Article

Generalisation of the Signed Distance

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Abstract: This paper presents a comprehensive study of the signed distance metric for fuzzy numbers. Due to the property of directionality, this measure has been widely used. However, it has a main drawback in handling asymmetry and irregular shapes in fuzzy numbers. To overcome this rather bad feature, we introduce two new distances, the Balanced signed distance (BSGD) and the Generalised signed distance (GSGD), seen as generalisations of the classical signed distance. The developed distances successfully and effectively take into account the shape, the asymmetry and the overlap of fuzzy numbers. The GSGD is additionally directional, while the BSGD satisfies the requirements for being a metric of fuzzy quantities. Analytical simplifications of both distances in the case of often-used particular types of fuzzy numbers are provided to simplify the computation process, making them as simple as the classical signed distance but more realistic and precise. We empirically analyse the sensitivity of these distances. Considering several scenarios of fuzzy numbers, we also compare numerically these distances against established metrics, highlighting the advantages of the BSGD and the GSGD in capturing the shape properties of fuzzy numbers. One main finding of this research is that the defended distances capture with great precision the distance between fuzzy numbers, they are theoretically appealing and are computationally easy for traditional fuzzy numbers such as triangular, trapezoidal, Gaussian, etc., making these metrics promising.

Keywords: Asymmetrical fuzzy numbers; Balanced signed distance; Fuzzy metrics; Fuzzy numbers; Fuzzy statistics; Generalised signed distance; L^2 metric; Signed distance.

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1. Introduction

Calculating the difference between fuzzy numbers is generally considered a complex task due to the semi-linear characteristics of the fuzzy space $\mathbb{F}(\mathbb{R})$. One of the problems is that there is no universally accepted approach for achieving a complete ordering within the fuzzy number space. A practical way to address these challenges is by adopting an appropriate metric having in mind that suitable metrics or distances should be well-defined within the space of fuzzy sets to offer a structured means of comparison.

The Hausdorff distance δ_H has been one of the most used distances as a basis for developing metrics in the fuzzy space. For example, Kaleva [1987] used it in fuzzy differential equations and Puri and Ralescu [1985] in treating fuzzy random variables. But, from one side, the Hausdorff distance has been said by Klement et al. [1986] to have handicaps when used in the fuzzy environment since it fails to be separable. From another one, the measure is known as "disadvantaging" fuzzy sets compared to interval ones.

Researchers have then proposed new distances and metrics to better account for these handicaps. Diamond and Kloeden [1990] introduced two families of metrics defined on the space of fuzzy numbers, one denoted by d_p, and another one by ρ_p. These metrics are established on extending the Hausdorff distance and are widely used in practice, particularly the metric ρ_2.

Bertoluzza et al. [1995] highlighted significant limitations of the Hausdorff-based metric as well as of those developed by Diamond and Kloeden [1990]. The authors observed that measuring the distance between fuzzy numbers using these tools is less suitable" compared to an application to crisp numbers. For this reason, they introduced a metric $d_{\text{Bertoluzza}}$ that considers the distance as a weighted mean of the partial distances between α-levels of fuzzy numbers.

Trutschnig et al. [2009] revisited and extended the metric $d_{\text{Bertoluzza}}$. Their approach introduced a distance denoted by $d_{\text{mid/spr}}$ based on the central location of a fuzzy number, referred to as the "mid," and its spread, also known as the spread or "spr". The authors considered their metric to be a generalisation of the Hausdorff δ_H, the Bertoluzza $d_{\text{Bertoluzza}}$ and the ρ_2 of Diamond and Kloeden [1990] in an L^2 space.

Building on the metrics developed by Bertoluzza et al. [1995] and Trutschnig et al. [2009], Sinova et al. [2014] introduced a new family of metrics called "ϕ-wabl/ldev/rdev". These metrics focus on capturing the "deviations in shape" of fuzzy numbers at different levels, written "ldev" (left deviation) and "rdev" (right deviation), while also accounting for the central location of the fuzzy numbers via the "wabl" (weighted average of the left and right bounds).

Another type of distance, such as the signed distance, has emerged. This distance has been introduced by Dubois and Prade [1987] under the name of "expected value of a particular fuzzy number". Yao and Wu [2000] were the first scholars to describe the signed distance in its current form, notably for ranking fuzzy numbers. This distance has been known as simple and interesting because of its directionality property. Such distances can give negative values, contrary to the aforementioned metrics.

The classical signed distance (SGD) as proposed by Yao and Wu [2000] has long served as a foundational tool for comparing fuzzy numbers. This is because the SGD provides a straightforward and interpretable approach, mainly as mentioned above, due to its directionality property. This property permits negative or positive distances indicating the "travel direction" between two fuzzy numbers. It is a simple and flexible distance that is computationally easy. Thus, it is not surprising that it is considered a broad application-based tool and, indeed, has been used for many purposes. For example, first, it has been used as a decision-making tool by Chen [2012], who used it as an aggregation method for handling fuzzy multiple-criteria group decision-making problems. Berkachy and Donzé [2016] considered it in the context of ranking to evaluate linguistic questionnaires or to define ordering between fuzzy numbers as seen in Yao and Lin [2000], since there is no unique and natural ordering in the family of all fuzzy numbers. It has also been utilised as a defuzzification operator as shown in Berkachy [2021] and Berkachy and Donzé [2019]. Finally, Berkachy and Donzé [2019] focused on the use of the SGD in the development of fuzzy hypotheses testing approaches.

Although the signed distance appears to have great properties, it shows two serious drawbacks. First, the SGD has limitations when dealing with asymmetrical or complex-shaped fuzzy numbers. This distance coincides with a central location measure. Therefore, it effectively focuses on the core differences between fuzzy numbers, not their spreads and shapes. This leads to oversimplifying calculations, especially when the shapes or ranges of the fuzzy numbers are irregular. Second, the SGD as a directional distance lacks symmetry and separability since it can have negative values. This leads to a lack of defining this distance as a metric of fuzzy quantities.

To address these somewhat limitations of the classical signed distance, we introduce the Generalised signed distance (GSGD) and a L^2 metric, the Balanced signed distance (BSGD). The BSGD expands the classical measure by effectively capturing variations in the spread and shape of fuzzy numbers. It introduces sensitivity to both the left and right sides of fuzzy numbers, thus effectively capturing asymmetry. Flexibility is then added for the measure, it can treat with precision both symmetrical and asymmetrical fuzzy numbers. In addition, the BSGD is of the L^2 class and satisfies the fundamental properties required to be considered a valid metric for fuzzy quantities. This solves the theoretical limitations of symmetry and separability, showing that this improvement presents a mathematically sound tool for comparing fuzzy quantities. Yet, the directionality property is lost in this case.

We propose the GSGD measure to recapture this property. It is built on the framework of the BSGD by incorporating the directionality emerging from the SGD. Thus, the GSGD ensures the directionality of the classical signed distance and, in addition, provides flexibility in handling irregularities in fuzzy numbers of the BSGD.

By balancing the treatment of the core, the shape and the spread, the BSGD and GSGD measures are particularly well-suited for calculating distances between fuzzy numbers in cases where symmetry or irregularities in shape play a critical role. This balance is reflected by a parameter $\theta^\star$, which has the role of weighting shape and spread. The sensitivity of the BSGD and the GSGD to these aspects gives them a distinct advantage in practical applications where more complex calculations are needed, such as in decision-making under uncertainty or systems involving imprecise data.

To understand the effect of the weighting parameter $\theta^\star$, we compute the measures for different scenarios of fuzzy numbers. The package Berkachy and Donzé [2024] from the `R` statistical software R Core Team [2024] is used for these computations. As we will show, a finding is that the presence or absence of overlap between two fuzzy numbers has an essential impact on the obtained distances, depending on the tuning given by $\theta^\star$. In addition, we provide a comparison between the BSGD (GSGD) and other established fuzzy distance measures. As a result of all analyses, it appears that our distances are particularly robust when applied to fuzzy numbers with irregular shapes.

Analytical simplifications of the BSGD and the GSGD measures are also proposed, making these distances more general, adaptable, and easy-to-use than the SGD one.

The remainder of the paper proceeds as follows. In Section 2, we present the definition of the classical signed distance measure. Section 3 is devoted to the definitions of the Balanced signed distance (BSGD) and the Generalised signed distance (GSGD). In Section 4, we present and discuss the simulations where we study the influence of the variation of the weighting parameter $\theta^\star$ in multiple scenarios. We end this paper with Section 5 illustrating a simulation study comparing different computed distances.

2. Signed distance

Dubois and Prade [1987] are the first to introduce a particular measure as an expected value of a specific fuzzy number. This measure is nowadays known as the signed distance (SGD), popularised by Yao and Wu [2000], who presented it in the context of ranking fuzzy numbers. This measure has been widely applied in various fields, including evaluating linguistic questionnaires (Berkachy and Donzé[2016]) and hypothesis testing (Berkachy[2021] and Berkachy and Donzé[2019]), among others. This distance is known for its simplicity to be calculated. It is also known as directional because the measure can yield a negative value between two fuzzy numbers. This characteristic has made the signed distance a compelling subject for research, particularly in fuzzy ranking methods, as highlighted by Abbasbandy and Asady [2006]. Since no general ordering rule for fuzzy numbers exists, such a measure can be seen as very advantageous. Finally, the computational aspects of this distance metric add to its appeal.

Let us define the signed distance between two fuzzy numbers $\tilde{X}$ and $\tilde{Y}$ and enumerate some fundamental properties. In the following, we denote by $\tilde{X}^\alpha$ the α-cuts of the fuzzy number $\tilde{X}$. In interval form, the α-cuts are given by $(\tilde{X}^{L,\alpha}, \tilde{X}^{R,\alpha})$. The left and right alpha-cuts can also be given as a function of α, i.e. $\tilde{X}^{L,\alpha}(\alpha)$ and $\tilde{X}^{R,\alpha}(\alpha)$, integrable on their domain. In practice, triangular or trapezoidal fuzzy numbers are expressed as triplets or quadruplets, e.g. $\tilde{X} = (p, q, r)$ or $\tilde{X} = (p, q, r, s)$.

Definition 1 (Signed distance between two fuzzy sets)**.**

Consider $\tilde{X}$ and $\tilde{Y}$ two fuzzy sets of the class of the non-empty compact, convex and normal fuzzy sets $\mathbb{F}^{\star}_{c}(\mathbb{R})$ on $\mathbb{R}$, with their respective left and right α-cuts $\tilde{X}^{L,\alpha}$, $\tilde{X}^{R,\alpha}$, $\tilde{Y}^{L,\alpha}$ and $\tilde{Y}^{R,\alpha}$. The signed distance between $\tilde{X}$ and $\tilde{Y}$ denoted by d_{SGD} is the mapping

$$\begin{array}{rcl} d_{SGD} : \mathbb{F}^{\star}_{c}(\mathbb{R}) \times \mathbb{F}^{\star}_{c}(\mathbb{R}) & \to & \mathbb{R} \\ \tilde{X} \times \tilde{Y} & \mapsto & d_{SGD}(\tilde{X}, \tilde{Y}), \end{array}$$

such that

$$d_{SGD}(\tilde{X}, \tilde{Y}) = \frac{1}{2} \int_0^1 \left[\tilde{X}^{L,\alpha}(\alpha) + \tilde{X}^{R,\alpha}(\alpha) - \tilde{Y}^{L,\alpha}(\alpha) - \tilde{Y}^{R,\alpha}(\alpha) \right] \mathrm{d}\alpha. \quad (1)$$

It is important to consider the fuzzy origin $\tilde{0}$. Its support and core sets are reduced to the singleton $\{0\}$. Calculating the signed distance of a fuzzy set measured from the fuzzy origin $\tilde{0}$ is interesting for ranking fuzzy numbers. This distance is written as:

Definition 2 (Signed distance of a fuzzy set)**.**

The signed distance of the fuzzy set $\tilde{X}$ measured from the fuzzy origin $\tilde{0}$ is

$$d_{SGD}(\tilde{X}, \tilde{0}) = \frac{1}{2} \int_0^1 \left[\tilde{X}^{L,\alpha}(\alpha) + \tilde{X}^{R,\alpha}(\alpha) \right] \mathrm{d}\alpha. \quad (2)$$

The signed distance between two fuzzy numbers can be written in terms of their distance to the fuzzy origin $\tilde{0}$. This relationship is represented as follows:

Proposition 1.
For $\tilde{X}$ and $\tilde{Y} \in \mathbb{F}^{\star}_{c}(\mathbb{R})$, we have that

$$d_{SGD}(\tilde{X}, \tilde{Y}) = d_{SGD}(\tilde{X}, \tilde{0}) - d_{SGD}(\tilde{Y}, \tilde{0}). \quad (3)$$

Proof. The proof is direct from Definitions 1 and 2. □

As seen in Definitions 1 and 2, this distance can take negative values. This shows the directional property of this distance. Unlike traditional measures, which only provide magnitude seen as positive values, the sign of the signed distance reveals the direction between the two fuzzy numbers. This directional information is valuable for ranking because it clarifies the relative ordering of the fuzzy numbers, especially in the fuzzy field, where a clear ordering of fuzzy numbers does not exist.

The signed distance takes its value on $\mathbb{R}$, unlike most known distances, which take their value on $\mathbb{R}^+$. This is due to the directionality of the signed distance. A direct consequence is that the signed distance does not ensure the separability and the symmetry properties. Berkachy [2021] called it a "partial" symmetry. However, this "partial" symmetry is unfortunately not enough to define this distance as a metric of fuzzy quantities.

2.1. Signed distance of particular fuzzy numbers

The signed distance $d_{SGD}(\tilde{X}, \tilde{0})$ of a fuzzy set $\tilde{X}$ measured from the fuzzy origin $\tilde{0}$ (Berkachy [2021]) is:

- for a triangular fuzzy number $\tilde{X} = (p, q, r)$ such that $p < q < r$,

$$d_{SGD}(\tilde{X}, \tilde{0}) = \frac{1}{2} \int_0^1 \left[p + (q-p)\alpha + r - (r-q)\alpha \right] \mathrm{d}\alpha = \frac{1}{4}(p + 2q + r); \quad (4)$$

- for a trapezoidal fuzzy number $\tilde{X} = (p, q, r, s)$ such that $p < q < r < s$,

$$d_{SGD}(\tilde{X}, \tilde{0}) = \frac{1}{2}\int_0^1 \Big[p + (q-p)\alpha + s - (s-r)\alpha\Big]\mathrm{d}\alpha = \frac{1}{4}(p+q+r+s); \tag{5}$$

- for a Gaussian fuzzy number $\tilde{X}$ given by the mean μ and the variance σ,

$$d_{SGD}(\tilde{X}, \tilde{0}) = \frac{1}{2}\int_0^1 \Big[\tilde{X}^{L,\alpha}(\alpha) + \tilde{X}^{R,\alpha}(\alpha)\Big]\mathrm{d}\alpha = \frac{1}{2}\int_0^1 2\mu\mathrm{d}\alpha = \mu; \tag{6}$$

- for a two-sided Gaussian fuzzy number $\tilde{X}$ given by the mean μ_1 and variance σ_1 for the left part, the mean μ_2 and the variance σ_2 for the right part,

$$\begin{aligned} d_{SGD}(\tilde{X}, \tilde{0}) &= \frac{1}{2}\int_0^1 \Big[\tilde{X}^{L,\alpha}(\alpha) + \tilde{X}^{R,\alpha}(\alpha)\Big]\mathrm{d}\alpha \\ &= \frac{1}{2}\int_0^1 \Big[\mu_1 - \sqrt{-2\sigma_1^2 \ln\alpha} + \mu_2 + \sqrt{-2\sigma_2^2 \ln\alpha}\Big]\mathrm{d}\alpha \\ &= \frac{1}{2}(\mu_1 + \mu_2) - \frac{1}{2}\sigma_1 \int_0^1 \sqrt{-2\ln\alpha}\mathrm{d}\alpha + \frac{1}{2}\sigma_2 \int_0^1 \sqrt{-2\ln\alpha}\mathrm{d}\alpha \end{aligned} \tag{7}$$

and since $\int_0^1 \sqrt{-2\ln\alpha}\mathrm{d}\alpha = \sqrt{\frac{\pi}{2}}$, we have that

$$d_{SGD}(\tilde{X}, \tilde{0}) = \frac{1}{2}(\mu_1 + \mu_2) + \frac{1}{2}\sqrt{\frac{\pi}{2}}(\sigma_2 - \sigma_1). \tag{8}$$

It is also important to develop the signed distance between two particular fuzzy numbers.

Proposition 2.
Let $\tilde{X}$ and $\tilde{Y}$ be two fuzzy numbers of the class of non-empty compact and bounded fuzzy numbers.

1. *Consider two triangular fuzzy numbers given by their triplets $\tilde{X} = (p_1, q_1, r_1)$ with $p_1 < q_1 < r_1$, and $\tilde{Y} = (p_2, q_2, r_2)$ with $p_2 < q_2 < r_2$. Their left and right alpha-cuts are given by:*

$$\begin{aligned} &\tilde{X}^{L,\alpha}(\alpha) = p_1 + (q_1 - p_1)\alpha, \quad \text{and} \quad \tilde{X}^{R,\alpha}(\alpha) = r_1 - (r_1 - q_1)\alpha; \\ &\tilde{Y}^{L,\alpha}(\alpha) = p_2 + (q_2 - p_2)\alpha, \quad \text{and} \quad \tilde{Y}^{R,\alpha}(\alpha) = r_2 - (r_2 - q_2)\alpha. \end{aligned}$$

The signed distance between $\tilde{X}$ and $\tilde{Y}$ is

$$\begin{aligned} d_{SGD}(\tilde{X}, \tilde{Y}) &= \frac{1}{2}\int_0^1 \Big[\tilde{X}^{L,\alpha}(\alpha) + \tilde{X}^{R,\alpha}(\alpha) - \tilde{Y}^{L,\alpha}(\alpha) - \tilde{Y}^{R,\alpha}(\alpha)\Big]\mathrm{d}\alpha \\ &= \frac{1}{2}\int_0^1 \Big[p_1 + (q_1 - p_1)\alpha + r_1 - (r_1 - q_1)\alpha \\ &\qquad - p_2 - (q_2 - p_2)\alpha - r_2 + (r_2 - q_2)\alpha\Big]\mathrm{d}\alpha \\ &= \frac{1}{4}(p_1 - p_2 + 2q_1 - 2q_2 + r_1 - r_2). \end{aligned} \tag{9}$$

2. *Consider two trapezoidal fuzzy numbers given by their quadruplets $\tilde{X} = (p_1, q_1, r_1, s_1)$ with $p_1 < q_1 < r_1 < s_1$, and $\tilde{Y} = (p_2, q_2, r_2, s_2)$ with $p_2 < q_2 < r_2 < s_2$. Their left and right alpha-cuts are given by:*

$$\begin{aligned} &\tilde{X}^{L,\alpha}(\alpha) = p_1 + (q_1 - p_1)\alpha, \quad \text{and} \quad \tilde{X}^{R,\alpha}(\alpha) = s_1 - (s_1 - r_1)\alpha; \\ &\tilde{Y}^{L,\alpha}(\alpha) = p_2 + (q_2 - p_2)\alpha, \quad \text{and} \quad \tilde{Y}^{R,\alpha}(\alpha) = s_2 - (s_2 - r_2)\alpha. \end{aligned}$$

The signed distance between $\tilde{X}$ and $\tilde{Y}$ is

$$\begin{aligned} d_{SGD}(\tilde{X},\tilde{Y}) &= \frac{1}{2}\int_0^1 \Big[\tilde{X}^{L,\alpha}(\alpha)+\tilde{X}^{R,\alpha}(\alpha)-\tilde{Y}^{L,\alpha}(\alpha)-\tilde{Y}^{R,\alpha}(\alpha)\Big]\mathrm{d}\alpha \\ &= \frac{1}{2}\int_0^1 \Big[p_1+(q_1-p_1)\alpha+s_1-(s_1-r_1)\alpha \\ &\qquad -p_2-(q_2-p_2)\alpha-s_2+(s_2-r_2)\alpha\Big]\mathrm{d}\alpha \\ &= \frac{1}{4}\big(p_1-p_2+q_1-q_2+r_1-r_2+s_1-s_2\big). \end{aligned} \tag{10}$$

3. *Consider two Gaussian fuzzy numbers given by their couples of mean μ and standard deviation σ such that $\tilde{X}$ given by (μ_1,σ_1), and $\tilde{Y}$ given by (μ_2,σ_2). Their left and right alpha-cuts are given by:*

$$\begin{aligned} \tilde{X}^{L,\alpha}(\alpha) &= \mu_1-\sqrt{-2\sigma_1^2\ln\alpha}, \qquad \textit{and} \quad \tilde{X}^{R,\alpha}(\alpha)=\mu_1+\sqrt{-2\sigma_1^2\ln\alpha}; \\ \tilde{Y}^{L,\alpha}(\alpha) &= \mu_2-\sqrt{-2\sigma_2^2\ln\alpha}, \qquad \textit{and} \quad \tilde{Y}^{R,\alpha}(\alpha)=\mu_2+\sqrt{-2\sigma_2^2\ln\alpha}. \end{aligned}$$

The signed distance between $\tilde{X}$ and $\tilde{Y}$ is

$$\begin{aligned} d_{SGD}(\tilde{X},\tilde{Y}) &= \frac{1}{2}\int_0^1 \Big[\tilde{X}^{L,\alpha}(\alpha)+\tilde{X}^{R,\alpha}(\alpha)-\tilde{Y}^{L,\alpha}(\alpha)-\tilde{Y}^{R,\alpha}(\alpha)\Big]\mathrm{d}\alpha \\ &= \frac{1}{2}\int_0^1 \Big[\mu_1-\sqrt{-2\sigma_1^2\ln\alpha}+\mu_1+\sqrt{-2\sigma_1^2\ln\alpha} \\ &\qquad -\mu_2+\sqrt{-2\sigma_2^2\ln\alpha}-\mu_2-\sqrt{-2\sigma_2^2\ln\alpha}\Big]\mathrm{d}\alpha \\ &= \mu_1-\mu_2. \end{aligned} \tag{11}$$

4. *Consider two two-sided Gaussian fuzzy numbers given by their couples of means μ and standard deviations σ such that $\tilde{X}$ given by $(\mu_{X_1},\sigma_{X_1},\mu_{X_2},\sigma_{X_2})$, and $\tilde{Y}$ given by $(\mu_{Y_1},\sigma_{Y_1},\mu_{Y_2},\sigma_{Y_2})$. Their left and right alpha-cuts are given by:*

$$\begin{aligned} \tilde{X}^{L,\alpha}(\alpha) &= \mu_{X_1}-\sqrt{-2\sigma_{X_1}^2\ln\alpha}, \qquad \textit{and} \quad \tilde{X}^{R,\alpha}(\alpha)=\mu_{X_2}+\sqrt{-2\sigma_{X_2}^2\ln\alpha}; \\ \tilde{Y}^{L,\alpha}(\alpha) &= \mu_{Y_1}-\sqrt{-2\sigma_{Y_1}^2\ln\alpha}, \qquad \textit{and} \quad \tilde{Y}^{R,\alpha}(\alpha)=\mu_{Y_2}+\sqrt{-2\sigma_{Y_2}^2\ln\alpha}. \end{aligned}$$

The signed distance between $\tilde{X}$ and $\tilde{Y}$ is

$$\begin{aligned} d_{SGD}(\tilde{X},\tilde{Y}) &= \frac{1}{2}\int_0^1 \Big[\tilde{X}^{L,\alpha}(\alpha)+\tilde{X}^{R,\alpha}(\alpha)-\tilde{Y}^{L,\alpha}(\alpha)-\tilde{Y}^{R,\alpha}(\alpha)\Big]\mathrm{d}\alpha \\ &= \frac{1}{2}\int_0^1 \Big[\mu_{X_1}-\sqrt{-2\sigma_{X_1}^2\ln\alpha}+\mu_{X_2}+\sqrt{-2\sigma_{X_2}^2\ln\alpha} \\ &\qquad -\mu_{Y_1}+\sqrt{-2\sigma_{Y_1}^2\ln\alpha}-\mu_{Y_2}-\sqrt{-2\sigma_{Y_2}^2\ln\alpha}\Big]\mathrm{d}\alpha \\ &= \frac{1}{2}\big(\mu_{X_1}-\sigma_{X_1}\sqrt{\frac{\pi}{2}}+\mu_{X_2}+\sigma_{X_2}\sqrt{\frac{\pi}{2}}-\mu_{Y_1}+\sigma_{Y_1}\sqrt{\frac{\pi}{2}}-\mu_{Y_2}-\sigma_{Y_2}\sqrt{\frac{\pi}{2}}\big) \\ &= \frac{1}{2}\Big(\mu_{X_1}+\mu_{X_2}-\mu_{Y_1}-\mu_{Y_2}+\sqrt{\frac{\pi}{2}}\big(-\sigma_{X_1}+\sigma_{X_2}+\sigma_{Y_1}-\sigma_{Y_2}\big)\Big). \end{aligned} \tag{12}$$

3. Generalized signed distance

While the signed distance is an appealing measure as it offers valuable directional information, it comes with serious drawbacks. First, this measure essentially captures a central location, similar to measures like the simple mean. It primarily relies on the extreme values, i.e. the minima and the maxima, of fuzzy numbers to calculate the distance. Consequently, it overlooks the shape and the potential irregularities of fuzzy numbers. Since fuzzy numbers represent imprecise data, the information contained within the fuzzy set, beyond the extremes, can be essential for accurate analysis. Secondly, the signed distance lacks in total separability and symmetry, preventing it from being considered as a complete metric and, consequently, from defining a metric space. For these two reasons, using the signed distance in statistical analysis can be considered inappropriate even though it is simple and quick to compute. Addressing these issues requires additional steps. To fill this gap, introducing a new metric based on the signed distance is essential, leveraging its benefits while addressing its shortcomings.

Based on the original signed distance and on Berkachy [2021], we introduce an L^2 metric denoted by d_{BSGD} or $d_{SGD}^{\theta^\star}$ and called the balanced signed distance. This new metric incorporates a weight parameter $\theta^\star$. It has the great advantage of considering the central location tendency in addition to variations and irregularities in the shapes of the fuzzy numbers. And this metric fulfils the criteria for defining a metric for fuzzy quantities.

To construct $d_{SGD}^{\theta^\star}$, we must define the functions corresponding to the deviations of the shape first. They are written as:

Definition 3 (Left and right deviations of the shape of a fuzzy number)**.**
Let $\tilde{X}$ be a a fuzzy number of $\mathbb{F}_c^\star(\mathbb{R})$ with its α-level set $\tilde{X}^\alpha = [\tilde{X}^{L,\alpha}, \tilde{X}^{R,\alpha}]$. The left and right deviations of the shape of $\tilde{X}$ denoted by $dev^L\tilde{X}$ and $dev^R\tilde{X}$ are such that:

$$dev^L\tilde{X}(\alpha) = d_{SGD}(\tilde{X}, \tilde{0}) - \tilde{X}^{L,\alpha}, \tag{13}$$
$$dev^R\tilde{X}(\alpha) = \tilde{X}^{R,\alpha} - d_{SGD}(\tilde{X}, \tilde{0}). \tag{14}$$

The new L^2 class of distances is defined as follows:

Definition 4 (The balanced $d_{SGD}^{\theta^\star}$ distance (BSGD))**.**
Let $\tilde{X}$ and $\tilde{Y}$ be two fuzzy numbers of the class of non-empty compact and bounded fuzzy numbers. The L^2 metric $d_{SGD}^{\theta^\star}$ is the mapping

$$\begin{array}{rcl} d_{SGD}^{\theta^\star} : \mathbb{F}_c^\star(\mathbb{R}) \times \mathbb{F}_c^\star(\mathbb{R}) & \rightarrow & \mathbb{R}^+ \\ \tilde{X} \times \tilde{Y} & \mapsto & d_{SGD}^{\theta^\star}(\tilde{X}, \tilde{Y}), \end{array}$$

such that

$$d_{SGD}^{\theta^\star}(\tilde{X}, \tilde{Y}) = \Big(\big(d_{SGD}(\tilde{X}, \tilde{Y})\big)^2 + \theta^\star \Big(\int_0^1 \max\big(dev^R\tilde{Y}(\alpha) - dev^L\tilde{X}(\alpha), dev^R\tilde{X}(\alpha) - dev^L\tilde{Y}(\alpha)\big) \mathrm{d}\alpha \Big)^2 \Big)^{\frac{1}{2}}, \tag{15}$$

where:

- *$d_{SGD}(\tilde{X}, \tilde{Y})$ is the signed distance between $\tilde{X}$ and $\tilde{Y}$;*
- *$\theta^\star$ is the weight on the shape of the fuzzy numbers such that $0 \leq \theta^\star \leq 1$;*
- *$dev^L\tilde{X}$, $dev^R\tilde{X}$, $dev^L\tilde{Y}$, $dev^R\tilde{Y}$ are the left and right deviations of the shape of $\tilde{X}$ and $\tilde{Y}$ respectively.*

The construction of this metric is done in a way covering the complete surface between $\tilde{X}$ and $\tilde{Y}$. Using the maximum between the differences in the deviations of the two fuzzy numbers is intentional since we do not know in advance the potential ordering between these fuzzy numbers. Furthermore, we introduced a parameter $\theta^\star$, which represents the weight assigned to the shape of the fuzzy numbers compared to their signed distance. In other terms, if $\theta^\star$ is equal to 1, it means that the shape and the central location measure are weighted equally. Conversely, if $\theta^\star$ is equal to 0, the measure reduces to the absolute value of the signed distance.

It is important to mention that the $d^{\theta^\star}_{SGD}$ metric depends on the $\int_0^1 \max\big(\mathrm{dev}^R\tilde{Y}(\alpha) - \mathrm{dev}^L\tilde{X}(\alpha), \mathrm{dev}^R\tilde{X}(\alpha) - \mathrm{dev}^L\tilde{Y}(\alpha)\big)\mathrm{d}\alpha$. Based on Definitions 1, 2 and 3, we can write the following proposition:

Proposition 3.
Let $\tilde{X}$ and $\tilde{Y}$ be two fuzzy numbers of the class of non-empty compact and bounded fuzzy numbers. We get:

$$\int_0^1 \big(dev^R\tilde{Y}(\alpha) - dev^L\tilde{X}(\alpha)\big)\mathrm{d}\alpha = \frac{1}{2}\int_0^1 \big(\tilde{X}^{L,\alpha} - \tilde{X}^{R,\alpha} - \tilde{Y}^{L,\alpha} + \tilde{Y}^{R,\alpha}\big)\mathrm{d}\alpha; \tag{16}$$

$$\int_0^1 \big(dev^R\tilde{X}(\alpha) - dev^L\tilde{Y}(\alpha)\big)\mathrm{d}\alpha = -\frac{1}{2}\int_0^1 \big(\tilde{X}^{L,\alpha} - \tilde{X}^{R,\alpha} - \tilde{Y}^{L,\alpha} + \tilde{Y}^{R,\alpha}\big)\mathrm{d}\alpha. \tag{17}$$

Thus,

$$\int_0^1 \big(dev^R\tilde{Y}(\alpha) - dev^L\tilde{X}(\alpha)\big)\mathrm{d}\alpha = -\int_0^1 \big(dev^R\tilde{X}(\alpha) - dev^L\tilde{Y}(\alpha)\big)\mathrm{d}\alpha. \tag{18}$$

The proof of Proposition 3 is given in Appendix A.1.

Corollary 1.

$$\begin{aligned}
&\Big(\int_0^1 max\big(dev^R\tilde{Y}(\alpha) - dev^L\tilde{X}(\alpha), dev^R\tilde{X}(\alpha) - dev^L\tilde{Y}(\alpha)\big)\Big)^2 \\
&= \Big(\int_0^1 \big(dev^R\tilde{Y}(\alpha) - dev^L\tilde{X}(\alpha)\big)\mathrm{d}\alpha\Big)^2 \\
&= \Big(\int_0^1 \big(dev^R\tilde{X}(\alpha) - dev^L\tilde{Y}(\alpha)\big)\mathrm{d}\alpha\Big)^2 \\
&= \frac{1}{4}\Big(\int_0^1 \big(\tilde{X}^{L,\alpha} - \tilde{X}^{R,\alpha} - \tilde{Y}^{L,\alpha} + \tilde{Y}^{R,\alpha}\big)\mathrm{d}\alpha\Big)^2.
\end{aligned} \tag{19}$$

Proof. The proof of this corollary is trivial and will not be produced. □

3.1. Properties of the BSGD

The distance $d^{\theta^\star}_{SGD}$ is a metric of fuzzy quantities. This metric presents the following properties:

Proposition 4.
$(\mathbb{F}^\star_c(\mathbb{R}), d^{\theta^\star}_{SGD})$ is a metric space, with $d^{\theta^\star}_{SGD}$ of type L^2, such that:

1. *$d^{\theta^\star}_{SGD}$ is reflexive: $d^{\theta^\star}_{SGD}(\tilde{X}, \tilde{X}) = 0$.*
2. *$d^{\theta^\star}_{SGD}$ is non-negative (separability): $d^{\theta^\star}_{SGD}(\tilde{X}, \tilde{Y}) \geq 0$, $\forall \tilde{X}, \tilde{Y} \in \mathbb{F}^\star_c(\mathbb{R})$.*
3. *$d^{\theta^\star}_{SGD}$ is non-degenerate: $d^{\theta^\star}_{SGD}(\tilde{X}, \tilde{Y}) = 0 \Leftrightarrow \tilde{X} = \tilde{Y}$.*
4. *$d^{\theta^\star}_{SGD}$ is symmetric: $d^{\theta^\star}_{SGD}(\tilde{X}, \tilde{Y}) = d^{\theta^\star}_{SGD}(\tilde{Y}, \tilde{X})$.*
5. *The triangular inequality holds: $d^{\theta^\star}_{SGD}(\tilde{X}, \tilde{Y}) \leq d^{\theta^\star}_{SGD}(\tilde{X}, \tilde{Z}) + d^{\theta^\star}_{SGD}(\tilde{Z}, \tilde{Y})$.*

Proposition 5.
$d^{\theta^\star}_{SGD}$ is translation invariant and scale invariant, which means that

$$d^{\theta^\star}_{SGD}\big((\tilde{X}+k), (\tilde{Y}+k)\big) = d^{\theta^\star}_{SGD}(\tilde{X}, \tilde{Y}),$$

$$\textit{and } d^{\theta^\star}_{SGD}\big((k\cdot\tilde{X}), (k\cdot\tilde{Y})\big) = |k|\cdot d^{\theta^\star}_{SGD}(\tilde{X}, \tilde{Y}), \forall k \in \mathbb{R}.$$

Proposition 6.

Let $\theta_1^\star < \theta_2^\star$. Then, $d_{SGD}^{\theta_1^\star} < d_{SGD}^{\theta_2^\star}$.

Furthermore, although the signed distance presents clear deficiencies, its directionality property is greatly appreciated in ranking studies. For this reason, we develop a more suitable version of this measure for the ranking procedures by defining a general form of it and maintaining its directional characteristic while enhancing the way it's calculated to consider the irregularities in fuzzy numbers. This new ranking distance is the Generalised signed distance referred to as d_{GSGD}. This measure should take into account the directional property of the signed distance. We will incorporate the sign of the travel between the two considered fuzzy numbers in the following way. We first define a sign function indicating the direction of travel between $\tilde{X}$ and $\tilde{Y}$ as follows:

Definition 5 (The sign function)**.**
Let $\tilde{X}$ and $\tilde{Y}$ be two fuzzy numbers in $\mathbb{F}_c^\star(\mathbb{R})$. The sign function denoted by δ_{SGD} is given by:

$$\delta_{SGD}(\tilde{X},\tilde{Y}) = \begin{cases} 1 & \text{if} \quad d_{SGD}(\tilde{X},\tilde{Y}) \geq 0, \\ -1 & \text{if} \quad d_{SGD}(\tilde{X},\tilde{Y}) < 0. \end{cases} \tag{20}$$

The metric d_{BSGD} will now be re-written using the sign function to get the d_{GSGD} distance:

Definition 6 (The generalized signed distance)**.**
Consider two fuzzy numbers $\tilde{X}$ and $\tilde{Y}$ in $\mathbb{F}_c^\star(\mathbb{R})$. The generalized signed distance is the mapping

$$\begin{aligned} d_{GSGD} : \mathbb{F}_c^\star(\mathbb{R}) \times \mathbb{F}_c^\star(\mathbb{R}) &\to \mathbb{R} \\ \tilde{X} \times \tilde{Y} &\mapsto d_{GSGD}(\tilde{X},\tilde{Y}), \end{aligned}$$

such that

$$d_{GSGD}(\tilde{X},\tilde{Y}) = \delta_{SGD}(\tilde{X},\tilde{Y}) \cdot d_{SGD}^{\theta^\star}(\tilde{X},\tilde{Y}). \tag{21}$$

It is simple to see that the d_{GSGD} is directly linked to the d_{BSGD} since the d_{GSGD} could practically take either the value $-d_{BSGD}$, or the value d_{BSGD} depending on the direction of travel.

Inheriting the properties of the metric d_{BSGD}, the metric d_{GSGD} is reflexive, non-degenerate, and satisfies the triangular inequality. The proofs are straightforward. The design of the distance is aimed at maintaining the connection between the signed distance and its generalised version. Therefore, the original distance's properties, including direction, remain preserved while considering the shape of the fuzzy numbers involved.

Proposition 7. *The generalized signed distance d_{GSGD} is partially separable and partially symmetric.*

The proofs of Propositions 4, 5 and 7 can be found in Berkachy [2021]. Proposition 6 is evident.

3.2. *Balanced signed distance of particular fuzzy numbers*

Although the metric d_{BSGD} is conceptually appealing, computing it remains heavy. This is mainly due to the arithmetic operations performed from one side and to the choice of complex fuzzy numbers from another. It would be then interesting to know the d_{BSGD} distance for well-known fuzzy numbers, i.e. for triangular, trapezoidal, Gaussian and two-sided Gaussian fuzzy numbers as done in Subsection 2.1 for the original signed distance.

Proposition 8.

Let $\tilde{X}$ and $\tilde{Y}$ be two fuzzy numbers of the class of non-empty compact and bounded fuzzy numbers.

1. ***For triangular fuzzy numbers*** *given by their triplets such that $\tilde{X} = (p_1, q_1, r_1)$ with $p_1 < q_1 < r_1$, and $\tilde{Y} = (p_2, q_2, r_2)$ with $p_2 < q_2 < r_2$, the L^2 metric $d^{\theta^\star}_{SGD}$ is*

$$d^{\theta^\star}_{SGD}(\tilde{X}, \tilde{Y}) = \left(\frac{1}{16} (p_1 - p_2 + 2q_1 - 2q_2 + r_1 - r_2)^2 + \theta^\star \frac{1}{16} (p_1 - p_2 - r_1 + r_2)^2 \right)^{\frac{1}{2}}; \tag{22}$$

2. ***For trapezoidal fuzzy numbers*** *given by their quadruplets such that $\tilde{X} = (p_1, q_1, r_1, s_1)$ with $p_1 < q_1 < r_1 < s_1$, and $\tilde{Y} = (p_2, q_2, r_2, s_2)$ with $p_2 < q_2 < r_2 < s_2$, the L^2 metric $d^{\theta^\star}_{SGD}$ is*

$$\begin{aligned} d^{\theta^\star}_{SGD}(\tilde{X}, \tilde{Y}) = & \Big(\frac{1}{16} (p_1 - p_2 + q_1 - q_2 + r_1 - r_2 + s_1 - s_2)^2 \\ & + \theta^\star \frac{1}{16} (p_1 - p_2 + q_1 - q_2 - r_1 + r_2 - s_1 + s_2)^2 \Big)^{\frac{1}{2}}. \end{aligned} \tag{23}$$

   Additionally, for $\theta^\star = 1$, we have

$$d^{\theta^\star=1}_{SGD}(\tilde{X}, \tilde{Y}) = \frac{\sqrt{(p_1 - p_2 + q_1 - q_2)^2 + (r_1 - r_2 + s_1 - s_2)^2}}{2\sqrt{2}}; \tag{24}$$

3. ***For Gaussian fuzzy numbers*** *given by their couples of mean μ and standard deviation σ such that $\tilde{X}$ given by (μ_1, σ_1), and $\tilde{Y}$ given by (μ_2, σ_2), the L^2 metric $d^{\theta^\star}_{SGD}$ is*

$$d^{\theta^\star}_{SGD}(\tilde{X}, \tilde{Y}) = \left((\mu_1 - \mu_2)^2 + \theta^\star (\sigma_2 - \sigma_1)^2 \, \frac{\pi}{2} \right)^{\frac{1}{2}}; \tag{25}$$

4. ***For two-sided Gaussian fuzzy numbers*** *given by their couples of means μ and standard deviations σ such that $\tilde{X}$ given by $(\mu_{X_1}, \sigma_{X_1}, \mu_{X_2}, \sigma_{X_2})$, and $\tilde{Y}$ given by $(\mu_{Y_1}, \sigma_{Y_1}, \mu_{Y_2}, \sigma_{Y_2})$, the L^2 metric $d^{\theta^\star}_{SGD}$ is*

$$\begin{aligned} d^{\theta^\star}_{SGD}(\tilde{X}, \tilde{Y}) \;\; = \;\; & \Big(\frac{1}{4} (\mu_{X_1} + \mu_{X_2} - \mu_{Y_1} - \mu_{Y_2} + \sqrt{\frac{\pi}{2}} (-\sigma_{X_1} + \sigma_{X_2} + \sigma_{Y_1} - \sigma_{Y_2}))^2 \\ + \;\; & \theta^\star \frac{1}{4} (\mu_{X_1} - \mu_{X_2} - \mu_{Y_1} + \mu_{Y_2} + \sqrt{\frac{\pi}{2}} (-\sigma_{X_1} - \sigma_{X_2} + \sigma_{Y_1} + \sigma_{Y_2}))^2 \Big)^{\frac{1}{2}}. \end{aligned} \tag{26}$$

The proof of Proposition 8 is given in Appendix A.2.

4. Analysis on the influence of $\theta^\star$

4.1. Description and set up of the empirical analysis

Measuring distances between two fuzzy numbers can be a critical aspect that can significantly influence the outcomes of fuzzy-based applications. Many known distances do not assume irregularities in the shape of fuzzy numbers, which might not accurately reflect the inherent irregularities of real-world data. To address this issue, the Balanced and Generalised signed distances incorporate the coefficient $\theta^\star \in [0, 1]$ explicitly considering the irregularity in the shape of the considered fuzzy numbers. This coefficient then serves as a tuning parameter that adjusts the sensitivity of the distances to these irregularities. By changing $\theta^\star$, the defended distances can either emphasise or de-emphasise these irregularities. For instance, if $\theta^\star = 0$, both distances do not consider the irregular shapes, returning exactly the classical signed distance measure or its negative depending on the position of the fuzzy numbers. In the same way, if $\theta^\star = 1$, both distances fully consider the irregular shapes. This coefficient gives our distances a more flexible and accurate advantage for effectively representing the dissimilarity between fuzzy numbers. To gain insights into the behaviour of both distances in addition to the classical signed distance, we conducted a systematic analysis based on calculating the distances between pairs of fuzzy numbers $\tilde{X}$ and $\tilde{Y}$ for different values of $\theta^{\star}$. The objective of this analysis is to explore how modifying $\theta^{\star}$ influences the computed distances and to interpret the implications of these variations. Since the shapes and the positions of fuzzy numbers could also have an important impact, we have intentionally considered various shapes, classical, such as the triangular and trapezoidal, and less classical ones, such as the Gaussian and two-sided Gaussian, and different positions, often seen as more complex and demanding.

Pairs of fuzzy numbers are chosen with varying degrees of shape irregularity. For the selection of fuzzy numbers, the following characteristics are considered:

1. Type of membership functions:
   - Classical membership functions such as the triangular or the trapezoidal ones are considered.
   - More complex shapes such as the Gaussian or the two-sided Gaussian are also considered.
2. Symmetry of membership functions: For each type of membership functions, symmetrical and non-symmetrical shapes are considered.
3. Skewness of membership functions: For each type of membership functions, flat as well as peaked membership functions are considered.
4. Position of membership functions: In each case, overlapping and non-overlapping fuzzy numbers are considered.
5. For sake of simplicity, the calculated distances are always between $\tilde{X}$ and $\tilde{Y}$, and not $\tilde{Y}$ and $\tilde{X}$.

For each pair of fuzzy numbers, we computed the classical signed distance and the Generalised signed distance for a range of $\theta^{\star}$ values. The values of $\theta^{\star}$ were chosen to cover a broad spectrum of cases, from low values that minimise the impact of irregularities, i.e. $\theta^{\star} = 0$, $\theta^{\star} = \frac{1}{3}$, $\theta^{\star} = \frac{1}{2}$, to high values that emphasise these irregularities, i.e. $\theta^{\star} = 1$. The results of these analyses are in Table 1 for triangular fuzzy numbers, in Table 2 for trapezoidal fuzzy numbers, in Table 3 for Gaussian fuzzy numbers, and in Table 4 for two-sided Gaussian fuzzy numbers. The computed distances are analysed to observe the relationship between $\theta^{\star}$ and the resulting values. The analysis of the four tables reveals significant insights into how the shape in terms of asymmetry and range and overlap between fuzzy numbers influence the computed distances with different $\theta^{\star}$ levels.

4.2. Results: Sensitivity analysis

4.2.1. Effect of $\theta^{\star}$

Overall, the GSGD distance demonstrates noticeable variation with different values of $\theta^{\star}$, especially when the fuzzy numbers exhibit irregularity, asymmetry or differ in range of their support or core sets. Changes in the weight $\theta^{\star}$ lead to subtle but important adjustments in the calculated distances, capturing shape-related nuances. The only cases where varying $\theta^{\star}$ does not impact the computed distances are when the fuzzy numbers have symmetrical shapes with identical ranges (for the support set in the case of triangular and trapezoidal fuzzy numbers), or exhibit the same dispersion (for the Gaussian and the two-sided Gaussian fuzzy numbers). Additionally, in the case of two-sided Gaussian fuzzy numbers, it is important that the range of the core set remains consistent between the pair of fuzzy numbers to ensure $\theta^{\star}$ to have no influence on the GSDG distance.

In summary, the effect of varying $\theta^{\star}$ can be interpreted as:

- Smaller $\theta^\star$ values give lesser weight to the irregularities or shape differences between the fuzzy numbers. Consequently, the distance closely resembles more traditional fuzzy distances, notably the classical signed distance, where the shape of fuzzy numbers is of lesser relevance.
- Larger $\theta^\star$ values give greater weight to irregularities, making shape differences more pronounced. In this case, the distance increases when fuzzy numbers present more complex or irregular shapes, highlighting the influence of shape variations.
- In the case of symmetrical shapes, i.e. triangular, trapezoidal or Gaussian fuzzy numbers exhibiting identical ranges or identical dispersions, the weight $\theta^\star$ does not affect the computed distance. This indicates that these shapes are treated as regular or non-irregular, resulting in identical outcomes for both the GSGD and the classical signed distance SGD.

As a consequence of Proposition 6, the calculated distances are then ordered as follows:

$$|d_{GSGD}^{\theta^\star=0}| = |d_{SGD}| \leq |d_{GSGD}^{\theta^\star=\frac{1}{3}}| \leq |d_{GSGD}^{\theta^\star=\frac{1}{2}}| \leq |d_{GSGD}^{\theta^\star=1}|.$$

4.2.2. Effect of Overlap

The first clear observation is that the presence or absence of overlap is an important factor that affects distances. This impact depends mainly on the chosen type of fuzzy numbers.

For triangular and trapezoidal fuzzy numbers considered classical and often used in fuzzy-based studies, when there is overlap between two of them, meaning when the membership functions intersect, the GSGD distance tends to be less sensitive to changes when varying the coefficient $\theta^\star$. This is because these fuzzy numbers have simpler, symmetrical structures where the overlap significantly reduces the influence of shape irregularities. A combination of symmetrical and overlapping fuzzy numbers, for instance, exhibits consistent distances across different $\theta^\star$ values since the shape of both numbers remains overall unchanged regardless of different variations in weighting irregularities.

Contrariwise, for Gaussian and two-sided Gaussian fuzzy numbers, overlapping do not fully neutralise the effect of $\theta^\star$. Gaussian shapes are generally seen as more complex than triangles and trapezoids due to their construction. Furthermore, Gaussian shapes are considered to be more sensitive to changes in their spread, in other words, in their dispersion. Thus, when fuzzy numbers overlap, the subtle differences in shape, such as different standard deviations between fuzzy numbers and/or asymmetry in the two-sided numbers, become more pronounced. In this case, increasing $\theta^\star$ means, theoretically but also empirically, as seen in this analysis, that these irregularities, even if small, would potentially lead to larger variations in the calculated distance. This behaviour can be understood as a reflection of the greater complexity and sensitivity of Gaussian numbers compared to the more structured, simpler triangular and trapezoidal shapes. Consequently, overlapped Gaussian-shaped fuzzy numbers are great cases for showcasing the GSGD distance since the influence of irregularities is more pronounced when $\theta^\star$ is larger.

A key finding here is that the GSGD distance proves to be particularly effective when dealing with Gaussian or two-sided Gaussian fuzzy numbers or simply exhibiting complexities and irregularities. These shapes intrinsically capture more uncertainty nuances, allowing the GSGD distance to consider meaningful differences when $\theta^\star$ varies. This finding confirms the foundation of the GSGD and its design principle: it has been intentionally formulated to better account for irregular and complex fuzzy numbers, for which conventional metrics may or would neglect essential features. This emphasises the strength and adaptability of the GSGD in scenarios where fuzzy numbers do not present simple or symmetrical shapes.

Table 1. Triangular fuzzy numbers.

$\tilde{X}$	$\tilde{Y}$	**Description**	**SGD**	**GSGD with** $\theta^\star = 1$	**GSGD with** $\theta^\star = \frac{1}{2}$	**GSGD with** $\theta^\star = \frac{1}{3}$	**GSGD with** $\theta^\star = 0$
(1, 2, 3)	(2, 3, 4)	sym., over., same range	-1	-1	-1	-1	-1
(1, 2, 3)	(4, 5, 6)	sym., no over., same range	-3	-3	-3	-3	-3
(1, 2, 3)	(2.5,3, 3.5)	sym., over., diff. range	-1	-1.0308	-1.0155	-1.0104	-1
(0.5,2, 3)	(1.75,4,4.5)	non-sym., over., diff. range	-1.6875	-1.6887	-1.6881	-1.6879	-1.6875
(0.5,2, 3)	(4, 5,5.75)	non-sym., non-over., diff. range	-3.0625	-3.0682	-3.0654	-3.0644	-3.0625
(1.8,2,2.1)	(1.9,2.2,2.3)	non-sym., over., peaked	-0.1749	-0.1768	-0.1759	-0.1756	-0.1749
(1.8,2,2.1)	(2.9,3,3.3)	non-sym., non-over., peaked	-1.0750	-1.0753	-1.0751	-1.0751	-1.0750
(0.1,2,4.3)	(0.2,3,5.2)	non-sym., over., flat	-0.7500	-0.7762	-0.7632	-0.7588	-0.7500
(0.1,2,4.3)	(4.4,6,8)	non-sym., non-over., flat	-4	-4.0028	-4.0014	-4.001	-4

Table 2. Trapezoidal fuzzy numbers.

$\tilde{X}$	$\tilde{Y}$	**Description**	**SGD**	**GSGD with** $\theta^\star = 1$	**GSGD with** $\theta^\star = \frac{1}{2}$	**GSGD with** $\theta^\star = \frac{1}{3}$	**GSGD with** $\theta^\star = 0$
(1, 2, 3,4)	(2, 3, 4,5)	sym., over., same range	-1	-1	-1	-1	-1
(1, 2, 3,4)	(6,7,8,9)	sym., no over., same range	-5	-5	-5	-5	-5
(1, 2, 3,4)	(3.5,4,4.5,5)	sym., over., diff. range	-1.75	-1.8200	-1.785	-1.7736	-1.75
(0.5,2, 3,3.25)	(1.75,4,4.5,5.5)	non-sym., over., diff. range	-1.75	-1.7544	-1.7522	-1.7515	-1.75
(0.5,2, 3,3.25)	(4, 5,5.75,6)	non-sym., non-over., diff. range	-3	-3.0104	-3.0052	-3.0035	-3
(1.8,2,2.1,2.15)	(1.9,2.2,2.3,2.4)	non-sym., over., peaked	-0.1875	-0.1912	-0.1894	-0.1887	-0.1875
(1.8,2,2.1,2.15)	(2.9,3,3.3,3.4)	non-sym., non-over., peaked	-1.1375	-1.1409	-1.1392	-1.1386	-1.1375
(0.1,2,4.3,7)	(0.2,3,5.2,8)	non-sym., over., flat	-0.7500	-0.7762	-0.7632	-0.7588	-0.7500
(0.1,2,4.3,7)	(7.4,8,10,12)	non-sym., non-over., flat	-6	-6.0351	-6.0176	-6.0117	-6

Table 3. Gaussian fuzzy numbers.

$\tilde{X} = (\mu_X, \sigma_X)$	$\tilde{Y} = (\mu_Y, \sigma_Y)$	**Description**	**SGD**	**GSGD with** $\theta^\star = 1$	**GSGD with** $\theta^\star = \frac{1}{2}$	**GSGD with** $\theta^\star = \frac{1}{3}$	**GSGD with** $\theta^\star = 0$
(1,2)	(2,2)	over., same sigma	-1	-1	-1	-1	-1
(1,2)	(15,2)	no over., same sigma	-14	-14	-14	-14	-14
(1,2)	(15,1)	over., diff. sigma	-14	-14.0560	-14.0280	-14.0187	-14
(2,0.5)	(3,0.25)	over., peaked	-1	-1.0479	-1.0242	-1.0162	-1
(2,0.5)	(15,0.25)	non-over., peaked	-13	-13.0038	-13.0019	-13.0013	-13
(2,4)	(3,5)	over., flat	-1	-1.60337	-1.3362	-1.2343	-1
(2,4)	(25,5)	non-over., flat	-23	-23.0341	-23.0171	-23.0114	-23

Table 4. Two-sided Gaussian fuzzy numbers.

$\tilde{X} = (\mu_{X_1}, \sigma_{X_1}, \mu_{X_2}, \sigma_{X_2})$	$\tilde{Y} = (\mu_{Y_1}, \sigma_{Y_1}, \mu_{Y_2}, \sigma_{Y_2})$	**Description**	**SGD**	**GSGD with** $\theta^\star = 1$	**GSGD with** $\theta^\star = \frac{1}{2}$	**GSGD with** $\theta^\star = \frac{1}{3}$	**GSGD with** $\theta^\star = 0$
(1,2,3,2)	(2,2,4,2)	sym., over., same sigma	-1	-1	-1	-1	-1
(1,2,3,2)	(2,2,10,2)	sym., over., same sigma	-4	-5	-4.5277	-4.3589	-4
(1,2,3,2)	(14,2,16,2)	sym., non-over., same sigma	-13	-13	-13	-13	-13
(1,2,3,2)	(2,1,4,1)	sym., over., same sigma	-1	-1.6034	-1.3362	-1.2343	-1
(2,1,3,2)	(4,1.75,5,1.5)	non-sym., over., diff. sigma	-1.2156	-1.2267	-1.2217	-1.2200	-1.2156
(2,1,3,2)	(14,1.5,15,1)	non-sym., non-over., diff. sigma	-11.06	-11.0645	-11.0622	-11.0615	-11.06
(2,0.25,2.3,0.5)	(3,0.1,3.2,0.2)	non-sym., over., peaked	-0.8560	-0.9181	-0.8876	-0.8772	-0.8560
(2,0.25,2.3,0.5)	(15,0.2,15.1,0.15)	non-sym., non-over., peaked	-12.7117	-12.7168	-12.7144	-12.7136	-12.7117
(2,5,5,6)	(4,4,10,5)	non-sym., over., flat	-3.5	-3.5087	-3.5043	-3.5029	-3.5
(2,5,5,6)	(40,6,50,5)	non-sym., non-over., flat	-40.245	-40.3986	-40.3227	-40.2974	-40.245

5. Comparison of the GSGD with other metrics

This section proposes an empirical comparison between the GSGD with other established fuzzy distance measures such as BSGD proposed in this paper, the classical SGD, the wabl metric presented by Sinova et al. [2014], the Bertoluzza metric seen in Bertoluzza et al. [1995], the Mid Spread of Trutschnig et al. [2009] and the known ρ_2 metric. The distances are computed using the same protocol characteristics as Section 4. Results for different types of fuzzy numbers – triangular, trapezoidal, Gaussian and two-sided Gaussian – are analysed to understand the behaviour of these distances. A detailed interpretation for each type of fuzzy number is given below.

First of all, it is simple to interpret the difference between the GSGD and the BSGD. The main difference between these two distances is that GSGD can yield negative distances, whereas BSGD always yields positive distances. This is because GSGD is intentionally constructed to show the direction of travel between the considered fuzzy numbers. The BSGD, however, is the balanced metric for which the sign is removed to only reflect the magnitude of the distance between the fuzzy numbers. For the sake of comparability, the BSGD will be used for this analysis.

The BSGD differs considerably from other distances such as wabl, Bertoluzza, Mid-Spr, and ρ_2 because of its unique way of handling overlapping and shape. These differences stem from how each measure mathematically operates fuzzy numbers, each focuses on different features of fuzzy numbers, such as core values, spread, asymmetry, or shape. Although they are all L^2 metrics, meaning they all measure distance based on squared differences, their formulations emphasise different aspects of the considered fuzzy numbers. This naturally leads to variations in how they capture the distances.

A general observation is that the BSGD is overall between Bertoluzza, Mid-Spr and ρ_2. This results from how these distances treat overlapping, shape and asymmetry. The order is as follows:

$$|d_{SGD}| \leq d_{\text{wabl/ldev/rdev}} \leq d_{\text{Bertoluzza}} = d_{\text{Mid-Spr}} \leq d_{BSGD} = |d_{GSGD}| \leq d_{\rho_2}.$$

5.1. Effect of overlap

Overlapping significantly reduces the distances in all metrics in general. Non-overlapping fuzzy numbers lead to much larger distances, particularly for the ρ_2, BSGD, followed by the Bertoluzza and the wabl metrics.

Regarding the comparison between metrics, the BSGD appears to be more sensitive to the overlapping of fuzzy numbers than the other metrics. For instance, the other metrics are generally designed to take account of core values, spreads, or supports such as the wabl, Bertoluzza, or Mid-Spr. In contrast, ρ_2 is conceived differently since it is an Euclidean-based measure. The BSGD (and, of course, the GSGD, the directional one) emphasises the irregularities of the fuzzy numbers rather than simply comparing their support or the spread. It considers how the slopes of the left and right alpha-cuts behave.

5.2. Effect of shape

BSGD is highly sensitive to the shape of fuzzy numbers by construction. It captures the asymmetry more directly than other measures, which means it can respond to both the non-linearity of the shape and the irregularity of the membership functions. When comparing two asymmetrical shapes, for example, two fuzzy numbers with one side steeper than the other, the BSGD can potentially yield a larger distance due to the variation in the position of points across their support sets. Thus, this metric effectively evaluates how the shape of fuzzy numbers changes at different alpha-cuts. The more pronounced the irregularities in the shapes, the more insightful this metric becomes. This is seen in the cases of Gaussian and two-sided Gaussian fuzzy numbers.

On the other hand, the wabl, Bertoluzza, Mid-Spr and ρ_2 focus on shape characteristics like the spread, the width, the support and the core sets. The metric wabl/ldev/rdev captures the average distance between the left and the right deviations of the fuzzy numbers and, therefore, tends to give a smaller distance than Bertoluzza and Mid-Spr, which focus more on shape and dispersion. Bertoluzza and Mid-Spr are sensitive to the overall width of the fuzzy numbers, making them slightly more sensitive to shape differences than wabl/ldev/rdev. The Bertoluzza and the Mid-Spr metrics provide similar results, focusing on the dispersion and overall shape of the fuzzy numbers.

ρ_2 is a measure based on Euclidean distance and is sensitive to variation in support and core values. It directly compares the entire fuzzy number, including its kernel and type, resulting in more considerable distances in the case of important differences in shape or position. ρ_2 measures the direct differences of shape in an L^2 norm, which often results in larger distances than BSGD and GSGD, particularly when the fuzzy numbers have irregularities or extend significantly over their supports.

Overall, it is clear to see that the BSGD and the GSGD have their place among these measures and can be effective in practice, particularly in scenarios where:

- The fuzzy numbers present irregularities in the shape or simply when non-classical fuzzy numbers are used. This makes them suitable for complex data sets where other metrics might smooth out these variations or fail to consider them.
- These metrics also handle asymmetry by considering both the left and right sides of their membership functions. Such fuzzy numbers are common in real-world applications.
- These metrics are effective for overlapping fuzzy numbers. They can capture the granular differences between overlapped fuzzy numbers.
- The GSGD metric has the advantage of being directional, leading to a negative or a positive value.

Table 5. Comparison of distances for Triangular fuzzy numbers.

$\tilde{X}$	$\tilde{Y}$	Description	GSGD with $\theta^\star = 1$	BSGD with $\theta^\star = 1$	SGD	wabl	Bertoluzza	Mid.Spr	ρ_2
(1, 2, 3)	(2, 3, 4)	sym., over., same range	-1	1	-1	1	1	1	1
(1, 2, 3)	(4, 5, 6)	sym., no over., same range	-3	3	-3	3	3	3	3
(1, 2, 3)	(2.5,3, 3.5)	sym., over., diff. range	-1.0308	1.0308	-1	1.0138	1.0138	1.0138	1.0408
(0.5,2, 3)	(1.75,4,4.5)	non-sym., over., diff. range	-1.6887	1.6887	-1.6875	1.6912	1.6976	1.6976	1.6987
(0.5,2, 3)	(4, 5,5.75)	non-sym., non-over., diff. range	-3.0682	3.0682	-3.0625	3.0651	3.0653	3.0653	3.0704
(1.8,2,2.1)	(1.9,2.2,2.3)	non-sym., over., peaked	-0.1768	0.1768	-0.175	0.1760	0.1764	0.1764	0.1780
(1.8,2,2.1)	(2.9,3,3.3)	non-sym., non-over., peaked	-1.0753	1.0753	-1.075	1.0754	1.0760	1.0760	1.0763
(0.1,2,4.3)	(0.2,3,5.2)	non-sym., over., flat	-0.7762	0.7762	-0.75	0.7663	0.7753	0.7753	0.7979
(0.1,2,4.3)	(4.4,6,8)	non-sym., non-over., flat	-4.0028	4.0028	-4	4.0012	4.0012	4.0012	4.0037

Table 6. Comparison of distances for Trapezoidal fuzzy numbers.

$\tilde{X}$	$\tilde{Y}$	Description	GSGD with $\theta^\star = 1$	BSGD with $\theta^\star = 1$	SGD	wabl	Bertoluzza	Mid.Spr	ρ_2
(1, 2, 3,4)	(2, 3, 4,5)	sym., over., same range	-1	1	-1	1	1	1	1
(1, 2, 3,4)	(6,7,8,9)	sym., no over., same range	-5	5	-5	5	5	5	5
(1, 2, 3,4)	(3.5,4,4.5,5)	sym., over., diff. range	-1.8200	1.8200	-1.75	1.7756	1.7756	1.7756	1.8257
(0.5,2, 3,3.25)	(1.75,4,4.5,5.5)	non-sym., over., diff. range	-1.7545	1.7545	-1.75	1.7559	1.7559	1.7559	1.7678
(0.5,2, 3,3.25)	(4, 5,5.75,6)	non-sym., non-over., diff. range	-3.0104	3.0104	-3	3.0040	3.0046	3.0046	3.0121
(1.8,2,2.1,2.15)	(1.9,2.2,2.3,2.4)	non-sym., over., peaked	-0.1912	0.1912	-0.1875	0.1892	0.1893	0.1893	0.1926
(1.8,2,2.1,2.15)	(2.9,3,3.3,3.4)	non-sym., non-over., peaked	-1.1409	1.1409	-1.1375	1.1387	1.1388	1.1388	1.1411
(0.1,2,4.3,7)	(0.2,3,5.2,8)	non-sym., over., flat	-0.7762	0.7762	-0.75	0.7663	0.7721	0.7721	0.7979
(0.1,2,4.3,7)	(7.4,8,10,12)	non-sym., non-over., flat	-6.0351	6.0351	-6	6.0142	6.0147	6.0147	6.0426

Table 7. Comparison of distances for Gaussian fuzzy numbers.

$\tilde{X} = (\mu_X, \sigma_X)$	$\tilde{Y} = (\mu_Y, \sigma_Y)$	Description	GSGD with $\theta^\star = 1$	BSGD with $\theta^\star = 1$	SGD	wabl	Bertoluzza	Mid.Spr	ρ_2
(1,2)	(2,2)	over., same sigma	-1	1	-1	1	1	1	1
(1,2)	(15,2)	no over., same sigma	-14	14	-14	14	14	14	14
(1,2)	(15,1)	over., diff. sigma	-14.0561	14.0561	-14	14.0240	14.0240	14.0240	14.0718
(2,0.5)	(3,0.25)	over., peaked	-1.0481	1.0481	-1	1.0208	1.0208	1.0208	1.0611
(2,0.5)	(15,0.25)	non-over., peaked	-13.0038	13.0038	-13	13.0016	13.0016	13.0016	13.0048
(2,4)	(3,5)	over., flat	-1.6047	1.6047	-1	1.2929	1.2929	1.2929	1.7363
(2,4)	(25,5)	non-over., flat	-23.0342	23.0342	-23	23.0146	23.0146	23.0146	23.0438

Table 8. Comparison of distances for Two-sided Gaussian fuzzy numbers.

$\tilde{X} = (\mu_{X_1}, \sigma_{X_1}, \mu_{X_2}, \sigma_{X_2})$	$\tilde{Y} = (\mu_{Y_1}, \sigma_{Y_1}, \mu_{Y_2}, \sigma_{Y_2})$	**Description**	**GSGD with** $\theta^\star = 1$	**BSGD with** $\theta^\star = 1$	**SGD**	**wabl**	**Bertoluzza**	**Mid.Spr**	ρ_2
(1,2,3,2)	(2,2,4,2)	sym., over., same sigma	-1	1	-1	1	1	1	1
(1,2,3,2)	(2,2,10,2)	sym., over., same sigma	-5	5	-4	4.3589	4.3589	4.3589	5
(1,2,3,2)	(14,2,16,2)	sym., non-over., same sigma	-13	13	-13	13	13	13	13
(1,2,3,2)	(2,1,4,1)	sym., over., same sigma	-1.6047	1.6047	-1	1.2929	1.2929	1.2929	1.7363
(2,1,3,2)	(4,1.75,5,1.5)	non-sym., over., diff. sigma	-1.2257	1.2257	-1.2156	1.2432	1.2884	1.2884	1.2965
(2,1,3,2)	(14,1.5,15,1)	non-sym., non-over., diff. sigma	-11.0632	11.0632	-11.0588	11.0644	11.0718	11.0718	11.0756
(2,0.25,2.3,0.5)	(3,0.1,3.2,0.2)	non-sym., over., peaked	-0.9181	0.9181	-0.8559	0.8818	0.8827	0.8827	0.9315
(2,0.25,2.3,0.5)	(15,0.2,15.1,0.15)	non-sym., non-over., peaked	-12.7166	12.7166	-12.7118	12.7137	12.7140	12.7140	12.7177
(2,5,5,6)	(4,4,10,5)	non-sym., over., flat	-3.5086	3.5086	-3.5	3.5237	3.5237	3.5237	3.5707
(2,5,5,6)	(40,6,50,5)	non-sym., non-over., flat	-40.3969	40.3969	-40.2450	40.2975	40.3012	40.3012	40.4024

6. Conclusion

In this paper, we have provided a comprehensive exploration of the classical signed distance (SGD). We have developed the Generalised signed distance (GSGD), as well as the Balanced signed distance (BSGD), to address the drawbacks of the signed distance. The new distances then consider the irregularities in the shape of fuzzy numbers while preserving the directionality for the GSGD. Additionally to the flexibility in treating asymmetrical fuzzy numbers, the introduction of BSGD solves the problem of symmetry and separability. It satisfies the needed conditions to define a metric of fuzzy quantities. Analytical simplifications of the definitions of both distances are also provided to facilitate the computation procedure.

We have also shown simulation analyses first to understand the effect of the weighting coefficient $\theta^\star$ and second to compare our distances to different metrics using a variety of scenarios of fuzzy numbers with different spreads and shapes (triangular, trapezoidal, Gaussian, and two-sided Gaussian). Such analysis provided critical insights into how these measures perform in practice. Through both theoretical development and practical simulations, we have demonstrated the strengths and applications of these metrics, particularly in their ability to capture the nuanced characteristics of fuzzy numbers, such as asymmetry, shape, and overlapping. The results revealed that the defended distances consistently offered a detailed and accurate representation of the differences between fuzzy numbers, especially in cases involving asymmetry or overlap. For example, our distances performed well in scenarios where the other metrics might have overlooked critical irregularities due to their focus on either core values or linearity. This places the BSGD and GSGD uniquely positioned as practical and effective tools for situations where shape and irregularity are essential.

Following this research, future work could be aimed at applying these new distance measures in real-world problems involving uncertain data, for instance, in decision-making or machine learning, in order to assess the validity of their practical usage and further refine their theoretical foundations.

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Abbreviations

The following abbreviations are used in this manuscript:

BSGD	Balanced Signed Distance
GSGD	Generalized Signed Distance
SGD	Signed Distance

Appendix A. Proofs of propositions

Appendix A.1. Proposition 3

Proof. Let us first develop $\text{dev}^R\tilde{Y}(\alpha) - \text{dev}^L\tilde{X}(\alpha)$:

$$\begin{aligned}
\text{dev}^R\tilde{Y}(\alpha) - \text{dev}^L\tilde{X}(\alpha) \quad &= \tilde{Y}^{R,\alpha} - d_{SGD}(\tilde{Y},\tilde{0}) - d_{SGD}(\tilde{X},\tilde{0}) + \tilde{X}^{L,\alpha} \\
&= \frac{1}{2}\tilde{Y}^{R,\alpha} + \frac{1}{2}\tilde{Y}^{R,\alpha} + \frac{1}{2}\tilde{Y}^{L,\alpha} - \frac{1}{2}\tilde{Y}^{L,\alpha} \\
&\quad \frac{1}{2}\tilde{X}^{L,\alpha} + \frac{1}{2}\tilde{X}^{L,\alpha} + \frac{1}{2}\tilde{X}^{R,\alpha} - \frac{1}{2}\tilde{X}^{R,\alpha} \\
&\quad -d_{SGD}(\tilde{Y},\tilde{0}) - d_{SGD}(\tilde{X},\tilde{0}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^1 \big(\text{dev}^R\tilde{Y}(\alpha) - \text{dev}^L\tilde{X}(\alpha)\big)\mathrm{d}\alpha \quad &= \frac{1}{2}\int_0^1 \big(\tilde{Y}^{L,\alpha} + \tilde{Y}^{R,\alpha}\big)\mathrm{d}\alpha + \frac{1}{2}\int_0^1 \big(\tilde{X}^{L,\alpha} + \tilde{X}^{R,\alpha}\big)\mathrm{d}\alpha \\
&\quad + \frac{1}{2}\int_0^1 \big(\tilde{X}^{L,\alpha} - \tilde{X}^{R,\alpha} - \tilde{Y}^{L,\alpha} + \tilde{Y}^{R,\alpha}\big)\mathrm{d}\alpha \\
&\quad - \int_0^1 d_{SGD}(\tilde{Y},\tilde{0})\mathrm{d}\alpha - \int_0^1 d_{SGD}(\tilde{X},\tilde{0})\mathrm{d}\alpha \\
&= d_{SGD}(\tilde{Y},\tilde{0}) + d_{SGD}(\tilde{X},\tilde{0}) \\
&\quad + \frac{1}{2}\int_0^1 \big(\tilde{X}^{L,\alpha} - \tilde{X}^{R,\alpha} - \tilde{Y}^{L,\alpha} + \tilde{Y}^{R,\alpha}\big)\mathrm{d}\alpha \\
&\quad - d_{SGD}(\tilde{Y},\tilde{0}) - d_{SGD}(\tilde{X},\tilde{0}) \\
&= \frac{1}{2}\int_0^1 \big(\tilde{X}^{L,\alpha} - \tilde{X}^{R,\alpha} - \tilde{Y}^{L,\alpha} + \tilde{Y}^{R,\alpha}\big)\mathrm{d}\alpha.
\end{aligned}$$

Let us now develop $\text{dev}^R\tilde{X}(\alpha) - \text{dev}^L\tilde{Y}(\alpha)$:

$$\begin{aligned}
\text{dev}^R\tilde{X}(\alpha) - \text{dev}^L\tilde{Y}(\alpha) \quad &= \tilde{X}^{R,\alpha} - d_{SGD}(\tilde{X},\tilde{0}) - d_{SGD}(\tilde{Y},\tilde{0}) + \tilde{Y}^{L,\alpha} \\
&= \frac{1}{2}\tilde{X}^{R,\alpha} + \frac{1}{2}\tilde{X}^{R,\alpha} + \frac{1}{2}\tilde{X}^{L,\alpha} - \frac{1}{2}\tilde{X}^{L,\alpha} \\
&\quad \frac{1}{2}\tilde{Y}^{L,\alpha} + \frac{1}{2}\tilde{Y}^{L,\alpha} + \frac{1}{2}\tilde{Y}^{R,\alpha} - \frac{1}{2}\tilde{Y}^{R,\alpha} \\
&\quad - d_{SGD}(\tilde{X},\tilde{0}) - d_{SGD}(\tilde{Y},\tilde{0}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^1 \big(\text{dev}^R\tilde{X}(\alpha) - \text{dev}^L\tilde{Y}(\alpha)\big)\mathrm{d}\alpha \quad &= \frac{1}{2}\int_0^1 \big(\tilde{X}^{L,\alpha} + \tilde{X}^{R,\alpha}\big)\mathrm{d}\alpha + \frac{1}{2}\int_0^1 \big(\tilde{Y}^{L,\alpha} + \tilde{Y}^{R,\alpha}\big)\mathrm{d}\alpha \\
&\quad + \frac{1}{2}\int_0^1 \big(-\tilde{X}^{L,\alpha} + \tilde{X}^{R,\alpha} + \tilde{Y}^{L,\alpha} - \tilde{Y}^{R,\alpha}\big)\mathrm{d}\alpha \\
&\quad - \int_0^1 d_{SGD}(\tilde{X},\tilde{0})\mathrm{d}\alpha - \int_0^1 d_{SGD}(\tilde{Y},\tilde{0})\mathrm{d}\alpha \\
&= d_{SGD}(\tilde{X},\tilde{0}) + d_{SGD}(\tilde{Y},\tilde{0}) \\
&\quad - \frac{1}{2}\int_0^1 \big(\tilde{X}^{L,\alpha} - \tilde{X}^{R,\alpha} - \tilde{Y}^{L,\alpha} + \tilde{Y}^{R,\alpha}\big)\mathrm{d}\alpha \\
&\quad - d_{SGD}(\tilde{X},\tilde{0}) - d_{SGD}(\tilde{Y},\tilde{0}) \\
&= -\frac{1}{2}\int_0^1 \big(\tilde{X}^{L,\alpha} - \tilde{X}^{R,\alpha} - \tilde{Y}^{L,\alpha} + \tilde{Y}^{R,\alpha}\big)\mathrm{d}\alpha.
\end{aligned}$$

□

Appendix A.2. Proposition 8

Proof.

1. The fuzzy numbers $\tilde{X} = (p_1, q_1, r_1)$ and $\tilde{Y} = (p_2, q_2, r_2)$ are triangular. Their left and right alpha-cuts are written as:

$$\begin{aligned}
\tilde{X}^L(\alpha) &= p_1 + (q_1 - p_1)\alpha, \qquad \text{and} \quad \tilde{X}^R(\alpha) = r_1 - (r_1 - q_1)\alpha; \\
\tilde{Y}^L(\alpha) &= p_2 + (q_2 - p_2)\alpha, \qquad \text{and} \quad \tilde{Y}^R(\alpha) = r_2 - (r_2 - q_2)\alpha.
\end{aligned}$$

Let us first calculate $\frac{1}{4}\big(\int_0^1 \big(\tilde{X}^{L,\alpha} - \tilde{X}^{R,\alpha} - \tilde{Y}^{L,\alpha} + \tilde{Y}^{R,\alpha}\big)\mathrm{d}\alpha\big)^2$:

$$\begin{aligned}\frac{1}{4}\Big(\int_0^1 \big(\tilde{X}^{L,\alpha} - \tilde{X}^{R,\alpha} - \tilde{Y}^{L,\alpha} + \tilde{Y}^{R,\alpha}\big)\mathrm{d}\alpha\Big)^2 &= \frac{1}{4}\Big(\int_0^1 \big[p_1 + (q_1 - p_1)\alpha - r_1 + (r_1 - q_1)\alpha \\ &\qquad -p_2 - (q_2 - p_2)\alpha + r_2 - (r_2 - q_2)\alpha\big]\mathrm{d}\alpha\Big)^2 \\ &= \frac{1}{4}\big(p_1 + \frac{1}{2}(q_1 - p_1) - r_1 + \frac{1}{2}(r_1 - q_1) \\ &\qquad -p_2 - \frac{1}{2}(q_2 - p_2) + r_2 - \frac{1}{2}(r_2 - q_2)\big)^2 \\ &= \frac{1}{16}\big(p_1 - p_2 - r_1 + r_2\big)^2.\end{aligned}$$

Then, using Equations 9 and 15, and Corollary 1, the $d_{SGD}^{\theta^\star}$ measure between $\tilde{X}$ and $\tilde{Y}$ is written as:

$$d_{SGD}^{\theta^\star}(\tilde{X}, \tilde{Y}) = \Big(\frac{1}{16}\big(p_1 - p_2 + 2q_1 - 2q_2 + r_1 - r_2\big)^2 + \theta^\star \frac{1}{16}\big(p_1 - p_2 - r_1 + r_2\big)^2\Big)^{\frac{1}{2}}.$$

2. The fuzzy numbers $\tilde{X} = (p_1, q_1, r_1, s_1)$ and $\tilde{Y} = (p_2, q_2, r_2, s_2)$ are trapezoidal. Their left and right alpha-cuts are written as:

$$\begin{aligned}\tilde{X}^{L,\alpha}(\alpha) &= p_1 + (q_1 - p_1)\alpha, \quad &\text{and}\quad \tilde{X}^{R,\alpha}(\alpha) = s_1 - (s_1 - r_1)\alpha; \\ \tilde{Y}^{L,\alpha}(\alpha) &= p_2 + (q_2 - p_2)\alpha, \quad &\text{and}\quad \tilde{Y}^{R,\alpha}(\alpha) = s_2 - (s_2 - r_2)\alpha.\end{aligned}$$

We calculate now $\frac{1}{4}\big(\int_0^1 \big(\tilde{X}^{L,\alpha} - \tilde{X}^{R,\alpha} - \tilde{Y}^{L,\alpha} + \tilde{Y}^{R,\alpha}\big)\mathrm{d}\alpha\big)^2$:

$$\begin{aligned}\frac{1}{4}\Big(\int_0^1 \big(\tilde{X}^{L,\alpha} - \tilde{X}^{R,\alpha} - \tilde{Y}^{L,\alpha} + \tilde{Y}^{R,\alpha}\big)\mathrm{d}\alpha\Big)^2 &= \frac{1}{4}\Big(\int_0^1 \big[p_1 + (q_1 - p_1)\alpha - s_1 + (s_1 - r_1)\alpha \\ &\qquad -p_2 - (q_2 - p_2)\alpha + s_2 - (s_2 - r_2)\alpha\big]\mathrm{d}\alpha\Big)^2 \\ &= \frac{1}{4}\big(p_1 + \frac{1}{2}(q_1 - p_1) - s_1 + \frac{1}{2}(s_1 - r_1) \\ &\qquad -p_2 - \frac{1}{2}(q_2 - p_2) + s_2 - \frac{1}{2}(s_2 - r_2)\big)^2 \\ &= \frac{1}{16}\big(p_1 - p_2 + q_1 - q_2 - r_1 + r_2 - s_1 + s_2\big)^2.\end{aligned}$$

Then, using Equations 10 and 15, and Corollary 1, the $d_{SGD}^{\theta^\star}$ measure between $\tilde{X}$ and $\tilde{Y}$ is written as:

$$\begin{aligned}d_{SGD}^{\theta^\star}(\tilde{X}, \tilde{Y}) = \quad &\Big(\frac{1}{16}\big(p_1 - p_2 + q_1 - q_2 + r_1 - r_2 + s_1 - s_2\big)^2 \\ &+\theta^\star \frac{1}{16}\big(p_1 - p_2 + q_1 - q_2 - r_1 + r_2 - s_1 + s_2\big)^2\Big)^{\frac{1}{2}}.\end{aligned}$$

In addition, if $\theta^\star = 1$, this expression becomes using Wolfram Alpha software:

$$d_{SGD}^{\theta^\star=1}(\tilde{X}, \tilde{Y}) = \frac{\sqrt{\big(p_1 - p_2 + q_1 - q_2\big)^2 + \big(r_1 - r_2 + s_1 - s_2\big)^2}}{2\sqrt{2}}.$$

3. The fuzzy numbers $\tilde{X}$ and $\tilde{Y}$ are Gaussian. They are written by their couples of mean μ and standard deviation σ such that $\tilde{X}$ given by (μ_1, σ_1), and $\tilde{Y}$ given by (μ_2, σ_2). Their left and right alpha-cuts are written as:

$$\tilde{X}^{L,\alpha}(\alpha) = \mu_1 - \sqrt{-2\sigma_1^2 \ln \alpha}, \quad \text{and} \quad \tilde{X}^{R,\alpha}(\alpha) = \mu_1 + \sqrt{-2\sigma_1^2 \ln \alpha};$$
$$\tilde{Y}^{L,\alpha}(\alpha) = \mu_2 - \sqrt{-2\sigma_2^2 \ln \alpha}, \quad \text{and} \quad \tilde{Y}^{R,\alpha}(\alpha) = \mu_2 + \sqrt{-2\sigma_2^2 \ln \alpha}.$$

We calculate now $\frac{1}{4}\left(\int_0^1 (\tilde{X}^{L,\alpha} - \tilde{X}^{R,\alpha} - \tilde{Y}^{L,\alpha} + \tilde{Y}^{R,\alpha})\mathrm{d}\alpha\right)^2$:

$$\begin{aligned} \frac{1}{4}\left(\int_0^1 (\tilde{X}^{L,\alpha} - \tilde{X}^{R,\alpha} - \tilde{Y}^{L,\alpha} + \tilde{Y}^{R,\alpha})\mathrm{d}\alpha\right)^2 &= \frac{1}{4}\Big(\int_0^1 (\mu_1 - \sqrt{-2\sigma_1^2 \ln \alpha} \\ &\qquad -\mu_1 - \sqrt{-2\sigma_1^2 \ln \alpha} \\ &\qquad -\mu_2 + \sqrt{-2\sigma_2^2 \ln \alpha} \\ &\qquad +\mu_2 + \sqrt{-2\sigma_2^2 \ln \alpha})\mathrm{d}\alpha\Big)^2. \end{aligned}$$

Since $\int_0^1 \sqrt{-2 \ln \alpha}\mathrm{d}\alpha = \sqrt{\frac{\pi}{2}}$, then

$$\begin{aligned} \frac{1}{4}\left(\int_0^1 (\tilde{X}^{L,\alpha} - \tilde{X}^{R,\alpha} - \tilde{Y}^{L,\alpha} + \tilde{Y}^{R,\alpha})\mathrm{d}\alpha\right)^2 &= \frac{1}{4}\Big(\mu_1 - \sigma_1\sqrt{\frac{\pi}{2}} - \mu_1 - \sigma_1\sqrt{\frac{\pi}{2}} \\ &\qquad -\mu_2 + \sigma_2\sqrt{\frac{\pi}{2}} + \mu_2 + \sigma_2\sqrt{\frac{\pi}{2}}\Big)^2 \\ &= (\sigma_2 - \sigma_1)^2 \frac{\pi}{2}. \end{aligned}$$

Then, using Equations 11 and 15, and Corollary 1, the $d_{SGD}^{\theta^\star}$ measure between $\tilde{X}$ and $\tilde{Y}$ is written as:

$$d_{SGD}^{\theta^\star}(\tilde{X}, \tilde{Y}) = \left((\mu_1 - \mu_2)^2 + \theta^\star (\sigma_2 - \sigma_1)^2 \frac{\pi}{2}\right)^{\frac{1}{2}};$$

4. The fuzzy numbers $\tilde{X}$ and $\tilde{Y}$ are two-sided Gaussian. They are written by their couples of means μ and standard deviations σ such that $\tilde{X}$ given by $(\mu_{X_1}, \sigma_{X_1}, \mu_{X_2}, \sigma_{X_2})$, and $\tilde{Y}$ given by $(\mu_{Y_1}, \sigma_{Y_1}, \mu_{Y_2}, \sigma_{Y_2})$. Their left and right alpha-cuts are written as:

$$\tilde{X}^{L,\alpha}(\alpha) = \mu_{X_1} - \sqrt{-2\sigma_{X_1}^2 \ln \alpha}, \quad \text{and} \quad \tilde{X}^{R,\alpha}(\alpha) = \mu_{X_2} + \sqrt{-2\sigma_{X_2}^2 \ln \alpha};$$
$$\tilde{Y}^{L,\alpha}(\alpha) = \mu_{Y_1} - \sqrt{-2\sigma_{Y_1}^2 \ln \alpha}, \quad \text{and} \quad \tilde{Y}^{R,\alpha}(\alpha) = \mu_{Y_2} + \sqrt{-2\sigma_{Y_2}^2 \ln \alpha}.$$

We calculate now $\frac{1}{4}\left(\int_0^1 (\tilde{X}^{L,\alpha} - \tilde{X}^{R,\alpha} - \tilde{Y}^{L,\alpha} + \tilde{Y}^{R,\alpha})\mathrm{d}\alpha\right)^2$:

$$\begin{aligned} \frac{1}{4}\left(\int_0^1 (\tilde{X}^{L,\alpha} - \tilde{X}^{R,\alpha} - \tilde{Y}^{L,\alpha} + \tilde{Y}^{R,\alpha})\mathrm{d}\alpha\right)^2 &= \frac{1}{4}\Big(\int_0^1 (\mu_{X_1} - \sqrt{-2\sigma_{X_1}^2 \ln \alpha} \\ &\qquad -\mu_{X_2} - \sqrt{-2\sigma_{X_2}^2 \ln \alpha} \\ &\qquad -\mu_{Y_1} + \sqrt{-2\sigma_{Y_1}^2 \ln \alpha} \\ &\qquad +\mu_{Y_2} + \sqrt{-2\sigma_{Y_2}^2 \ln \alpha})\mathrm{d}\alpha\Big)^2. \end{aligned}$$

Since $\int_0^1 \sqrt{-2\ln\alpha}\mathrm{d}\alpha = \sqrt{\frac{\pi}{2}}$, then

$$\begin{aligned}\frac{1}{4}\Big(\int_0^1 (\tilde{X}^{L,\alpha} - \tilde{X}^{R,\alpha} - \tilde{Y}^{L,\alpha} + \tilde{Y}^{R,\alpha})\mathrm{d}\alpha\Big)^2 &= \frac{1}{4}\Big(\mu_{X_1} - \sigma_{X_1}\sqrt{\frac{\pi}{2}} - \mu_{X_2} - \sigma_{X_2}\sqrt{\frac{\pi}{2}} \\ &\quad -\mu_{Y_1} + \sigma_{Y_1}\sqrt{\frac{\pi}{2}} + \mu_{Y_2} + \sigma_{Y_2}\sqrt{\frac{\pi}{2}}\Big)^2 \\ &= \frac{1}{4}\Big(\mu_{X_1} - \mu_{X_2} - \mu_{Y_1} + \mu_{Y_2} \\ &\quad + \sqrt{\frac{\pi}{2}}(-\sigma_{X_1} - \sigma_{X_2} + \sigma_{Y_1} + \sigma_{Y_2})\Big)^2.\end{aligned}$$

Then, using Equations 12 and 15, and Corollary 1, the $d_{SGD}^{\theta^\star}$ measure between $\tilde{X}$ and $\tilde{Y}$ is written as:

$$\begin{aligned} d_{SGD}^{\theta^\star}(\tilde{X},\tilde{Y}) &= \Big(\frac{1}{4}(\mu_{X_1} + \mu_{X_2} - \mu_{Y_1} - \mu_{Y_2} + \sqrt{\frac{\pi}{2}}(-\sigma_{X_1} + \sigma_{X_2} + \sigma_{Y_1} - \sigma_{Y_2}))^2 \\ &+ \theta^\star\frac{1}{4}(\mu_{X_1} - \mu_{X_2} - \mu_{Y_1} + \mu_{Y_2} + \sqrt{\frac{\pi}{2}}(-\sigma_{X_1} - \sigma_{X_2} + \sigma_{Y_1} + \sigma_{Y_2}))^2\Big)^{\frac{1}{2}}.\end{aligned}$$

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