Crude Monte Carlo estimation

Monte Carlo (MC) integration or estimation is a method used to approximate integrals. Suppose that we wish to evaluate the integral

$$\theta = E_f[h(x)] = \int h(x) f(x) dx$$

A natural estimator is given by

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} h(x_i)$$

$\hat{\theta}$ is an unbiased estimator for $\theta = E_f[h(x)]$.

Example of crude Monte Carlo estimation

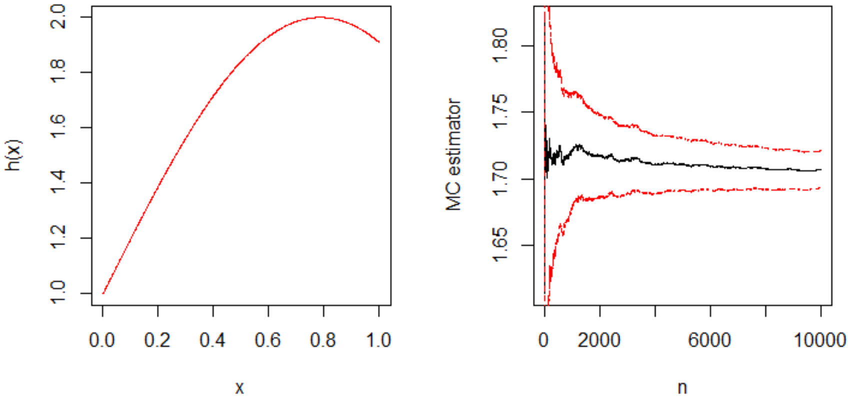
Suppose that we wish to approximate the following integral

$$\theta = \int_0^1 (cos(x) + sin(x))^2$$

First we generate $x_1, x_2, ..., x_n \sim U_{[0,1]}$, then we apply the formula $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} h(x_i)$ with $h(x) = (cos(x) + sin(x))^2$. In R, we get

```
# crude Monte Carlo estimation
h <- function(x) {
  (cos(x) + sin(x))^2
}

n = 10000
xi <- runif(n = n)

set.seed(1986)
theta_hat <- mean(h(xi))
theta_hat
# [1] 1.707224
```

Visualizing the function and convergence



Introduction to importance sampling

Importance sampling is a method used to approximate integrals while using observations generated from a different distribution than the distribution of interest, that we call here $\tilde{f}$. The method was introduced by T. Kloek and H. K. van Dijk in 1978.

We introduce the PDF $\tilde{f}(x)$ so that the original integral becomes

$$\theta = E_{\tilde{f}}\left[\frac{h(x)f(x)}{\tilde{f}(x)}\right] = \int \frac{h(x)f(x)}{\tilde{f}(x)}\tilde{f}(x)dx$$

with $\tilde{f}(x) \geq 0$ and $\int \tilde{f}(x)dx = 1$.

Importance sampling estimator

A Monte Carlo estimator is given by

$$\hat{\theta}_{IS} = \frac{1}{n}\sum_{i=1}^{n} h(x_i)w(x_i)$$

where $w(x_i) = \frac{f(x)}{\tilde{f}(x)}$ and x_i are r.v. with PDF $\tilde{f}$. The quantity $w(x_i)$ is called the importance sampling weight of the point x_i. We have

$$E[\hat{\theta}_{IS}] = E_{\tilde{f}}\left[\frac{h(x)f(x)}{\tilde{f}(x)}\right] = \theta$$

and

$$var(\hat{\theta}_{IS}) = \frac{1}{n}\int \left(\frac{h(x)f(x)}{\tilde{f}(x)} - \theta\right)^2 \tilde{f}(x)dx$$

Importance sampling algorithm

To implement importance sampling, we proceed as follows:

1) Choose the PDF $\tilde{f}$ proportional to $h(x)f(x)$.

2) Generate iid observations $x_1, x_2, ..., x_n$ with PDF $\tilde{f}$.

3) Compute the estimator $\hat{\theta}_{IS} = \frac{1}{n}\sum_{i=1}^{n} h(x_i)w(x_i)$ with $w(x_i) = \frac{f(x)}{\tilde{f}(x)}, i = 1, ..., n$.

Importance sampling example

(see M. Rizzo, p. 140)

We wish to estimate the following integral

$$\int_0^1 \frac{e^{-x}}{1-x^2} dx$$

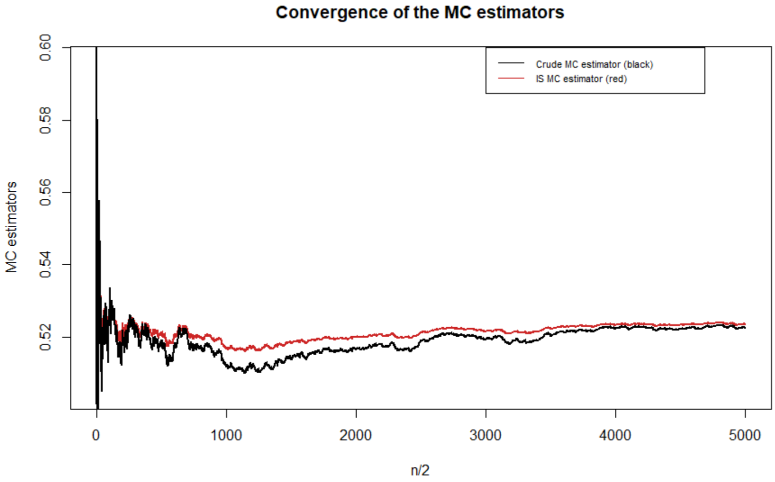
We will use $\tilde{f} = \frac{4}{\pi(1-x^2)}$ for $0 < x < 1$. This function is found using the inverse transform method.

In this example, we will compare the importance sampling estimator with the crude Monte Carlo estimator. We set $h(x) = \frac{e^{-x}}{1-x^2}$ and $f(x) = 1$.

Importance sampling example in R

```
set.seed(1986)
n <- 10000
theta.hat <- se <- numeric(2)
h <- function(x) {exp(-x)/(1+x^2)}  # function to integrate
# crude MC estimation
u <- runif(n)
fg_1 <- h(u)

theta.hat[1] <- mean(fg_1)  # the crude MC estimator
se[1] <- sd(fg_1)/sqrt(n)  # its estimated standard error

x <- tan(pi * u / 4)   # x's are transformation of u's via the choosen function
fg_2 <- h(x) / (4 / ((1 + x^2) * pi))  # integration of f_tilde

theta.hat[2] <- mean(fg_2)   # the importance sampling MC estimator
se[2] <- sd(fg_2)/sqrt(n)  # its estimated standard error

table=as.data.frame(rbind(theta.hat, se))
names(table)=c("crude_MC","IS_MC")
table  # ISMC estimator has a smaller variance
#                crude MC        IS MC
# theta.hat 0.525188327 0.524929528
# se        0.002453147 0.001414611
```

Visualizing convergence



References

Rizzo, M.L. (2019). Statistical Computing with R, Second Edition (2nd ed.). Chapman and Hall/CRC.
https://doi.org/10.1201/9780429192760

The R Project for Statistical Computing:
https://www.r-project.org/

Python:
https://www.python.org/

course notes