Moment Generating Function (MGF): definition (multivariate case)

We are interested in finding the Moment Generating Function (MGF) of an n-dimentional random vector having a Multivariate Normal distribution, that is $\mathbf{X} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. This function, which is denoted $M_{\mathbf{X}}(\mathbf{t})$ is an alternative specification of its PDF and defined as

$$M_{\mathbf{X}}(\mathbf{t}) = E[e^{\mathbf{t}^T \mathbf{X}}]$$

Then n^{th} moment will be given by $E[\mathbf{X}^n] = \frac{d^n M_{\mathbf{X}}(\mathbf{t})}{d\mathbf{t}^n}\Big|_{\mathbf{t}=\mathbf{0}}$, that is the n^{th} derivative of the MGF evaluated at $\mathbf{t} = \mathbf{0}$.

Multivariate Normal random vectors

An n-dimentional random vector $\mathbf{X}$ has a Multivariate Normal distribution if its density is given by

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{n/2} \mid \boldsymbol{\Sigma} \mid^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$

for $\mathbf{x} \in \mathbb{R}^n$ and where $\boldsymbol{\mu} = (\mu_1, \mu_2, ..., \mu_n)^T$ and $\boldsymbol{\Sigma}$ is a $n \times n$ symmetric positive definite covariance matrix.

The Moment Generating Function (MGF) of $\mathbf{X}$ is then

$$M_{\mathbf{X}}(\mathbf{t}) = \exp\left(\mathbf{t}^T\boldsymbol{\mu} + \frac{1}{2}\mathbf{t}^T\boldsymbol{\Sigma}\mathbf{t}\right)$$

MGF of a Multivariate Normal random vector

Indeed, we have

$$\begin{aligned}
M_{\mathbf{X}}(\mathbf{t}) &= E[e^{\mathbf{t}^T\mathbf{X}}] \\
&= E[e^{\mathbf{t}^T(\boldsymbol{\mu}+\mathbf{Z}(\boldsymbol{\Sigma}^{1/2})^T)}] \\
&= e^{\mathbf{t}^T\boldsymbol{\mu}} E[e^{\mathbf{Z}(\mathbf{t}\boldsymbol{\Sigma}^{1/2})^T}] \\
&= e^{\mathbf{t}^T\boldsymbol{\mu}} M_{\mathbf{Z}}(\mathbf{t}\boldsymbol{\Sigma}^{1/2}) \\
&= e^{\mathbf{t}^T\boldsymbol{\mu}} e^{\frac{1}{2}||\mathbf{t}\boldsymbol{\Sigma}^{1/2}||^2} \\
&= e^{\mathbf{t}^T\boldsymbol{\mu}+\frac{1}{2}\mathbf{t}^T\boldsymbol{\Sigma}^{1/2T}\boldsymbol{\Sigma}^{1/2}\mathbf{t}} \\
&= e^{\mathbf{t}^T\boldsymbol{\mu}+\frac{1}{2}\mathbf{t}^T\boldsymbol{\Sigma}\mathbf{t}}
\end{aligned}$$

First moment of a Multivariate Normal random vector

For the first moment, we have

$$
\begin{aligned}
E[\mathbf{X}] = \frac{dM_{\mathbf{X}}(\mathbf{t})}{d\mathbf{t}}\Big|_{\mathbf{t}=\mathbf{0}} &= M'_{\mathbf{X}}(\mathbf{0}) \\
&= (\boldsymbol{\mu} + \boldsymbol{\Sigma}\mathbf{t})e^{\mathbf{t}^T\boldsymbol{\mu} + \frac{1}{2}\mathbf{t}^T\boldsymbol{\Sigma}\mathbf{t}} \\
&= \boldsymbol{\mu}
\end{aligned}
$$

So we conclude that the first moment of a Multivariate Normal random vector is just $\boldsymbol{\mu} = (\mu_1, \mu_2, ..., \mu_n)^T$. Moments of higher order can be similarly derived from the MGF.

Multivariate Normal MGF in R

```
## parameters

rho_12 <- -0.75; rho_13 <- 0.4; rho_23 <- -0.1
mu1 <- 0 ; mu2 <- 0 ; mu3 <- 2
s1 <- 2; s2 <- 1; s3 <- 1

# mean
mu <- c(mu1, mu2, mu3)

# covariance matrix
# cov_12 = s1*s2*rho_12
Sigma <- matrix(c(s1^2, s1*s2*rho_12, s1*s3*rho_13,
                  s2*s1*rho_12, s2^2, s2*s3*rho_23,
                  s1*s3*rho_13, s2*s3*rho_23, s3^2), nrow = 3)


t <- c(0,0,0)

mgf <- exp((t(t) %*% mu) + 0.5* t(t) %*% Sigma %*% t )
dmgf.dt <- (mu +  Sigma %*% t) %*%  exp((t(t) %*% mu) + 0.5* t(t) %*% Sigma %*% t )
dmgf.dt

#       [,1]
# [1,]     0
# [2,]     0
# [3,]     2
```