

# Poisson Likelihood

Suppose that a r.v.  $X$  obeys a Poisson distribution  $\mathcal{P}(\lambda)$ , characterized by the following Probability Mass Function (PMF)

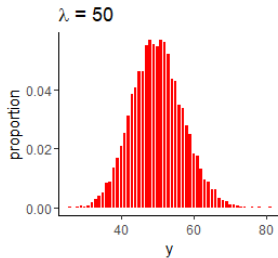
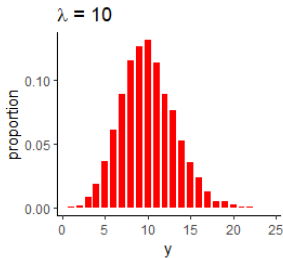
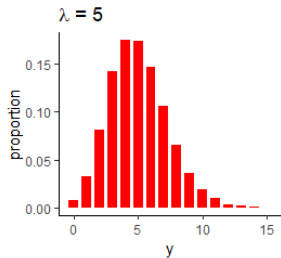
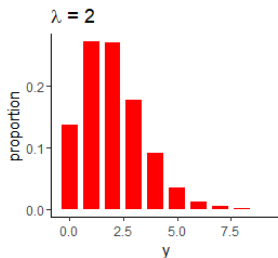
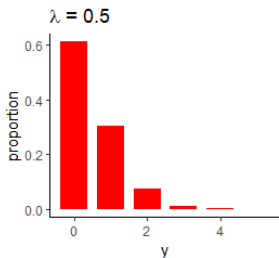
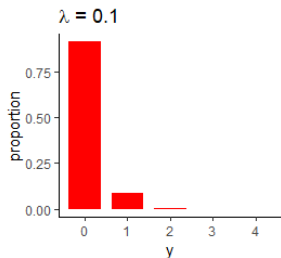
$$f_{\lambda}(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

If we observe a sample  $x_1, \dots, x_n$ , then the likelihood is just the product of the individual PMF

$$\begin{aligned}\pi(x_1, \dots, x_n \mid \lambda) &= \frac{\lambda^{\sum_{i=1}^n x_i} e^{-n\lambda}}{\prod_{i=1}^n (x_i!)} \\ &= \frac{\lambda^{n\bar{x}} e^{-n\lambda}}{\prod_{i=1}^n (x_i!)}\end{aligned}$$

where the model parameter  $\lambda$  is the count of some event of interest and  $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$ .

# Some examples of Poisson samples



# Gamma prior

A convenient choice to model the uncertainty about  $\lambda$  is a Gamma distribution as prior, since the support of such a distribution is the interval  $[0, \infty[$ . A Gamma prior with shape parameter  $\alpha$  and scale parameter  $\beta$  has the following form

$$\pi(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda\beta}$$

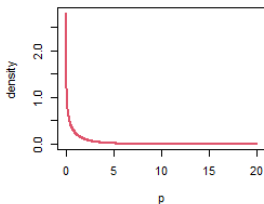
where  $\Gamma(\alpha)$  is the Gamma function, defined as  $\int_0^\infty x^{\alpha-1} e^{-x} dx = (\alpha - 1)!$ .

Let  $X \sim \text{Gamma}(\alpha, \beta)$ . Then  $E[X] = \alpha/\beta$  and  $\text{var}(X) = \alpha/\beta^2$ .

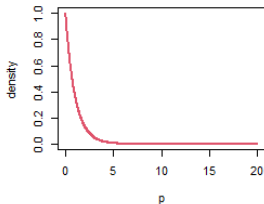
It is a convenient and flexible choice since the Gamma distribution can take a wide variety of shapes.

# Some examples of the Gamma family

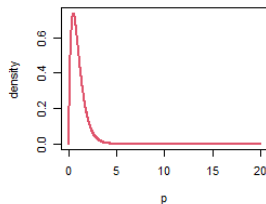
Ga(1/2, 1/2)



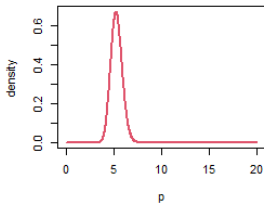
Ga(1, 1)



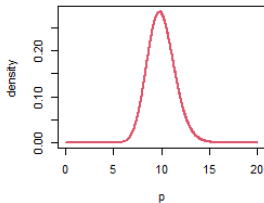
Ga(2, 2)



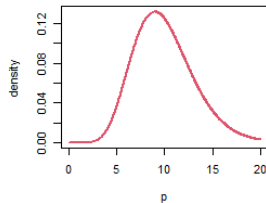
Ga(80, 15)



Ga(50, 5)



Ga(10, 1)



# Gamma posterior (1/2)

The posterior distribution for  $\lambda$  if our data  $x_1, \dots, x_n$  is modeled with a Poisson likelihood and a Gamma prior is chosen for  $\lambda$  will also have the functional form of a Gamma r.v. Using the Bayes theorem, we have that

$$\begin{aligned}\pi(\lambda \mid x_1, \dots, x_n) &= \frac{\pi(x_1, \dots, x_n \mid \lambda)\pi(\lambda)}{\pi(x_1, \dots, x_n)} \\&= \frac{\lambda^{n\bar{x}} e^{-n\lambda}}{\prod_{i=1}^n (x_i!)} \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda\beta} \\&= \underbrace{\frac{1}{\prod_{i=1}^n (x_i!)} \frac{\beta^\alpha}{\Gamma(\alpha)}}_{\text{do NOT depend on } \lambda} \lambda^{n\bar{x}} e^{-n\lambda} \lambda^{\alpha-1} e^{-\lambda\beta} \\&\propto \lambda^{\alpha+n\bar{x}-1} e^{-(n+\beta)\lambda}\end{aligned}$$

## Gamma posterior (2/2)

So we find that  $\pi(\lambda \mid x_1, \dots, x_n) \propto \text{Gamma}(\alpha + n\bar{x}, n + \beta)$ . The posterior mean and variance are given by

$$E[\lambda] = \frac{\alpha + n\bar{x}}{n + \beta} \quad \text{var}(\lambda) = \frac{\alpha + n\bar{x}}{(n + \beta)^2}$$

where  $n\bar{x} = \sum_{i=1}^n x_i$  is the sum of the counts and  $n$  is the sample size. Let's take a few examples and plot the likelihood, a possible prior and the posterior, all at once in R.

# General remark on Bayesian inference

Unlike in the traditional Frequentist framework, the Bayesian approach views parameters as random variables rather than fixed, unknown quantities. Given a Poisson sample  $x_1, \dots, x_n$  and a Poisson parameter  $\lambda$ , from the Bayes theorem, we can write

$$\pi(\lambda \mid x_1, \dots, x_n) = \frac{\pi(x_1, \dots, x_n \mid \lambda) \pi(\lambda)}{\pi(x_1, \dots, x_n)}$$

Adopting the 'proportional' notation, the constant term in the denominator is dropped so that the above expression is rewritten as  $\pi(\lambda \mid x_1, \dots, x_n) \propto \pi(x_1, \dots, x_n \mid \lambda) \pi(\lambda)$

When conjugate models are used (as in the case of a Poisson-Gamma model), the posterior distribution can be identified and closed-form quantities of interest like a mean, a variance or quantiles can be computed. Most of the time in practice, the posterior distribution is intractable so that it is necessary to resort to MCMC techniques.

## Working example

Suppose that we record the number of a specific bacteria present in 20 water samples taken in the Mekong Delta (Vietnam) so that we have the following data at hand:

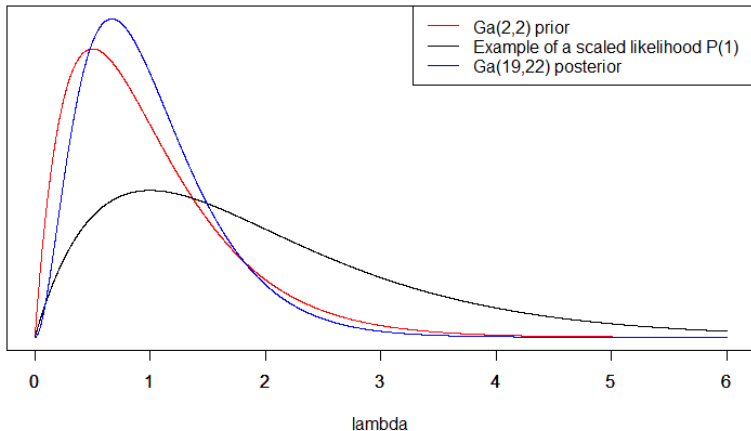
$$x_i = 1, 0, 0, 1, 0, 0, 1, 1, 0, 1, 2, 0, 0, 5, 2, 0, 0, 2, 0, 1$$

So assuming a Poisson likelihood with parameter  $\lambda = 1$ , namely a  $\mathcal{P}(1)$  likelihood for the data and using a  $\text{Gamma}(2, 2)$  prior, with mean  $2/2 = 1$  and variance  $2/2^2 = 0.5$ , what is the posterior mean and the 95% credible interval for the model parameter?



# Working example: posterior

Posterior distribution in blue -  $\text{Ga}(19, 22)$



# Posterior quantities obtained from direct sampling

```
1 # Posterior mean, posterior variance and 95% Credible Interval including the
  sample median
2 set.seed(2023)
3 data1 = rpois(n = n, lambda = lambda1)
4 alpha_posterior = round(alpha1 + n*mean(data1), 2) # 19
5 beta_posterior = n + beta1 # 22
6
7 pmean = alpha_posterior / beta_posterior
8 pmean
9 # [1] 0.8636364
10
11 pvariance = alpha_posterior / beta_posterior^2
12 pvariance
13 # [1] 0.0392562
14
15 # 95% Credible Interval obtained by direct sampling (simulation)
16 set.seed(2023)
17 round(quantile(rgamma(n = 10^8, alpha_posterior, beta_posterior), probs = c
  (0.025, 0.5, 0.975)),4)
18 #      2.5%      50%     97.5%
19 # 0.5200 0.8486 1.2931
20
21 # Posterior mean obtained from direct sampling
22 set.seed(2023)
23 mean(rgamma(n = 10^8, alpha_posterior, beta_posterior))
24 # [1] 0.8928863
```

## Working example: in conclusion

So the theoretical posterior mean is given by

$$E[\lambda] = \frac{\alpha + n\bar{x}}{n + \beta} = \frac{2 + 20 * 0.85}{20 + 2} = 19/22 = 0.8636364$$

By direct sampling, using  $10^8$  number of simulations, the posterior sample mean is 0.8636725

By direct sampling, a 95% Credible Interval is given by

$$[-0.5200, 1.2931]$$

So, combining modeling and simulations, we are now able to generalize and infer to the whole population of bacteria in the Mekong Delta those values from a sample of size 20.

## Further reading and code

The R Project for Statistical Computing:

<https://www.r-project.org/>

Accessing the R code:

<https://github.com/JRigh/Poisson-Gamma-example-in-R/blob/main/Poisson-Gamma>