The problem

Example of a multi-stage Gibbs sampler for reliability analysis. We want to set up a Bayesian analysis to eventually draw from the posterior distribution of the number of failures for a given time of observation of nuclear plant pumps based on some initial data.

Keywords: Reliability, Poisson process, Gibbs sampling, Poisson-Gamma model, Empirical Bayes

Pump data

The data: number of failures and times of observation of 10 pumps in a nuclear plant water system (Source: Gaver and O'Muircheartaigh, 1987)

	Pump	Failures	Time
1	1	5.00	94.32
2	2	1.00	15.72
3	3	5.00	62.88
4	4	14.00	125.76
5	5	3.00	5.24
6	6	19.00	31.44
7	7	1.00	1.05
8	8	1.00	1.05
9	9	4.00	2.10
10	10	22.00	10.48

Model and assumptions

The failure of the ith pump follow a Poisson process with parameter λ_i, for $i = 1, 2, ..., 10$. For an observed time t_i the number of failures p_i is thus a Poisson $\mathcal{P}(\lambda_i t_i)$ r.v.

Likelihood of failure

$$y_i \sim \mathcal{P}(\lambda_i t_i)$$

Prior on λ_i and prior on β

$$\lambda_i \sim \mathcal{G}a(\alpha, \beta)$$

$$\beta \sim \mathcal{G}a(\gamma, \delta)$$

with $\alpha = 1.8$, $\gamma = 0.01$ and $\delta = 1$ (see. Gaver and O'Muircheartaigh 1987 for a motivation of these numerical values)

Joint and conditional distributions

Joint posterior distribution

$$\pi(\lambda_1, ..., \lambda_{10}, \beta \mid t_1, ..., t_{10}, p_1, ..., p_{10})$$
$$\propto \prod_{i=1}^{10} \left((\lambda_i t_i)^{p_i} e^{-\lambda_i t_i} \lambda_i^{\alpha-1} e^{-\beta\lambda_i} \right) \beta^{10\alpha} \beta^{\gamma-1} e^{-\delta\beta}$$
$$\propto \prod_{i=1}^{10} \left((\lambda_i)^{p_i+\alpha-1} e^{-(t_i+\beta)\lambda_i} e^{-\beta\lambda_i} \right) \beta^{10\alpha+\gamma-1} e^{-\delta\beta}$$

and a natural decomposition of π in conditional distributions is

$$\lambda_i \mid \beta, t_i, p_i \sim \mathcal{G}a(p_i + \alpha . t_i + \beta)$$

$$\beta \mid \lambda_1, ..., \lambda_{10} \sim \mathcal{G}a\left(\gamma + 10\alpha, \delta \sum_{i=1}^{10} \lambda_i \right)$$

Multistage Gibbs sampler for Poisson-Gamma model

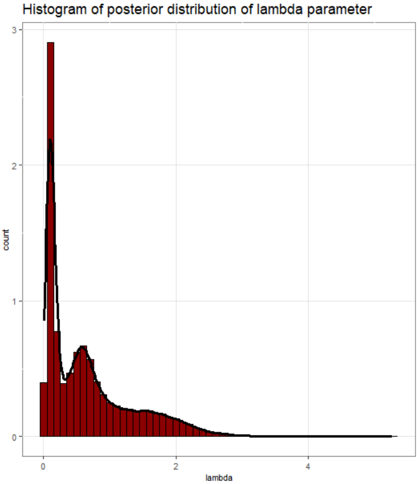
```r
Gibbs_sampler_PG <- function(nsim, beta, alpha, gamma, delta, y, t, burnin) {

  X = matrix(0, nrow = nsim, ncol = length(y)+1) # empty matrix to record the simulated values
  X[1,1] = beta  # beta prior parameter
  X[1,c(2:(length(y)+1))] = rgamma(length(y), y + alpha, t + X[1,1]) # initial lambda

  for(i in 2:nsim) {

    X[i,c(2:(length(y)+1))] = rgamma(length(y), y + alpha, t + X[i-1,1]) # update lambda
    X[i,1] = rgamma(1, length(y) * alpha + gamma, delta + sum(X[i-1,c(2:(length(y)+1))])) # update beta
  }

  b <- burnin + 1 # record the burn in period (observations to be discarded)
  x <- X[b:nsim, ]

  return(list('lambda' = as.numeric(x[,c(2:(length(y)+1))]), 'beta' = x[,1] ))
}

# posterior
set.seed(2023)
posterior <- Gibbs_sampler_PG(nsim = 10000, beta = 1, alpha = 1.8, gamma = 0.01,
                              delta = 1, y = dataset[,2], t = dataset[,3], burnin = 1000)
```

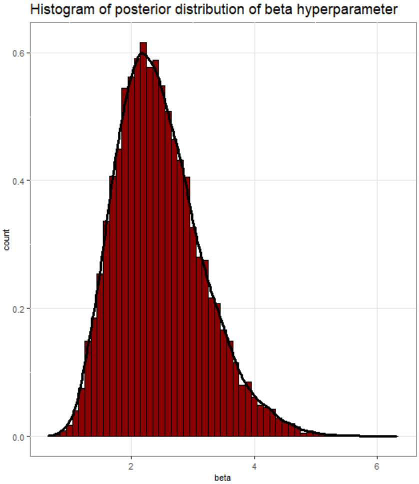
Posterior quantities obtained from Gibbs sampling

	lambda	beta
post mean	0.6510	2.4598
post sd	0.6534	0.7101
2.5%	0.0431	1.3196
50%	0.4626	2.3690
97.5%	2.2652	4.0806

We see that the distribution of the number of failures given time is bimodal and right skewed. We can now draw our conclusions not only based on point estimates but we have the whole distribution.

Posterior distributions





References

Donald P. Gaver I. G. O'Muircheartaigh (1987) Robust Empirical Bayes Analyses of Event Rates, Technometrics, 29:1, 1-15, DOI: 10.1080/00401706.1987.10488178

Robert, C. P., Casella, G. (1999). Monte Carlo statistical methods (Vol. 2). New York: Springer.

The R Project for Statistical Computing: https://www.r-project.org/