CHAOS IN RANDOM NEURAL NETWORKS

Introduction to Computational Neuroscience.

Quantitative Life Sciences

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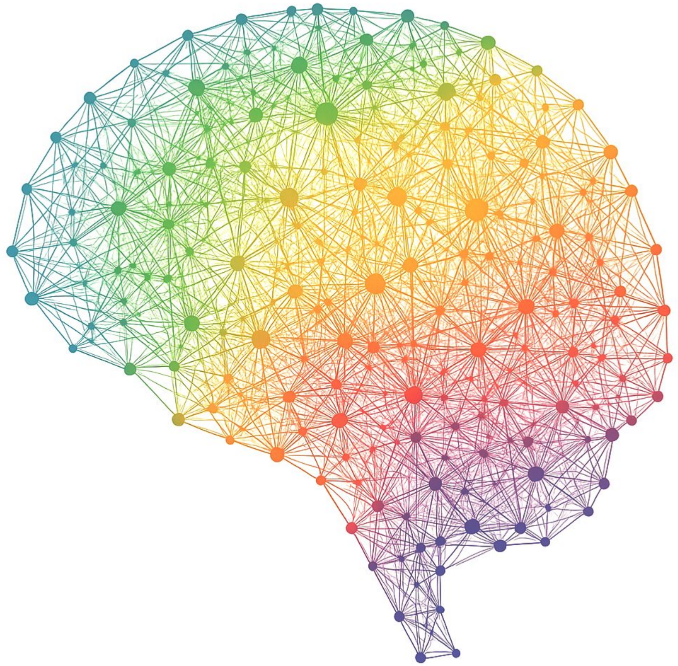
Outline

“I accept the principle of cerebral localization, though its limits are uncertain. The mind has distinct faculties and the brain distinct convolutions. But to know whether each particular faculty has its seat in a particular convolution, is a question which seems completely insoluble at the present state of science.

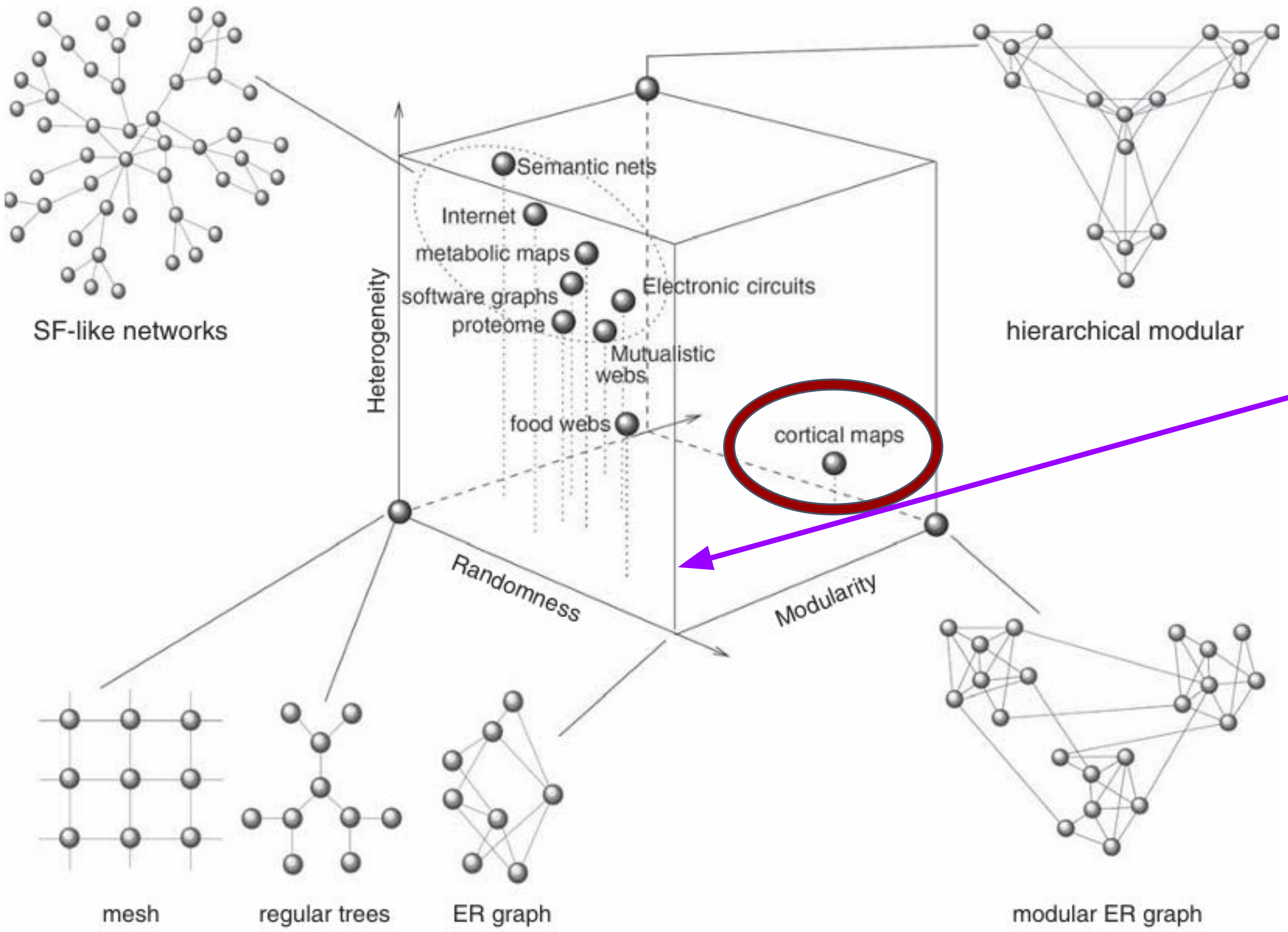
— Paul Broca, 1861.

1. **Introduction**
2. **Model**
3. **Dynamic Mean Field Theory**
4. **Results**
5. **Simulations**
6. **Conclusions**



Sompolinsky, H., Crisanti, A., & Sommers, H. J. (1988). Chaos in random neural networks. *Physical review letters*, *61*(3), 259.

- Networks: Way to capture what each neuron do.



Model of Sompolinsky (1988)

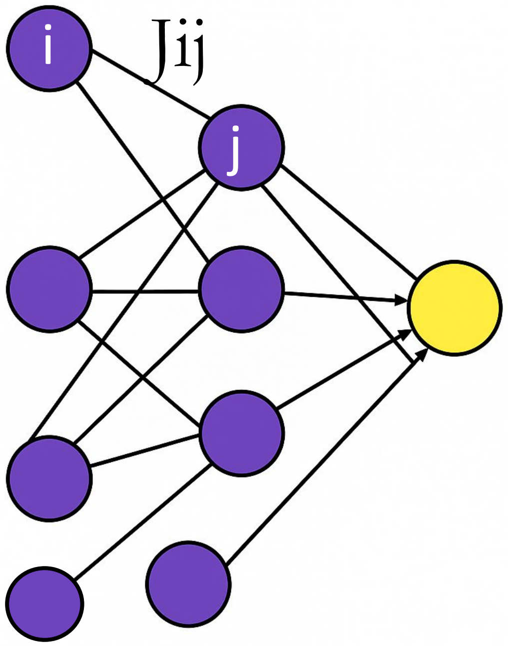
Sporns, Olaf. *Networks of the Brain*. MIT press, 2016.

Model

- Our model is composed of N **localized continuous variables** $\{S_i(t)\}$ **, where** $i = 1, \ldots, N,$ and $-1 \leq S_i(t) \leq 1$, each neuron has a local field $h_i(t)$,

Observable output ← $S_i(t) = \phi(h_i(t))$ → Total input received of neuron i, membrane potential.

in this paper, $\phi(x) = \tanh(gx)$, g show us the degree of nonlinearity.



🧠 Dynamics, N-coupled 1st ODE.

$$\frac{dh_i}{dt} = -h_i + \sum_{j=1}^{N} J_{ij}\phi(h_j)$$

- Time and states are continuous variables.
- Here J_{ij} is a **synaptic efficacy** from neuron j (presynaptic) to neuron i (postsynaptic).
- $J_{ij} \sim \mathcal{N}(0, J^2/N)$, i.i.d.

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- In all the paper we will understand the **properties** of this model, in specific:

$$\frac{dh_i}{dt} = -h_i + \sum_{j=1}^{N} J_{ij}\phi(h_j)$$

1) Solve this equation.
   a) x=0. Always a solution.
2) Stability. Linearize the dynamics

$$M_{ij} = -\delta_{ij} + J_{ij}$$

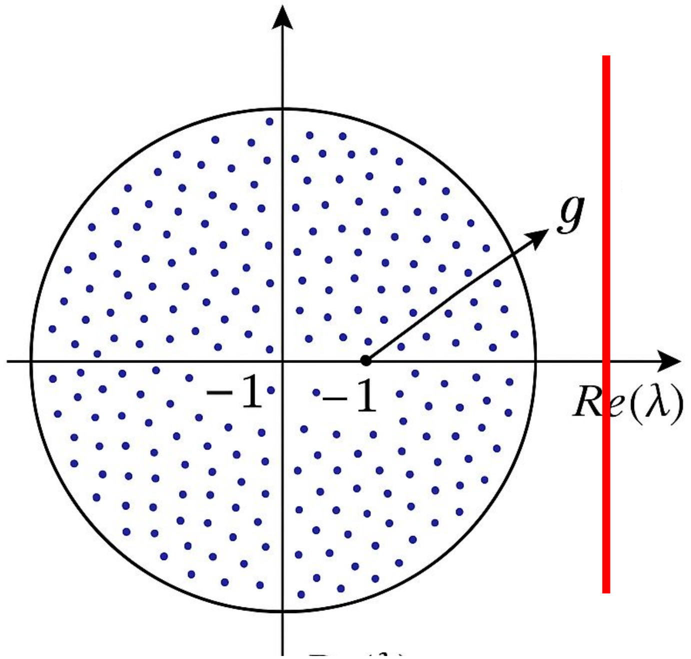
3) Since the connections are random, we don't have a unique path, so in general, J is a Random matrix.

- Mean: $\langle J_{ij} \rangle = 0$
- Variance: $\langle J_{ij}^2 \rangle = \frac{J^2}{N}$

Girko's Circular Law

Eigenvalues of J are uniform distributed on a circle of radius, g:

$$R = \sqrt{N} \cdot \sigma_J$$



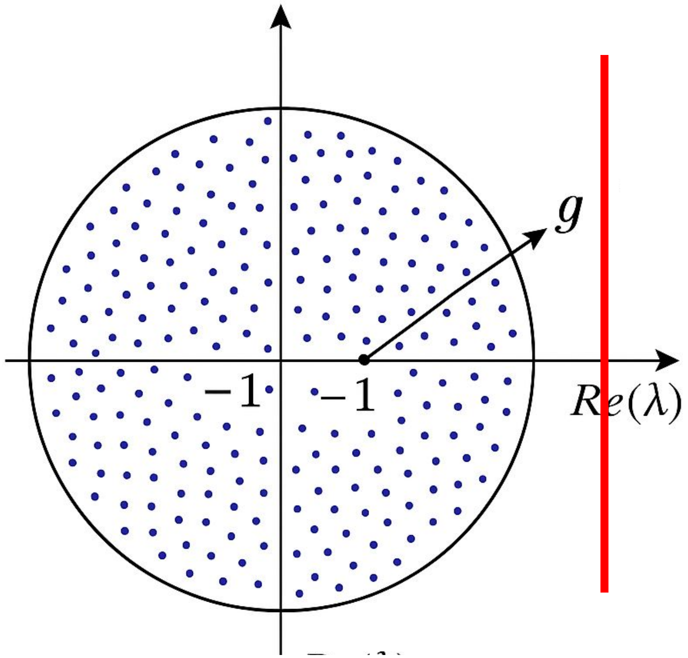
Girko, V. L. (1985). Circular law. *Theory of Probability & Its Applications*, *29*(4), 694-706.

Model: First Analysis

$$N \to \infty$$

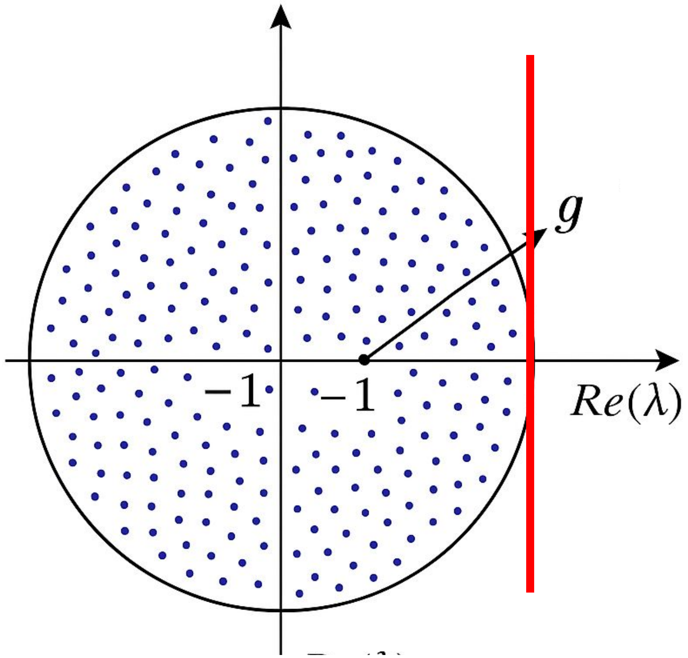
Fix Point, g<1

$$R = \sqrt{N} \cdot \sigma_J$$



Chaotic Behavior, g>1

$$R = \sqrt{N} \cdot \sigma_J$$



As soon as our circle touches gJ > 1, the transition to chaos occurs immediately

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Dynamic Mean-Field Theory

- The long-time properties of the solutions, exact theory when $N \to \infty$
- **Part 1:** Self-consistent equation of a single neuron (**REDUCE COMPLEXITY**)

$$\dot{h}_i(t) = -h_i(t) + \eta_i(t)$$

- $\eta_i(t)$: Gaussian random input, cumulative effect of all the neurons in the network because of CLT.
- X(t), weakly correlated random variables

(1)
$$X_j(t) = J_{ij}\phi(h_j(t)) \quad \Rightarrow \quad \eta_i(t) = \sum_{j=1}^{N} X_j(t)$$

(2)
$$\langle X_j(t) \rangle = 0 \quad \text{(mean zero)}$$

$$\mathrm{Var}(X_j(t)) = \langle X_j(t)^2 \rangle = \frac{J^2}{N}\langle \phi(h_j(t))^2 \rangle \approx \frac{J^2}{N}$$

(3)
$$\mathrm{Var}[\eta_i(t)] = \mathrm{Var}\left[\sum_{j=1}^{N} X_j(t)\right] :$$

$$= \sum_{j=1}^{N} \mathrm{Var}[X_j(t)] + \sum_{j \neq k} \mathrm{Cov}[X_j(t), X_k(t)]$$

(4)
$$\mathrm{Var}[\eta_i(t)] \approx N \cdot \frac{J^2}{N} = J^2$$

- Then, finite mean and finite variance converge to a Gaussian RV.
- The collective effect of many random inputs converges to a **Gaussian RV.**

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Deductions

- **Part 2:** Moments, effective input

$$\eta_i(t) = \sum_{j=1}^{N} J_{ij}\,\phi(h_j(t))$$

- First moment

$$\langle \eta_i(t) \rangle = 0$$

- Second moment: Autocorrelation

$$\langle \eta_i(t)\eta_i(t+\tau)\rangle = \sum_{j=1}^{N} \frac{J^2}{N} \langle \phi(h_j(t))\phi(h_j(t+\tau))\rangle$$

$$\boxed{\langle \eta_i(t)\eta_i(t+\tau)\rangle = J^2 C(\tau)}$$

- Where the **autocorrelation of Neural Activity**

$$C(\tau) = \left[\frac{1}{N}\sum_i S_i(t)S_i(t+\tau)\right]_J = \langle \phi(h_i(t))\phi(h_i(t+\tau))\rangle$$

- **Part 3:** Solve it,

$$\dot{h}_i(t) = -h_i(t) + \eta_i(t)$$

- Integrating factor,

$$e^t \dot{h}_i(t) + e^t h_i(t) = e^t \eta_i(t)$$

$$h_i(t) = e^{-t} h_i(0) + \int_0^t dt'\, e^{-(t-t')}\eta_i(t')$$

- Then, at long time behavior,

$$\boxed{h_i(t) = \int_{-\infty}^{t} dt'\, e^{-(t-t')}\eta_i(t')}$$

- The neuron "remembers" past inputs $\eta_i(t')$,
- This solution is a convolution of $\eta_i(t')$, with a exponential decay.

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Self consistent: the **neural outputs** determine the **temporal structure of the noise**.

$$\underbrace{\eta_i(t)}_{\text{Gaussian noise}} \longrightarrow \underbrace{h_i(t)}_{\text{input current}} \longrightarrow \underbrace{S_i(t) = \phi(h_i(t))}_{\text{neural output}} \longrightarrow \underbrace{C(\tau)}_{\text{autocorrelation}} \longrightarrow \underbrace{\langle \eta_i(t)\eta_i(t+\tau)\rangle}_{\text{noise correlation}}$$

Then, instead of solve $C(\tau)$, it is better to focus on local field autocorrelations.

$$\boxed{\Delta(\tau) = \langle h_i(t)h_i(t+\tau)\rangle}$$

Then, kind of DMFT master equation,

$$\langle \eta_i(t)\eta_i(t+\tau)\rangle = \langle [\dot{h}_i(t) + h_i(t)][\dot{h}_i(t+\tau) + h_i(t+\tau)]\rangle$$

$$\boxed{\langle \eta_i(t)\eta_i(t+\tau)\rangle = \Delta(\tau) - \ddot{\Delta}(\tau) = J^2 C(\tau)}$$

Sompolinsky, H., Crisanti, A., & Sommers, H. J. (1988). Chaos in random neural networks. *Physical review letters*, *61*(3), 259.

Dynamic Mean-Field Theory

The idea of find an expression with the second derivative is useful because:

- Second ODE for a particle in a potential

$$\boxed{\ddot{\Delta}(\tau) = -\frac{\partial V}{\partial \Delta}} \quad (5)$$

- Solving from Master DMFT,

$$\ddot{\Delta}(\tau) = \Delta(\tau) - J^2 C(\tau)$$

$$\frac{\partial V(\Delta)}{\partial \Delta} = J^2 C(\tau) - \Delta(\tau)$$

- Integrating,

$$V(\Delta) = -\frac{1}{2}\Delta^2 + J^2 \int^{\Delta} C(\tau)\, d\tau$$

- Rewrite $C(\tau)$,
- Understand that $(h_i(t), h_i(t+\tau))$ is a bivariate gaussian with variance $\Delta(0)$ and covariance $\Delta(\tau)$
- Rewriting,

$$h_i(t) = \sqrt{\Delta(0) - |\Delta(\tau)|}\, x$$

$$h_i(t+\tau) = \sqrt{\Delta(0) - |\Delta(\tau)|}\, x + \sqrt{|\Delta(\tau)|}\, z$$

- We find the effective potential (6)

$$\boxed{V(\Delta) = -\frac{1}{2}\Delta^2 + \int \mathcal{D}z \left(\int \mathcal{D}x\, \Phi\left(\sqrt{\Delta(0) - |\Delta|}\, x + \sqrt{|\Delta|}\, z\right)\right)^2}$$

$$\mathcal{D}x = \frac{dx}{\sqrt{2\pi}} e^{-x^2/2}$$

- **We reduce the network to a potential!**

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Brief Note: Generating Functional Formalism

- Just to mention that there is another way to deduce:

$$\boxed{\ddot{\Delta}(\tau) = -\frac{\partial V}{\partial \Delta}}$$

$$\boxed{V(\Delta) = -\frac{1}{2}\Delta^2 + \int \mathcal{D}z \left(\int \mathcal{D}x \, \Phi \left(\sqrt{\Delta(0) - |\Delta|}\, x + \sqrt{|\Delta|}\, z \right) \right)^2}$$

- Given Langevin equation, H introduces quenched disorder with J

$$\dot{x}(t) = -\frac{\partial H}{\partial x} + \eta(t)$$

- Introduce generating functional, with action

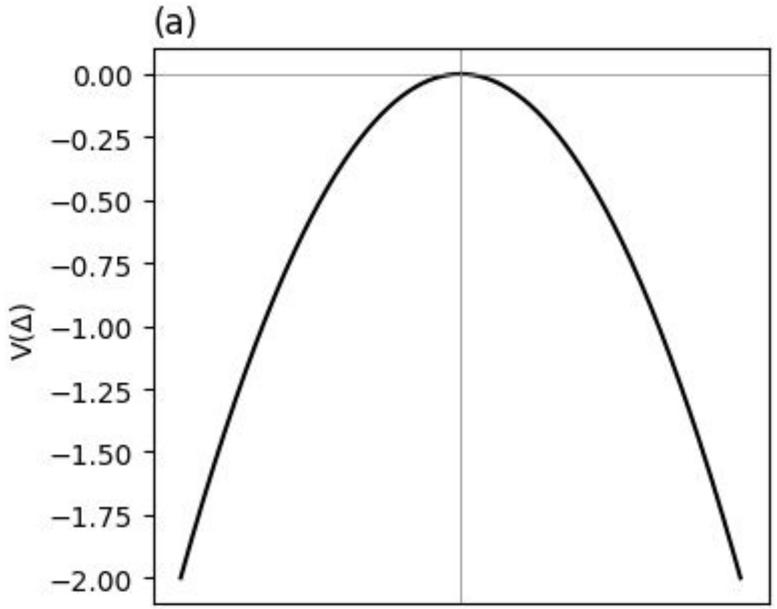
$$Z = \int \mathcal{D}x \, \mathcal{D}\hat{x} \, e^{S[x,\hat{x}]} \qquad S = -\frac{1}{2}\hat{x}D\hat{x} + i\hat{x}(\dot{x} + \partial_x H)$$

- By S, we d**erive the saddle-point equation for** $\Delta(\tau)$

Schuecker, J., Goedeke, S., Dahmen, D., & Helias, M. (2016). Functional methods for disordered neural networks. *arXiv preprint arXiv:1605.06758*.

Potential Solutions

Case 1: g . J<1→ **Zero Fixed Point**

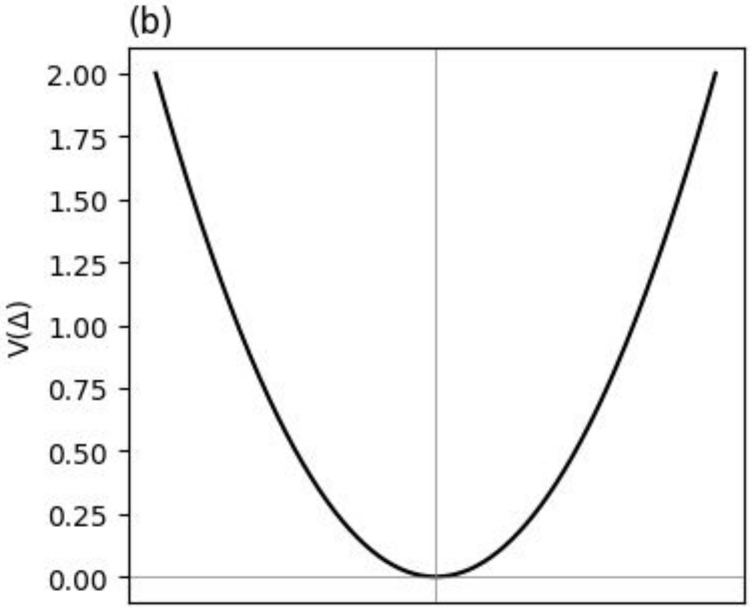


Given boundary conditions,

$$\Delta(\tau) = 0 \quad \Rightarrow \quad h_i(t) = 0$$

Fix point stable, no memory, no activity.

Case 2: g . J>1→ **Oscillatory**

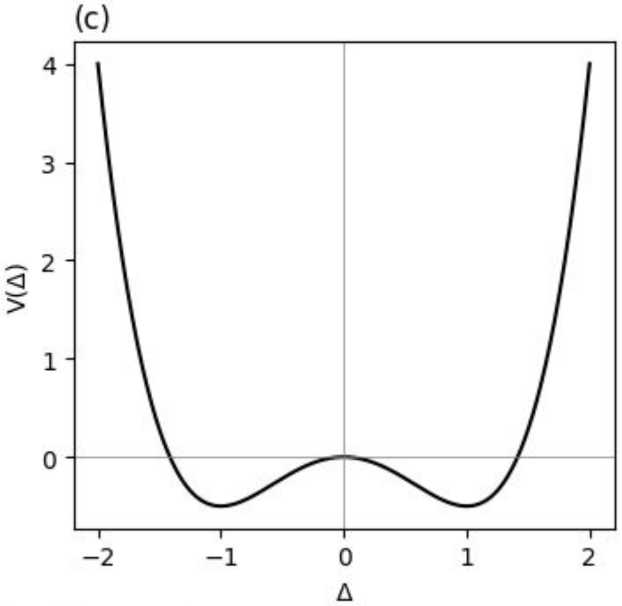


Thanks to conservative potential, oscillates if $\Delta(0) < \Delta_1$,

where $\Delta(0) = \langle h_i(t)^2 \rangle$, are the total variance

- Δ_1 is the minimal escape energy.

Case 3: g . J>1→ **Chaotic.**



if $\Delta(0) > \Delta_1$ **,** you explore multiple minima.

- Non trivial solution and dependence to ICS.

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Proof of Chaos: Lyapunov Exponent

To do it, we have to focus on quadratic mean of all the perturbations,

$$\chi^2(t) \equiv \lim_{\tau\to\infty} \frac{1}{N}\left[\sum_{ij} \chi_{ij}(\tau+t,\tau)\right]$$

where,

$$\chi_{ij}(t,t') = \frac{\delta x_i(t)}{\delta h_j(t')}$$

measures how much changes $x_i(t)$ given perturbation $h_j(t')$

- **Lyapunov exponent:**

$$\lambda = \lim_{t\to\infty} \frac{\ln \chi^2(t)}{2t}$$

- **where if λ >0, chaotic system.**
- **if λ < 0, system is stable.**

- The fluctuations can be rewritten as

$$\chi^2(t) = \sum_{n=0}^{\infty} \chi_n e^{2\omega_n t}$$

- where from DMFT,

$$\omega_n = -1 + \sqrt{1 - E_n}$$

- Curiously, E_n are the eigenvalues of 1D-Schrodinger equation,

$$\left[-\frac{d^2}{dt^2} + W(t)\right]\psi_n(t) = E_n\psi_n(t)$$

and W(t) are the fluctuation potential

$$W(t) = -\frac{\partial^2 V(\Delta)}{\partial \Delta^2}$$

Finally, in essence Lyapunov exponent is given,

$$\lambda = \omega_0 = -1 + \sqrt{1 - E_0}$$

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Conclusions

- We saw the power of DMFT by reducing the behavior in a potential.
- We found chaotic behavior iff $gJ > 1$
- We found a smart way to deduce Lyapunov Exponent
- We found a way to give sense of highly complex network in one analytical solution

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THANKS