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ORIGINAL ARTICLE

A parametric likelihood measure with beta distributions for Pythagorean fuzzy decision-making

Chueh-Yung Tsao[1] · Ting-Yu Chen[1]

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Abstract

The objective of this research is to introduce a parametric likelihood measure based on the beta distribution and develop a likelihood-oriented methodology for solving multiple criteria decision analysis (MCDA) problems with Pythagorean fuzzy (PF) sets. With the rapid advancement of PF theory, exploring an effective approach to compare PF information is indispensable in resolving MCDA issues. The beta distribution is one the most commonly used distributions to simulate the theoretical distribution. By changing the parameter values, the beta distribution can generate symmetrical or asymmetrical patterns and various shapes, including flat or steep. Due to its flexibility and adaptability, the beta distribution is able to effectively solve complex real-world problems. To make a major contribution to the technical development of decision support applications, this paper utilizes beta distributions as a parameterization tool to introduce a new parametric likelihood measure for evaluating the outranking relationships among PF information (signified by Pythagorean membership grades). Based on the evolved likelihood-based concepts (e.g., mean outranking indices, weighted outranking grades, and comprehensive outranking measures and indices), this paper proposes a pragmatic PF likelihood-oriented method to prioritize competing alternatives under uncertain and ambiguous Pythagorean fuzzy conditions. To carefully examine the practicality and suitability of the proposed methodology in realistic decision-making environments, this paper utilizes the evolved methods to solve a realistic MCDA problem of selecting pilot hospitals in relation to postacute care. The main results that are generated by the practical application and subsequent experimental analysis and comparative study demonstrate the effectivity and superiority of the developed technique and can be used for practical purposes in flexible and convenient ways. This most important conclusion of this paper is the great aptitude and dominance of the proposed methodology based on the corroboration of the experimental and comparative results of the application. Furthermore, this study has a noticeable originality in the utilization of the generic beta distribution-based approach and the construction of an effective PF likelihood-oriented decision model, which enriches the development of decision-making applications with PF theory.

Keywords Parametric likelihood measure · Beta distribution · Likelihood-oriented methodology · Multiple criteria decision analysis (MCDA) · Pythagorean fuzzy (PF) set

1 Introduction

In real-world environments, decision-makers in enterprises, organizations, and governments often face many unknowns and an extensive range of uncertainties [13, 25, 33]. Pythagorean fuzzy (PF) sets were initially introduced by Yager [40] and Yager and Abbasov [42] and deliver a constructive manner to manifest equivocal information in realistic decision environments. The premise of the PF model is that the sum of the squared degrees of membership and nonmembership is less than or equal to 1. Such

✉ Ting-Yu Chen
tychen@mail.cgu.edu.tw

Chueh-Yung Tsao
cytsao@mail.cgu.edu.tw

[1] Department of Industrial and Business Management, Graduate Institute of Management, Chang Gung University, No. 259, Wenhua 1st Rd., Guishan District, Taoyuan City 33302, Taiwan

prerequisites provide PF theory with greater flexibility and adaptability than intuitionistic fuzzy sets to reflect the ambiguity and uncertainty of human subjectivity and management in practical applications [1, 28, 46]. The PF sets are capable of permitting broader spaces to more effectively represent levels of agreement, disagreement, and hesitancy [10, 28]. Moreover, PF theory has gradually emerged as a powerful model for imprecise knowledge representation that is applicable in managing complicated real-world decisions [23, 27, 29]. Consequently, PF sets have received widespread acceptance as a beneficial method to depict higher-order fuzzy information during uncertain multiple criteria decision analysis (MCDA) processes [4, 7, 10, 14]. With the establishment of PF theory, research on the comparison of PF information is very active in addressing MCDA issues, such as score and accuracy functions [22, 35], closeness indices [20, 27], and scalar functions [9, 38]. In addition, the valuable concept of likelihood measures can be employed to determine the outranking relationship for PF information [17, 26]. In promoting the growth of MCDA approaches in PF situations, complicated and uncertain information can usually be managed because the employment of likelihood measures in MCDA processes ensures the determination of outranking relationships among PF information [9, 11, 17, 26, 37].

The concept of likelihood measures is useful for determining the outranking relationships between PF assessments in MCDA processes. Nevertheless, only a few studies in recent years have proposed beneficial likelihood measures to determine the outranking relationships for addressing ambiguous and vague decision-making activities (e.g., choosing, ranking, and sorting) with Pythagorean fuzziness. Garg [18] utilized the likelihood between two interval numbers to form a possibility degree matrix and then proposed an interval-valued MCDA approach based on exponential operational laws and aggregation operators,however, this likelihood measure was only applicable for ordinary interval numbers, not PF or interval-valued PF numbers. Fei et al. [17] utilized the soft likelihood function of PF sets and created an MCDA approach based on an ordered weighted average. Nonetheless, the soft likelihood function was designed to promote the growth of aggregation methods, but not the identification of outranking relationships between PF sets. Liang et al. [26] identified a likelihood of preference relationship by making use of a uniform distribution and further established a PF linear assignment methodology for decision support. However, the suitability of this probability density function is unproven, and thus, a uniform distribution-based likelihood may be limited in practice. Chen [11] utilized the scalar function and an admissible upper approximation of PF information to introduce a new likelihood measure for group decision-making. This measurement technique is easy to implement. However, using only an admissible upper approximation to delineate a PF likelihood measure is too simple,therefore, this process cannot make full use of PF information. Chen [9] proposed a new likelihood measure using dual point operators in PF contexts to introduce an advantageous preference ranking organization approach. However, the specification of dual PF point operators was not determined from the statistical or probabilistic perspectives, which led to a lack of practicality in real decisions. In accordance with these considerations, the likelihood measures within PF environments have not been thoroughly investigated, which creates the first motivation of this research.

In advocating the widespread development of PF MCDA approaches, there exists an evident limitation, namely, the lack of an appropriate parameterization tools to determine a PF likelihood of outranking relationships. In particular, the PF likelihood measure proposed by Liang et al. [26] was based on the uniform density assumption. They employed the uniform distribution to randomly choose the approval and disapproval behaviors for allocating the indeterminacy composition of membership and nonmembership, respectively. Although the appropriateness of the probability density function is unproven, this assumption still makes this research valuable. One of the most suitable candidates for modeling dependent variables valued from zero to one is the class of beta distributions [6, 15, 34, 36, 43]. The uniform density function is a special example of a beta distribution [15, 34]. In real-world decisions, uncertain, imprecise, and vague results commonly occur [16]. It would be highly advantageous to utilize beta distributions as a parameterization tool to develop an innovative likelihood measure under uncertain PF conditions to more comprehensively compare Pythagorean membership grades and determine outranking relationships. However, previous studies involving likelihood measurements in PF settings, such as Chen [9, 11], Garg [18], Fei et al. [17], and Liang et al. [26], have rarely explored this subject. Tsao and Chen [37] addressed questions regarding the applicability of a symmetric beta distribution to identify a promising likelihood function in PF settings. To our knowledge, this is the first work that shows how to incorporate a beta function into the determination of the possibility degrees for outranking relationships under PF conditions. From a pragmatic perspective, the mechanism of the current PF likelihood measures is deficient in the consideration of parameterization tools as a result of limited suitability and flexibility in real-life applications. These limitations form the second motivation of this research.

In probability and statistical theory, beta distributions possess several interesting properties, and a range of uncertainties can be effectively modeled by them [21, 24]. Specifically, beta distributions are able to describe random changes in uncertain information [36], shape other probability distributions through minimal information loss [21], and approximate an assortment of symmetric and biased distributions containing the normal distribution by regulating relevant parameters [6]. Beta distributions intrinsically incorporate features such as skewness and heteroscedasticity [19]. More importantly, the beta distribution is differentiable over its entire domain, making it very versatile and adjustable for processing uncertain estimates [16, 21]. For these reasons, beta distributions are one of the most frequently utilized methods to produce theoretical distributions. The above-mentioned efficacy and flexibility encourage the empirical use of beta distributions in a great variety of applications, especially in fuzzy settings. A large amount of effort has been made to model distributions, fuzzy estimations, and practical applications through beta distributions [6, 12, 19, 34, 43]. However, there has been only one study, i.e., Tsao and Chen [37], until now that used a symmetrically standard beta distribution to elucidate a PF likelihood function. They presumed that the likelihood estimations would conform to a symmetrically standard beta distribution. However, their limited parameter settings led to a narrow scope of application. As a general rule, beta distributions can be symmetric, left-skewed, or right-skewed through changes in various parameter settings, which can align with neutral, optimistic, or pessimistic, respectively, attitudes regarding decision environments. Nevertheless, except for Tsao and Chen [37], the parameterization model as a result of the beta distribution seems to be rarely investigated in manipulating uncertain Pythagorean fuzzy information. This issue establishes the last motivation of this research.

The above-mentioned motivational considerations introduce challenges to the theory and practice of PF sets. Due to the sophisticated uncertainties involved in real-world environments, the key research question in addressing these issues is how to utilize beta distributions for carrying out pairwise comparisons of PF information and determining the probability of the outranking relationships between Pythagorean membership grades. The necessity of this research is manifested in three aspects. First, it is of vital significance to establish a PF likelihood measure of outranking relationships due to the minimal progress in developing a possibility formula in PF contexts. Second, few studies have focused on developing an appropriate parameterization tool for the measurement of PF likelihood functions for better flexibility and adaptability. Third, minimal research has focused on the combination of beta distributions and likelihood measures in PF settings and the development of a PF likelihood-oriented methodology. Based on these aspects, this paper makes an effort to demonstrate the suitability and practicality of beta distributions in MCDA to introduce a flexible parametric likelihood measure for Pythagorean membership grades, which can perform the effective differentiation of PF assessment information.

The research objectives of this paper are to introduce an original parametric PF likelihood measure based on beta distributions and develop an effective PF likelihood-oriented methodology for managing MCDA issues involving the intricate uncertainty of Pythagorean fuzziness. This paper takes advantage of beta distributions to measure the possibility of an outranking relationship between PF evaluative ratings. Beta distributions can be symmetric, left-skewed, or right-skewed through changes in various parameter settings, which can accommodate decision-makers' neutral, optimistic, or pessimistic attitudes, respectively. Moreover, the mean value associated with a beta distribution depends on the relative ratio of parameters. In particular, when relevant parameters increase proportionally, the mean remains constant, the variance decreases, and the distribution favors the standard normal distribution. In consideration of the aforementioned characteristics, beta distributions possess adjustability and versatility in practical applications, and thus, this paper delivers a beta distribution-based technique to introduce a new parametric likelihood measure within PF decision environments. Nevertheless, no closed-form solution exists in constructing a possibility formula because the beta distribution function is in the form of a polynomial. Therefore, this paper investigates the closed-form solution for the measurement of PF likelihoods for more flexibly and adaptively estimating the probability of the outranking relationships between assessment information in PF formats. Through the advanced likelihood measure, an innovative PF likelihood-oriented method is proposed to address multiple criteria assessment tasks involving uncertain PF information. Moreover, this paper makes use of the current parametric likelihood measure to develop several beneficial likelihood-based concepts, including mean outranking indices, weighted outranking grades, and comprehensive outranking measures and indices. Additionally, a convenient-multiple criteria evaluation model is established to address MCDA issues in PF conditions. This paper utilizes the advanced methodology to address a pragmatic problem regarding hospital-based postacute care. Finally, the results generated by the practical application and subsequent experimental analysis and comparative study also demonstrate the effective and favorable aspects of the current PF likelihood-oriented method using a beta distribution-based likelihood measure.

The originality of the proposed methodology in comparison with the existing techniques is unique in its establishment of parametric PF likelihood measurements based on a standard beta distribution and the oriented MCDA methods. The main reasons for utilizing a standard beta distribution in this study are threefold. First, to evaluate the likelihood measure over two Pythagorean membership grades, this study introduces two random variables with domains of [0, 1]. These variables are used as proportions to distribute the degree of indeterminacy to the degree of membership. However, most of the widely used continuous distributions, such as normal, gamma, and chi-square, are unbounded. The domain of a standard beta distribution is bounded and exactly the unit interval [0, 1], which supports the utilization of a standard beta distribution.

Second, the choice of the densities for the two random variables plays an important role in processing PF information for practical applications. The densities should be flexible enough to capture the variability of human subjective judgments. For example, when an individual has a tendency of optimistically or pessimistically distributing indeterminacy into the degree of membership, one should choose a left-skewed or right-skewed density, respectively, for the two random variables to evaluate the likelihood measure. By altering parameter values, a standard beta distribution is capable of generating symmetric or asymmetric patterns and a variety of shapes, including a flat or a steep. In a general sense, beta distributions are regarded as the most frequently used distribution to model theoretical distributions [6, 21, 24]. By designating appropriate parameters, the standard beta distribution is capable of handling complex evaluation information in practical applications due to its flexibility and adaptability, which provides the support and approval of the standard beta distribution.

Third, the likelihood measure based on a standard beta distribution is general enough to aggregate the existing likelihood measure in the literature, as shown by Liang et al. [26] and Tsao and Chen [37]. Liang et al. [26] also made use of a uniform distribution and established a uniform distribution-based likelihood measure. Tsao and Chen [37] employed a symmetrically standard beta distribution to elucidate randomness and obtained a symmetry beta distribution-based likelihood measure. Unlike Liang et al. [26] and Tsao and Chen [37], this study proposes a likelihood measure based on a standard beta distribution. The shapes of a standard beta distribution could be symmetric or asymmetric and flat or steep. Moreover, it is easy to verify that the results of Liang et al. [26] and Tsao and Chen [37] are both special cases of our proposed method, which produces evidence to support the use of standard beta distributions.

To give a wide overall view of the initiated techniques, the PF likelihood-oriented methodology utilizes beta distributions as a parameterization tool to propose a new parametric likelihood measure for appraising the outranking relationships toward Pythagorean membership grades. Then, an advantageous multiple criteria evaluation model for prioritizing competing alternatives and treating an MCDA issue involving intricate and uncertain information with Pythagorean fuzziness is developed. Notably, the developed PF likelihood-oriented methodology has remarkable originality in the development of a parametric likelihood measure based on beta distributions. Moreover, central to the evolved parametric PF likelihood measure is the utilization of a standard beta distribution. To measure the possibility of an outranking relationship between PF evaluative ratings, one needs to employ a random variable that has the following properties. First, the domain of the selected random variable needs to be between 0 and 1 because this variable is employed as a percentage to partition the degree of indeterminacy. Second, the distribution of the selected random variable must be flexible enough to reflect the decision-maker's decision-making attitude, such as neutral, optimistic, or pessimistic. In probability and statistical theory, beta distributions possess several interesting properties, and a range of uncertainties can be productively modeled by them. A standard beta distribution has a range of 0 and 1 and is the most frequently used to model theoretical distributions. Beta distributions can be symmetric, left-skewed, or right-skewed through various parameter settings, which can align with decision-makers' neutral, optimistic, or pessimistic attitudes, respectively. For these reasons, it is extremely beneficial to utilize the standard beta distribution as a density function in developing the proposed parametric likelihood function. Using a standard beta distribution to delineate the PF parameter likelihood measure has become the most important technological innovations of the advanced methods in this study.

The evolved research methodology of MCDA and its practicality within PF environments has made certain worthwhile contributions to the theoretical advantages and pragmatic merits over the existing techniques. More specifically, the innovative contributions of the PF likelihood-oriented methodology are fivefold. First, unlike the current measurement of PF likelihoods, the proposed methodology is capable of manipulating preference differences through the advanced parametric likelihood measure, which emphasizes flexibility and adaptability. Second, the theoretical focus in this study is toward the realization and incorporation of the beta distributions in uncertain PF circumstances, in contrast to the empirical use of beta distributions in ordinary fuzzy settings. Third, this paper specifically establishes outranking relationships between PF evaluative ratings and thus eliminates the difficulty of manipulating sophisticated Pythagorean membership grades. Fourth, this paper utilizes the PF likelihood-oriented method, an easy-to-use multiple criteria evaluation model, which contains practical parametric likelihood-based concepts, such as mean outranking indices, weighted outranking grades, and comprehensive outranking measures and indices. Finally, the proposed methodology provides decision-makers broader and more flexible spaces to manifest human subjectivity and facilitate the management of the ambiguity and uncertainty that arise in practical applications. Furthermore, with the establishment of the PF likelihood-oriented methodology, a demonstrative case study concerning hospital-based postacute care is implemented to investigate the feasibility and practicality of the advanced methods and techniques in uncertain real-world conditions. The results also support the value and strength of the proposed methodology.

The structure of this paper is given as follows. Section 2 presents the preliminary concept and operation of PF sets to form an essential foundation. Section 3 introduces a useful parametric likelihood measure concerning PF evaluative ratings through the utility of beta distributions. Furthermore, this section provides technical explanations to strengthen the rationality of the debates and arguments. Section 4 develops an innovative PF likelihood-oriented method based on the advanced parametric likelihood measure for manipulating MCDA problems within PF environments. Section 5 carries out a real-world application of selecting pilot hospitals for postacute care and conducts an experimental analysis and comparative study to justify the value and strength of this method. Section 6 summarizes the results of the analysis, contributions of the study, relevance of the main findings, limitations of the proposed methods, scientific value of this work, and research topics of future merit.

2 Preliminary concepts

This section briefly presents select fundamental concepts with relevance to PF sets. Selected operations of Pythagorean membership grades that are advantageous in subsequent research are further reviewed. Finally, this section relates the definition to a standard beta distribution.

Definition 1 [40] A PF set P defined in a finite universe of discourse U is expressed as follows:

$$P = \{\langle u, (\mu_P(u), \nu_P(u))\rangle | u \in U\},\ \ \mu_P(u), \nu_P(u) : U \to [0,1], \tag{1}$$

where $\mu_P(u)$ and $\nu_P(u)$ are the degrees of membership and nonmembership, respectively, of an element $u \in U$ related to P and subject to the following constraint:

$$0 \leq (\mu_P(u))^2 + (\nu_P(u))^2 \leq 1\ \forall u \in U. \tag{2}$$

Definition 2 [41, 44] A Pythagorean membership grade p of u belonging to the PF set P is defined as follows:

$$p = (\mu_P(u), \nu_P(u)). \tag{3}$$

The complement of this expression is $p^c = (\nu_P(u), \mu_P(u))$. The degree of indeterminacy of p is given by:

$$\tau_P(u) = \sqrt{1 - (\mu_P(u))^2 - (\nu_P(u))^2}. \tag{4}$$

Definition 3 [44] Some basic operational laws related to the three Pythagorean membership grades p, p_1 $(= (\mu_P(u_1), \nu_P(u_1)))$, and p_2 $(= (\mu_P(u_2), \nu_P(u_2)))$ in U are indicated below:

$$p_1 \oplus p_2 = \left(\sqrt{(\mu_P(u_1))^2 + (\mu_P(u_2))^2 - (\mu_P(u_1))^2 (\mu_P(u_2))^2}, \nu_P(u_1)\nu_P(u_2)\right), \tag{5}$$

$$p_1 \otimes p_2 = \left(\mu_P(u_1)\mu_P(u_2), \sqrt{(\nu_P(u_1))^2 + (\nu_P(u_2))^2 - (\nu_P(u_1))^2 (\nu_P(u_2))^2}\right), \tag{6}$$

$$\lambda \odot p = \left(\sqrt{1 - \left(1 - (\mu_P(u))^2\right)^{\lambda}}, (\nu_P(u))^{\lambda}\right),\ \lambda > 0, \tag{7}$$

$$p^{\lambda} = \left((\mu_P(u))^{\lambda}, \sqrt{1 - \left(1 - (\nu_P(u))^2\right)^{\lambda}}\right),\ \lambda > 0. \tag{8}$$

Herein, $\oplus$, $\otimes$, and $\odot$ represent the addition, multiplication, and multiplication by an ordinary number λ operations, respectively, where λ is a nonnegative real-value constant.

Definition 4 [24] The density function associated with a standard beta distribution is defined as follows:

$$f(y; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} y^{\alpha-1}(1-y)^{\beta-1},\ 0 \leq y \leq 1, \tag{9}$$

where $\alpha > 0$ and $\beta > 0$ are two parameters, and $B(\alpha, \beta)$ is a beta function in the following form:

$$B(\alpha, \beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1} dy. \tag{10}$$

Let beta(α, β) signify a random variable abiding by a standard beta distribution with α and β. Note that a standard beta distribution has mean $\alpha(\alpha+\beta)^{-1}$ and variance $\alpha\beta(\alpha+\beta)^{-2}(\alpha+\beta+1)^{-1}$. The skewness of a standard beta distribution is $2(\beta-\alpha)(\alpha^{-1}+\beta^{-1}+(\alpha\beta)^{-1})^{0.5}(\alpha+\beta+2)^{-1}$.

A beta distribution can be symmetric, left-skewed, or right-skewed depending on its embedded parameters α and β. In particular, when $\alpha<\beta$, the skewness is positive, and the distribution skews to the right, and when $\alpha>\beta$, the skewness is negative, and the distribution skews to the left. A symmetry beta distribution is recognized when $\alpha=\beta$. The mean value $\alpha(\alpha+\beta)^{-1}$ depends on the ratio of α and β. If α and β increase proportionally, the mean remains constant, the variance decreases, and the distribution favors the standard normal distribution. By varying the parameters α and β, a standard beta distribution possesses great capability to handle complicated scenarios in practical applications due to its flexibility and adaptability.

3 Parametric likelihood measure based on beta distributions

Using beta distributions, this section proposes an innovative parametric likelihood measure to estimate the probability of the outranking relationships regarding PF evaluative ratings within a Pythagorean fuzziness decision environment. With the assistance of beta density functions, this section utilizes a parameterization tool to construct a possibility formula for the measurement of PF likelihoods. Furthermore, several favorable and attractive properties of the proposed parametric likelihood measure are investigated to validate the strengths of the PF likelihood measurements supported by a beta distribution-based approach.

Under uncertain PF decision conditions, an MCDA problem is formulated using the means of m candidate alternatives appraised with n evaluative criteria, where m and n are positive integers and $m,n\geq 2$. Specifically, let $A=\{a_1,a_2,\cdots,a_m\}$ precisely represent a discrete collection of candidate alternatives and $C=\{c_1,c_2,\cdots,c_n\}$ indicate a finite collection of evaluative criteria. To facilitate the construction of PF evaluative ratings, the assessment concerning alternative $a_i\in A$ pertaining to criterion $c_j\in C$ can be effectively performed through satisfaction and dissatisfaction surveys. Depending on human subjective judgments, the grades that a_i satisfies and c_j dissatisfies are portrayed using the degrees of membership μ_{ij} and nonmembership ν_{ij}, respectively, such that $0\leq(\mu_{ij})^2+(\nu_{ij})^2\leq 1$ and $\mu_{ij},\nu_{ij}\in[0,1]$. As a result, the PF evaluative rating p_{ij} is manifested by the Pythagorean membership grade $p_{ij}=(\mu_{ij},\nu_{ij})$ in regard to $i\in\{1,2,\cdots,m\}$ and $j\in\{1,2,\cdots,n\}$; therefore, its degree of indeterminacy $\tau_{ij}=\sqrt{1-(\mu_{ij})^2-(\nu_{ij})^2}$. Moreover, let a PF set P_i explain the PF characteristic relative to each alternative a_i, which is constructed from an accumulation of p_{ij} across the n criteria as follows:

$$\begin{aligned}P_i&=\{\langle c_j,p_{ij}\rangle|c_j\in C\}\\&=\{\langle c_1,(\mu_{i1},\nu_{i1})\rangle,\langle c_2,(\mu_{i2},\nu_{i2})\rangle,\cdots,\langle c_n,(\mu_{in},\nu_{in})\rangle\}.\end{aligned}\tag{11}$$

The following is the definition of the likelihood measure of $p_{ij}\geq p_{\kappa j}$.

Definition 5 Consider two PF evaluative ratings $p_{ij}\in P_i$ and $p_{\kappa j}\in P_\kappa$. Let $x_i=(\mu_{ij})^2+y(\tau_{ij})^2$ and $x_\kappa=(\mu_{\kappa j})^2+z(\tau_{\kappa j})^2$, in which y and z are two independent random variables bound in the domain of [0, 1]. The likelihood measure of $p_{ij}\geq p_{\kappa j}$ is defined as the probability measure $\Pr(x_i\geq x_\kappa)$. Therefore, $\Pr(x_i\geq x_\kappa)=\Pr((\mu_{ij})^2+y(\tau_{ij})^2\geq(\mu_{\kappa j})^2+z(\tau_{\kappa j})^2)$.

An example of Definition 5 is as follows. An individual evaluates the likelihood of $p_{ij}\geq p_{\kappa j}$ depending on two kinds of grades. The first grade is the degrees of membership (μ_{ij} and $\mu_{\kappa j}$), and the second grade is the degrees of indeterminacy (τ_{ij} and $\tau_{\kappa j}$). A higher value of $(\mu_{ij})^2$ relative to $(\mu_{\kappa j})^2$ causes a higher degree of likelihood with certainty. However, the effect of indeterminacy on the likelihood measure is uncertain. Therefore, two random variables y and z with a domain of [0, 1] are employed as a mechanism to model this uncertainty. The variables y and z can be regarded as the proportions to distribute $(\tau_{ij})^2$ and $(\tau_{\kappa j})^2$ to the degrees of membership $(\mu_{ij})^2$ and $(\mu_{\kappa j})^2$, respectively. The likelihood measure can then be evaluated by calculating the probability that $(\mu_{ij})^2+y(\tau_{ij})^2\geq(\mu_{\kappa j})^2+z(\tau_{\kappa j})^2$.

Obviously, the choice of the densities of y and z plays an important role in practical applications. As discussed earlier, Liang et al. [26] assumed that y and z are uniformly distributed to present a uniform distribution-based likelihood measure. Tsao and Chen [37] extended the uniform distribution by introducing a beta distribution beta(α,β). By assuming that $\alpha=\beta$, Tsao and Chen [37] obtained a symmetry beta-based likelihood measure. One can easily examine whether the beta distribution becomes a uniform distribution when the two parameters α and β are all equal to one. Although the extended distribution in Tsao and Chen [37] increased the flexibility to utilize likelihood measures in practice, it cannot fully capture the variability of human subjective judgments. The restriction of symmetry distributions ($\alpha=\beta$) limits the use when an individual has a tendency of optimistically or pessimistically distributing the indeterminacy to the degrees of membership. When evaluating the likelihood measure, an individual with an optimistic rationale will choose the values of $\alpha > \beta$ and an individual with a pessimistic rationale will choose the values of $\alpha < \beta$.

In this paper, a more general assumption is made for the distributions of y and z. In particular, this paper assumes that y and z follow independent beta distributions beta(α, β) and that α and β can be any positive integers. Therefore, the likelihood functions of $p_{ij} \geq p_{\kappa j}$ developed by Liang et al. [26] and Tsao and Chen [37] are both special cases of our proposal. For instance, Fig. 1 depicts the beta density functions with parameters α and β ranging from integers 1 to 4. As illustrated in Fig. 1, the beta distribution is left-skewed in the case of $\alpha = 4$ and $\beta = 2$, which fits with the optimistic rationale. In contrast, the beta distribution is right-skewed in the case of $\alpha = 2$ and $\beta = 4$, which fits with the pessimistic rationale. By the designation of parameters α and β, the proposed beta distribution-based approach can be extended to methodically manipulate the decision-maker's disposition in regard to the decision environment. When the decision-maker possesses an optimistic attitude, it is appropriate to employ the condition of $\alpha > \beta$ to secure a left-skewed distribution. In contrast, when the decision-maker has a pessimistic attitude, it is appropriate to utilize the condition of $\alpha < \beta$ to acquire a right-skewed distribution. However, when the parameters α and β are fixed to a value of 1, as in Liang et al. [26], or α is restricted to be equal to β, as in Tsao and Chen [37], the model is limited use in modeling the decision-maker's varying attitudes.

Based on the above discussion, the main reasons for employing a standard beta distribution in this study are threefold. First, the random variables y and z in Definition 5 must be bound between 0 and 1. This condition contrasts the most widely used continuous distributions, such as normal, gamma, and chi-square, which are unbounded. Second, by altering parameters, the beta distribution is capable of generating symmetric or asymmetric patterns and a variety of shapes, including a flat or steep. Beta distributions are regarded as the most frequently used theoretical distribution models [24]. Third, the likelihood measure based on a standard beta distribution is general enough to aggregate existing likelihood measures in the literature, such as Liang et al. [26] and Tsao and Chen [37].

The likelihood function proposed by Liang et al., denoted as $lik(p_{ij} \geq p_{\kappa j})$, is described as follows:

$$lik(p_{ij} \geq p_{\kappa j}) = \begin{cases} 0 & \textit{if}\ 1-(v_{ij})^2 \leq (\mu_{\kappa j})^2, \\ \dfrac{\left(1-(\mu_{\kappa j})^2-(v_{ij})^2\right)^2}{2\left(1-(\mu_{ij})^2-(v_{ij})^2\right)\left(1-(\mu_{\kappa j})^2-(v_{\kappa j})^2\right)} & \textit{if}\ (\mu_{ij})^2 \leq (\mu_{\kappa j})^2 \leq 1-(v_{ij})^2 \leq 1-(v_{\kappa j})^2, \\ \dfrac{1-2(v_{ij})^2-(\mu_{\kappa j})^2+(v_{\kappa j})^2}{2\left(1-(\mu_{ij})^2-(v_{ij})^2\right)} & \textit{if}\ (\mu_{ij})^2 \leq (\mu_{\kappa j})^2 \leq 1-(v_{\kappa j})^2 \leq 1-(v_{ij})^2, \\ 0.5 & \textit{if}\ (\mu_{ij})^2 = (\mu_{\kappa j})^2\ \textit{and}\ (v_{ij})^2 = (v_{\kappa j})^2, \\ \dfrac{1+(\mu_{ij})^2-(v_{ij})^2-2(\mu_{\kappa j})^2}{2\left(1-(\mu_{\kappa j})^2-(v_{\kappa j})^2\right)} & \textit{if}\ (\mu_{\kappa j})^2 \leq (\mu_{ij})^2 \leq 1-(v_{ij})^2 \leq 1-(v_{\kappa j})^2, \\ 1-\dfrac{\left(1-(\mu_{ij})^2-(v_{\kappa j})^2\right)^2}{2\left(1-(\mu_{ij})^2-(v_{ij})^2\right)\left(1-(\mu_{\kappa j})^2-(v_{\kappa j})^2\right)} & \textit{if}\ (\mu_{\kappa j})^2 \leq (\mu_{ij})^2 \leq 1-(v_{\kappa j})^2 \leq 1-(v_{ij})^2, \\ 1 & \textit{if}\ 1-(v_{\kappa j})^2 \leq (\mu_{ij})^2. \end{cases} \quad (12)$$

The above equation is the solution of $\Pr(x_i \geq x_\kappa)$ when the random variables y and z are uniformly distributed. The proof of this result is provided by Liang et al. [26]. As shown from Eq. (12), there are seven scenarios in the solution, and they depend on the relative magnitudes of μ_{ij}, $\mu_{\kappa j}$, v_{ij}, and $v_{\kappa j}$. Among them, there are three trivial scenarios: when $1-(v_{ij})^2 \leq (\mu_{\kappa j})^2$, $(\mu_{ij})^2 = (\mu_{\kappa j})^2$ and $(v_{ij})^2 = (v_{\kappa j})^2$, and $1-(v_{\kappa j})^2 \leq (\mu_{ij})^2$, in which the corresponding likelihoods are 0, 0.5, and 1, respectively. Consider the following numerical examples of the likelihood function $lik(p_{ij} \geq p_{\kappa j})$ initiated by Liang et al. [26]. Let $p_{ij} = (0.2,\ 0.8)$ and $p_{\kappa j} = (0.4,\ 0.6)$, which satisfies the condition of $(\mu_{ij})^2 \leq (\mu_{\kappa j})^2 \leq 1-(v_{ij})^2 \leq 1-(v_{\kappa j})^2$. Therefore,

$$lik(p_{ij} \geq p_{\kappa j}) = \frac{\left(1-(0.4)^2-(0.8)^2\right)^2}{2\left(1-(0.2)^2-(0.8)^2\right)\left(1-(0.4)^2-(0.6)^2\right)} = 0.1302.$$

When p_{ij} = (0.2, 0.6) and $p_{\kappa j}$ = (0.4, 0.8), the third scenario is applicable, and the result is given below:

$$lik(p_{ij} \geq p_{\kappa j}) = \frac{1-2(0.6)^2-(0.4)^2+(0.8)^2}{2\left(1-(0.2)^2-(0.6)^2\right)} = 0.6333.$$

Moreover, when p_{ij} = (0.4, 0.8) and $p_{\kappa j}$ = (0.2, 0.6), the fifth condition holds and $lik(p_{ij} \geq p_{\kappa j}) = 0.3667$. Finally, when p_{ij} = (0.4, 0.6) and $p_{\kappa j}$ = (0.2, 0.8), the sixth condition is satisfied; thus, $lik(p_{ij} \geq p_{\kappa j}) = 0.8698$.

Definition 6 Let $x_i = (\mu_{ij})^2 + y(\tau_{ij})^2$ and $x_\kappa = (\mu_{\kappa j})^2 + z(\tau_{\kappa j})^2$, in which y and z are two independent random variables in [0, 1]. The parametric likelihood measure $L^{(\alpha,\beta)}$ is defined as the probability measure $\Pr(x_i \geq x_\kappa)$ when y and z follow independent standard beta distributions. The parametric likelihood function $L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j})$ for $p_{ij} \in P_i$ and $p_{\kappa j} \in P_\kappa$ is represented as:

$$\begin{aligned} L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) &= \Pr(x_i \geq x_\kappa) \\ &= \iint_{\left\{\begin{array}{c} (\mu_{\kappa j})^2+z(\tau_{\kappa j})^2 \leq (\mu_{ij})^2+y(\tau_{\kappa j})^2 \\ 0 \leq y \leq 1 \\ 0 \leq z \leq 1 \end{array}\right\}} \\ &\left(\frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1} z^{\alpha-1}(1-z)^{\beta-1}\right) dydz. \end{aligned} \tag{13}$$

This paper uses $L^{(\alpha,\beta)}$ to represent the parametric likelihood measure, in contrast to the lik function used in Liang et al. [26] and the lik^β function in Tsao and Chen [37], to emphasize that the proposed parametric likelihood function value will depend on the parameters α and β. However, in the proposed beta distribution-based approach, two issues need to be solved before obtaining the closed-form solution for the likelihood function of $p_{ij} \geq p_{\kappa j}$. The first issue is determining the domain of the double integration in Eq. (13). The second issue is solving the double integration problem of beta distribution functions. Lemma 1 studies the first issue and sets explicit bounds of the double

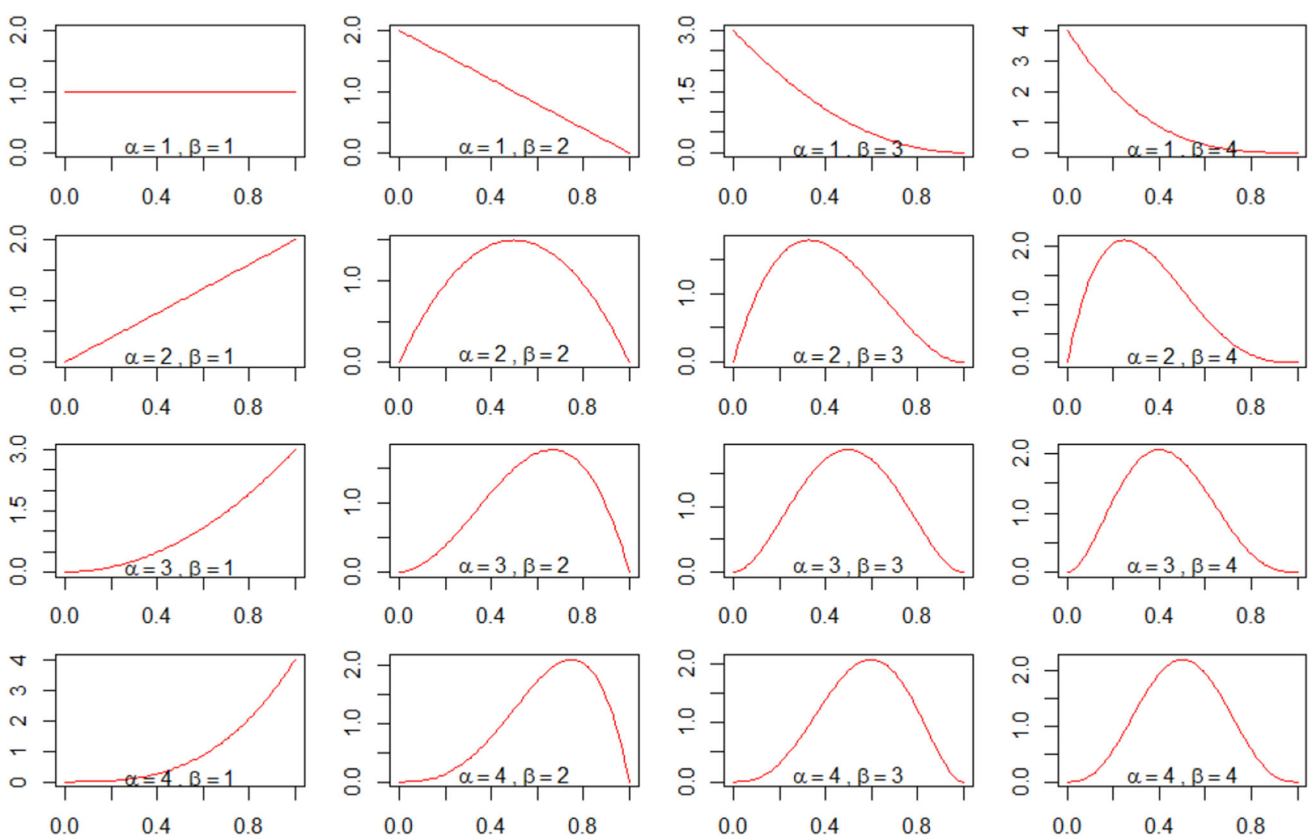
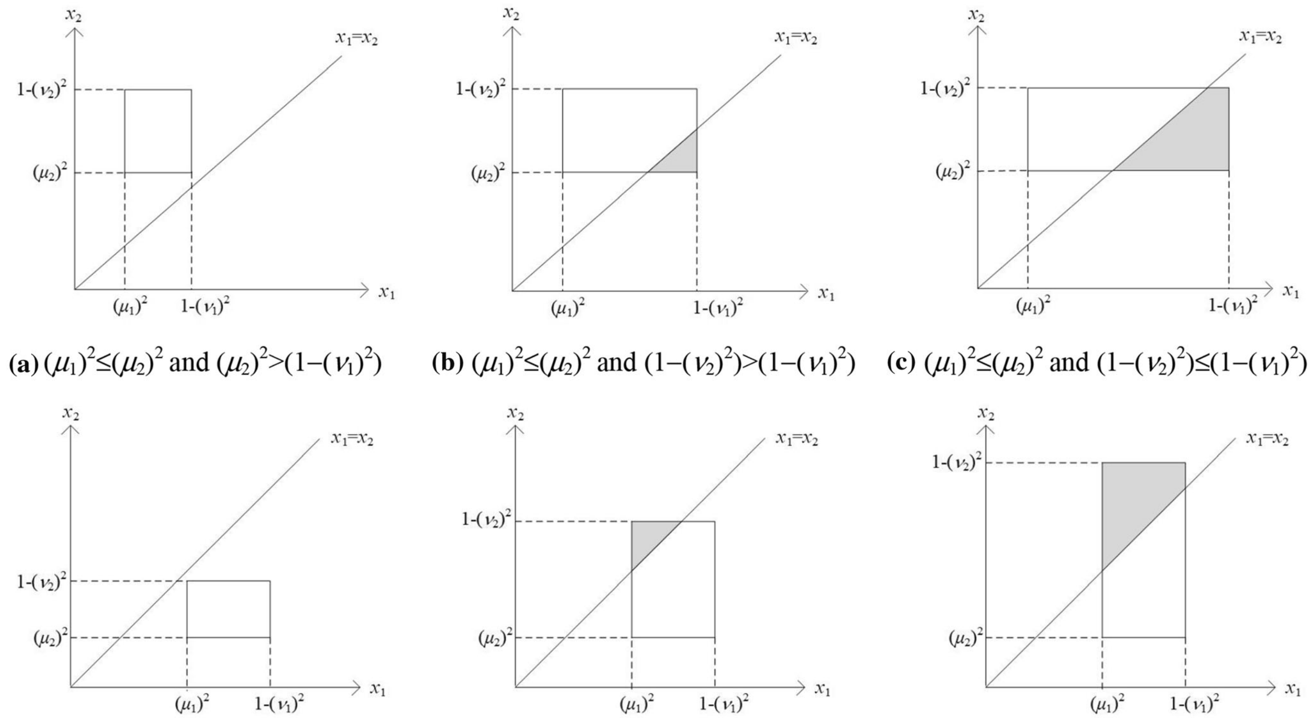


Fig. 1 Symmetric and asymmetric shapes of the beta density function



(a) $(\mu_1)^2 \le (\mu_2)^2$ and $(\mu_2)^2 > (1-(\nu_1)^2)$ **(b)** $(\mu_1)^2 \le (\mu_2)^2$ and $(1-(\nu_2)^2) > (1-(\nu_1)^2)$ **(c)** $(\mu_1)^2 \le (\mu_2)^2$ and $(1-(\nu_2)^2) \le (1-(\nu_1)^2)$

(d) $(\mu_1)^2 > (\mu_2)^2$ and $(\mu_1)^2 > (1-(\nu_2)^2)$ **(e)** $(\mu_1)^2 > (\mu_2)^2$ and $(1-(\nu_1)^2) > (1-(\nu_2)^2)$ **(f)** $(\mu_1)^2 > (\mu_2)^2$ and $(1-(\nu_1)^2) \le (1-(\nu_2)^2)$

Fig. 2 Six integration domains when solving the likelihood function of $p_1 \geq p_2$

integration. Figure 2 depicts the possible integration domains when solving the parametric likelihood function of $p_{ij} \geq p_{\kappa j}$ to facilitate the proof.

Lemma 1 *The analytical solution of* $L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j})$ *is determined as follows*:

$$L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) = \begin{cases} \int_0^{k_3} \int_{k_1+k_2 z}^{1} \left(\frac{1}{B(\alpha,\beta)^2} y^{\alpha-1}(1-y)^{\beta-1} z^{\alpha-1}(1-z)^{\beta-1} \right) dydz \, if \, (\mu_{ij})^2 \leq (\mu_{\kappa j})^2, \\ 1 - \int_0^{k_6} \int_{k_4+k_5 y}^{1} \left(\frac{1}{B(\alpha,\beta)^2} y^{\alpha-1}(1-y)^{\beta-1} z^{\alpha-1}(1-z)^{\beta-1} \right) dzdy \, if \, (\mu_{ij})^2 (\mu_{\kappa j})^2. \end{cases} \tag{14}$$

where $k_1 = ((\mu_{\kappa j})^2 - (\mu_{ij})^2)/(1-(\mu_{ij})^2-(v_{ij})^2)$, $k_2 = (1-(\mu_{\kappa j})^2-(v_{\kappa j})^2)/(1-(\mu_{ij})^2-(v_{ij})^2)$, $k_3 = \min\{1, [(1-(\mu_{\kappa j})^2-(v_{ij})^2)/(1-(\mu_{\kappa j})^2-(v_{\kappa j})^2)]^+\}$, $k_4 = ((\mu_{ij})^2-(\mu_{\kappa j})^2)/(1-(\mu_{\kappa j})^2-(v_{\kappa j})^2)$, $k_5 = (1-(\mu_{ij})^2-(v_{ij})^2)/(1-(\mu_{\kappa j})^2-(v_{\kappa j})^2)$, $k_6 = \min\{1, [(1-(\mu_{ij})^2-(v_{\kappa j})^2)/\ (1-(\mu_{ij})^2-(v_{ij})^2)]^+\}$, and $[a]^+$ is defined as $\max\{0, a\}$, where a is a real number.

Proof Consider the case when $(\mu_{ij})^2 \leq (\mu_{\kappa j})^2$. The first two inequalities of the domain in Eq. (13) imply $(k_1 + k_2 z) \leq y \leq 1$, where $k_1 = ((\mu_{\kappa j})^2 - (\mu_{ij})^2)/(1 - (\mu_{ij})^2 - (v_{ij})^2)$ and $k_2 = (1-(\mu_{\kappa j})^2-(v_{\kappa j})^2)/(1-(\mu_{ij})^2-(v_{ij})^2)$. The domain of z has three possible ranges, which depend on the values of $(v_{ij})^2$, $(\mu_{\kappa j})^2$, and $(v_{\kappa j})^2$. First, the integration domain is null when $(\mu_{\kappa j})^2 > (1-(v_{ij})^2)$, and it can be determined by $0 \leq z \leq 0$ (or equivalently, when $z = 0$) (see Fig. 2a). Second, when $(1-(v_{\kappa j})^2) > (1-(v_{ij})^2)$, the upper bound of z is limited to $(1-(\mu_{\kappa j})^2-(v_{ij})^2)/(1-(\mu_{\kappa j})^2-(v_{\kappa j})^2)$ (see Fig. 2b). Finally, when $(1-(v_{\kappa j})^2) \leq (1-(v_{ij})^2)$, the domain of z is $0 \leq z \leq 1$ (see Fig. 2c). These three possible ranges of z can be simply represented as $0 \leq z \leq k_3$, where $k_3 = \min\{1, [(1-(\mu_{\kappa j})^2-(v_{ij})^2)/(1-(\mu_{\kappa j})^2-(v_{\kappa j})^2)]^+\}$. Therefore, when $(\mu_{ij})^2 \leq (\mu_{\kappa j})^2$, Eq. (13) becomes:

$$L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) = \Pr(x_i \geq x_\kappa) = \int_0^{k_3} \int_{k_1+k_2 z}^{1} \left(\frac{1}{B(\alpha,\beta)^2} y^{\alpha-1} (1-y)^{\beta-1} z^{\alpha-1} (1-z)^{\beta-1} \right) dydz.$$

Next, the proof considers the case when $(\mu_{ij})^2 > (\mu_{\kappa j})^2$. Equation (13) is rewritten as follows:

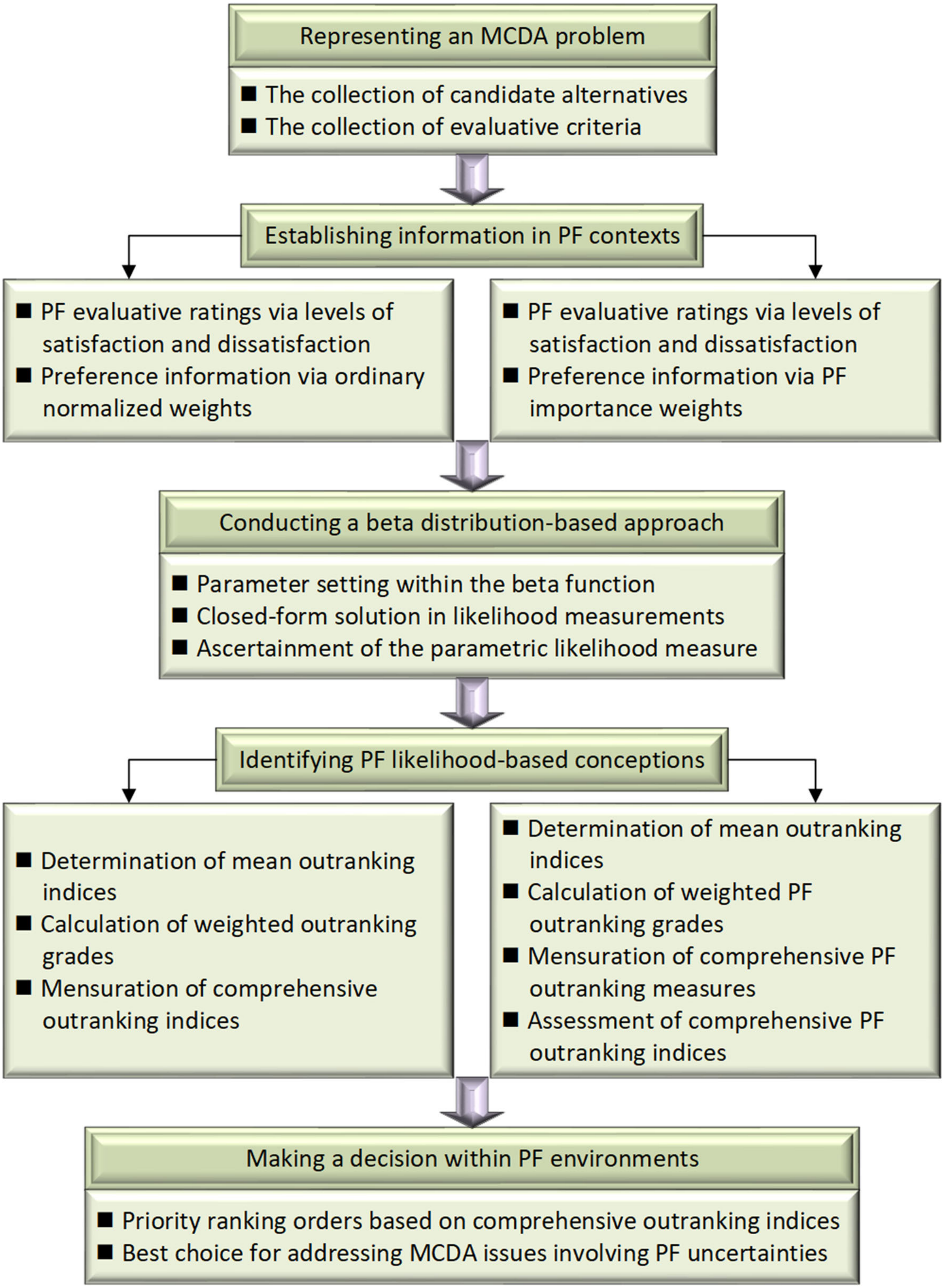


Fig. 3 Framework of the proposed methodology

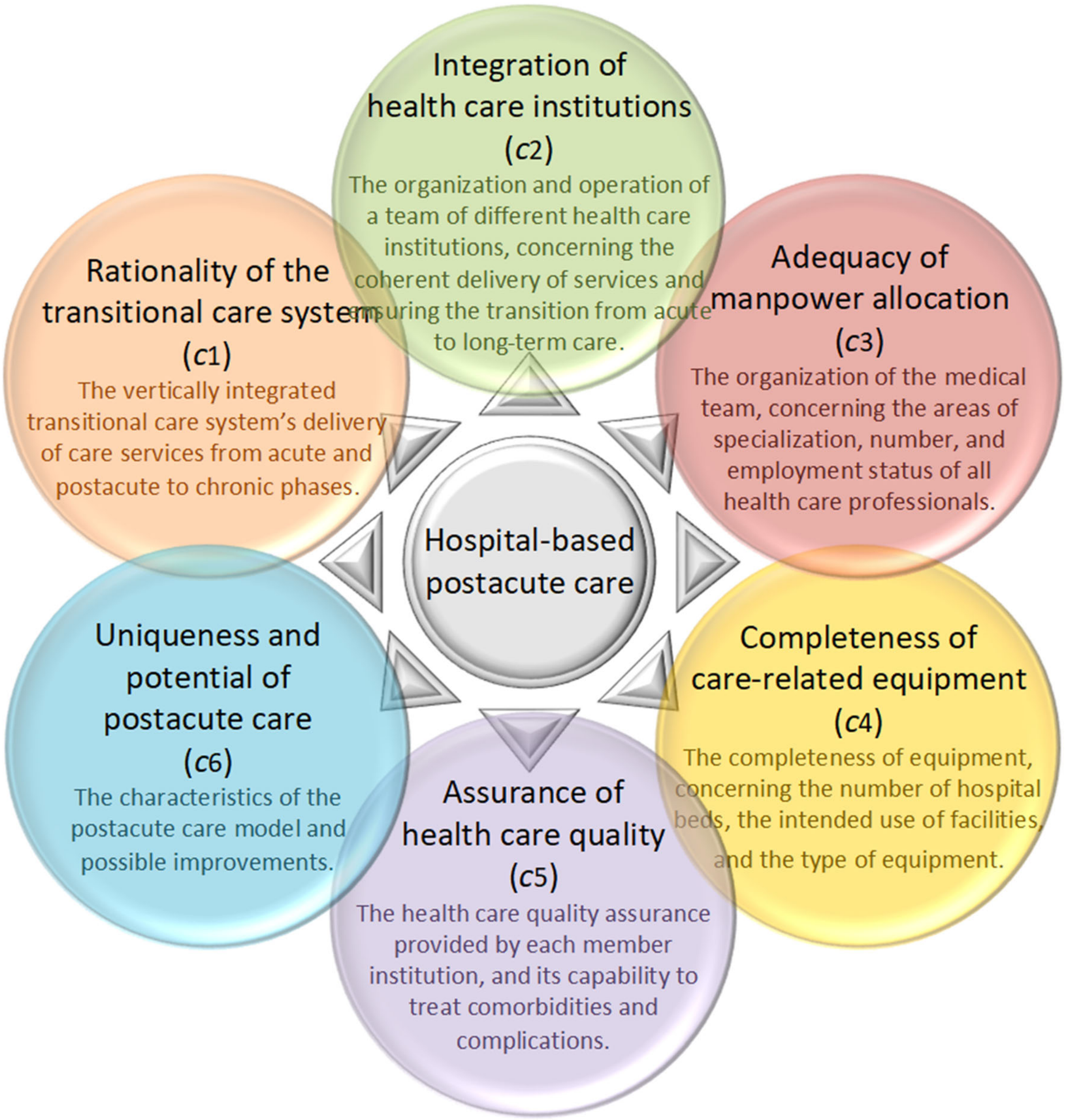


Fig. 4 Profile of the pilot hospital selection problem for postacute care

$$L^{(\alpha,\beta)}(p_{ij} \ge p_{\kappa j}) = \Pr(x_i \ge x_\kappa) = 1 - \Pr(x_\kappa > x_i) = 1 - \iint_{\left\{\begin{array}{c} (\mu_{\kappa j})^2 + z(\tau_{\kappa j})^2 > (\mu_{ij})^2 + y(\tau_{ij})^2 \\ 0 \le y \le 1 \\ 0 \le z \le 1 \end{array}\right\}} \left(\frac{1}{B(\alpha,\beta)^2} y^{\alpha-1}(1-y)^{\beta-1} z^{\alpha-1}(1-z)^{\beta-1}\right) dz\,dy. \tag{15}$$

Double integration is easily solved by integrating the z variable. The first and third inequalities in the integration domain imply $(k_4 + k_5 y) \le z \le 1$, in which $k_4 = ((\mu_{ij})^2 - (\mu_{\kappa j})^2)/(1 - (\mu_{\kappa j})^2 - (\nu_{\kappa j})^2)$ and $k_5 = (1 - (\mu_{ij})^2 - (\nu_{ij})^2)/(1 - (\mu_{\kappa j})^2 - (\nu_{\kappa j})^2)$. The domain of y then has three possible ranges and depends on the values of $(\mu_{ij})^2$, $(\nu_{ij})^2$, and $(\nu_{\kappa j})^2$. First, the integration domain is null when $(\mu_{ij})^2 > (1 - (\nu_{\kappa j})^2)$, and it can be determined by $0 \le y \le 0$ (or equivalently, when $y = 0$) (see Fig. 2(d)). Second, when $(1 - (\nu_{ij})^2) > (1 - (\nu_{\kappa j})^2)$, y has an upper bound of $(1 - (\mu_{ij})^2 - (\nu_{\kappa j})^2)/(1 - (\mu_{ij})^2 - (\nu_{ij})^2)$ (see Fig. 2e). Finally, when $(1 - (\nu_{ij})^2) \le (1 - (\nu_{\kappa j})^2)$, y has a full range, i.e., $0 \le y \le 1$ (see Fig. 2f). This paper aggregates the three possible ranges of y as $0 \le y \le k_6$, where $k_6 = \min\{1, [(1 - (\mu_{ij})^2 - (\nu_{\kappa j})^2)/(1 - (\mu_{ij})^2 - (\nu_{ij})^2)]^+\}$. Therefore, when $(\mu_{ij})^2 > (\mu_{\kappa j})^2$, Eq. (13) becomes:

$$L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) = \Pr(x_i \geq x_\kappa) = 1 - \Pr(x_\kappa > x_i)$$
$$= 1 - \int_0^{k_6} \int_{k_4+k_5 y}^{1} \left(\frac{1}{B(\alpha,\beta)^2} y^{\alpha-1}(1-y)^{\beta-1} z^{\alpha-1}(1-z)^{\beta-1} \right) dzdy.$$

This completes the proof.

This paper now addresses the second issue. Note that the beta distribution function is in the form of a polynomial. Therefore, there is generally no closed-form solution to the double integration problem. However, a closed-form solution may result when the polynomial orders are integers. Specifically, the double integration problem can be solved by applying a binomial expansion to the polynomial. Theorem 1 provides the closed-form solution of $L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j})$ when α and β are integers.

Theorem 1 *Let two parameters α and β be positive integers within the beta function $B(\alpha, \beta)$. When two independent random variables y and z $(y, z \in [0, 1])$ follow the independent standard beta distributions, the parametric likelihood function $L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j})$ is obtained in the following manner:*

$$L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) = \begin{cases} \dfrac{1}{B(\alpha,\beta)^2}\left(\displaystyle\sum_{k=0}^{\beta-1}\sum_{l=0}^{\beta-1}\left[C_k^{\beta-1} C_l^{\beta-1} \frac{(-1)^{k+l}}{l+\alpha}\frac{(k_3)^{l+\alpha}}{k+\alpha}\right] \right. \\ \left. - \displaystyle\sum_{k=0}^{\beta-1}\sum_{l=0}^{\beta-1}\sum_{b=0}^{k+\alpha}\left[C_k^{\beta-1} C_l^{\beta-1} C_b^{k+\alpha}\frac{(-1)^{k+l}}{k+\alpha}(k_1)^{k+\alpha-b}(k_2)^b \frac{(k_3)^{l+\alpha+b}}{l+\alpha+b}\right]\right) & \text{if } (\mu_{ij})^2 \leq (\mu_{\kappa j})^2, \\ 1 - \dfrac{1}{B(\alpha,\beta)^2}\left(\displaystyle\sum_{k=0}^{\beta-1}\sum_{l=0}^{\beta-1}\left[C_k^{\beta-1} C_l^{\beta-1} \frac{(-1)^{k+l}}{l+\alpha}\frac{(k_6)^{k+\alpha}}{k+\alpha}\right] \right. \\ \left. - \displaystyle\sum_{k=0}^{\beta-1}\sum_{l=0}^{\beta-1}\sum_{b=0}^{l+\alpha}\left[C_k^{\beta-1} C_l^{\beta-1} C_b^{l+\alpha}\frac{(-1)^{k+l}}{l+\alpha}(k_4)^{l+\alpha-b}(k_5)^b \frac{(k_6)^{k+\alpha+b}}{k+\alpha+b}\right]\right) & \text{if } (\mu_{ij})^2 > (\mu_{\kappa j})^2, \end{cases} \tag{16}$$

where k_1, k_2, k_3, k_4, k_5, and k_6 are defined in Lemma 1.

Proof In the case of $(\mu_{ij})^2 \leq (\mu_{\kappa j})^2$, Lemma 1 gives:

$$L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j})$$
$$= \int_0^{k_3} \int_{k_1+k_2 z}^{1} \left(\frac{1}{B(\alpha,\beta)^2} y^{\alpha-1}(1-y)^{\beta-1} z^{\alpha-1}(1-z)^{\beta-1} \right) dydz.$$

Because β is an integer, a binomial expansion can be applied to $(1-y)^{\beta-1}$ to gain a polynomial form of y. Then, $L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j})$ is rewritten as:

$$L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) = \frac{1}{B(\alpha,\beta)^2} \int_0^{k_3} z^{\alpha-1}(1-z)^{\beta-1}$$
$$\left(\int_{k_1+k_2 z}^{1} \sum_{k=0}^{\beta-1} \left[C_k^{\beta-1}(-1)^k y^{k+\alpha-1} \right] dy \right) dz$$
$$= \frac{1}{B(\alpha,\beta)^2} \int_0^{k_3} z^{\alpha-1}(1-z)^{\beta-1} \sum_{k=0}^{\beta-1}$$
$$\left[C_k^{\beta-1} \frac{(-1)^k}{k+\alpha}\left(1-(k_1+k_2 z)^{k+\alpha}\right)\right] dz.$$

Applying a binomial expansion to both $(1-z)^{\beta-1}$ and $(k_1+k_2 z)^{k+\alpha}$ yields the following:

$$L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j})$$
$$= \frac{1}{B(\alpha,\beta)^2} \int_0^{k_3} \sum_{l=0}^{\beta-1} \left[C_l^{\beta-1}(-1)^l z^{l+\alpha-1} \right]$$
$$\sum_{k=0}^{\beta-1} \left[C_k^{\beta-1} \frac{(-1)^k}{k+\alpha} \left(1 - \sum_{b=0}^{k+\alpha} \left[C_b^{k+\alpha}(k_1)^{k+\alpha-b}(k_2)^b z^b \right] \right) \right] dz$$
$$= \frac{1}{B(\alpha,\beta)^2} \sum_{k=0}^{\beta-1}\sum_{l=0}^{\beta-1} \left[C_k^{\beta-1} C_l^{\beta-1} \frac{(-1)^{l+k}}{k+\alpha} \left(\int_0^{k_3} z^{l+\alpha-1} dz \right.\right.$$
$$\left.\left. - \sum_{b=0}^{k+\alpha} \left[C_b^{k+\alpha}(k_1)^{k+\alpha-b}(k_2)^b \int_0^{k_3} z^{l+\alpha+b-1} dz \right] \right) \right]$$
$$= \frac{1}{B(\alpha,\beta)^2} \left(\sum_{k=0}^{\beta-1}\sum_{l=0}^{\beta-1} \left[C_k^{\beta-1} C_l^{\beta-1} \frac{(-1)^{k+l}}{l+\alpha}\frac{(k_3)^{l+\alpha}}{k+\alpha} \right] \right.$$
$$\left. - \sum_{k=0}^{\beta-1}\sum_{l=0}^{\beta-1}\sum_{b=0}^{k+\alpha} \left[C_k^{\beta-1} C_l^{\beta-1} C_b^{k+\alpha} \frac{(-1)^{k+l}}{k+\alpha}(k_1)^{k+\alpha-b}(k_2)^b \frac{(k_3)^{l+\alpha+b}}{l+\alpha+b} \right] \right).$$

Next, in the case of $(\mu_{ij})^2 > (\mu_{\kappa j})^2$, Eq. (14) gives: $L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) = 1 - \int_0^{k_6} \int_{k_4+k_5 y}^1 \left(\frac{1}{B(\alpha,\beta)^2} y^{\alpha-1}(1-y)^{\beta-1} z^{\alpha-1}(1-z)^{\beta-1}\right) dzdy$. Similarly, when a binomial expansion is applied to $(1-z)^{\beta-1}$, the parametric likelihood function becomes:

$$
\begin{aligned}
L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) &= 1 - \frac{1}{B(\alpha,\beta)^2} \int_0^{k_6} y^{\alpha-1}(1-y)^{\beta-1} \left(\int_{k_4+k_5 y}^1 \sum_{l=0}^{\beta-1} \left[C_l^{\beta-1} (-1)^l z^{l+\alpha-1} \right] dz \right) dy \\
&= 1 - \frac{1}{B(\alpha,\beta)^2} \int_0^{k_6} y^{\alpha-1}(1-y)^{\beta-1} \sum_{l=0}^{\beta-1} \left[C_l^{\beta-1} \frac{(-1)^l}{l+\alpha} \left(1 - (k_4 + k_5 y)^{l+\alpha} \right) \right] dy.
\end{aligned}
$$

After applying a binomial expansion to both $(1-y)^{\beta-1}$ and $(k_4 + k_5 y)^{l+\alpha}$, $L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j})$ becomes:

$$
\begin{aligned}
&L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) \\
&= 1 - \frac{1}{B(\alpha,\beta)^2} \int_0^{k_6} \sum_{k=0}^{\beta-1} \left[C_k^{\beta-1} (-1)^k y^{k+\alpha-1} \right] \\
&\quad \sum_{l=0}^{\beta-1} \left[C_l^{\beta-1} \frac{(-1)^l}{l+\alpha} \left(1 - \sum_{b=0}^{l+\alpha} \left[C_b^{l+\alpha} (k_4)^{l+\alpha-b} (k_5)^b y^b \right] \right) \right] dy \\
&= 1 - \frac{1}{B(\alpha,\beta)^2} \sum_{k=0}^{\beta-1} \sum_{l=0}^{\beta-1} \left[C_k^{\beta-1} C_l^{\beta-1} \frac{(-1)^{k+l}}{l+\alpha} \left(\int_0^{k_6} y^{k+\alpha-1} dy \right.\right. \\
&\quad \left.\left. - \sum_{b=0}^{l+\alpha} \left[C_b^{l+\alpha} (k_4)^{l+\alpha-b} (k_5)^b \int_0^{k_6} y^{k+\alpha+b-1} dy \right] \right) \right] \\
&= 1 - \frac{1}{B(\alpha,\beta)^2} \left(\sum_{k=0}^{\beta-1} \sum_{l=0}^{\beta-1} \left[C_k^{\beta-1} C_l^{\beta-1} \frac{(-1)^{k+l}}{l+\alpha} \frac{(k_6)^{k+\alpha}}{k+\alpha} \right] \right. \\
&\quad \left. - \sum_{k=0}^{\beta-1} \sum_{l=0}^{\beta-1} \sum_{b=0}^{l+\alpha} \left[C_k^{\beta-1} C_l^{\beta-1} C_b^{i+\alpha} \frac{(-1)^{k+l}}{l+\alpha} (k_4)^{l+\alpha-b} (k_5)^b \frac{(k_6)^{k+\alpha+b}}{k+\alpha+b} \right] \right).
\end{aligned}
$$

This completes the proof.

Two numerical examples are presented to demonstrate how Eq. (16) works. Consider $\alpha = 4$ and $\beta = 2$ in both cases. For the scenario of $(\mu_1)^2 \leq (\mu_2)^2$, p_{ij} = (0.2, 0.8) and $p_{\kappa j}$ = (0.4, 0.6). Then, $k_1 = ((0.4)^2 - (0.2)^2)/(1-(0.2)^2-(0.8)^2) = 0.375$, $k_2 = (1-(0.4)^2 - (0.6)^2)/(1-(0.2)^2-(0.8)^2) = 1.5$, and $k_3 = \min\{1, [(1-(0.4)^2-(0.8)^2)/(1-(0.4)^2-(0.6)^2)]^+\}$ $= 0.4167$. The likelihood function is evaluated as follows:

$$
\begin{aligned}
L^{(4,2)}(p_{ij} \geq p_{\kappa j}) &= \frac{1}{B(4,2)^2} \left(\sum_{k=0}^{1} \sum_{l=0}^{1} \left[C_k^1 C_l^1 \frac{(-1)^{k+l}}{l+4} \frac{(0.4167)^{l+4}}{k+4} \right] \right. \\
&\quad \left. - \sum_{k=0}^{1} \sum_{l=0}^{1} \sum_{b=0}^{k+4} \left[C_k^1 C_l^1 C_b^{k+4} \frac{(-1)^{k+l}}{k+4} (0.375)^{k+4-b} (1.5)^b \frac{(0.4167)^{l+4+b}}{l+4+b} \right] \right) \\
&= 0.0169.
\end{aligned}
$$

For the scenario of $(\mu_{ij})^2 > (\mu_{\kappa j})^2$, p_{ij} = (0.4, 0.6) and $p_{\kappa j}$ = (0.2, 0.8). Then, $k_4 = ((0.4)^2 - (0.2)^2)/(1-(0.2)^2-(0.8)^2) = 0.375$, $k_5 = (1-(0.4)^2-(0.6)^2)/(1-(0.2)^2-(0.8)^2) = 1.5$, and $k_6 = \min\{1, [(1-(0.4)^2-(0.8)^2)/(1-(0.4)^2 - (0.6)^2)]^+\} = 0.4167$. The likelihood function is evaluated as shown:

$$
\begin{aligned}
L^{(4,2)}(p_{ij} \geq p_{\kappa j}) &= 1 - \frac{1}{B(4,2)^2} \left(\sum_{k=0}^{1} \sum_{l=0}^{1} \left[C_k^1 C_l^1 \frac{(-1)^{k+l}}{l+4} \frac{(0.4167)^{k+4}}{k+4} \right] \right. \\
&\quad \left. - \sum_{k=0}^{1} \sum_{l=0}^{1} \sum_{b=0}^{l+4} \left[C_k^1 C_l^1 C_b^{l+4} \frac{(-1)^{k+l}}{l+4} (0.375)^{l+4-b} (1.5)^b \frac{(0.4167)^{k+4+b}}{k+4+b} \right] \right) \\
&= 0.9831.
\end{aligned}
$$

Note that p_{ij} and $p_{\kappa j}$ in the two examples can swap with one another. In other words, the two examples evaluate $L^{(4,2)}(p_{ij} \geq p_{\kappa j})$ and $L^{(4,2)}(p_{\kappa j} \geq p_{ij})$. Furthermore, the summation of $L^{(4,2)}(p_{ij} \geq p_{\kappa j})$ and $L^{(4,2)}(p_{\kappa j} \geq p_{ij})$ is 0.0169 + 0.9831 = 1. This is not a coincidence but a complementary property that is commonly exhibited in a parametric likelihood measure. Theorem 2 demonstrates the properties of the proposed parametric likelihood measure $L^{(\alpha,\beta)}$. First, it is shown in Corollary 1 that the proposed parametric likelihood function is an extension of Liang et al.'s likelihood function.

Corollary 1 *The parametric likelihood function $L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j})$ in Theorem 1 reduces to the likelihood function introduced by Liang et al. in Eq. (12) when $\alpha = \beta = 1$. That is, $L^{(1,1)}(p_{ij} \geq p_{\kappa j}) = lik(p_{ij} \geq p_{\kappa j})$.*

Proof Consider the case of $(\mu_{ij})^2 \leq (\mu_{\kappa j})^2$. When $\alpha = \beta = 1$, the parametric likelihood function $L^{(1,1)}$ of $p_{ij} \geq p_{\kappa j}$ is obtained as follows:

$$L^{(1,1)}(p_{ij} \geq p_{\kappa j}) = k_3 - \sum_{b=0}^{1}\left[C_b^1 (k_1)^{1-b}(k_2)^b \frac{(k_3)^{1+b}}{1+b}\right] = k_3\left(1 - k_1 - \frac{k_2 k_3}{2}\right).$$

Note that k_3 can be acquired through the following three formulas depending on the relative magnitudes of μ_{ij}, $\mu_{\kappa j}$, v_{ij}, and $v_{\kappa j}$:

$$k_3 = \begin{cases} 0 & \mathit{if}\ 1-(\mu_{\kappa j})^2-(v_{ij})^2<0, \\ \dfrac{1-(\mu_{\kappa j})^2+(v_{ij})^2}{1-(\mu_{\kappa j})^2-(v_{\kappa j})^2} & \mathit{if}\ 0 \leq 1-(\mu_{\kappa j})^2-(v_{ij})^2 \leq 1-(\mu_{\kappa j})^2-(v_{\kappa j})^2, \\ 1 & \mathit{if}\ 1-(\mu_{\kappa j})^2-(v_{\kappa j})^2<1-(\mu_{\kappa j})^2-(v_{ij})^2. \end{cases}$$

Accordingly, the parametric likelihood function $L^{(1,1)}$ can be determined under the three scenarios:

$$L^{(1,1)}(p_{ij} \geq p_{\kappa j}) = k_3\left(1 - k_1 - \frac{k_2 k_3}{2}\right)$$
$$= \begin{cases} 0 & \mathit{if}\ 1-(\mu_{\kappa j})^2-(v_{ij})^2<0, \\ \dfrac{1-(\mu_{\kappa j})^2-(v_{ij})^2}{1-(\mu_{\kappa j})^2-(v_{\kappa j})^2}\left(1-k_1-\dfrac{k_2}{2}\left(\dfrac{1-(\mu_{\kappa j})^2-(v_{ij})^2}{1-(\mu_{\kappa j})^2-(v_{\kappa j})^2}\right)\right) & \mathit{if}\ 0 \leq 1-(\mu_{\kappa j})^2-(v_{ij})^2 \leq 1-(\mu_{\kappa j})^2-(v_{\kappa j})^2, \\ 1-k_1-\dfrac{k_2}{2} & \mathit{if}\ 0 \leq 1-(\mu_{\kappa j})^2-(v_{\kappa j})^2<1-(\mu_{\kappa j})^2-(v_{ij})^2. \end{cases}$$

After plugging k_1 and k_2 into the function and attaching the condition $(\mu_{ij})^2 \leq (\mu_{\kappa j})^2$ into the formula, the following can be determined:

$$L^{(1,1)}(p_{ij} \geq p_{\kappa j}) = \begin{cases} 0 & \mathit{if}\ 1-(v_{ij})^2 \leq (\mu_{\kappa j})^2, \\ \dfrac{\left(1-(\mu_{\kappa j})^2-(v_{ij})^2\right)^2}{2\left(1-(\mu_{ij})^2-(v_{ij})^2\right)\left(1-(\mu_{\kappa j})^2-(\mu_{\kappa j})^2\right)} & \mathit{if}\ (\mu_{ij})^2 \leq (\mu_{\kappa j})^2 \leq 1-(v_{ij})^2 \leq 1-(v_{\kappa j})^2, \\ \dfrac{1-2(v_{ij})^2-(\mu_{\kappa j})^2+(v_{\kappa j})^2}{2\left(1-(\mu_{ij})^2-(v_{ij})^2\right)} & \mathit{if}\ (\mu_{ij})^2 \leq (\mu_{\kappa j})^2 \leq 1-(v_{\kappa j})^2<1-(v_{ij})^2. \end{cases}$$

Next, consider the case of $(\mu_{ij})^2 > (\mu_{\kappa j})^2$. When $\alpha = \beta = 1$, the parametric likelihood function $L^{(1,1)}$ of $p_{ij} \geq p_{\kappa j}$ is determined as follows:

$$L^{(1,1)}(p_{ij} \geq p_{\kappa j}) = 1 - k_6\left(1 - k_4 - \frac{k_5 k_6}{2}\right).$$

Then, k_6 can be derived through the following three formulas depending on the relative magnitudes of μ_{ij}, $\mu_{\kappa j}$, v_{ij}, and $v_{\kappa j}$:

$$k_6 = \begin{cases} 0 & \mathit{if}\ 1-(\mu_{ij})^2-(v_{\kappa j})^2<0, \\ \dfrac{1-(\mu_{ij})^2+(v_{\kappa j})^2}{1-(\mu_{ij})^2-(v_{ij})^2} & \mathit{if}\ 0 \leq 1-(\mu_{ij})^2-(v_{\kappa j})^2 \leq 1-(\mu_{ij})^2-(v_{ij})^2, \\ 1 & \mathit{if}\ 1-(\mu_{ij})^2-(v_{ij})^2<1-(\mu_{ij})^2-(v_{\kappa j})^2. \end{cases}$$

Therefore, the parametric likelihood function $L^{(1,1)}$ can be acquired under the three scenarios:

$$L^{(1,1)}(p_{ij} \geq p_{\kappa j}) = 1 - k_6\left(1 - k_4 - \frac{k_5 k_6}{2}\right)$$

$$= \begin{cases} 1 & \textit{if } 1 - (\mu_{ij})^2 - (v_{\kappa j})^2 < 0, \\ 1 - \left(\frac{1 - (\mu_{ij})^2 + (v_{\kappa j})^2}{1 - (\mu_{ij})^2 - (v_{ij})^2}\right)\left(1 - k_4 - \frac{k_5}{2}\left(\frac{1 - (\mu_{ij})^2 + (v_{\kappa j})^2}{1 - (\mu_{ij})^2 - (v_{ij})^2}\right)\right) & \textit{if } 0 \leq 1 - (\mu_{ij})^2 - (v_{\kappa j})^2 \leq 1 - (\mu_{ij})^2 - (v_{ij})^2, \\ k_4 + \frac{k_5}{2} & \textit{if } 0 \leq 1 - (\mu_{ij})^2 - (v_{ij})^2 < 1 - (\mu_{ij})^2 - (v_{\kappa j})^2. \end{cases}$$

After plugging k_4 and k_5 into the function and attaching the condition $(\mu_{ij})^2 > (\mu_{\kappa j})^2$ into the formula, the following can be determined:

$$L^{(1,1)}(p_{ij} \geq p_{\kappa j}) = \begin{cases} 1 & \textit{if } 1 - (v_{\kappa j})^2 < (\mu_{ij})^2, \\ 1 - \frac{\left(1 - (\mu_{ij})^2 - (v_{\kappa j})^2\right)^2}{2\left(1 - (\mu_{ij})^2 - (v_{ij})^2\right)\left(1 - (\mu_{\kappa j})^2 - (v_{\kappa j})^2\right)} & \textit{if } (\mu_{\kappa j})^2 < (\mu_{ij})^2 \leq 1 - (v_{\kappa j})^2 \leq 1 - (v_{ij})^2, \\ \frac{1 + (\mu_{ij})^2 - (v_{ij})^2 - 2(\mu_{\kappa j})^2}{2\left(1 - (\mu_{\kappa j})^2 - (v_{\kappa j})^2\right)} & \textit{if } (\mu_{\kappa j})^2 < (\mu_{ij})^2 \leq 1 - (v_{ij})^2 < 1 - (v_{\kappa j})^2. \end{cases}$$

Combining the obtained results in the two cases (i.e., $(\mu_{ij})^2 \leq (\mu_{\kappa j})^2$ and $(\mu_{ij})^2 > (\mu_{\kappa j})^2$), the following is derived:

$$L^{(1,1)}(p_{ij} \geq p_{\kappa j}) = \begin{cases} 0 & \textit{if } 1 - (v_{ij})^2 \leq (\mu_{\kappa j})^2, \\ \frac{\left(1 - (\mu_{\kappa j})^2 - (v_{ij})^2\right)^2}{2\left(1 - (\mu_{ij})^2 - (v_{ij})^2\right)\left(1 - (\mu_{\kappa j})^2 - (v_{\kappa j})^2\right)} & \textit{if } (\mu_{ij})^2 \leq (\mu_{\kappa j})^2 \leq 1 - (v_{ij})^2 \leq 1 - (v_{\kappa j})^2, \\ \frac{1 - 2(v_{ij})^2 - (\mu_{\kappa j})^2 + (v_{\kappa j})^2}{2\left(1 - (\mu_{ij})^2 - (v_{ij})^2\right)} & \textit{if } (\mu_{ij})^2 \leq (\mu_{\kappa j})^2 \leq 1 - (v_{\kappa j})^2 < 1 - (v_{ij})^2, \\ 1 & \textit{if } 1 - (v_{\kappa j})^2 < (\mu_{ij})^2, \\ 1 - \frac{\left(1 - (\mu_{ij})^2 - (v_{\kappa j})^2\right)^2}{2\left(1 - (\mu_{ij})^2 - (v_{ij})^2\right)\left(1 - (\mu_{\kappa j})^2 - (v_{\kappa j})^2\right)} & \textit{if } (\mu_{\kappa j})^2 < (\mu_{ij})^2 \leq 1 - (v_{\kappa j})^2 \leq 1 - (v_{ij})^2, \\ \frac{1 + (\mu_{ij})^2 - (v_{ij})^2 - 2(\mu_{\kappa j})^2}{2\left(1 - (\mu_{\kappa j})^2 - (v_{\kappa j})^2\right)} & \textit{if } (\mu_{\kappa j})^2 < (\mu_{ij})^2 \leq 1 - (v_{ij})^2 < 1 - (v_{\kappa j})^2. \end{cases} \tag{17}$$

In the special case that $(\mu_{ij})^2 = (\mu_{\kappa j})^2$ and $(v_{ij})^2 = (v_{\kappa j})^2$, the second or the fifth scenario of Eq. (17) must be satisfied. It is easy to verify that $L^{(1,1)}(p_{ij} \geq p_{\kappa j}) = 0.5$ in both cases. Based on the above results, $L^{(1,1)}(p_{ij} \geq p_{\kappa j}) = lik(p_{ij} \geq p_{\kappa j})$. This completes the proof.

Lemma 2 *Let x_i, x_κ, and x_η be three continuous random variables. Tsao and Chen [37] showed that the probability measure Pr fulfills the following properties*:

(L2.1) $0 \leq \Pr(x_i \geq x_\kappa) \leq 1$;

(L2.2) $\Pr(x_i \geq x_\kappa) + \Pr(x_\kappa > x_i) = 1$;

(L2.3) $\Pr(x_i \geq x_\kappa) = \Pr(x_\kappa > x_i) = 0.5$ if $\Pr(x_i \geq x_\kappa) = \Pr(x_\kappa > x_i)$;

(L2.4) $\Pr(x_i \geq x_\kappa) \geq 0.5$ if $\Pr(x_i \geq x_\eta) \geq 0.5$ and $\Pr(x_\eta \geq x_\kappa) \geq 0.5$.

Proof Refer to Tsao and Chen [37].

Theorem 2 *For the three Pythagorean PF evaluative ratings p_{ij}, $p_{\kappa j}$, and $p_{\eta j}$, the parametric likelihood measure $L^{(\alpha,\beta)}$ possesses several useful properties below*:

(T2.1) $0 \leq L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) \leq 1$;

(T2.2) $L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) + L^{(\alpha,\beta)}(p_{\kappa j} > p_{ij}) = 1$;

(T2.3) $L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) = L^{(\alpha,\beta)}(p_{\kappa j} > p_{ij}) = 0.5$ if $L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) = L^{(\alpha,\beta)}(p_{\kappa j} > p_{ij})$;

(T2.4) $L^{(\alpha,\beta)}(p_{ij} \geq p_{ij}) = 0.5$;

(T2.5) $L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) \geq 0.5$ if $L^{(\alpha,\beta)}(p_{ij} \geq p_{\eta j}) \geq 0.5$ and $L^{(\alpha,\beta)}(p_{\eta j} \geq p_{\kappa j}) \geq 0.5$.

Proof Properties (T2.1)–(T2.3) and (T2.5) are inferred from Lemma 2, and (T2.4) can be deduced via (T2.3).

4 PF likelihood-oriented methodology for MCDA

This section utilizes the proposed parametric likelihood measure to estimate the probability of an outranking relationship between PF evaluative ratings and further develop a straightforward multiple criteria evaluation method with the assistance of a new PF likelihood-oriented approach. Some advantageous concepts relevant to beta distribution-based likelihoods, such as the mean outranking indices, weighted outranking grades, and comprehensive outranking measures and indices, are utilized to assist the decision-maker in the management of MCDA issues in uncertain PF environments.

4.1 Proposed methods

The parametric likelihood measure supported by the beta distribution-based approach has several constructive and appealing properties. In particular, the proposed measure has the ability to discriminate the outranking relationships between uncertain PF information and deliver the priorities among candidate alternatives corresponding to each evaluative criterion. Considering the merits of the proposed parametric likelihood measure, this subsection utilizes this measure to establish a novel PF likelihood-oriented method for prioritizing available alternatives and resolving MCDA issues under uncertain PF conditions.

Mathematically, uncertain MCDA conditions containing assessment opinions based on Pythagorean fuzziness can be introduced as a PF evaluation matrix P in the following manner:

$$P = \left[p_{ij}\right]_{m\times n} = \left[(\mu_{ij}, \nu_{ij})\right]_{m\times n} = \begin{bmatrix} (\mu_{11},\nu_{11}) & (\mu_{12},\nu_{12}) & \cdots & (\mu_{1n},\nu_{1n}) \\ (\mu_{21},\nu_{21}) & (\mu_{22},\nu_{22}) & \cdots & (\mu_{2n},\nu_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{m1},\nu_{m1}) & (\mu_{m2},\nu_{m2}) & \cdots & (\mu_{mn},\nu_{mn}) \end{bmatrix}. \tag{18}$$

Consider that the parametric likelihood function $L^{(\alpha,\beta)}$ can effectively capture the outranking relationship between PF evaluative ratings with the assistance of beta density functions. This paper further introduces the likelihood-based concepts of mean outranking indices, weighted outranking grades, and comprehensive outranking measures and indices to establish a PF likelihood-oriented method for addressing MCDA issues. Specifically, for any pair of PF evaluative ratings p_{ij} and $p_{\kappa j}$ in P, the measure $L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j})$ describes the possibility of alternative a_i outranking a_κ about criterion c_j. On the one hand, the decision-maker is convinced that p_{ij} is better than $p_{\kappa j}$ based on the likelihood-based comparison mechanism. On the other hand, $L^{(\alpha,\beta)}(p_{\kappa j} \geq p_{ij})$ expresses that the possibility that a_i is outranked by a_κ with reference to c_j. Therefore, the decision-maker is convinced that p_{ij} is worse than $p_{\kappa j}$ based on the parametric likelihood measures. Supported by the proposed parametric likelihood measure, this study aims to identify appropriate likelihood-based indices for establishing an effective multiple criteria evaluation method from the perspective of PF likelihood orientation. Considering the strength of the evolved parametric likelihood measure, this study introduces a practical mean outranking index as a substantial foundation to estimate the extent of an alternative taking precedence over another one in connection with a specific criterion.

Definition 7 Let $\mathrm{L}_j^{(\alpha,\beta)}$ denote the parametric likelihood matrix associated with each c_j, and it is concisely represented as:

$$L_j^{(\alpha,\beta)} = \left[L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j})\right]_{m\times m} = \begin{bmatrix} 0.5 & L^{(\alpha,\beta)}(p_{1j} \geq p_{2j}) & \cdots & L^{(\alpha,\beta)}(p_{1j} \geq p_{mj}) \\ L^{(\alpha,\beta)}(p_{2j} \geq p_{1j}) & 0.5 & \cdots & L^{(\alpha,\beta)}(p_{2j} \geq p_{mj}) \\ \vdots & \vdots & \ddots & \vdots \\ L^{(\alpha,\beta)}(p_{mj} \geq p_{1j}) & L^{(\alpha,\beta)}(p_{mj} \geq p_{2j}) & \cdots & 0.5 \end{bmatrix}, \tag{19}$$

in which $L^{(\alpha,\beta)}(p_{ij} \geq p_{ij}) = 0.5$ from Theorem 2.

Lemma 3 *With respect to the evaluative criterion $c_j \in C$, the parametric likelihood matrix $\mathrm{L}_j^{(\alpha,\beta)}$ given by Eq. (19) is fuzzy complementary.*

Proof The boundedness property (T2.1) yields $0 \le L^{(\alpha,\beta)}(p_{ij} \ge p_{\kappa j}) \le 1$ for each $c_j \in C$, in which $i, \kappa \in \{1, 2, \cdots, m\}$. Thus, the parametric likelihood matrix $\mathrm{L}_j^{(\alpha,\beta)}$ is a fuzzy matrix. Based on the complementary property in (T2.2), $L^{(\alpha,\beta)}(p_{ij} \ge p_{\kappa j}) + L^{(\alpha,\beta)}(p_{\kappa j} \ge p_{ij}) = 1$ can be generated for each c_j. Thus, the fuzzy matrix $\mathrm{L}_j^{(\alpha,\beta)}$ is proven to be fuzzy complementary.

Definition 8 Let $\overline{L}^{(\alpha,\beta)}(p_{ij})$ denote the total parametric likelihood concerning an alternative $a_i \in A$ in regard to criterion c_j. Related to the ith row of the matrix $\mathrm{L}_j^{(\alpha,\beta)}$, $\overline{L}^{(\alpha,\beta)}(p_{ij})$ is determined using the sum of the parametric likelihoods $L^{(\alpha,\beta)}(p_{ij} \ge p_{\kappa j})$ for all $\kappa \in \{1, 2, \cdots, m\}$, as shown:

$$\overline{L}^{(\alpha,\beta)}\left(p_{ij}\right) = \sum_{\kappa=1}^{m} L^{(\alpha,\beta)}\left(p_{ij} \ge p_{\kappa j}\right). \tag{20}$$

To establish an effective approach for determining the outranking relationship associated with a PF evaluative rating p_{ij} with regard to another rating $p_{\kappa j}$, a linear transformation $\Psi^{(\alpha,\beta)}(p_{ij} \ge p_{\kappa j})$ can be made with the assistance of a couple of $\overline{L}^{(\alpha,\beta)}(p_{ij})$ and $\overline{L}^{(\alpha,\beta)}(p_{\kappa j})$ $(i, \kappa = 1, 2, \cdots, m)$ as follows:

$$\Psi^{(\alpha,\beta)}(p_{ij} \ge p_{\kappa j}) = \frac{1}{2} + \frac{\overline{L}^{(\alpha,\beta)}(p_{ij}) - \overline{L}^{(\alpha,\beta)}(p_{\kappa j})}{2(m-1)}. \tag{21}$$

Moreover, the collection of all transformed values can be concisely expressed in the following transformation matrix:

$$\Psi_j^{(\alpha,\beta)} = \left[\Psi^{(\alpha,\beta)}(p_{ij} \ge p_{\kappa j})\right]_{m \times m} = \begin{bmatrix} 0.5 & \Psi^{(\alpha,\beta)}(p_{1j} \ge p_{2j}) & \cdots & \Psi^{(\alpha,\beta)}(p_{1j} \ge p_{mj}) \\ \Psi^{(\alpha,\beta)}(p_{2j} \ge p_{1j}) & 0.5 & \cdots & \Psi^{(\alpha,\beta)}(p_{2j} \ge p_{mj}) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi^{(\alpha,\beta)}(p_{mj} \ge p_{1j}) & \Psi^{(\alpha,\beta)}(p_{mj} \ge p_{2j}) & \cdots & 0.5 \end{bmatrix}, \tag{22}$$

in which $\Psi^{(\alpha,\beta)}(p_{ij} \ge p_{ij}) = 0.5$ based on Eq. (21).

Lemma 4 *For each parametric likelihood matrix* $\mathrm{L}_j^{(\alpha,\beta)}$, *the transformation matrix* $\Psi_j^{(\alpha,\beta)}$ *is given by Eqs. (20)-(22). Then, the matrix* $\Psi_j^{(\alpha,\beta)}$ *is both fuzzy complementary and additive consistent.*

Proof First, it is known that $\mathrm{L}_j^{(\alpha,\beta)}$ is fuzzy complementary in accordance with Lemma 3, which indicates that $0 \le L^{(\alpha,\beta)}(p_{ij} \ge p_{\kappa j}) \le 1$ and $L^{(\alpha,\beta)}(p_{ij} \ge p_{\kappa j}) + L^{(\alpha,\beta)}(p_{\kappa j} \ge p_{ij}) = 1$ for each $i, \kappa \in \{1, 2, \cdots, m\}$. Moreover, $L^{(\alpha,\beta)}(p_{ij} \ge p_{ij}) = 0.5$. Accordingly, $\overline{L}^{(\alpha,\beta)}(p_{ij}) = \sum_{\imath=1, \imath \ne i}^{m} L^{(\alpha,\beta)}(p_{ij} \ge p_{\imath j}) + L^{(\alpha,\beta)}(p_{ij} \ge p_{ij}) = \sum_{\imath=1,\imath\ne i}^{m} L^{(\alpha,\beta)}(p_{ij} \ge p_{\imath j}) + 0.5$. In this manner, it is given that $0.5 \le \overline{L}^{(\alpha,\beta)}(p_{ij}) \le (m-1) + 0.5$ (for $i = 1, 2, \cdots, m$). Therefore, $-(m-1) \le \overline{L}^{(\alpha,\beta)}(p_{ij}) - \overline{L}^{(\alpha,\beta)}(p_{\kappa j}) \le (m-1)$, which infers that $-0.5 \le (\overline{L}^{(\alpha,\beta)}(p_{ij}) - \overline{L}^{(\alpha,\beta)}(p_{\kappa j}))/2(m-1) \le 0.5$. Hence, $0 \le \Psi^{(\alpha,\beta)}(p_{ij} \ge p_{\kappa j}) \le 1$. Accordingly, $\Psi_j^{(\alpha,\beta)}$ is a fuzzy matrix. Using Eq. (21), the following outcome can be derived:

$$\begin{aligned} &\Psi^{(\alpha,\beta)}(p_{ij} \ge p_{\kappa j}) + \Psi^{(\alpha,\beta)}(p_{\kappa j} \ge p_{ij}) \\ &\quad = \left[\frac{1}{2} + \frac{\overline{L}^{(\alpha,\beta)}(p_{ij}) - \overline{L}^{(\alpha,\beta)}(p_{\kappa j})}{2(m-1)}\right] \\ &\qquad + \left[\frac{1}{2} + \frac{\overline{L}^{(\alpha,\beta)}(p_{\kappa j}) - \overline{L}^{(\alpha,\beta)}(p_{ij})}{2(m-1)}\right] = 1. \end{aligned}$$

As a consequence, $\Psi_j^{(\alpha,\beta)}$ is a complementary matrix. Finally, this paper investigates the property of additive transitivity, which is the condition that a consistent matrix should satisfy. From Eq. (21):

$$\begin{aligned} &\left(\Psi^{(\alpha,\beta)}(p_{ij} \ge p_{\imath j}) - \frac{1}{2}\right) + \left(\Psi^{(\alpha,\beta)}(p_{\imath j} \ge p_{\kappa j}) - \frac{1}{2}\right) \\ &\quad = \frac{\overline{L}^{(\alpha,\beta)}(p_{ij}) - \overline{L}^{(\alpha,\beta)}(p_{\imath j})}{2(m-1)} + \frac{\overline{L}^{(\alpha,\beta)}(p_{\imath j}) - \overline{L}^{(\alpha,\beta)}(p_{\kappa j})}{2(m-1)} \\ &\quad = \frac{\overline{L}^{(\alpha,\beta)}(p_{ij}) - \overline{L}^{(\alpha,\beta)}(p_{\kappa j})}{2(m-1)} = \Psi^{(\alpha,\beta)}(p_{ij} \ge p_{\kappa j}) - \frac{1}{2}. \end{aligned}$$

Thus, the transformation matrix $\Psi_j^{(\alpha,\beta)}$ is proven to be additive transitive, which satisfies the condition of the consistent matrix. As a result, $\Psi_j^{(\alpha,\beta)}$ is not only fuzzy complementary but also additive consistent.

Based on the transformed values of $\Psi^{(\alpha,\beta)}(p_{ij} \ge p_{\kappa j})$ in the transformation matrix $\Psi_j^{(\alpha,\beta)}$, this study attempts to identify a mean outranking index of alternative a_i in terms of criterion c_j. Specifically, the sum of all $\Psi^{(\alpha,\beta)}(p_{ij} \ge p_{\kappa j})$ values of each row in $\Psi_j^{(\alpha,\beta)}$ is computed, i.e., $\sum_{\kappa=1}^{m} \Psi^{(\alpha,\beta)}(p_{ij} \ge p_{\kappa j})$.

Definition 9 The mean outranking index $MO^{(\alpha,\beta)}(p_{ij})$ of a_i in relation to c_j is defined through the normalized value of $\sum_{\kappa=1}^{m} \Psi^{(\alpha,\beta)}(p_{ij} \ge p_{\kappa j})$ as follows:

$$MO^{(\alpha,\beta)}(p_{ij}) = \frac{\sum_{\kappa=1}^{m} \Psi^{(\alpha,\beta)}(p_{ij} \ge p_{\kappa j})}{\sum_{i=1}^{m} \sum_{\kappa=1}^{m} \Psi^{(\alpha,\beta)}(p_{ij} \ge p_{\kappa j})}. \tag{23}$$

Theorem 3 *Based on the parametric likelihood matrix* $\mathrm{L}_j^{(\alpha,\beta)}$ $(= [L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j})]_{m\times m})$, *the mean outranking index* $MO^{(\alpha,\beta)}(p_{ij})$ *of an alternative* $a_i \in A$ *in connection with each criterion* $c_j \in C$ *is determined in the following manner*:

$$MO^{(\alpha,\beta)}(p_{ij}) = \frac{m + 2\left(\sum_{\kappa=1}^{m} L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) - 1\right)}{2m(m-1)}. \quad (24)$$

Proof The transformation matrix $\Psi_j^{(\alpha,\beta)}$ $(= [\Psi^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j})]_{m\times m})$ can be established by Eqs. (20) and (21). As proven in Lemma 4, $\Psi_j^{(\alpha,\beta)}$ is a fuzzy complementary and consistent matrix, which indicates that $0 \leq \Psi^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) \leq 1$, $\Psi^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) + \Psi^{(\alpha,\beta)}(p_{\kappa j} \geq p_{ij}) = 1$, and $\Psi^{(\alpha,\beta)}(p_{ij} \geq p_{ij}) = 0.5$. For the parametric likelihood matrix $\mathrm{L}_j^{(\alpha,\beta)} = [L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j})]_{m\times m}$, one obtains:

$$\begin{aligned}\sum_{i=1}^{m}\sum_{\kappa=1}^{m} L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) &= \sum_{i=1}^{m} L^{(\alpha,\beta)}(p_{ij} \geq p_{ij}) \\ &\quad + \sum_{1\leq i<\kappa\leq m}\left(L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) + L^{(\alpha,\beta)}(p_{\kappa j} \geq p_{ij})\right) \\ &= \frac{m}{2} + \frac{m(m-1)}{2} = \frac{m^2}{2}.\end{aligned} \quad (25)$$

Combining this expression with Eqs. (20) and (21), the following outcome results:

$$\begin{aligned}MO^{(\alpha,\beta)}(p_{ij}) &= \frac{\sum_{\kappa=1}^{m}\Psi^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j})}{\sum_{1\leq i<\kappa\leq m}^{m}\left(\Psi^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) + \Psi^{(\alpha,\beta)}(p_{\kappa j} \geq p_{ij})\right) + \frac{m}{2}} \\ &= \frac{\sum_{\kappa=1}^{m}\left[\frac{1}{2} + \frac{\overline{L}^{(\alpha,\beta)}(p_{ij}) - \overline{L}^{(\alpha,\beta)}(p_{\kappa j})}{2(m-1)}\right]}{\frac{m(m-1)}{2} + \frac{m}{2}} = \frac{m\overline{L}^{(\alpha,\beta)}(p_{ij}) - \sum_{\kappa=1}^{m}\overline{L}^{(\alpha,\beta)}(p_{\kappa j}) + m(m-1)}{m^2(m-1)} \\ &= \frac{m\overline{L}^{(\alpha,\beta)}(p_{ij}) - \sum_{i=1}^{m}\sum_{\kappa=1}^{m} L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) + m(m-1)}{m^2(m-1)} = \frac{m\overline{L}^{(\alpha,\beta)}(p_{ij}) - \frac{m^2}{2} + m(m-1)}{m^2(m-1)} \\ &= \frac{\overline{L}^{(\alpha,\beta)}(p_{ij}) + \frac{m}{2} - 1}{m(m-1)} = \frac{\sum_{\kappa=1}^{m} L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) + \frac{m}{2} - 1}{m(m-1)} = \frac{m + 2\left(\sum_{\kappa=1}^{m} L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) - 1\right)}{2m(m-1)}.\end{aligned}$$

This sets up the theorem.

Theorem 4 Concerned with each PF evaluative rating p_{ij} contained in the evaluation matrix P, its mean outranking index $MO^{(\alpha,\beta)}(p_{ij})$ meets the subsequent properties:

(T4.1) $1/2m \leq MO^{(\alpha,\beta)}(p_{ij}) \leq 3/2m$ for all $a_i \in A$ and $c_j \in C$;

(T4.2) $\sum_{i=1}^{m} MO^{(\alpha,\beta)}(p_{ij}) = 1$ for each $c_j \in C$;

(T4.3) $\sum_{j=1}^{n}\sum_{i=1}^{m} MO^{(\alpha,\beta)}(p_{ij}) = n$.

Proof It is realized that $\sum_{\kappa=1}^{m} L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) = \sum_{\kappa=1,\kappa\neq i}^{m} L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) + L^{(\alpha,\beta)}(p_{ij} \geq p_{ij})$. From (T2.1) and (T2.4), $0 \leq L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) \leq 1$ and $L^{(\alpha,\beta)}(p_{ij} \geq p_{ij}) = 0.5$, respectively, which follows that $0.5 \leq \sum_{\kappa=1}^{m} L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) \leq (m-1) + 0.5$. It is recognized that:

$$\begin{aligned}MO^{(\alpha,\beta)}(p_{ij}) &= \frac{m + 2\left(\sum_{\kappa=1,\kappa\neq i}^{m} L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) + L^{(\alpha,\beta)}(p_{ij} \geq p_{ij}) - 1\right)}{2m(m-1)} \\ &\leq \frac{m + 2[(m-1) + 0.5 - 1]}{2m(m-1)} = \frac{3}{2m},\end{aligned}$$

$$MO^{(\alpha,\beta)}(p_{ij}) = \frac{m + 2\left(\sum_{\kappa=1,\kappa\neq i}^{m} L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) + L^{(\alpha,\beta)}(p_{ij} \geq p_{ij}) - 1\right)}{2m(m-1)}$$
$$\geq \frac{m + 2(0 + 0.5 - 1)}{2m(m-1)} = \frac{1}{2m}.$$

Therefore, the inequality $1/2m \leq MO^{(\alpha,\beta)}(p_{ij}) \leq 3/2m$ holds, i.e., (T4.1) is correct. For (T4.2), based on Eq. (25), it is considered that:

$$\begin{aligned}\sum_{i=1}^{m} MO^{(\alpha,\beta)}(p_{ij}) &= \sum_{i=1}^{m} \frac{m + 2\left(\sum_{\kappa=1}^{m} L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) - 1\right)}{2m(m-1)} \\ &= \sum_{i=1}^{m} \frac{m-2}{2m(m-1)} + \frac{2}{2m(m-1)} \sum_{i=1}^{m} \sum_{\kappa=1}^{m} L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j}) \\ &= m \cdot \frac{m-2}{2m(m-1)} + \frac{2}{2m(m-1)} \cdot \frac{m^2}{2} \\ &= \frac{m-2}{2(m-1)} + \frac{m}{2(m-1)} = 1.\end{aligned}$$

As a result, (T4.2) is fulfilled. Property (T4.3) can be determined from (T4.2), which establishes the theorem.

Another essential task is to quantify the weights of evaluative criteria. In making preference judgments, the decision-maker can exploit widely used normalized weights or PF importance weights in relation to the relative meaning and numerical priorities of criteria. Therefore, the relative importance of a specific criterion can be represented by either an ordinary (i.e., nonfuzzy) normalized weight or a PF importance weight. Let $\overline{W} = (\overline{w}_1, \overline{w}_2, \cdots, \overline{w}_n)$ signify a vector of ordinary normalized weights related to all n criteria with the restrictions $\sum_{j=1}^{n} \overline{w}_j = 1$ and $0 \leq \overline{w}_j \leq 1$ for each c_j.

Definition 10 By utilizing the ordinary normalized weight $\overline{w}_j$, the weighted outranking grade $\overline{WO}^{(\alpha,\beta)}(p_{ij})$ related to each p_{ij} in the PF characteristic P_i is given by:

$$\overline{WO}^{(\alpha,\beta)}(p_{ij}) = \overline{w}_j \cdot MO^{(\alpha,\beta)}(p_{ij}). \tag{26}$$

The comprehensive outranking index $\overline{CI}_i^{(\alpha,\beta)}$ of each alternative a_i is determined as follows:

$$\overline{CI}_i^{(\alpha,\beta)} = \sum_{j=1}^{n} \overline{WO}^{(\alpha,\beta)}(p_{ij}) = \sum_{j=1}^{n} \overline{w}_j \cdot MO^{(\alpha,\beta)}(p_{ij}). \tag{27}$$

Theorem 5 The weighted outranking grade $\overline{WO}^{(\alpha,\beta)}(p_{ij})$ and the comprehensive outranking index $\overline{CI}_i^{(\alpha,\beta)}$ fulfill the following properties:

(T5.1) $\bar{w}_j/2m \leq \overline{WO}^{(\alpha,\beta)}(p_{ij}) \leq 3\bar{w}_j/2m$ for all $a_i \in A$ and $c_j \in C$;

(T5.2) $1/2m \leq \overline{CI}_i^{(\alpha,\beta)} \leq 3/2m$ for each $a_i \in A$;

(T5.3) $\sum_{i=1}^{m} \overline{WO}^{(\alpha,\beta)}(p_{ij}) = \overline{w}_j$ for each $c_j \in C$;

(T5.4) $\sum_{i=1}^{m} \overline{CI}_i^{(\alpha,\beta)} = 1$.

Proof Property (T5.1) is insignificant but known for each PF evaluative rating p_{ij} based on the criterion weights $0 \leq \overline{w}_j \leq 1$ and $1/2m \leq MO^{(\alpha,\beta)}(p_{ij}) \leq 3/2m$ from (T4.1). From (T5.1), $\sum_{j=1}^{n} \bar{w}_j/2m \leq \sum_{j=1}^{n} \overline{WO}^{(\alpha,\beta)}(p_{ij}) \leq \sum_{j=1}^{n} 3\bar{w}_j/2m$ is generated. The normalization condition $\sum_{j=1}^{n} \overline{w}_j = 1$ and Eq. (27) result in $1/2m \leq \overline{CI}_i^{(\alpha,\beta)} \leq 3/2m$ for each alternative a_i, i.e., (T5.2) is confirmed. For (T5.3), in accordance with the property $\sum_{i=1}^{m} MO^{(\alpha,\beta)}(p_{ij}) = 1$ in (T4.2), $\sum_{i=1}^{m} \overline{WO}^{(\alpha,}\beta)(p_{ij}) = \sum_{i=1}^{m} \overline{w}_j \cdot MO^{(\alpha,\beta)}(p_{ij}) = \overline{w}_j \sum_{i=1}^{m} MO^{(\alpha,\beta)}(p_{ij}) = \overline{w}_j$ for each c_j. The property (T5.3) indicates that $\sum_{i=1}^{m} \overline{CI}_i^{(\alpha,\beta)} = \sum_{j=1}^{n} \sum_{i=1}^{m} \overline{WO}^{(\alpha,\beta)}(p_{ij}) = \sum_{j=1}^{n} \overline{w}_j = 1$, i.e., (T5.4) is confirmed.

Alternatively, let a Pythagorean membership grade $w_j = (\omega_j, \varpi_j)$ depict the PF importance weight related to criterion c_j. Moreover, under the constraints $0 \leq (\omega_j)^2 + (\varpi_j)^2 \leq 1$ and $\omega_j, \varpi_j \in [0, 1]$ over each c_j, the collection involving all PF importance weights is represented as a PF set, i.e.,

$$\begin{aligned}W &= \left\{\langle c_j, w_j \rangle | c_j \in C\right\} \\ &= \left\{\langle c_1, (\omega_1, \varpi_1)\rangle, \langle c_2, (\omega_2, \varpi_2)\rangle, \ldots, \langle c_n, (\omega_n, \varpi_n)\rangle\right\}.\end{aligned} \tag{28}$$

Definition 11 By utilizing the PF importance weight w_j, the weighted PF outranking grade $WO^{(\alpha,\beta)}(p_{ij})$ related to each p_{ij} in the PF characteristic P_i is given by:

$$\begin{aligned}WO^{(\alpha,\beta)}(p_{ij}) &= MO^{(\alpha,\beta)}(p_{ij}) \odot w_j \\ &= \left(\sqrt{1 - \left(1 - (\omega_j)^2\right)^{MO^{(\alpha,\beta)}(p_{ij})}}, (\varpi_j)^{MO^{(\alpha,\beta)}(p_{ij})}\right).\end{aligned} \tag{29}$$

The comprehensive PF outranking measure $CO^{(\alpha,\beta)}(P_i)$ of each PF characteristic P_i is given by:

$$CO^{(\alpha,\beta)}(P_i) = WO^{(\alpha,\beta)}(p_{i1}) \oplus WO^{(\alpha,\beta)}(p_{i2}) \oplus \cdots \oplus WO^{(\alpha,\beta)}(p_{in}) = \mathop{\oplus}_{j=1}^{n} WO^{(\alpha,\beta)}(p_{ij}). \quad (30)$$

It is worth mentioning that $WO^{(\alpha,\beta)}(p_{ij})$ and $CO^{(\alpha,\beta)}(P_i)$ are Pythagorean membership grades instead of ordinary numbers. To facilitate calculation efficiency, this paper provides the subsequent theorem by addition and multiplication operational laws.

Theorem 6 Consider the mean outranking index $MO^{(\alpha,\beta)}(p_{ij})$ and the PF importance weight $w_j\ (=(\omega_j, \varpi_j))$ for each $p_{ij} \in P_i$ and $c_j \in C$, respectively. The comprehensive PF outranking measure $CO^{(\alpha,\beta)}(P_i)$ in view of the PF characteristic P_i $(i = 1, 2, \cdots, m)$ is determined by:

$$CO^{(\alpha,\beta)}(P_i) = \left(\sqrt{1 - \prod_{j=1}^{n}\left(1 - (\omega_j)^2\right)^{MO^{(\alpha,\beta)}(p_{ij})}}, \prod_{j=1}^{n}(\varpi_j)^{MO^{(\alpha,\beta)}(p_{ij})}\right), \quad (31)$$

Proof This theorem is demonstrated by mathematical induction about n. On the occasion that $n = 2$, it is received that $CO^{(\alpha,\beta)}(P_i) = WO^{(\alpha,\beta)}(p_{i1}) \oplus WO^{(\alpha,\beta)}(p_{i2}) = (MO^{(\alpha,\beta)}(p_{i1}) \odot w_1) \oplus (MO^{(\alpha,\beta)}(p_{i2}) \odot w_2)$. Applying the addition and multiplication operations, $CO^{(\alpha,\beta)}(P_i)$ is calculated as follows:

$$\begin{aligned}
CO^{(\alpha,\beta)}(P_i) &= \left(MO^{(\alpha,\beta)}(p_{i1}) \odot w_1\right) \oplus \left(MO^{(\alpha,\beta)}(p_{i2}) \odot w_2\right) \\
&= \left(\sqrt{1 - \left(1 - (\omega_1)^2\right)^{MO^{(\alpha,\beta)}(p_{i1})}}, (\varpi_1)^{MO^{(\alpha,\beta)}(p_{i1})}\right) \oplus \left(\sqrt{1 - \left(1 - (\omega_2)^2\right)^{MO^{(\alpha,\beta)}(p_{i2})}}, (\varpi_2)^{MO^{(\alpha,\beta)}(p_{i2})}\right) \\
&= \left(\left(1 - \left(1 - (\omega_1)^2\right)^{MO^{(\alpha,\beta)}(p_{i1})} + 1 - \left(1 - (\omega_2)^2\right)^{MO^{(\alpha,\beta)}(p_{i2})} - \left(1 - \left(1 - (\omega_1)^2\right)^{MO^{(\alpha,\beta)}(p_{i1})}\right)\right.\right. \\
&\quad \left.\left.\cdot\left(1 - \left(1 - (\omega_2)^2\right)^{MO^{(\alpha,\beta)}(p_{i2})}\right)\right)^{0.5}, (\varpi_1)^{MO^{(\alpha,\beta)}(p_{i1})} \cdot (\varpi_2)^{MO^{(\alpha,\beta)}(p_{i2})}\right) \\
&= \left(\sqrt{1 - \left(1 - \left(1 - \left(1 - (\omega_1)^2\right)^{MO^{(\alpha,\beta)}(p_{i1})}\right)\right) \cdot \left(1 - \left(1 - \left(1 - (\omega_2)^2\right)^{MO^{(\alpha,\beta)}(p_{i2})}\right)\right)},\right. \\
&\quad \left.(\varpi_1)^{MO^{(\alpha,\beta)}(p_{i1})} \cdot (\varpi_2)^{MO^{(\alpha,\beta)}(p_{i2})}\right) \\
&= \left(\sqrt{1 - \prod_{j=1}^{2}\left(1 - (\omega_j)^2\right)^{MO^{(\alpha,\beta)}(p_{ij})}}, \prod_{j=1}^{2}(\varpi_j)^{MO^{(\alpha,\beta)}(p_{ij})}\right).
\end{aligned}$$

It is deduced that Eq. (31) is fulfilled if $n = 2$. Suppose that Eq. (31) is satisfied when $n = \chi$, i.e.,

$$CO^{(\alpha,\beta)}(P_i) = \left(\sqrt{1 - \prod_{j=1}^{\chi}\left(1 - (\omega_j)^2\right)^{MO^{(\alpha,\beta)}(p_{ij})}}, \prod_{j=1}^{\chi}(\varpi_j)^{MO^{(\alpha,\beta)}(p_{ij})}\right).$$

When $n = \chi + 1$, the following outcome is acquired: $CO^{(\alpha,\beta)}(P_i) = \left(\mathop{\oplus}_{j=1}^{\chi}\left(MO^{(\alpha,\beta)}(p_{ij}) \odot w_j\right)\right) \oplus \left(MO^{(\alpha,\beta)}(p_{i,\chi+1}) \odot w_{\chi+1}\right)$.

By use of the arithmetic operations with respect to Pythagorean membership grades, one obtains:

$$CO^{(\alpha,\beta)}(P_i) = \left(\sqrt{1-\prod_{j=1}^{\chi}\left(1-(\omega_j)^2\right)^{MO^{(\alpha,\beta)}(p_{ij})}}, \prod_{j=1}^{\chi}(\varpi_j)^{MO^{(\alpha,\beta)}(p_{ij})}\right)$$
$$\oplus\left(\sqrt{1-\left(1-(\omega_{\chi+1})^2\right)^{MO^{(\alpha,\beta)}(p_{i,\chi+1})}}, (\varpi_{\chi+1})^{MO^{(\alpha,\beta)}(p_{i,\chi+1})}\right)$$
$$=\left(\left(\left(1-\prod_{j=1}^{\chi}\left(1-(\omega_j)^2\right)^{MO^{(\alpha,\beta)}(p_{ij})}+1-\left(1-\left(1-\left(1-(\omega_{\chi+1})^2\right)^{MO^{(\alpha,\beta)}(p_{i,\chi+1})}\right)\right)\right.\right.\right.$$
$$\left.\left.-\left(1-\prod_{j=1}^{\chi}\left(1-(\omega_j)^2\right)^{MO^{(\alpha,\beta)}(p_{ij})}\right)\cdot\left(1-\left(1-\left(1-\left(1-(\omega_{\chi+1})^2\right)^{MO^{(\alpha,\beta)}(p_{i,\chi+1})}\right)\right)\right)\right)^{0.5},$$
$$\left.\left(\prod_{j=1}^{\chi}(\varpi_j)^{MO^{(\alpha,\beta)}(p_{ij})}\right)\cdot(\varpi_{\chi+1})^{MO^{(\alpha,\beta)}(p_{i,\chi+1})}\right)$$
$$=\left(\sqrt{1-\prod_{j=1}^{\chi+1}\left(1-(\omega_j)^2\right)^{Pen(p_{ij})\cdot|\Phi_{ij}-\phi|}}, \prod_{j=1}^{\chi+1}(\varpi_j)^{Pen(p_{ij})\cdot|\Phi_{ij}-\phi|}\right).$$

It reveals that Eq. (31) is satisfied when $n=\chi+1$, which follows that Eq. (31) is valid for all n. This accomplishes the proof.

Because the comprehensive PF outranking measure $CO^{(\alpha,\beta)}(P_i)$ is a Pythagorean membership grade, it is necessary to utilize an effective and credible approach to the identification of eventual priority ranks regarding competing alternatives. With the aim of systematically prioritizing m candidate alternatives based on their overall outranking relationships, this paper applies the likelihood-based concept to derive the parametric likelihood between comprehensive PF outranking measures and then determines the comprehensive outranking index for each alternative. Based on Theorem 1, let $L^{(\alpha',\beta')}(CO^{(\alpha,\beta)}(P_i)\geq CO^{(\alpha,\beta)}(P_\kappa))$ represent the parametric likelihood measure of the relation $CO^{(\alpha,\beta)}(P_i)\geq CO^{(\alpha,\beta)}(P_\kappa)$ for $i,\kappa\in\{1,2,\ldots,m\}$. In this study, assume that $\alpha'=\alpha$ and $\beta'=\beta$ form a consistent basis. The parametric likelihood of $CO^{(\alpha,\beta)}(P_i)\geq CO^{(\alpha,\beta)}(P_\kappa)$ is denoted as $L^{(\alpha,\beta)}(CO(P_i)\geq CO(P_\kappa))$ for notational convenience. Let $\mathrm{L}^{(\alpha,\beta)}$ represent the collection of the obtained $L^{(\alpha,\beta)}(CO(P_i)\geq CO(P_\kappa))$, which is expressed in the following matrix format:

$$L^{(\alpha,\beta)} = \left[L^{(\alpha,\beta)}(CO(P_i)\geq CO(P_\kappa))\right]_{m\times m}$$
$$=\begin{bmatrix} 0.5 & L^{(\alpha,\beta)}(CO(P_1)\geq CO(P_2)) & \cdots & L^{(\alpha,\beta)}(CO(P_1)\geq CO(P_m)) \\ L^{(\alpha,\beta)}(CO(P_2)\geq CO(P_1)) & 0.5 & \cdots & L^{(\alpha,\beta)}(CO(P_2)\geq CO(P_m)) \\ \vdots & \vdots & \ddots & \vdots \\ L^{(\alpha,\beta)}(CO(P_m)\geq CO(P_1)) & L^{(\alpha,\beta)}(CO(P_m)\geq CO(P_2)) & \cdots & 0.5 \end{bmatrix},$$

where $L^{(\alpha,\beta)}(CO(P_i)\geq CO(P_i))=0.5$. Next, analogous to the mean outranking index in Theorem 3, the notion of comprehensive PF outranking indices is exhibited below.

Definition 12 The comprehensive PF outranking index $CI_i^{(\alpha,\beta)}$ of an alternative a_i is explained by:

$$CI_i^{(\alpha,\beta)} = \frac{m+2\left(\sum_{\kappa=1}^{m}L^{(\alpha,\beta)}(CO(P_i)\geq CO(P_\kappa))-1\right)}{2m(m-1)}. \quad (33)$$

Theorem 7 The comprehensive PF outranking index $CI_i^{(\alpha,\beta)}$ of an alternative a_i in the alternative set A meets the following properties:

(T7.1) $1/2m \le CI_i^{(\alpha,\beta)} \le 3/2m$ for each $a_i \in A$;

(T7.2) $\sum_{i=1}^{m} CI_i^{(\alpha,\beta)} = 1$.

Proof The proof is similar to the processes in (T4.1) and (T4.2).

The concept of comprehensive outranking indices by using the proposed parametric likelihood measure contributes a practical approach to differentiating the overall performance of the candidate alternatives and prioritizing them on the evidence of total dominance over competing alternatives. For certain cases, an alternative a_i is better than a_κ (denoted as $a_i \succ a_\kappa$) if $\overline{CI}_i^{(\alpha,\beta)} > \overline{CI}_\kappa^{(\alpha,\beta)}$ and $CI_i^{(\alpha,\beta)} > CI_\kappa^{(\alpha,\beta)}$ with respect to the preference structure involving ordinary normalized weights and PF importance weights, respectively. Moreover, a_i is as good as a_κ (denoted as $a_i \sim a_\kappa$) if $\overline{CI}_i^{(\alpha,\beta)} = \overline{CI}_\kappa^{(\alpha,\beta)}$ and $CI_i^{(\alpha,\beta)} = CI_\kappa^{(\alpha,\beta)}$ in terms of normalized weights and PF importance weights, respectively. Last, a_i is worse than a_κ (denoted as $a_i \prec a_\kappa$) if $\overline{CI}_i^{(\alpha,\beta)} < \overline{CI}_\kappa^{(\alpha,\beta)}$ and $CI_i^{(\alpha,\beta)} < CI_\kappa^{(\alpha,\beta)}$ on the subject of normalized weights and PF importance weights, respectively. Therefore, the priority ranking among m candidate alternatives will be presented in descending order of the $\overline{CI}_i^{(\alpha,\beta)}$ (or $CI_i^{(\alpha,\beta)}$) outcomes.

4.2 Proposed algorithms

Supported by the parametric likelihood measure based on beta distributions, the proposed PF likelihood-oriented methodology can be summarized using several essential concepts and manipulations, as shown in Fig. 3. Utilizing different preference structures relating to criteria, the multiple criteria evaluation tasks under PF uncertainty can be precisely delineated by two algorithmic procedures, namely, Algorithms I and II. These two algorithms address different preference information related to criteria. More precisely, Algorithm I is applicable to MCDA problems involving ordinary normalized weights (i.e., $\overline{w}_j$) and PF evaluative ratings (i.e., p_{ij}). Algorithm II is applicable to MCDA problems containing PF importance weights (i.e., w_j) and PF evaluative ratings.

Algorithm I for MCDA involving normalized weights and PF evaluative ratings

Step I.1: The decision-making problem using collection of evaluative criteria $C = \{c_1, c_2, \cdots, c_n\}$ and candidate alternatives $A = \{a_1, a_2, \cdots, a_m\}$ is designed. Apposite positive integers for the parameters α and β within the beta function $B(\alpha, \beta)$ are designated.

Step I.2: The PF evaluative rating $p_{ij} = (\mu_{ij}, \nu_{ij})$ is determined through an investigation of satisfaction and dissatisfaction toward a_i in relation to c_j. The PF characteristic P_i in Eq. (11) is accounted for and the PF evaluation matrix $\mathrm{P} = [p_{ij}]_{m\times n}$ in Eq. (18) is generated.

Step I.3: A vector containing ordinary weights for n evaluative criteria $W = (w_1, w_2, \cdots, \overline{w}_n)$ with the restrictions $\sum_{j=1}^{n} w_j = 1$ and $0 \le w_j \le 1$ $(j = 1, 2, \cdots, n)$ is established.

Step I.4: The parametric likelihood measure $L^{(\alpha,\beta)}(p_{ij} \ge p_{\kappa j})$ $(i, \kappa = 1, 2, \cdots, m)$ by Eq. (15) is considered to form the parametric likelihood matrix $\mathrm{L}_j^{(\alpha,\beta)} = [L^{(\alpha,\beta)}(p_{ij} \ge p_{\kappa j})]_{m\times m}$ in Eq. (19) with regard to each c_j.

Step I.5: The function $\mathrm{L}_j^{(\alpha,\beta)}$ is used as a basis to determine the mean outranking index $MO^{(\alpha,\beta)}(p_{ij})$ associated with each p_{ij} in the PF evaluation matrix P by Eq. (24).

Step I.6: The weight vector W and the $MO^{(\alpha,\beta)}(p_{ij})$ values in the PF characteristic P_i are integrated to derive the comprehensive outranking index $CI_i^{(\alpha,\beta)}$ of each alternative a_i using Eq. (27).

Step I.7: The priority ranking for all candidate alternatives in collection A supported by a descending order of the $CI_i^{(\alpha,\beta)}$ outcomes are identified. The best choice refers to the alternative enjoying the largest $CI_i^{(\alpha,\beta)}$ value.

Algorithm II for MCDA involving PF importance weights and PF evaluative ratings

Steps II.1 and II.2: See Steps I.1 and I.2.

Step II.3: A PF importance weight $w_j = (\omega_j, \varpi_j)$ related to criterion c_j, in which $0 \le (\omega_j)^2 + (\varpi_j)^2 \le 1$ and $\omega_j, \varpi_j \in [0,1]$ is developed. The collection $W = \{\langle c_j, w_j \rangle \mid c_j \in C\}$ is composed.

Steps II.4 and II.5: See Steps I.4 and I.5.

Step II.6: The PF importance weight w_j and the $MO^{(\alpha,\beta)}(p_{ij})$ values for each p_{ij} are combined to derive the comprehensive PF outranking measure $CO^{(\alpha,\beta)}(P_i)$ related to each P_i by use of Eq. (31).

Step II.7: The parametric likelihood $L^{(\alpha,\beta)}(CO(P_i) \ge CO(P_\kappa))$ of the PF outranking relation $CO^{(\alpha,\beta)}(P_i) \ge CO^{(\alpha,\beta)}(P_\kappa)$ is computed to determine the comprehensive PF outranking index $CI_i^{(\alpha,\beta)}$ of an alternative a_i using Eq. (33).

Step II.8: The priority ranking for all candidate alternatives in collection A depending on the descending order of the $CI_i^{(\alpha,\beta)}$ outcomes are identified. The best choice refers to the alternative having the largest $CI_i^{(\alpha,\beta)}$ value.

5 Realistic application and comparative studies

This section utilizes the advanced PF likelihood-oriented method to manage a realistic MCDA problem of selecting pilot hospitals in relation to postacute care. Moreover, this section conducts an experimental analysis and compares the results to other approaches to demonstrate the superiority of the proposed approaches to be used for practical purposes in flexible and accessible ways.

5.1 Problem description and decision process

This real-world case study was introduced by Chen [8] and concerns a postacute care program that targets patients with cerebrovascular diseases. Postacute care focuses on a collaborative health care service that aims to support homeward bound patients after prescribing acute medical treatment. Pilot hospitals play an important role in creating a transitional care framework with vertical integration, especially in the acute, postacute, and chronic stages. In this regard, Chen [8] formulated a pilot hospital selection problem for cerebrovascular diseases to facilitate the establishment of a postacute care model. Figure 4 illustrates the meaning of the evaluative criteria related to the pilot hospital selection problem for postacute care. The MCDA method attempts to evaluate and prioritize available pilot hospitals to improve the rehabilitation quality of patients and reduce the rehospitalization rate with respect to medical centers and hospitals.

Notably, when there are a large number of uncertainties in real-life applications, it is crucial to appropriately structure the MCDA problem and demonstratively evaluate multiple criteria. In judging how to give precedence to candidate hospitals for postacute care and determining whether to select an appropriate pilot hospital, there are not only very compounded considerations related to multiple criteria, but there are also sophisticated assessment information involving Pythagorean fuzziness that may affect the consequences of decision-making. In this subsection, this paper utilizes the proposed PF likelihood-oriented method to address the MCDA problem concerning hospital selection for postacute care and illustrates the implementation processes of the developed algorithmic procedures.

It is noted that the hospital selection problem for postacute care, which Chen [8] initiated, was formulated within the interval-valued PF decision environment. In this regard, this paper applies the midpoint approach to convert interval-valued PF data into PF data. Moreover, the preference structure of evaluative criteria is represented as

Table 1 Experimental results of mean outranking indices, weighted outranking grades, and comprehensive outranking indices in the cases of $\alpha = 1$ and $\beta = 1,2,\ldots,5$

c_j	a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5
	Results of $MO^{(1,1)}(p_{ij})$					Results of $MO^{(1,2)}(p_{ij})$					Results of $MO^{(1,3)}(p_{ij})$				
c_1	0.1990	0.1573	0.1573	0.1990	0.2874	0.2020	0.1503	0.1503	0.2020	0.2956	0.2065	0.1443	0.1443	0.2065	0.2985
c_2	0.2926	0.1049	0.1893	0.1893	0.2239	0.2976	0.1022	0.1866	0.1866	0.2270	0.2992	0.1009	0.1842	0.1842	0.2315
c_3	0.1897	0.2705	0.2304	0.1197	0.1897	0.1857	0.2815	0.2344	0.1129	0.1857	0.1829	0.2887	0.2378	0.1078	0.1829
c_4	0.2145	0.2145	0.2564	0.2145	0.1001	0.2106	0.2106	0.2682	0.2106	0.1000	0.2074	0.2074	0.2777	0.2074	0.1000
c_5	0.1398	0.2765	0.2256	0.1791	0.1791	0.1292	0.2885	0.2321	0.1751	0.1751	0.1212	0.2943	0.2374	0.1735	0.1735
c_6	0.1025	0.2145	0.1799	0.2885	0.2145	0.1011	0.2145	0.1742	0.2958	0.2145	0.1004	0.2161	0.1688	0.2986	0.2161
	Results of $\overline{WO}^{(1,1)}(p_{ij})$					Results of $\overline{WO}^{(1,2)}(p_{ij})$					Results of $\overline{WO}^{(1,3)}(p_{ij})$				
c_1	0.0358	0.0283	0.0283	0.0358	0.0517	0.0364	0.0270	0.0270	0.0364	0.0532	0.0372	0.0260	0.0260	0.0372	0.0537
c_2	0.0644	0.0231	0.0416	0.0416	0.0493	0.0655	0.0225	0.0411	0.0411	0.0499	0.0658	0.0222	0.0405	0.0405	0.0509
c_3	0.0303	0.0433	0.0369	0.0192	0.0303	0.0297	0.0450	0.0375	0.0181	0.0297	0.0293	0.0462	0.0380	0.0172	0.0293
c_4	0.0300	0.0300	0.0359	0.0300	0.0140	0.0295	0.0295	0.0375	0.0295	0.0140	0.0290	0.0290	0.0389	0.0290	0.0140
c_5	0.0280	0.0553	0.0451	0.0358	0.0358	0.0258	0.0577	0.0464	0.0350	0.0350	0.0242	0.0589	0.0475	0.0347	0.0347
c_6	0.0103	0.0215	0.0180	0.0289	0.0215	0.0101	0.0214	0.0174	0.0296	0.0214	0.0100	0.0216	0.0169	0.0299	0.0216
	Results of $\overline{CI}_i^{(1,1)}$					Results of $\overline{CI}_i^{(1,2)}$					Results of $\overline{CI}_i^{(1,3)}$				
	0.1988	0.2015	0.2058	0.1913	0.2026	0.1970	0.2032	0.2070	0.1895	0.2033	0.1956	0.2039	0.2078	0.1885	0.2042
Rank	4	3	1	5	2	4	3	1	5	2	4	3	1	5	2
	Results of $CO^{(1,1)}(P_i)$					Results of $CO^{(1,2)}(P_i)$					Results of $CO^{(1,3)}(P_i)$				
	(0.4639, 0.5248)	(0.4468, 0.5291)	(0.4594, 0.5174)	(0.4473, 0.5357)	(0.4619, 0.5199)	(0.4628, 0.5273)	(0.4483, 0.5266)	(0.4601, 0.5159)	(0.4454, 0.5385)	(0.4627, 0.5187)	(0.4617, 0.5295)	(0.4488, 0.5256)	(0.4604, 0.5150)	(0.4442, 0.5402)	(0.4641, 0.5169)
	Results of $CI_i^{(1,1)}$					Results of $CI_i^{(1,2)}$					Results of $CI_i^{(1,3)}$				
	0.2019	0.1971	0.2028	0.1955	0.2027	0.2027	0.1966	0.2034	0.1933	0.2041	0.2031	0.1955	0.2044	0.1904	0.2066
Rank	3	4	1	5	2	3	4	2	5	1	3	4	2	5	1

a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5
Results of $MO^{(1,4)}(p_{ij})$					Results of $MO^{(1,5)}(p_{ij})$				
0.2106	0.1397	0.1397	0.2106	0.2995	0.2139	0.1361	0.1361	0.2139	0.2998
0.2997	0.1003	0.1822	0.1822	0.2356	0.2999	0.1001	0.1805	0.1805	0.2389
0.1810	0.2929	0.2404	0.1047	0.1810	0.1796	0.2955	0.2424	0.1028	0.1796
0.2052	0.2052	0.2845	0.2052	0.1000	0.2036	0.2036	0.2892	0.2036	0.1000
0.1156	0.2971	0.2412	0.1731	0.1731	0.1116	0.2984	0.2438	0.1731	0.1731
0.1002	0.2179	0.1645	0.2995	0.2179	0.1001	0.2195	0.1611	0.2998	0.2195
Results of $\overline{WO}^{(1,4)}(p_{ij})$					Results of $\overline{WO}^{(1,5)}(p_{ij})$				

Table 1 (continued)

a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5
0.0379	0.0251	0.0251	0.0379	0.0539	0.0385	0.0245	0.0245	0.0385	0.0540
0.0659	0.0221	0.0401	0.0401	0.0518	0.0660	0.0220	0.0397	0.0397	0.0526
0.0290	0.0469	0.0385	0.0167	0.0290	0.0287	0.0473	0.0388	0.0165	0.0287
0.0287	0.0287	0.0398	0.0287	0.0140	0.0285	0.0285	0.0405	0.0285	0.0140
0.0231	0.0594	0.0482	0.0346	0.0346	0.0223	0.0597	0.0488	0.0346	0.0346
0.0100	0.0218	0.0165	0.0300	0.0218	0.0100	0.0220	0.0161	0.0300	0.0220
Results of $\overline{CI}_i^{(1,4)}$					Results of $\overline{CI}_i^{(1,5)}$				
0.1947	0.2040	0.2082	0.1880	0.2051	0.1941	0.2040	0.2084	0.1878	0.2059
4	3	1	5	2	4	3	1	5	2
Results of $CO^{(1,4)}(P_i)$					Results of $CO^{(1,5)}(P_i)$				
(0.4609, 0.5309)	(0.4490, 0.5253)	(0.4605, 0.5147)	(0.4435, 0.5412)	(0.4653, 0.5152)	(0.4603, 0.5319)	(0.4489, 0.5253)	(0.4604, 0.5146)	(0.4431, 0.5417)	(0.4663, 0.5137)
Results of $CI_i^{(1,4)}$					Results of $CI_i^{(1,5)}$				
0.2034	0.1943	0.2054	0.1875	0.2094	0.2037	0.1929	0.2062	0.1848	0.2124
3	4	2	5	1	3	4	2	5	1

Table 2 Experimental results of mean outranking indices, weighted outranking grades, and comprehensive outranking indices in the cases of $\alpha = 2$ and $\beta = 1,2,\ldots,5$

c_j	a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5
	Results of $MO^{(2,1)}(p_{ij})$					Results of $MO^{(2,2)}(p_{ij})$					Results of $MO^{(2,3)}(p_{ij})$				
c_1	0.2010	0.1532	0.1532	0.2010	0.2915	0.2007	0.1515	0.1515	0.2007	0.2956	0.2033	0.1477	0.1477	0.2033	0.2981
c_2	0.2953	0.1016	0.1885	0.1885	0.2260	0.2977	0.1011	0.1878	0.1878	0.2257	0.2990	0.1006	0.1861	0.1861	0.2283
c_3	0.1891	0.2756	0.2345	0.1118	0.1891	0.1872	0.2803	0.2347	0.1106	0.1872	0.1850	0.2858	0.2364	0.1078	0.1850
c_4	0.2135	0.2135	0.2596	0.2135	0.1000	0.2118	0.2118	0.2646	0.2118	0.1000	0.2094	0.2094	0.2718	0.2094	0.1000
c_5	0.1337	0.2803	0.2292	0.1784	0.1784	0.1305	0.2870	0.2312	0.1757	0.1757	0.1251	0.2922	0.2347	0.1740	0.1740
c_6	0.1008	0.2149	0.1774	0.2919	0.2149	0.1006	0.2139	0.1759	0.2957	0.2139	0.1003	0.2146	0.1724	0.2981	0.2146
	Results of $\overline{WO}^{(2,1)}(p_{ij})$					Results of $\overline{WO}^{(2,2)}(p_{ij})$					Results of $\overline{WO}^{(2,3)}(p_{ij})$				
c_1	0.0362	0.0276	0.0276	0.0362	0.0525	0.0361	0.0273	0.0273	0.0361	0.0532	0.0366	0.0266	0.0266	0.0366	0.0537
c_2	0.0650	0.0224	0.0415	0.0415	0.0497	0.0655	0.0222	0.0413	0.0413	0.0497	0.0658	0.0221	0.0409	0.0409	0.0502
c_3	0.0303	0.0441	0.0375	0.0179	0.0303	0.0300	0.0448	0.0375	0.0177	0.0300	0.0296	0.0457	0.0378	0.0173	0.0296
c_4	0.0299	0.0299	0.0363	0.0299	0.0140	0.0296	0.0296	0.0371	0.0296	0.0140	0.0293	0.0293	0.0381	0.0293	0.0140
c_5	0.0267	0.0561	0.0458	0.0357	0.0357	0.0261	0.0574	0.0462	0.0351	0.0351	0.0250	0.0584	0.0469	0.0348	0.0348
c_6	0.0101	0.0215	0.0177	0.0292	0.0215	0.0101	0.0214	0.0176	0.0296	0.0214	0.0100	0.0215	0.0172	0.0298	0.0215
	Results of $\overline{CI}_i^{(2,1)}$					Results of $\overline{CI}_i^{(2,2)}$					Results of $\overline{CI}_i^{(2,3)}$				
	0.1981	0.2015	0.2065	0.1903	0.2036	0.1974	0.2028	0.2070	0.1895	0.2034	0.1963	0.2036	0.2076	0.1887	0.2037
Rank	4	3	1	5	2	4	3	1	5	2	4	3	1	5	2
	Results of $CO^{(2,1)}(P_i)$					Results of $CO^{(2,2)}(P_i)$					Results of $CO^{(2,3)}(P_i)$				
	(0.4636, 0.5257)	(0.4465, 0.5293)	(0.4599, 0.5165)	(0.4463, 0.5373)	(0.4630, 0.5182)	(0.4631, 0.5267)	(0.4477, 0.5273)	(0.4602, 0.5158)	(0.4455, 0.5385)	(0.4626, 0.5188)	(0.4623, 0.5283)	(0.4485, 0.5260)	(0.4605, 0.5151)	(0.4446, 0.5398)	(0.4633, 0.5180)
	Results of $CI_i^{(2,1)}$					Results of $CI_i^{(2,2)}$					Results of $CI_i^{(2,3)}$				
	0.2010	0.1967	0.2050	0.1927	0.2046	0.2016	0.1973	0.2041	0.1932	0.2038	0.2017	0.1971	0.2044	0.1920	0.2048
Rank	3	4	1	5	2	3	4	1	5	2	3	4	2	5	1

a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5
Results of $MO^{(2,4)}(p_{ij})$					Results of $MO^{(2,5)}(p_{ij})$				
0.2064	0.1440	0.1440	0.2064	0.2992	0.2095	0.1407	0.1407	0.2095	0.2997
0.2996	0.1003	0.1844	0.1844	0.2314	0.2998	0.1001	0.1828	0.1828	0.2345
0.1832	0.2900	0.2382	0.1055	0.1832	0.1818	0.2929	0.2398	0.1037	0.1818
0.2073	0.2073	0.2781	0.2073	0.1000	0.2056	0.2056	0.2832	0.2056	0.1000
0.1203	0.2953	0.2379	0.1732	0.1732	0.1164	0.2972	0.2406	0.1729	0.1729
0.1001	0.2159	0.1689	0.2992	0.2159	0.1001	0.2173	0.1656	0.2997	0.2173
Results of $\overline{WO}^{(2,4)}(p_{ij})$					Results of $\overline{WO}^{(2,5)}(p_{ij})$				

Table 2 (continued)

a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5
0.0372	0.0259	0.0259	0.0372	0.0539	0.0377	0.0253	0.0253	0.0377	0.0539
0.0659	0.0221	0.0406	0.0406	0.0509	0.0660	0.0220	0.0402	0.0402	0.0516
0.0293	0.0464	0.0381	0.0169	0.0293	0.0291	0.0469	0.0384	0.0166	0.0291
0.0290	0.0290	0.0389	0.0290	0.0140	0.0288	0.0288	0.0397	0.0288	0.0140
0.0241	0.0591	0.0476	0.0346	0.0346	0.0233	0.0594	0.0481	0.0346	0.0346
0.0100	0.0216	0.0169	0.0299	0.0216	0.0100	0.0217	0.0166	0.0300	0.0217
Results of $\overline{CI}_i^{(2,4)}$					Results of $\overline{CI}_i^{(2,5)}$				
0.1955	0.2040	0.2080	0.1882	0.2043	0.1948	0.2042	0.2083	0.1878	0.2049
4	3	1	5	2	4	3	1	5	2
Results of $CO^{(2,4)}(P_i)$					Results of $CO^{(2,5)}(P_i)$				
(0.4616, 0.5296)	(0.4489, 0.5253)	(0.4606, 0.5147)	(0.4439, 0.5408)	(0.4641, 0.5168)	(0.4610, 0.5307)	(0.4491, 0.5251)	(0.4606, 0.5145)	(0.4434, 0.5414)	(0.4650, 0.5156)
Results of $CI_i^{(2,4)}$					Results of $CI_i^{(2,5)}$				
0.2018	0.1967	0.2049	0.1904	0.2062	0.2019	0.1961	0.2053	0.1889	0.2078
3	4	2	5	1	3	4	2	5	1

Table 3 Experimental results of mean outranking indices, weighted outranking grades, and comprehensive outranking indices in the cases of $\alpha = 3$ and $\beta = 1,2,\ldots,5$

c_j	a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5
	Results of $MO^{(3,1)}(p_{ij})$					Results of $MO^{(3,2)}(p_{ij})$					Results of $MO^{(3,3)}(p_{ij})$				
c_1	0.2041	0.1485	0.1485	0.2041	0.2947	0.2022	0.1496	0.1496	0.2022	0.2965	0.2035	0.1474	0.1474	0.2035	0.2982
c_2	0.2972	0.1005	0.1866	0.1866	0.2291	0.2982	0.1004	0.1871	0.1871	0.2272	0.2991	0.1003	0.1861	0.1861	0.2285
c_3	0.1873	0.2812	0.2378	0.1065	0.1873	0.1870	0.2823	0.2364	0.1072	0.1870	0.1855	0.2859	0.2370	0.1061	0.1855
c_4	0.2116	0.2116	0.2651	0.2116	0.1000	0.2113	0.2113	0.2661	0.2113	0.1000	0.2097	0.2097	0.2710	0.2097	0.1000
c_5	0.1271	0.2847	0.2334	0.1774	0.1774	0.1278	0.2879	0.2329	0.1757	0.1757	0.1246	0.2918	0.2349	0.1743	0.1743
c_6	0.1002	0.2158	0.1733	0.2949	0.2158	0.1002	0.2144	0.1744	0.2966	0.2144	0.1001	0.2147	0.1723	0.2982	0.2147
	Results of $\overline{WO}^{(3,1)}(p_{ij})$					Results of $\overline{WO}^{(3,2)}(p_{ij})$					Results of $\overline{WO}^{(3,3)}(p_{ij})$				
c_1	0.0367	0.0267	0.0267	0.0367	0.0531	0.0364	0.0269	0.0269	0.0364	0.0534	0.0366	0.0265	0.0265	0.0366	0.0537
c_2	0.0654	0.0221	0.0411	0.0411	0.0504	0.0656	0.0221	0.0412	0.0412	0.0500	0.0658	0.0221	0.0409	0.0409	0.0503
c_3	0.0300	0.0450	0.0380	0.0170	0.0300	0.0299	0.0452	0.0378	0.0172	0.0299	0.0297	0.0457	0.0379	0.0170	0.0297
c_4	0.0296	0.0296	0.0371	0.0296	0.0140	0.0296	0.0296	0.0373	0.0296	0.0140	0.0294	0.0294	0.0379	0.0294	0.0140
c_5	0.0254	0.0569	0.0467	0.0355	0.0355	0.0256	0.0576	0.0466	0.0351	0.0351	0.0249	0.0584	0.0470	0.0349	0.0349
c_6	0.0100	0.0216	0.0173	0.0295	0.0216	0.0100	0.0214	0.0174	0.0297	0.0214	0.0100	0.0215	0.0172	0.0298	0.0215
	Results of $\overline{CI}_i^{(3,1)}$					Results of $\overline{CI}_i^{(3,2)}$					Results of $\overline{CI}_i^{(3,3)}$				
	0.1972	0.2020	0.2070	0.1894	0.2045	0.1971	0.2028	0.2072	0.1891	0.2039	0.1964	0.2035	0.2076	0.1886	0.2039
Rank	4	3	1	5	2	4	3	1	5	2	4	3	1	5	2
	Results of $CO^{(3,1)}(P_i)$					Results of $CO^{(3,2)}(P_i)$					Results of $CO^{(3,3)}(P_i)$				
	(0.4629, 0.5271)	(0.4469, 0.5286)	(0.4601, 0.5160)	(0.4453, 0.5387)	(0.4641, 0.5167)	(0.4629, 0.5272)	(0.4477, 0.5273)	(0.4603, 0.5156)	(0.4451, 0.5391)	(0.4633, 0.5178)	(0.4624, 0.5282)	(0.4484, 0.5262)	(0.4605, 0.5151)	(0.4445, 0.5400)	(0.4635, 0.5176)
	Results of $CI_i^{(3,1)}$					Results of $CI_i^{(3,2)}$					Results of $CI_i^{(3,3)}$				
	0.1998	0.1966	0.2073	0.1889	0.2074	0.2009	0.1974	0.2052	0.1915	0.2050	0.2011	0.1974	0.2052	0.1911	0.2053
Rank	3	4	2	5	1	3	4	1	5	2	3	4	2	5	1

a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5
Results of $MO^{(3,4)}(p_{ij})$					Results of $MO^{(3,5)}(p_{ij})$				
0.2057	0.1448	0.1448	0.2057	0.2992	0.2081	0.1421	0.1421	0.2081	0.2996
0.2996	0.1001	0.1848	0.1848	0.2307	0.2998	0.1001	0.1835	0.1835	0.2331
0.1839	0.2892	0.2381	0.1048	0.1839	0.1826	0.2919	0.2393	0.1036	0.1826
0.2080	0.2080	0.2761	0.2080	0.1000	0.2065	0.2065	0.2806	0.2065	0.1000
0.1212	0.2946	0.2372	0.1735	0.1735	0.1180	0.2964	0.2394	0.1731	0.1731
0.1001	0.2155	0.1697	0.2992	0.2155	0.1000	0.2166	0.1671	0.2996	0.2166
Results of $\overline{WO}^{(3,4)}(p_{ij})$					Results of $\overline{WO}^{(3,5)}(p_{ij})$				
0.0370	0.0261	0.0261	0.0370	0.0538	0.0375	0.0256	0.0256	0.0375	0.0539
0.0659	0.0220	0.0407	0.0407	0.0507	0.0660	0.0220	0.0404	0.0404	0.0513

Table 3 (continued)

a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5
0.0294	0.0463	0.0381	0.0168	0.0294	0.0292	0.0467	0.0383	0.0166	0.0292
0.0291	0.0291	0.0386	0.0291	0.0140	0.0289	0.0289	0.0393	0.0289	0.0140
0.0242	0.0589	0.0474	0.0347	0.0347	0.0236	0.0593	0.0479	0.0346	0.0346
0.0100	0.0216	0.0170	0.0299	0.0216	0.0100	0.0217	0.0167	0.0300	0.0217
Results of $\overline{CI}_i^{(3,4)}$					Results of $\overline{CI}_i^{(3,5)}$				
0.1957	0.2040	0.2079	0.1882	0.2043	0.1951	0.2041	0.2081	0.1879	0.2047
4	3	1	5	2	4	3	1	5	2
Results of $CO^{(3,4)}(P_i)$					Results of $CO^{(3,5)}(P_i)$				
(0.4618, 0.5293)	(0.4488, 0.5255)	(0.4606, 0.5148)	(0.4440, 0.5407)	(0.4640, 0.5169)	(0.4613, 0.5302)	(0.4490, 0.5251)	(0.4606, 0.5146)	(0.4436, 0.5413)	(0.4647, 0.5160)
Results of $CI_i^{(3,4)}$					Results of $CI_i^{(3,5)}$				
0.2011	0.1972	0.2054	0.1902	0.2061	0.2012	0.1968	0.2056	0.1891	0.2072
3	4	2	5	1	3	4	2	5	1

Table 4 Experimental results of mean outranking indices, weighted outranking grades, and comprehensive outranking indices in the cases of $\alpha = 4$ and $\beta = 1,2,\ldots,5$

c_j	a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5
	Results of $MO^{(4,1)}(p_{ij})$					Results of $MO^{(4,2)}(p_{ij})$					Results of $MO^{(4,3)}(p_{ij})$				
c_1	0.2071	0.1445	0.1445	0.2071	0.2968	0.2042	0.1471	0.1471	0.2042	0.2975	0.2046	0.1461	0.1461	0.2046	0.2985
c_2	0.2983	0.1001	0.1847	0.1847	0.2321	0.2987	0.1002	0.1860	0.1860	0.2292	0.2993	0.1001	0.1855	0.1855	0.2296
c_3	0.1853	0.2855	0.2403	0.1036	0.1853	0.1862	0.2848	0.2382	0.1047	0.1862	0.1852	0.2870	0.2381	0.1044	0.1852
c_4	0.2099	0.2099	0.2704	0.2099	0.1000	0.2104	0.2104	0.2688	0.2104	0.1000	0.2094	0.2094	0.2719	0.2094	0.1000
c_5	0.1218	0.2880	0.2367	0.1767	0.1767	0.1245	0.2893	0.2349	0.1756	0.1756	0.1230	0.2921	0.2359	0.1745	0.1745
c_6	0.1001	0.2169	0.1694	0.2968	0.2169	0.1001	0.2152	0.1720	0.2975	0.2152	0.1001	0.2152	0.1711	0.2985	0.2152
	Results of $\overline{WO}^{(4,1)}(p_{ij})$					Results of $\overline{WO}^{(4,2)}(p_{ij})$					Results of $\overline{WO}^{(4,3)}(p_{ij})$				
c_1	0.0373	0.0260	0.0260	0.0373	0.0534	0.0367	0.0265	0.0265	0.0367	0.0535	0.0368	0.0263	0.0263	0.0368	0.0537
c_2	0.0656	0.0220	0.0406	0.0406	0.0511	0.0657	0.0220	0.0409	0.0409	0.0504	0.0658	0.0220	0.0408	0.0408	0.0505
c_3	0.0297	0.0457	0.0384	0.0166	0.0297	0.0298	0.0456	0.0381	0.0167	0.0298	0.0296	0.0459	0.0381	0.0167	0.0296
c_4	0.0294	0.0294	0.0379	0.0294	0.0140	0.0295	0.0295	0.0376	0.0295	0.0140	0.0293	0.0293	0.0381	0.0293	0.0140
c_5	0.0244	0.0576	0.0473	0.0353	0.0353	0.0249	0.0579	0.0470	0.0351	0.0351	0.0246	0.0584	0.0472	0.0349	0.0349
c_6	0.0100	0.0217	0.0169	0.0297	0.0217	0.0100	0.0215	0.0172	0.0297	0.0215	0.0100	0.0215	0.0171	0.0299	0.0215
	Results of $\overline{CI}_i^{(4,1)}$					Results of $\overline{CI}_i^{(4,2)}$					Results of $\overline{CI}_i^{(4,3)}$				
	0.1963	0.2024	0.2072	0.1889	0.2052	0.1966	0.2029	0.2073	0.1887	0.2044	0.1962	0.2035	0.2076	0.1884	0.2043
Rank	4	3	1	5	2	4	3	1	5	2	4	3	1	5	2
	Results of $CO^{(4,1)}(P_i)$					Results of $CO^{(4,2)}(P_i)$					Results of $CO^{(4,3)}(P_i)$				
	(0.4622, 0.5284)	(0.4473, 0.5279)	(0.4600, 0.5159)	(0.4447, 0.5397)	(0.4650, 0.5154)	(0.4625, 0.5279)	(0.4478, 0.5271)	(0.4603, 0.5155)	(0.4446, 0.5398)	(0.4640, 0.5169)	(0.4622, 0.5285)	(0.4483, 0.5262)	(0.4604, 0.5152)	(0.4443, 0.5403)	(0.4639, 0.5170)
	Results of $CI_i^{(4,1)}$					Results of $CI_i^{(4,2)}$					Results of $CI_i^{(4,3)}$				
	0.1980	0.1967	0.2095	0.1851	0.2107	0.2001	0.1974	0.2065	0.1895	0.2065	0.2005	0.1976	0.2060	0.1898	0.2062
Rank	3	4	2	5	1	3	4	2	5	1	3	4	2	5	1

a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5
Results of $MO^{(4,4)}(p_{ij})$					Results of $MO^{(4,5)}(p_{ij})$				
0.2061	0.1443	0.1443	0.2061	0.2992	0.2080	0.1422	0.1422	0.2080	0.2996
0.2996	0.1001	0.1846	0.1846	0.2311	0.2998	0.1000	0.1836	0.1836	0.2330
0.1841	0.2895	0.2387	0.1037	0.1841	0.1830	0.2916	0.2395	0.1030	0.1830
0.2080	0.2080	0.2759	0.2080	0.1000	0.2068	0.2068	0.2797	0.2068	0.1000
0.1205	0.2944	0.2375	0.1738	0.1738	0.1180	0.2961	0.2392	0.1733	0.1733
0.1000	0.2157	0.1692	0.2992	0.2157	0.1000	0.2166	0.1672	0.2996	0.2166
Results of $\overline{WO}^{(4,4)}(p_{ij})$					Results of $\overline{WO}^{(4,5)}(p_{ij})$				
0.0371	0.0260	0.0260	0.0371	0.0539	0.0374	0.0256	0.0256	0.0374	0.0539

Table 4 (continued)

a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5
0.0659	0.0220	0.0406	0.0406	0.0508	0.0660	0.0220	0.0404	0.0404	0.0513
0.0294	0.0463	0.0382	0.0166	0.0294	0.0293	0.0467	0.0383	0.0165	0.0293
0.0291	0.0291	0.0386	0.0291	0.0140	0.0289	0.0289	0.0392	0.0289	0.0140
0.0241	0.0589	0.0475	0.0348	0.0348	0.0236	0.0592	0.0478	0.0347	0.0347
0.0100	0.0216	0.0169	0.0299	0.0216	0.0100	0.0217	0.0167	0.0300	0.0217
Results of $\overline{CI}_i^{(4,4)}$					Results of $\overline{CI}_i^{(4,5)}$				
0.1957	0.2039	0.2078	0.1881	0.2045	0.1952	0.2041	0.2080	0.1879	0.2048
4	3	1	5	2	4	3	1	5	2
Results of $CO^{(4,4)}(P_i)$					Results of $CO^{(4,5)}(P_i)$				
(0.4618, 0.5293)	(0.4487, 0.5256)	(0.4605, 0.5149)	(0.4439, 0.5408)	(0.4642, 0.5165)	(0.4614, 0.5300)	(0.4490, 0.5252)	(0.4606, 0.5147)	(0.4436, 0.5413)	(0.4647, 0.5159)
Results of $CI_i^{(4,4)}$					Results of $CI_i^{(4,5)}$				
0.2006	0.1974	0.2060	0.1893	0.2067	0.2007	0.1972	0.2061	0.1885	0.2075
3	4	2	5	1	3	4	2	5	1

Table 5 Experimental results of mean outranking indices, weighted outranking grades, and comprehensive outranking indices in the cases of $\alpha = 5$ and $\beta = 1,2,\ldots,5$

c_j	a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5
	Results of $MO^{(5,1)}(p_{ij})$					Results of $MO^{(5,2)}(p_{ij})$					Results of $MO^{(5,3)}(p_{ij})$				
c_1	0.2099	0.1411	0.1411	0.2099	0.2980	0.2063	0.1446	0.1446	0.2063	0.2982	0.2061	0.1445	0.1445	0.2061	0.2988
c_2	0.2990	0.1000	0.1831	0.1831	0.2349	0.2991	0.1001	0.1848	0.1848	0.2313	0.2994	0.1000	0.1847	0.1847	0.2311
c_3	0.1836	0.2887	0.2422	0.1020	0.1836	0.1851	0.2871	0.2397	0.1029	0.1851	0.1847	0.2883	0.2393	0.1031	0.1847
c_4	0.2084	0.2084	0.2749	0.2084	0.1000	0.2094	0.2094	0.2717	0.2094	0.1000	0.2088	0.2088	0.2735	0.2088	0.1000
c_5	0.1176	0.2906	0.2393	0.1763	0.1763	0.1214	0.2907	0.2369	0.1755	0.1755	0.1210	0.2926	0.2371	0.1746	0.1746
c_6	0.1000	0.2179	0.1661	0.2980	0.2179	0.1000	0.2161	0.1696	0.2982	0.2161	0.1000	0.2158	0.1695	0.2988	0.2158
	Results of $\overline{WO}^{(5,1)}(p_{ij})$					Results of $\overline{WO}^{(5,2)}(p_{ij})$					Results of $\overline{WO}^{(5,3)}(p_{ij})$				
c_1	0.0378	0.0254	0.0254	0.0378	0.0536	0.0371	0.0260	0.0260	0.0371	0.0537	0.0371	0.0260	0.0260	0.0371	0.0538
c_2	0.0658	0.0220	0.0403	0.0403	0.0517	0.0658	0.0220	0.0407	0.0407	0.0509	0.0659	0.0220	0.0406	0.0406	0.0508
c_3	0.0294	0.0462	0.0388	0.0163	0.0294	0.0296	0.0459	0.0384	0.0165	0.0296	0.0296	0.0461	0.0383	0.0165	0.0296
c_4	0.0292	0.0292	0.0385	0.0292	0.0140	0.0293	0.0293	0.0380	0.0293	0.0140	0.0292	0.0292	0.0383	0.0292	0.0140
c_5	0.0235	0.0581	0.0479	0.0353	0.0353	0.0243	0.0581	0.0474	0.0351	0.0351	0.0242	0.0585	0.0474	0.0349	0.0349
c_6	0.0100	0.0218	0.0166	0.0298	0.0218	0.0100	0.0216	0.0170	0.0298	0.0216	0.0100	0.0216	0.0170	0.0299	0.0216
	Results of $\overline{CI}_i^{(5,1)}$					Results of $\overline{CI}_i^{(5,2)}$					Results of $\overline{CI}_i^{(5,3)}$				
	0.1956	0.2027	0.2074	0.1886	0.2057	0.1961	0.2031	0.2074	0.1885	0.2049	0.1959	0.2035	0.2076	0.1883	0.2047
Rank	4	3	1	5	2	4	3	1	5	2	4	3	1	5	2
	Results of $CO^{(5,1)}(P_i)$					Results of $CO^{(5,2)}(P_i)$					Results of $CO^{(5,3)}(P_i)$				
	(0.4616, 0.5295)	(0.4476, 0.5274)	(0.4600, 0.5158)	(0.4442, 0.5402)	(0.4658, 0.5143)	(0.4621, 0.5286)	(0.4479, 0.5268)	(0.4602, 0.5155)	(0.4443, 0.5402)	(0.4646, 0.5159)	(0.4620, 0.5289)	(0.4484, 0.5262)	(0.4604, 0.5152)	(0.4441, 0.5406)	(0.4644, 0.5162)
	Results of $CI_i^{(5,1)}$					Results of $CI_i^{(5,2)}$					Results of $CI_i^{(5,3)}$				
	0.1960	0.1968	0.2116	0.1815	0.2141	0.1991	0.1975	0.2077	0.1874	0.2083	0.1998	0.1976	0.2069	0.1883	0.2075
Rank	4	3	2	5	1	3	4	2	5	1	3	4	2	5	1

a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5
Results of $MO^{(5,4)}(p_{ij})$					Results of $MO^{(5,5)}(p_{ij})$				
0.2070	0.1433	0.1433	0.2070	0.2994	0.2085	0.1417	0.1417	0.2085	0.2997
0.2997	0.1000	0.1841	0.1841	0.2320	0.2998	0.1000	0.1834	0.1834	0.2335
0.1838	0.2901	0.2395	0.1028	0.1838	0.1829	0.2918	0.2400	0.1023	0.1829
0.2078	0.2078	0.2766	0.2078	0.1000	0.2067	0.2067	0.2798	0.2067	0.1000
0.1194	0.2945	0.2382	0.1740	0.1740	0.1174	0.2960	0.2395	0.1736	0.1736
0.1000	0.2162	0.1683	0.2994	0.2162	0.1000	0.2168	0.1667	0.2997	0.2168
Results of $\overline{WO}^{(5,4)}(p_{ij})$					Results of $\overline{WO}^{(5,5)}(p_{ij})$				
0.0373	0.0258	0.0258	0.0373	0.0539	0.0375	0.0255	0.0255	0.0375	0.0539

Table 5 (continued)

a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5
0.0659	0.0220	0.0405	0.0405	0.0510	0.0660	0.0220	0.0403	0.0403	0.0514
0.0294	0.0464	0.0383	0.0164	0.0294	0.0293	0.0467	0.0384	0.0164	0.0293
0.0291	0.0291	0.0387	0.0291	0.0140	0.0289	0.0289	0.0392	0.0289	0.0140
0.0239	0.0589	0.0476	0.0348	0.0348	0.0235	0.0592	0.0479	0.0347	0.0347
0.0100	0.0216	0.0168	0.0299	0.0216	0.0100	0.0217	0.0167	0.0300	0.0217
Results of $\overline{CI}_i^{(5,4)}$					Results of $\overline{CI}_i^{(5,5)}$				
0.1956	0.2038	0.2078	0.1880	0.2048	0.1952	0.2040	0.2080	0.1878	0.2050
4	3	1	5	2	4	3	1	5	2
Results of $CO^{(5,4)}(P_i)$					Results of $CO^{(5,5)}(P_i)$				
(0.4617, 0.5295)	(0.4487, 0.5257)	(0.4604, 0.5150)	(0.4438, 0.5410)	(0.4646, 0.5160)	(0.4613, 0.5301)	(0.4489, 0.5253)	(0.4605, 0.5148)	(0.4435, 0.5414)	(0.4649, 0.5156)
Results of $CI_i^{(5,4)}$					Results of $CI_i^{(5,5)}$				
0.2000	0.1975	0.2067	0.1882	0.2076	0.2001	0.1974	0.2067	0.1877	0.2082
3	4	2	5	1	3	4	2	5	1

ordinary normalized weights. To demonstrate the implementation process of Algorithm II, this paper additionally provides a set of PF importance weights in line with the original priority orders of ordinary normalized weights. In the following, the realistic case of the hospital selection problem to demonstrate the two algorithmic procedures and validate the applicability of the PF likelihood-oriented methodology is presented.

In Step I.1 of Algorithm I and Step II.1 of Algorithm II, five hospitals served as available alternative pilot hospitals in the postacute care program for cerebrovascular diseases. The collection of candidate alternatives was given by $A = \{a_1$ (Hospital #1), a_2 (Hospital #2),…, a_5 (Hospital #5)$\}$, in which the number of alternatives $m = 5$. Six evaluative criteria were considered to appraise the alternatives in connection with the effectiveness of hospital-based postacute care for acute stroke management. The collection of evaluative criteria was denoted as $C = \{c_1$ (rationality of the transitional care system), c_2 (integration of health care institutions), c_3 (adequacy of manpower allocation), c_4 (completeness of care-related equipment), c_5 (assurance of health care quality), c_6 (uniqueness and potential of postacute care)$\}$, where the number of criteria $n = 6$. Afterward, the parameter values were designated within the beta function $B(\alpha, \beta)$ as follows: $\alpha = 1$ and $\beta = 1$.

In Step I.2 of Algorithm I (as well as Step II.2 of Algorithm II), the degrees μ_{ij} and v_{ij} embedded in p_{ij} can be determined through satisfaction and dissatisfaction surveys, respectively. Because the pilot hospital selection problem was adopted from Chen [8], PF evaluative ratings were obtained by utilizing the conversion of interval-valued PF evaluation data by the midpoint approach. More precisely, Chen [8] utilized a seven-point linguistic rating scale to gather the linguistic assessment data provided by the National Health Insurance Administration and then externalized these assessments by interval-valued PF numbers. This study utilized a convenient midpoint approach to convert interval-valued PF evaluations into PF evaluative ratings. Using p_{11} as an example, the interval-valued PF evaluation conveyed by Chen [8] was ([0.6, 0.7], [0.3, 0.4]). With the aid of the midpoint approach, $p_{11} = ((0.6 + 0.7)/2,\ (0.3 + 0.4)/2) = (0.65,\ 0.35)$. Next, the collection of PF evaluative ratings was represented as the following PF evaluation matrix:

$$P = \left[p_{ij}\right]_{5\times 6} = \left[(\mu_{ij}, \nu_{ij})\right]_{5\times 6} = \begin{bmatrix} (0.65, 0.35) & (0.85, 0.15) & (0.55, 0.45) & (0.65, 0.35) & (0.55, 0.45) & (0.25, 0.75) \\ (0.55, 0.45) & (0.25, 0.75) & (0.75, 0.25) & (0.65, 0.35) & (0.85, 0.15) & (0.65, 0.35) \\ (0.55, 0.45) & (0.55, 0.45) & (0.65, 0.35) & (0.75, 0.25) & (0.75, 0.25) & (0.55, 0.45) \\ (0.65, 0.35) & (0.55, 0.45) & (0.35, 0.65) & (0.65, 0.35) & (0.65, 0.35) & (0.85, 0.15) \\ (0.85, 0.15) & (0.65, 0.35) & (0.55, 0.45) & (0.25, 0.75) & (0.65, 0.35) & (0.65, 0.35) \end{bmatrix}.$$

In addition, the PF characteristic P_i can be rendered by concentering p_{i1}, $p_{i2}, \ldots$, p_{i6}. More precisely, $P_i = \{\langle c_j, p_{ij}\rangle | c_j \in C\} = \{\langle c_1, (\mu_{i1}, \nu_{i1})\rangle, \langle c_2, (\mu_{i2}, \nu_{i2})\rangle, \cdots, \langle c_6, (\mu_{i6}, \nu_{i6})\rangle\}$. On the one hand, in Step I.3 of Algorithm I, the preference information of the six evaluative criteria was expressed as a set of ordinary normalized weights. In accordance with Chen [8], the weight vector $\overline{W}$ =(0.18, 0.22, 0.16, 0.14, 0.20, 0.10), in which $\sum_{j=1}^{6} \overline{w}_j = 1$. On the other hand, in Step II.3 of Algorithm II, the preference information of criteria was represented as a PF set W of Pythagorean membership grades $w_j = (\omega_j, \varpi_j)$ for $j \in \{1, 2, \cdots, 6\}$, i.e., $W = \{\langle c_j, w_j\rangle | c_j \in C\} = \{\langle c_j, (\omega_j, \varpi_j)\rangle | c_j \in C\}$. The collection of six PF importance weights, conforming to the relative importance of the nonfuzzy weights more or less, was determined as follows: $W = \{\langle c_1, (0.41, 0.59)\rangle, \langle c_2, (0.55, 0.45)\rangle, \langle c_3, (0.38, 0.62)\rangle, \langle c_4, (0.35, 0.65)\rangle, \langle c_5, (0.45, 0.55)\rangle, \langle c_6, (0.32, 0.68)\rangle\}$.

In Step I.4 of Algorithm I and Step II.4 of Algorithm II, the parametric likelihood measures $L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j})$ $(i, \kappa = 1, 2, \ldots, 5$ and $j = 1, 2, \ldots, 6)$ were determined under the setting of parameters α = 1 and β = 1. Consider $L^{(1,1)}(p_{11} \geq p_{21})$ between p_{11} (= (0.65, 0.35)) and p_{21} (= (0.55, 0.45)) as an example. Because α = β = 1, Eq. (16) can be directly applied instead of Eq. (15) for convenience. It can be observed that the fifth condition $(\mu_{21})^2 < (\mu_{11})^2 \leq 1 - (\nu_{21})^2 \leq 1 - (\nu_{11})^2$ of Eq. (16) holds, i.e., $0.55^2 < 0.65^2 \leq 1 - 0.45^2 \leq 1 - 0.35^2$. Based on Eq. (16),

$$L^{(1,1)}(p_{11} \geq p_{21}) = 1 - \frac{\left(1 - (\mu_{11})^2 - (\nu_{21})^2\right)^2}{2\left(1 - (\mu_{11})^2 - (\nu_{11})^2\right)\left(1 - (\mu_{21})^2 - (\nu_{21})^2\right)}$$
$$= 1 - \frac{(1 - 0.65^2 - 0.45^2)^2}{2(1 - 0.65^2 - 0.35^2)(1 - 0.55^2 - 0.45^2)} = 0.6878.$$

By collecting the results of $L^{(1,1)}(p_{ij} \geq p_{\kappa j})$ for all $a_i, a_\kappa \in A$ (= $\{a_1, a_2, \ldots, a_5\}$) on the subject of each $c_j \in C$ (= $\{c_1, c_2, \cdots, c_6\}$), the parametric likelihood matrices $\mathrm{L}_j^{(1,1)}$ $(j = 1, 2, \ldots, 6)$ were constructed as follows:

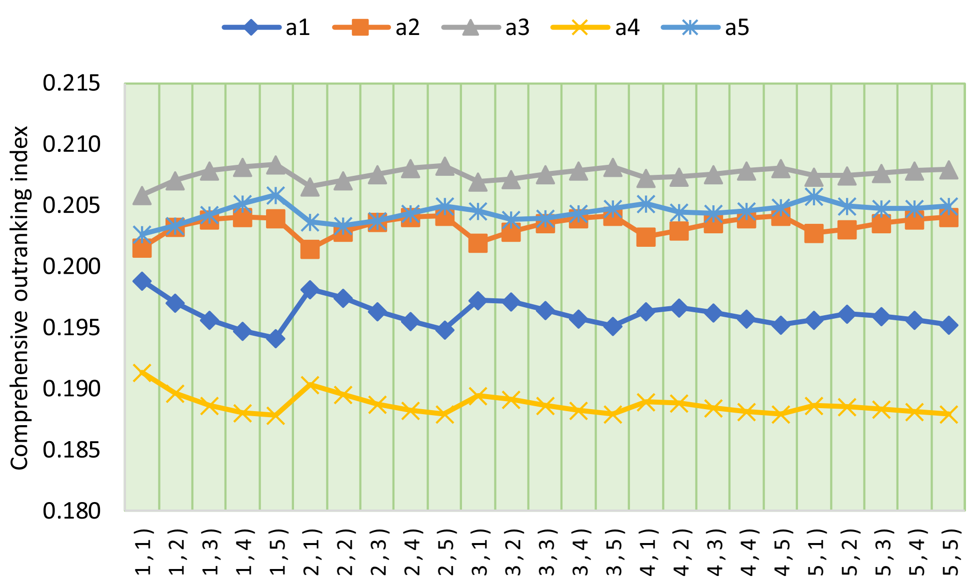
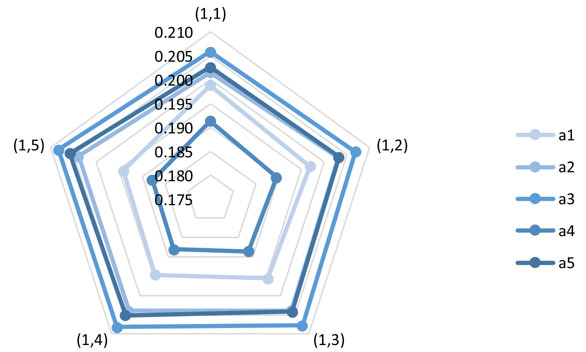
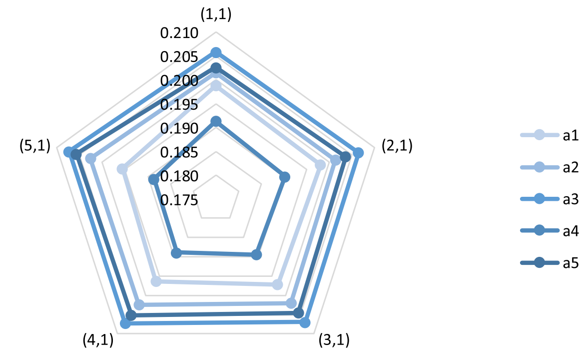


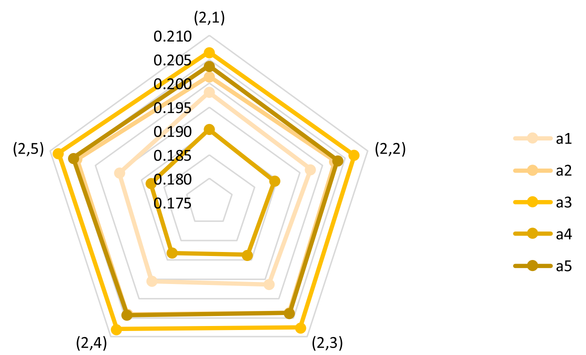
Fig. 5 Comparison of comprehensive outranking indices under various settings of (α, β)



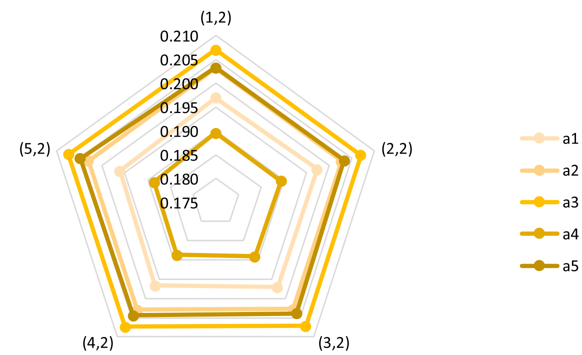
(a) Contrast when α=1 and β=1,2,$\cdots$,5



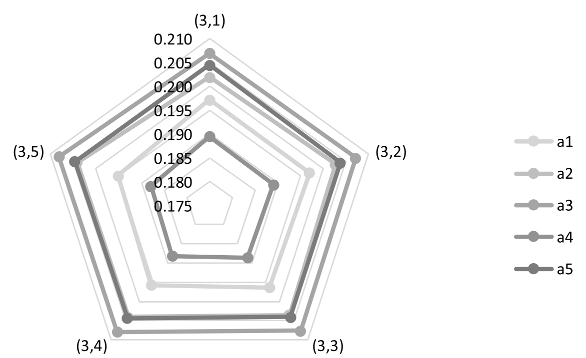
(b) Contrast when β=1 and α=1,2,$\cdots$,5



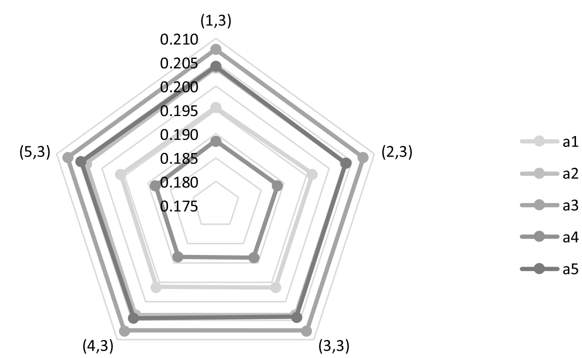
(c) Contrast when α=2 and β=1,2,$\cdots$,5



(d) Contrast when β=2 and α=1,2,$\cdots$,5

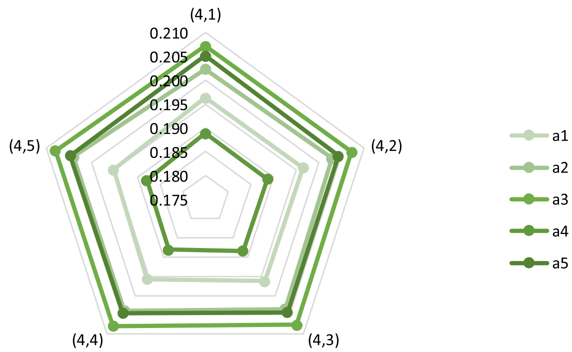


(e) Contrast when α=3 and β=1,2,$\cdots$,5

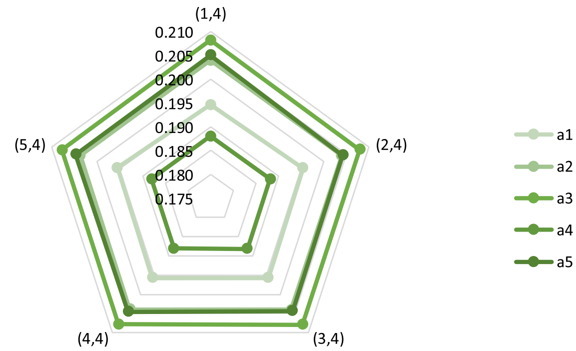


(f) Contrast when β=3 and α=1,2,$\cdots$,5

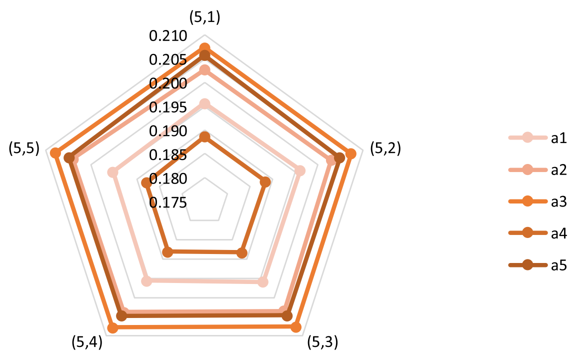
Fig. 6 Contrast of the $\overline{CI}_i^{(\alpha,\beta)}$ values among all alternatives yielded by Algorithm I



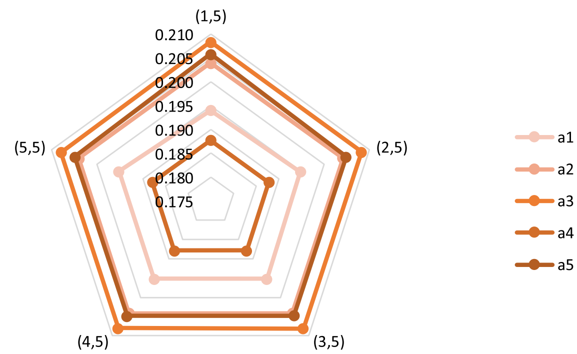
(g) Contrast when α=4 and β=1,2,$\cdots$,5



(h) Contrast when β=4 and α=1,2,$\cdots$,5



(i) Contrast when α=5 and β=1,2,$\cdots$,5



(j) Contrast when β=5 and α=1,2,$\cdots$,5

Fig. 6 continued

$$\mathrm{L}_1^{(1,1)} = \left[L^{(1,1)}(p_{i1} \geq p_{\kappa 1})\right]_{5\times 5} = \begin{bmatrix} 0.5000 & 0.6878 & 0.6878 & 0.5000 & 0.1035 \\ 0.3122 & 0.5000 & 0.5000 & 0.3122 & 0.0223 \\ 0.3122 & 0.5000 & 0.5000 & 0.3122 & 0.0223 \\ 0.5000 & 0.6878 & 0.6878 & 0.5000 & 0.1035 \\ 0.8965 & 0.9777 & 0.9777 & 0.8965 & 0.5000 \end{bmatrix},$$

$$\mathrm{L}_2^{(1,1)} = \left[L^{(1,1)}(p_{i2} \geq p_{\kappa 2})\right]_{5\times 5} = \begin{bmatrix} 0.5000 & 1.0000 & 0.9777 & 0.9777 & 0.8965 \\ 0.0000 & 0.5000 & 0.0491 & 0.0491 & 0.0007 \\ 0.0223 & 0.9509 & 0.5000 & 0.5000 & 0.3122 \\ 0.0223 & 0.9509 & 0.5000 & 0.5000 & 0.3122 \\ 0.1035 & 0.9993 & 0.6878 & 0.6878 & 0.5000 \end{bmatrix},$$

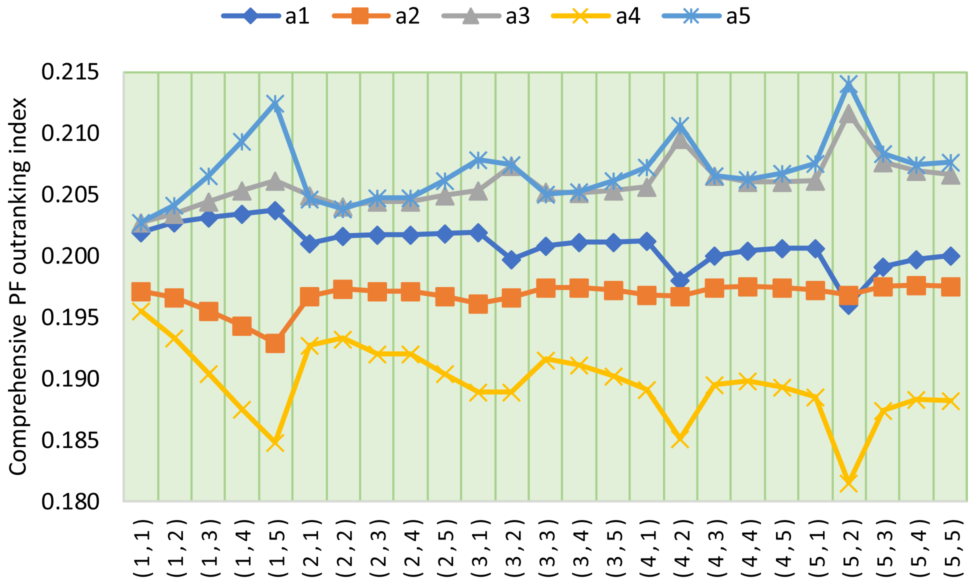


Fig. 7 Comparison of comprehensive PF outranking indices under various settings of (α, β)

$$\mathrm{L}_3^{(1,1)} = \left[L^{(1,1)}(p_{i3} \geq p_{\kappa 3})\right]_{5\times 5} = \begin{bmatrix} 0.5000 & 0.1488 & 0.3122 & 0.8321 & 0.5000 \\ 0.8512 & 0.5000 & 0.7092 & 0.9993 & 0.8512 \\ 0.6878 & 0.2908 & 0.5000 & 0.9420 & 0.6878 \\ 0.1679 & 0.0007 & 0.0580 & 0.5000 & 0.1679 \\ 0.5000 & 0.1488 & 0.3122 & 0.8321 & 0.5000 \end{bmatrix},$$

$$\mathrm{L}_4^{(1,1)} = \left[L^{(1,1)}(p_{i4} \geq p_{\kappa 4})\right]_{5\times 5} = \begin{bmatrix} 0.5000 & 0.5000 & 0.2908 & 0.5000 & 0.9993 \\ 0.5000 & 0.5000 & 0.2908 & 0.5000 & 0.9993 \\ 0.7092 & 0.7092 & 0.5000 & 0.7092 & 1.0000 \\ 0.5000 & 0.5000 & 0.2908 & 0.5000 & 0.9993 \\ 0.0007 & 0.0007 & 0.0000 & 0.0007 & 0.5000 \end{bmatrix},$$

$$\mathrm{L}_5^{(1,1)} = \left[L^{(1,1)}(p_{i5} \geq p_{\kappa 5})\right]_{5\times 5} = \begin{bmatrix} 0.5000 & 0.0223 & 0.1488 & 0.3122 & 0.3122 \\ 0.9777 & 0.5000 & 0.7583 & 0.8965 & 0.8965 \\ 0.8512 & 0.2417 & 0.5000 & 0.7092 & 0.7092 \\ 0.6878 & 0.1035 & 0.2908 & 0.5000 & 0.5000 \\ 0.6878 & 0.1035 & 0.2908 & 0.5000 & 0.5000 \end{bmatrix},$$

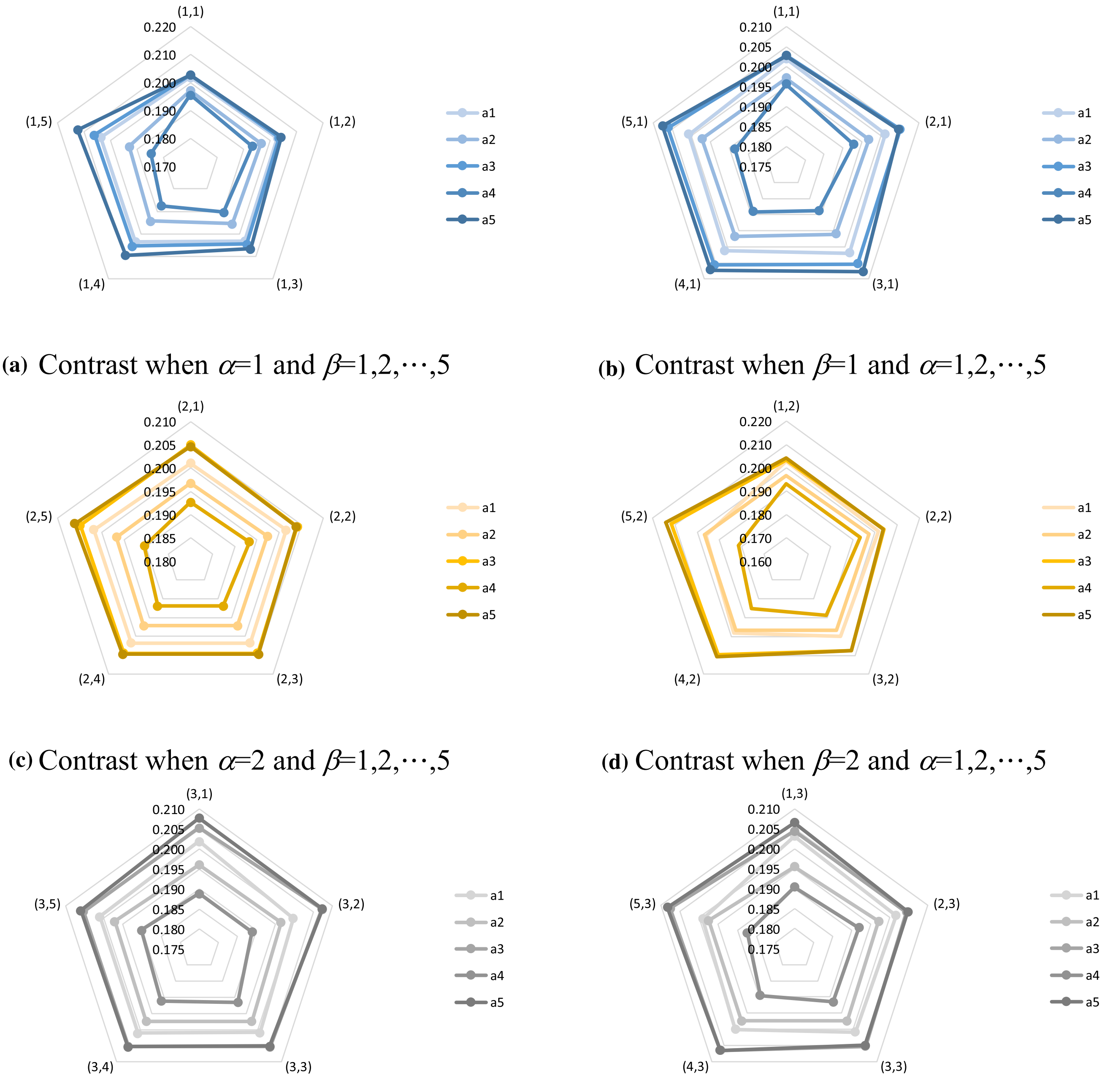
$$\mathrm{L}_6^{(1,1)} = \left[L^{(1,1)}(p_{i6} \geq p_{\kappa 6})\right]_{5\times 5} = \begin{bmatrix} 0.5000 & 0.0007 & 0.0491 & 0.0000 & 0.0007 \\ 0.9993 & 0.5000 & 0.6878 & 0.1035 & 0.5000 \\ 0.9509 & 0.3122 & 0.5000 & 0.0223 & 0.3122 \\ 1.0000 & 0.8965 & 0.9777 & 0.5000 & 0.8965 \\ 0.9993 & 0.5000 & 0.6878 & 0.1035 & 0.5000 \end{bmatrix}.$$

In Step I.5 of Algorithm I (as well as Step II.5 of Algorithm II), the mean outranking index $MO^{(1,1)}(p_{ij})$ relating to each p_{ij} can be derived based on the obtained parametric likelihood matrices. By using Eq. (24) in Theorem 3, the mean outranking indices of each PF evaluative rating p_{1j} for alternative a_1 were yielded as follows:

$$MO^{(1,1)}(p_{11}) = \frac{m+2\left(\sum_{\kappa=1}^{5} L^{(1,1)}(p_{11} \geq p_{\kappa 1}) - 1\right)}{2m(m-1)} = \frac{5+2(0.5000+0.6878+0.6878+0.5000+0.1035-1)}{2\cdot 5(5-1)} = 0.1990,$$

$MO^{(1,1)}(p_{12}) = 0.2926$, $MO^{(1,1)}(p_{13}) = 0.1897$, $MO^{(1,1)}(p_{14}) = 0.2145$, $MO^{(1,1)}(p_{15}) = 0.1398$, and $MO^{(1,1)}(p_{16}) = 0.1025$. In relation to the mean outranking index $MO^{(1,1)}(p_{2j})$ for alternative a_2, the following results were acquired: 0.1573, 0.1049, 0.2705, 0.2145, 0.2765, and 0.2145 for $c_1, c_2, \ldots, c_6$, respectively. Concerning $MO^{(1,1)}(p_{3j})$ for alternative a_3, the following results were acquired: 0.1573, 0.1893, 0.2304, 0.2564, 0.2256, and 0.1799 for $c_1, c_2, \ldots, c_6$, respectively. In view of $MO^{(1,1)}(p_{4j})$ for alternative a_4, the results yielded 0.1990, 0.1893, 0.1197, 0.2145, 0.1791, and 0.2885 for $c_1, c_2, \ldots, c_6$, respectively. In relation to $MO^{(1,1)}(p_{5j})$ for alternative a_5, the results were 0.2874, 0.2239, 0.1897, 0.1001, 0.1791, and 0.2145 for $c_1, c_2, \ldots, c_6$, respectively.

Consider the vector of ordinary normalized weights, i.e., $\overline{W} = (0.18, 0.22, 0.16, 0.14, 0.20, 0.10)$. In Step I.6 of Algorithm I, Eq. (27) was utilized to calculate the comprehensive outranking index $\overline{CI}_i^{(1,1)}$ for each alternative a_i. The below results were acquired:



(a) Contrast when α=1 and β=1,2,⋯,5

(b) Contrast when β=1 and α=1,2,⋯,5

(c) Contrast when α=2 and β=1,2,⋯,5

(d) Contrast when β=2 and α=1,2,⋯,5

(e) Contrast when α=3 and β=1,2,⋯,5

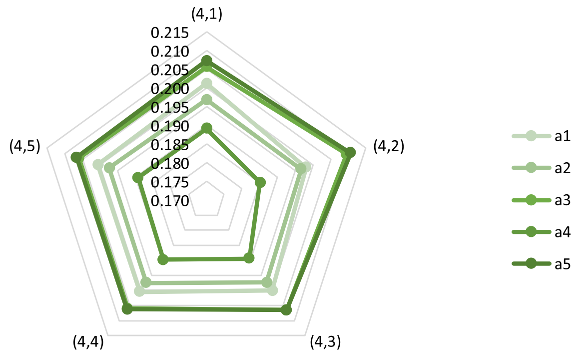
(f) Contrast when β=3 and α=1,2,⋯,5

Fig. 8 Contrast of the $CI_i^{(\alpha,\beta)}$ values among all alternatives yielded by Algorithm II

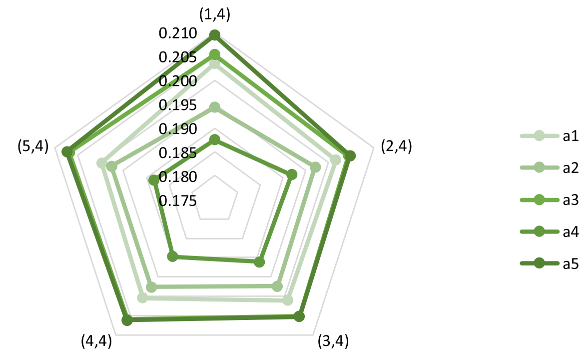
$$\overline{CI}_1^{(1,1)} = \sum_{j=1}^{6} \overline{w}_j \cdot MO^{(1,1)}(p_{1j})$$
$$= 0.18 \cdot 0.1990 + 0.22 \cdot 0.2926 + 0.16 \cdot 0.1897$$
$$+ 0.14 \cdot 0.2145 + 0.20 \cdot 0.1398 + 0.10 \cdot 0.1025 = 0.1988,$$

$\overline{CI}_2^{(1,1)} = 0.2015$, $\overline{CI}_3^{(1,1)} = 0.2058$, $\overline{CI}_4^{(1,1)} = 0.1913$, and $\overline{CI}_5^{(1,1)} = 0.2026$. In Step I.7 of Algorithm I, the final ranking $a_3 \succ a_5 \succ a_2 \succ a_1 \succ a_4$ of the five candidate hospitals was identified in descending order of the $\overline{CI}_i^{(1,1)}$ outcomes; furthermore, a_3 was the best choice under a setting of $\alpha = 1$ and $\beta = 1$.

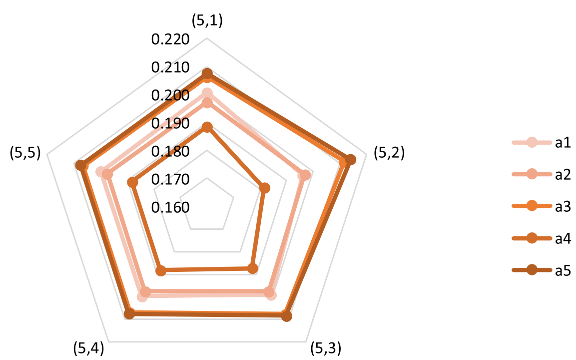
Considering the PF importance weights, the collection $W = \{\langle c_1, (0.41, 0.59)\rangle, \langle c_2, (0.55, 0.45)\rangle, \langle c_3, (0.38, 0.62)\rangle, \langle c_4, (0.35, 0.65)\rangle, \langle c_5, (0.45, 0.55)\rangle, \langle c_6, (0.32, 0.68)\rangle\}$. Applying Step II.6 in Algorithm II, by combining



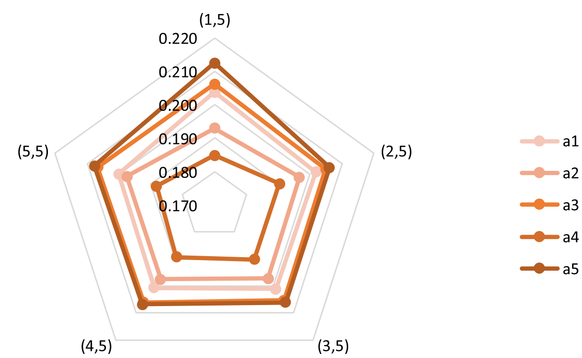
(g) Contrast when α=4 and β=1,2,$\cdots$,5



(h) Contrast when β=4 and α=1,2,$\cdots$,5



(i) Contrast when α=5 and β=1,2,$\cdots$,5



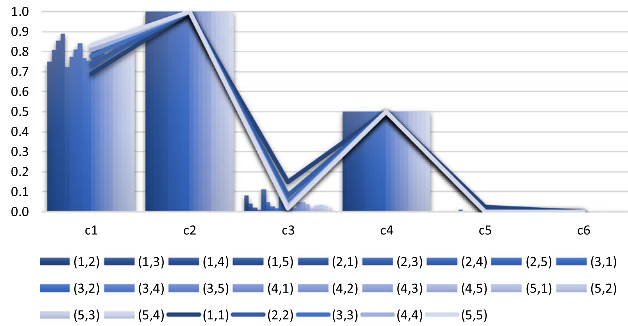
(j) Contrast when β=5 and α=1,2,$\cdots$,5

Fig. 8 continued

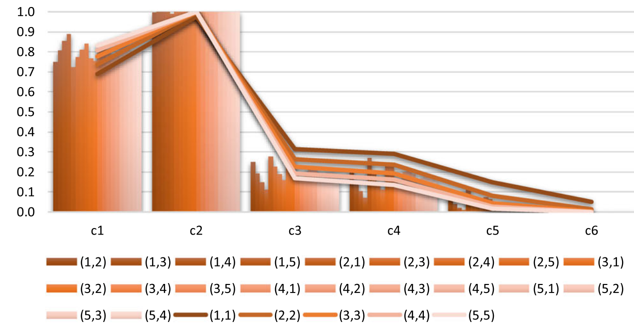
w_j and $MO^{(1,1)}(p_{ij})$, Eq. (31) was utilized to determine the comprehensive PF outranking measures. The measure $CO^{(1,1)}(P_1)$ of the PF characteristic P_i for alternative a_1 was derived along as follows:

$$CO^{(1,1)}(P_1) = \left(\sqrt{1 - \prod_{j=1}^{6}\left(1-(\omega_j)^2\right)^{MO^{(1,1)}(p_{1j})}}, \prod_{j=1}^{6}(\varpi_j)^{MO^{(1,1)}(p_{1j})}\right)$$
$$= \left(\left[1-\left(1-0.41^2\right)^{0.1990}\cdot\left(1-0.55^2\right)^{0.2926}\cdot\left(1-0.38^2\right)^{0.1897}\right.\right.$$
$$\left.\cdot\left(1-0.35^2\right)^{0.2145}\cdot\left(1-0.45^2\right)^{0.1398}\cdot\left(1-0.32^2\right)^{0.1025}\right]^{0.5},$$
$$\left.0.59^{0.1990}\cdot 0.45^{0.2926}\cdot 0.62^{0.1897}\cdot 0.65^{0.2145}\cdot 0.55^{0.1398}\cdot 0.68^{0.1025}\right) = (0.4639, 0.5248).$$

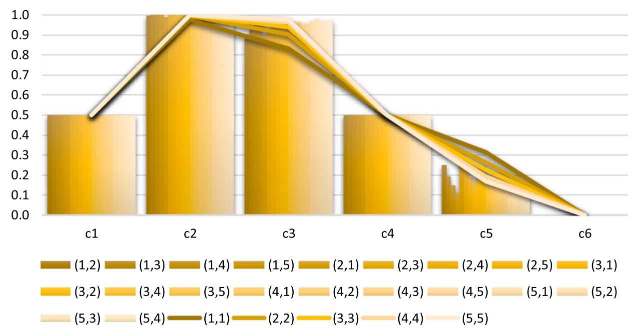
In a similar manner, $CO^{(1,1)}(P_2) = (0.4468, 0.5291)$, $CO^{(1,1)}(P_3) = (0.4594, 0.5174)$, $CO^{(1,1)}(P_4) = (0.4473, 0.5357)$, and $CO^{(1,1)}(P_5) = (0.4619, 0.5199)$.



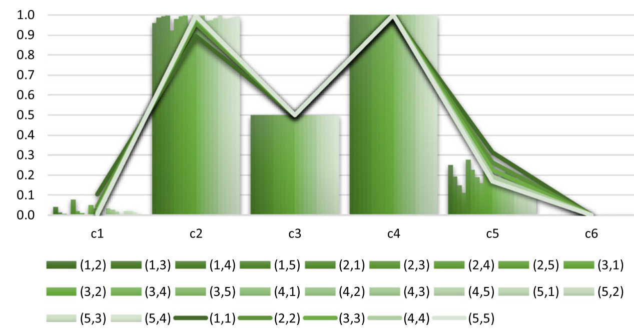
(a) Comparison of $L^{(\alpha,\beta)}(p_{1j} \geq p_{2j})$



(b) Comparison of $L^{(\alpha,\beta)}(p_{1j} \geq p_{3j})$

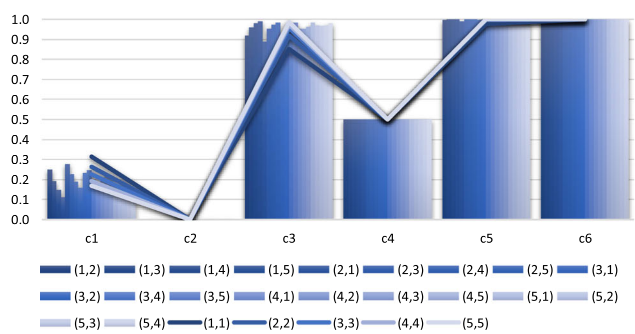


(c) Comparison of $L^{(\alpha,\beta)}(p_{1j} \geq p_{4j})$

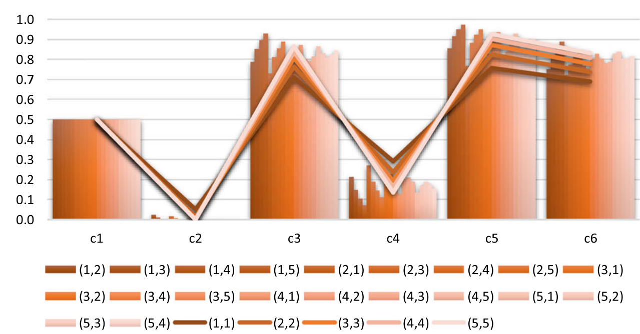


(d) Comparison of $L^{(\alpha,\beta)}(p_{1j} \geq p_{5j})$

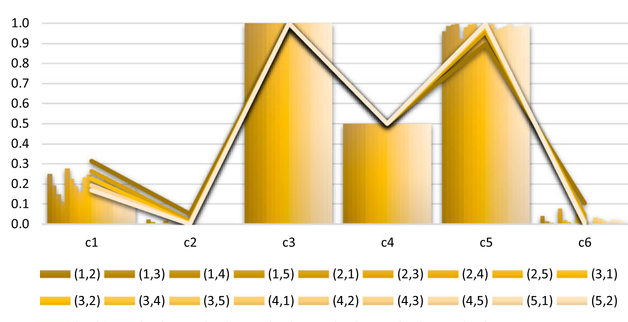
Fig. 9 Contrast of the parametric PF likelihood measure $L^{(\alpha,\beta)}(p_{1j} \geq p_{\kappa j})$ (κ = 2, 3, 4, 5)



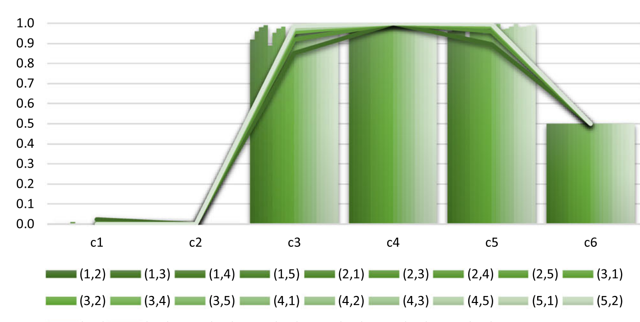
(a) Comparison of $L^{(\alpha,\beta)}(p_{2j} \geq p_{1j})$



(b) Comparison of $L^{(\alpha,\beta)}(p_{2j} \geq p_{3j})$

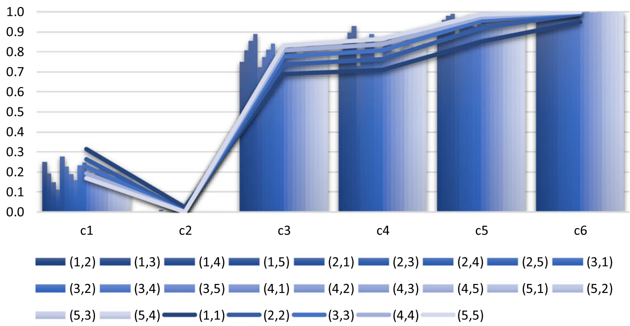


(c) Comparison of $L^{(\alpha,\beta)}(p_{2j} \geq p_{4j})$

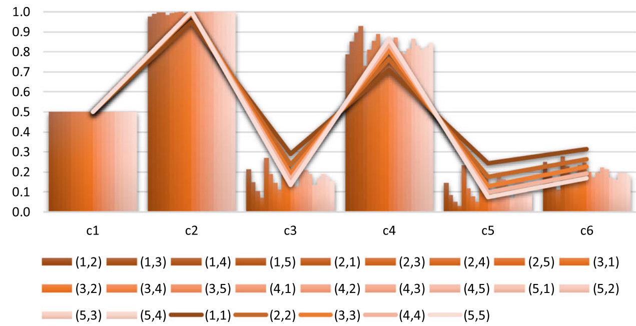


(d) Comparison of $L^{(\alpha,\beta)}(p_{2j} \geq p_{5j})$

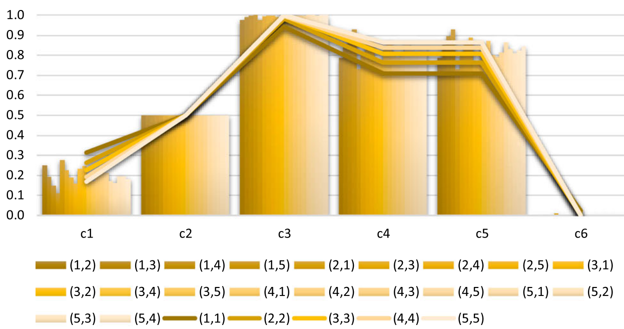
Fig. 10 Contrast of the parametric PF likelihood measure $L^{(\alpha,\beta)}(p_{2j} \geq p_{\kappa j})$ (κ = 1, 3, 4, 5)



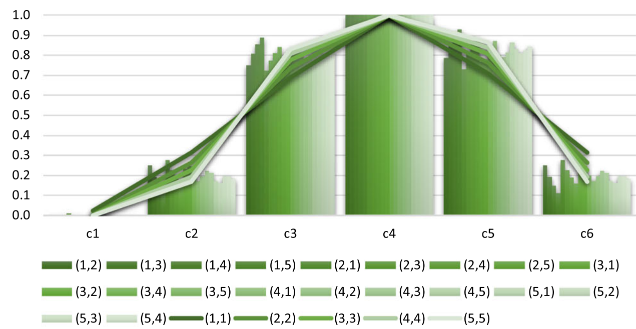
(a) Comparison of $L^{(\alpha,\beta)}(p_{3_j} \geq p_{1_j})$



(b) Comparison of $L^{(\alpha,\beta)}(p_{3_j} \geq p_{2_j})$



(c) Comparison of $L^{(\alpha,\beta)}(p_{3_j} \geq p_{4_j})$



(d) Comparison of $L^{(\alpha,\beta)}(p_{3_j} \geq p_{5_j})$

Fig. 11 Contrast of the parametric PF likelihood measure $L^{(\alpha,\beta)}(p_{3j} \geq p_{\kappa j})$ (κ = 1, 2, 4, 5)

In Step II.7 of Algorithm II, the parametric likelihood $L^{(1,1)}(CO(P_i) \geq CO(P_\kappa))$ of the PF outranking relation was calculated by $CO^{(1,1)}(P_i) \geq CO^{(1,1)}(P_\kappa)$. The collection of the computed results was represented by the following matrix:

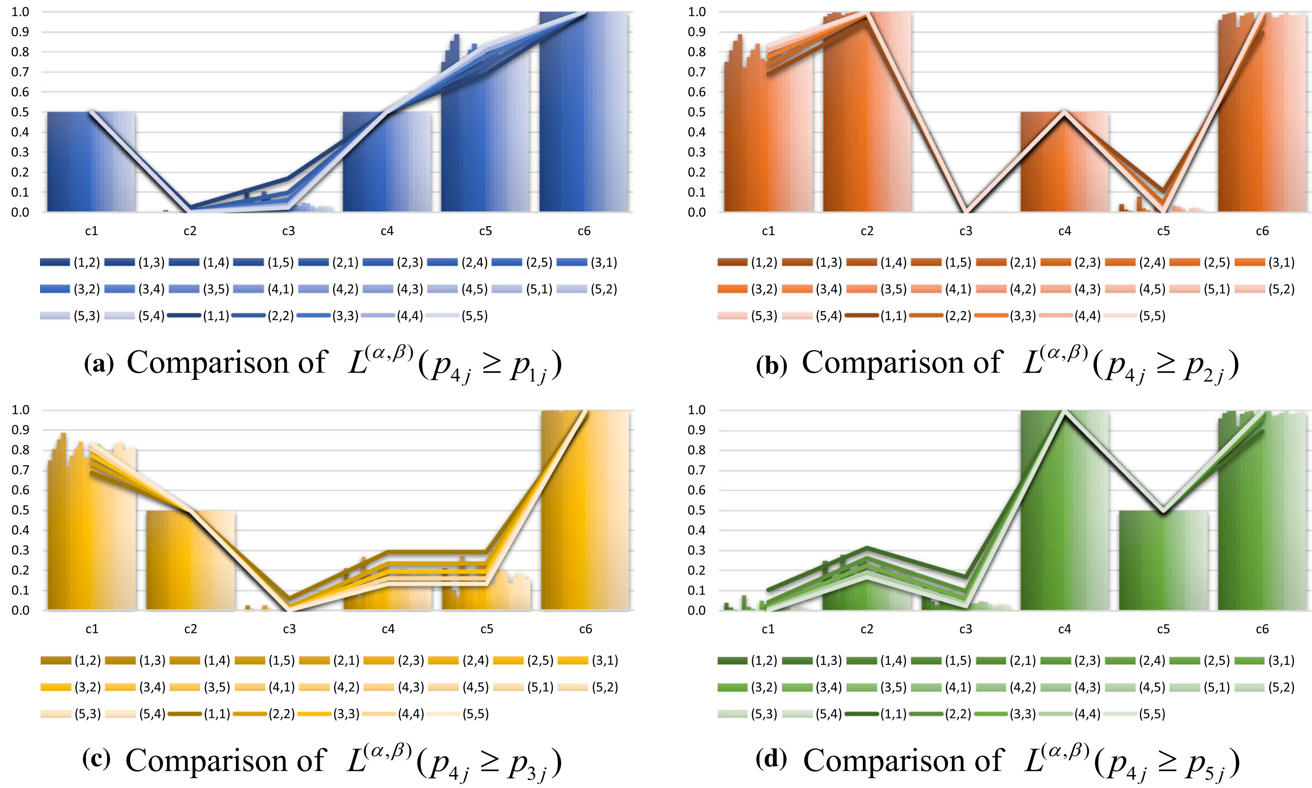
$$L^{(1,1)} = \left[L^{(1,1)}(CO(P_i) \geq CO(P_\kappa))\right]_{5\times 5} = \begin{bmatrix} 0.5000 & 0.5193 & 0.4966 & 0.5258 & 0.4968 \\ 0.4807 & 0.5000 & 0.4775 & 0.5063 & 0.4777 \\ 0.5034 & 0.5225 & 0.5000 & 0.5288 & 0.5003 \\ 0.4742 & 0.4937 & 0.4712 & 0.5000 & 0.4713 \\ 0.5032 & 0.5223 & 0.4997 & 0.5287 & 0.5000 \end{bmatrix}.$$

The comprehensive PF outranking indices can be acquired using Eq. (33) as follows: $CI_1^{(1,1)}$ =[5 + 2 (0.5000 + 0.5193 + 0.4966 + 0.5258 + 0.4968 − 1)]/ [2·5(5 − 1)] = 0.2019, $CI_2^{(1,1)}$ = 0.1971, $CI_3^{(1,1)}$ = 0.2028, $CI_4^{(1,1)}$ = 0.1955, and $CI_5^{(1,1)}$ = 0.2027. In Step II.8 of Algorithm II, the priority ranking $a_3 \succ a_5 \succ a_1 \succ a_2 \succ a_4$ of the candidate alternatives was identified as stated by descending orders of the $CI_i^{(1,1)}$ outcomes. Accordingly, a_3 was the best choice under the settings of α = 1 and β = 1.

5.2 Experimental analysis and discussions

This subsection attempts to perform an experimental study to compare the application solutions and examine the superiority and merits of the advanced methodology. Consider that beta distributions are utilized as a parameterization tool to create an innovative parametric likelihood measure. In this regard, the experiment analyzes the influences of various parameter settings on the outranking relationships among PF information and the relative priority of candidate alternatives yielded by the advanced PF likelihood-oriented approach.

Because the relative effectiveness of the various settings regarding parameters α and β is not known, two comparative studies are executed to explore the parameterization mechanism and verify the given results in the pilot hospital selection for postacute care. Several reasonable and acceptable values of parameters α and β within the beta distribution-based likelihood measure were designated in the following manner: α and β ranging from integers 1 to 5 separately. Accordingly, 25 instances of the (α, β) pairs were investigated, i.e., the 5×5 combinations of α = 1,2,…,5 and β = 1,2,…,5. Through the developed algorithm, indispensable outranking-related concepts were



(a) Comparison of $L^{(\alpha,\beta)}(p_{4j} \geq p_{1j})$

(b) Comparison of $L^{(\alpha,\beta)}(p_{4j} \geq p_{2j})$

(c) Comparison of $L^{(\alpha,\beta)}(p_{4j} \geq p_{3j})$

(d) Comparison of $L^{(\alpha,\beta)}(p_{4j} \geq p_{5j})$

Fig. 12 Contrast of the parametric PF likelihood measure $L^{(\alpha,\beta)}(p_{4j} \geq p_{\kappa j})$ (κ = 1, 2, 3, 5)

determined for the 25 instances of the (α, β) pairs. The results of mean outranking indices, weighted outranking grades, comprehensive outranking indices, comprehensive PF outranking measures, and comprehensive PF outranking indices were tabulated to analyze the consequences of parameter settings. More precisely, having relevance for each $\beta \in \{1, 2, \ldots, 5\}$, the experimental results of $MO^{(\alpha,\beta)}(p_{ij})$, $\overline{WO}^{(\alpha,\beta)}(p_{ij})$, $\overline{CI}_i^{(\alpha,\beta)}$ (with the priority rank), $CO^{(\alpha,\beta)}(P_i)$, and $CI_i^{(\alpha,\beta)}$ (with the priority rank) in the cases of α = 1,2,…,5 are indicated in Tables 1, 2, 3, 4, 5, respectively.

The first experimental analysis examines the application results of different parameter settings through Algorithm I. The results of priority ranking orders generated by the comprehensive outranking indices are shown in Tables 1, 2, 3, 4, 5. In consideration of the preference structure involving ordinary normalized weights, the identical priority ranking $a_3 \succ a_5 \succ a_2 \succ a_1 \succ a_4$ was acquired in the 25 instances of the (α, β) pairs. In particular, the ranking outcomes of the candidate alternatives were very stable and reliable in relation to the ultimate rankings generated by Chen [8].

To fully realize the influences of parameters α and β on the comprehensive outranking index $\overline{CI}_i^{(\alpha,\beta)}$, a line graph and ten radar charts generated are shown in Figs. 5 and 6, respectively, to contradistinguish the results obtained under various changes of (α, β). More precisely, Fig. 5 displays an overall comparison of comprehensive PF outranking indices in the 25 instances of the (α, β) pairs. On the one hand, Fig. 6a, c, e, g, and i compare the consequences of $\overline{CI}_i^{(1,\beta)}$, $\overline{CI}_i^{(2,\beta)}$,…, $\overline{CI}_i^{(5,\beta)}$, respectively, among the five alternatives for $a_i \in \{a_1, a_2, \cdots, a_5\}$ and $\beta \in \{1, 2, \ldots, 5\}$. On the other hand, Fig. 6b, d, f, h, and j contradistinguish the consequences of $\overline{CI}_i^{(\alpha,1)}$, $\overline{CI}_i^{(\alpha,2)}$,…, $\overline{CI}_i^{(\alpha,5)}$, respectively, among the five alternatives for $a_i \in \{a_1, a_2, \cdots, a_5\}$ and $\alpha \in \{1, 2, \ldots, 5\}$. The priority rankings of the five alternatives were unchangeable based on varied settings of (α, β). Nevertheless, these figures showed significant variation in the obtained comprehensive outranking indices. The $\overline{CI}_i^{(\alpha,\beta)}$ values are moderately adaptable to changes in parameters α and β under the guidance of the likelihood measure based on beta distributions. Accordingly, the proposed PF likelihood-oriented methodology is able to adapt to different settings of α and β; thus, it has the

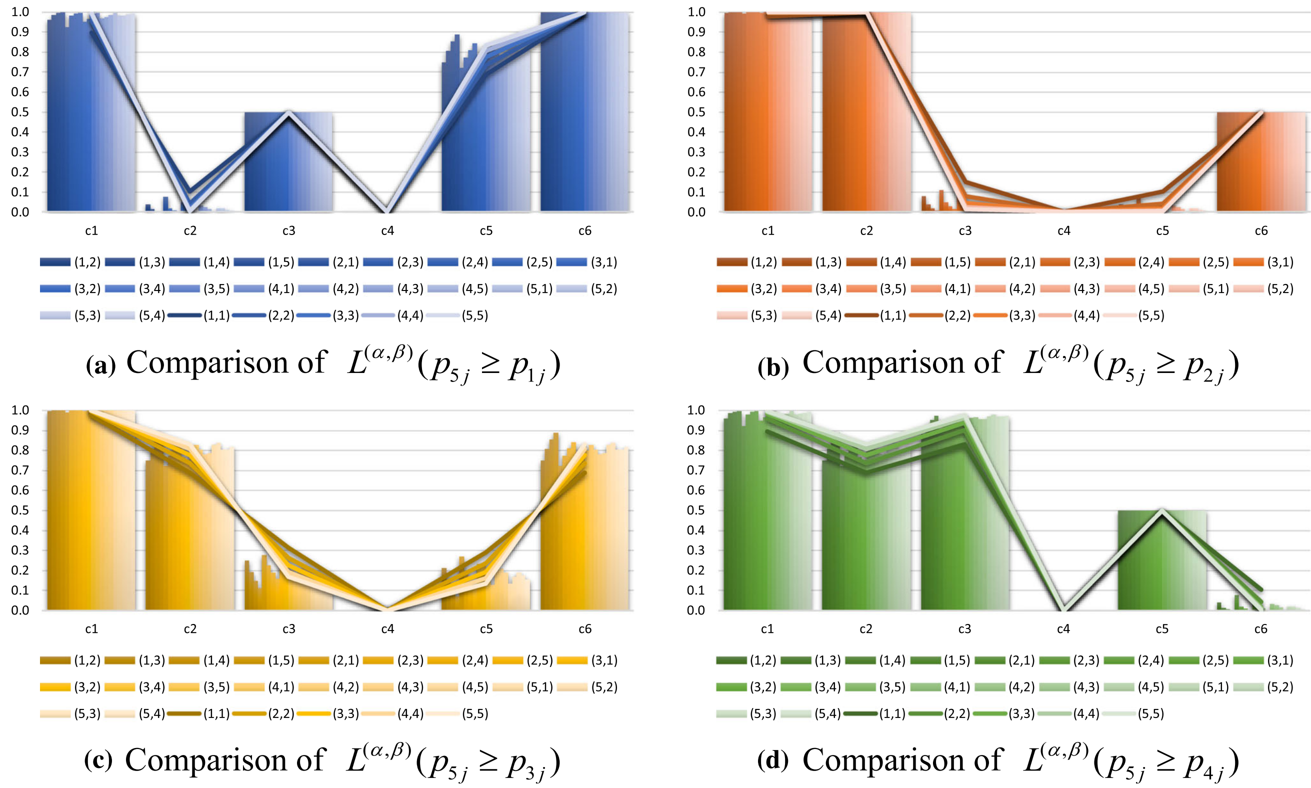
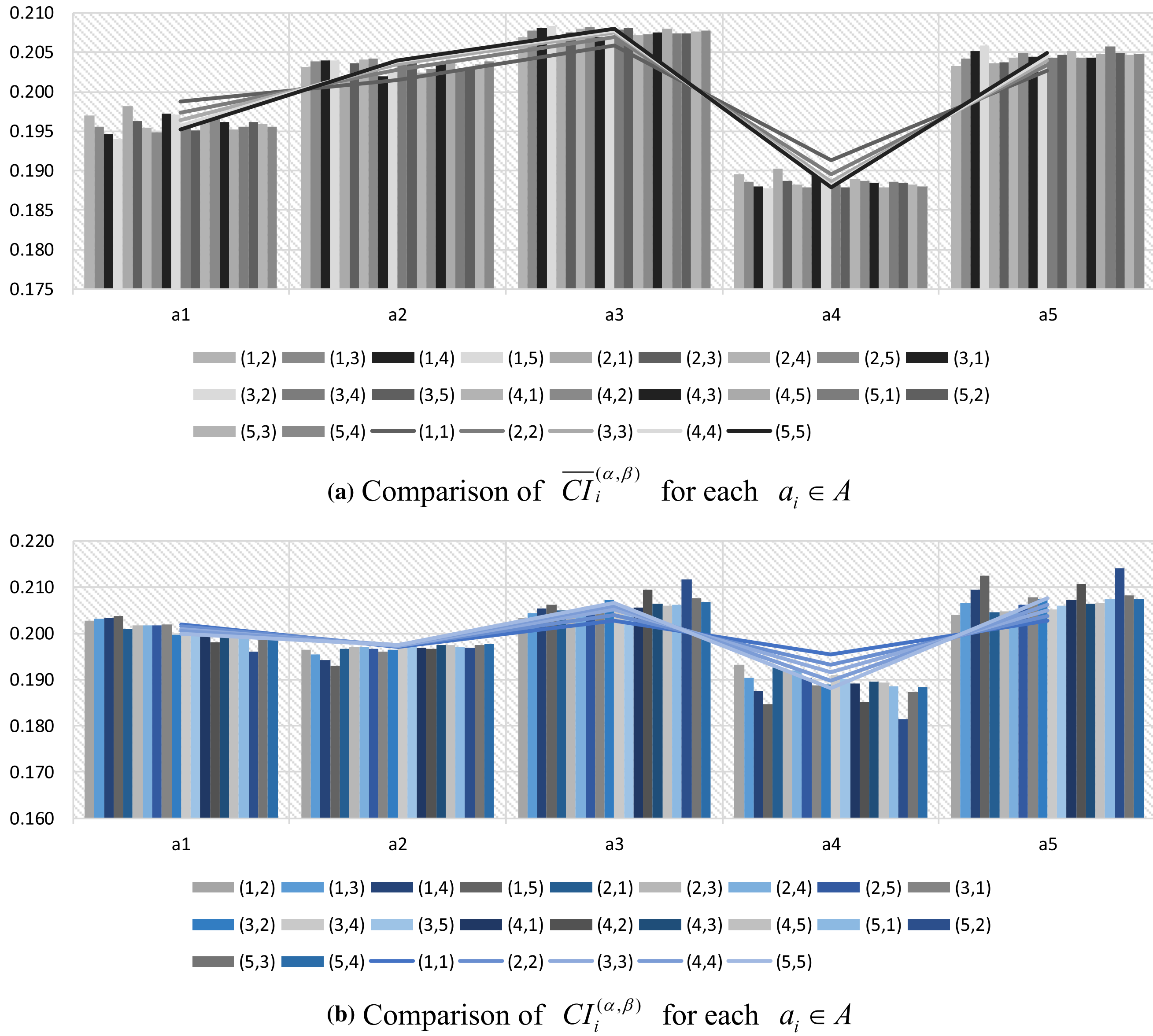


Fig. 13 Contrast of the parametric PF likelihood measure $L^{(\alpha,\beta)}(p_{5j} \geq p_{\kappa j})$ (κ = 1, 2, 3, 4)

capability of rendering considerably adjustable but relatively stable ranking consequences that can convey optimistic and pessimistic attitudes by $\alpha > \beta$ and $\alpha < \beta$, respectively.

The second experimental analysis investigates the solution outcomes of the distinct parameter settings by proposed Algorithm II. Let us observe the priority ranking orders rendered by the comprehensive PF outranking index $CI_i^{(\alpha,\beta)}$, as shown in Tables 1, 2, 3, 4, 5. When considering the preference structure involving PF importance weights, the current PF likelihood-oriented methodology produced three priority rankings $a_3 \succ a_5 \succ a_1 \succ a_2 \succ a_4$, $a_5 \succ a_3 \succ a_2 \succ a_1 \succ a_4$, and $a_5 \succ a_3 \succ a_1 \succ a_2 \succ a_4$. More precisely, when (α, β) =(1, 1), (2, 1), (2, 2), and (3, 2), the same ranking $a_3 \succ a_5 \succ a_1 \succ a_2 \succ a_4$ of the candidate alternatives was given, and a_3 was the best choice. In the case that (α, β) =(5, 1), the ranking $a_5 \succ a_3 \succ a_2 \succ a_1 \succ a_4$ was generated, and a_5 was the best choice. In the remaining 20 instances of the (α, β) pairs (namely, excluding (1, 1), (2, 1), (2, 2), (3, 2), and (5, 1)), the identical ranking $a_5 \succ a_3 \succ a_1 \succ a_2 \succ a_4$ was acquired; moreover, a_5 was the best alternative. Note that the same ranking results were obtained when β = 3, 4, and 5. As a result, a consistent ranking $a_5 \succ a_3 \succ a_1 \succ a_2 \succ a_4$ was obtained concerning the settings of parameter pairs $(\alpha, 3)$, $(\alpha, 4)$, and $(\alpha, 5)$, where α = 1,2,…,5. When β = 1, the most diverse consequences were determined as follows: $a_3 \succ a_5 \succ a_1 \succ a_2 \succ a_4$, $a_5 \succ a_3 \succ a_1 \succ a_2 \succ a_4$, and $a_5 \succ a_3 \succ a_2 \succ a_1 \succ a_4$ in the cases where α = 1,2, α = 3,4, and α = 5, respectively. When β = 2, the two rankings $a_3 \succ a_5 \succ a_1 \succ a_2 \succ a_4$ and $a_5 \succ a_3 \succ a_1 \succ a_2 \succ a_4$ in the cases regarding α = 2,3 and α = 1,4,5, respectively. As a whole, a_5 was the best alternative for most instances. This result was generally in accordance with the ultimate compromise solution yielded by Chen [8], even though the data manipulated by Chen [8] involved interval-valued PF evaluative ratings and ordinary normalized weights.

Furthermore, to determine the influences of the parameterization mechanism on comprehensive PF outranking indices, this paper delineated a line graph in Fig. 7 and several radar charts in Fig. 8 to contrast the acquired results of the $CI_i^{(\alpha,\beta)}$ values under diverse settings of parameters α and β. A complete contraposition with respect to the five alternatives in the 25 instances of the (α, β) pairs is sketched in Fig. 7. Moreover, Fig. 8a, 8c, e, g, and i contradistinguish the consequences of $CI_i^{(1,\beta)}$, $CI_i^{(2,\beta)}$,…,



(a) Comparison of $\overline{CI}_i^{(\alpha,\beta)}$ for each $a_i \in A$

(b) Comparison of $CI_i^{(\alpha,\beta)}$ for each $a_i \in A$

Fig. 14 Contrast of the comprehensive (PF) outranking indices

$CI_i^{(5,\beta)}$, respectively, among the five alternatives for $a_i \in \{a_1, a_2, \ldots, a_5\}$ and $\beta \in \{1, 2, \ldots, 5\}$. In contrast, Fig. 8b, d, f, h, and j compare the consequences over $CI_i^{(\alpha,1)}$, $CI_i^{(\alpha,2)}$,…, $CI_i^{(\alpha,5)}$, respectively, among the five alternatives for $a_i \in \{a_1, a_2, \cdots, a_5\}$ and $\alpha \in \{1, 2, \ldots, 5\}$. These figures, analogous to Figs. 5 and 6, displayed considerable variation in the resulting comprehensive PF outranking indices and their related priority orders in distinct surroundings of (α, β). Observable differences in the obtained $CI_i^{(\alpha,\beta)}$ values were found in Figs. 7 and 8, which led to the adaptability and flexibility of the current PF likelihood-oriented technique. As evidenced by the above discussions, the experimental analysis and subsequent comparisons have proven the effectiveness and accomplishments of the advanced methodology.

5.3 Comparative analysis and discussion

This subsection investigates the application outcomes and compares the results of the present research with previous studies. The practical experimental results yielded by the developed methods are further compared with the outcomes using other approaches. These investigation and comparison studies can prove the strength and justification

Table 6 Summary of the comparison analysis in the pilot hospital selection problem

Methodology	Source	Preference structure	Ultimate ranking outcome	Parameter setting	Best choice
The extended VIKOR method	Chen [8]	Normalized weight	$a_4 \sim a_5 \succ a_3 \succ a_1 \succ a_2$	VIKOR parameter: 0.0	a_4, a_5
			$a_5 \succ a_4 \succ a_3 \succ a_1 \succ a_2$	VIKOR parameter: 0.1, 0.2	a_5
			$a_5 \succ a_3 \succ a_4 \succ a_2 \succ a_1$	VIKOR parameter: 0.3, 0.4	a_5
			$a_3 \sim a_5 \succ a_2 \succ a_4 \succ a_1$	VIKOR parameter: 0.5	a_3, a_5
			$a_2 \sim a_3 \sim a_5 \succ a_1 \succ a_4$	VIKOR parameter: 0.6, 0.7, 0.8, 0.9	a_2, a_3, a_5
			$a_2 \sim a_3 \succ a_5 \succ a_1 \succ a_4$	VIKOR parameter: 1.0	a_2, a_3
The uniform distribution- based likelihood approach	Liang et al. [26]	Normalized weight	$a_3 \succ a_5 \succ a_2 \succ a_1 \succ a_4$	$(\alpha, \beta) = (1, 1)$	a_3
		PF importance weight	$a_3 \succ a_5 \succ a_1 \succ a_2 \succ a_4$	$(\alpha, \beta) = (1, 1)$	a_3
The symmetry beta distribution-based likelihood approach	Tsao and Chen [37]	Normalized weight	$a_3 \succ a_5 \succ a_2 \succ a_1 \succ a_4$	$(\alpha, \beta) = (1, 1), (2, 2), \cdots, (5, 5)$	a_3
		PF importance weight	$a_3 \succ a_5 \succ a_1 \succ a_2 \succ a_4$	$(\alpha, \beta) = (1, 1), (2, 2)$	a_3
			$a_5 \succ a_3 \succ a_1 \succ a_2 \succ a_4$	$(\alpha, \beta) = (3, 3), (4, 4), (5, 5)$	a_5
The parametric PF likelihood-oriented methodology	Current paper	Normalized weight	$a_3 \succ a_5 \succ a_2 \succ a_1 \succ a_4$	$(\alpha, \beta) = (1, 1), (1, 2), \cdots, (1, 5),$ $(2, 1), (2, 2), \cdots, (2, 5), \cdots,$ $(5, 1), (5, 2), \cdots, (5, 5)$	a_3
		PF importance weight	$a_3 \succ a_5 \succ a_1 \succ a_2 \succ a_4$	$(\alpha, \beta) = (1, 1), (2, 1), (2, 2), (3, 2)$	a_3
			$a_5 \succ a_3 \succ a_1 \succ a_2 \succ a_4$	$(\alpha, \beta) = (1, 2), (1, 3), (1, 4), (1, 5), (2, 3),$ $(2, 4), (2, 5), (3, 1), (3, 3), (3, 4),$ $(3, 5), (4, 1), (4, 2), \cdots, (4, 5),$ $(5, 2), (5, 3), (5, 4), (5, 5)$	a_5
			$a_5 \succ a_3 \succ a_2 \succ a_1 \succ a_4$	$(\alpha, \beta) = (5, 1)$	a_5

of the evolved methodology in manipulating multiple criteria evaluation tasks.

The first comparative analysis investigates the ultimate ranking outcomes with the results from previous studies. Chen [8] introduced an extended VlseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) method for treating a decision-making issue in interval-valued PF circumstances. Because of the interval-valued PF evaluative ratings, Chen's extended VIKOR methodology produced diverse ranking results under distinct settings of the VIKOR parameter within [0, 1]. Specifically, the following ultimate ranking results were identified using the extended VIKOR procedure: $a_4 \sim a_5 \succ a_3 \succ a_1 \succ a_2$ when the VIKOR parameter was equal to 0, $a_5 \succ a_4 \succ a_3 \succ a_1 \succ a_2$ when the parameter was equal to 0.1 and 0.2, $a_5 \succ a_3 \succ a_4 \succ a_2 \succ a_1$ when the parameter was equal to 0.3 and 0.4, $a_3 \sim a_5 \succ a_2 \succ a_4 \succ a_1$ when the parameter was equal to 0.5, $a_2 \sim a_3 \sim a_5 \succ a_1 \succ a_4$ when the parameter was equal to 0.6, 0.7, 0.8, and 0.9, and $a_2 \sim a_3 \succ a_5 \succ a_1 \succ a_4$ when the parameter was equal to 1. Accordingly, miscellaneous consequences of the best compromise solutions were obtained: $\{a_4, a_5\}$ with the VIKOR parameter 0, $\{a_5\}$ with the parameter 0.1, 0.2, 0.3, and 0.4, $\{a_3, a_5\}$ with the parameter 0.5, $\{a_2, a_3, a_5\}$ with the parameter 0.6, 0.7, 0.8, and 0.9, and $\{a_2, a_3\}$ with the parameter 1. It can be observed that the outcomes yielded by Chen [8] were unstable and unsettled. As can be expected, an MCDA process seeks to discover the best compromise solution and near an agreement for settling a matter of conflicting criteria. However, the best compromise solutions provided by the extended VIKOR methodology did not appear to resolve the decision conflict. In contrast, these results were confusing and troublesome for the decision-maker. The decision-maker cannot confidently accept the diverse consequences of the best compromise solutions because such solutions are not feasible or practical for reaching an agreement. Unlike the acquired results via the extended VIKOR approach, the advanced PF likelihood-oriented methodology generated reasonable and steady results for aiding in decision-making. Alternative a_3 was validated to be the best choice under various settings of the (α, β) pair. Thus, it is anticipated that such an MCDA solution can be easily accepted by the decision-maker with confidence. Because of the comparative discussions, it was verified that the advanced methodology could reliably resolve the MCDA problem and provide stable and established consequences.

The second comparative analysis investigates the PF likelihood measurements determined by the uniform distribution-based approach [26], the symmetric beta distribution-based approach [37], and the current comprehensive approach in this study. As stated previously, the beta distribution utilized by the proposed parametric PF likelihood measure is considered to be a generic structure of the uniform distribution (i.e., both α and β are equal to 1) and the symmetric beta distribution (i.e., α and β are equal positive integers). In this comparison study, the settings of the (α, β) pair were designated in this manner: $(\alpha, \beta) = (1, 1)$, $(\alpha, \beta) = (1, 1), (2, 2), \cdots, (5, 5)$, and $(\alpha, \beta) = (1, 1), (1, 2), \ldots, (1, 5), (2, 1), (2, 2), \ldots, (2, 5), \ldots, (5, 1), (5, 2), \ldots, (5, 5)$ in connection with the approaches by Liang et al. [26], Tsao and Chen [37], and the current study, respectively. The comparison results of the parametric PF likelihood measure $L^{(\alpha,\beta)}(p_{ij} \geq p_{\kappa j})$ of each alternative a_i with regard to the other four alternatives for each criterion c_j under varied settings of (α, β) are represented in Figs. 9, 10, 11, 12, 13. More specifically, the contrasting results of the determination outcomes of $L^{(\alpha,\beta)}(p_{1j} \geq p_{\kappa j})$, $L^{(\alpha,\beta)}(p_{2j} \geq p_{\kappa j}), \ldots,$ and $L^{(\alpha,\beta)}(p_{5j} \geq p_{\kappa j})$ (κ = 1,2,…, 5, $\kappa \neq i$, j = 1,2,…,6) are revealed in Figs. 9, 10, 11, 12, 13. Herein, the line charts in Figs. 9, 10, 11, 12, 13 show the PF likelihood measures yielded by the uniform distribution-based approach (i.e., the setting of $(\alpha, \beta) = (1, 1)$) and the symmetric beta distribution-based approach (i.e., the settings of $(\alpha, \beta) = (1, 1), (2, 2), \ldots, (5, 5)$). These line charts display relatively consistent patterns between paired PF evaluative ratings in connection with evaluative criteria. In contrast to the relatively fixed patterns yielded by the approaches proposed by Liang et al. [26] and Tsao and Chen [37], the proposed parametric PF likelihood measures produced moderately variable determination results, as revealed in the group bar graphs (i.e., the situations that $\alpha \neq \beta$) and the line charts (i.e., the situations that $\alpha = \beta$). Therefore, the proposed technique is more versatile and flexible for processing uncertain PF estimates than the existing approaches.

The third comparative analysis explores the outcomes of the comprehensive (PF) outranking indices predicated on the PF likelihood measurements by the uniform distribution-based approach, the symmetric beta distribution-based approach, and the current comprehensive approach, as shown in Fig. 14. Using Algorithm I, the comparison outcomes of the comprehensive outranking index $\overline{CI}_i^{(\alpha,\beta)}$ established on three likelihood-assessing approaches are shown in Fig. 14a. Next, based on Algorithm II, the contrasting consequences of the comprehensive PF outranking index $CI_i^{(\alpha,\beta)}$ generated by three likelihood-assessing approaches are depicted in Fig. 14b. In the same way, the line charts in Fig. 14a and b display the comprehensive (PF) outranking indices provided by the uniform distribution-based approach in the setting of $(\alpha, \beta) = (1, 1)$ and by the symmetric beta distribution-based procedure in the settings of $(\alpha, \beta) = (1, 1), (2, 2), \ldots, (5, 5)$. The outcomes of $\overline{CI}_i^{(\alpha,\beta)}$ yielded by the PF likelihood-oriented methodology are depicted by the group bar graphs and the line charts when $\alpha \neq \beta$ and $\alpha = \beta$, respectively. As displayed in Fig. 14a, the proposed methodology can generate relatively flexible consequences that reveal certain variations relative to distinct parameter settings. The patterns of the obtained $\overline{CI}_i^{(\alpha,\beta)}$ can reflect the influences of different pairs of parameters α and β embedded in the beta distribution. The proposed methodology is capable of delivering adjustable but relatively stable results; such findings indicate optimistic and pessimistic attitudes through the utility of $\alpha > \beta$ and $\alpha < \beta$, respectively. From Fig. 14b, similar observations can be found on the subject of the comparative outcomes of the comprehensive PF outranking index $CI_i^{(\alpha,\beta)}$. The contrasting consequences of the comprehensive (PF) outranking indices give substance to the adaptability and flexibility of the evolved techniques. Based on the comparison results, it appears clear that the advanced PF likelihood-oriented methodology is superior in generating reasonable and adjustable solutions.

Finally, Table 6 provides a summary of the comparative analysis in the pilot hospital selection problem for postacute care. The four comparison methods include the extended VIKOR method [8], the uniform distribution-based likelihood approach [26], the symmetry beta distribution-based likelihood approach [37], and the proposed parametric PF likelihood-oriented methodology in the current paper. As revealed in this table, the ultimate ranking outcomes were unstable under different setting values of the VIKOR parameter, which yielded different best compromise solutions. Except a_1, the candidate alternatives a_2, a_3, a_4, and a_5 were the best compromise solutions in various parameter values. To be specific, a_2 was the best choice when the VIKOR parameter was designated as 0.6, 0.7,, 1, a_3 under the setting values of 0.5, 0.6,, 1, a_4 under the setting value of 0, and a_5 under the setting values of 0, 0.1,, 0.9. These uncertain consequences of the ultimate rankings and the best compromise solutions would confuse the decision maker about how to choose the final solution. On the other side, based on the summary results in Table 6, it can be recognized that the solution outcomes of the uniform distribution-based likelihood approach and the symmetry beta distribution-based likelihood approach were included in the consequence produced by the proposed methodology. This finding means that the parametric PF likelihood-oriented methodology is regarded as a generalization of Liang et al.'s [26] and Tsao and Chen's [37] approaches, which give substance to its relative merits to the comparative methods.

6 Overall discussion and conclusions

This paper has utilized beta density functions to provide a parameterization tool in likelihood measurements and initiate an innovative parametric likelihood measure via a beta distribution-based approach. Finding an appropriate specification of likelihood measures has many difficulties in addressing complex decision-making issues. Uncertainties in intricate and unpredictable decision circumstances add to the difficulty of addressing MCDA issues credibly and plausibly. In particular, sophisticated and complicated methods are currently utilized because the outranking relationships between alternatives are poorly determined with multiple conflicting criteria under uncertainty and vagueness. Accordingly, the outranking relationship for assessment information containing Pythagorean fuzziness can be effectively analyzed using the measurement of likelihood functions. In contrast with the existing literature, this paper has delivered an adaptable and flexible parameterization model on the strength of a standard beta distribution for ascertaining PF likelihood functions and manipulating uncertain information with Pythagorean fuzziness. To address the research gap in the existing related literature, this paper has adopted a different technique to identify an innovative PF likelihood measure. Instead of utilizing some commonly used concepts such as score functions, accuracy functions, and scalar functions, this paper has employed beta density functions to present a creative likelihood measurement for identifying a possibility formula and manipulating intricate uncertain information involving Pythagorean fuzziness. As revealed in the aforementioned theorems, the parametric likelihood measure is capable of representing the outranking relationship between PF evaluative ratings in a credible and effective manner. Consider that the proposed parameterized likelihood possesses formidable capability to describe the outranking relationship and handle PF information efficaciously.

This study has utilized an innovative parametric likelihood measure to develop some beneficial likelihood-based concepts of mean outranking indices, weighted outranking grades, and comprehensive outranking measures and indices. Based on the above indices, a user-friendly PF likelihood-oriented method has been introduced to characterize the priority orders among the PF characteristics of alternatives to assist decision-makers is making the most preferred choice. This paper has executed a realistic application of selecting pilot hospitals for postacute care and has demonstrated the practicality and reasonability of the proposed techniques in decision situations. Moreover, this paper has implemented experimental analysis and comparative studies to investigate the parameterization mechanism for realizing the relative effectiveness of varied parameter specifications. The experimental outcomes and discussions have substantiated the strengths and stability of the evolved PF likelihood-oriented method in circumstances under PF uncertainties.

To ascertain the applicable merits of the proposed methodology in the field of complex decision-making, there is a need to discuss the practical and academic implications for a more generic parametric likelihood measure. In real-world applications, the employment of the proposed parametric likelihood measure should be aware the decision-maker's disposition about the uncertain MCDA contexts before making a judgment of parameter settings. The specification of parameters α and β is able to portray neutral, optimistic, and pessimistic attitudes by symmetric, left-skewed, and right-skewed beta distributions, respectively. Considering the academic implications of the proposed PF likelihood-oriented methodology, this paper has investigated how a standard beta distribution could be used for building a parameter likelihood function. One potential direction of future extension is the generalization of the standard beta distribution. The beta distributions can be generalized in an attempt to generate more flexible probability density functions and cover many shapes for minimal information loss. Available generalized examples include the Gaussian hypergeometric distribution, the confluent hypergeometric distribution, the five-parameter generalized beta distribution, and the six-parameter generalized beta distribution [5, 31, 32]. These generalized versions of beta distributions can be recognized as valuable generic notions for developing more comprehensive parametric PF likelihood measures.

Nevertheless, the proposed methodology has limitations. These limitations appear due to constraints on the parameterization design. To yield a closed-form solution for the

PF likelihood function, the parameters α and β have been restricted to be integers in this study. Although integer-valued α and β can still generate rich distribution shapes and are flexible enough for realistic applications, the diversity of the distribution is inevitably reduced. When applying advanced techniques to analyze practical MCDA problems, practitioners (e.g., decision-makers or analysts) must recognize some of the realistic limitations of the proposed approach. That is, due to limitations on the determination of a closed-form solution, noninteger values of parameters α and β have been left out. Fortunately, the main feature of the initiated parametric likelihood measure is that its closed-form solution will be very clear and easily understood by the decision-maker, even though there are limitations on the specification of parameter values.

The scientific value and technical implication of the proposed parametric PF likelihood measure via a beta distribution-based approach can provide important inspiration and reference significance for the field of complex and uncertain decision-making. Based on real-world examples, many long-established decision-making methods have also broadened the applicability and scope to PF environments, such as the VIKOR [7, 46], technique for order preference by similarity to ideal solutions (TOPSIS) [23, 39], linear programming technique for multidimensional analysis of preference (LINMAP) [10, 45], elimination et choice translating reality (ELECTRE) [2, 3], and preference ranking organization method for enrichment evaluations (PROMETHEE) [9, 30]. The (comprehensive) parametric PF likelihood measures can be combined with these methods to assist various decision-making models by determining the outranking relationships for intricate uncertain information. Moreover, the utilization of such measurements can provide a solid and justified basis for decision support and contribute to a sustainable innovation of scientific technology in the field of decision-making.

Finally, based on the overall discussion considering all results, the recommendations for future research directions are fourfold: (i) assigning appropriate values of α and β to secure left-skewed and right-skewed distributions for adapting to optimistic and pessimistic attitudes, respectively, about the decision environment, (ii) spreading the applicability range of the advanced PF likelihood-oriented method through the medium of other PF likelihood measures, such as the soft likelihood function and interval-valued PF likelihoods, (iii) incorporating the proposed parametric likelihood measure into other MCDA methodology, such as TOPSIS, VIKOR, LINMAP, ELECTRE, and PROMETHEE, for wider suitability and appropriateness in the area of decision aiding and support, and (iv) increasing practicality and justiciability of the proposed methodology to manipulate MCDA issues in interval-valued PF contexts or q-rung orthopair fuzzy circumstances.

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Data availability The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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