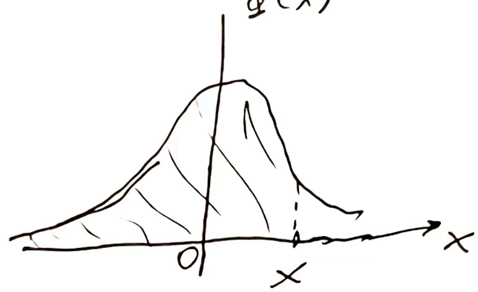
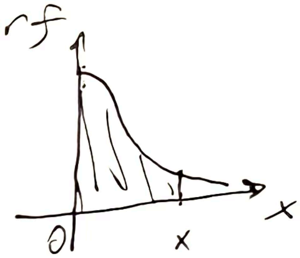
Relationship between erf and $\Phi(x)$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \qquad (1)$$

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt \qquad (2)$$

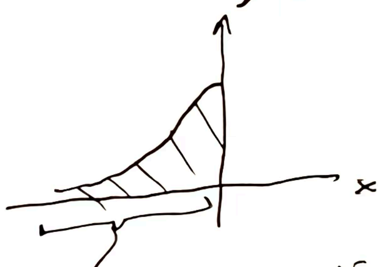
$\Phi(x)$ is





If $x > 0$, we can break the distribution into two halves

$\Phi(x)$ below $x = 0$.



Because the pdf is symmetric about zero, $\frac{1}{2}$ of the probability mass is below zero and one half is above.

Thus for $x > 0$,

$$\cancel{\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt}$$

$$\Phi(x) = \frac{1}{2} + \underbrace{\frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2}\, dt}_{\text{Looks more like } \mathrm{erf}(x)}$$

let $u = \frac{t}{\sqrt{2}}$, $t = \sqrt{2}\,u$

$du = \frac{dt}{\sqrt{2}}$, $dt = \sqrt{2}\,du$

For $x \geq 0$

For $x \geq 0$, $\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_{t=0}^{t=x} e^{-u^2} \sqrt{2}\, du$

For $x \geq 0$, $\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2}u=0}^{\sqrt{2}u=x} e^{-u^2} \sqrt{2}\, du$

For $x \geq 0$, $\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sqrt{2} \int_{u=0}^{u=\frac{x}{\sqrt{2}}} e^{-u^2}\, du$

$$\frac{\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)}{2} = \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x}{\sqrt{2}}} e^{-u^2}\, du$$

For $x \geq 0$, $\Phi(x) = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)$

To evaluate $P[X \leq x]$ when $X \sim N(\mu, \sigma)$

For $x \geq 0$

$F(\cancel{X}\, x) = P[X < x] = \Phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{\frac{x-\mu}{\sigma}}{\sqrt{2}}\right)\right)$

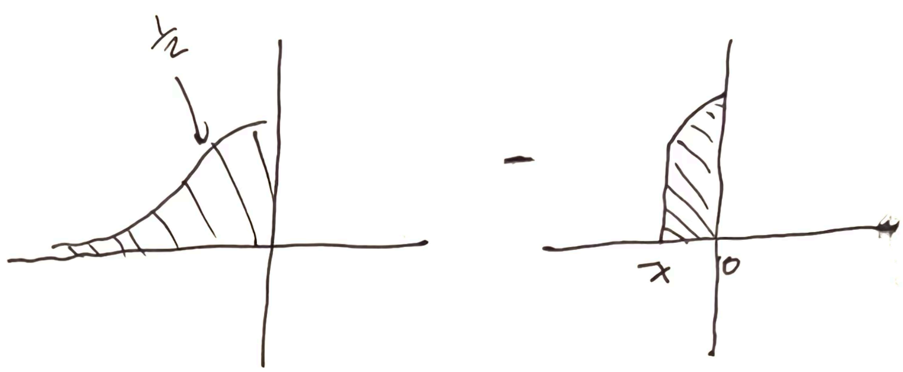
lower case x

$F(x)$

But what about for $x < 0$?

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

$$\mathrm{erf}(-x) = \frac{2}{\sqrt{\pi}} \int_{t=0}^{t=-x} e^{-t^2}\,dt = -\mathrm{erf}(x)$$



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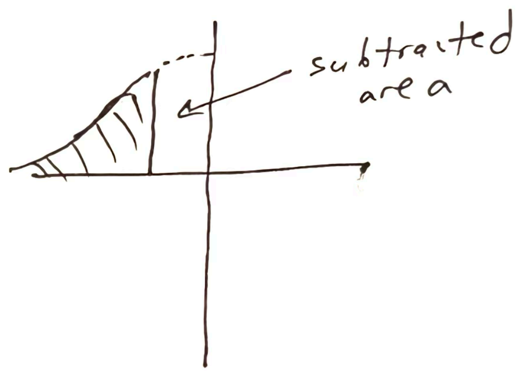

so

$$F(x) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{x-\mu}{\sqrt{2}}\right)\right)$$

applies for all x and not just $x > 0$.