Lab4Q2-report

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11/25/2021

Assignment2

We are given the following:

$$n = number\ of\ observations$$

$$\vec{\mu} = (\mu_1, ..., \mu_n)\ are\ unknown\ parameters$$

$$Y_i \sim N(\mu_i, variance = 0.2),\ i = 1, ..., n$$

$$Prior:\ p(\mu_1) = 1\ ;\ p(\mu_{i+1}|\mu_i) = N(\mu_i, 0.2),\ i = 1, ..., (n-1)$$

We'll be interested in deriving the posterior i.e. P(μ|Y) using the Bayes' theorem as follows:

$$posterior = likelihood\ *\ prior$$
$$P(\vec{\mu}|Y) = \frac{P(Y|\vec{\mu}) * P(\vec{\mu})}{\int_\mu P(Y|\vec{\mu})P(\vec{\mu})d\mu}$$

Since the denominator is constant w.r.t μ, we can drop it in favour of proportionality

$$posterior \propto likelihood\ *\ prior$$
$$P(\vec{\mu}|Y) \propto P(Y|\vec{\mu}) * P(\vec{\mu})$$

First we define the likelihood:

$$L(Y_i|\mu_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} exp\Big(-\frac{1}{2\sigma^2}(Y_i - \mu_i)^2\Big)$$

Since, we are only interested in the terms dependent on the parameter μ we can drop the term at the start of the expression and the likelihood then becomes:

$$L(Y_i|\mu_i) \propto exp\Big(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - \mu_i)^2\Big)$$

Similarly, now for the prior using chain rule,

$$P(\vec{\mu}) = P(\mu_1).P(\mu_2|\mu_1).P(\mu_3|\mu_2)...P(\mu_n|\mu_{n-1}) \qquad ...(a)$$

$$= \prod_{i=1}^{n-1} P(\mu_{i+1}|\mu_i) \;\sim\; N(\mu_i, \sigma^2_{\mu_i})$$

$$= \prod_{i=1}^{n-1} \frac{1}{\sqrt{2\pi\sigma^2}} exp\Big(- \frac{1}{2\sigma^2_\mu} (\mu_{i+1} - \mu_i)^2 \Big)$$

$$\propto exp\Big(- \frac{1}{2\sigma^2_\mu} \sum_{i=1}^{n-1} (\mu_{i+1} - \mu_i)^2 \Big)$$

Hence, we can now write our posterior as the following:

$$P(\vec{\mu}|Y) \propto exp\Big(- \frac{1}{2\sigma^2} \sum_{i=1}^{n} (Y_i - \mu_i)^2 \Big) \; * \; exp\Big(- \frac{1}{2\sigma^2_\mu} \sum_{i=1}^{n-1} (\mu_{i+1} - \mu_i)^2 \Big)$$

$$\propto exp\Big[- \frac{1}{2\sigma^2} \sum_{i=1}^{n} (Y_i - \mu_i)^2 - \frac{1}{2\sigma^2_\mu} \sum_{i=1}^{n-1} (\mu_{i+1} - \mu_i)^2 \Big] \qquad ...(b)$$

Since, σ is given to be same everywhere in the problem we can use common notation across the expression.

Moving to the next part to find conditional probability $(\mu_k|\mu_{-k})$, we know that by the definition of conditional probability, the joint probability of the prior, P($\vec{\mu}$), can be expanded into the conditional components of each individual μ_k and can be expressed as

$$P(\vec{\mu}) \propto P(\mu_k|\vec{\mu}) * P(\mu_1, ..., \mu_{k-1}, \mu_{k+1}, ..., \mu_n)$$

Or in a more condensed form as,

$$P(\mu_i|\vec{\mu}_{-i}, \vec{Y}) = \frac{P(\vec{\mu}|\vec{Y})}{P(\vec{\mu}_{-i}|\vec{Y})} \qquad ...(c)$$

$$= \frac{P(\vec{\mu}|\vec{Y})}{\int P(\vec{\mu}|\vec{Y}) d\mu}$$

We now solve for μ_1

$$P(\mu_1|\vec{\mu}_{-1}, \vec{Y}) = \frac{P(\vec{\mu}, \vec{Y})}{\int P(\vec{\mu}, \vec{Y}) d\mu_1}$$

$$= \frac{\prod_{i=2}^{n} P(Y_i|\mu_i) \prod_{i=3}^{n} P(\mu_i|\mu_{i-1}) P(\mu_2|\mu_1) P(Y_1|\mu_1) P(\mu_1)}{\prod_{i=2}^{n} P(Y_i|\mu_i) \prod_{i=3}^{n} P(\mu_i|\mu_{i-1}) \int P(\mu_2|\mu_1) P(Y_1|\mu_1) P(\mu_1) d\mu_1}$$

Common product terms cancel out in numerator and denominator, and the integral is constant w.r.t. μ_1

$$P(\mu_1|\vec{\mu}_{-1}, \vec{Y}) \propto P(\mu_2|\mu_1) P(Y_1|\mu_1) P(\mu_1)$$

$$\propto exp\Big(- \frac{1}{2\sigma^2} (\mu_2 - \mu_1)^2 \Big) \; * \; exp\Big(- \frac{1}{2\sigma^2} (Y_1 - \mu_1)^2 \Big) \; * \; 1$$

$$\propto exp\Big(- \Big(\frac{1}{2\sigma^2}\Big) \Big((\mu_2 - \mu_1)^2 + (Y_1 - \mu_1)^2 \Big) \Big)$$

using *Hint B*

$$P(\mu_1|\vec{\mu}_{-1}, \vec{Y}) \propto exp\Big(- \frac{(\mu_1 - (Y_1 + \mu_2)/2)^2}{\sigma^2} \Big) \;\sim\; N\Big((\frac{Y_1 + \mu_2}{2}), \frac{\sigma^2}{2} \Big)$$

Similarly, we now check the case for μ_n. Here we use the posterior we defined in (b) and the relationship we established in (c)

$$P(\mu_n|\vec{\mu}_{-n},\vec{Y}) = \frac{P(\vec{\mu}|\vec{Y})}{P(\vec{\mu}_{-n}|\vec{Y})}$$
$$\propto \frac{exp\Big(-\sum_{i=1}^{n-1}\frac{(\mu_{i+1}-\mu_i)^2}{2\sigma^2}-\sum_{i=1}^{n}\frac{(Y_i-\mu_i)^2}{2\sigma^2}\Big)}{exp\Big(-\sum_{i=1}^{n-2}\frac{(\mu_{i+1}-\mu_i)^2}{2\sigma^2}-\sum_{i=1}^{n-1}\frac{(Y_i-\mu_i)^2}{2\sigma^2}\Big)}$$
$$\propto exp\Big(-\Big(\frac{1}{2\sigma^2}\Big)\Big((\mu_n-\mu_{n-1})^2+(Y_n-\mu_n)^2\Big)\Big)$$

again using *hint B*

$$P(\mu_n|\vec{\mu}_{-n},\vec{Y}) \propto exp\Big(-\frac{(\mu_n-(Y_n+\mu_{n-1})/2)^2}{\sigma^2}\Big) \;\sim\; N\Big((\frac{Y_n+\mu_{n-1}}{2}),\frac{\sigma^2}{2}\Big)$$

Now for the tedious case of μ_i where is i $\notin$ (1,n)

$$P(\mu_i|\vec{\mu}_{-i},\vec{Y}) \propto P(\mu_{i+1}|\mu_i)P(Y_i|\mu_i)P(\mu_i|\mu_{i-1}) \propto exp\Big[-\Big(\frac{1}{2\sigma^2}\Big)\Big((\mu_{i+1}-\mu_i)^2+(Y_i-\mu_i)^2+(\mu_i-\mu_{i-1})^2\Big)\Big]$$

Using *hint C*

$$P(\mu_i|\vec{\mu}_{-i},\vec{Y}) \propto exp\Big[-\frac{(\mu_i-(Y_i+\mu_{i-1}+\mu_{i+1})/3)^2}{2\sigma^2/3}\Big] \;\sim\; N\Big((\frac{Y_i+\mu_{i-1}+\mu_{i+1}}{3}),\frac{\sigma^2}{3}\Big)$$