

# CS-Lab4Group6-report

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## Statement of contribution

This lab was done by 3 people in the group.

The primary responsibility for the 1st problem was taken on by Filip, while the 2nd problem was solved by Jaskirat with Dinuke's help.

## Assignment1

We start out by finding the constant for this function. If we realize this is a Gamma function finding the constant is quite easy. Looking at  $x^5 e^{-x}$ ,  $x > 0$ . If we have a look at the pdf for a Gamma PDF we have  $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ . Looking at this we realise that for  $\beta = 1$  and  $\alpha = 6$  we get  $x^5 e^{-x}$  and we now just need to find the constant multiplier. Since we now know  $\beta$  and  $\alpha$  we can use the constant multiplier formula for a gamma function which we know to be

$$\frac{\beta^\alpha}{\Gamma(\alpha)}$$

, by plugging in our values we get:  $\frac{1^6}{\Gamma(6)} = \frac{1}{5!}$

This gives  $f(x) = \frac{1}{5!} x^5 e^{-x}$

### 1.1

In this assignment we will be creating a Metropolis-Hastings algorithm. It works as follows.

First we select a initial  $\theta_0$  value and then we do the following. For  $i = 1, \dots, n$  we draw a candidate value  $\theta^* \sim g(\theta^* | \theta_{i-1})$ . After this we calculate our  $\alpha$  value.

$$\alpha = \frac{g(\theta^*)/g(\theta^* | \theta_{i-1})}{g(\theta_{i-1})/g(\theta_{i-1} | \theta^*)} = \frac{g(\theta^*)g(\theta_{i-1} | \theta^*)}{g(\theta_{i-1})g(\theta^* | \theta_{i-1})}$$

If  $\alpha \geq 1$  we accept the  $\theta^*$  and set that to our current  $\theta_i$ . For  $0 < \alpha < 1$  we accept the  $\theta$  and assign  $\theta_i = \theta^*$  with the probability  $\alpha$ . If none of these happens we reject  $\theta^*$  and we assign  $\theta_i = \theta^*$  with the probability  $1 - \alpha$ . This is what our code below does.

Bellow our code can be seen. We first create a function that contains our targeted distribution given in the assignment. Then we create the main function which takes three arguments , the numbers of steps , a starting point and probability P. The rest of the code does the things stated above.

```
library(ggplot2)
```

```
## Warning: package 'ggplot2' was built under R version 4.1.1
```

```
library(coda)
```

```
## Warning: package 'coda' was built under R version 4.1.2
```

```
# Target distribution function
```

```
pdf_function <- function(x) x^5*exp(-x)
```

```

# Initiate Metropolis-Hastings algorithm
Metro_hast <- function(steps ,starting_point, p ){
  #variable to keep track of our current point.
  current_point <- starting_point

  #Loop to do our calulations
  for(i in 2:steps){
    currentX <- current_point[i-1]
    X_to_evaluate <- rlnorm(1, meanlog = log(currentX) , p)
    Uniform_var <- runif(1)

    ratio_numerator <- (pdf_function(X_to_evaluate)* dlnorm(currentX, meanlog = log(X_to_evaluate), p) )
    denominator <- (pdf_function(currentX)*dlnorm(X_to_evaluate,meanlog=log(X_to_evaluate), p))

    alpha <- min(1 , ratio_numerator / denominator)

    if(Uniform_var <= alpha)current_point[i] <- X_to_evaluate

    else current_point[i] <- currentX
  }
  # Return the points
  current_point
}

```

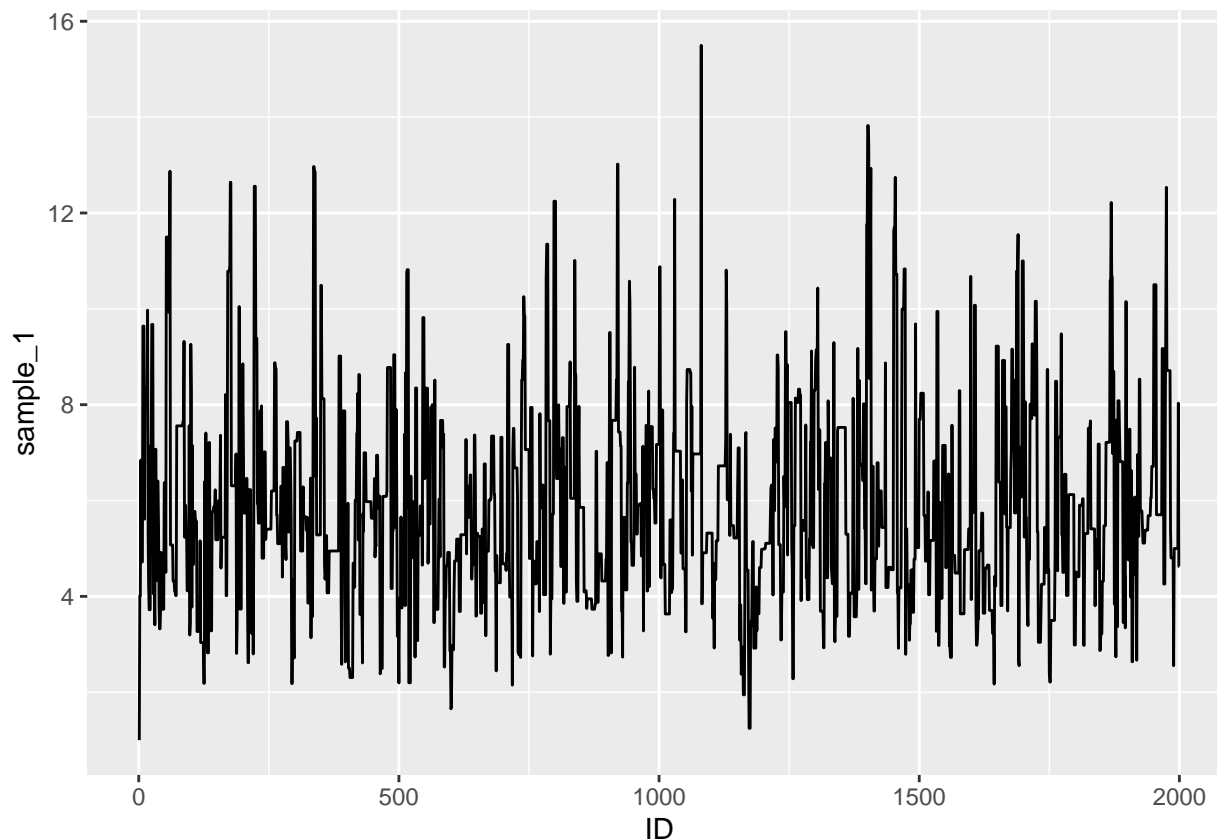
Now we want to visualize the data generated by our function.

```

sample_1<-Metro_hast(2000,1,1)
df_1 <- as.data.frame(sample_1)
df_1$ID <- 1:2000

ggplot(df_1 , aes(x = ID , y = sample_1))+geom_line()

```



We set the numbers of steps to 2000, the starting point to one and the probability to one and get the plot above. Looking at the plot it does look like the mcmc is converging.

##1.2

```
Metro_hast_chi <- function(steps ,starting_point ){
  #create our vectors
  current_point <- rep(starting_point, steps)

  for(i in 2:steps) {

    currentX <- current_point[i-1]

    X_to_evaluate <- rchisq(1, df=floor(currentX +1))

    Uniform_var <- runif(1)

    ratio_numerator <- pdf_function(X_to_evaluate)*dchisq(currentX, df=floor(X_to_evaluate+1))

    denominator <- pdf_function(currentX)*dchisq(X_to_evaluate, df=floor(currentX+1))

    alpha <- min(1 , ratio_numerator / denominator)

    if(Uniform_var <= alpha)current_point[i] <- X_to_evaluate
    else current_point[i] <- currentX
  }
}
```

```

    }
    current_point

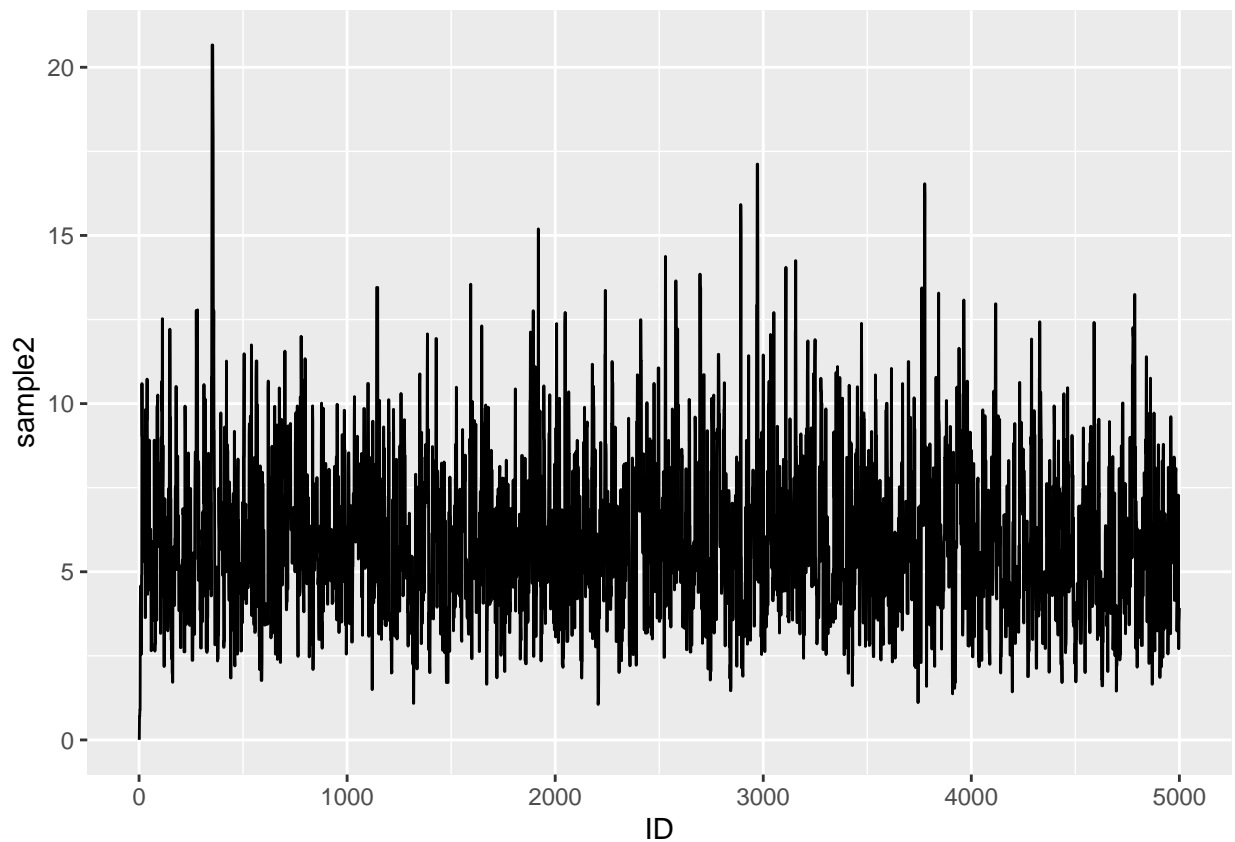
}

sample2<-Metro_hast_chi(5000,rchisq(1,1))

df_2 <- as.data.frame(sample2)
df_2$ID <- 1:5000

ggplot(df_2 , aes(x = ID , y = sample2))+geom_line()

```



Above we create another Metropolis-Hastings algorithm that uses  $\chi^2([X_t + 1])$  as the proposal distribution. The code works in the same way as our function in 1.1. Looking at the plot it does seem like this mcm also converges. ### 1.3

```

one_to_ten <- 1:10

Matrix_1<-matrix(,10,2000)
set.seed(1234)
for(i in 1:10){
  Matrix_1[one_to_ten,] <- Metro_hast_chi(2000, i )
}

Gelman_list <- list()

```

```

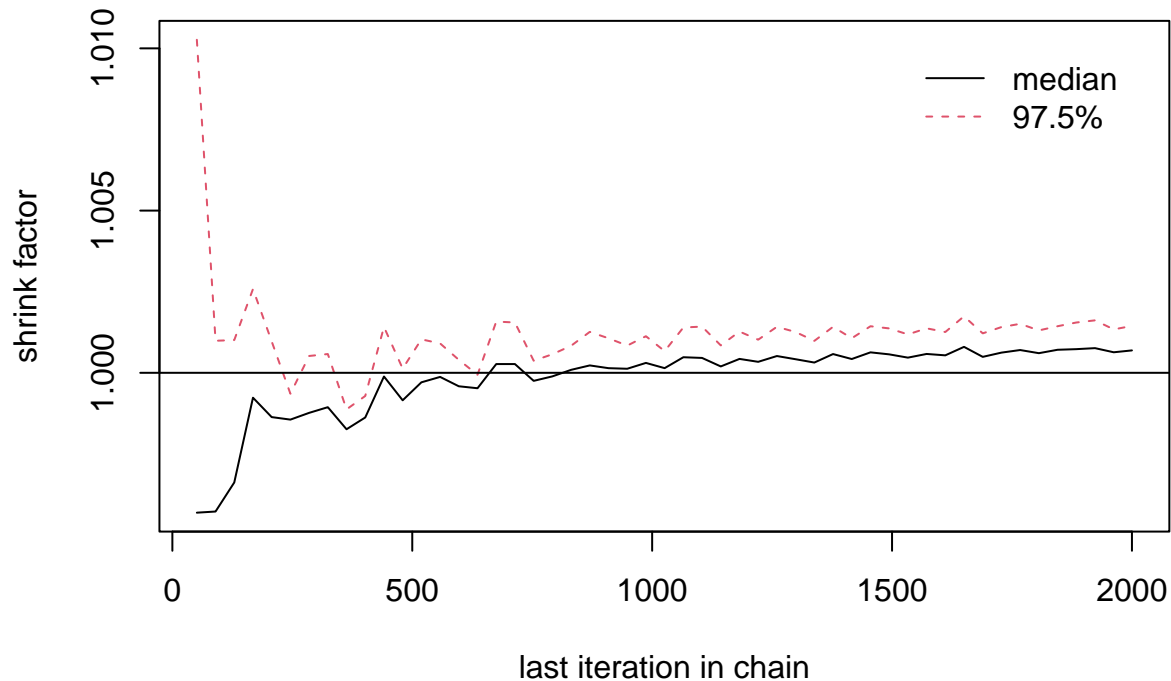
for (i in 1:10) {

  Gelman_list[[i]]<- as.mcmc(Matrix_1[i,])

}
gelman.diag(Gelman_list, confidence = 0.95, transform=FALSE, autoburnin=TRUE,
            multivariate=TRUE)

## Potential scale reduction factors:
##
##      Point est. Upper C.I.
## [1,]          1          1
gelman.plot(Gelman_list, bin.width = 10, max.bins = 50,
            confidence = 0.95)

```



Above we see the Gelman–Rubin method and get a value of one which should be considered very good. This means that the variance within chains and between chains variance are equal. Looking at the plot showing a Gelman plot we can also see that the chains seems to converge after around 300 steps. After this point there is a few upticks but it seems to mostly converge.

## 1.4

We now want to investigate the following integral.  $\int_0^\infty xf(x)dx$  where  $f(x) = \frac{1}{5!}x^5e^{-x}$ . We can immediately realise that  $\int_0^\infty f(x)dx = 1$  since integrating over a Probability density function gives us the cumulative distribution function which always equals one. We now set  $g(x) = x$  and obtain the following expression.

$$\int_0^\infty g(x)f(x)dx = E[g(X)] = \frac{1}{n} \sum_{i=1}^n E[X_i] = \frac{1}{n} \sum_{i=1}^n x_i$$

We now calculate the mean from the data generated by our two functions.

```
set.seed(1234)
draw_1 <- Metro_hast(100000, rlnorm(1, 0, 1), 1)
mean(draw_1)
```

```
## [1] 6.004613
```

```
set.seed(1234)
draw_2 <- Metro_hast_chi(100000, rchisq(1, floor(1)))
mean(draw_2)
```

```
## [1] 6.001918
```

We can see here that both functions has a mean  $\approx 6$  for a big sample. If we increase the size of the samples we should get even closer to 6.

**1.5** The Gamma pdf is as follows:  $f_x(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$  To find the expected value we do the following  $E[X] = \int_0^\infty x f(x) dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-\beta x} dx$ . We now set  $t = \beta x$  giving us a easier expression.  $\frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \left(\frac{t}{\beta}\right)^\alpha e^{-t} \frac{dt}{\beta} = \frac{\beta^\alpha}{\beta^{\alpha+1} \Gamma(\alpha)} \int_0^\infty t^\alpha e^{-t} dt = \frac{\Gamma(\alpha+1)}{\beta \Gamma(\alpha)}$  Here we now use the definition of the Gamma function to get:  $\frac{\alpha \Gamma(\alpha)}{\beta \Gamma(\alpha)} = \frac{\alpha}{\beta} \square$

Now we plug in your values for  $\alpha$  and  $\beta$  and get  $\frac{6}{1} = 6$ . This is in line with what we got in 1.4

## Assignment2

We are given the following:

$n = \text{number of observations}$

$\vec{\mu} = (\mu_1, \dots, \mu_n)$  are unknown parameters

$Y_i \sim N(\mu_i, \text{variance} = 0.2), i = 1, \dots, n$

*Prior* :  $p(\mu_1) = 1$  ;  $p(\mu_{i+1}|\mu_i) = N(\mu_i, 0.2), i = 1, \dots, (n-1)$

We'll be interested in deriving the posterior i.e.  $P(\mu|Y)$  using the Bayes' theorem as follows:

*posterior* = *likelihood* \* *prior*

$$P(\vec{\mu}|Y) = \frac{P(Y|\vec{\mu}) * P(\vec{\mu})}{\int_{\mu} P(Y|\vec{\mu})P(\vec{\mu})d\mu}$$

Since the denominator is constant w.r.t  $\mu$ , we can drop it in favour of proportionality

*posterior*  $\propto$  *likelihood* \* *prior*

$$P(\vec{\mu}|Y) \propto P(Y|\vec{\mu}) * P(\vec{\mu})$$

First we define the likelihood:

$$L(Y_i|\mu_i) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(Y_i - \mu_i)^2\right)$$

Since, we are only interested in the terms dependent on the parameter  $\mu$  we can drop the term at the start of the expression and the likelihood then becomes:

$$L(Y_i|\mu_i) \propto \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (Y_i - \mu_i)^2\right)$$

Similarly, now for the prior using chain rule,

$$\begin{aligned} P(\vec{\mu}) &= P(\mu_1) \cdot P(\mu_2|\mu_1) \cdot P(\mu_3|\mu_2) \dots P(\mu_n|\mu_{n-1}) \quad \dots(a) \\ &= \prod_{i=1}^{n-1} P(\mu_{i+1}|\mu_i) \quad \sim N(\mu_i, \sigma_{\mu_i}^2) \\ &= \prod_{i=1}^{n-1} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma_{\mu}^2}(\mu_{i+1} - \mu_i)^2\right) \\ &\propto \exp\left(-\frac{1}{2\sigma_{\mu}^2} \sum_{i=1}^{n-1} (\mu_{i+1} - \mu_i)^2\right) \end{aligned}$$

Hence, we can now write our posterior as the following:

$$\begin{aligned}
P(\vec{\mu}|Y) &\propto \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (Y_i - \mu_i)^2\right) * \exp\left(-\frac{1}{2\sigma_\mu^2} \sum_{i=1}^{n-1} (\mu_{i+1} - \mu_i)^2\right) \\
&\propto \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^n (Y_i - \mu_i)^2 - \frac{1}{2\sigma_\mu^2} \sum_{i=1}^{n-1} (\mu_{i+1} - \mu_i)^2\right] \quad \dots(b)
\end{aligned}$$

Since,  $\sigma$  is given to be same everywhere in the problem we can use common notation across the expression.

Moving to the next part to find conditional probability  $(\mu_k|\mu_{-k})$ , we know that by the definition of conditional probability, the joint probability of the prior,  $P(\vec{\mu})$ , can be expanded into the conditional components of each individual  $\mu_k$  and can be expressed as

$$P(\vec{\mu}) \propto P(\mu_k|\vec{\mu}) * P(\mu_1, \dots, \mu_{k-1}, \mu_{k+1}, \dots, \mu_n)$$

Or in a more condensed form as,

$$\begin{aligned}
P(\mu_i|\vec{\mu}_{-i}, \vec{Y}) &= \frac{P(\vec{\mu}|\vec{Y})}{P(\vec{\mu}_{-i}|\vec{Y})} \quad \dots(c) \\
&= \frac{P(\vec{\mu}|\vec{Y})}{\int P(\vec{\mu}|\vec{Y}) d\mu}
\end{aligned}$$

We now solve for  $\mu_1$

$$\begin{aligned}
P(\mu_1|\vec{\mu}_{-1}, \vec{Y}) &= \frac{P(\vec{\mu}, \vec{Y})}{\int P(\vec{\mu}, \vec{Y}) d\mu_1} \\
&= \frac{\prod_{i=2}^n P(Y_i|\mu_i) \prod_{i=3}^n P(\mu_i|\mu_{i-1}) P(\mu_2|\mu_1) P(Y_1|\mu_1) P(\mu_1)}{\prod_{i=2}^n P(Y_i|\mu_i) \prod_{i=3}^n P(\mu_i|\mu_{i-1}) \int P(\mu_2|\mu_1) P(Y_1|\mu_1) P(\mu_1) d\mu_1}
\end{aligned}$$

Common product terms cancel out in numerator and denominator, and the integral is constant w.r.t.  $\mu_1$

$$\begin{aligned}
P(\mu_1|\vec{\mu}_{-1}, \vec{Y}) &\propto P(\mu_2|\mu_1) P(Y_1|\mu_1) P(\mu_1) \\
&\propto \exp\left(-\frac{1}{2\sigma^2} (\mu_2 - \mu_1)^2\right) * \exp\left(-\frac{1}{2\sigma^2} (Y_1 - \mu_1)^2\right) * 1 \\
&\propto \exp\left(-\left(\frac{1}{2\sigma^2}\right) \left((\mu_2 - \mu_1)^2 + (Y_1 - \mu_1)^2\right)\right)
\end{aligned}$$

using *Hint B*

$$P(\mu_1|\vec{\mu}_{-1}, \vec{Y}) \propto \exp\left(-\frac{(\mu_1 - (Y_1 + \mu_2)/2)^2}{\sigma^2}\right) \simeq N\left(\frac{Y_1 + \mu_2}{2}, \frac{\sigma^2}{2}\right)$$

Similarly, we now check the case for  $\mu_n$ . Here we use the posterior we defined in (b) and the relationship we established in (c)

$$\begin{aligned}
P(\mu_n|\vec{\mu}_{-n}, \vec{Y}) &= \frac{P(\vec{\mu}|\vec{Y})}{P(\vec{\mu}_{-n}|\vec{Y})} \\
&\propto \frac{\exp\left(-\sum_{i=1}^{n-1} \frac{(\mu_{i+1} - \mu_i)^2}{2\sigma^2} - \sum_{i=1}^n \frac{(Y_i - \mu_i)^2}{2\sigma^2}\right)}{\exp\left(-\sum_{i=1}^{n-2} \frac{(\mu_{i+1} - \mu_i)^2}{2\sigma^2} - \sum_{i=1}^{n-1} \frac{(Y_i - \mu_i)^2}{2\sigma^2}\right)} \\
&\propto \exp\left(-\left(\frac{1}{2\sigma^2}\right) \left((\mu_n - \mu_{n-1})^2 + (Y_n - \mu_n)^2\right)\right)
\end{aligned}$$



again using *hint B*

$$P(\mu_n | \vec{\mu}_{-n}, \vec{Y}) \propto \exp\left(-\frac{(\mu_n - (Y_n + \mu_{n-1})/2)^2}{\sigma^2}\right) \sim N\left(\frac{Y_n + \mu_{n-1}}{2}, \frac{\sigma^2}{2}\right)$$

Now for the tedious case of  $\mu_i$  where  $i \notin (1, n)$

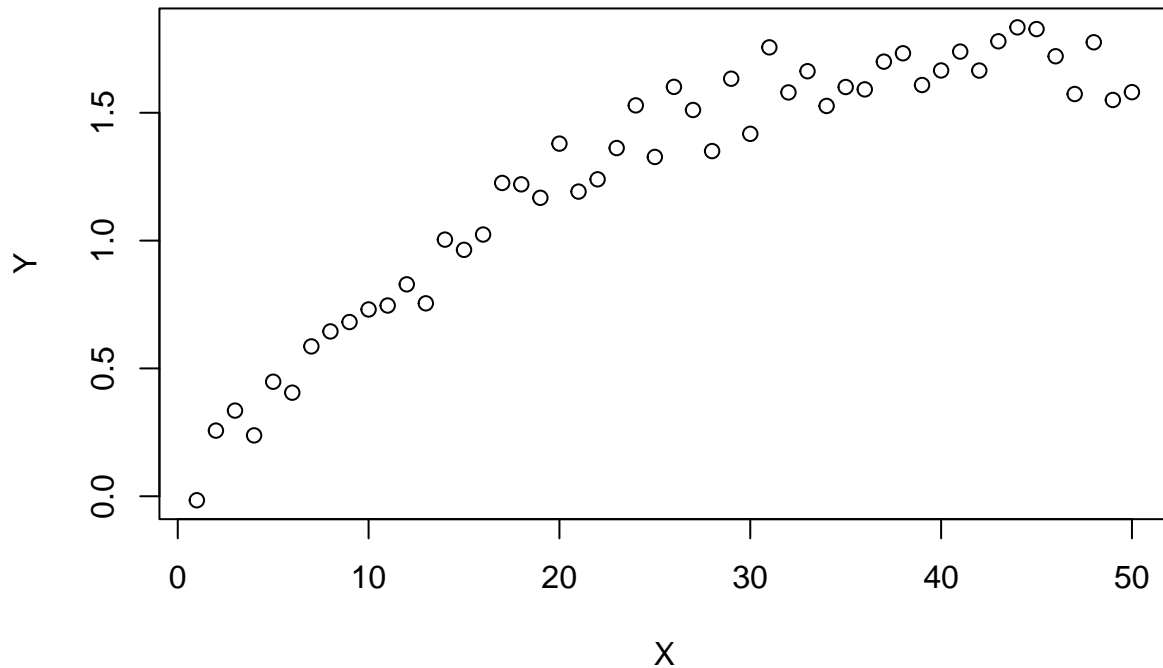
$$P(\mu_i | \vec{\mu}_{-i}, \vec{Y}) \propto P(\mu_{i+1} | \mu_i) P(Y_i | \mu_i) P(\mu_i | \mu_{i-1}) \propto \exp\left[-\left(\frac{1}{2\sigma^2}\right)\left((\mu_{i+1} - \mu_i)^2 + (Y_i - \mu_i)^2 + (\mu_i - \mu_{i-1})^2\right)\right]$$

Using *hint C*

$$P(\mu_i | \vec{\mu}_{-i}, \vec{Y}) \propto \exp\left[-\frac{(\mu_i - (Y_i + \mu_{i-1} + \mu_{i+1})/3)^2}{2\sigma^2/3}\right] \sim N\left(\frac{Y_i + \mu_{i-1} + \mu_{i+1}}{3}, \frac{\sigma^2}{3}\right)$$

We now use these derivations to design a Gibbs sampler which will estimate  $\vec{\mu}$  for 1000 values of  $\mu$

```
#load the data
load("chemical.RData")
#converting data into a DF
chemical_df <- data.frame(X, Y)
#initial plot of data to understand distribution
plot(X, Y)
```



```
#3 inputs to sampler: # of time steps, Y to be estimated, known sigma
mc_gibbs <- function(step, Y, sig) {
  mu_i <- 0
```

```

#initialize matrix to store results of sampler in each iteration
result_matrix <- matrix(0, nrow = 50, ncol = step)
for (tstep in 1:(step-1)) {
  for (i in 1:50) {
    #Case for mu_1
    if (i==1) {
      mu_i <- mean(result_matrix[i+1,tstep],Y[i])
      result_matrix[i,tstep+1] <- rnorm(1, mean = mu_i, sd = sig/sqrt(2))
    }
    #Case for mu_n
    else if (i == 50) {
      mu_i <- mean(result_matrix[i-1,tstep+1],Y[i])
      result_matrix[i,tstep+1] <- rnorm(1, mean = mu_i, sd = sig/sqrt(2))
    }
    #Case for mu_i
    else {
      mu_i <- mean(Y[i],result_matrix[i-1,tstep+1], result_matrix[i+1,tstep])
      result_matrix[i,tstep+1] <- rnorm(1, mean = mu_i, sd = sig/sqrt(3))
    }
  }
}
return(result_matrix)
}

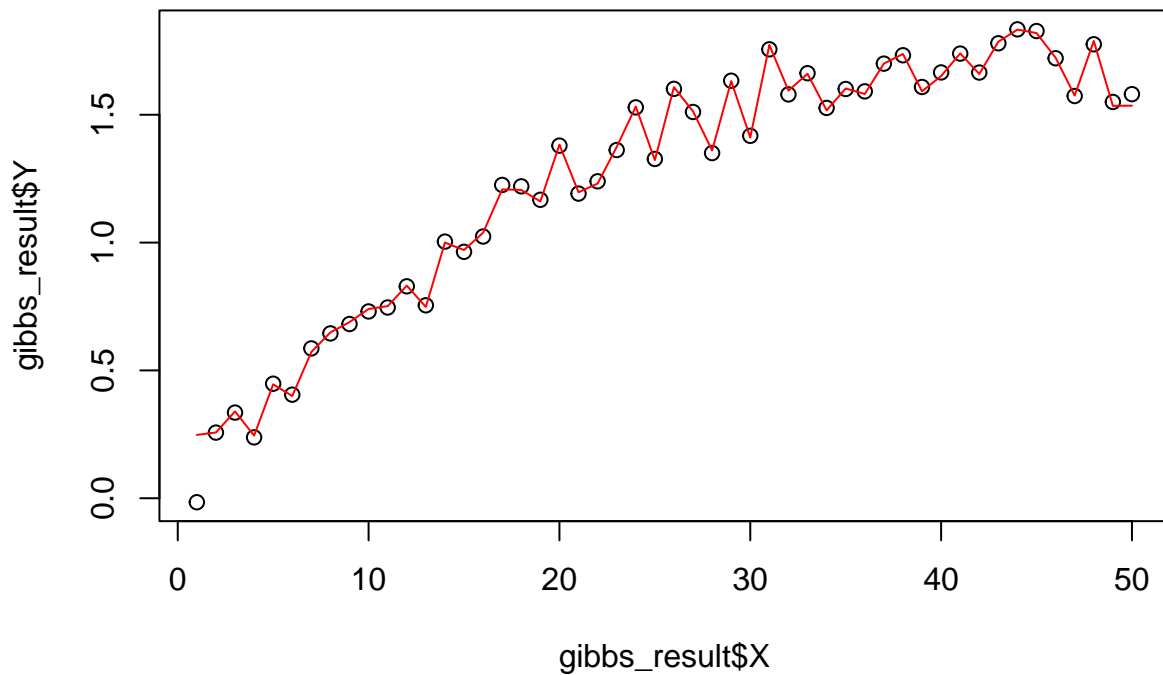
```

Plotting the results of the sampler:

```

#function call with 1000 steps and variance 0.2
gibbs_output <- mc_gibbs(1000, chemical_df$Y, sqrt(0.2))
#append expected values of mu to original DF
gibbs_result <- cbind(chemical_df, rowMeans(gibbs_output))
names(gibbs_result) <- c("X", "Y", "expected_mu")
#Plot original points
plot(gibbs_result$X, gibbs_result$Y)
lines(gibbs_result$expected_mu, col = "red")

```

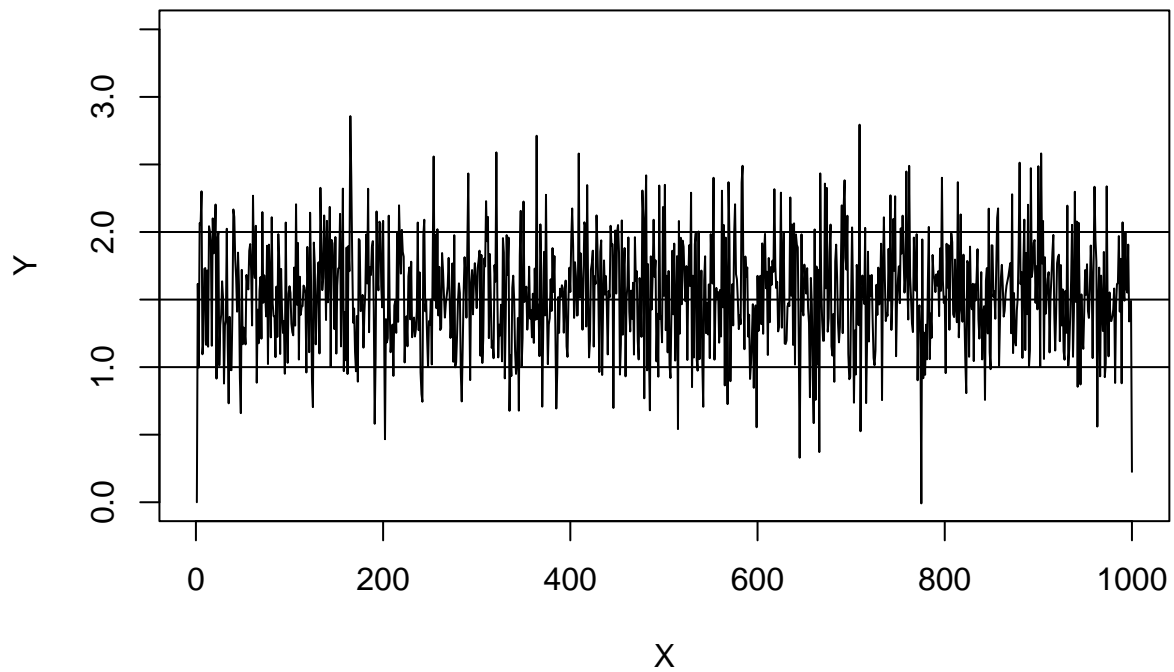


```
#Expected values as a line
```

The result shows that our expected values have followed a similar result as the original data. Implying that **not much improvement has happened** in terms of reducing the measurement noise of the concentration. But the noise reduction at the start and end is quite pronounced, especially at the start, thus, suggesting that our burning period may have been very short.

Trace Plot:

```
vN <- 1:ncol(gibbs_output)
vX <- gibbs_output
plot(vN,vX[50,],pch=19,cex=0.3,col="black",xlab="X",ylab="Y", ylim=c(0,3.5), type = 'l')
abline(h=1)
abline(h=2)
abline(h=1.5)
```



Our above conclusion is supported by the trace plot, where we see that **the chain doesn't converge** and keeps producing noisy values, but the burning period is evidently quite small. But this can also be because the data set that we are working with is **quite small (only 50 observations)**. To expect a good reduction in noise, we would ideally want a larger data set.