Moment Generating Function (MGF): definition

We are interested in finding the Moment Generating Function (MGF) of a random variable $X \sim N(\mu, \sigma^2)$, which is denoted $M_X(t)$ and is an alternative specification of its PDF, and defined as

$$M_X(t) = E[e^{tX}] = \int\limits_{-\infty}^{+\infty} e^{tX} \ f(x) \ dx$$

for X being a continuous random variable.

Then n^{th} moment will be given by $E[X^n] = \frac{d^n M_X(t)}{dt^n}\Big|_{t=0}$, that is the n^{th} derivative of the MGF evaluated at $t = 0$.

MGF of a standard Normal r.v.

To derive the MGF of an arbitrary Normal random variable, we first begin with the MGF of the random variable $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$. We have

$$
\begin{aligned}
M_Z(t) = E[e^{tZ}] = \int_{-\infty}^{+\infty} e^{tZ}\ f_{\mu,\sigma}(z)\ dz &= \int_{-\infty}^{+\infty} e^{tZ}\ \frac{1}{\sqrt{2\pi}} e^{\frac{-Z^2}{2}}\ dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-(Z^2-2tZ)}{2}} dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{(Z-t)^2}{2}+\frac{t^2}{2}} dz \\
&= e^{t^2/2} \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{+\infty} e^{-\frac{(Z-t)^2}{2}} dz}_{=\sqrt{2\pi}} \\
&= e^{t^2/2}
\end{aligned}
$$

MGF of a Normal r.v.

Then, to obtain the MGF of the random variable $X = \mu + \sigma Z \sim N(\mu, \sigma^2)$, we proceed as follows:

$$
\begin{aligned}
M_X(t) &= E[e^{tX}] \\
&= E[e^{t(\mu+\sigma Z)}] \\
&= E[e^{\mu t} e^{t\sigma Z)}] \\
&= e^{\mu t} E[e^{t\sigma Z)}] \\
&= e^{\mu t} M_Z(t\sigma) \\
&= e^{\mu t} e^{(t\sigma)^2/2} \\
&= e^{\frac{t^2\sigma^2}{2}+\mu t}
\end{aligned}
$$

First and second moments of a Normal random variable

For instance the first and second moment are respectively given by

$$E[X] = \frac{dM_X(t)}{dt}\Big|_{t=0} = M_X'(0)$$
$$= (t\sigma^2 + \mu)e^{\frac{t^2\sigma^2}{2}+\mu t}$$
$$= \mu$$

$$E[X^2] = \frac{d^2M_X(t)}{dt^2}\Big|_{t=0} = M_X''(0)$$
$$= (t\sigma^2 + \mu)^2 e^{\frac{t^2\sigma^2}{2}+\mu t} + \sigma^2 e^{\frac{t^2\sigma^2}{2}+\mu t}$$
$$= \mu^2 + \sigma^2$$

So in our case, the variance of $X \sim N(\mu, \sigma^2)$ is given by $E[X^2] - (E[X])^2 = \mu^2 + \sigma^2 - (\mu)^2 = \sigma^2$.

MGF in R

```
mgf.normal <- function(t, sigma, mu){

  mgf <- expression(exp(((t^2 * sigma^2)/2) + mu*t))
  dmgf.dt <- D(mgf, "t")
  ddmgt.dtt <- D(dmgf.dt, "t")
  dddmgt.dttt <- D(ddmgt.dtt, "t")
  first.moment <- eval(dmgf.dt)
  second.moment <- eval(ddmgt.dtt)
  third.moment <- eval(dddmgt.dttt)
  output <- NULL
  output$first.moment <- first.moment
  output$second.moment <- second.moment
  output$third.moment <- third.moment
  return(output)
}

mgf.normal( t= 0, sigma = 1, mu = 0)
# $first.moment
# [1] 0
# $second.moment
# [1] 1
# $third.moment
# [1] 0

mgf.normal(t = 0, sigma = 2, mu = 4)
# $first.moment
# [1] 4
# $second.moment
# [1] 20
# $third.moment
# [1] 112
```