Linear Regression (Review)

assumptions

1. linearity: $E[Y] = X\beta$ linear combination of the predictors
2. independence: $cov(\varepsilon_i, \varepsilon_j) = 0$
   $i \neq j$
3. Homoscedasticity: $var(Y_j) = var(\varepsilon_j) = 0$
4. Normality: $\varepsilon_j \overset{iid}{\sim} N(0, \sigma^2)$

What if assumptions are not met? We can't use linear regression

case study:
- response is not Normal
- constant variance (homoscedasticity) is not met

If the response is Binomial

$$Y_i \sim Bin(n, p_i) \qquad i = 1, \dots, n$$

1. PMF $P(Y_i = y_i) = \binom{n}{y} p_i^{y} (1-p_i)^{n-y}$
2. mean $E[Y_i] = \mu_i = np_i$ from definition of expectation and variance for discrete r.v.
3. variance $var(Y_i) = \sigma_i^2 = np_i(1-p_i)$

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This variance can change for each y_i, so assumption 3 is not met.

On residuals vs fitted values we can see if assumptions are met. Here, not the case.

GLM (Generalized Linear Models)

if there is a correlation structure, mixed models can be used.

ACF of the AR(p)

$$\rho(h) - \phi \rho(h-1) - \dots - \phi_p \rho(h-p) = 0$$

$\alpha_1, \dots, \alpha_r$: reciprocal roots

$\downarrow$

$m_1, \dots, m_r$: multiplicity $\qquad \sum_{i=1}^{r} m_i = p$

in this case

$$\rho(h) = \alpha_1^h p_1(h) + \dots + \alpha_r^h p_r(h)$$

$p_j(h)$ polynomial of order m

Exponential Family

Y is from exponential family if it can be written as:

$$f(y;\theta,\phi) = \exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right\}$$

θ : the location (or natural) parameter

ϕ : the dispersion parameter

example : Binomial

$$Y \sim Bin(n,p)$$

$$\begin{aligned}
f(y;n,p) &= P(Y=y;n,p) = \binom{n}{y} p^y (1-p)^{n-y} \\
&= \exp\left\{\log \binom{n}{y} p^y (1-p)^{n-y}\right\} \\
&= \exp\left\{\log\binom{n}{y} + y\log(p) + (n-y)\log(1-p)\right\} \\
&= \exp\left\{\log\binom{n}{y} + \underbrace{y\log(p) - y\log(1-p)} + n\log(1-p)\right\} \\
&= \exp\left\{y \underbrace{\log\left(\frac{p}{1-p}\right)}_{\theta} + \underbrace{n\log(1-p)}_{b(\theta)} + \underbrace{\log\binom{n}{y}}_{c(y,\phi)}\right\}
\end{aligned}$$

Note that $\theta = \log\left(\frac{p}{1-p}\right)$, $\phi = 1$

$$\Leftrightarrow \quad p = \frac{e^{\theta}}{1+e^{\theta}}$$

So the Binomial distribution is a member of the Exponential family.

$$\mu = E[Y] = b'(\theta)$$

$$\sigma^2 = var(Y) = b''(\theta)\,\phi$$

Introduction to Binomial Regression

Each y_i is a realization from a Y_i, Binomial

$$Y_i \sim Bin(1, p_i) \qquad p_i \in [0,1]$$

Binary Logistic Regression

$$Y_i \sim Bin(n_i, p_i)$$

assumption: Y_i are independent

Goal: predict or explain p_i using covariates $x_{i,1}, \dots, x_{i,p}$

Then predict $E[Y_i] = \mu_i = n_i p_i$

<u>Systematic component</u>:

$$\eta_i = \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \dots + \beta_p x_{i,p}$$

how to link the linear predictor to the probability of success ?

$g(\mu_i)$?

<u>Binomial Regression</u>: link functions

1. Logit (logistic link):

$$\eta_i \overset{\text{set}}{=} g(p_i) = \log\left(\frac{p_i}{1-p_i}\right) = \theta \quad \text{natural parameter}$$

2. Probit:

$$g(p_i) = \Phi^{-1}(p_i)$$

Φ^{-1}: inverse $N(0,1)$ CDF

Binomial Regression Parameter Estimation

maximum likelihood estimation

n_i = number of trials

1. marginal p.m.f. :

$$P(Y_i = y_i) = \binom{n_i}{y_i} p_i^{y_i} (1-p_i)^{n_i - y_i}$$

$$p_i = \frac{e^{\theta_i}}{1+e^{\theta_i}} \quad , \quad \theta_i = \log\left(\frac{p_i}{1-p_i}\right) = \eta_i$$

$$= \beta_0 + \beta_1 x_{i,1} + \dots + \beta_p x_{i,p}$$

2. joint p.m.f. :

because we have independence

$$f(y_i; n_i, p_i) = \prod_{i=1}^{n} \binom{n_i}{p_i} p_i^{y_i} (1-p_i)^{n_i - y_i}$$

$$= \prod_{i=1}^{n} \binom{n_i}{p_i} \left(\frac{e^{\eta_i}}{1+e^{\eta_i}}\right)^{y_i} \left(1 - \frac{e^{\eta_i}}{1+e^{\eta_i}}\right)^{n_i - y_i}$$

3. likelihood function :

$$L(\beta) = \prod_{i=1}^{n} \binom{n_i}{y_i} \left(\frac{e^{\eta_i}}{1+e^{\eta_i}}\right)^{y_i} \left(1 - \frac{e^{\eta_i}}{1+e^{\eta_i}}\right)^{n_i - y_i}$$

4. log-likelihood function:

$$\ell(\underline{\beta}) = \sum_{i=1}^{n} \left(y_i \underline{\eta_i} - n_i \log\left(1 + e^{\eta_i}\right) + \log\binom{n_i}{y_i} \right)$$

not linear. We use iterative technique to maximize the log-likelihood function.

5. maximize!

Interpretation of Binomial Regression

$$\eta_i = \hat{\beta}_0 = \hat{\beta}_1 x_{i,1} + \hat{\beta}_2 x_{i,2} = \log\left(\frac{\hat{p}_i}{1-\hat{p}_i}\right)$$

2 predictors: $x_{i,1}$ and $x_{i,2}$

we have ml estimates $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\beta}_2$

Definition: Let event E have probability p of occurence. Then the odds in favor of E is:

$$O_E = \frac{p}{1-p}$$

biased coin: $P(H) = 3/4$ and $P(T) = 1/4$

$$O_E = \frac{p}{1-p} = \frac{(3/4)^2}{1-(3/4)^2} = \frac{0.5625}{0.375} \approx \underline{1.29}$$

that we get H twice in a row.

It only holds for the logit link function.

β_0 : log odds of success when all predictors are equal to 0.

β_j : $X_{:,j}$ increase by 1 unit and all other predictors are held constant, the log-odds of success increases by β_j.

Odds of success increases by e^{β_j}

Binomial Regression in R

Occupancy : 0: not occupied

1: occupied

Temperature : in Celcius

Relative Humidity :

Light :

CO_2 measurement :

use "RCurl" package

"glm" function , family = "binomial"

format of the response : factor or
as a two-column matrix
(factor here)

Estimates computed using maximum likelihood estimation us IRWLS.

$\hat{\beta}_0 = -29.31$:
- on the scale of the linear predictor (not exponentiated)
- average log-odds of an app. being occupied, everything else is 0, is about 29,3 %.
- $e^{\hat{\beta}_0} \approx 0$ (odd)

$\hat{\beta}_3$ (light) :
- $e^{\hat{\beta}_3} \approx 1.02$ (odd)
- increase is multiplicative

$$
\begin{aligned}
e^{\hat{\eta}+1} &= e^{\hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \hat{\beta}_3 (x_3+1) + \hat{\beta}_4 x_4} \\
&= e^{\hat{\beta}_3} \underbrace{e^{\hat{\beta}_0 + \hat{\beta} x_1 + \hat{\beta}_2 x_2 + \hat{\beta}_3 x_3 + \hat{\beta}_4 x_4}}_{e^{\hat{\eta}}} \\
&= e^{\hat{\beta}_3} e^{\hat{\eta}}
\end{aligned}
$$

$e^{\hat{\beta}_3} \approx 1.02$

Poisson Regression : model for count data

Y_i , $i = 1, \dots, n$ response variable , Poisson

$x_{i,j}$, $j = 1, \dots, p$ systematic component

$Y_i \sim Poi(\lambda_i)$

$$P(Y_i = y_i) = \frac{e^{-\lambda_i} \lambda_i^{y_i}}{y_i!}$$

$y_i = 0, 1, 2, \dots$

$\lambda_i > 0$ potentially different for diff. measurement

mean : $\mu_i = E[Y_i] = \lambda_i$

variance : $\sigma_i^2 = var(Y_i) = \lambda_i$

canonical parameter : $\theta_i = \log(\mu_i) = \log(\lambda_i)$

canonical link function : $\log(\lambda)$

systematic component :

$$\eta = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p$$

has to be positive. The mean (positive) should be linked to the linear predictor by a link function

$$g(\lambda_i) = \log(\lambda_i)$$ mean in a rate

$$\Rightarrow \quad \eta_i \overset{\text{set}}{=} g(\lambda_i) = \log(\lambda_i) = \theta \quad \left(\begin{smallmatrix}\text{canonical}\\ \text{parameter}\end{smallmatrix}\right)$$

$$\lambda_i = e^{\eta_i}$$

estimation by maximum likelihood.

<u>Rate response</u> : $\mu_i = \lambda_i = \dfrac{\text{count}}{\text{exposure time}} = \dfrac{y_i}{e_i}$

e_i = Known period of time

<u>offset term</u> : (log link)

$$g(\lambda_i) = \log(\lambda_i) = \log\left(\frac{y_i}{e_i}\right) = \log(y_i) - \log(e_i)$$

offset in glm formula. For different period of time of exposure.

family = "poisson"

Poisson Regression : parameter estimation

maximum likelihood estimation

1. marginal pmf :

$$P(Y_i = y_i) = \frac{e^{-\lambda_i} \lambda^{y_i}}{y_i!} \quad , y_i = 0, 1, 2, \ldots$$

2. joint pmf. :

$$f(y_i ; \lambda_i) = \prod_{i=1}^{n} \left(\frac{e^{-\lambda_i} \lambda_i^{y_i}}{y_i!} \right)$$

3. likelihood function

$$L(\boldsymbol{\beta}) = \prod_{i=1}^{n} \left(\frac{e^{\lambda_i} \lambda_i^{y_i}}{y_i!} \right)$$

$$\eta_i = \log(\lambda_i)$$

$$e^{\eta_i} = e^{\beta_0 + \beta_1 x_{i1} + \ldots}$$

$$= \prod_{i=1}^{n} \left(\frac{e^{e^{-\eta_i}} e^{y_i \eta_i}}{y_i!} \right) = \prod_{i=1}^{n} \left(\frac{e^{y_i \eta_i - e^{\eta_i}}}{y_i!} \right)$$

4. log-likelihood function

$$\ell(\boldsymbol{\beta}) = \sum_{i=1}^{n} \left(y_i \eta_i - e^{\eta_i} - \log(y_i!) \right)$$

5. maximize! using iterative techniques.

Interpreting the Poisson Regression model

remember : we use the log link $\quad \log(\lambda_i) = \eta_i$

- β_0 : e^{β_0} can be interpreted as the mean of the response when each predictor is set to 0.

- β_j : e^{β_j} can be interpreted as the multiplicative increase in the mean of the response for a one unit increase in $x_{i,j}$, holding all other predictors constant.

$$y_i^{+1} = \hat{\lambda}_i^{+1} = e\left\{ \hat{\beta}_0 + \hat{\beta}_1 x_{i,1} + \ldots + \hat{\beta}_j (x_{i,j} + 1) + \ldots + \hat{\beta}_p x_{i,p} \right\}$$

$$= \exp\{\hat{\beta}_j\} \exp\left\{\hat{\beta}_0 + \hat{\beta}_1 x_{i,1} + \ldots + \hat{\beta}_j (x_{i,j}) + \ldots + \hat{\beta}_p x_{i,p}\right\}$$

$$= \exp\{\hat{\beta}_j\} \hat{\lambda}_i$$