CSCI 443 Lecture 16

Slide 19, 20, 21

Power law $\alpha = 2$, $x_{min} = 1$

$$\underbrace{f}_{\text{PDF}}(x;\ \overset{\alpha}{2},\ \overset{x_{min}}{1}) = \frac{\alpha - 1}{x_{min}} \left(\frac{x}{x_{min}}\right)^{-\alpha} = \frac{2-1}{1}\left(\frac{x}{1}\right)^{-2} = \frac{1}{x^2}$$

$$E[X] = \int_{x=1}^{\infty} x\, f(x;2,1)\, dx = \int_{x=1}^{\infty} x \cdot \frac{1}{x^2}\, dx = \int_{x=1}^{\infty} \frac{1}{x}\, dx = \ln x \Big|_{x=1}^{\infty}$$

$$E[X] = \ln \infty - \ln 1 = \infty - 0 \qquad \boxed{E[X] = \infty}$$

$$Var[X] = E[(x-\mu)^2] = E[X^2] - \mu^2 = \int_{x=1}^{\infty} x^2 \cdot \frac{1}{x^2}\, dx - \mu^2 = \int_1^{\infty} dx - \mu^2$$

$$Var[X] = x\Big|_1^{\infty} - \mu^2 = \infty - \infty = ?$$

~~the variance is not defined.~~

Depends on which term dominates

$\cancel{Var[X] = E[x^2] - \mu^2 = \int_1 x^2 f(x;2,1)\, dx - \mu^2}$

$$x\Big|_1^{\infty} - \ln x\Big|_1^{\infty} = ?$$

(see slide 21)

CSCI 443 Lecture 16
slide 22
Power law $\alpha = 3$

$$f(x; 3, 1) = \frac{\alpha - 1}{x_{min}} \left(\frac{x}{x_{min}} \right)^{-\alpha} = \frac{3-1}{1} \left(\frac{x}{1} \right)^{-3} = 2x^{-3}$$

$$F(\infty) = \int_{x=1}^{\infty} \frac{2}{x^3} dx = -\frac{1}{x^2} \Big|_{x=1}^{\infty} = 0 - (-1) = 1 \checkmark$$

This is a valid distribution, because

a) $f(x; 3, 1) \geq 0$ for all $x \geq x_{min}$

b) $\int_{x_{min}}^{\infty} f(x; 3, 1) = 1$. Total area under the PDF = 1.

$$E[X] = \int_{x=1}^{\infty} x f(x; \alpha = 3, x_{min} = 1) dx = \int_{x=1}^{\infty} x \cdot \frac{2}{x^3} dx = -\frac{2}{x} \Big|_{x=1}^{\infty}$$

$$E[X] = -\frac{2}{\infty} - \left(-\frac{2}{1} \right) = \boxed{2 = E[X]}$$

$$Var[X] = E[(X - \mu)^2] = \int_{x=1}^{\infty} x^2 f(x) dx = \int_{x=1}^{\infty} x^2 \frac{2}{x^3} dx - \mu^2$$

$$Var[X] = \int_{x=1}^{\infty} \frac{2}{x} dx - \mu^2 = 2 \ln x \Big|_{x=1}^{\infty} - 2^2 = (2 \ln \infty - 0) - 4$$

$$\boxed{Var[X] = \infty}$$