, we compare the weighted censored *cdf* over $[0, z_u]$, $\int_{z_l}^{z_u} \rho(z) F_z(z-y)dz$ and $\int_{z_l}^{z_u} \rho(z) G_z(z-y)dz$. If the poverty border region degenerates to a single point z, i.e., $z_u \to z_l$, condition (4.8) collapses to $F_z(z-y) < G_z(z-y)$ which is equivalent to $F(x) < G(x)$ for all $x \in [0, z]$ - thus (4.8) includes the ordinary first-order condition as a special case.

For the relative classes of poverty measures, we have:

Proposition 4.3 *For a given membership density function $\rho(x)$ with $x \in [z_l, z_u]$, condition (4.7) holds for all poverty measures in $\mathfrak{P}_1^r$ if and only if*

$$\int_{z_l}^{z_u} \rho(z) F_z(yz)dz \leq \int_{z_l}^{z_u} \rho(z) G_z(yz)dz \textit{ for all } y \in [0, 1] \tag{4.8a}$$

where $F_z(x)$ is $F(x)$ censored at z: $F_z(x) = F(x)$ for $x < z$ and $F_z(x) = 1$ for $x \geq z$ ($G_z(x)$ is similarly defined); and condition (4.7) holds for all poverty measures in $\mathfrak{P}_2^r$ if and only if

$$\int_0^y \int_{z_l}^{z_u} \rho(z) F_z(tz)dzdt \leq \int_0^y \int_{z_l}^{z_u} \rho(z) G_z(tz)dzdt \textit{ for all } y \in [0, 1]. \tag{4.9a}$$

Proof Denote $y \equiv \frac{x}{z}$ for $x < z$ (and $y \equiv 1$ for $x \geq z$) and y's cumulative distribution functions $\tilde{F}_z(y)$ and $\tilde{G}_z(y)$, respectively. Then $\tilde{F}_z(y)$ is related to $F(x)$ through

$$\tilde{F}_z(y_0) = P\left\{y \leq y_0\right\} = P\left\{\frac{x}{z} \leq y_0\right\} = P\{x \leq y_0 z\} = F_z(y_0 z)$$

where again $F_z(x) = F(x)$ for $x < z$ and $F_z(x) = 1$ for $x \geq z$. The rest of the proof is similar to that for the case of absolute poverty measures (but noting that now $\tilde{p}'(y) < 0$ and $\tilde{p}''(y) > 0$). □

It is interesting to note one more difference between crisp poverty orderings and fuzzy poverty orderings: for a given (and same) poverty line, both the relative class and the absolute class of poverty measures yield the same dominance condition; the condition for the class of relative fuzzy poverty measures and the class of absolute poverty measures could be quite different.

4.3 Poverty-membership-measure partial ordering

Finally one may naturally be interested in establishing dominance conditions for all possible additively separable fuzzy poverty measures—that is, for all possible crisp decomposable poverty measures and all possible membership functions. Given the results obtained above, we can achieve this by following two routes: considering all membership functions first, then requiring the results to hold for all poverty measures;

or considering all poverty measures first, then requiring the results to hold for all membership functions.

For example, if we wish to derive dominance conditions for all membership functions in $\mathfrak{M}_1$ and for all absolute poverty measures in $\mathfrak{P}_1^a$, we could start from Proposition 4.1 and require the results to hold for all absolute poverty measures. This approach amounts to requiring (4.3) to hold for all absolute poverty measures with $\tilde{p}'(x) > 0$ and for all $z \in [z_l, z_u]$, which yields $F(x) \leq G(x)$ for all $x \in [0, z_u]$. We could also start from Proposition 4.2 and require the results to hold for all membership functions in $\mathfrak{M}_1$. This alternative approach amounts to requiring (4.8) to be valid for all possible membership density functions $\rho(z)$. We would have $F_z(z-y) \leq G_z(z-y)$ for all $y \in [0, z_u]$ and all $z \in [z_l, z_u]$. Clearly, by setting $x = z - y$, this route also leads to $F(x) \leq G(x)$ for all $x \in [0, z_u]$.

As another example, if we want to establish dominance conditions for all membership functions in $\mathfrak{M}_2$ and all absolute poverty measures in $\mathfrak{P}_1^a$, we could start from (4.8) and require $\rho'(z)$ to change sign from positive to negative at $z = \xi$. Following the proof of the second part of Proposition 4.1, we would be able to establish the following condition

$$\int_s^\xi F_z(z-y)dz \leq \int_s^\xi G_z(z-y)dz \text{ for all } y \in [0, \xi] \text{ and } s \in [z_l, \xi], \text{ and}$$
$$\int_\xi^s F_z(z-y)dz \leq \int_\xi^s G_z(z-y)dz \text{ for all } y \in [0, z_u] \text{ and } s \in [\xi, z_u].$$

But the first part of the condition can be simplified to $F_\xi(\xi - y) \leq G_\xi(\xi - y)$ for all $y \in [0, \xi]$ since it implies $F_s(s-y) \leq G_s(s-y)$ for any $s \leq \xi$. The condition $F_\xi(\xi - y) \leq G_\xi(\xi - y)$ further turns out to be a part of the second condition (by letting $s \to \xi$). The second part of the condition, however, cannot be simplified any further. The alternative route—starting from (4.4) and requiring them to hold for all $\tilde{p}(y)$s with $\tilde{p}'(y) \geq 0$—would also lead to the same conclusion (by applying crisp poverty ordering results to each part of (4.4)). Note that the dominance condition now consists of only the right half of the bifurcated device in Fig. 2.

The following two propositions summarize the results of partial poverty orderings for all membership functions in $\mathfrak{M}_1$ and $\mathfrak{M}_2$, and for all poverty measures in the relative and absolute classes, respectively. The additional results beyond what we have demonstrated above can also be easily verified.

Proposition 4.4 *For any distributions F and G: (i) $\Pi(F; m) \leq \Pi(G; m)$ for all membership functions in $\mathfrak{M}_1$ and all poverty measures in $\mathfrak{P}_1^a$ if and only if*

$$F(x) \leq G(x) \text{ for all } x \in [0, z_u]; \tag{4.14}$$

(ii) $\Pi(F; m) \leq \Pi(G; m)$ for all membership functions in $\mathfrak{M}_1$ and all poverty measures in $\mathfrak{P}_2^a$ if and only if

$$\int_0^x F(t)dt \leq \int_0^x G(t)dt \text{ for all } x \in [0, z_u]; \tag{4.15}$$

(iii) $\Pi(F;m) \leq \Pi(G;m)$ *for all membership functions in* $\mathfrak{M}_2$ *and all poverty measures in* $\mathfrak{P}_1^a$ *if and only if*

$$\int_{\xi}^{s} F_z(z-y)dz \leq \int_{\xi}^{s} G_z(z-y)dz \text{ for all } y \in [0, z_u] \text{ and } s \in [\xi, z_u]; \qquad (4.16)$$

and (iv) $\Pi(F;m) \leq \Pi(G;m)$ *for all membership functions in* $\mathfrak{M}_2$ *and all poverty measures in* $\mathfrak{P}_2^a$ *if and only if*

$$\int_{0}^{y}\int_{\xi}^{s} F_z(z-t)dzdt \leq \int_{0}^{y}\int_{\xi}^{s} G_z(z-t)dzdt \text{ for all } y \in [0, z_u] \text{ and all } s \in [\xi, z_u]. \qquad (4.17)$$

Proposition 4.5 *For any distributions* F *and* G*: (i)* $\Pi(F;m) \leq \Pi(G;m)$ *for all membership functions in* $\mathfrak{M}_1$ *and all poverty measures in* $\mathfrak{P}_1^r$ *if and only if*

$$F(x) \leq G(x) \text{ for all } x \in [0, z_u];$$

(ii) $\Pi(F;m) \leq \Pi(G;m)$ *for all membership functions in* $\mathfrak{M}_1$ *and all poverty measures in* $\mathfrak{P}_2^r$ *if and only if*

$$\int_{0}^{x} F(t)dt \leq \int_{0}^{x} G(t)dt \text{ for all } x \in [0, z_u];$$

(iii) $\Pi(F;m) \leq \Pi(G;m)$ *for all membership functions in* $\mathfrak{M}_2$ *and all poverty measures in* $\mathfrak{P}_1^r$ *if and only if*

$$\int_{\xi}^{s} F_z(yz)dz \leq \int_{\xi}^{s} G_z(yz)dz \text{ for all } y \in [0, 1] \text{ and all } s \in [\xi, z_u]; \qquad (4.15a)$$

and (iv) $\Pi(F;m) \leq \Pi(G;m)$ *for all membership functions in* $\mathfrak{M}_2$ *and all poverty measures in* $\mathfrak{P}_2^r$ *if and only if*

$$\int_{0}^{y}\int_{\xi}^{s} F_z(xz)dzdt \leq \int_{0}^{y}\int_{\xi}^{s} G_z(xz)dzdt \text{ for all } y \in [0, 1] \text{ and all } s \in [\xi, z_u]. \qquad (4.16a)$$

4.4 An application to the US data

To illustrate the approach proposed, we apply the dominance conditions derived in Propositions 4.1, 4.4 and 4.5 to the US current population survey (CPS) income data.[7] In the illustration, we consider US poverty in 1995, 2000, 2005 and 2010. In each year, the poverty line is not a single number but a fuzzy set. In this illustration, we

[7] We do not illustrate Propositions 4.2 and 4.3 since they require the estimation of poverty membership functions which is not feasible with the CPS data.

Table 1 Fuzzy poverty orderings for a given P and all m in $\mathfrak{M}_1$

	H			HI			HI^2			W		
	00	05	10	00	05	10	00	05	10	00	05	10
1995	+	+	+	×	×	×	+	+	×	+	×	×
2000		×	−		−	−		×	−		−	−
2005			−			×			−			×

Table 2 Fuzzy poverty orderings for a given P and all m in $\mathfrak{M}_2$

	H			HI			HI^2			W		
	00	05	10	00	05	10	00	05	10	00	05	10
1995	+	+	+	+	+	+	+	+	×	+	+	×
2000		×	−		−	−		×	−		−	−
2005			−			−			−			−

assume that the center of the fuzzy poverty-line set is the official poverty line and the lower and upper boundaries of the fuzzy set equal 50 % of the official poverty line and 150 % of the official poverty line, respectively. That is, we assume ξ = the official poverty line z, $z_l = 0.5z$ and $z_u = 1.5z$.

Table 1 reports the pairwise comparisons of poverty among the four years for all fuzzy membership functions in $\mathfrak{M}_1$, i.e., m with $m' < 0$, and a given poverty measure. We consider four different poverty measures: the headcount ratio (H), the poverty gap ratio (HI), the squared poverty gap ratio (HI^2), and the Watts measure (W). In the table, each year in the first column is compared with all other 3 years in the second row (00, 05 and 10 stand for 2000, 2005 and 2010, respectively). A plus (+) indicates that the column year has more poverty than the row year compared, a minus (−) indicates less poverty, and a cross (×) indicates no clear ranking between the 2 years. Of the six pairwise comparisons, H can rank order five, HI ranks two, HI^2 ranks four and W ranks three. The only comparison that all four measures agree on is that between 2000 and 2010: 2010 has more poverty than 2000.

When fuzzy poverty membership functions are limited to $\mathfrak{M}_2$, as Table 2 indicates, HI is able to rank order all six pairs of comparison and W can also rank more pairs. All four measures now also agree on the rankings of 1995 with 2000 and 2005, and 2005 with 2010: 1995 has more poverty than both 2000 and 2005, and 2005 has less poverty than 2010.

Tables 3, 4 and 5 report fuzzy poverty orderings for a class of poverty measures and a class of poverty membership functions. In Table 3, poverty membership functions are limited to $\mathfrak{M}_1$. The first half of the table (columns 2, 3 and 4) summarizes the comparisons by all poverty measures in $\mathfrak{P}_1$ ($\mathfrak{P}_1^a$ or $\mathfrak{P}_1^r$) and the second half (columns 5, 6 and 7) summarizes the comparisons by all poverty measures in $\mathfrak{P}_2$ ($\mathfrak{P}_2^a$ or $\mathfrak{P}_2^r$). All poverty measures in $\mathfrak{P}_1$ with all poverty membership functions are able to rank

Table 3 Fuzzy poverty orderings for all P and all m in $\mathfrak{M}_1$

	$\mathfrak{P}_1$			$\mathfrak{P}_2$		
	2000	2005	2010	2000	2005	2010
1995	+	×	×	+	×	×
2000		×	−		−	−
2005			−			−

Table 4 Fuzzy poverty orderings for all P^a and all m in $\mathfrak{M}_2$

	$\mathfrak{P}_1^a$			$\mathfrak{P}_2^a$		
	2000	2005	2010	2000	2005	2010
1995	+	+	+	+	+	+
2000		−	−		−	−
2005			−			−

Table 5 Fuzzy poverty orderings for all P^r and all m in $\mathfrak{M}_2$

	$\mathfrak{P}_1^r$			$\mathfrak{P}_2^r$		
	2000	2005	2010	2000	2005	2010
1995	+	+	×	+	+	×
2000		−	−		−	−
2005			−			−

three out of six comparisons: 1995 has more poverty than 2000, both 2000 and 2005 have less poverty than 2010. When poverty measures are limited to $\mathfrak{P}_2$, 2000 is seen to have less poverty than 2005.

In Tables 4 and 5, poverty membership functions are required to be of the inverted-S shape. Table 4 shows that all absolute poverty measures (in $\mathfrak{P}_1^a$ and $\mathfrak{P}_2^a$, respectively) can unanimously rank order poverty among the four years: 1995 has the highest level of poverty and 2000 has the lowest level of poverty; 2005 and 2010 are somewhere in between with 2010 having more poverty than 2005.

When all relative poverty measures (in $\mathfrak{P}_1^r$ and $\mathfrak{P}_2^r$, respectively) are considered (Table 5), the pattern of poverty ranking is similar to the case of absolute poverty measures except now 1995 and 2010 cannot be ranked. This no-ranking stands even when poverty measures are restricted from $\mathfrak{P}_1^r$ to $\mathfrak{P}_2^r$.

5 Summary and conclusion

Deciding an individual "to be" or "not to be" in poverty is an important first question in poverty measurement. The almost four-decade deep poverty measurement research started from Sen (1976), however, has not paid sufficient attention to this important question. The dichotomous treatment of the poor and the non-poor has largely remained the "rule of the land" in empirical poverty research. But the concept of poverty, as many have argued, is not a crisp one and there exist "borderline" situations where it is impossible to proclaim an individual to be definitely poor or definitely non-poor. Poverty, like many linguistic variables such as "tall" and "pretty," is a fuzzy predicate and poverty measurement must take this fuzziness nature into account. The fuzzy set theory, developed over the last-half century, can be fruitfully employed to extend crisp poverty measurement into fuzzy poverty measurement.

At the heart of fuzzy poverty measurement is the poverty membership function—it quantifies the degree of an individual's belonging to the poverty population. The calibration of a membership function, however, depends on the source of fuzziness. In the paper we argued that a plausible source is that people (or voters) have different perceptions about what constitutes poverty. This difference-in-perception characterization of fuzziness renders a direct estimation of the poverty membership function: the degree of poverty membership for an individual with income x is simply the cumulative percentage of voters who believe that the poverty line should be at least x. In the paper, we considered two sets of membership functions: the first set contains all functions decreasing in income while the second set contains all inverted-S shape membership functions (Fig. 1). To provide a meaningful interpretation for the membership function, we further defined a "density function" which is the proportion of the voters who believe the poverty line to be at a given income level. With the help of an intuitive equal voting-right axiom, the difference-in-perception approach also provides an axiomatic characterization to the class of decomposable fuzzy poverty measures that Shorrocks and Subramanian (1994) defined. Compared with their characterization, ours is more elementary and intuitive.

The other major contribution of the paper was to provide a set of dominance conditions for fuzzy partial poverty orderings. Since a fuzzy poverty measure involves the choices of a crisp poverty measure and a poverty membership function, there are three types of partial poverty orderings we need to consider: the orderings by all possible membership functions with a given crisp poverty measure, the orderings by all crisp poverty measures with a given membership function, and the orderings by all possible crisp poverty measures *and* all membership functions. The first-order dominance condition we derived for all membership functions is akin to a first-order stochastic dominance applied to vectors of sorted poverty values. But the second-order condition is a bifurcated Lorenz-type dominance with origin being at the "switching point" of the membership functions (Fig. 2). The dominance conditions for all crisp poverty measures that we established are for two types of decomposable poverty measures, namely the absolute classes and the relative classes. All these conditions can be easily applied to real income data and we illustrate some of the conditions with the US CPS data.

We hope that the discussions presented and the results derived in the paper would be useful to poverty measurement researchers. We also hope that the fact that poverty is a fuzzy predicate will be taken more seriously in future poverty measurement research. Looking forward, we see a number of specific issues remain to be topics for future research. First, the empirical estimation of a poverty membership function. Although our approach makes it possible to estimate a membership function from empirical data, no such empirical studies have been done. To do the estimation we may need to collect specific data on people's perception of poverty.[8] Second, the axiomatic characterization we developed in Sect. 2 is only for decomposable (additively separable) fuzzy poverty measures. It would be useful to develop a similar characterization for all rank-based poverty measures such as the Sen measure. Third, the dominance conditions we derived for crisp poverty measures are only for the absolute and relative subclasses. It would certainly be helpful to know whether similar dominance conditions exist for all poverty measures. Finally, in considering different membership functions of the inverted-S shape we have maintained a same "switching point" assumption—the same ξ value for all distributions. This assumption would certainly be viewed as too restrictive as different societies or different populations may have different "switching points." A more general question along this line would be poverty rankings between societies with different "poverty membership functions" that may stem from different sets of voters. It would be useful to extend our analysis to these new situations.

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[8] The author of this paper is currently engaging in collaborations in estimating the poverty membership functions using the World Bank's LSMS data and EU's SILC data.

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