Poisson Likelihood

Suppose that a r.v. X obeys a Poisson distribution $\mathcal{P}(\lambda)$, characterized by the following Probability Mass Function (PMF)

$$f_\lambda(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

If we observe a sample $x_1, ..., x_n$, then the likelihood is just the product of the individual PMF

$$\begin{aligned}\pi(x_1, ..., x_n \mid \lambda) &= \frac{\lambda^{\sum_{i=1}^n x_i} e^{-n\lambda}}{\prod_{i=1}^n (x_i!)} \\ &= \frac{\lambda^{n\bar{x}} e^{-n\lambda}}{\prod_{i=1}^n (x_i!)}\end{aligned}$$

where the model parameter λ is the count of some event of interest and $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$.

Pump data

Numbers of failures and times of observation of 10 pumps in a nuclear plant (Source: Gaver and O'Muircheartaigh, 1987)

	Pump	Failures	Time
1	1	5.00	94.32
2	2	1.00	15.72
3	3	5.00	62.88
4	4	14.00	125.76
5	5	3.00	5.24
6	6	19.00	31.44
7	7	1.00	1.05
8	8	1.00	1.05
9	9	4.00	2.10
10	10	22.00	10.48

Model and assumptions

The failure of the ith pump follow a Poisson process with parameter λ_i, for $i = 1, 2, ..., 10$. For an observed time t_i the number of failures p_i is thus a Poisson $\mathcal{P}(\lambda_i t_i)$ r.v.

Likelihood of failure

$$y_i \sim \mathcal{P}(\lambda_i t_i)$$

Prior on λ_i and prior on β

$$\lambda_i \sim \mathcal{G}a(\alpha, \beta)$$

$$\beta \sim \mathcal{G}a(\gamma, \delta)$$

with $\alpha = 1.8$, $\gamma = 0.01$ and $\delta = 1$.

Joint and conditional distributions

Joint posterior distribution

$$\begin{aligned}
&\pi(\lambda_1, ..., \lambda_{10}, \beta \mid t_1, ..., t_{10}, p_1, ..., p_{10}) \\
&\propto \prod_{i=1}^{10} \left((\lambda_i t_i)^{p_i} e^{-\lambda_i t_i} \lambda_i^{\alpha-1} e^{-\beta\lambda_i} \right) \beta^{10\alpha} \beta^{\gamma-1} e^{-\delta\beta} \\
&\propto \prod_{i=1}^{10} \left((\lambda_i)^{p_i+\alpha-1} e^{-(t_i+\beta)\lambda_i} e^{-\beta\lambda_i} \right) \beta^{10\alpha+\gamma-1} e^{-\delta\beta}
\end{aligned}$$

and a natural decomposition of π in conditional distributions is

$$\lambda_i \mid \beta, t_i, p_i \sim \mathcal{G}a(p_i + \alpha . t_i + \beta)$$

$$\beta \mid \lambda_1, ..., \lambda_{10} \sim \mathcal{G}a\left(\gamma + 10\alpha, \delta \sum_{i=1}^{10} \lambda_i\right)$$

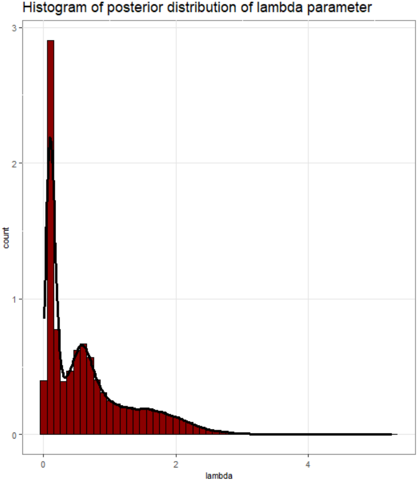
Multistage Gibbs sampler for Poisson-Gamma model

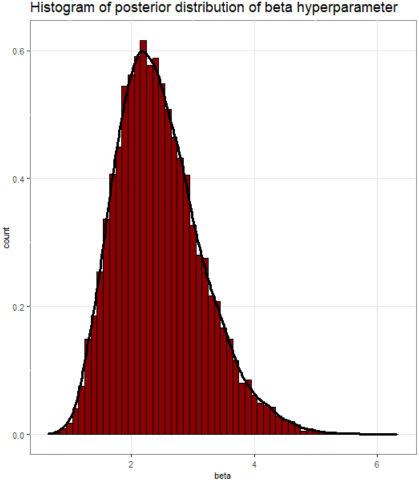
```
twostageGibbsSamplerPG <- function(nsim, beta.start, alpha, gamma, delta, y, t,
    burnin = 0,
                                   thin = 1) {
  beta.draws <- c()
  lambda.draws <- matrix(NA, nrow = nsim, ncol = length(y))
  beta.cur <- beta.start

  lambda.update <- function(alpha, beta, y, t) {
    rgamma(length(y), y + alpha, t + beta)
  }

  beta.update <- function(alpha, gamma, delta, lambda, y) {
    rgamma(1, length(y) * alpha + gamma, delta + sum(lambda))
  }
  for (i in 1:nsim) {
    lambda.cur <- lambda.update(alpha = alpha, beta = beta.cur, y = y, t = t)
    beta.cur <- beta.update(alpha = alpha, gamma = gamma, delta = delta,
                            lambda = lambda.cur, y = y)
    if (i > burnin & (i - burnin)% == 0) {
      lambda.draws[(i - burnin), ] <- lambda.cur
      beta.draws[(i - burnin)] <- beta.cur
    }
  }
  return(list(lambda.draws = lambda.draws, beta.draws = beta.draws))
}
```

Posterior quantities obtained from Gibbs sampling

	lambda	beta
post mean	0.6498	2.4641
post sd	0.6515	0.7062
2.5%	0.0432	1.3195
50%	0.4625	2.3792
97.5%	2.2563	4.0827

Posterior distributions





Further reading and code

Robert, C. P., Casella, G., Casella, G. (1999). Monte Carlo statistical methods (Vol. 2). New York: Springer.

The R Project for Statistical Computing:
https://www.r-project.org/