Simple Monte Carlo integration with example

Suppose that we want to approximate some integral I using simulations. I has the form

$$I = \int g(x)f(x)dx$$

where $g(x)$ is some function of a continuous r.v. X and $f(x)$ is its density. We can draw a sample of size m from the density $f(x)$ and then use the empirical average to get a Monte Carlo estimate of I, that is

$$\hat{I} = \frac{1}{m}\sum_{i=1}^{m} g(x_i)$$

The standard error $se(\hat{I})$ is given by

$$se(\hat{I}) = \sqrt{\frac{1}{m^2}\sum_{i=1}^{m}\Big(g(x_i) - \hat{I}\Big)^2}$$

so we clearly see that the larger the number of simulations, the lower the standard error. Indeed, by the Law of Large Numbers (LLN), we have that $\lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} g(x_i) = E_f[g(X)]$, or equivalently $\lim_{m\to\infty} \hat{I} = I$. Then, by the Central Limit Theorem (CLT), we have that $\hat{I} \to N(I, se(\hat{I}))$. We can then construct a $(1-\alpha)$ Confidence Interval for I, given by

$$\Big[\hat{I} - z_{1-\alpha/2}se(\hat{I}), \hat{I} + z_{1-\alpha/2}se(\hat{I})\Big]$$

Example We want to approximate the following integral

$$I = \int_0^{\pi} g(x)dx \qquad \text{with } g(x) = sin(x) + cos(x)$$

We can compute it analytically and we will find that $I = 2$. Using Monte Carlo integration, we will generate $m = 10,000$ realizations of $X \sim U[0, \pi]$ and then use the fact that the PDF of X is

$$f(x) = \begin{cases} 1/(\pi - 0) & \text{if } x \in [0, \pi] \\ 0 & \text{otherwise} \end{cases}$$

In our case, we will have

$$I = \int_0^{\pi} g(x)dx = \pi \int_0^{\pi} g(x)\frac{1}{\pi}dx = \pi E_f[g(X)]$$

$$\hat{I} = \frac{\pi}{10,000}\sum_{i=1}^{10{,}000} g(x_i) \approx 2.03$$

$$se(\hat{I}) = \frac{\pi}{10,000}\sqrt{\sum_{i=1}^{10{,}000}\Big(g(x_i) - \hat{I}\Big)^2} \approx 0.05$$

$$I \in [1.94, 2.13]$$

$$\pi(x \mid \lambda) = \frac{\lambda^{\sum_{i=1}^{n} x_i} e^{-n\lambda}}{\prod_{i=1}^{n}(x_i!)} = \frac{\lambda^{n\bar{x}} e^{-n\lambda}}{\prod_{i=1}^{n}(x_i!)}$$

where λ, the model parameter, is the count of some event of interest and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$. A convenient choice to model the uncertainty about λ is a Gamma distribution as prior, since the support of such a distribution is the interval $[0, \infty[$. A Gamma distribution with shape parameter α and scale parameter β has the following form

$$f_{\alpha,\beta}(\lambda) = \pi(\lambda) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda\beta}$$

where $\Gamma(\alpha)$ is the Gamma function, defined as $\int_0^{\infty} x^{\alpha-1} e^{-x} dx = (\alpha - 1)!$.

The posterior distribution for λ if our data x is modeled with a Poisson distribution and a Gamma prior is chosen for λ will also have the functional form of a Gamma r.v. Using the Bayes theorem, we have that

$$\begin{aligned}
\pi(\lambda \mid x) &= \frac{\pi(x \mid \lambda)\pi(\lambda)}{\pi(x)} \\
&= \frac{\lambda^{n\bar{x}} e^{-n\lambda}}{\prod_{i=1}^{n}(x_i!)} \frac{\beta^{\alpha}}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda\beta} \\
&= \underbrace{\frac{1}{\prod_{i=1}^{n}(x_i!)} \frac{\beta^{\alpha}}{\Gamma(\alpha)}}_{\text{do NOT depend on } \lambda} \quad \lambda^{n\bar{x}} e^{-n\lambda} \lambda^{\alpha-1} e^{-\lambda\beta} \\
&\propto \lambda^{\alpha+n\bar{x}-1} e^{-(n+\beta)\lambda}
\end{aligned}$$

and we find that $\lambda \mid x \propto Gamma(\alpha + n\bar{x}, n + \beta)$. The posterior mean and variance are given by

$$E[\lambda \mid x] = \frac{\alpha + n\bar{x}}{n + \beta} \qquad var(\lambda \mid x) = \frac{\alpha + n\bar{x}}{(n + \beta)^2}$$

where $n\bar{x} = \sum_{i=1}^{n} x_i$ is the sum of the counts and n is the sample size.