Option pricing through statistical simulations

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Goals of the study

(*i*) Estimation of European and Asian call options prices using:

- Monte Carlo approximations
- Bootstrap approximations

(*ii*) Assumptions are either:

- Homoskedastic errors (constant volatility over time)
- GARCH(1,1) errors (non-constant volatility)

(*iii*) Compare and discuss estimates.

Notes: *A call option is a right, but not an obligation to buy an underlying asset - i.e. a stock or a bond - at a given time in the future (at maturity) and a a prespecified price (strike price). A European call allows the holder to buy at a fixed price at maturity whereas an Asian call allows the holder to buy at a price based on the average price over time, anytime before maturity.*

Stochastic models

Assume that the **generating process** of S&P 500 returns data is:

$$\ln\left(\frac{P_t}{P_{t-1}}\right) = \beta_0 + \mu_t, \qquad \mu_t = \sigma_t \epsilon_t, \qquad \epsilon_t \sim N(0,1)$$

We consider the two following distinct cases:

- Homoskedastic: $\sigma_t^2 = \alpha_0$
- GARCH(1,1): $\sigma_t^2 = \alpha_0 + \alpha_1 \mu_{t-1}^2 + \alpha_2 \sigma_{t-1}^2$

Notes: *The daily returns are the log of the ratio of stock price at time t, P_t divided by the stock price at time t-1, P_{t-1} and are equal to some constant term β_0 plus one time varying component μ_t, with the latter being equal to the standard deviation at time t multiplied by a gaussian noise.*

Option characteristics

- Data: daily S&P500 prices (1.12.2006–22.11.2019)
- Initial price: $P_0 = \$3110.29$
- Strike price: $K = \$3100$
- Maturity: $T = 20$ trading days

Notes: *The strike price is the price against which we will compare our simulated option price. It's the price at which we can buy the stock at maturity. And time to maturity is the lifetime of our option. In other words, a call option is a contract between two parties to possibly exchange a stock at a strike price by a predetermined date. Basically, we will simulate the process up to 20 days ahead and compare the stock price to the strike price to determine whether we have a gain or a not. If not, the option price will be* 0.

Cases assuming homoskedastic errors

The base formula for **Monte carlo** approximations is:

$$P_t = P_{t-1} e^{U + SE\sigma}$$

where U is the drift (or mean) and SE the standard deviation, after log differenciation of the series. σ is a random normal stock, i.e. a Gaussian noise.

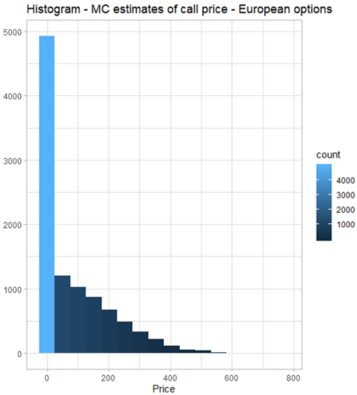
For **Bootstrap** approximations, we sample with replacement the log of stock returns using *sample()* in R.

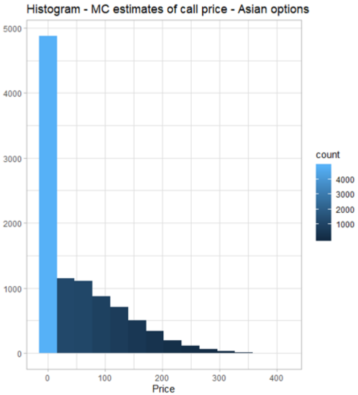
MC approximations with homoskedastic errors

```r
K = 3100 # strike price
n = 20 # time to maturity
R = 10000 # number of MC replicates
return.mean = mean(SP500returns)  # mean
Sd =  sd(SP500returns) # sd
P_T_1 = c(3110.29, rep(0,n-1))
P_T  = rep(0,20)
price_European_HE = rep(0,R)
price_Asian_HE = rep(0,R)

set.seed(2019)
for(i in 1:R){
  for(j in 1:n) {

    # compute stock price at time t and do it for t = 1,...,20
    P_T[j] = P_T_1[j]*exp(return.mean + (Sd * rnorm(1, 0, 1))) # simulated stock prices
    P_T_1[j+1] <- P_T[j]
  }
  price_European_HE[i] = max(P_T[j] - K, 0)
  price_Asian_HE[i]    = max(mean(P_T) - K, 0)

  # compute call price
  call_European_HE_MC = mean(price_European_HE)
  call_Asian_HE_MC    = mean(price_Asian_HE)
}
```

Notes: *Excerpt of the program that we implemented for this specific case.*

Histograms of simulated call prices





Note: *Typical for simulated option prices. About half of the time, we get a price of* 0 *(simulated stock prices are lower than the strike price) so we don't use the option. In other cases (simulated stock price is above the stike price) we use the option, buy the stock and have positive gains.*

Cases assuming GARCH(1,1) errors

The base formula for **Monte carlo** approximations is:

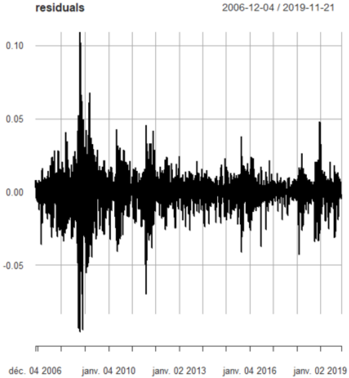
$$P_t = P_{t-1} e^{U + SE^* \sigma}$$

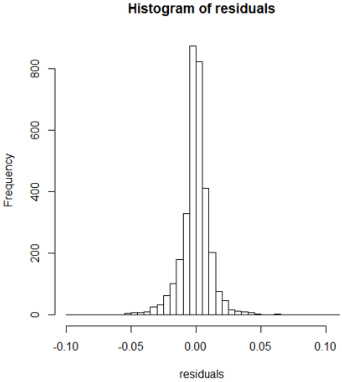
where U is the drift (or mean) and $SE^* = \sqrt{\alpha_0 + \alpha_1 \mu_{t-1}^2 + \alpha_2 \sigma_2^2}$ and σ is a random Gaussian stock.

For **Bootstrap** approximations, again we use the function *sample()* in R.

Note: *For bootsrap approximations, we replace the exponentiated expression by ressampled daily returns (without replacement). Using bootstrap allows us not to assume a particular distribution for daily returns, whereas for Monte Carlo, we assume normality.*

Residuals analysis





Note: *Using GARCH(1,1), we now have a model for which we can compute residuals. The series seem to exhibit heteroskedasticity since the variance of the residuals seem not to be constant over time. In addition, the residuals exhibit excess kurtosis.*

Bootstrap approximations with GARCH(1,1) errors

```
set.seed(2019)
for(i in 1:R){
  for(j in 1:n) {

    # compute stock price at time t and do it for t = 1, ..., 20
    Se[j] =  sqrt(alpha0 + (alpha1 * (resid_1[j] )^2) +
    (alpha2 * (sigmas[length(sigmas)-20+j])^2  ))
    resid_1[j] = rnorm(1,0,Se[j])
    nus=(resid_1/sigmas)
    P_T[j] = P_T_1[j] * exp(beta0+(Se[j] * sample(nus, size=1))) # sim. stock prices
    P_T_1[j+1] <- P_T[j]
  }
  price_European_BT_GA[i] = max(P_T[j] - K, 0)
  price_Asian_BT_GA[i]    = max(mean(P_T) - K, 0)

  # compute call price
  call_European_BT_GA = mean(price_European_BT_GA)
  call_Asian_BT_GA    = mean(price_Asian_BT_GA)
}
```

Note: *In the inner loop, SE^* is obtained using the GARCH formula. The residuals are then updated using the previous SE^*. In the outer loop, call option prices are computed then averaged.*

Results

		Homoskedastic errors	GARCH(1,1) errors
Monte Carlo	*European*	85.04	73.99
	Asian	52.48	42.55
Bootstrap	*European*	84.00	84.84
	Asian	51.40	47.36

Note: *For both methods, in the case where we assume homoskedastic errors, the call prices are quite comparable. When GARCH errors are assumed, MC approximations of prices are slightly lower than the approximations obtained by bootstrap.*

Discussion of the results

(*i*) The two approaches yield slightly different, however comparable results.

(*ii*) European option prices are higher than Asian option prices because for the later, we computed an average over the lifetime of the option

(*iii*) Assuming homoskedasticity, option prices approximations are higher. However, earlier returns may not be relevant to option price approximation and it can be argued that they should be discarded.

(*iv*) If errors are non normal, bootstrap approximations should be preferred for pricing.

References

Martin PAŽICKÝ, 2017. Stock Price Simulation Using Bootstrap And Monte Carlo.

Christian P. ROBERT, George CASELLA, 2005. Monte Carlo Statistical Methods.

Maria L. RIZZO, 2008. Statistical Computing With R.